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MAXIMAL SUBGROUPS OF NON-TORSION GRIGORCHUK-GUPTA-SIDKI GROUPS

DOMINIK FRANCOEUR AND ANITHA THILLAISUNDARAM

ABSTRACT. A Grigorchuk-Gupta-Sidki (GGS-)group is a subgroup of the automorphism group of the p -adic tree for an odd prime p , generated by one rooted automorphism and one directed automorphism. Pervova proved that all torsion GGS-groups do not have maximal subgroups of infinite index. Here we extend the result to non-torsion GGS-groups, which include the weakly regular branch, but not branch, GGS-group.

1. INTRODUCTION

The automorphism group of an infinite spherically homogeneous rooted tree is well established as a source of interesting finitely generated infinite groups, such as finitely generated groups of intermediate word growth, finitely generated infinite torsion groups, finitely generated amenable but not elementary amenable groups, and finitely generated just infinite groups. Early constructions were produced by Grigorchuk [10] and Gupta and Sidki [13] in the 1980s, which then led to a generalised family of so-called GGS-groups.

An important type of subgroup of the automorphism group of an infinite spherically homogeneous rooted tree is one having subnormal subgroup structure similar to the corresponding structure in the full group of automorphisms of the tree. These subgroups are termed branch groups; see Section 2 for the definition.

The study of maximal subgroups of finitely generated branch groups began with the work of Pervova [15, 16], who proved that the Grigorchuk groups and the torsion GGS-groups do not contain maximal subgroups of infinite index. Bondarenko [3] gave the first example of a finitely generated branch group that does have maximal subgroups of infinite index. His method does not apply to groups acting on the binary and ternary trees. However, recently Francoeur and Garrido [8] provided the first examples of finitely generated branch groups, acting on the binary tree, with maximal subgroups of infinite index. Their examples are the non-torsion Sunic groups. For an extensive introduction to the subject of maximal subgroups of finitely generated branch groups, we refer the reader to [8].

Francoeur and Garrido (see [6]) have further shown that certain non-torsion GGS-groups, namely the generalised Fabrykowski-Gupta groups, one for each odd prime p , do not have maximal subgroups of infinite index. In this paper, we extend this result to all non-torsion GGS-groups. We recall that a GGS-group acts on the p -adic tree for p an odd prime, with generators a and b , where a cyclically permutes the p maximal subtrees rooted at the first-level vertices, whereas b fixes the first-level vertices pointwise and is recursively defined by the tuple $(a^{e_1}, \dots, a^{e_{p-1}}, b)$ which corresponds to the action of b on the maximal subtrees, for some exponents $e_1, \dots, e_{p-1} \in \mathbb{F}_p$.

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Theorem 1.1. *Let G be a GGS-group acting on the p -adic tree, for p an odd prime. Then every maximal subgroup of G is of finite index. Furthermore, if G is branch, then every maximal subgroup is normal and of index p in G .*

There is only one GGS-group that is weakly branch but not branch; this is the GGS-group \mathcal{G} where the automorphism b is defined by (a, \dots, a, b) . For the group \mathcal{G} , it turns out that there are maximal subgroups that are not normal, nor of index p in \mathcal{G} . Also there are infinitely many maximal subgroups; see Proposition 5.11.

We recall that two groups are *commensurable* if they have isomorphic subgroups of finite index. The following result is a consequence of the work of Grigorchuk and Wilson [12], and the proof follows exactly as in [1, Cor. 1.3]:

Corollary 1.2. *Let G be a branch GGS-group acting on the p -adic tree, for p an odd prime. If H is a group commensurable with G , then every maximal subgroup of H has finite index in H .*

Organisation. Section 2 contains preliminary material on groups acting on the p -adic tree. In Section 3, we formally define the GGS-groups and state some of their basic properties. In Section 4 we set up the scene to prove Theorem 1.1, whose proof we complete in Section 5.

2. PRELIMINARIES

In the present section we recall the notion of branch groups and establish prerequisites for the rest of the paper. For more information, see [2, 11].

2.1. The p -adic tree and its automorphisms. Let T be the p -adic tree, for an odd prime p . Let $X = \{1, 2, \dots, p\}$ be an alphabet on p letters. The set of vertices of T can be identified with the free monoid X^* , and we will freely use this identification without special mention. The root of T corresponds to the empty word \emptyset , and for each word $v \in X^*$ and letter x , an edge connects v to vx . There is a natural length function on X^* , and the words w of length $|w| = n$, representing vertices that are at distance n from the root, form the n th layer of the tree. The boundary ∂T consisting of all infinite simple rooted paths is in one-to-one correspondence with the p -adic integers.

For u a vertex, we write T_u for the full rooted subtree of T that has its root at u and includes all vertices v with u a prefix of v . For any two vertices u and v the subtrees T_u and T_v are isomorphic under the map that deletes the prefix u and replaces it by the prefix v . We refer to this identification as the *natural identification of subtrees* and write T_n to denote the subtree rooted at a generic vertex of level n .

We observe that every automorphism of T fixes the root and that the orbits of $\text{Aut } T$ on the vertices of the tree T are precisely its layers. Let $f \in \text{Aut } T$ be an automorphism of T . The image of a vertex u under f is denoted by u^f . For a vertex u , considered as a word over X , and a letter $x \in X$ we have $(ux)^f = u^f x'$ where $x' \in X$ is uniquely determined by u and f . This gives a permutation $f(u)$ of X so that

$$(ux)^f = u^f x^{f(u)}.$$

The automorphism f is called *rooted* if $f(u) = 1$ for $u \neq \emptyset$. The automorphism f is called *directed*, with directed path $\ell \in \partial T$, if the support $\{u \mid f(u) \neq 1\}$ of its labelling is infinite and contains only vertices at distance 1 from ℓ .

The *section* of f at a vertex u is the unique automorphism f_u of $T \cong T_{|u|}$ given by the condition $(uv)^f = u^f v^{f_u}$ for $v \in X^*$.

2.2. Subgroups of $\text{Aut } T$. Let $G \leq \text{Aut } T$. The *vertex stabiliser* $\text{St}_G(u)$ is the subgroup consisting of elements in G that fix the vertex u . For $n \in \mathbb{N}$, the *n th level stabiliser* $\text{St}_G(n) = \bigcap_{|v|=n} \text{St}_G(v)$ is the subgroup of automorphisms that fix all vertices at level n . Note that elements in $\text{St}_G(n)$ fix all vertices up to level n and that $\text{St}_G(n)$ has finite index in G .

The full automorphism group $\text{Aut } T$ is a profinite group:

$$\text{Aut } T = \varprojlim_{n \rightarrow \infty} \text{Aut } T_{[n]},$$

where $T_{[n]}$ denotes the subtree of T on the finitely many vertices up to level n . The topology of $\text{Aut } T$ is defined by the open subgroups $\text{St}_{\text{Aut } T}(n)$, for $n \in \mathbb{N}$. For $G \leq \text{Aut } T$, we say that the subgroup G has the *congruence subgroup property* if for every subgroup H of finite index in G , there exists some $n \in \mathbb{N}$ such that $\text{St}_G(n) \leq H$.

For $n \in \mathbb{N}$, every $g \in \text{St}_{\text{Aut } T}(n)$ can be identified with a collection g_1, \dots, g_{p^n} of elements of $\text{Aut } T_n$, where p^n is the number of vertices at level n . Denoting the vertices of T at level n by u_1, \dots, u_{p^n} , there is a natural isomorphism

$$\text{St}_{\text{Aut } T}(n) \cong \prod_{i=1}^{p^n} \text{Aut } T_{u_i} \cong \text{Aut } T_n \times \cdots \times \text{Aut } T_n.$$

Recall that $\text{Aut } T_n$ is isomorphic to $\text{Aut } T$ via the natural identification of subtrees. Therefore the decomposition (g_1, \dots, g_{p^n}) of g defines an embedding

$$\psi_n: \text{St}_{\text{Aut } T}(n) \rightarrow \prod_{i=1}^{p^n} \text{Aut } T_{u_i} \cong \text{Aut } T \times \cdots \times \text{Aut } T.$$

For convenience, we will write $\psi = \psi_1$.

