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# The exit from a metastable state: concentration of the exit point distribution on the low energy saddle points, part 2

Tony Lelièvre\*, Dorian Le Peutrec† and Boris Nectoux ‡

## Abstract

We consider the first exit point distribution from a bounded domain  $\Omega$  of the stochastic process  $(X_t)_{t \geq 0}$  solution to the overdamped Langevin dynamics

$$dX_t = -\nabla f(X_t)dt + \sqrt{h} dB_t$$

starting from deterministic initial conditions in  $\Omega$ , under rather general assumptions on  $f$  (for instance,  $f$  may have several critical points in  $\Omega$ ). This work is a continuation of the previous paper [14] where the exit point distribution from  $\Omega$  is studied when  $X_0$  is initially distributed according to the quasi-stationary distribution of  $(X_t)_{t \geq 0}$  in  $\Omega$ . The proofs are based on analytical results on the dependency of the exit point distribution on the initial condition, large deviation techniques and results on the genericity of Morse functions.

## 1 Introduction and main results

The aim of this article is to extend the results of [14] on the concentration of the first exit point distribution from a domain to general initial conditions within the domain. For the sake of consistency, we first recall in Section 1.1 the motivation for such a study, some related works in the literature, and an informal presentation of our results. Section 1.2 then gives precise statements of the results we prove.

### 1.1 Motivation and informal presentation of the results

We are interested in the overdamped Langevin dynamics

$$dX_t = -\nabla f(X_t)dt + \sqrt{h} dB_t, \tag{1}$$

where  $X_t \in \mathbb{R}^d$  is a vector in  $\mathbb{R}^d$ ,  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  is a  $C^\infty$  function,  $h$  is a positive parameter and  $(B_t)_{t \geq 0}$  is a standard  $d$ -dimensional Brownian motion. The process (1) can be used to model the evolution of molecular systems, for example. In this case,  $f$  is the potential function and  $h > 0$  is proportional to the temperature of the system. Let us consider a domain  $\Omega \subset \mathbb{R}^d$  and the associated exit event from  $\Omega$ . More precisely, let us introduce

$$\tau_\Omega = \inf\{t \geq 0 | X_t \notin \Omega\} \tag{2}$$

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the first exit time from  $\Omega$ . We are interested in the limit of the law of  $X_{\tau_\Omega}$  when  $h \rightarrow 0$ . Under some assumptions which will be made precise below, this law concentrates on a subset of  $\partial\Omega$ , concentration being defined in the following sense.

**Definition 1.** *Let  $\mathcal{Y} \subset \partial\Omega$  and let us consider a family of random variable  $(Y_h)_{h>0}$  which admits a limit in distribution when  $h \rightarrow 0$ . The law of  $Y_h$  concentrates on  $\mathcal{Y}$  in the limit  $h \rightarrow 0$  if for every neighborhood  $\mathcal{V}_\mathcal{Y}$  of  $\mathcal{Y}$  in  $\partial\Omega$*

$$\lim_{h \rightarrow 0} \mathbb{P}[Y_h \in \mathcal{V}_\mathcal{Y}] = 1,$$

and if for all  $x \in \mathcal{Y}$  and for all neighborhood  $\mathcal{V}_x$  of  $x$  in  $\partial\Omega$

$$\lim_{h \rightarrow 0} \mathbb{P}[Y_h \in \mathcal{V}_x] > 0.$$

In other words, the law of  $Y_h$  concentrates on  $\mathcal{Y}$  if  $\mathcal{Y}$  is the support of the law of  $Y_h$  in the limit  $h \rightarrow 0$ .

In this work, we investigate the concentration of the law of  $X_{\tau_\Omega}$  on a subset of  $\partial\Omega$  when  $X_0 = x \in \Omega$ , under general assumptions on the function  $f : \bar{\Omega} \rightarrow \mathbb{R}$ . This is of practical interest in order to predict where the process (1) is more likely to leave  $\Omega$  in the zero-noise limit. The study of the exit event in the small temperature regime has interesting theoretical and numerical counterparts, to relate continuous state space dynamics such as (1) to discrete state space dynamics (jump Markov model), and to accelerate the sampling of metastable trajectories, see [12, 31].

**Review of the literature.** Let us mention the main results from the mathematical literature on the exit problem related to our problem. We refer to [6] for a more comprehensive review.

The concentration of the law of  $X_{\tau_\Omega}$  on  $\arg \min_{\partial\Omega} f$  in the small temperature regime ( $h \rightarrow 0$ ) has been studied in [35, 37, 43] through formal computations, see also [33]. Many of these results have been rigorously proven either by studying the underlying partial differential equations, or by using large deviations techniques. In particular, when it holds

$$\partial_n f > 0 \text{ on } \partial\Omega, \tag{3}$$

where  $\partial_n f$  is the normal derivative of  $f$  on  $\partial\Omega$  ( $n$  is the unit outward normal vector to  $D$ ), and

$$\{x \in \Omega, |\nabla f(x)| = 0\} = \{x_0\} \text{ with } f(x_0) = \min_{\Omega} f \text{ and } \det \text{Hess}f(x_0) > 0, \tag{4}$$

and  $f$  attains its minimum on  $\partial\Omega$  at one single point  $y_0$ , it is proved in [17, Theorem 2.1] that the law of  $X_{\tau_\Omega}$  concentrates on  $y_0$  in the limit  $h \rightarrow 0$ , when  $X_0 = x \in \Omega$ . This result has been extended in [7, 8, 23, 24, 41] when only (3) and (4) are satisfied: it is proved there that the the law of  $X_{\tau_\Omega}$  concentrates in this case on  $\arg \min_{\partial\Omega} f$  in the limit  $h \rightarrow 0$ , when  $X_0 = x \in \Omega$ .

In [17, Theorem 5.1], for  $\Sigma \subset \partial\Omega$ , the limit of  $h \ln \mathbb{P}[X_{\tau_\Omega} \in \Sigma]$  when  $h \rightarrow 0$  is related to a minimization problem involving the quasipotential of the process (1). Let us mention two limitations of [17, Theorem 5.1]. First, this theorem requires to be able to compute the quasipotential in order to get useful information: this is trivial under the assumptions (3) and (4) but more complicated under more general assumptions on  $f$  (in particular when

$f$  has several critical points in  $\Omega$ ). Second, even when the quasipotential is analytically known, this result only gives the subset of  $\partial\Omega$  where exit will not occur on an exponential scale in the limit  $h \rightarrow 0$ . It does not allow to exclude exit points with probability which goes to zero polynomially in  $h$  (this may indeed occur, see Remark 6 below and examples in [11, Section 1.4]), and, when the exit point distribution concentrates on more than one point, it does not give the relative probabilities to exit through the various exit points in the limit  $h \rightarrow 0$ . However, let us emphasize that the strength of large deviation theory is that it is very general:  $f$  may have several local minima in  $\Omega$ , and the theory actually applies to a much wider class of dynamics (in particular for non-gradient drift and non-isotropic noise) and in a broader geometric setting [17, 40]. See for example [1, 25, 26] for recent contributions to the non reversible case. Other references where the exit problem appears as an intermediate tool to study spectral properties of the infinitesimal generator are [2, 3, 34, 36].

**Objective of this work.** Our work aims at generalizing in the reversible case the results of [17, Theorem 2.1] and [7, 8, 23, 24, 41], when  $f$  has several critical points in  $\Omega$ . First, we exhibit a general set of assumptions on  $f$  and an ensemble of initial conditions for which the law of  $X_{\tau_\Omega}$  concentrates on points belonging to  $\arg \min_{\partial\Omega} f$  and we compute the relative probabilities to leave the domain through each of them (see Theorem 1): in this case, the limiting exit point distribution is the same as starting from the quasi-stationary distribution in  $\Omega$ , and we thus rely on our previous work [14]. Second, using this first result, we identify the exit points when the process starts more generally from initial conditions contained in a potential well which touches the boundary of  $\Omega$ , in a sense to be made precise (see Theorem 2 and Theorem 4).

Concerning the assumptions on  $f$ ,  $\partial_n f$  is not assumed to be positive on  $\partial\Omega$  and  $f$  may have several critical points in  $\Omega$  with larger energies than  $\min_{\partial\Omega} f$ . However, we do not consider the case when  $f$  has critical points on  $\partial\Omega$ .

Here are representative examples of outputs of this work, which are new to the best of our knowledge. Let us assume that  $f : \bar{\Omega} \rightarrow \mathbb{R}$  and its restriction  $f|_{\partial\Omega}$  to the boundary of  $\Omega$  are smooth Morse functions, and that  $f$  has no critical point on  $\partial\Omega$  (see **(A0)** below).

- We prove that if  $\{y \in \Omega, f(y) < \min_{\partial\Omega} f\}$  is connected and contains all the local minima of  $f$  in  $\Omega$ , and if  $\partial_n f > 0$  on  $\arg \min_{\partial\Omega} f$ , then the law of  $X_{\tau_\Omega}$  concentrates on  $\arg \min_{\partial\Omega} f$  when  $X_0 = x \in \{y \in \Omega, f(y) < \min_{\partial\Omega} f\}$  (see Theorem 1).
- Besides, when  $\{y \in \Omega, f(y) < \min_{\partial\Omega} f\}$  is not connected (we denote in this case by  $C_1, \dots, C_N$  its connected components, with  $N \geq 2$ ) and if, for some  $j \in \{1, \dots, N\}$ ,  $\partial C_j \cap \partial\Omega \neq \emptyset$  and  $|\nabla f| \neq 0$  on  $\partial C_j$ , then, when  $X_0 = x \in C_j$ , the law of  $X_{\tau_\Omega}$  concentrates on  $\partial C_j \cap \partial\Omega$  in the limit  $h \rightarrow 0$  (see Theorem 2).
- Furthermore, when  $X_0 = x \in C_j$  and  $z \in \arg \min_{\partial\Omega} f \setminus \partial C_j$ , for all sufficiently small neighborhood  $\Sigma_z$  of  $z$  in  $\partial\Omega$ ,  $\mathbb{P}[X_{\tau_\Omega} \in \Sigma_z] = O(e^{-\frac{c}{h}})$  for  $h$  small enough (see item 1 in Theorem 2).

Let us mention that the preprint [11] concatenates most of the results of this manuscript and [14]. A simplified version of the results of this work is also presented in [32].

**On the results from [14]: metastability and the quasi-stationary distribution.** As explained above, this article generalizes to a broader class of initial conditions in  $\Omega$  the

results of [14] where it is assumed that  $X_0$  is distributed according to the quasi-stationary distribution in  $\Omega$ . Let us quickly recall what is the quasi-stationary distribution, and why it is relevant to study the exit event starting from this distribution.

Let us assume that  $\Omega \subset \mathbb{R}^d$  is smooth, open, bounded and connected.

**Definition 2.** *A quasi-stationary distribution for the dynamics (1) in  $\Omega$  is a probability measure  $\nu_h$  supported in  $\Omega$  such that for all measurable sets  $A \subset \Omega$  and for all  $t \geq 0$*

$$\nu_h(A) = \frac{\int_{\Omega} \mathbb{P}_x [X_t \in A, t < \tau_{\Omega}] \nu_h(dx)}{\int_{\Omega} \mathbb{P}_x [t < \tau_{\Omega}] \nu_h(dx)}. \quad (5)$$

Here and in the following, the subscript  $x$  indicates that the stochastic process starts from  $x \in \mathbb{R}^d$  ( $X_0 = x$ ). In words, (5) means that if  $X_0$  is distributed according to  $\nu_h$ , then for all  $t > 0$ ,  $X_t$  is still distributed according to  $\nu_h$  conditionally on  $X_s \in \Omega$  for all  $s \in [0, t]$ .

The quasi-stationary distribution is related to an eigenvalue problem on the infinitesimal generator of the dynamics (1), namely the differential operator

$$L_{f,h} = -\frac{h}{2} \Delta + \nabla f \cdot \nabla. \quad (6)$$

Let us define

$$L_w^2(\Omega) = \left\{ u : \Omega \rightarrow \mathbb{R}, \int_{\Omega} u^2 e^{-\frac{2}{h}f} < \infty \right\}.$$

The weighted Sobolev spaces  $H_w^k(\Omega)$  are defined similarly. The operator  $L_{f,h}$  with homogeneous Dirichlet boundary conditions on  $\partial\Omega$  is denoted by  $L_{f,h}^D$ . Its domain is  $\mathcal{D}(L_{f,h}^D) = H_{w,0}^1(\Omega) \cap H_w^2(\Omega)$ , where  $H_{w,0}^1(\Omega) = \{u \in H_w^1(\Omega), u = 0 \text{ on } \partial\Omega\}$ . It is well known that  $-L_{f,h}^D$  is self adjoint on  $L_w^2(\Omega)$ , positive and has compact resolvent. Moreover, from standard results on elliptic operator (see for example [16,19]), its smallest eigenvalue  $\lambda_h$  (a.k.a. the principal eigenvalue) is non degenerate and its associated eigenfunction  $u_h$  has a sign on  $\Omega$  and is in  $C^\infty(\bar{\Omega})$ . Without loss of generality, one can then assume that:

$$u_h > 0 \text{ on } \Omega \quad \text{and} \quad \int_{\Omega} u_h^2 e^{-\frac{2}{h}f} = 1. \quad (7)$$

The following result (see for example [28]) relates the quasi-stationary distribution  $\nu_h$  to the principal eigenfunction  $u_h$ .

**Proposition 3.** *The unique quasi-stationary distribution  $\nu_h$  associated with the dynamics (1) and the domain  $\Omega$  is given by:*

$$\nu_h(dx) = \frac{u_h(x) e^{-\frac{2}{h}f(x)}}{\int_{\Omega} u_h(y) e^{-\frac{2}{h}f(y)} dy} dx. \quad (8)$$

Moreover, whatever the law of the initial condition  $X_0$  with support in  $\Omega$ , it holds:

$$\lim_{t \rightarrow \infty} \|\text{Law}(X_t | t < \tau_{\Omega}) - \nu_h\|_{TV} = 0. \quad (9)$$

Here,  $\text{Law}(X_t | t < \tau_\Omega)$  denotes the law of  $X_t$  conditional to the event  $\{t < \tau_\Omega\}$ . For a given initial distribution of the process (1), if the convergence in (9) is much quicker than the exit from  $\Omega$ , the exit from the domain  $\Omega$  is said to be metastable. In [14], we have investigated the concentration of the law of  $X_{\tau_\Omega}$  on  $\arg \min_{\partial\Omega} f$  in the limit  $h \rightarrow 0$  when  $X_0 \sim \nu_h$ , namely when the exit is metastable. In this work, we extend this study to the general case:  $X_0 = x \in \Omega$ .

## 1.2 Main results

In all this work, we assume that  $\Omega \subset \mathbb{R}^d$  is smooth, open, bounded and connected, and that  $f : \bar{\Omega} \rightarrow \mathbb{R}$  is a  $C^\infty$  function. This section is dedicated to the statement of the main result of this work.

### 1.2.1 Assumptions

Let us now introduce the basic assumption which is used throughout this work:

$$\left. \begin{array}{l} \text{The function } f : \bar{\Omega} \rightarrow \mathbb{R} \text{ is a } C^\infty \text{ Morse function.} \\ \text{For all } x \in \partial\Omega, |\nabla f(x)| \neq 0. \\ \text{The function } f : \{x \in \partial\Omega, \partial_n f(x) > 0\} \rightarrow \mathbb{R} \text{ is a Morse function.} \\ \text{The function } f \text{ has at least one local minimum in } \Omega. \end{array} \right\} \quad (\mathbf{A0})$$

**Remark 4.** We recall that a function  $\phi : \bar{\Omega} \rightarrow \mathbb{R}$  is a Morse function if all its critical points are non degenerate (which implies in particular that  $\phi$  has a finite number of critical points since  $\bar{\Omega}$  is compact and a non degenerate critical point is isolated from the other critical points). Let us recall that a critical point  $z \in \bar{\Omega}$  of  $\phi$  is non degenerate if the hessian matrix of  $\phi$  at  $z$ , denoted by  $\text{Hess } \phi(z)$ , is invertible. We refer for example to [22, Definition 4.3.5] for a definition of the hessian matrix on a manifold (see also [13, Remark 10] for explicit formulas). A non degenerate critical point  $z \in \bar{\Omega}$  of  $\phi$  is said to have index  $p \in \{0, \dots, d\}$  if  $\text{Hess } \phi(z)$  has precisely  $p$  negative eigenvalues (counted with multiplicity). In the case  $p = 1$ ,  $z$  is called a saddle point.

In order to introduce the remaining assumptions, we need to introduce three notations. First, the following notation will be used for the level sets of  $f$ : for  $a \in \mathbb{R}$ ,

$$\{f < a\} = \{x \in \bar{\Omega}, f(x) < a\}, \quad \{f \leq a\} = \{x \in \bar{\Omega}, f(x) \leq a\},$$

and

$$\{f = a\} = \{x \in \bar{\Omega}, f(x) = a\}.$$

Second, for any local minimum  $x$  of  $f$  in  $\Omega$ , one defines

$$H_f(x) := \inf \left\{ \max_{t \in [0,1]} f(\gamma(t)) \mid \gamma \in C^0([0,1], \bar{\Omega}), \gamma(0) = x, \text{ and } \gamma(1) \in \partial\Omega \right\}, \quad (10)$$

where  $C^0([0,1], \bar{\Omega})$  is the set of continuous paths from  $[0,1]$  to  $\bar{\Omega}$ . Intuitively,  $H_f(x)$  is the minimal energy any path connecting  $x$  to the boundary  $\partial\Omega$  has to cross. This energy is necessarily either the energy of a saddle point  $z$  in  $\Omega$  (see e.g.  $\partial\mathcal{C}_2$  on Figure 1), or the energy of a generalized saddle point  $z$  on  $\partial\Omega$  (see e.g.  $\partial\Omega \cap \partial\mathcal{C}_3$  and  $\partial\Omega \cap \partial\mathcal{C}_{\max}$  on

Figure 1; see also Equation (15) for a proper definition of a generalized saddle point). Third, for a local minimum  $x$  of  $f$  in  $\Omega$ ,

$$\mathcal{C}(x) \text{ is the connected component of } \{f < \mathbf{H}_f(x)\} \text{ containing } x. \quad (\mathbf{11})$$

Finally, set

$$\mathcal{C} := \{\mathcal{C}(x), x \text{ is a local minimum of } f \text{ in } \Omega\}. \quad (\mathbf{12})$$

Let us mention that when **(A0)** holds, one has: for all local minima  $x$  of  $f$  in  $\Omega$ ,  $\mathcal{C}(x)$  is an open subset of  $\Omega$  (see [11, Remark 7]).