For $\omega \in X^*$, we further define

$$\varphi_\omega: \text{St}_{\text{Aut } T}(\omega) \rightarrow \text{Aut } T_\omega \cong \text{Aut } T$$

to be the natural restriction of $f \in \text{St}_{\text{Aut } T}(\omega)$ to its section f_ω .

We write $G_u = \varphi_u(\text{St}_G(u))$ for the restriction of the vertex stabiliser $\text{St}_G(u)$ to the subtree rooted at a vertex u . We say that G is *self-similar* if G_u is contained in G for every vertex u , and we say that G is *fractal* if G_u equals G for every vertex u , after the natural identification of subtrees.

The subgroup $\text{Rist}_G(u)$, consisting of all automorphisms in G that fix all vertices v of T not having u as a prefix, is called the *rigid vertex stabiliser* of u in G . The *rigid n th level stabiliser* is the product

$$\text{Rist}_G(n) = \prod_{i=1}^{p^n} \text{Rist}_G(u_i) \trianglelefteq G$$

of the rigid vertex stabilisers of the vertices u_1, \dots, u_{p^n} at level n .

Let G be a subgroup of $\text{Aut } T$ acting *spherically transitively*, i.e. transitively on every layer of T . Then G is a *branch group*, if $\text{Rist}_G(n)$ has finite index in G for every $n \in \mathbb{N}$. If, in addition, the group G is self-similar and $1 \neq K \leq G$ with $K \times \cdots \times K \subseteq \psi(K \cap \text{St}_G(1))$ and $|G : K| < \infty$, then G is said to be *regular branch over K* . If the condition $|G : K| < \infty$ in the previous definition is omitted, then G is said to be *weakly regular branch over K* .

3. THE GGS-GROUPS

By a we denote the rooted automorphism, corresponding to the p -cycle $(1\ 2 \cdots p) \in \text{Sym}(p)$, that cyclically permutes the vertices u_1, \dots, u_p at the first level. Recall the coordinate map

$$\psi: \text{St}_{\text{Aut } T}(1) \rightarrow \text{Aut } T_{u_1} \times \cdots \times \text{Aut } T_{u_p} \cong \text{Aut } T \times \cdots \times \text{Aut } T.$$

Given a non-zero vector $\mathbf{e} = (e_1, e_2, \dots, e_{p-1}) \in (\mathbb{F}_p)^{p-1}$, we recursively define a directed automorphism $b \in \text{St}_{\text{Aut}T}(1)$ via

$$\psi(b) = (a^{e_1}, a^{e_2}, \dots, a^{e_{p-1}}, b).$$

We call the subgroup $G = G_{\mathbf{e}} = \langle a, b \rangle$ of $\text{Aut}T$ the *GGS-group* associated to the defining vector \mathbf{e} . We observe that $\langle a \rangle \cong \langle b \rangle \cong C_p$ are cyclic groups of order p . Hence, whenever we refer to an exponent of a or of b , it goes without saying that it is an element of \mathbb{F}_p .

A GGS-group G acts spherically transitively on the tree T , and every section of every element of G is contained in G . Moreover, a GGS-group G is fractal.

If $G_{\mathbf{e}} = \langle a, b \rangle$ is a GGS-group corresponding to the defining vector \mathbf{e} , then G is an infinite p -group if and only if $\sum_{j=1}^{p-1} e_j = 0$ in \mathbb{F}_p ; compare [11, 17].

We also call $G = \langle a, b \rangle$ a *generalised Fabrykowski-Gupta group* if

$$\psi(b) = (a, 1, \dots, 1, b);$$

see [6]. By [5, Thm. 2.16], it follows that a GGS-group whose defining vector contains only one non-trivial element is conjugate in $\text{Aut}T$ to a generalised Fabrykowski-Gupta group.

We write $\mathcal{G} = \langle a, b \rangle$ with $\psi(b) = (a, \dots, a, b)$, for the GGS-group arising from the constant defining vector $(1, \dots, 1)$. It is known that \mathcal{G} is weakly regular branch [5, Lem. 4.2] but not branch [4, Thm. 3.7].

For all other GGS-groups $G \neq \mathcal{G}$, we have from [5] that G is regular branch over $\gamma_3(G)$. Further, from [4], a GGS-group G has the congruence subgroup property and is just infinite if and only if $G \neq \mathcal{G}$; we recall that an infinite group G is *just infinite* if all its proper quotients are finite.

The following result is useful.

Lemma 3.1. *For G a GGS-group, let $g \in G'$ and write $\psi(g) = (g_1, \dots, g_p)$. Then $g_1 g_2 \cdots g_p \in G'$.*

Proof. It suffices to show the statement for $g = [a, b]$. As $[a, b] = (b^{-1})^a b$, we have that $\psi([a, b]) = (b^{-1} a^{e_1}, a^{e_2 - e_1}, \dots, a^{e_{p-1} - e_{p-2}}, a^{-e_{p-1}} b)$, and the result follows. \square

As evident in the proof of the above result, all actions of group elements on the tree T will be taken on the right.

3.1. Length function. Here we recall the following items from [16]: the abelianisation G/G' of a GGS-group G and a natural length function on elements of G .

Let $G = \langle a, b \rangle$ be a GGS-group acting on the p -adic tree T . We consider

$$H = \langle \hat{a}, \hat{b} \mid \hat{a}^p = \hat{b}^p = 1 \rangle,$$

the free product $\langle \hat{a} \rangle * \langle \hat{b} \rangle$ of cyclic groups $\langle \hat{a} \rangle \cong C_p$ and $\langle \hat{b} \rangle \cong C_p$. There is a unique epimorphism $\pi: H \rightarrow G$ such that $\hat{a} \mapsto a$ and $\hat{b} \mapsto b$, which induces an isomorphism from $H/H' \cong \langle \hat{a} \rangle \times \langle \hat{b} \rangle \cong C_p^2$ to G/G' ; see [16].

Let $h \in H$. Recall that h can be uniquely represented in the form

$$h = \hat{a}^{\alpha_1} \cdot \hat{b}^{\beta_1} \cdot \hat{a}^{\alpha_2} \cdot \hat{b}^{\beta_2} \cdots \hat{a}^{\alpha_m} \cdot \hat{b}^{\beta_m} \cdot \hat{a}^{\alpha_{m+1}},$$

where $m \in \mathbb{N} \cup \{0\}$, $\alpha_1, \dots, \alpha_{m+1} \in \mathbb{F}_p$ with $\alpha_i \neq 0$ for $i \in \{2, \dots, m\}$, and $\beta_1, \dots, \beta_m \in \mathbb{F}_p^*$.

We denote by $|h| = m$ the *length* of h , with respect to the factor $\langle \hat{b} \rangle$. Clearly, for $h_1, h_2 \in H$ we have

$$(3.1) \quad |h_1 h_2| \leq |h_1| + |h_2|.$$

Then, for $G = \langle a, b \rangle$ a GGS-group, the *length* of $g \in G$ is

$$|g| = \min\{|h| \mid h \in \pi^{-1}(g)\}.$$

Based on (3.1), we deduce that for $g_1, g_2 \in G$,

$$(3.2) \quad |g_1 g_2| \leq |g_1| + |g_2|.$$

We end this section with the following result from [1], specialised to the setting of GGS-groups:

Lemma 3.2. [1, Lem. 4.4] *Let G be a GGS-group, and $g \in \text{St}_G(1)$ with $\psi(g) = (g_1, \dots, g_p)$. Then $\sum_{j=1}^p |g_j| \leq |g|$, and $|g_j| \leq \frac{|g|+1}{2}$ for each $j \in \{1, \dots, p\}$.*

In particular, if $|g| > 1$ then $|g_j| < |g|$ for every $j \in \{1, \dots, p\}$.

3.2. The GGS-group defined by the constant vector. Due to the group \mathcal{G} not being branch, some further properties of \mathcal{G} need to be established for the next section.

By [5, Lem. 4.2], the group \mathcal{G} is weakly regular branch over K' , where $K = \langle (ba^{-1})^{a^i} \mid i \in \{0, \dots, p-1\} \rangle^{\mathcal{G}}$.

Lemma 3.3. *Let \mathcal{G} and K be as above. Then K' is a subdirect product of $K \times \cdots \times K$ and $\psi(K'') \geq \gamma_3(K) \times \cdots \times \gamma_3(K)$.*

Proof. This is similar to [9, Proof of Prop. 17]. We first show that K' is a subdirect product of $K \times \cdots \times K$. Indeed, writing $y_0 = ba^{-1}$ and $y_i = (ba^{-1})^{a^i}$ for $1 \leq i \leq p-1$, we have

$$\psi([y_0, y_1]) = \psi((b^{-2})^{a^{-1}} b^{a^{-2}} b) = (1, \dots, 1, y_2, (y_0^{-1} y_1^{-1})^a, y_1),$$

hence the result.

Thus, for every $k_1 \in K$ there is some $g_1 \in K'$ with $\psi(g_1) = (k_1, *, \dots, *)$. As \mathcal{G} is weakly regular branch over K' , for every $k_2 \in K'$, there is some $g_2 \in K'$ with $\psi(g_2) = (k_2, 1, \dots, 1)$. Therefore, $\psi([g_1, g_2]) = ([k_1, k_2], 1, \dots, 1)$, and the second part of the lemma follows. \square

Proposition 3.4. *Let \mathcal{G} be the GGS-group defined by the constant vector. Then every proper quotient of \mathcal{G} is virtually nilpotent and has maximal subgroups only of finite index.*

Proof. For the first statement, using [7, Thm. 4.10], it suffices to show that \mathcal{G}/K'' is virtually nilpotent. As \mathcal{G}'/K'' is of finite index in \mathcal{G}/K'' , it suffices to show that \mathcal{G}'/K'' is nilpotent. Since $\psi(\mathcal{G}') \leq K \times \cdots \times K$ by [5, Lem. 4.2(iii)], the result follows from the second part of the previous lemma.

The second statement is immediate, because every virtually nilpotent group has maximal subgroups only of finite index: indeed, a nilpotent group G has $\Phi(G)$ containing G' , see [18]. Then every maximal subgroup of a nilpotent group G is normal and hence of finite index. As the property of having only maximal subgroups of finite index passes to finite extensions [6, Cor. 5.1.3], we have that virtually nilpotent groups have maximal subgroups only of finite index. \square

4. DENSE SUBGROUPS

In the present section we lay out the strategy for proving Theorem 1.1, where it suffices to restrict to non-torsion GGS-groups in light of [16].

We recall that a subgroup H of G is *prodense* if $G = NH$ for every non-trivial normal subgroup N of G . Let G be a finitely generated group such that every proper quotient of G has maximal subgroups only of finite index. Then by [7, Prop. 2.21], every maximal subgroup

of infinite index in G is prodense and every proper prodense subgroup is contained in a maximal subgroup of infinite index.

We now recall a key result concerning prodense subgroups of weakly branch groups.

Proposition 4.1. [7, Lem. 3.1 and Thm. 3.3] *Let T be a spherically homogeneous rooted tree and G a weakly branch group acting on T . Suppose that every proper quotient of G has maximal subgroups only of finite index. If H is a prodense subgroup of G , then H_u is a prodense subgroup of G_u , for every vertex u of T . Furthermore, if $M < G$ is a maximal subgroup of G of infinite index, then M_u is a maximal subgroup of G_u , for every vertex u of T .*

Note that every proper quotient of a branch GGS-group has maximal subgroups only of finite index, since proper quotients of branch groups are virtually abelian; cf. [7, Prop. 2.22]. This, and Proposition 3.4, allows us to use the above proposition to show that non-torsion GGS-groups G do not possess proper prodense subgroups: for a prodense subgroup M of G , we will show that there exists a vertex u of T such that $M_u = G$. This then shows that M is not proper.

For G a non-torsion GGS-group and M a prodense subgroup of G , it remains to show that there exists a vertex u of T such that $M_u = G$. The following result was proved in a more general setting in [1], but it was stated for *dense* instead of prodense subgroups M . However, the proof, and hence the result, still holds for prodense subgroups.

Proposition 4.2. [1, Prop. 5.4] *Let $G = \langle a, b \rangle$ be a GGS-group that is not conjugate to a generalised Fabrykowski-Gupta group and let M be a prodense subgroup of G . If $b \in M$, then there exists a vertex u of T such that $M_u = G$.*

It follows from the above result that it suffices to show that $b \in M_u$ for M a prodense subgroup of G and u a vertex of T . This will be done in the next section.

5. OBTAINING b .

For $G = \langle a, b \rangle$ a non-torsion GGS-group with defining vector $\mathbf{e} = (e_1, \dots, e_{p-1})$, we write $\lambda := \sum_{i=1}^{p-1} e_i \neq 0$. Further, we write $g \equiv_{G'} h$ when $gh^{-1} \in G'$, for elements $g, h \in G$.

Lemma 5.1. *Let $G = \langle a, b \rangle$ be a non-torsion GGS-group and let $g \in G$ be such that $g \equiv_{G'} b^t$ for some $t \neq 0$. For $u \in X$, write $\varphi_u(g) \equiv_{G'} a^{n_u} b^{m_u}$ for some $n_u, m_u \in \mathbb{F}_p$. Then, exactly one of the following cases is true.*

- (1) *There exist $u \in X$ and $j_0 \neq 0$ such that $\varphi_u(g) \equiv_{G'} b^{j_0}$.*
- (2) *If $m_u \neq 0$ for some $u \in X$, then $n_u \neq 0$. Furthermore, there exist at least two vertices $u_1, u_2 \in X$ such that $n_{u_k} \neq \lambda m_{u_k}$ for $k \in \{1, 2\}$, and there exist at least two vertices $v_1, v_2 \in X$ such that $m_{v_1}, m_{v_2} \neq 0$.*

Proof. As $g \equiv_{G'} b^t$, it follows that

$$g = (b^{j_1})^{a^{l_1}} (b^{j_2})^{a^{l_2}} \dots (b^{j_n})^{a^{l_n}}$$

for some $n \in \mathbb{N}$, $l_1, \dots, l_n \in \mathbb{F}_p$ with $l_k \neq l_{k+1}$ for $1 \leq k \leq n-1$, and $j_1, \dots, j_n \in \mathbb{F}_p^*$ with $\sum_{k=1}^n j_k = t$ in \mathbb{F}_p .

Now, for $u \in X$, we have $\varphi_u(g) \equiv_{G'} a^{n_u} b^{m_u}$, where

$$(5.1) \quad n_u = \sum_{\substack{i=0 \\ i \neq u}}^{p-1} e_{u-i} m_i = \sum_{j=1}^{p-1} e_j m_{u-j},$$

and

$$m_u = \sum j_k$$

where the sum is taken over k such that $l_k = u$.

It is clear that Cases 1 and 2 are mutually exclusive. Thus, it suffices to show that if Case 1 does not hold, then Case 2 does. So we assume now that Case 1 does not hold. Suppose that $n_u = \lambda m_u$ for all $u \in X$. Then

$$\sum_{j=1}^{p-1} e_j m_{u-j} - \lambda m_u = 0 \quad \text{for all } u \in X.$$

This is equivalent to

$$\begin{aligned} (e_1 \quad \cdots \quad e_{p-1} \quad -\lambda) \cdot \begin{pmatrix} m_p \\ m_{p-1} \\ \vdots \\ m_2 \\ m_1 \end{pmatrix} &= (e_1 \quad \cdots \quad e_{p-1} \quad -\lambda) \cdot \begin{pmatrix} m_{p-1} \\ m_{p-2} \\ \vdots \\ m_1 \\ m_p \end{pmatrix} = \cdots \\ &\cdots = (e_1 \quad \cdots \quad e_{p-1} \quad -\lambda) \cdot \begin{pmatrix} m_1 \\ m_p \\ \vdots \\ m_3 \\ m_2 \end{pmatrix} = 0. \end{aligned}$$

In other words,

$$(5.2) \quad \begin{pmatrix} m_p \\ m_{p-1} \\ \vdots \\ m_2 \\ m_1 \end{pmatrix}, \begin{pmatrix} m_{p-1} \\ m_{p-2} \\ \vdots \\ m_1 \\ m_p \end{pmatrix}, \dots, \begin{pmatrix} m_1 \\ m_p \\ \vdots \\ m_3 \\ m_2 \end{pmatrix}$$

are all in the subspace orthogonal to $(e_1, \dots, e_{p-1}, -\lambda)$. However the vectors in (5.2) form the rows of a circulant matrix. The rank of this circulant matrix is less than p if and only if $\sum_{i=1}^p m_i = 0$; compare [5, Lem. 2.7(i)]. However $\sum_{i=1}^p m_i = \sum_{k=1}^n j_k = t \neq 0$. Thus, there is at least one vertex $u_1 \in X$ such that $n_{u_1} \neq \lambda m_{u_1}$.