We are now in position to state the assumptions we will use in addition to **(A0)**:

- First geometric assumption:

$$\mathbf{(A0)} \text{ holds and } \exists! \mathcal{C}_{\max} \in \mathcal{C} \text{ s.t. } \max_{\mathcal{C} \in \mathcal{C}} \left\{ \max_{\mathcal{C}} f - \min_{\mathcal{C}} f \right\} = \max_{\mathcal{C}_{\max}} f - \min_{\mathcal{C}_{\max}} f. \quad (\mathbf{A1})$$

- Second geometric assumption:

$$\mathbf{(A1)} \text{ holds and } \partial \mathcal{C}_{\max} \cap \partial \Omega \neq \emptyset. \quad (\mathbf{A2})$$

- Third geometric assumption:

$$\mathbf{(A1)} \text{ holds and } \partial \mathcal{C}_{\max} \cap \partial \Omega \subset \arg \min_{\partial \Omega} f. \quad (\mathbf{A3})$$

Assumption **(A1)** implies that there is a unique deepest well, namely  $\mathcal{C}_{\max}$ . Assumptions **(A2)** and **(A3)** mean that the closure of this deepest well intersects  $\partial \Omega$  at points where  $f$  reaches its minimum on  $\partial \Omega$ . Notice that these assumptions imply that  $\mathcal{C}_{\max}$  contains the global minima of  $f$  in  $\Omega$ . Assumptions **(A2)** and **(A3)** ensure that when  $X_0 \sim \nu_h$  ( $\nu_h$  being the quasi-stationary distribution introduced in Definition 2) or  $X_0 = x \in \mathcal{C}_{\max}$ , the law of  $X_{\tau_\Omega}$  concentrates on the set  $\partial \mathcal{C}_{\max} \cap \partial \Omega$ , (see [14, Theorem 1] and items 1 and 2 in Theorem 1 below). Finally, the last assumption is:

- Fourth geometric assumption:

$$\mathbf{(A1)} \text{ holds and } \partial \mathcal{C}_{\max} \cap \Omega \text{ contains no } \textit{separating saddle point} \text{ of } f, \quad (\mathbf{A4})$$

where the proper definition of a separating saddle point of  $f$  is introduced below in Section 1.2.2.

Assumption **(A4)** is equivalent to the fact that any minimal energy path connecting a point in  $\mathcal{C}_{\max}$  to the boundary  $\partial \Omega$  remains necessarily within  $\overline{\mathcal{C}_{\max}}$ . In particular, such a path leaves  $\Omega$  on  $\partial \mathcal{C}_{\max} \cap \partial \Omega$ . Notice indeed that Assumption **(A4)** implies  $\partial \mathcal{C}_{\max} \cap \partial \Omega \neq \emptyset$  (this is a consequence of [14, Proposition 15]). The assumptions **(A4)** together with **(A0)**, **(A1)**, **(A2)**, and **(A3)**, ensure that the probability that the process (1) starting from  $x \in \mathcal{C}_{\max}$  leaves  $\Omega$  through any sufficiently small neighborhood of  $z \in \partial \Omega \setminus \partial \mathcal{C}_{\max}$  in  $\partial \Omega$  is exponentially small when  $h \rightarrow 0$ , see item 3 in Theorem 1 below. We refer to Remark 6 below for a discussion on the necessity of the assumptions **(A0)**, **(A1)**, **(A2)**, **(A3)**, and **(A4)** to get these results. Figure 1 gives a one-dimensional example where **(A1)**, **(A2)**, **(A3)**, and **(A4)** are satisfied, and Figure 2 gives a typical example where **(A1)**, **(A2)**, and **(A3)** are satisfied but not **(A4)**.

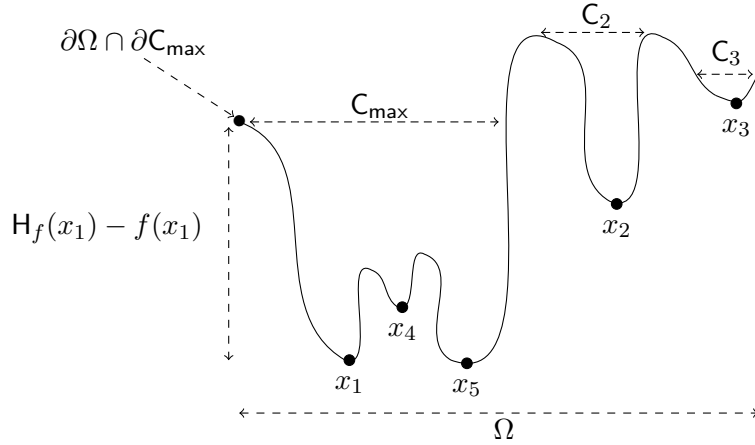


Figure 1: A one-dimensional case where **(A1)**, **(A2)**, **(A3)** and **(A4)** are satisfied. On the figure,  $f(x_1) = f(x_5)$ ,  $H_f(x_1) = H_f(x_4) = H_f(x_5)$ ,  $\mathcal{C} = \{C_{\max}, C_2, C_3\}$ ,  $\partial C_2 \cap \partial C_{\max} = \emptyset$  and  $\partial C_3 \cap \partial C_{\max} = \emptyset$ .

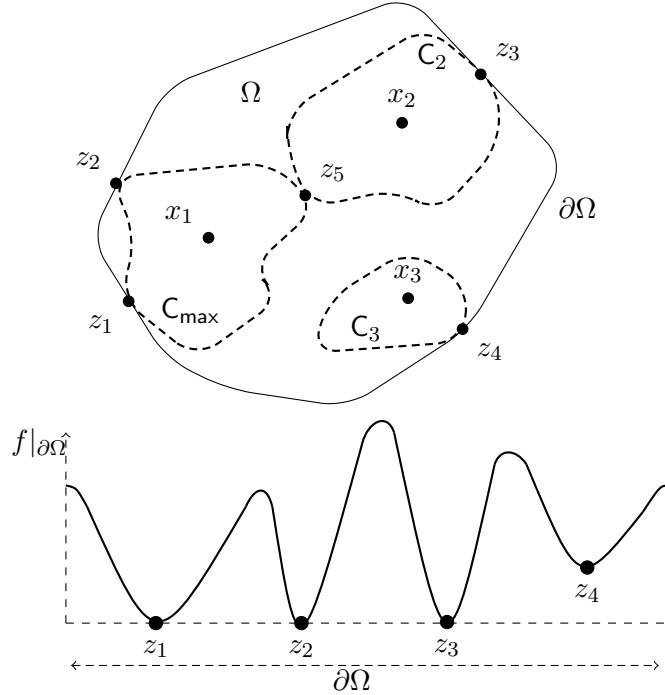


Figure 2: Schematic representation of  $\mathcal{C}$  (see (12)) and  $f|_{\partial\Omega}$  when the assumptions **(A0)**, **(A1)**, **(A2)** and **(A3)** are satisfied. In this representation,  $x_1 \in \Omega$  is the global minimum of  $f$  in  $\Omega$  and the other local minima of  $f$  in  $\Omega$  are  $x_2$  and  $x_3$ . Moreover,  $\min_{\partial\Omega} f = f(z_1) = f(z_2) = f(z_3) = H_f(x_1) = H_f(x_2) < H_f(x_3) = f(z_4)$ ,  $\{f < H_f(x_1)\}$  has two connected components:  $C_{\max}$  (see **(A1)**) which contains  $x_1$  and  $C_2$  which contains  $x_2$ . Thus, one has  $\mathcal{C} = \{C_{\max}, C_2, C_3\}$ . In addition,  $\{z_1, z_2, z_3\} = \arg \min_{\partial\Omega} f$  ( $\kappa_1^{\partial\Omega} = 3$ ),  $\{z_5\} = \overline{C_{\max}} \cap \overline{C_2}$  is a separating saddle point of  $f$  (thus **(A4)** is not satisfied), and  $\partial C_{\max} \cap \partial\Omega = \{z_1, z_2\}$  ( $\kappa_1^{\partial C_{\max}} = 2$ ).



### 1.2.2 Notation associated with the function $f$

In order to state our main result, we will need a few more notation associated with the function  $f$ .

**Domain of attraction  $\mathcal{A}(D)$ .** The domain of attraction of a subset  $D$  of  $\Omega$  for the dynamics  $\dot{x} = -\nabla f(x)$  is defined as follows. Let  $x \in \Omega$  and denote by  $\varphi_t(x)$  the solution to the ordinary differential equation

$$\frac{d}{dt}\varphi_t(x) = -\nabla f(\varphi_t(x)) \text{ with } \varphi_0(x) = x, \quad (13)$$

on the interval  $t \in [0, t_x]$ , where  $t_x > 0$  is defined by

$$t_x = \inf\{t \geq 0, \varphi_t(x) \notin \Omega\}.$$

Let  $x \in \Omega$  be such that  $t_x = +\infty$ . The  $\omega$ -limit set of  $x$ , denoted by  $\omega(x)$ , is defined by

$$\omega(x) = \{y \in \bar{\Omega}, \exists (s_n)_{n \in \mathbb{N}} \in (\mathbb{R}_+)^{\mathbb{N}}, \lim_{n \rightarrow \infty} s_n = +\infty, \lim_{n \rightarrow \infty} \varphi_{s_n}(x) = y\}.$$

Let us recall that the  $\omega$ -limit set  $\omega(x)$  is included in the set of the critical points of  $f$  in  $\bar{\Omega}$ . Moreover, when  $f$  has a finite number of critical points in  $\bar{\Omega}$ ,  $\omega(x)$  is either empty (if  $t_x < \infty$ ) or of cardinality one (if  $t_x = \infty$ ). Let  $D$  be a subset of  $\Omega$ . The domain of attraction of a subset  $D$  of  $\Omega$  is defined by

$$\mathcal{A}(D) = \{x \in \Omega, t_x = +\infty \text{ and } \omega(x) \subset D\}. \quad (14)$$

**Remark 5.** Recall that when **(A0)** holds, one has: for all local minima  $x$  of  $f$  in  $\Omega$ ,  $\mathcal{C}(x) \subset \Omega$ , where  $\mathcal{C}(x)$  is defined in (11) (see [11, Remark 7]). This implies that for all  $y \in \mathcal{C}(x)$ ,  $t_y = +\infty$  and then,  $\mathcal{C}(x) \subset \mathcal{A}(\mathcal{C}(x))$ .

**Generalized saddle points of  $f$ .** We introduce in this paragraph an ensemble of points on  $\partial\Omega$ , which will contain the exit points of the process  $(X_t)_{t \geq 0}$  from  $\Omega$ . Let us define

$$\mathcal{U}_1^{\partial\Omega} := \{z \in \partial\Omega, z \text{ is a local minimum of } f|_{\partial\Omega} \text{ but not a local minimum of } f \text{ in } \bar{\Omega}\}.$$

Notice that an equivalent definition of  $\mathcal{U}_1^{\partial\Omega}$  is

$$\mathcal{U}_1^{\partial\Omega} = \{z \in \partial\Omega, z \text{ is a local minimum of } f|_{\partial\Omega} \text{ and } \partial_n f(z) > 0\}, \quad (15)$$

which follows from the fact that  $|\nabla f(x)| \neq 0$  for all  $x \in \partial\Omega$ . The set  $\mathcal{U}_1^{\partial\Omega}$  contains the so-called *generalized saddle points* of  $f$  on the boundary  $\partial\Omega$  [20, 29]. These generalized saddle points are indeed geometrically saddle points of the function  $f$  extended by  $-\infty$  outside  $\bar{\Omega}$ , which is consistent with the fact that homogeneous Dirichlet boundary conditions are used to define the operator  $L_{f,h}^D$ .

Let us introduce the following notation for the  $k_1^{\partial\Omega}$  elements of  $\mathcal{U}_1^{\partial\Omega} \cap \arg \min_{\partial\Omega} f$ :

$$\{z_1, \dots, z_{k_1^{\partial\Omega}}\} = \mathcal{U}_1^{\partial\Omega} \cap \arg \min_{\partial\Omega} f. \quad (16)$$

Let us assume that the assumptions **(A1)**, **(A2)**, and **(A3)** are satisfied. Let us recall that  $\mathcal{C}_{\max}$  is defined in **(A1)**. Moreover, in this case, one has  $k_1^{\partial\Omega} \geq 1$  and

$$\partial\mathcal{C}_{\max} \cap \partial\Omega \subset \{z_1, \dots, z_{k_1^{\partial\Omega}}\}.$$

Indeed, by assumption  $\partial\mathbf{C}_{\max} \cap \partial\Omega \subset \{f = \min_{\partial\Omega} f\}$  (see **(A3)**) and there is no local minimum of  $f$  in  $\bar{\Omega}$  on  $\partial\mathbf{C}_{\max}$  (since  $\mathbf{C}_{\max}$  is a connected component of a sublevel set of  $f$ ). We assume lastly that the set  $\{z_1, \dots, z_{k_1^{\partial\Omega}}\}$  is ordered such that:

$$\{z_1, \dots, z_{k_1^{\partial\mathbf{C}_{\max}}}\} = \{z_1, \dots, z_{k_1^{\partial\Omega}}\} \cap \partial\mathbf{C}_{\max}. \quad (17)$$

Notice that  $k_1^{\partial\mathbf{C}_{\max}} \in \mathbb{N}^*$  and  $k_1^{\partial\mathbf{C}_{\max}} \leq k_1^{\partial\Omega}$ .

**Separating saddle points.** Let us finally introduce the notion of *separating saddle point*, following [21, Section 4.1]. To this end, let us first recall that according to [20, Section 5.2], for any non critical point  $z \in \Omega$ , for  $r > 0$  small enough

$$\{f < f(z)\} \cap B(z, r) \text{ is connected,}$$

and for any critical point  $z \in \Omega$  of index  $p$  of the Morse function  $f$ , for  $r > 0$  small enough, one has the three possible cases:

- either  $p = 0$  (i.e.  $z$  is a local minimum of  $f$ ) and  $\{f < f(z)\} \cap B(z, r) = \emptyset$ ,
- or  $p = 1$  and  $\{f < f(z)\} \cap B(z, r)$  has exactly two connected components,
- or  $p \geq 2$  and  $\{f < f(z)\} \cap B(z, r)$  is connected,

where  $B(z, r) := \{x \in \bar{\Omega} \text{ s.t. } |x - z| < r\}$ . A separating saddle point of  $f$  in  $\Omega$  is a saddle point of  $f$  in  $\Omega$  such that for any  $r > 0$  small enough, the two connected components of  $\{f < f(z)\} \cap B(z, r)$  are contained in different connected components of  $\{f < f(z)\}$ .

Figure 2 gives an illustration of the notations introduced in this section.

### 1.2.3 Main results on the exit point distribution

In Theorem 1, we first make explicit a geometric setting and an ensemble of initial conditions for which the exit distribution is the same as when starting from the quasi-stationary distribution.

**Theorem 1.** *Let us assume that the assumptions **(A0)**, **(A1)**, **(A2)**, and **(A3)** are satisfied. Let  $F \in L^\infty(\partial\Omega, \mathbb{R})$  and  $(\Sigma_i)_{i \in \{1, \dots, k_1^{\partial\Omega}\}}$  be a family of pairwise disjoint open subsets of  $\partial\Omega$  (i.e. such that  $\Sigma_i \cap \Sigma_j = \emptyset$  whenever  $i \neq j$ ) such that*

$$\text{for all } i \in \{1, \dots, k_1^{\partial\Omega}\}, z_i \in \Sigma_i,$$

where we recall that  $\{z_1, \dots, z_{k_1^{\partial\Omega}}\} = \mathbf{U}_1^{\partial\Omega} \cap \arg \min_{\partial\Omega} f$  (see (16)). Let  $K$  be a compact subset of  $\Omega$  such that  $K \subset \mathcal{A}(\mathbf{C}_{\max})$  (see **(A1)** and (14) for the definitions of  $\mathbf{C}_{\max}$  and  $\mathcal{A}$ ). Let  $x \in K$ . Then:

1. There exists  $c > 0$  such that in the limit  $h \rightarrow 0$ :

$$\mathbb{E}_x [F(X_{\tau_\Omega})] = \sum_{i=1}^{k_1^{\partial\Omega}} \mathbb{E}_x [(1_{\Sigma_i} F)(X_{\tau_\Omega})] + O(e^{-\frac{c}{h}}) \quad (18)$$

and

$$\sum_{i=k_1^{\partial\mathbf{C}_{\max}}+1}^{k_1^{\partial\Omega}} \mathbb{E}_x [(1_{\Sigma_i} F)(X_{\tau_\Omega})] = O(h^{\frac{1}{4}}), \quad (19)$$

where we recall that  $\{z_1, \dots, z_{k_1^{\partial\mathbf{C}_{\max}}}\} = \partial\mathbf{C}_{\max} \cap \partial\Omega$  (see (17)).

2. When for some  $i \in \{1, \dots, k_1^{\partial C_{\max}}\}$  the function  $F$  is  $C^\infty$  in a neighborhood of  $z_i$ , one has when  $h \rightarrow 0$ :

$$\mathbb{E}_x[(\mathbf{1}_{\Sigma_i} F)(X_{\tau_\Omega})] = F(z_i) a_i + O(h^{\frac{1}{4}}), \quad (20)$$

where

$$a_i = \frac{\partial_n f(z_i)}{\sqrt{\det \text{Hess} f|_{\partial\Omega}(z_i)}} \left( \sum_{j=1}^{k_1^{\partial C_{\max}}} \frac{\partial_n f(z_j)}{\sqrt{\det \text{Hess} f|_{\partial\Omega}(z_j)}} \right)^{-1}. \quad (21)$$

3. When **(A4)** is satisfied the remainder term  $O(h^{\frac{1}{4}})$  in (19) is of the order  $O(e^{-\frac{c}{h}})$  for some  $c > 0$  and the remainder term  $O(h^{\frac{1}{4}})$  in (20) is of the order  $O(h)$  and admits a full asymptotic expansion in  $h$  (see (22) below).

Finally, the constants involved in the remainder terms in (18), (19), and (20) are uniform with respect to  $x \in K$ .