Suppose that there is only one such u_1 . Then $n_u = \lambda m_u$ for all $u \neq u_1$, hence using $\sum_{u \in X} n_u = \lambda \sum_{u \in X} m_u$, we have

$$n_{u_1} + \lambda \sum_{u \neq u_1} m_u = \lambda \sum_{u \in X} m_u,$$

which yields $n_{u_1} = \lambda m_{u_1}$; a contradiction.

For the final statement, clearly $m_u \neq 0$ for at least one $u \in X$, since $\sum_{u \in X} m_u = t \neq 0$. Since we are not in Case 1, it follows that $n_u \neq 0$. From (5.1), the result follows. \square

Lemma 5.2. *For G a non-torsion GGS-group, let $x \in G$ be such that $x \equiv_{G'} b^t$ for some $t \neq 0$, and write $\psi(x) = (x_1, x_2, \dots, x_p)$. If x is in Case 2 of Lemma 5.1 and if $|x| = 2\mu$ for some $\mu \in \mathbb{N}$, then there exists $u \in X$ such that $x_u \neq 1$ and $|x_u| < \mu = \frac{|x|}{2}$.*

Proof. Suppose for the sake of contradiction that this is not the case. By Lemma 3.2 and using the fact that x is in Case 2 of Lemma 5.1, there exist two vertices $v_1, v_2 \in X$ such that $|x_{v_1}| = |x_{v_2}| = \frac{|x|}{2}$ and $x_u = 1$ for all $u \in X \setminus \{v_1, v_2\}$. This means that

$$x = (b^{i_1})^{a^{v_1}} (b^{j_1})^{a^{v_2}} \dots (b^{i_\mu})^{a^{v_1}} (b^{j_\mu})^{a^{v_2}}$$

for some $i_1, \dots, i_\mu, j_1, \dots, j_\mu \in \mathbb{F}_p$. Since the result is true for x if and only if it is true for $x^{a^{-v_1}}$, we can suppose without loss of generality that

$$x = b^{i_1} (b^{j_1})^{a^v} \dots b^{i_\mu} (b^{j_\mu})^{a^v}$$

for some $v \in X$. Let us set $i = \sum_{k=1}^\mu i_k$ and $j = \sum_{k=1}^\mu j_k$. Notice that $i + j = t$. Furthermore, since x is in Case 2 of Lemma 5.1, there must exist two vertices with non-zero powers of b , and our assumptions then imply that i and j are both non-zero.

For all $u \in X \setminus \{p, v\}$, the elements $\varphi_u(b)$ and $\varphi_u(b^{a^v})$ are both powers of a , and thus commute. Therefore, for $u \in X \setminus \{p, v\}$, we have

$$x_u = \varphi_u(x) = \varphi_u(b^i (b^j)^{a^v}) = a^{ie_u + je_{u-v}}.$$

Since $x_u = 1$ for all $u \in X \setminus \{p, v\}$, we have $ie_u + je_{u-v} = 0$ for all $u \in X \setminus \{p, v\}$. In other words, for all $2 \leq k \leq p-1$, we have $ie_{kv} + je_{(k-1)v} = 0$. By induction, we see that for all $1 \leq k \leq p-1$, we have $e_{kv} = r^{k-1}e_v$, where $r = -\frac{j}{i}$, which is well-defined, since $i \neq 0$. Notice that $r \neq 0$, since $j \neq 0$, and $r \neq 1$, since $i + j = t \neq 0$. Therefore, we have

$$\lambda = \sum_{k=1}^{p-1} e_k = \sum_{k=1}^{p-1} e_{kv} = \sum_{k=1}^{p-1} r^{k-1}e_v = e_v \sum_{k=0}^{p-2} r^k = e_v \frac{r^{p-1} - 1}{r - 1} = 0,$$

a contradiction. The conclusion follows. \square

Lemma 5.3. *Let $G = \langle a, b \rangle$ be a non-torsion GGS-group with $\lambda = \sum_{i=1}^{p-1} e_i$. Let $g \in G$ be such that $g \equiv_{G'} a^i b^j$ with $i \neq 0$ and $j \in \mathbb{F}_p$, and write $g = a^i \cdot \psi^{-1}((g_1, g_2, \dots, g_p))$. Then, for all $u \in X$, we have $\varphi_u(g^p) \equiv_{G'} a^{\lambda j} b^j$, with $|\varphi_u(g^p)| \leq \sum_{k=1}^p |g_k| \leq |g|$.*

Proof. This follows as in [6, Lem. 7.2.2]. \square

We will now establish the following in several steps.

Proposition 5.4. *Let $G = \langle a, b \rangle$ be a non-torsion GGS-group, and let H be a prodense subgroup. Then there exists $v_0 \in X^*$ such that $b \in H_{v_0}$.*

First, let us consider the set

$$U = \left\{ g \in \bigsqcup_{v \in X^*} H_v \mid g \equiv_{G'} a^{\lambda i} b^i \text{ for some } i \neq 0 \right\}.$$

Since H is a prodense subgroup of G and G' is a non-trivial normal subgroup in G , the set U is non-empty. Let $y \in U$ be an element of minimal length in U . Let $v \in X^*$ be such that $y \in H_v$. By Proposition 4.1, the subgroup H_v is prodense in $G_v = G$. Hence if we prove the above proposition for H_v , we will also have proved it for H . Thus, without loss of generality, we may assume that v is the root of the tree X^* . Further, in light of Lemma 5.3, we have “ y ”, that is, an element of U of minimal length, everywhere.

Now let us consider the set

$$V = \left\{ g \in \bigsqcup_{v \in X^*} H_v \mid g \equiv_{G'} b^i \text{ for some } i \neq 0 \right\},$$

and let $x \in V$ be an element of minimal length in V . Likewise, such an element exists as V is non-empty by the prodensity of H . Let $w \in X^*$ be such that $x \in H_w$. As before, we may assume that w is the root of the tree. Hence, using the fact that we get y everywhere, we have y and x at the root.

If $|x| = 1$, then $x = a^{-j}b^i a^j$ with $i \neq 0$ and $j \in \{1, \dots, p\}$. Thus, we have that $\varphi_j(x) = b^i$. As i is invertible modulo p , we obtain $b \in H_j$.

We will now show that $|x|$ must be equal to 1. The case $|x| = 0$ is not possible, so we assume that $|x| > 1$. In this case, there cannot exist $u \in X$ such that $\varphi_u(x) \equiv_{G'} b^j$ with $j \neq 0$. Indeed, since $|\varphi_u(x)| < |x|$, this would contradict the minimality of x ; compare Lemma 3.2. Therefore according to Lemma 5.1, there exists $u \in X$ such that $\varphi_u(x) \equiv_{G'} a^{i_u} b^{j_u}$ with $i_u \neq 0$ and $i_u \neq \lambda j_u$. For this $u \in X$ we write $x_u = \varphi_u(x)$.

Furthermore, also by Lemma 5.1, there exist two vertices $u_1, u_2 \in X$ such that $j_{u_1}, j_{u_2} \neq 0$ with $i_{u_1}, i_{u_2} \neq 0$. Now by Lemma 5.3, we see that $\varphi_v(x_{u_i}^p) \in U$ for $i \in \{1, 2\}$ and $v \in X$, and thus

$$|y| \leq |\varphi_v(x_{u_i}^p)| \leq |x_{u_i}|.$$

Hence, using $\sum_{u \in X} |x_u| \leq |x|$, it follows that

$$(5.3) \quad 2|y| \leq |x|.$$

Remark 5.5. We observe that if there is a $v \in X^*$ with $a^n \in H_v$ for some $n \neq 0$, then writing $y \equiv_{G'} a^{\lambda j} b^j$ for some $j \neq 0$, the product of y with $a^{-\lambda j}$ yields

$$|x| \leq |y|,$$

which contradicts (5.3), and so we are done in this case. Hence, in what follows, we will assume freely without special mention that $a \notin H_v$ for all $v \in X^*$.

Lemma 5.6. *In the set-up above, in particular, assuming that $|x| > 1$, let $v \in X^*$ be any vertex and let $z \in H_v$. If $z \neq 1$, then $|z| \geq |y|$.*

Proof. Without loss of generality, we may assume that $z \notin \text{St}_G(1)$. Indeed, otherwise, since z is non-trivial, there must exist some $w \in X^*$ such that $z \in \text{St}_G(w)$ and $\varphi_w(z) \notin \text{St}_G(1)$. As we have $|z| \geq |\varphi_w(z)|$, it is sufficient to show that $|\varphi_w(z)| \geq |y|$. Thus, we may assume that $z \notin \text{St}_G(1)$. Hence, we have $z \equiv_{G'} a^{i_z} b^{j_z}$ for some $i_z, j_z \in \mathbb{F}_p$ with $i_z \neq 0$.

Notice that for all $u \in X$, we have $\varphi_u(z^p) \equiv_{G'} a^{\lambda j_z} b^{j_z}$ by Lemma 5.3. In particular, if $j_z \neq 0$, by the minimality of $|y|$, we must have $|\varphi_u(z^p)| \geq |y|$. Since Lemma 5.3 also says that $|z| \geq |\varphi_u(z^p)|$, the conclusion follows in this case.

Thus, it only remains to treat the case where $j_z = 0$. Let us assume for the sake of contradiction that we have $|z| < |y|$, and let $j_y \neq 0$ be such that $y \equiv_{G'} a^{\lambda j_y} b^{j_y}$. Since $j_z = 0$, there exists $k \in \{1, 2, \dots, p-1\}$ such that $z^k y \equiv_{G'} b^{j_y}$. By Lemmata 3.2 and 5.3 together with (3.2), for all $u \in X$ we have $|\varphi_u(z^k y)| \leq |z| + \frac{|y|+1}{2} < 2|y|$. If there were some $u \in X$ with $\varphi_u(z^k y) \equiv_{G'} b^l$ for some $l \neq 0$, by the minimality of $|x|$ we would have that $|x| \leq |\varphi_u(z^k y)| < 2|y|$, a contradiction to (5.3). Thus, by Lemma 5.1, there must exist two vertices $u_1, u_2 \in X$ such that $\varphi_{u_l}(z^k y) \equiv_{G'} a^{i_{u_l}} b^{j_{u_l}}$ with $i_{u_l}, j_{u_l} \neq 0$, for $l \in \{1, 2\}$.

Let us write $y = a^{i_y} \cdot \psi^{-1}((y_1, \dots, y_p))$ and $z = a^{i_z} \cdot \psi^{-1}((z_1, \dots, z_p))$. For any $u \in X$, we have, remembering that we act on the tree T on the right,

$$\begin{aligned} \varphi_u(z^k y) &= \varphi_u((z_1, \dots, z_p)^{a^{-i_z}} (z_1, \dots, z_p)^{a^{-2i_z}} \cdots (z_1, \dots, z_p)^{a^{-ki_z}} (y_1, \dots, y_p)) \\ &= z_{u+i_z} z_{u+2i_z} \cdots z_{u+ki_z} y_u. \end{aligned}$$

We also have

$$\begin{aligned}\varphi_u(z^{k-p}y) &= \varphi_u((z_1^{-1}, \dots, z_p^{-1})(z_1^{-1}, \dots, z_p^{-1})^{a^{iz}} \cdots (z_1^{-1}, \dots, z_p^{-1})^{a^{(p-k-1)iz}}(y_1, \dots, y_p)) \\ &= z_u^{-1} z_{u-iz}^{-1} \cdots z_{u+(k+1)iz}^{-1} y_u.\end{aligned}$$

As $\varphi_u(z^p) \equiv_{G'} 1$, we have

$$\varphi_u(z^{k-p}y) \equiv_{G'} \varphi_u(z^p)\varphi_u(z^{k-p}y) \equiv_{G'} \varphi_u(z^k y).$$

In particular, this means that for $u \in \{u_1, u_2\}$, we have

$$\varphi_u(z^{k-p}y) \equiv_{G'} \varphi_u(z^k y) \equiv_{G'} a^{iu} b^{ju}$$

with $i_u, j_u \neq 0$. We have seen above that this implies that $|\varphi_u(z^{k-p}y)|, |\varphi_u(z^k y)| \geq |y|$.

Since $\sum_{n=1}^p |y_n| \leq |y|$, there must exist some $u \in \{u_1, u_2\}$ such that $|y_u| \leq \frac{|y|}{2}$. For this u , we have

$$|\varphi_u(z^k y)| \leq |y_u| + \sum_{n=1}^k |z_{u+ni_z}|$$

and

$$|\varphi_u(z^{k-p}y)| \leq |y_u| + \sum_{n=0}^{p-k-1} |z_{u-ni_z}|.$$

Since $\sum_{n=0}^{p-1} |z_{u-ni_z}| \leq |z|$, we must have that one of $\sum_{n=1}^k |z_{u+ni_z}|$ or $\sum_{n=0}^{p-k-1} |z_{u-ni_z}|$ is at most $\frac{|z|}{2}$. Consequently, we either have

$$|\varphi_u(z^k y)| \leq |y_u| + \frac{|z|}{2} \leq \frac{|y| + |z|}{2} < |y|$$

or $|\varphi_u(z^{k-p}y)| < |y|$. In either case, we get a contradiction, as required. \square

The following result is essential. First let us write $y = a^i \cdot \psi^{-1}((y_1, \dots, y_p)) \equiv_{G'} a^{\lambda j} b^j$, for some $j \neq 0$ and $i = \lambda j$. By abuse of notation, we will still write y for $\varphi_u(y^p)$ for any $u \in X$; compare Lemma 5.3. Additionally, for notational convenience, we sometimes write z_v or $(z_1, \dots, z_p)_v$ for $\varphi_v(z)$, where $z = \psi^{-1}((z_1, \dots, z_p)) \in \text{St}_G(v)$ and $v \in X^* \setminus \{\emptyset\}$.

Lemma 5.7. *In the set-up above, in particular assuming that $|x| > 1$, there is an element $w \in H_v$, for some $v \in X^*$, with $w \equiv_{G'} a^n b^m$ for $n \neq \lambda m$, $n \neq 0$ and $|w| = |y|$.*

Proof. Let

$$W = \left\{ g \in \bigsqcup_{v \in X^*} H_v \mid g \equiv_{G'} a^{\lambda n} b^m \text{ for some } m, n \text{ with } n \neq 0, n \neq \lambda m \right\}$$

and let $w \in W$ be an element of minimal length. We need to show that $|w| = |y|$. Let us first remark that $|w| \leq \frac{|x|}{2}$. Indeed, since $|x| > 1$, it follows from Lemma 5.1 that there exist $u_1, u_2 \in X$ such that $x_{u_1}, x_{u_2} \in W$. Since $|x_{u_1}| + |x_{u_2}| \leq |x|$, we conclude that $|w| \leq \min\{|x_{u_1}|, |x_{u_2}|\} \leq \frac{|x|}{2}$.