Let us recall that for  $\alpha > 0$ ,  $(r(h))_{h>0}$  admits a full asymptotic expansion in  $h^\alpha$  if there exists a sequence  $(a_k)_{k \geq 0} \in \mathbb{R}^{\mathbb{N}}$  such that for any  $N \in \mathbb{N}$ , it holds in the limit  $h \rightarrow 0$ :

$$r(h) = \sum_{k=0}^N a_k h^{\alpha k} + O(h^{\alpha(N+1)}). \quad (22)$$

According to (18), when the function  $F$  belongs to  $C^\infty(\partial\Omega, \mathbb{R})$  and  $x \in \mathcal{A}(C_{\max})$ , one has in the limit  $h \rightarrow 0$ :

$$\mathbb{E}_x[F(X_{\tau_\Omega})] = \sum_{i=1}^{k_1^{\partial C_{\max}}} a_i F(z_i) + O(h^{\frac{1}{4}}) = \frac{\sum_{i=1}^{k_1^{\partial C_{\max}}} \int_{\Sigma_i} F \partial_n f e^{-\frac{2}{h}f}}{\sum_{i=1}^{k_1^{\partial C_{\max}}} \int_{\Sigma_i} \partial_n f e^{-\frac{2}{h}f}} + o_h(1),$$

where the order in  $h$  of the remainder term  $o_h(1)$  depends on the support of  $F$  and on whether or not the assumption **(A4)** is satisfied. This is reminiscent of previous results obtained in [7, 8, 23, 24, 41].

Theorem 1 implies that when  $X_0 = x \in \mathcal{A}(C_{\max})$ , the law of  $X_{\tau_\Omega}$  concentrates on  $\{z_1, \dots, z_{k_1^{\partial C_{\max}}}\} = \partial\Omega \cap \partial C_{\max}$  in the limit  $h \rightarrow 0$ , with explicit formulas for the probabilities to exit through each of the  $z_i$ 's. Moreover, the probability to exit through a global minimum  $z$  of  $f|_{\partial\Omega}$  which satisfies  $\partial_n f(z) < 0$  is exponentially small in the limit  $h \rightarrow 0$  (see (18)) and when assuming **(A4)**, the probability to exit through  $z_{k_1^{\partial C_{\max}+1}}, \dots, z_{k_1^{\partial\Omega}}$  is also exponentially small even though all these points belong to  $\arg \min_{\partial\Omega} f$ . Theorem 1 is thus a generalization of [14, Theorem 1] to other initial conditions than the quasi-stationary distribution  $\nu_h$ .

**Remark 6.** Assumptions **(A0)** and **(A1)** ensure that  $C_{\max}$  appearing in Theorem 1 is well defined. Concerning the assumption **(A2)**, there exist functions  $f$  satisfying **(A0)** and **(A1)** but not **(A2)** such that when  $X_0 = x \in C_{\max}$ , the law of  $X_{\tau_\Omega}$  concentrates on points which do not belong to  $\arg \min_{\partial\Omega} f$  (see indeed [11, Figure 4]). The same holds for the assumption **(A3)**: there exist functions  $f$  which satisfy **(A0)**, **(A1)**, and **(A2)** but not **(A3)** such that when  $X_0 = x \in C_{\max}$ , the law of  $X_{\tau_\Omega}$  concentrates on points which do

not belong to  $\arg \min_{\partial\Omega} f$  (see Figure 5 and [11, Figure 5]). Thus, **(A2)** and **(A3)** are necessary to ensure that the law of  $X_{\tau_\Omega}$  concentrates on points belonging to  $\arg \min_{\partial\Omega} f$ . Finally, assumption **(A4)** is necessary to get item 3 in Theorem 1. Indeed, in [11, Section 1.4], we provide examples of functions  $f$  satisfying **(A0)**, **(A1)**, **(A2)**, and **(A3)** but not **(A4)**, such that for all sufficiently small neighborhood  $\Sigma_z$  of  $z \in \arg \min_{\partial\Omega} f \setminus \partial\mathbf{C}_{\max}$  in  $\partial\Omega$ ,  $\mathbb{P}[X_{\tau_\Omega} \in \Sigma_z] = C\sqrt{h}(1 + o(1))$  in the limit  $h \rightarrow 0$ , for some  $C > 0$  independent of  $h$  and when  $X_0 = x \in \mathbf{C}_{\max}$ .

**Remark 7.** When  $x \in \Omega$  is such that  $t_x < +\infty$ , it is a simple consequence of the large deviations estimate (52) below that, in the limit  $h \rightarrow 0$ , the process (1) almost surely exits  $\Omega$  through any neighborhood of  $\varphi_{t_x}(x)$  in  $\partial\Omega$ .

It is also possible to describe the exit point distribution when  $X_0 = x \in \mathcal{A}(\mathbf{C})$  and  $\mathbf{C} \in \mathcal{C}$  is not necessarily  $\mathbf{C}_{\max}$  (we recall that  $\mathcal{C}$  is defined in (12)). This is the objective of Theorem 2, whose proof uses Theorem 1 applied to a suitable subdomain of  $\Omega$  containing  $\mathbf{C}$ .

**Theorem 2.** Let us assume that **(A0)** holds. Let  $\mathbf{C} \in \mathcal{C}$ . Let us assume that

$$\partial\mathbf{C} \cap \partial\Omega \neq \emptyset \quad \text{and} \quad |\nabla f| \neq 0 \quad \text{on} \quad \partial\mathbf{C}. \quad (23)$$

Recall that  $\partial\mathbf{C} \cap \partial\Omega \subset \mathbf{U}_1^{\partial\Omega}$  (see (15) for a definition of  $\mathbf{U}_1^{\partial\Omega}$ ). Let  $F \in L^\infty(\partial\Omega, \mathbb{R})$ . For all  $z \in \partial\mathbf{C} \cap \partial\Omega$ , let  $\Sigma_z$  be an open subset of  $\partial\Omega$  such that  $z \in \Sigma_z$ . Let  $K$  be a compact subset of  $\Omega$  such that  $K \subset \mathcal{A}(\mathbf{C})$ . Then:

1. There exists  $c > 0$  such that for  $h$  small enough,

$$\sup_{x \in K} \mathbb{E}_x \left[ \left( F \mathbf{1}_{\partial\Omega \setminus \bigcup_{z \in \partial\mathbf{C} \cap \partial\Omega} \Sigma_z} \right) (X_{\tau_\Omega}) \right] \leq e^{-\frac{c}{h}}.$$

Assume moreover that the sets  $(\Sigma_z)_{z \in \partial\mathbf{C} \cap \partial\Omega}$  are pairwise disjoint (i.e. such that  $\Sigma_z \cap \Sigma_{z'} = \emptyset$  whenever  $z \neq z'$ ). Let  $z \in \partial\mathbf{C} \cap \partial\Omega$ .

2. If  $F$  is  $C^\infty$  in a neighborhood of  $z$ , it holds for all  $x \in K$ ,

$$\mathbb{E}_x[(F \mathbf{1}_{\Sigma_z})(X_{\tau_\Omega})] = F(z) \frac{\partial_n f(z)}{\sqrt{\det \text{Hess} f|_{\partial\Omega}(z)}} \left( \sum_{y \in \partial\mathbf{C} \cap \partial\Omega} \frac{\partial_n f(y)}{\sqrt{\det \text{Hess} f|_{\partial\Omega}(y)}} \right)^{-1} + O(h),$$

in the limit  $h \rightarrow 0$  and uniformly in  $x \in K$ .

Theorem 2 implies that when  $\mathbf{C} \in \mathcal{C}$  satisfies (23) (for instance, this is the case for  $\mathbf{C}_3$  on Figure 2), the law of  $X_{\tau_\Omega}$  when  $X_0 = x \in \mathcal{A}(\mathbf{C})$  concentrates when  $h \rightarrow 0$  on  $\partial\mathbf{C} \cap \partial\Omega$ .

Theorems 1 and 2 are actually special cases of a more general result which will be stated and illustrated in Section 3.3 (see indeed Theorem 4). The proof of Theorem 4 is a simple extension of the proof of Theorem 2. For pedagogical purposes, we prefer to first present Theorem 1 and 2, before stating the more general and abstract result of Theorem 4.

### 1.3 Organization of the paper

The rest of this paper is dedicated to the proofs of Theorem 1, in Section 2 and of Theorems 2 and 4, in Section 3. The proof of Theorem 1 heavily relies on results from [14] (recalled in Theorem 3 below), together with a so-called leveling result on  $x \mapsto \mathbb{E}_x[F(X_{\tau_\Omega})]$ . The proof of Theorem 2 uses Theorem 1 applied to a domain  $\Omega_{\mathbb{C}}$  which contains  $\mathbb{C}$ . The construction of this domain uses tools from differential topology related to the genericity of Morse functions. Finally, Section 4 gives conclusions and perspectives.

## 2 Proof of Theorem 1

After recalling some results from [14] in Section 2.1, we prove a so-called leveling result (as initially introduced in [4]) in  $\mathbb{C}_{\max}$  for  $x \mapsto \mathbb{E}_x[F(X_{\tau_\Omega})]$ , in Section 2.2. Then, combining the results of these two sections, one proves Theorem 1 in Section 2.3 for a smooth function  $F$  and finally for a measurable bounded  $F$  in Section 2.4.

### 2.1 Previous results on the principal eigenfunction of $L_{f,h}^D$ and on $\nu_h$

Let us recall the following result from [14, Theorem 4] on the spectral gap of  $L_{f,h}^D$ . We recall that  $\lambda_h$  is the principal eigenvalue of  $-L_{f,h}^D$  (see Section 1.1).

**Proposition 8.** *Assume that the assumptions **(A0)** and **(A1)** are satisfied. Let us denote by  $f_{\max}$  the value of  $f$  on  $\partial\mathbb{C}_{\max}$  and  $x_{\max}$  a minimum point of  $f$  in  $\overline{\mathbb{C}_{\max}}$ :*

$$f \equiv f_{\max} \text{ on } \partial\mathbb{C}_{\max} \text{ and } x_{\max} \in \arg \min_{\overline{\mathbb{C}_{\max}}} f. \quad (24)$$

Notice that  $f_{\max} = \max_{\overline{\mathbb{C}_{\max}}} f$  and  $x_{\max} \in \mathbb{C}_{\max}$ . Then, there exists  $C > 1$  and  $\gamma \in \mathbb{R}$ , such that, for  $h$  small enough,

$$C^{-1} h^\gamma e^{-\frac{2}{h}(f_{\max} - f(x_{\max}))} \leq \lambda_h \leq C h^\gamma e^{-\frac{2}{h}(f_{\max} - f(x_{\max}))}.$$

Moreover, there exists  $c > 0$  such that for  $h$  small enough,  $\min \sigma(L_{f,h}^D) \setminus \{\lambda_h\} \geq e^{\frac{c}{h}} \lambda_h$ .

We now give a series of three results which are consequences of the previous proposition. These results can all be found in [14]. We provide the proofs for these results since they are short and they are opportunities to introduce some notation which will be needed later on.

A direct corollary of Proposition 8 is the following.

**Corollary 9.** *Let us assume that the assumptions **(A0)** and **(A1)** are satisfied. Then, there exists  $\beta_0 > 0$  such that for all  $\beta \in (0, \beta_0)$ , there exists  $h_0 > 0$  such that for all  $h \in (0, h_0)$ , the orthogonal projector in  $L_w^2(\Omega)$*

$$\pi_h := \pi_{[0, e^{-\frac{2}{h}(f_{\max} - f(x_{\max}) - \beta)}]}(L_{f,h}^D) \text{ has rank 1.}$$

Here and in the following, for a Borel set  $E \subset \mathbb{R}$ ,  $\pi_E(L_{f,h}^D)$  is the spectral projector of  $L_{f,h}^D$  on  $E$ .

For  $\alpha > 0$ , one defines

$$\mathbb{C}_{\max}(\alpha) = \mathbb{C}_{\max} \cap \{f < f_{\max} - \alpha\}. \quad (25)$$

For  $\alpha > 0$ , let  $\chi \in C^\infty(\Omega)$  be such that

$$\chi = 0 \text{ on } \overline{\Omega} \setminus \mathbf{C}_{\max}(\alpha) \text{ and } \chi = 1 \text{ on } \mathbf{C}_{\max}(2\alpha).$$

Using Corollary 9, we obtain a good approximation of  $u_h$  using the function  $\chi$ .

**Proposition 10.** *Let us assume that the assumptions (A0) and (A1) are satisfied. Let us define*

$$\psi := \frac{\chi}{\|\chi\|_{L_w^2(\Omega)}}.$$

*Then, for all  $\alpha > 0$  small enough, there exists  $c > 0$  such that in the limit  $h \rightarrow 0$ :*

$$u_h = \psi + O(e^{-\frac{c}{h}}) \text{ in } L_w^2(\Omega).$$

*Proof.* Let us recall that for all  $u \in H_{w,0}^1(\Omega)$  and  $b > 0$ ,

$$\|\pi_{[b,+\infty)}(L_{f,h}^D)u\|^2 \leq \frac{\frac{h}{2} \int_{\Omega} |\nabla u|^2 e^{-\frac{2}{h}f}}{b}.$$

Thus, one has:

$$\|(1 - \pi_h)\psi\|_{L_w^2(\Omega)}^2 \leq e^{\frac{2}{h}(f_{\max} - f(x_{\max}) - \beta)} \frac{h}{2} \|\nabla \psi\|_{L_w^2(\Omega)}^2. \quad (26)$$

Moreover, by definition of the function  $\psi$ , it holds,

$$\|\nabla \psi\|_{L_w^2(\Omega)}^2 = \frac{\int_{\Omega} |\nabla \chi|^2 e^{-\frac{2}{h}f}}{\int_{\Omega} \chi^2 e^{-\frac{2}{h}f}}. \quad (27)$$

Since  $\chi = 0$  on  $\overline{\Omega} \setminus \mathbf{C}_{\max}(\alpha)$ , it holds,  $\int_{\Omega} \chi^2 e^{-\frac{2}{h}f} = \int_{\mathbf{C}_{\max}(\alpha)} \chi^2 e^{-\frac{2}{h}f}$ . Since  $f$  has finite number of global minima in  $\overline{\mathbf{C}_{\max}}$  which are all included in  $\mathbf{C}_{\max}$ , one deduces that for all  $\alpha > 0$  small enough,

$$\arg \min_{\overline{\mathbf{C}_{\max}}} f \subset \mathbf{C}_{\max}(2\alpha). \quad (28)$$

Consequently, using a Laplace's method together with the fact that  $\chi = 1$  on  $\mathbf{C}_{\max}(2\alpha)$ , one has in the limit  $h \rightarrow 0$ ,

$$\int_{\Omega} \chi^2 e^{-\frac{2}{h}f} = h^{\frac{d}{2}} \pi^{\frac{d}{2}} e^{-\frac{2}{h}f(x_{\max})} \sum_{x \in \arg \min_{\mathbf{C}_{\max}} f} (\det \text{Hess} f(x))^{-\frac{1}{2}} (1 + O(h)). \quad (29)$$

Since the support of  $\nabla \chi$  is included in  $\overline{\mathbf{C}_{\max}(\alpha)} \setminus \mathbf{C}_{\max}(2\alpha)$  (thus  $f \geq f_{\max} - 2\alpha$  on the support of  $\nabla \chi$  it holds for some  $C > 0$ :

$$\int_{\Omega} |\nabla \chi|^2 e^{-\frac{2}{h}f} \leq C h^{1-\frac{d}{2}} e^{-\frac{2}{h}(f_{\max} - 2\alpha)}.$$

Plugging these estimates in (27) and using (26), one finally deduces that for

$$\|(1 - \pi_h)\psi\|_{L_w^2(\Omega)}^2 \leq C e^{-\frac{2}{h}(\beta - 2\alpha)}.$$

Choosing  $\alpha < \beta/4$ , this implies that for  $h$  small enough,  $\|(1 - \pi_h)\psi\|_{L_w^2(\Omega)} \leq e^{-\frac{\beta}{h}}$ . Therefore, using Corollary 9 and the fact that the functions  $u_h$  and  $\psi$  are non negative,

$$u_h = \frac{\pi_h \psi}{\|\pi_h \psi\|_{L_w^2(\Omega)}} = \psi + O(e^{-\frac{c}{h}}) \text{ in } L_w^2(\Omega)$$

for some positive  $c$ . This concludes the proof of Proposition 10. ■

We end this section, by the following consequence of Proposition 10.

**Corollary 11.** *Let us assume that the assumptions (A0) and (A1) are satisfied. Let us moreover assume that*

$$\frac{\min}{\mathcal{C}_{\max}} f = \min_{\bar{\Omega}} f.$$

Then, when  $h \rightarrow 0$ :

$$\int_{\Omega} u_h e^{-\frac{2}{h}f} = h^{\frac{d}{4}} \pi^{\frac{d}{4}} e^{-\frac{1}{h} \min_{\bar{\Omega}} f} \left( \sum_{x \in \arg \min_{\mathcal{C}_{\max}} f} (\det \text{Hess} f(x))^{-\frac{1}{2}} \right)^{\frac{1}{2}} (1 + O(h)).$$

Moreover, if  $\mathcal{O}$  is an open subset of  $\Omega$  such that  $\bar{\mathcal{O}} \cap \arg \min_{\mathcal{C}_{\max}} f = \emptyset$ , then, there exists  $c > 0$  such that for  $h$  small enough,

$$\int_{\mathcal{O}} u_h e^{-\frac{2}{h}f} = O(e^{-\frac{1}{h}(\min_{\bar{\Omega}} f + c)}).$$

*Proof.* Let  $\mathcal{O}$  be an open subset of  $\Omega$ . Using Proposition 10 and thanks to the Cauchy-Schwarz inequality, one obtains in the limit  $h \rightarrow 0$ :

$$\int_{\mathcal{O}} u_h e^{-\frac{2}{h}f} = \int_{\mathcal{O}} \psi e^{-\frac{2}{h}f} + O(e^{-\frac{c}{h}}) \sqrt{\int_{\mathcal{O}} e^{-\frac{2}{h}f}} = \int_{\mathcal{O}} \psi e^{-\frac{2}{h}f} + O(e^{-\frac{1}{h}(\min_{\bar{\Omega}} f + c)}).$$

Then, the first statement in Corollary 11 follows choosing  $\mathcal{O} = \Omega$  in the previous equality and using (29) (notice that the same estimate holds replacing  $\chi^2$  by  $\chi$  in (29)), the definition of  $\psi$  (see Proposition 10) together with the fact that by assumption  $f(x_{\max}) = \min_{\mathcal{C}_{\max}} f = \min_{\bar{\Omega}} f$ . The second statement in Corollary 11 follows using (29) and the fact that when  $\bar{\mathcal{O}} \cap \arg \min_{\mathcal{C}_{\max}} f = \emptyset$ , there exists  $c > 0$  such that for  $h$  small enough,  $\int_{\mathcal{O}} \chi e^{-\frac{2}{h}f} = O(e^{-\frac{2}{h}(\min_{\bar{\Omega}} f + c)})$ .  $\blacksquare$

We end this section by recalling the results of [14, Theorem 1] on the law of  $X_{\tau_{\Omega}}$  when  $X_0$  is initially distributed according to the quasi-stationary distribution  $\nu_h$  of the process (1).