Let us write $w := x_{u_1} = a^n \cdot \psi^{-1}((w_1, \dots, w_p))$. There exists $k \in \{1, 2, \dots, p-1\}$ with $y^k \equiv_{G'} y^{k-p} \equiv_{G'} a^n b^{jk}$. Using y , or more precisely the projection of some power of y to the

vertex where w lives, to cancel a^n in w yields

$$\begin{aligned}
 |(y^{-k}w)_v| &= |((y_1^{-1}, \dots, y_p^{-1})(y_1^{-1}, \dots, y_p^{-1})^{a^i} \cdots (y_1^{-1}, \dots, y_p^{-1})^{a^{(k-1)i}}(w_1, \dots, w_p))_v| \\
 &= |y_v^{-1}y_{v-i}^{-1} \cdots y_{v-(k-1)i}^{-1}w_v| \\
 (5.4) \quad &\leq |y_v^{-1}y_{v-i}^{-1} \cdots y_{v-(k-1)i}^{-1}| + |w_v| \\
 &\leq \sum_{d=0}^{k-1} |y_{v-di}| + |w_v|
 \end{aligned}$$

and

$$\begin{aligned}
 |(y^{p-k}w)_v| &= |y_{v-(p-1)i} \cdots y_{v-i}y_v y_{v-i}^{-1}y_{v-i}^{-1} \cdots y_{v-(k-1)i}^{-1}w_v| \\
 (5.5) \quad &\leq |y_{v-(p-1)i} \cdots y_{v-ki}| + |w_v| \\
 &\leq \sum_{d=k}^{p-1} |y_{v-di}| + |w_v|
 \end{aligned}$$

for $v \in X$.

Since $\sum_{d=0}^{p-1} |y_{v-di}| \leq |y|$ for $v \in X$, we have

$$\min \left\{ |(y^{-k}w)_v|, |(y^{p-k}w)_v| \right\} \leq \frac{|y|}{2} + |w_v| \leq \frac{|y|}{2} + \frac{|x|}{4} + \frac{1}{2} \leq \frac{|x|+1}{2}$$

and

$$\max \left\{ |(y^{-k}w)_v|, |(y^{p-k}w)_v| \right\} \leq |y| + |w_v| \leq \frac{|x|}{2} + \frac{|x|}{4} + \frac{1}{2} \leq \frac{3|x|+2}{4},$$

making use of (5.3).

As we will see, this implies that there exists a $w_1 \in \{(y^{-k}w)_v, (y^{p-k}w)_v \mid v \in X\}$ such that $w_1 \in W$ and $|w_1| \leq \frac{|y|+|w|}{2}$.

Indeed, if $y^{-k}w$ is in Case 1 of Lemma 5.1, then there exists $v \in X$ such that $(y^{-k}w)_v \equiv_{G'} b^{t'}$ for some $t' \neq 0$. By the minimality of $|x|$ and the estimations above, we must have $|x| \leq |(y^{-k}w)_v| \leq \frac{3|x|+2}{4}$. This is only possible if $|x| \leq 2$, and since we assume that $|x| > 1$, this implies $|x| = 2$, and so $|y| = 1$ by (5.3). In this case, we must have $(y^{p-k}w)_v \equiv_{G'} a^{\lambda j} b^{j+t'}$ with $|(y^{p-k}w)_v| \leq \frac{|x|+1}{2} = \frac{3}{2}$. Since the length must be an integer, we have $|(y^{p-k}w)_v| \leq 1$, and so $|(y^{p-k}w)_v| = 1 \leq \frac{|y|+|w|}{2}$ by Lemma 5.6 and the fact that $\lambda j \neq 0$. We then set $w_1 = (y^{p-k}w)_v$.

By symmetry, if $y^{p-k}w$ is in Case 1 of Lemma 5.1, then we can also find some w_1 of the required form. Thus, it only remains to check the case when both $y^{-k}w$ and $y^{p-k}w$ are in Case 2 of Lemma 5.1. In this case, there exist at least two vertices $v_1, v_2 \in X$ such that $(y^{-k}w)_{v_1}$ and $(y^{-k}w)_{v_2}$ have non-zero total a -exponent differing from λ times their total b -exponent, and it follows from Lemma 5.3 that $(y^{p-k}w)_{v_1}$ and $(y^{p-k}w)_{v_2}$ satisfy the same property. Since $|w_{v_1}| + |w_{v_2}| \leq |w|$, we can assume without loss of generality that $|w_{v_1}| \leq \frac{|w|}{2}$. Then, we have

$$\min \left\{ |(y^{-k}w)_{v_1}|, |(y^{p-k}w)_{v_1}| \right\} \leq \frac{|y|}{2} + |w_{v_1}| \leq \frac{|y|+|w|}{2},$$

and we set w_1 as the smallest of these two elements.

We have thus found some $w_1 \in W$ with $|w_1| \leq \frac{|y|+|w|}{2}$. By the minimality of $|w|$, we have $|w| \leq |w_1|$, and so $|w| \leq |y|$. As we have $|y| \leq |w|$ from Lemma 5.6, we conclude that $|w| = |y|$. \square

Finally, to prove Proposition 5.4, we return to our assumption that $|x| > 1$. We will now obtain a contradiction using the above results. Hence $|x| = 1$, as required.

We will use w from Lemma 5.7 to show that there exists a $g \in \bigsqcup_{v \in X^*} H_v$ with $0 < |g| < |y|$, which will contradict Lemma 5.6.

As before, we write $y = a^i \cdot \psi^{-1}((y_1, \dots, y_p)) \equiv_{G'} a^{\lambda j} b^j$ for some $j \neq 0$ and set $i = \lambda j$. From Lemma 5.7, we have $w \in H_v$ for some $v \in X^*$ with $w \equiv_{G'} a^n b^m$ for $n \neq \lambda m$, $n \neq 0$ and $|w| = |y|$. Let $k \in \{1, 2, \dots, p-1\}$ be such that $y^k \equiv_{G'} a^n b^k$. There must exist at least two vertices $v_1, v_2 \in X$ such that $(y^{-k}w)_{v_1}$ and $(y^{-k}w)_{v_2}$ have non-zero total a -exponent differing from λ times their total b -exponent. Indeed, as we saw in the proof of the previous lemma, both $y^{-k}w$ and $y^{p-k}w$ must be in Case 2 of Lemma 5.1, unless possibly when $|x| = 2$ and $|y| = 1$. As we will now see, this last case is impossible. Certainly, if $|x| = 2$, $|y| = 1$ and one of $y^{-k}w$ or $y^{p-k}w$ is in Case 1 of Lemma 5.1, then there exists $u \in X$ such that $(y^{-k}w)_u \equiv_{G'} b^{t'}$ or $(y^{p-k}w)_u \equiv_{G'} b^{t'}$ for some $t' \neq 0$. For concreteness, let us assume that $(y^{-k}w)_u \equiv_{G'} b^{t'}$; the other case is similar. From the fact that $|y| = |w| = 1$, we quickly see that $t' = m - j$. By considerations of length, and keeping the first line of (5.5) in mind, if $(y^{-k}w)_u \equiv_{G'} b^{t'}$, we must have $(y^{p-k}w^{1-p})_u = a^{\lambda(j-m)}$. As $j - m \neq 0$, we get a contradiction with Lemma 5.6. We conclude that both $y^{-k}w$ and $y^{p-k}w$ must be in Case 2 of Lemma 5.1.

Let $u \in \{v_1, v_2\}$. We may assume that $|w_u| = \frac{|w|}{2}$, else we are done. Indeed, otherwise, one of $|w_{v_1}|$ or $|w_{v_2}|$ must be strictly smaller than $\frac{|w|}{2}$. Let us suppose without loss of generality that $|w_{v_2}| < \frac{|w|}{2}$. Then either $|(y^{-k}w)_{v_2}| < \frac{|y|}{2} + \frac{|w|}{2} = |y|$ or $|(y^{p-k}w)_{v_2}| < \frac{|y|}{2} + \frac{|w|}{2} = |y|$. As this contradicts Lemma 5.6, we have that

$$(5.6) \quad |w_{v_1}| = |w_{v_2}| = \frac{|w|}{2}$$

and hence $|y| = |w| = 2\mu$ for some $\mu \in \mathbb{N}$.

By symmetry, that is, by considering $w^{k'}y$ for some k' , it likewise follows that only two first-level sections of y are of non-zero length and that the sum of their length must be $|y|$. Let $i_1, i_2 \in X$ be such that

$$|y_{i_1}| = |y_{i_2}| = \frac{|y|}{2} = \mu.$$

If $k = 1$, then $|x| \leq |y^{-1}w| \leq |y| + |w| = 2|y|$ and hence $|y^{-1}w| = 2|y|$ by (5.3). Therefore, it follows from Lemma 5.2 that there exists $u \in X$ such that $(y^{-1}w)_u \neq 1$ and $|(y^{-1}w)_u| < |y|$. As this contradicts Lemma 5.6, we conclude that the case $k = 1$ is impossible. Likewise, we see that the case $k = p - 1$ is impossible by looking at $y^{p-(p-1)}w = yw$.