**Theorem 3.** *Let us assume that the assumptions (A0), (A1), (A2), and (A3) are satisfied. Let  $F \in L^{\infty}(\partial\Omega, \mathbb{R})$  and  $(\Sigma_i)_{i \in \{1, \dots, k_1^{\partial\Omega}\}}$  be a family of pairwise disjoint open subsets of  $\partial\Omega$  such that*

$$\text{for all } i \in \{1, \dots, k_1^{\partial\Omega}\}, z_i \in \Sigma_i,$$

where we recall that  $\{z_1, \dots, z_{k_1^{\partial\Omega}}\} = \mathcal{U}_1^{\partial\Omega} \cap \arg \min_{\partial\Omega} f$  (see (16)). Then:

1. *There exists  $c > 0$  such that when  $h \rightarrow 0$ :*

$$\mathbb{E}_{\nu_h} [F(X_{\tau_{\Omega}})] = \sum_{i=1}^{k_1^{\partial\Omega}} \mathbb{E}_{\nu_h} [(1_{\Sigma_i} F)(X_{\tau_{\Omega}})] + O(e^{-\frac{c}{h}}) \quad (30)$$

and

$$\sum_{i=k_1^{\partial\mathcal{C}_{\max}}+1}^{k_1^{\partial\Omega}} \mathbb{E}_{\nu_h} [(1_{\Sigma_i} F)(X_{\tau_{\Omega}})] = O(h^{\frac{1}{4}}), \quad (31)$$

where we recall that  $\{z_1, \dots, z_{k_1^{\partial\mathcal{C}_{\max}}}\} = \partial\mathcal{C}_{\max} \cap \partial\Omega$  (see (17)).

2. When for some  $i \in \{1, \dots, k_1^{\partial C_{\max}}\}$  the function  $F$  is  $C^\infty$  in a neighborhood of  $z_i$ , one has when  $h \rightarrow 0$ :

$$\mathbb{E}_{\nu_h}[(\mathbf{1}_{\Sigma_i} F)(X_{\tau_\Omega})] = F(z_i) a_i + O(h^{\frac{1}{4}}) \quad \text{where } a_i \text{ is defined by (21)}. \quad (32)$$

3. When **(A4)** is satisfied the remainder term  $O(h^{\frac{1}{4}})$  in (19) is of the order  $O(e^{-\frac{c}{h}})$  for some  $c > 0$  and the remainder term  $O(h^{\frac{1}{4}})$  in (20) is of the order  $O(h)$  and admits a full asymptotic expansion in  $h$ .

## 2.2 Leveling results

To go from Theorem 3 to Theorem 1, the basic idea is to write

$$\mathbb{E}_{\nu_h}[F(X_{\tau_\Omega})] = \int_{\Omega} \mathbb{E}_x[F(X_{\tau_\Omega})] \nu_h(dx)$$

and to use the fact that the function  $x \mapsto \mathbb{E}_x[F(X_{\tau_\Omega})]$  is “more and more constant” as  $h \rightarrow 0$ : this is called a *leveling property*.

**Definition 12.** Let  $K$  be a compact subset of  $\Omega$  and  $F \in C^\infty(\partial\Omega, \mathbb{R})$ . We say that  $x \mapsto \mathbb{E}_x[F(X_{\tau_\Omega})]$  satisfies a leveling property on  $K$  if

$$\lim_{h \rightarrow 0} |\mathbb{E}_x[F(X_{\tau_\Omega})] - \mathbb{E}_y[F(X_{\tau_\Omega})]| = 0 \quad (33)$$

and this limit holds uniformly with respect to  $(x, y) \in K \times K$ .

The leveling property (33) has been widely studied in the literature in various geometrical settings, see for example [4, 10, 15, 17, 23, 24, 41]. We prove the following proposition which is a leveling property in our framework.

**Proposition 13.** Let us assume that the assumption **(A0)** holds. Let  $\lambda \in \mathbb{R}$  and  $\mathbf{C}$  be a connected component of  $\{f < \lambda\}$  such that  $\mathbf{C} \subset \Omega$ . Then, for any path-connected compact set  $K \subset \mathbf{C}$  and for any  $F \in C^\infty(\partial\Omega, \mathbb{R})$ , there exist  $c > 0$  and  $M > 0$ , such that for all  $(x, y) \in K \times K$ ,

$$|\mathbb{E}_x[F(X_{\tau_\Omega})] - \mathbb{E}_y[F(X_{\tau_\Omega})]| \leq M e^{-\frac{c}{h}}. \quad (34)$$

*Proof.* The proof is inspired from techniques used in [10]. The proof of Proposition 13 is divided into two steps. In the following  $C > 0$  is a constant which can change from one occurrence to another and which does not depend on  $h$ .

**Step 1.** Let  $F \in C^\infty(\partial\Omega, \mathbb{R})$ . Let us denote by  $v_h \in H^1(\Omega)$  the unique weak solution to the elliptic boundary value problem

$$\begin{cases} \frac{h}{2} \Delta v_h - \nabla f \cdot \nabla v_h = 0 & \text{on } \Omega \\ v_h = F & \text{on } \partial\Omega. \end{cases} \quad (35)$$

Then, we will prove in this step that  $v_h$  belongs to  $C^\infty(\overline{\Omega}, \mathbb{R})$  and that for all  $k \in \mathbb{N}$ , there exist  $C > 0$ ,  $n \in \mathbb{N}$  and  $h_0 > 0$  such that for all  $h \in (0, h_0)$ , it holds

$$\|v_h\|_{H^{k+2}(\Omega)} \leq \frac{C}{h^n} (\|\nabla v_h\|_{L^2(\Omega)} + 1). \quad (36)$$



Here  $L^2(\Omega) = \{u : \Omega \rightarrow \mathbb{R}, \int_{\Omega} u^2 < \infty\}$ , and, for  $k \geq 1$ ,

$$H^k(\Omega) = \{u : \Omega \rightarrow \mathbb{R}, \forall \alpha \in \mathbb{N}^k \text{ such that } |\alpha| \leq k, \int_{\Omega} |\partial^\alpha u|^2 < \infty\}.$$

Moreover, the Dynkin's formula implies that

$$\forall x \in \bar{\Omega}, v_h(x) = \mathbb{E}_x [F(X_{\tau_{\Omega}})]. \quad (37)$$

Let us prove that  $v_h$  belongs to  $C^\infty(\bar{\Omega}, \mathbb{R})$  and (36). Since  $F$  is  $C^\infty$ , for all  $k \geq 1$ , there exists  $\tilde{F} \in H^k(\Omega)$  such that  $\tilde{F} = F$  on  $\partial\Omega$  and

$$\|\tilde{F}\|_{H^k} \leq C \|F\|_{H^{k-\frac{1}{2}}(\partial\Omega)}.$$

From (35), the function  $\tilde{v}_h = v_h - \tilde{F} \in H^1(\Omega)$  is the weak solution to

$$\begin{cases} \Delta \tilde{v}_h = \frac{2}{h} \nabla f \cdot \nabla v_h - \Delta \tilde{F} \text{ on } \Omega \\ \tilde{v}_h = 0 \text{ on } \partial\Omega. \end{cases} \quad (38)$$

Thus, using [16, Theorem 5, Section 6.3],  $\tilde{v}_h \in H^2(\Omega)$  (and thus  $v_h \in H^2(\Omega)$ ) and there exist  $C > 0$  and  $h_0 > 0$  such that for all  $h \in (0, h_0)$

$$\|\tilde{v}_h\|_{H^2(\Omega)} \leq C \left( \frac{1}{h} \|\nabla v_h\|_{L^2(\Omega)} + \|\tilde{F}\|_{H^2(\Omega)} \right) \leq \frac{C}{h} \left( \|\nabla v_h\|_{L^2(\Omega)} + \|F\|_{H^{\frac{3}{2}}(\partial\Omega)} \right),$$

and thus

$$\|v_h\|_{H^2(\Omega)} \leq \frac{C}{h} \left( \|\nabla v_h\|_{L^2(\Omega)} + 1 \right). \quad (39)$$

This proves (36) for  $k = 0$ . The inequality (36) is then obtained by a bootstrap argument, by induction on  $k$ . This implies by Sobolev embeddings that  $v_h$  belongs to  $C^\infty(\bar{\Omega}, \mathbb{R})$ .

Let us now prove that there exist  $\alpha > 0$  and  $C > 0$  such that

$$\|v_h\|_{L^\infty(\Omega)} + \|\nabla v_h\|_{L^\infty(\Omega)} \leq Ch^{-\alpha}. \quad (40)$$

Notice that from (37), one has that for all  $h > 0$ ,  $\|v_h\|_{L^\infty(\Omega)} \leq \|F\|_{L^\infty(\partial\Omega)}$ . Using this bound, (35) and (39), there exists  $C > 0$  such that for any  $\varepsilon > 0$  and  $\varepsilon' > 0$ ,

$$\begin{aligned} h \int_{\Omega} |\nabla v_h|^2 &\leq C \left( h \int_{\partial\Omega} |F \partial_n v_h| d\sigma + \|F\|_{L^\infty(\partial\Omega)} \int_{\Omega} |\nabla f \cdot \nabla v_h| \right) \\ &\leq C \left( h \frac{\|F\|_{L^2(\partial\Omega)}^2}{4\varepsilon} + h\varepsilon \|v_h\|_{H^2(\Omega)}^2 + \frac{\|\nabla f\|_{L^2(\Omega)}^2}{4\varepsilon'} + \varepsilon' \|\nabla v_h\|_{L^2(\Omega)}^2 \right) \\ &\leq Ch \frac{\|F\|_{L^2(\partial\Omega)}^2}{4\varepsilon} + Ch\varepsilon C_1 h^{-2} \left( \|\nabla v_h\|_{L^2(\Omega)}^2 + 1 \right) \\ &\quad + C \frac{\|\nabla f\|_{L^2(\Omega)}^2}{4\varepsilon'} + C\varepsilon' \|\nabla v_h\|_{L^2(\Omega)}^2. \end{aligned}$$

Choosing  $\varepsilon = \frac{h^2}{4(CC_1+1)}$  and  $\varepsilon' = \frac{h}{4(C+1)}$  we get

$$\|\nabla v_h\|_{L^2(\Omega)} \leq \frac{C}{h}.$$

Therefore, using (36), one obtains that for all  $k \geq 0$ , there exist  $C > 0$ ,  $n \in \mathbb{N}$  and  $h_0 > 0$  such that for all  $h \in (0, h_0)$

$$\|v_h\|_{H^k(\Omega)} \leq \frac{C}{h^n}.$$

Let  $k \geq 0$  such that  $k - \frac{d}{2} > 1$ . Then, one obtains (40) from the continuous Sobolev injection  $H^k(\Omega) \subset W^{1,\infty}(\Omega)$ .

**Step 2.** Let us assume that **(A0)** holds. Let  $\lambda \in \mathbb{R}$  and  $\mathbf{C}$  be a connected component of  $\{f < \lambda\}$  such that  $\mathbf{C} \subset \Omega$ . Let us now define the set  $\mathbf{C}_r$  by

$$\mathbf{C}_r = \{f < \lambda - r\} \cap \mathbf{C} \subset \Omega \quad (41)$$

which is not empty and  $C^\infty$  for all  $r \in (0, r_1)$ , for some  $r_1 > 0$ . Indeed, the boundary of  $\mathbf{C}_r$  is the set  $\{f = \lambda - r\} \cap \mathbf{C}$  (since  $\mathbf{C} \subset \Omega$ ) which contains no critical points of  $f$  for  $r \in (0, r_1)$ , with  $r_1 > 0$  small enough (since there is a finite number of critical points under the assumption **(A0)**). In this step, we will prove that for all  $r_0 \in (0, r_1)$  there exists  $\alpha_0 > 0$  such that

$$\|\nabla v_h\|_{L^\infty(\mathbf{C}_{r_0})} \leq e^{-\frac{\alpha_0}{h}}. \quad (42)$$

Equation (42) implies that for any compact subset  $K$  of  $\mathbf{C}$ , there exist  $c > 0$  and  $C > 0$  such that

$$\forall (x, y) \in K \times K, |v_h(x) - v_h(y)| \leq C e^{-\frac{c}{h}},$$

which will then conclude the proof of (34). This follows from the fact that there exists  $r_0 \in (0, r_1)$  such that  $K \subset \mathbf{C}_{r_0}$ .

Let us now prove (42). Let  $r$  be such that  $2^n r = r_0$  where  $n \in \mathbb{N}^*$  will be fixed later (since  $r \leq r_0$ ,  $r \in (0, r_1)$ ). Equation (35) rewrites

$$\begin{cases} \operatorname{div} (e^{-\frac{2}{h}f} \nabla v_h) = 0 & \text{on } \Omega \\ v_h = F & \text{on } \partial\Omega. \end{cases}$$

Using (40), there exist  $C > 0$  and  $\alpha > 0$  such that,

$$\left| \int_{\mathbf{C}_{r/2}} |\nabla v_h|^2 e^{-\frac{2}{h}f} \right| = \left| \int_{\partial\mathbf{C}_{r/2}} e^{-\frac{2}{h}f} v_h \partial_n v_h d\sigma \right| \leq \frac{C}{h^\alpha} e^{-\frac{2}{h}(\lambda - \frac{r}{2})},$$

where we used the Green formula (valid since  $\mathbf{C}_r$  is  $C^\infty$  for all  $r \in (0, r_1)$ ) and the inclusion  $\partial\mathbf{C}_{r/2} \subset \{f = \lambda - \frac{r}{2}\}$ . In addition, since  $\mathbf{C}_r \subset \mathbf{C}_{r/2}$  it holds,

$$e^{-\frac{2}{h}(\lambda - r)} \int_{\mathbf{C}_r} |\nabla v_h|^2 \leq \int_{\mathbf{C}_r} |\nabla v_h|^2 e^{-\frac{2}{h}f} \leq \int_{\mathbf{C}_{r/2}} |\nabla v_h|^2 e^{-\frac{2}{h}f} \leq \frac{C}{h^\alpha} e^{-\frac{2}{h}(\lambda - \frac{r}{2})}.$$

Therefore, there exists  $\beta > 0$  such that for  $h$  small enough,

$$\int_{\mathbf{C}_r} |\nabla v_h|^2 \leq \frac{C}{h^\alpha} e^{-\frac{r}{h}} \leq C e^{-\frac{\beta}{h}},$$

and from (35), we then have  $\|\Delta v_h\|_{L^2(\mathbf{C}_r)} \leq C e^{-\frac{\beta}{h}}$  for some constant  $\beta > 0$ . In the following,  $\beta > 0$  is a constant which may change from one occurrence to another and does not depend on  $h$ . Let  $\chi_1 \in C_c^\infty(\mathbf{C}_r)$  be such that  $\chi_1 \equiv 1$  on  $\mathbf{C}_{2r}$ . Since  $\Delta(\chi_1 v_h) =$

$\chi_1 \Delta v_h + v_h \Delta \chi_1 + 2\nabla \chi_1 \cdot \nabla v_h$ , there exists  $C$ , such that  $\|\Delta(\chi_1 v_h)\|_{L^2(\mathbf{C}_r)} \leq C$  for  $h$  small enough. By elliptic regularity (see [16, Theorem 5, Section 6.3]) it comes

$$\|v_h\|_{H^2(\mathbf{C}_{2r})} \leq C.$$

Let  $\alpha \in (0, 1)$  be an irrational number such that  $p_1 = \frac{2d}{d-2\alpha} > 0$ . From the Gagliardo-Nirenberg interpolation inequality (see [39, Lecture II]), the following inequality holds

$$\|\nabla v_h\|_{L^{p_1}(\mathbf{C}_{2r})} \leq C \|v_h\|_{H^2(\mathbf{C}_{2r})}^\alpha \|\nabla v_h\|_{L^2(\mathbf{C}_{2r})}^{1-\alpha} + C \|\nabla v_h\|_{L^2(\mathbf{C}_{2r})} \leq C e^{-\frac{\beta}{h}}.$$

From (35),  $\|\Delta v_h\|_{L^{p_1}(\mathbf{C}_{2r})} \leq C e^{-\frac{\beta}{h}}$ . Using a cutoff function  $\chi_2 \in C_c^\infty(\mathbf{C}_{2r})$  such that  $\chi_2 \equiv 1$  on  $\mathbf{C}_{4r}$ , we get, as previously, from the elliptic regularity  $\|v_h\|_{W^{2,p_1}(\mathbf{C}_{4r})} \leq C$ . Let  $p_2 = \frac{2d}{d-4\alpha}$  (i.e.  $1/p_2 = 1/p_1 - \alpha/d$ ). If  $p_2 < 0$ , then [39, Lecture II] implies

$$\|\nabla v_h\|_{L^\infty(\mathbf{C}_{4r})} \leq C \|v_h\|_{W^{2,p_1}(\mathbf{C}_{4r})}^\alpha \|\nabla v_h\|_{L^{p_1}(\mathbf{C}_{4r})}^{1-\alpha} + C \|\nabla v_h\|_{L^{p_1}(\mathbf{C}_{4r})} \leq C e^{-\frac{\beta}{h}}.$$

Thus, (42) is proved (if one chooses  $n = 2$ , i.e.  $2^2 r = r_0$ ). Otherwise, we prove (42) by induction as follows. From the Gagliardo-Nirenberg interpolation inequality (see [39, Lecture II]), we get

$$\|\nabla v_h\|_{L^{p_2}(\mathbf{C}_{4r})} \leq C \|v_h\|_{W^{2,p_1}(\mathbf{C}_{4r})}^\alpha \|\nabla v_h\|_{L^{p_1}(\mathbf{C}_{4r})}^{1-\alpha} + C \|\nabla v_h\|_{L^{p_1}(\mathbf{C}_{4r})} \leq C e^{-\frac{\beta}{h}}.$$

We repeat this procedure  $n$  times where  $n$  is the first integer such that  $d - 2n\alpha < 0$  and the Gagliardo-Nirenberg interpolation inequality implies that  $\|\nabla v_h\|_{L^\infty(\mathbf{C}_{2^n r})} \leq C e^{-\frac{\beta}{h}}$  which ends the proof of (42). This concludes the proof of Proposition 13.  $\blacksquare$

### 2.3 Link between the law of $X_{\tau_\Omega}$ when $X_0 \sim \nu_h$ and $X_0 = x \in \mathcal{A}(\mathbf{C}_{\max})$

Using Proposition 13, one can now compare  $\mathbb{E}_{\nu_h}[F(X_{\tau_\Omega})]$  and  $\mathbb{E}_x[F(X_{\tau_\Omega})]$  for smooth functions  $F$ : the next proposition combined with Theorem 3 already gives the result of Theorem 1 for smooth functions  $F$ .