Let us now assume that $1 < k < p - 1$. For $v \in X$, recall that we have

$$|(y^{-k}w)_v| \leq \sum_{d=0}^{k-1} |y_{v-d}| + |w_v|$$

and

$$|(y^{p-k}w)_v| \leq \sum_{d=k}^{p-1} |y_{v-d}| + |w_v|.$$

Since we assume that $|w_{v_1}| = |w_{v_2}| = \frac{|y|}{2}$, it follows that unless $\sum_{d=0}^{k-1} |y_{v_l-d}| = \sum_{d=k}^{p-1} |y_{v_l-d}| = \frac{|y|}{2}$ for $l \in \{1, 2\}$, then one of $|(y^{-k}w)_{v_l}|$ or $|(y^{p-k}w)_{v_l}|$ is strictly smaller than $|y|$, which is impossible.

Notice also that if there exists $v \in X \setminus \{v_1, v_2\}$ such that $0 < |\prod_{d=0}^{k-1} y_{v-d}| < |y|$, then one of $(y^{-k}w)_v$ or $(y^{p-k}w)_v$ is a non-trivial element of length strictly smaller than $|y|$, since we

must have $|w_v| = 0$ when v is different from v_1 or v_2 . As this contradicts Lemma 5.6, for all $v \in X \setminus \{v_1, v_2\}$, we must have either $|\prod_{d=0}^{k-1} y_{v-di}| = 0$ or $|\prod_{d=0}^{k-1} y_{v-di}| = |y|$. Likewise, we must also have either $|\prod_{d=k}^{p-1} y_{v-di}| = 0$ or $|\prod_{d=k}^{p-1} y_{v-di}| = |y|$.

This implies strong restrictions on the form of y . Recall from above that there exist $i_1, i_2 \in X$ such that $|y_{i_1}| = |y_{i_2}| = \frac{|y|}{2}$. This implies that, up to renaming i_1 and i_2 , we must have

$$(5.7) \quad y = a^i (b^{s_1})^{a^{i_1}} (b^{s_2})^{a^{i_2}} \dots (b^{s_{2\mu-1}})^{a^{i_1}} (b^{s_{2\mu}})^{a^{i_2}}$$

where $s_1 + \dots + s_{2\mu} = j$. In particular, for all $u \in X \setminus \{i_1, i_2\}$, we have $|y_u| = 0$. This implies that if, for some $v \in X$, the set $\{v - di \mid 0 \leq d \leq k-1\}$ contains exactly one of either i_1 or i_2 , then $|\prod_{d=0}^{k-1} y_{v-di}| = \frac{|y|}{2}$. By the above considerations, there can be only two such sets, namely when $v = v_1$ or $v = v_2$. As the next lemma shows, this can only be the case if $i_2 = i_1 + i$ or $i_2 = i_1 - i$.

Lemma 5.8. *Let $i, k, i_1, i_2 \in \mathbb{F}_p$ be four elements of \mathbb{F}_p , with $i \neq 0$, $i_1 \neq i_2$ and $1 < k < p-1$. For all $v \in \mathbb{F}_p$, let*

$$I_v = \{v - di \mid 0 \leq d \leq k-1\}.$$

If there exist two elements $v_1, v_2 \in \mathbb{F}_p$ such that

$$|I_v \cap \{i_1, i_2\}| = 1 \quad \iff \quad v \in \{v_1, v_2\},$$

then either $i_2 = i_1 + i$ or $i_2 = i_1 - i$.

Proof. Let $f: \mathbb{F}_p \rightarrow \mathbb{F}_p$ be the map defined by $f(x) = i^{-1}(x - i_1)$. As f is a bijection, we have $|f(I_v) \cap \{f(i_1), f(i_2)\}| = |I_v \cap \{i_1, i_2\}|$, and since we have

$$f(I_v) = \{f(v) - d \mid 0 \leq d \leq k-1\}$$

for all $v \in \mathbb{F}_p$, it suffices to prove the result for $i = 1$ and $i_1 = 0$.

Suppose for the sake of contradiction that $i_2 \neq 1, -1$, and let $v_1 \in \{k, k+1, \dots, p-1\}$ be the smallest element, with respect to the standard order on $\{k, k+1, \dots, p-1\}$, such that $i_2 \in I_{v_1}$. Note that v_1 exists, since $\bigcup_{v=k}^{p-1} I_v = \{1, 2, \dots, p-1\}$. Note also that since $v_1 \in \{k, k+1, \dots, p-1\}$, we know that $0 \notin I_{v_1}$. We claim that $i_2 \in I_{v_1+1}$ and that $0 \notin I_{v_1+1}$. Indeed, we have $I_{v_1+1} \setminus I_{v_1} = \{v_1 + 1\}$. Thus, if we had $0 \in I_{v_1+1}$, this would imply that $v_1 = p-1$. Since $k < p-1$, by the minimality of v_1 , this would then imply that $i_2 \notin I_{p-2}$, and therefore $i_2 = p-1$, a contradiction. Similarly, since $I_{v_1} \setminus I_{v_1+1} = \{v_1 - k + 1\}$, if we had $i_2 \notin I_{v_1+1}$, this would imply that $i_2 = v_1 - k + 1$. Since $k > 1$, we have $i_2 < v_1$, and so $i_2 \in I_{v_1-1}$. By the minimality of v_1 , this implies that $v_1 = k$. However, this then means that $i_2 = 1$, a contradiction.

We conclude that $i_2 \in I_{v_1}, I_{v_1+1}$ and that $0 \notin I_{v_1}, I_{v_1+1}$. Likewise, we can also find $v_2 \in \{i_2 + k, i_2 + k + 1, \dots, i_2 + p - 1\}$ such that $0 \in I_{v_2}, I_{v_2+1}$ and $i_2 \notin I_{v_2}, I_{v_2+1}$. Clearly $v_1, v_1 + 1, v_2, v_2 + 1$ are four different elements, and we have $|I_v \cap \{0, i_2\}| = 1$ for all $v \in \{v_1, v_1 + 1, v_2, v_2 + 1\}$, which contradicts our assumptions. The result follows. \square

Coming back to our considerations on the form of y , we have

$$y = a^i (b^{s_1})^{a^{i_1}} (b^{s_2})^{a^{i_2}} \dots (b^{s_{2\mu-1}})^{a^{i_1}} (b^{s_{2\mu}})^{a^{i_2}},$$

where $i_2 = i_1 + i$ or $i_2 = i_1 - i$ by the previous lemma. If we had $i_2 = i_1 - i$, then we would have

$$\varphi_{i_2-i}(y^p) = y_{i_2} y_{i_1} y_{i_2+2i} \dots y_{i_2+(p-1)i}.$$

From the form of y above, we see that y_{i_2} ends with $b^{s_{2\mu}}$ and y_{i_1} begins with b^{s_1} , which implies that $|y_{i_2} y_{i_1}| \leq |y_{i_2}| + |y_{i_1}| - 1$. Consequently, we have $|\varphi_{i_2-i}(y^p)| < |y|$, a contradiction to the minimality of y by Lemma 5.3. We conclude that we must have $i_2 = i_1 + i$.

To finish the proof, it suffices to show that if y is of the above form, then one of $(y^p)_{v_1}$ or $(y^p)_{v_2}$ is not. Indeed, we can then repeat the whole argument above with $y' = (y^p)_v$ and $w' = (y^{-k}w)_v$ for the corresponding $v \in \{v_1, v_2\}$ and we will reach a contradiction.