**Proposition 14.** *Assume that the assumptions (A0) and (A1) are satisfied and that*

$$\frac{\min}{\mathbf{C}_{\max}} f = \min_{\Omega} f,$$

where we recall that  $\mathbf{C}_{\max}$  is introduced in (A1). Let  $K$  be a compact subset of  $\Omega$  such that  $K \subset \mathcal{A}(\mathbf{C}_{\max})$  and let  $F \in C^\infty(\partial\Omega, \mathbb{R})$ . Then, there exists  $c > 0$  such that for all  $x \in K$ :

$$\mathbb{E}_{\nu_h}[F(X_{\tau_\Omega})] = \mathbb{E}_x[F(X_{\tau_\Omega})] + O(e^{-\frac{c}{h}})$$

in the limit  $h \rightarrow 0$  and uniformly in  $x \in K$ .

*Proof.* Assume that the assumptions (A0) and (A1) are satisfied and that

$$\frac{\min}{\mathbf{C}_{\max}} f = \min_{\Omega} f.$$

**Step 1.** For  $\alpha > 0$  small enough, let  $\mathbf{C}_{\max}(\alpha)$  be as introduced in (25):

$$\mathbf{C}_{\max}(\alpha) = \mathbf{C}_{\max} \cap \{f < f_{\max} - \alpha\}.$$

Let  $F \in C^\infty(\partial\Omega, \mathbb{R})$ . In this first step, we will prove that  $\exists \alpha_0 > 0, \forall \alpha \in (0, \alpha_0), \exists c > 0, \forall y \in \overline{C_{\max}(\alpha)}$ :

$$\mathbb{E}_{\nu_h} [F(X_{\tau_\Omega})] = \mathbb{E}_y [F(X_{\tau_\Omega})] + O(e^{-\frac{c}{h}}) \quad (43)$$

in the limit  $h \rightarrow 0$  and uniformly in  $y \in \overline{C_{\max}(\alpha)}$ . Let us recall that from the notation of Proposition 13 (see (37)), for all  $x \in \overline{\Omega}$ :

$$v_h(x) = \mathbb{E}_x [F(X_{\tau_\Omega})].$$

From (8), one has:

$$\begin{aligned} \mathbb{E}_{\nu_h} [F(X_{\tau_\Omega})] &= \left( \int_{\Omega} u_h e^{-\frac{2}{h}f} \right)^{-1} \int_{\Omega} v_h u_h e^{-\frac{2}{h}f} \\ &= \frac{1}{Z_h(\Omega)} \int_{\overline{C_{\max}(\alpha)}} v_h u_h e^{-\frac{2}{h}f} + \frac{1}{Z_h(\Omega)} \int_{\Omega \setminus \overline{C_{\max}(\alpha)}} v_h u_h e^{-\frac{2}{h}f}, \end{aligned} \quad (44)$$

where

$$Z_h(\Omega) := \int_{\Omega} u_h e^{-\frac{2}{h}f}$$

and  $u_h$  is the principal eigenfunction of  $-L_{f,h}^D$  which satisfies (7). Let us first deal with the second term in (44). Since **(A0)** and **(A1)** hold, and because it is assumed that  $\min_{\overline{C_{\max}}} f = \min_{\overline{\Omega}} f$ , one obtains from Corollary 11 that there exists  $C > 0$  such that for  $h$  small enough:

$$\frac{1}{Z_h(\Omega)} \leq Ch^{-\frac{d}{4}} e^{\frac{1}{h} \min_{\overline{\Omega}} f}.$$

Let us recall that for  $\alpha > 0$  small enough, one has (see (28)),

$$\arg \min_{\overline{C_{\max}}} f \subset C_{\max}(\alpha).$$

Therefore, using the second statement in Corollary 11 with  $\mathbf{O} = \Omega \setminus \overline{C_{\max}(\alpha)}$ , for all  $\alpha > 0$  small enough, there exists  $c > 0$  such that when  $h \rightarrow 0$ :

$$\int_{\Omega \setminus \overline{C_{\max}(\alpha)}} u_h e^{-\frac{2}{h}f} = O\left(e^{-\frac{1}{h}(\min_{\overline{\Omega}} f + c)}\right).$$

Thus, there exists  $\alpha_0 > 0$  such that for all  $\alpha \in (0, \alpha_0)$  there exists  $c > 0$  such that when  $h \rightarrow 0$ :

$$\frac{1}{Z_h(\Omega)} \int_{\Omega \setminus \overline{C_{\max}(\alpha)}} u_h e^{-\frac{2}{h}f} = O(e^{-\frac{c}{h}}). \quad (45)$$

Then, since  $\|v_h\|_{L^\infty(\overline{\Omega})} \leq \|F\|_{L^\infty(\partial\Omega)}$ , one obtains that

$$\frac{1}{Z_h(\Omega)} \int_{\Omega \setminus \overline{C_{\max}(\alpha)}} v_h u_h e^{-\frac{2}{h}f} = O(e^{-\frac{c}{h}}). \quad (46)$$

Let us now deal with the first term in (44). Let us recall that  $C_{\max} \subset \Omega$  is a connected component of  $\{f < \max_{\overline{C_{\max}}} f\}$ . Moreover, for  $\alpha \in (0, \alpha_0)$  ( $\alpha_0 > 0$  small enough), the compact set  $\overline{C_{\max}(\alpha)}$  is connected and  $\overline{C_{\max}(\alpha)} \subset C_{\max}$ . Therefore, from Proposition 13 applied to  $K = \overline{C_{\max}(\alpha)}$  for  $\alpha \in (0, \alpha_0)$ , one obtains that there exists  $\delta_\alpha > 0$  such that for all  $y \in \overline{C_{\max}(\alpha)}$ ,

$$\frac{1}{Z_h(\Omega)} \int_{\overline{C_{\max}(\alpha)}} v_h u_h e^{-\frac{2}{h}f} = \frac{v_h(y)}{Z_h(\Omega)} \int_{\overline{C_{\max}(\alpha)}} u_h e^{-\frac{2}{h}f} + \frac{O(e^{-\frac{\delta_\alpha}{h}})}{Z_h(\Omega)} \int_{\overline{C_{\max}(\alpha)}} u_h e^{-\frac{2}{h}f} \quad (47)$$

in the limit  $h \rightarrow 0$  and uniformly with respect to  $y \in \overline{\mathbf{C}_{\max}(\alpha)}$ . Moreover, for all  $\alpha \in (0, \alpha_0)$  there exists  $c > 0$  such that in the limit  $h \rightarrow 0$ :

$$\frac{1}{Z_h(\Omega)} \int_{\mathbf{C}_{\max}(\alpha)} u_h e^{-\frac{2}{h}f} = 1 + O\left(e^{-\frac{c}{h}}\right). \quad (48)$$

which follows from the fact that

$$\frac{1}{Z_h(\Omega)} \int_{\mathbf{C}_{\max}(\alpha)} u_h e^{-\frac{2}{h}f} = 1 - \frac{1}{Z_h(\Omega)} \int_{\Omega \setminus \mathbf{C}_{\max}(\alpha)} u_h e^{-\frac{2}{h}f},$$

together with (45). Let us now fix  $\alpha \in (0, \alpha_0)$ . Then, using (47) and (48),  $\exists c > 0$ ,  $\exists \delta_\alpha > 0$ ,  $\forall y \in \overline{\mathbf{C}_{\max}(\alpha)}$ :

$$\frac{1}{Z_h(\Omega)} \int_{\mathbf{C}_{\max}(\alpha)} v_h u_h e^{-\frac{2}{h}f} = v_h(y) \left(1 + O\left(e^{-\frac{c}{h}}\right)\right) + O\left(e^{-\frac{\delta_\alpha}{h}}\right) \quad (49)$$

in the limit  $h \rightarrow 0$  and uniformly with respect to  $y \in \overline{\mathbf{C}_{\max}(\alpha)}$ . Therefore, using (44), (46) and (49),  $\exists \alpha_0 > 0$ ,  $\forall \alpha \in (0, \alpha_0)$ ,  $\exists c > 0$ ,  $\forall y \in \overline{\mathbf{C}_{\max}(\alpha)}$ :

$$\mathbb{E}_{\nu_h} [F(X_{\tau_\Omega})] = \mathbb{E}_y [F(X_{\tau_\Omega})] + O\left(e^{-\frac{c}{h}}\right),$$

in the limit  $h \rightarrow 0$  and uniformly with respect to  $y \in \overline{\mathbf{C}_{\max}(\alpha)}$ . This concludes the proof of (43).

**Step 2.** Let us now conclude the proof of Proposition 14 by considering a compact subset  $K$  of  $\Omega$  such that  $K \subset \mathcal{A}(\mathbf{C}_{\max})$ . Let us recall that (see (14)):

$$\mathcal{A}(\mathbf{C}_{\max}) = \{x \in \Omega, t_x = +\infty \text{ and } \omega(x) \subset \mathbf{C}_{\max}\}.$$

Since  $\mathbf{C}_{\max}$  is open and stable by the flow  $\varphi_t(\cdot)$  (defined by (13)), the continuity of  $\varphi_t(\cdot)$  implies that there exists  $T_K \geq 0$  such that for all  $x \in K$ ,

$$\varphi_{T_K}(x) \in \mathbf{C}_{\max}.$$

Moreover, since  $K$  is a compact subset of  $\Omega$  and for all  $x \in K$ ,  $t_x = +\infty$  (i.e.  $\varphi_t(x) \in \Omega$  for all  $t \geq 0$ ), there exists  $\delta > 0$  such that all continuous curves  $\gamma : [0, T_K] \rightarrow \overline{\Omega}$  such that

$$\exists x \in K, \sup_{t \in [0, T_K]} |\gamma(t) - \varphi_t(x)| \leq \delta,$$

satisfy:

$$\forall t \in [0, T_K], \gamma(t) \in \Omega. \quad (50)$$

Furthermore, up to choosing  $\delta > 0$  smaller, there exists  $\alpha_K > 0$  such that

$$\{\varphi_{T_K}(x) + z, x \in K \text{ and } |z| \leq \delta\} \subset \mathbf{C}_{\max}(\alpha_K) \quad (51)$$

where, we recall,  $\mathbf{C}_{\max}(\alpha_K)$  is defined by (25). Let us now recall the following estimate of Freidlin and Wentzell (see [17, Theorems 2.2 and 2.3 in Chapter 3, and Theorem 1.1 in Chapter 4], [5], [9, Theorem 3.5] and [18, Theorem 5.6.3]): for all  $x \in K$ , it holds:

$$\limsup_{h \rightarrow 0} h \ln \mathbb{P}_x \left[ \sup_{t \in [0, T_K]} |X_t - \varphi_t(x)| \geq \delta \right] \leq -I_{x, T_K}, \quad (52)$$

where

$$I_{x,T_K} = \frac{1}{2} \inf_{\gamma \in H_{x,T_K}^1(\delta)} \int_0^{T_K} \left| \frac{d}{dt} \gamma(t) + \nabla f(\gamma(t)) \right|^2 dt \in \mathbb{R}_+^* \cup \{+\infty\},$$

and  $H_{x,T_K}^1(\delta)$  is the set of curves  $\gamma : [0, T_K] \rightarrow \Omega$  of regularity  $H^1$  such that  $\gamma(0) = x$  and  $\sup_{t \in [0, T_K]} |\gamma(t) - \varphi_t(x)| \geq \delta$ . Since  $K$  is compact, there exists  $\eta_K > 0$  such that for  $h$  small enough, it holds:

$$\sup_{x \in K} \mathbb{P}_x \left[ \sup_{t \in [0, T_K]} |X_t - \varphi_t(x)| \geq \delta \right] \leq e^{-\frac{\eta_K}{h}}. \quad (53)$$

Notice that when  $X_0 = x \in K$  and  $\sup_{t \in [0, T_K]} |X_t - \varphi_t(x)| \leq \delta$ , it holds from (50) and (51):

$$\tau_\Omega > T_K \text{ and } X_{T_K} \in C_{\max}(\alpha_K). \quad (54)$$

Let us now consider  $F \in C^\infty(\partial\Omega, \mathbb{R})$ . Let  $x \in K$ . Then,

$$\mathbb{E}_x [F(X_{\tau_\Omega})] = \mathbb{E}_x \left[ F(X_{\tau_\Omega}) \mathbf{1}_{\sup_{t \in [0, T_K]} |X_t - \varphi_t(x)| \leq \delta} \right] + \mathbb{E}_x \left[ F(X_{\tau_\Omega}) \mathbf{1}_{\sup_{t \in [0, T_K]} |X_t - \varphi_t(x)| \geq \delta} \right].$$

Using (53), it holds for  $h$  small enough:

$$\left| \mathbb{E}_x \left[ F(X_{\tau_\Omega}) \mathbf{1}_{\sup_{t \in [0, T_K]} |X_t - \varphi_t(x)| \geq \delta} \right] \right| \leq \|F\|_{L^\infty} e^{-\frac{\eta_K}{h}}.$$

Using (54), (43) (with  $\alpha = \alpha_K$ ), (53), and the Markov property of the process (1), there exists  $c > 0$  such that for all  $x \in K$ , one has when  $h \rightarrow 0$ :

$$\begin{aligned} \mathbb{E}_x \left[ F(X_{\tau_\Omega}) \mathbf{1}_{\sup_{t \in [0, T_K]} |X_t - \varphi_t(x)| \leq \delta} \right] &= \mathbb{E}_x \left[ \mathbb{E}_{X_{T_K}} [F(X_{\tau_\Omega})] \mathbf{1}_{\sup_{t \in [0, T_K]} |X_t - \varphi_t(x)| \leq \delta} \right] \\ &= \left( \mathbb{E}_{\nu_h} [F(X_{\tau_\Omega})] + O(e^{-\frac{c}{h}}) \right) \mathbb{P}_x \left[ \sup_{t \in [0, T_K]} |X_t - \varphi_t(x)| \leq \delta \right] \\ &= \mathbb{E}_{\nu_h} [F(X_{\tau_\Omega})] + O(e^{-\frac{c}{h}}), \end{aligned}$$

uniformly in  $x \in K$ . This concludes the proof of Proposition 14. ■

## 2.4 From smooth functions $F$ to non-smooth functions $F$

Let us now complete the proof of Theorem 1.

*Proof.* In the following we assume that **(A0)**, **(A1)**, **(A2)** and **(A3)** are satisfied. We recall that this implies that  $\min_{\overline{C_{\max}}} f = \min_{\overline{\Omega}} f$  and thus, the results of Proposition 14 hold. Let  $K$  be a compact subset of  $\Omega$  such that

$$K \subset \mathcal{A}(C_{\max})$$

and let us assume that the process starts from  $X_0 = x \in K$ . Let  $F \in L^\infty(\partial\Omega, \mathbb{R})$ . The proof of Theorem 1 is divided into three steps.

**Step 1.** Proof of (18) and (19).

Let us first show that if  $\Sigma \subset \partial\Omega$  is open and there exists  $\beta > 0$  such that  $\Sigma \cap \bigcup_{i=1}^{k_\partial\Omega} B_{\partial\Omega}(z_i, \beta) = \emptyset$  (where  $B_{\partial\Omega}(z_i, \beta)$  is the open ball in  $\partial\Omega$  of radius  $\beta$  centered at  $z_i$ ), then, for all  $x \in K$ ,

$$\mathbb{P}_x [X_{\tau_\Omega} \in \Sigma] = O(e^{-\frac{c}{h}}) \quad (55)$$

in the limit  $h \rightarrow 0$  and uniformly in  $x \in K$ . To this end, let us consider  $\tilde{F} \in C^\infty(\partial\Omega, [0, 1])$  be such that

$$\tilde{F} = 1 \text{ on } \Sigma \text{ and } \tilde{F} = 0 \text{ on } \bigcup_{i=1}^{k_1^{\partial\Omega}} B_{\partial\Omega}(z_i, \frac{\beta}{2}).$$

Using Proposition 14, there exists  $c > 0$  such that for all  $x \in K$ :

$$\mathbb{P}_x [X_{\tau_\Omega} \in \Sigma] \leq \mathbb{E}_x [\tilde{F}(X_{\tau_\Omega})] = \mathbb{E}_{\nu_h} [\tilde{F}(X_{\tau_\Omega})] + O(e^{-\frac{c}{h}})$$

in the limit  $h \rightarrow 0$  and uniformly in  $x \in K$ . Then, Equation (55) follows from (30) applied to  $\tilde{F}$  and the family of sets  $\Sigma_i = B_{\partial\Omega}(z_i, \frac{\beta}{2})$  for  $i \in \{1, \dots, k_1^{\partial\Omega}\}$ .

Let us now prove (18) and (19). Let  $F \in L^\infty(\partial\Omega, \mathbb{R})$  and for all  $i \in \{1, \dots, k_1^{\partial\Omega}\}$ , let  $\Sigma_i \subset \partial\Omega$  be an open set which contains  $z_i$ . Let us assume in addition that  $\Sigma_i \cap \Sigma_j = \emptyset$  if  $i \neq j$ . One has for any  $x \in K$

$$\mathbb{E}_x [F(X_{\tau_\Omega})] = \sum_{i=1}^{k_1^{\partial\Omega}} \mathbb{E}_x [(\mathbf{1}_{\Sigma_i} F)(X_{\tau_\Omega})] + \mathbb{E}_x \left[ (\mathbf{1}_{\partial\Omega \setminus \bigcup_{i=1}^{k_1^{\partial\Omega}} \Sigma_i} F)(X_{\tau_\Omega}) \right].$$

Moreover, one has:

$$\left| \mathbb{E}_x \left[ (\mathbf{1}_{\partial\Omega \setminus \bigcup_{i=1}^{k_1^{\partial\Omega}} \Sigma_i} F)(X_{\tau_\Omega}) \right] \right| \leq \|F\|_{L^\infty} \mathbb{P}_x \left[ X_{\tau_\Omega} \in \partial\Omega \setminus \bigcup_{i=1}^{k_1^{\partial\Omega}} \Sigma_i \right].$$

Using (55) with  $\Sigma = \partial\Omega \setminus \bigcup_{i=1}^{k_1^{\partial\Omega}} \Sigma_i$ , one gets (18).

Let us now prove (19). Let  $j \in \{k_1^{\partial\Omega} + 1, \dots, k_1^{\partial\Omega}\}$ . Let  $\delta > 0$  be such that for any  $k \in \{1, \dots, k_1^{\partial\Omega}\}$  with  $k \neq j$ , the sets  $B_{\partial\Omega}(z_k, \delta)$  and  $\tilde{\Sigma}_j := \bigcup_{z \in \Sigma_j} B_{\partial\Omega}(z, \delta)$  are disjoint. Let us consider

$$G \in C_c^\infty(\tilde{\Sigma}_j, [0, 1]) \text{ such that } G = 1 \text{ on } \Sigma_j.$$

Using Proposition 14, there exists  $c > 0$  such that for all  $x \in K$ ,

$$\begin{aligned} \left| \mathbb{E}_x [(\mathbf{1}_{\Sigma_j} F)(X_{\tau_\Omega})] \right| &\leq \|F\|_{L^\infty} \mathbb{P}_x [X_{\tau_\Omega} \in \Sigma_j] \leq \|F\|_{L^\infty} \mathbb{E}_x [G(X_{\tau_\Omega})] \\ &= O(\mathbb{E}_{\nu_h} [G(X_{\tau_\Omega})]) + O(e^{-\frac{c}{h}}) \end{aligned}$$

in the limit  $h \rightarrow 0$  and uniformly in  $x \in K$ . Then, using (31) and item 3 in Theorem 3, it holds when  $h \rightarrow 0$ :

$$\mathbb{E}_x [(\mathbf{1}_{\Sigma_j} F)(X_{\tau_\Omega})] = O(h^{\frac{1}{4}}),$$

and when **(A4)** holds, one has when  $h \rightarrow 0$ :

$$\mathbb{E}_x [(\mathbf{1}_{\Sigma_j} F)(X_{\tau_\Omega})] = O(e^{-\frac{c}{h}}),$$

for some  $c > 0$ . This concludes the proof of (19).

**Step 2.** Proof of (20).

For all  $j \in \{1, \dots, k_1^{\partial\Omega}\}$ , let  $\Sigma_j$  be open subset of  $\partial\Omega$  such that  $z_j \in \Sigma_j$ . Let us assume that  $\Sigma_k \cap \Sigma_j = \emptyset$  if  $k \neq j$ . Let  $F \in L^\infty(\partial\Omega, \mathbb{R})$  be  $C^\infty$  in a neighborhood of  $z_i$  for

some  $i \in \{1, \dots, k_1^{\partial C_{\max}}\}$ . Let  $\beta > 0$  be such that  $F$  is  $C^\infty$  on  $B_{\partial\Omega}(z_i, 2\beta) \subset \Sigma_i$  and let  $\chi_i \in C^\infty(\partial\Omega, [0, 1])$  be such that

$$\text{supp } \chi_i \subset B_{\partial\Omega}(z_i, \beta) \text{ and } \chi_i = 1 \text{ on } B_{\partial\Omega}(z_i, \beta/2).$$

One has:

$$\mathbb{E}_x [(\mathbf{1}_{\Sigma_i} F)(X_{\tau_\Omega})] = \mathbb{E}_x [(\chi_i F)(X_{\tau_\Omega})] + \mathbb{E}_x [((\mathbf{1}_{\Sigma_i} - \chi_i)F)(X_{\tau_\Omega})].$$

Using Proposition 14 with  $\chi_i F \in C^\infty$  and (30)-(32) with  $X_0 \sim \nu_h$ ,  $F\chi_i$  and the family of pairwise disjoint open sets  $\{\Sigma_j, j = 1, \dots, k_1^{\partial\Omega}, j \neq i\} \cup \{B_{\partial\Omega}(z_i, \frac{\beta}{2})\}$ , there exists  $c > 0$  such that for all  $x \in K$ :

$$\begin{aligned} \mathbb{E}_x [(\chi_i F)(X_{\tau_\Omega})] &= \mathbb{E}_{\nu_h} [(\chi_i F)(X_{\tau_\Omega})] + O\left(e^{-\frac{c}{h}}\right) \\ &= \mathbb{E}_{\nu_h} \left[ (\mathbf{1}_{B_{\partial\Omega}(z_i, \frac{\beta}{2})} F)(X_{\tau_\Omega}) \right] + O\left(e^{-\frac{c}{h}}\right) = F(z_i) a_i + O(h^{\frac{1}{4}}) \end{aligned}$$

in the limit  $h \rightarrow 0$  and uniformly in  $x \in K$ , and where  $a_i$  is defined in (21). In addition, using item 3 in Theorem 3, when **(A4)** holds, one can replace  $O(h^{\frac{1}{4}})$  in the last computation by  $O(h)$ . Moreover, using (55) with  $\Sigma = \Sigma_i \setminus B_{\partial\Omega}(z_i, \frac{\beta}{2})$ : there exists  $c > 0$  such that for all  $x \in K$ :

$$|\mathbb{E}_x [((\mathbf{1}_{\Sigma_i} - \chi_i)F)(X_{\tau_\Omega})]| \leq \|F\|_{L^\infty} \mathbb{P}_x \left[ X_{\tau_\Omega} \in \Sigma_i \setminus B_{\partial\Omega}\left(z_i, \frac{\beta}{2}\right) \right] = O\left(e^{-\frac{c}{h}}\right)$$

in the limit  $h \rightarrow 0$  and uniformly in  $x \in K$ . Thus, one has when  $h \rightarrow 0$  and uniformly with respect to  $x \in K$ :

$$\mathbb{E}_x [(\mathbf{1}_{\Sigma_i} F)(X_{\tau_\Omega})] = F(z_i) a_i + O(h^{\frac{1}{4}}),$$

and when **(A4)** holds, one has:

$$\mathbb{E}_x [(\mathbf{1}_{\Sigma_i} F)(X_{\tau_\Omega})] = F(z_i) a_i + O(h).$$

This concludes the proof of (20). The proof of Theorem 1 is complete.  $\blacksquare$

### 3 On the exit point distribution when $X_0 = x \in \mathcal{A}(C)$ , where $C \in \mathcal{C}$

In this section, one proves Theorem 2 which aims at giving the concentration of the law of  $X_{\tau_\Omega}$  in the limit  $h \rightarrow 0$ , when  $X_0 = x \in \mathcal{A}(C)$ , where  $C \in \mathcal{C}$  (we recall that  $\mathcal{C}$  has been defined in (12)).

#### 3.1 Proof of Theorem 2

*Proof of Theorem 2.* Let us assume that **(A0)** holds. Let  $C \in \mathcal{C}$ . Assume that (see (23))

$$\partial C \cap \partial\Omega \neq \emptyset \text{ and } |\nabla f| \neq 0 \text{ on } \partial C.$$

To prove Theorem 2, the strategy consists in using Theorem 1 with a subdomain  $\Omega_C$  of  $\Omega$  containing  $C$  such that in the limit  $h \rightarrow 0$ , the most probable places of exit of the



process (1) from  $\Omega_{\mathbf{C}}$  when  $X_0 = x \in \mathbf{C}$  are the elements of  $\partial\mathbf{C} \cap \partial\Omega$ . This will imply (since the trajectories of the process (1) are continuous) that the most probable places of exit of the process (1) from  $\Omega$  when  $X_0 = x \in \mathbf{C}$  are the elements of  $\partial\mathbf{C} \cap \partial\Omega$ , which is the statement of Theorem 2. This result will be extended to initial conditions  $X_0 = x \in \mathcal{A}(\mathbf{C})$  using a large deviations method.

The proof of Theorem 2 is divided into two steps.

**Step 1:** Construction of a domain  $\Omega_{\mathbf{C}}$  containing  $\mathbf{C}$ .

In this step, one constructs a subset  $\Omega_{\mathbf{C}}$  of  $\Omega$  such that

$$\left\{ \begin{array}{l} \Omega_{\mathbf{C}} \text{ is a } C^\infty \text{ connected open subset of } \Omega \text{ containing } \mathbf{C}, \\ \partial\Omega_{\mathbf{C}} \cap \partial\Omega \text{ is a neighborhood of } \partial\mathbf{C} \cap \partial\Omega \text{ in } \partial\Omega, \\ \operatorname{argmin}_{\partial\Omega_{\mathbf{C}}} f = \partial\mathbf{C} \cap \partial\Omega, \\ \{x \in \overline{\Omega_{\mathbf{C}}}, f(x) < \min_{\partial\Omega_{\mathbf{C}}} f\} = \mathbf{C}, \\ \text{the critical points of } f \text{ in } \overline{\Omega_{\mathbf{C}}} \text{ are included in } \mathbf{C}, \end{array} \right. \quad (56)$$

and

$$f : \partial\Omega_{\mathbf{C}} \rightarrow \mathbb{R} \text{ is a Morse function.} \quad (57)$$

To construct a domain  $\Omega_{\mathbf{C}} \subset \Omega$  which satisfies (56) and (57), we first introduce a neighborhood  $V_{\mathbf{C}}$  of  $\overline{\mathbf{C}}$  in  $\overline{\Omega}$  as follows. Let  $\lambda \in \mathbb{R}$  be such that  $\mathbf{C}$  is a connected component of  $\{f < \lambda\}$  (see (11)). Then, for  $z \in \partial\mathbf{C}$ , we introduce a ball of radius  $\varepsilon_z > 0$  centred at  $z$  in  $\overline{\Omega}$  as follows:

1. If  $z \in \partial\mathbf{C} \cap \Omega$ : Since  $z \in \Omega$  and  $|\nabla f(z)| \neq 0$ , there exists  $\varepsilon_z > 0$  such that  $\overline{B(z, \varepsilon_z)} \subset \Omega$ ,  $|\nabla f(z)| \neq 0$  on  $\overline{B(z, \varepsilon_z)}$ , and, according to [20, Section 5.2],  $B(z, \varepsilon_z) \cap \{f < \lambda\}$  is connected and  $B(z, \varepsilon_z) \cap \partial\{f < \lambda\} = B(z, \varepsilon_z) \cap \{f = \lambda\}$  (where we recall that  $B(z, \varepsilon_z) = \{x \in \overline{\Omega} \text{ s.t. } |x - z| < \varepsilon_z\}$ ).
2. If  $z \in \partial\mathbf{C} \cap \partial\Omega$ : Recall that  $z \in \mathbf{U}_1^{\partial\Omega}$  (see (15)) and thus,  $\partial_n f(z) > 0$  and  $z$  is a non degenerate local minimum of  $f|_{\partial\Omega}$ . Thus, there exists  $\varepsilon_z > 0$ , such that  $|\nabla f(z)| \neq 0$  on  $\overline{B(z, \varepsilon_z)}$  and such that, according to [20, Section 5.2],  $B(z, \varepsilon_z) \cap \{f < \lambda\}$  is connected and included in  $\Omega$ . In addition,  $B(z, \varepsilon_z) \cap \partial\{f < \lambda\} = B(z, \varepsilon_z) \cap \{f = \lambda\}$ . Finally, up to choosing  $\varepsilon_z > 0$  smaller, one has:

$$\operatorname{argmin}_{B_{\partial\Omega}(z, \varepsilon_z)} f = \{z\}, \quad (58)$$

where we recall that  $B_{\partial\Omega}(z, \varepsilon_z)$  is the open ball of radius  $\varepsilon_z$  centred in  $z$  in  $\partial\Omega$ , and,

$$|\nabla_T f| \neq 0 \text{ on } \overline{B_{\partial\Omega}(z, \varepsilon_z)} \setminus \{z\} \text{ and } \partial_n f > 0 \text{ on } \overline{B(z, \varepsilon_z)} \cap \partial\Omega. \quad (59)$$

Items 1 and 2 above imply that for all  $z \in \partial\mathbf{C}$ , by definition of  $\mathbf{C}$  (see Theorem 2),

$$B(z, \varepsilon_z) \cap \mathbf{C} = B(z, \varepsilon_z) \cap \{f < \lambda\} \text{ and thus, } B(z, \varepsilon_z) \cap \partial\mathbf{C} = B(z, \varepsilon_z) \cap \{f = \lambda\}. \quad (60)$$

One then defines:

$$V_{\mathbf{C}} := \left( \bigcup_{z \in \partial\mathbf{C}} B(z, \varepsilon_z) \right) \cup \mathbf{C}.$$

The set  $V_C$  is an open neighborhood of  $\bar{C}$  in  $\bar{\Omega}$ . Moreover, according to items 1 and 2 above,

$$|\nabla f| \neq 0 \text{ on } \overline{V_C} \setminus C, \quad (61)$$

and using in addition (60),

$$\{f < \lambda\} \cap V_C = C \text{ and } \{f \leq \lambda\} \cap V_C = \bar{C}. \quad (62)$$

The second statement in (62) implies that  $\bar{C}$  is a connected component of  $\{f \leq \lambda\}$ . Thus, for  $r > 0$  small enough  $\overline{C(\lambda+r)} \subset V_C$ , where  $C(\lambda+r)$  is the connected component of  $\{f < \lambda+r\}$  which contains  $C$ . This suggests that a natural candidate to satisfy (56) and (57) is the domain  $C(\lambda+r)$ . However, for  $r > 0$  small enough, the boundary of  $C(\lambda+r)$  is not  $C^\infty$ : it is composed of two smooth pieces  $\overline{\partial C(\lambda+r) \cap \Omega} = \{x \in \partial C(\lambda+r), f(x) = \lambda+r\}$  and  $\partial C(\lambda+r) \cap \partial\Omega$ . The union of this two sets gives rise to "corners". Moreover, the function  $f|_{\partial C(\lambda+r) \cap \Omega}$  is not a Morse function since  $f \equiv \lambda+r$  on  $\overline{\partial C(\lambda+r) \cap \Omega}$ .

To justify the existence of a domain  $\Omega_C$  which satisfies (56) and (57), we now proceed in two steps, as follows.

- Domain  $D_C$  containing  $C$  which satisfies (56) and  $\partial_n f > 0$  on  $\partial D_C$ . The subdomain  $D_C$  of  $\Omega$  is constructed as a smooth regularization of the set  $C(\lambda+r)$  with  $r > 0$  such that  $\overline{C(\lambda+r)} \subset V_C$  by modifying  $C(\lambda+r)$  in a neighborhood of  $\{x \in \partial C(\lambda+r), f(x) = \lambda+r\} \cap \partial\Omega$  (where the two smooth pieces of  $\partial C(\lambda+r)$  intersect each other). Moreover,  $\partial_n f > 0$  on  $\overline{\partial C(\lambda+r) \cap \Omega}$  (since there is no critical point of  $f$  on  $\overline{\partial C(\lambda+r) \cap \Omega} = \{x \in \partial C(\lambda+r), f(x) = \lambda+r\}$ ) and on  $\partial C(\lambda+r) \cap \partial\Omega$  (since  $\overline{C(\lambda+r)} \subset V_C$  and  $\partial_n f > 0$  on  $V_C \cap \partial\Omega$ , see the second inequality in (59)).

Thus, using in addition (62) together with the fact that  $V_C$  is an open neighborhood of  $\bar{C}$  in  $\bar{\Omega}$ , there exists a  $C^\infty$  connected open subset  $D_C$  of  $\Omega$  such that

$$C \subset D_C, \quad \overline{D_C} \subset V_C, \quad (63)$$

and

$$\partial_n f > 0 \text{ on } \partial D_C, \quad (64)$$

which satisfies, for some  $\beta > 0$  and  $\Sigma_C \subset \Omega$ ,

$$\partial D_C = \left( \bigcup_{z \in \partial C \cap \partial\Omega} B_{\partial\Omega}(z, \varepsilon_z/2) \right) \cup \overline{\Sigma_C}, \text{ where, } f \geq \lambda + \beta \text{ on } \overline{\Sigma_C}. \quad (65)$$

Finally, according to the first statement in (59), there exists  $\delta_0 > 0$  such that for any open  $\delta$ -neighborhood  $U_{\partial\Omega}^\delta$  of  $\partial\Omega$  in  $\bar{\Omega}$ , with  $\delta \in (0, \delta_0)$ , one has

$$|\nabla_T f| \neq 0 \text{ on } \overline{\partial D_C \cap U_{\partial\Omega}^\delta} \setminus (\partial C \cap \partial\Omega), \quad (66)$$

where  $\nabla_T f$  is the tangential gradient of  $f$  on  $\partial D_C$ .

- Domain  $\Omega_C$  containing  $C$  which satisfies (56) and (57). The domain  $\Omega_C$  will be constructed as a perturbation of  $D_C$ , using an argument related to the genericity of Morse functions, and more precisely, a method due to René Thom based on Sard's theorem. All the details will be given in the next section. More precisely, we apply Proposition 17, which is stated in the next section, with  $D = D_C$ ,  $\mathcal{V}_- = C$ ,  $\mathcal{V}_+ = V_C$ , and, for a  $\delta \in (0, \delta_0)$  (see (66)):

- (i)  $S_1 = \partial D_C \cap U_{\partial\Omega}^{\delta/2}$ , which is such that  $f : \overline{S_1} \rightarrow \mathbb{R}$  is a Morse function with no critical point on  $\partial S_1$  (see (66) together with the fact that  $\partial C \cap \partial\Omega$  is the union of non degenerate critical points of  $f|_{\partial\Omega}$ ),
- (ii)  $S'_1 = \partial D_C \cap U_{\partial\Omega}^{\delta/4}$  which satisfies, according to (66),  $|\nabla_T f| \neq 0$  on  $\overline{S_1} \setminus S'_1$ .

Therefore, using in addition the fact that  $D_C$  satisfies (63)–(65), there exists a  $C^\infty$  connected open subset  $\Omega_C$  of  $\Omega$  such that  $C \subset \Omega_C$ ,  $\overline{\Omega_C} \subset V_C$ ,

$f : \partial\Omega_C \rightarrow \mathbb{R}$  is a Morse function,

and for some  $r > 0$  and  $\Gamma_C \subset \Omega$ ,

$$\partial\Omega_C = \left( \bigcup_{z \in \partial C \cap \partial\Omega} B_{\partial\Omega}(z, \varepsilon_z/2) \right) \bigcup \overline{\Gamma_C}, \text{ where, } f \geq \lambda + r \text{ on } \overline{\Gamma_C}. \quad (67)$$

It then remains to check that  $\Omega_C$  satisfies (56). From (67) and (58),  $\Omega_C$  satisfies the two first statements in (56) and  $\min_{\partial\Omega_C} f = \lambda$ . Since  $C \subset \Omega_C$  and  $\overline{\Omega_C} \subset V_C$ , one deduces from the first statement in (62), that

$$\{x \in \overline{\Omega_C}, f(x) < \lambda\} = C,$$

and from (61),

$$|\nabla f| \neq 0 \text{ on } \overline{\Omega_C} \setminus C.$$

This proves that  $\Omega_C$  satisfies the two last statements in (56). This concludes the construction of a domain  $\Omega_C$  which satisfies (56) and (57). A schematic representation of such a domain  $\Omega_C$  is given on Figure 3.