Let us then prove this last claim. Suppose that

$$(5.8) \quad y = a^i (b^{s_1})^{a^{i_1}} (b^{s_2})^{a^{i_1+i}} \dots (b^{s_{2\mu-1}})^{a^{i_1}} (b^{s_{2\mu}})^{a^{i_1+i}}.$$

For $u \in \{v_1, v_2\}$, we have

$$(y^p)_u = y_{u+i} y_{u+2i} \dots y_{u+pi},$$

and since we have supposed above that $\sum_{d=0}^{k-1} |y_{u-di}| = \frac{|y|}{2}$ and that $|y_{i_1}| = |y_{i_1+1}| = \frac{|y|}{2}$, we conclude that either $u = i_1$ or $u - ki = i_1$. Thus, up to renaming v_1 and v_2 , we can assume that $v_1 = i_1 + ki$ and $v_2 = i_1$. It follows that

$$(5.9) \quad (y^p)_{v_1} = a^{k_1} y_{i_1} y_{i_1+1} a^{k_2},$$

where $a^{k_1} = y_{v_1+i} \dots y_{v_1+(p-k-1)i}$ and $a^{k_2} = y_{v_1+(p-k+2)i} \dots y_{v_1+pi}$, and that

$$(5.10) \quad (y^p)_{v_2} = y_{i_1+1} a^{k_2+k_1} y_{i_1}.$$

From (5.8) and (5.9), we see that

$$\begin{aligned} (y^p)_{v_1} &= a^{k_1} b^{t_1} a^{r_1} b^{t_2} a^{r_2} \dots a^{r_{2\mu-1}} b^{t_{2\mu}} a^{k_2} \\ &= a^{k_1 + \sum_{d=1}^{2\mu-1} r_d + k_2} (b^{t_1})^{a^{\sum_{d=1}^{2\mu-1} r_d + k_2}} (b^{t_2})^{a^{\sum_{d=2}^{2\mu-1} r_d + k_2}} \dots (b^{t_{2\mu-1}})^{a^{r_{2\mu-1} + k_2}} (b^{t_{2\mu}})^{a^{k_2}} \end{aligned}$$

for some $t_1, \dots, t_{2\mu}$ and $r_1, \dots, r_{2\mu-1}$ in \mathbb{F}_p . Suppose that $(y^p)_{v_1}$ is of the same form as y , namely that there exists $i_3 \in X$ such that

$$(y^p)_{v_1} = a^i (b^{t_1})^{a^{i_3}} (b^{t_2})^{a^{i_3+i}} \dots (b^{t_{2\mu-1}})^{a^{i_3}} (b^{t_{2\mu}})^{a^{i_3+i}}.$$

This implies that $r_d = -i$ if d is odd and $r_d = i$ if d is even. Therefore, we obtain $\sum_{d=1}^{2\mu-1} r_d = -i$. Since we know from Lemma 5.3 that $k_1 + k_2 + \sum_{d=1}^{2\mu-1} r_d = i$, we conclude that $k_1 + k_2 = 2i$.

If we now turn our attention to $(y^p)_{v_2}$, it follows from (5.8) and (5.10) that

$$(y^p)_{v_2} = a^{r'_1} b^{t'_1} \dots a^{r'_\mu} b^{t'_\mu} a^{k_1+k_2} b^{t'_{\mu+1}} a^{r'_{\mu+1}} \dots b^{t'_{2\mu}} a^{r'_{2\mu}}$$

for some $t'_1, \dots, t'_{2\mu}$ and $r'_1, \dots, r'_{2\mu}$ in \mathbb{F}_p . Using the same reasoning as above, for $(y^p)_{v_2}$ to be of the same form as y , we would need either $k_1 + k_2 = i$ or $k_1 + k_2 = -i$, depending on the parity of μ .

We conclude that $(y^p)_{v_1}$ and $(y^p)_{v_2}$ cannot both be in the same form as y . Indeed, this would imply that either $i = 2i$ or $-i = 2i$. Since $i \neq 0$, the first equation is impossible, and the second can only be satisfied if $p = 3$. However, we assumed that there existed some $1 < k < p - 1$, which is impossible if $p = 3$.

Therefore, there is some $u \in \{v_1, v_2\}$ such that $y' = (y^p)_u$ is not in the form of (5.8). Setting $w' = (y^{-k}w)_u$ and repeating the whole argument above with y' and w' will thus necessarily yield a contradiction. We conclude that $|x| = 1$, and the result follows.

Theorem 5.9. *Let G be a GGS-group acting on the p -adic tree, for p an odd prime. Then G does not contain any proper prodense subgroups.*

Proof. By [16], it suffices to consider the non-torsion GGS-groups G . Further we may suppose that G is not conjugate to a generalised Fabrykowski-Gupta group, as otherwise the result follows by [6, Thm. 7.2.7]. Suppose on the contrary that M is a proper prodense subgroup of G . By Proposition 4.1, for every vertex $u \in T$ we have M_u is properly contained in G_u . However, by Propositions 4.2 and 5.4, there exists $v \in X^*$ such that the subgroup M_v is all of G . This gives the required contradiction. \square

The first statement of Theorem 1.1 is now proved. We show the second.

Proposition 5.10. *Let G be a branch GGS-group acting on the p -adic tree, for an odd prime p . Then every maximal subgroup of G is normal and of index p .*

Proof. Let M be a maximal subgroup of G . From the previous result, it follows that M has finite index in G . By [4], the group G has the congruence subgroup property, so there exists an $n \in \mathbb{N}$ such that $\text{St}_G(n) \leq M$. As $G/\text{St}_G(n)$ is a finite p -group, it follows that $G' \leq M \trianglelefteq_p G$. Hence the result. \square

For the constant GGS-group \mathcal{G} , the situation is different:

Proposition 5.11. *Let \mathcal{G} be the weakly branch, but not branch, GGS-group acting on the p -adic tree, for an odd prime p . Then there are infinitely many maximal subgroups. In particular, the group \mathcal{G} has maximal subgroups that are neither normal, nor of index p .*

Proof. By [4, Prop. 3.4] and using the notation of Section 3.2, we have $\mathcal{G}/K' \cong (\mathbb{Z}/p\mathbb{Z}) \times \mathbb{Z}^{p-1}$ where the action of $\mathbb{Z}/p\mathbb{Z}$ on \mathbb{Z}^{p-1} is given by the matrix

$$A = \left(\begin{array}{c|c} 0_{1 \times p-2} & -1 \\ \hline I_{p-2 \times p-2} & \begin{array}{c} -1 \\ \vdots \\ -1 \end{array} \end{array} \right).$$

We see that for all $q \in \mathbb{N}$, the subgroup $(q\mathbb{Z})^{p-1} \leq \mathbb{Z}^{p-1}$ is invariant under the action of $\mathbb{Z}/p\mathbb{Z}$. Therefore, for all $q \in \mathbb{N}$ different from 0 or 1, we can consider the subgroup $(\mathbb{Z}/p\mathbb{Z}) \times (q\mathbb{Z})^{p-1}$, which is non-trivial and proper in $(\mathbb{Z}/p\mathbb{Z}) \times \mathbb{Z}^{p-1}$. As $(\mathbb{Z}/p\mathbb{Z}) \times \mathbb{Z}^{p-1}$ is finitely generated, this subgroup is contained in a maximal subgroup $M_q \leq (\mathbb{Z}/p\mathbb{Z}) \times \mathbb{Z}^{p-1}$.

We notice that if $q_1, q_2 \in \mathbb{N}$ are two different prime numbers, then $M_{q_1} \neq M_{q_2}$. Indeed, otherwise, we would have $(q_1\mathbb{Z})^{p-1}, (q_2\mathbb{Z})^{p-1} \leq M_{q_1}$, which would imply $\mathbb{Z}^{p-1} \leq M_{q_1}$, since q_1 and q_2 are coprime. As we already had $(\mathbb{Z}/p\mathbb{Z}) \times (q_1\mathbb{Z})^{p-1} \leq M_{q_1}$, this means that $M_{q_1} = (\mathbb{Z}/p\mathbb{Z}) \times \mathbb{Z}^{p-1}$, which is absurd.

We have thus shown that we have an infinite number of maximal subgroups of \mathcal{G}/K' which are all pairwise distinct. By the correspondence theorem, the same is true for \mathcal{G} . This immediately implies that \mathcal{G} has maximal subgroups that are not of index p , since there can only be finitely many such subgroups. It also implies that \mathcal{G} admits maximal subgroups that are not normal. Indeed, otherwise, the Frattini subgroup of \mathcal{G} would contain \mathcal{G}' , since the quotient of any group by a normal maximal subgroup must be a cyclic group of prime order, and thus abelian, but \mathcal{G}/\mathcal{G}' is a p -group, which would imply that every maximal subgroup is of index p . \square

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