**Step 2:** End of the proof of Theorem 2.

For all  $z \in \partial C \cap \partial\Omega$ , let  $\Sigma_z$  be an open subset of  $\partial\Omega$  such that  $z \in \Sigma_z$ . Let  $K$  be a compact subset of  $\Omega$  such that  $K \subset \mathcal{A}(C)$ . Let us first consider the case when  $K \subset C$ .

Let  $\Omega_C$  be the  $C^\infty$  subdomain of  $\Omega$  constructed in the previous step and which, we recall, contains  $C$  and satisfies (56) and (57). Then, one easily deduces that when  $\Omega$  is replaced by  $\Omega_C$ , the function  $f : \overline{\Omega_C} \rightarrow \mathbb{R}$  satisfies **(A0)** and  $\mathcal{C} = \{C\}$  (see (12) for the definition of  $\mathcal{C}$ ). Thus, in this case  $C_{\max} = C$ . Moreover, using in addition the second and third statements in (56), one obtains that the assumptions **(A1)**, **(A2)**, **(A3)** and **(A4)** are satisfied for the function  $f : \overline{\Omega_C} \rightarrow \mathbb{R}$ . Thus, according to Theorem 1 applied to the function  $f : \overline{\Omega_C} \rightarrow \mathbb{R}$ , the most probable places of exit of the process (1) from  $\Omega_C$  when  $X_0 = x \in C$ , are  $\partial C \cap \partial\Omega$ , and the relative asymptotic probabilities to exit through each point in  $\partial C \cap \partial\Omega$  are given by item 2 in Theorem 1. In particular, from items 1 and 3 in Theorem 1, for any open subset  $\Sigma$  of  $\partial\Omega_C$  such that

$$\min_{\Sigma} f > \min_{\partial\Omega_C} f \text{ (where we recall } \arg \min_{\partial\Omega_C} f = \partial C \cap \partial\Omega, \text{ see (56))},$$

there exists  $c > 0$  such that for  $h$  small enough:

$$\sup_{x \in K} \mathbb{P}_x [X_{\tau_{\Omega_C}} \in \Sigma] \leq e^{-\frac{c}{h}}, \quad (68)$$

where  $\tau_{\Omega_C}$  is the first exit time from  $\Omega_C$  of the process (1).

**Step 2a:** Proof of the first asymptotic estimate in Theorem 2 when  $K \subset C$ .

Writing  $\partial\Omega = (\partial\Omega \cap \partial\Omega_C) \cup (\partial\Omega \setminus \partial\Omega_C)$ , it holds:

$$\left(\partial\Omega \setminus \bigcup_{z \in \partial C \cap \partial\Omega} \Sigma_z\right) \subset \left(\partial\Omega_C \cap \partial\Omega \setminus \bigcup_{z \in \partial C \cap \partial\Omega} \Sigma_z\right) \cup (\partial\Omega \setminus \partial\Omega_C). \quad (69)$$

To prove the first asymptotic estimate in Theorem 2, let us prove that when  $X_0 = x \in K$ , the probabilities that  $X_{\tau_\Omega}$  belongs to each of the two sets in the right-hand side of (69) are exponentially small when  $h \rightarrow 0$ . Let us recall that  $\tau_{\Omega_C}$  is the first exit time from  $\Omega_C$  of the process (1) and thus, when  $X_0 = x \in \Omega_C$ ,  $\tau_{\Omega_C} \leq \tau_\Omega$ , and

$$\tau_{\Omega_C} = \tau_\Omega \text{ if and only if } X_{\tau_{\Omega_C}} \in \partial\Omega_C \cap \partial\Omega. \quad (70)$$

Thus, from (70), when  $X_0 = x \in \Omega_C$ , it holds:

$$\left\{X_{\tau_\Omega} \in \partial\Omega_C \cap \partial\Omega \setminus \bigcup_{z \in \partial C \cap \partial\Omega} \Sigma_z\right\} \subset \left\{X_{\tau_{\Omega_C}} \in \partial\Omega_C \setminus \partial\Omega\right\} \cup \left\{X_{\tau_{\Omega_C}} \in \partial\Omega_C \cap \partial\Omega \setminus \bigcup_{z \in \partial C \cap \partial\Omega} \Sigma_z\right\}.$$

Using (68), there exists  $c > 0$  such that for  $h$  small enough:

$$\sup_{x \in K} \mathbb{P}_x \left[ X_{\tau_{\Omega_C}} \in \partial\Omega_C \cap \partial\Omega \setminus \bigcup_{z \in \partial C \cap \partial\Omega} \Sigma_z \right] \leq e^{-\frac{c}{h}}$$

and

$$\sup_{x \in K} \mathbb{P}_x \left[ X_{\tau_{\Omega_C}} \in \partial\Omega_C \setminus \partial\Omega \right] \leq e^{-\frac{c}{h}}. \quad (71)$$

Thus, there exists  $c > 0$  such that for  $h$  small enough:

$$\sup_{x \in K} \mathbb{P}_x \left[ X_{\tau_\Omega} \in \partial\Omega_C \cap \partial\Omega \setminus \bigcup_{z \in \partial C \cap \partial\Omega} \Sigma_z \right] \leq e^{-\frac{c}{h}}. \quad (72)$$

Let us now consider the case when  $X_{\tau_\Omega} \in \partial\Omega \setminus \partial\Omega_C$ . When  $X_0 = x \in K$ , it holds from (70):

$$\left\{X_{\tau_\Omega} \in \partial\Omega \setminus \partial\Omega_C\right\} \subset \left\{X_{\tau_{\Omega_C}} \in \partial\Omega_C \setminus \partial\Omega\right\}.$$

Therefore, from (71), there exists  $c > 0$  such that for  $h$  small enough:

$$\sup_{x \in K} \mathbb{P}_x \left[ X_{\tau_\Omega} \in \partial\Omega \setminus \partial\Omega_C \right] \leq e^{-\frac{c}{h}}. \quad (73)$$

In conclusion, from (69), (72) and (73), one obtains that there exists  $c > 0$  such that for  $h$  small enough:

$$\sup_{x \in K} \mathbb{P}_x \left[ X_{\tau_\Omega} \in \partial\Omega \setminus \bigcup_{z \in \partial C \cap \partial\Omega} \Sigma_z \right] \leq e^{-\frac{c}{h}}. \quad (74)$$

This proves the first asymptotic estimate in Theorem 2 when  $K \subset C$ .

**Step 2b:** Proof of the second asymptotic estimate in Theorem 2 when  $K \subset C$ .

Let us assume that the open sets  $(\Sigma_z)_{z \in \partial C \cap \partial\Omega}$  are pairwise disjoint. Let us consider  $z \in \partial C \cap \partial\Omega$  and  $\beta > 0$  such that (see indeed the second statement in (56)),

$$B_{\partial\Omega}(z, \beta) \subset \Sigma_z \cap \partial\Omega_C. \quad (75)$$

Then, one writes:

$$\mathbb{P}_x[X_{\tau_\Omega} \in \Sigma_z] = \mathbb{P}_x[X_{\tau_\Omega} \in B_{\partial\Omega}(z, \beta)] + \mathbb{P}_x[X_{\tau_\Omega} \in \Sigma_z \setminus B_{\partial\Omega}(z, \beta)]. \quad (76)$$

Let us first deal with the second term in the right-hand side of (76). It holds (since the sets  $(\Sigma_y)_{y \in \partial C \cap \partial\Omega}$  are pairwise disjoint and  $B_{\partial\Omega}(z, \beta) \subset \Sigma_z$ , see (75)), when  $X_0 = x \in \Omega$ :

$$\mathbb{P}_x[X_{\tau_\Omega} \in \Sigma_z \setminus B_{\partial\Omega}(z, \beta)] \leq \mathbb{P}_x \left[ X_{\tau_\Omega} \in \partial\Omega \setminus \left( B_{\partial\Omega}(z, \beta) \cup \bigcup_{y \in \partial C \cap \partial\Omega, y \neq z} \Sigma_y \right) \right].$$

Thus, from (74) (applied with  $B_{\partial\Omega}(z, \beta)$  instead of  $\Sigma_z$ ), one obtains that there exists  $c > 0$  such that for  $h$  small enough:

$$\sup_{x \in K} \mathbb{P}_x \left[ X_{\tau_\Omega} \in \Sigma_z \setminus B_{\partial\Omega}(z, \beta) \right] \leq e^{-\frac{c}{h}}. \quad (77)$$

Let us now deal with the first term in the right-hand side (76). It holds from (75), when  $X_0 = x \in K$ :

$$\begin{aligned} \mathbb{P}_x[X_{\tau_\Omega} \in B_{\partial\Omega}(z, \beta)] &= \mathbb{P}_x[X_{\tau_\Omega} \in B_{\partial\Omega}(z, \beta), \tau_{\Omega_C} < \tau_\Omega] + \mathbb{P}_x[X_{\tau_{\Omega_C}} \in B_{\partial\Omega}(z, \beta), \tau_{\Omega_C} = \tau_\Omega] \\ &= O(e^{-\frac{c}{h}}) + \mathbb{P}_x[X_{\tau_{\Omega_C}} \in B_{\partial\Omega}(z, \beta)], \end{aligned} \quad (78)$$

where we used the fact that  $\{\tau_{\Omega_C} < \tau_\Omega\} \subset \{X_{\tau_{\Omega_C}} \in \partial\Omega_C \setminus \partial\Omega\}$  (see (70)) and (71). Applying item 2 in Theorem 1 with the function  $f : \Omega_C \rightarrow \mathbb{R}$  and  $F = \mathbf{1}_{B_{\partial\Omega}(z, \beta)}$ , one has:

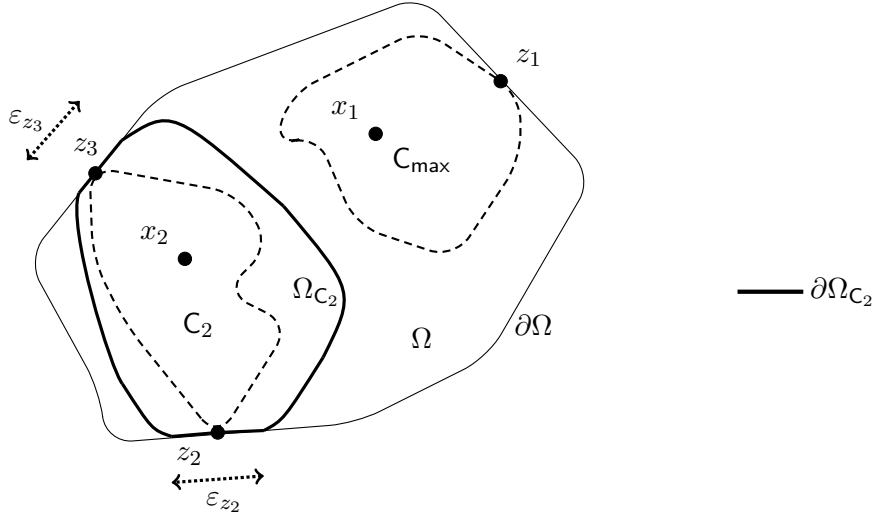


Figure 3: Schematic representation of  $\Omega_{C_2}$  satisfying (56) when  $C = C_2$ . On the figure,  $\partial C_2 \cap \partial\Omega = \{z_2, z_3\}$ ,  $x_2$  is the global minimum of  $f$  in  $C_2$  and  $C_{\max}$  is another element of  $\mathcal{C}$ .

$$\mathbb{P}_x[X_{\tau_{\Omega_C}} \in B_{\partial\Omega}(z, \beta)] = \frac{\partial_n f(z)}{\sqrt{\det \text{Hess} f|_{\partial\Omega}(z)}} \left( \sum_{y \in \partial C \cap \partial\Omega} \frac{\partial_n f(y)}{\sqrt{\det \text{Hess} f|_{\partial\Omega}(y)}} \right)^{-1} (1 + O(h)),$$

in the limit  $h \rightarrow 0$  and uniformly in  $x \in K$ . Together with (76), (77), and (78), this concludes the proof of the second asymptotic estimate in Theorem 2 for initial conditions

$X_0 = x \in K \subset \mathbb{C}$  and when  $F = 1$  on  $\partial\Omega$ . To extend the second asymptotic estimate in Theorem 2 to functions  $F \in L^\infty(\partial\Omega, \mathbb{R})$  which are smooth in a neighborhood of  $z$  in  $\partial\Omega$ , one uses the same procedure as the one made in the second step in Section 2.4.

Finally, the case when  $X_0 = x \in K \subset \mathcal{A}(\mathbb{C})$  is proved using the estimate of Freidlin and Wentzell (52), as in the second step of the proof of Proposition 14. This concludes the proof of Theorem 2.  $\blacksquare$

### 3.2 Proof of the existence of a domain $\Omega_{\mathbb{C}}$ satisfying (57)

In this section, we prove the existence of a domain  $\Omega_{\mathbb{C}}$  which satisfies (57) in addition to (56). To this end, we first give in Proposition 15 a simple perturbation result to present the main idea of the proof. Then, we extend this result to the setting we are interested in to prove the existence of such a domain  $\Omega_{\mathbb{C}}$  in Proposition 17.

**Proposition 15.** *Let  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  be a  $C^\infty$  function and  $D$  be a  $C^\infty$  open bounded and connected subset of  $\mathbb{R}^d$ . Let us assume that*

$$\forall x \in \partial D, \nabla f(x) \oplus T_x \partial D = \mathbb{R}^d.$$

*Then, for any open sets  $\mathcal{V}_-$  and  $\mathcal{V}_+$  such that  $\overline{\mathcal{V}_-} \subset D$  and  $\overline{\mathcal{V}_+} \subset \mathcal{V}_+$ , there exists a  $C^\infty$  open bounded and connected subset  $D'$  of  $\mathbb{R}^d$  such that*

$$\overline{\mathcal{V}_-} \subset D', \overline{D'} \subset \mathcal{V}_+, \text{ and } f|_{\partial D'} \text{ is a Morse function.}$$

**Remark 16.** *We are thankful to François Laudenbach who gave us the main ingredient of the proof of Proposition 15. The proof is inspired by a method due to René Thom [44] based on Sard's theorem [42], see [27, Section 5.6].*

*Proof.* Let  $\mathcal{V}_-$  and  $\mathcal{V}_+$  be two open subsets of  $\mathbb{R}^d$  such that  $\overline{\mathcal{V}_-} \subset D$  and  $\overline{\mathcal{V}_+} \subset \mathcal{V}_+$ . Let us denote by  $S$  the boundary of  $D$  which is a smooth compact hypersurface of  $\mathbb{R}^d$ . For  $r > 0$ , one denotes by  $B(0, r)$  the ball of radius  $r$  centred at 0 in  $\mathbb{R}^d$ . Let  $\mathcal{V}$  be a neighborhood of  $S$  in  $\mathbb{R}^d$ . By assumption on  $S$ , there exist  $\varepsilon_0 > 0$  and  $\varepsilon_1 > 0$  such that the map

$$(x, \lambda) \in S \times (-\varepsilon_0, \varepsilon_0) \mapsto x + \lambda \nabla f(x) \in \mathcal{V}$$

is well defined and is a diffeomorphism onto its image, and, for all  $(x, v) \in S \times B(0, \varepsilon_1)$ , there exists a unique  $\lambda(x, v) \in (-\varepsilon_0, \varepsilon_0)$  such that

$$f(x + \lambda(x, v) \nabla f(x)) = f(x) + v \cdot x.$$

Moreover, for every  $v \in B(0, \varepsilon_1)$ , according to the implicit function theorem, the map  $x \in S \mapsto \lambda(x, v) \in (-\varepsilon_0, \varepsilon_0)$  is smooth and then also is  $x \in S \mapsto x + \lambda(x, v) \nabla f(x) \in \mathbb{R}^d$ . The latter application is then an injective immersion and hence, since  $S$  is compact, it follows that  $S_v := \{x + \lambda(x, v) \nabla f(x)\}$  is a smooth compact hypersurface. Up to choosing  $\varepsilon_1 > 0$  smaller, for any  $v \in B(0, \varepsilon_1)$ ,  $S_v$  is the boundary of a  $C^\infty$  open bounded and connected subset  $D_v$  of  $\mathbb{R}^d$  such that

$$\overline{\mathcal{V}_-} \subset D_v \text{ and } \overline{D_v} \subset \mathcal{V}_+.$$

To prove Proposition 15, it remains to show that there exists  $v \in B(0, \varepsilon_1)$  such that  $f|_{S_v}$  is a Morse function. Let us introduce the function

$$F : (x, v) \in S \times B(0, \varepsilon_1) \mapsto f|_{S_v}(x + \lambda(x, v)\nabla f(x)) = f(x) + v \cdot x \in \mathbb{R}.$$

For all  $x \in S$  and for all  $v \in B(0, \varepsilon_1)$ , let  $v_x^T \in T_x S$  and  $v_x^N \in \mathbb{R}$  be such that

$$v = v_x^T + v_x^N n(x), \quad (79)$$

where we recall that  $n(x)$  is the unit outward normal vector to  $D$  at  $x \in \partial D$ . At  $(x, v) \in S \times B(0, \varepsilon_1)$ , it holds  $\partial_x F(x, v) : z \in T_x S \mapsto d_x f(x)z + v_x^T \cdot z$ , where  $\partial_x F(x, v)$  is the  $x$ -derivative of  $F$  at  $(x, v)$ . The function  $G : S \times B(0, \varepsilon_1) \rightarrow T^*S$  defined by

$$G : (x, v) \mapsto (x, \partial_x F(x, v))$$

is a submersion onto a small tube around the zero section of  $T^*S$ . This is obvious by considering the  $v$ -derivative of  $G$ . Hence,  $G$  is transverse to the zero section  $0_{T^*S}$  of  $T^*S$  (see [27, Chapitre 5.1] for the definition of transversality). Using the parametric transversality theorem (which is a consequence of Sard's theorem, see for instance [27, Chapitre 5.3.1]), one obtains that for almost every  $v \in B(0, \varepsilon_1)$ ,  $\partial_x(F|_{S \times \{v\}}) = d(f|_{S_v})$  is transverse to  $0_{T^*S}$ , which is equivalent to say that  $f|_{S_v}$  is a Morse function. This concludes the proof of Proposition 15.  $\blacksquare$

The next proposition gives sufficient conditions on  $D$  and  $f$  to modify the result of Proposition 15 so that the perturbed domain  $D'$  has the same boundary as  $D$  on a prescribed subset  $S'_1$  of  $\partial D$  on which  $f$  is already a Morse function.

**Proposition 17.** *Let  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  be a  $C^\infty$  function and  $D$  be a  $C^\infty$  open bounded and connected subset of  $\mathbb{R}^d$ . Let us assume that*

$$\forall x \in \partial D, \nabla f(x) \oplus T_x \partial D = \mathbb{R}^d.$$

*Furthermore, let us assume that there exists an open subset  $S_1$  of  $\partial D$  such that  $f : \overline{S_1} \rightarrow \mathbb{R}$  is a Morse function with no critical point on  $\partial S_1$ . Let us now consider an open set  $S'_1$  such that  $\overline{S'_1} \subset S_1$  and  $f|_{\partial D}$  has no critical point on  $\overline{S_1} \setminus S'_1$ . Then, for any open sets  $\mathcal{V}_-$  and  $\mathcal{V}_+$  such that  $\overline{\mathcal{V}_-} \subset D \cup S'_1$  and  $\overline{D} \setminus S'_1 \subset \mathcal{V}_+$ , there exists a  $C^\infty$  open bounded and connected subset  $D'$  of  $\mathbb{R}^d$  such that  $S'_1 \subset \partial D'$ ,*

$$\overline{\mathcal{V}_-} \subset D' \cup S'_1, \overline{D'} \setminus S'_1 \subset \mathcal{V}_+, \text{ and } f|_{\partial D'} \text{ is a Morse function.}$$

*Proof.* Let  $\mathcal{V}_-$  and  $\mathcal{V}_+$  be two open subsets of  $\mathbb{R}^d$  such that  $\overline{\mathcal{V}_-} \subset D \cup S'_1$  and  $\overline{D} \setminus S'_1 \subset \mathcal{V}_+$ . Let us denote by  $S$  the boundary of  $D$ . The submanifold  $S$  is a smooth compact hypersurface of  $\mathbb{R}^d$ . Let us introduce a function  $\chi \in C^\infty(S)$  such that  $\chi(x) = 1$  for all  $x \in S \setminus S_1$  and  $\chi(x) = 0$  for all  $x \in \mathcal{V}_{S'_1}$  where  $\mathcal{V}_{S'_1}$  is an open neighborhood of  $S'_1$  in  $S$  such that  $\overline{\mathcal{V}_{S'_1}} \subset S_1$ . To prove Proposition 17, one uses the cutoff function  $\chi$  in the definition of  $\lambda(x, t)$  to ensure that  $S'_1 \subset S_v$  (see the proof of Proposition 15 for the notation  $S_v$ ). This is made as follows. Let us first consider  $\varepsilon_0 > 0$  and  $\varepsilon_1 > 0$  such that the map

$$(x, \lambda) \in S \times (-\varepsilon_0, \varepsilon_0) \mapsto x + \lambda \nabla f(x) \in \mathcal{V}$$

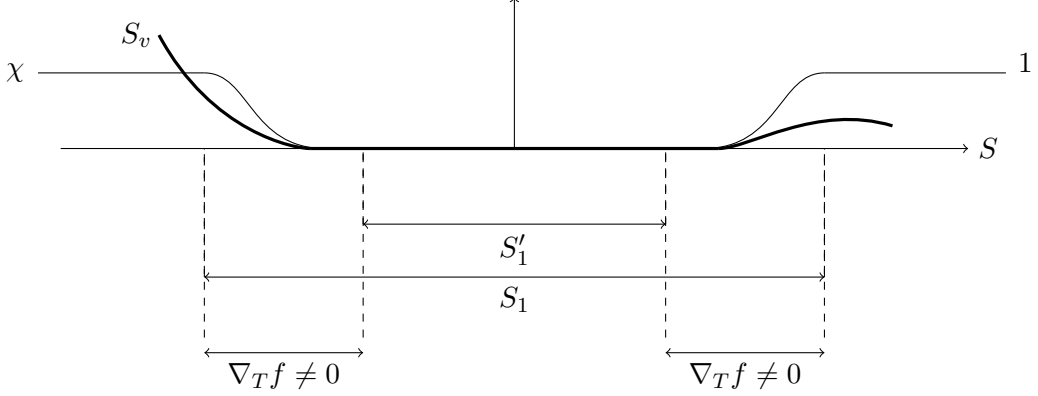


Figure 4: The support of  $\chi$  on  $S$ , the compact sets  $S_1$  and  $S'_1$ , and the hypersurface  $S_v$ .

is well defined and is a diffeomorphism onto its image, and, for all  $(x, v) \in S \times B(0, \varepsilon_1)$ , there exists a unique  $\lambda(x, v) \in (-\varepsilon_0, \varepsilon_0)$  such that

$$f(x + \lambda(x, v)\nabla f(x)) = f(x) + \chi(x)v \cdot x.$$

Notice that  $\lambda(x, v) = 0$  for all  $x \in \mathcal{V}_{S'_1}$  and  $v \in B(0, \varepsilon_1)$  (since  $\chi = 0$  on  $\mathcal{V}_{S'_1}$ ). Thus, for all  $v \in B(0, \varepsilon_1)$ ,  $\mathcal{V}_{S'_1} \subset S_v$  which implies that  $S'_1 \subset S_v$ . Again,  $S_v := \{x + \lambda(x, v)\nabla f(x)\}$  is a smooth compact hypersurface. A schematic representation of the function  $\chi$  and the hypersurface  $S_v$  are given in Figure 4. Up to choosing  $\varepsilon_1 > 0$  smaller, for any  $v \in B(0, \varepsilon_1)$ ,  $S_v$  is the boundary of a  $C^\infty$  open bounded and connected subset  $D_v$  of  $\mathbb{R}^d$  such that, since  $\mathcal{V}_{S'_1} \subset S_v$ ,

$$\overline{D_v} \subset D_v \cup S'_1, \text{ and } \overline{D_v} \setminus S'_1 \subset \mathcal{V}_+.$$

Let us now show that there exists  $v \in B(0, \varepsilon_1)$  such that  $f|_{S_v}$  is a Morse function. For that purpose, we consider the function

$$F : (x, v) \in S \times B(0, \varepsilon_1) \mapsto f|_{S_v}(x + \lambda(x, v)\nabla f(x)) = f(x) + \chi(x)v \cdot x \in \mathbb{R},$$

and the function  $G : S \times B(0, \varepsilon_1) \rightarrow T_x^*S$  defined by  $G : (x, v) \mapsto (x, \partial_x F(x, v))$ . Notice that for all  $v \in B(0, \varepsilon_1)$ ,  $x \in \overline{S'_1} \mapsto F(x, v) = f(x)$  is already, by assumption, a Morse function (with no critical point on  $\partial S'_1$ ). This implies that  $G$  is transverse to the zero section  $0_{T^*S}$  of  $T^*S$  along  $S'_1 \times B(0, \varepsilon_1)$ . Thus, to prove Proposition 17, it remains to study the function  $x \in S \setminus S'_1 \mapsto F(x, v)$ , for  $v \in B(0, \varepsilon_1)$ . For  $(x, v) \in \overline{S_1} \setminus \overline{S'_1} \times B(0, \varepsilon_1)$  and for all  $z \in T_x S$ , it holds:

$$\partial_x F(x, v)z = d_x f(x)z + O(\|v\|)z.$$

Since by assumption  $d_x f(x) \neq 0_{T_x^*S}$  for all  $x$  belonging to the compact set  $\overline{S_1} \setminus \overline{S'_1}$ , one has, up to choosing  $\varepsilon_1 > 0$  smaller: for all  $x \in \overline{S_1} \setminus \overline{S'_1}$  and  $v \in B(0, \varepsilon_1)$ ,  $\partial_x F(x, v) \neq 0_{T_x^*S}$ . Finally, for  $(x, v) \in S \setminus S_1 \times B(0, \varepsilon_1)$  and for all  $z \in T_x S$ , it holds:

$$G(x, v) = (x, d_x f(x)z + v_x^T \cdot z),$$

where  $v_x^T$  is defined by (79). Thus, the function  $G : S \setminus S_1 \times B(0, \varepsilon_1) \rightarrow T_x^*S$  is a submersion onto a small tube around the zero section  $0_{T^*S}$  of  $T^*S$ . This implies that  $G$  is



transverse to the zero section of  $T^*S$  along  $S \setminus S_1 \times B(0, \varepsilon_1)$ . In conclusion, the function  $G : S \times B(0, \varepsilon_1) \rightarrow T_x^*S$  is transverse to the zero section of  $T^*S$ . The parametric transversality theorem implies that for almost every  $v \in B(0, \varepsilon_1)$ ,  $\partial_x(F|_{S \times \{v\}}) = d(f|_{S_v})$  is transverse to  $0_{T^*S}$ , which is equivalent to  $f|_{S_v}$  is a Morse function. This concludes the proof of Proposition 17.  $\blacksquare$

### 3.3 Generalization of Theorems 1 and 2

In view of the proof of Theorem 2, we have the following generalization of Theorems 1 and 2.

**Theorem 4.** *Let us assume that (A0) holds. Let  $\lambda \in \mathbb{R}$  and  $C_1, \dots, C_m \in \mathcal{C}$  be  $m$  ( $m \geq 1$ ) connected components of  $\{f < \lambda\}$  such that:*

$$\bigcup_{j=1}^m \overline{C_j} \text{ is a connected component of } \{f \leq \lambda\} \quad (80)$$

and such that, up to reordering  $C_1, \dots, C_m$ ,

$$\partial C_1 \cap \partial \Omega \neq \emptyset \quad \text{and} \quad \forall j \in \{2, \dots, m\}, \quad \min_{C_1} f < \min_{C_j} f.$$

Let  $F \in L^\infty(\partial \Omega, \mathbb{R})$ . For all  $j \in \{1, \dots, m\}$  and  $z \in \partial C_j \cap \partial \Omega$ , let  $\Sigma_z$  be an open subset of  $\partial \Omega$  such that  $z \in \Sigma_z$  and such that the  $\Sigma_z$ 's are pairwise disjoint. Let  $K$  be a compact subset of  $\Omega$  such that  $K \subset \mathcal{A}(C_1)$  and  $x \in K$ . Then:

1. There exists  $c > 0$  such that in the limit  $h \rightarrow 0$ :

$$\mathbb{E}_x [F(X_{\tau_\Omega})] = \sum_{z \in \bigcup_{j=1}^m \partial \Omega \cap \partial C_j} \mathbb{E}_x [(1_{\Sigma_z} F)(X_{\tau_\Omega})] + O(e^{-\frac{c}{h}}) \quad (81)$$

and

$$\sum_{z \in \bigcup_{j=2}^m \partial \Omega \cap \partial C_j} \mathbb{E}_x [(1_{\Sigma_z} F)(X_{\tau_\Omega})] = O(h^{\frac{1}{4}}), \quad (82)$$

uniformly in  $x \in K$ .

2. When for some  $z \in \partial C_1 \cap \partial \Omega$  the function  $F$  is  $C^\infty$  in a neighborhood of  $z$ , one has when  $h \rightarrow 0$ :

$$\mathbb{E}_x [(1_{\Sigma_z} F)(X_{\tau_\Omega})] = \frac{F(z) \partial_n f(z)}{\sqrt{\det \text{Hess} f|_{\partial \Omega}(z)}} \left( \sum_{y \in \partial C_1 \cap \partial \Omega} \frac{\partial_n f(y)}{\sqrt{\det \text{Hess} f|_{\partial \Omega}(y)}} \right)^{-1} + O(h^{\frac{1}{4}}), \quad (83)$$

uniformly in  $x \in K$ .

3. When  $m = 1$ , the remainder term  $O(h^{\frac{1}{4}})$  in (83) is of order  $O(h)$  and admits a full asymptotic expansion in  $h$ .

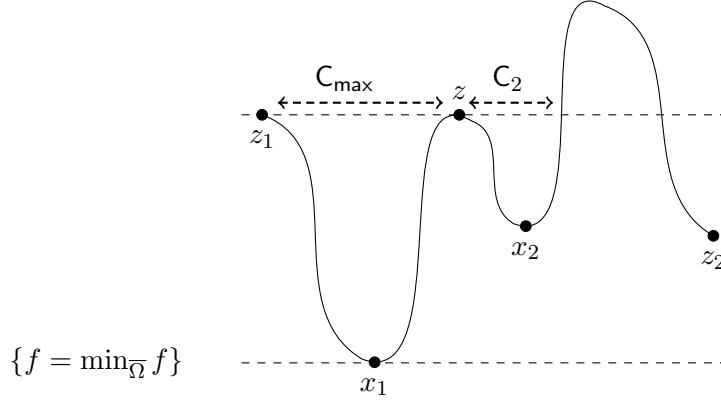
Theorem 4 is a generalization of Theorem 1 because we do not assume that  $\partial C_1 \cap \partial \Omega \subset \arg \min_{\partial \Omega} f$ . Theorem 4 is also a generalization of Theorem 2 since, in the framework of Theorem 4, when  $m \geq 2$ ,  $|\nabla f| = 0$  on  $\partial C_1 \cap \partial C_j$ , for  $j = 2, \dots, m$ , and thus Theorem 2 with  $C = C_1$  does not apply.

*Proof.* Let us denote by  $\tilde{C} = C_1 \cup \dots \cup C_m$ . The proof of Theorem 4 consists in applying Theorem 1 to a suitable subdomain  $\Omega_{\tilde{C}}$  of  $\Omega$  containing  $\tilde{C}$ . This proof is the same as the one made to prove Theorem 2 except that the justification of the existence of the neighborhood  $V_{\tilde{C}}$  of  $\tilde{C}$  in  $\bar{\Omega}$  such that

$$|\nabla f| \neq 0 \text{ on } \overline{V_{\tilde{C}}} \setminus \tilde{C}, \quad V_{\tilde{C}} \cap \{f < \lambda\} = \tilde{C} \quad \text{and} \quad V_{\tilde{C}} \cap \{f \leq \lambda\} = \tilde{C} \quad (84)$$

is slightly different (see more precisely (61) and (62) above). Actually, the only difference with the proof of Theorem 2 is that under the assumptions of Theorem 4, it may exist  $z \in \partial C_1 \cup \dots \cup \partial C_m$  such that  $|\nabla f|(z) = 0$ . Then, by assumption,  $z \in \Omega$ . Let  $i \in \{1, \dots, m\}$  be such that  $z \in \partial C_i$ . If  $z$  is not a separating saddle point of  $f$ , there exists  $\varepsilon_z > 0$  such that  $B(z, \varepsilon_z) \cap \{f < \lambda\} = B(z, \varepsilon_z) \cap C_i$  and  $B(z, \varepsilon_z) \cap \{f \leq \lambda\} = B(z, \varepsilon_z) \cap \overline{C_i}$ . Else, if  $z$  is a separating saddle point of  $f$ , from (80), there exists  $j \neq i$  ( $j \in \{1, \dots, m\}$ ),  $z \in \partial C_i \cap \partial C_j$ . Thus, there exists  $\varepsilon_z > 0$  such that  $B(z, \varepsilon_z) \cap \{f < \lambda\} = B(z, \varepsilon_z) \cap (C_i \cup C_j)$ . Moreover,  $B(z, \varepsilon_z) \cap \{f \leq \lambda\} = B(z, \varepsilon_z) \cap (\overline{C_i} \cup \overline{C_j})$ . Together with the analysis of the local structure of  $\{f < \lambda\}$  near the points  $z \in \partial C_1 \cup \dots \cup \partial C_m$  such that  $|\nabla f|(z) \neq 0$  (see items 1 and 2 just before (60)), this justifies the existence of neighborhood  $V_{\tilde{C}}$  of  $\tilde{C}$  in  $\bar{\Omega}$  satisfying (84). The end of the proof then follows exactly the same lines as the proof of Theorem 2.  $\blacksquare$

As an illustration, here are some simple examples of results which can be obtained by Theorem 4 but not by Theorems 1 and 2. In Figure 5, Theorems 1 and 2 do not apply. However, in the example depicted in Figure 5, item 1 in Theorem 4 implies that in the limit  $h \rightarrow 0$ , for all  $x \in C_{\max} = (z_1, z)$ , there exists  $c > 0$  such that  $\mathbb{P}_x[X_{\tau_\Omega} = z_2] = O(e^{-\frac{c}{h}})$ , and thus  $\mathbb{P}_x[X_{\tau_\Omega} = z_1] = 1 + O(e^{-\frac{c}{h}})$ . In Figure 6, item 1 in Theorem 1 only implies that



*Figure 5: A one dimensional case when (A0), (A1), and (A2) are satisfied with  $C = \{C_{\max}, C_2\}$ . The assumption (A3) is not satisfied because  $\partial C_{\max} \cap \partial\Omega = \{z_1\}$  and  $f(z_1) > f(z_2) = \min_{\partial\Omega} f$ .*

when  $h \rightarrow 0$  and for all  $x \in C_{\max} = (z_1, z)$ ,  $\mathbb{P}_x[X_{\tau_\Omega} = z_2] = O(h^{1/4})$  whereas item 1 in Theorem 4 implies that there exists  $c > 0$  such that  $\mathbb{P}_x[X_{\tau_\Omega} = z_2] = O(e^{-\frac{c}{h}})$ .

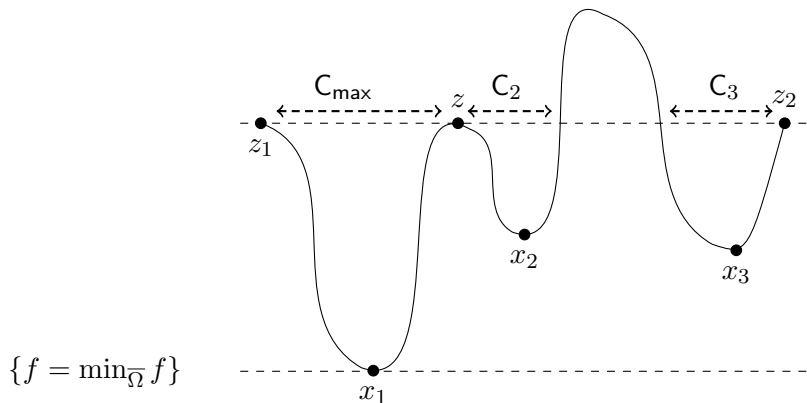


Figure 6: A one dimensional case when **(A0)**, **(A1)**, **(A2)**, and **(A3)** are satisfied with  $\mathcal{C} = \{\mathcal{C}_{\max}, \mathcal{C}_2, \mathcal{C}_3\}$ . The assumption **(A4)** is not satisfied because  $\partial\mathcal{C}_{\max} \cap \partial\mathcal{C}_2 = \{z\} \neq \emptyset$ .

## 4 Conclusion and perspectives

In conclusion, the objective of this work was to identify the exit points from  $\Omega$  of  $(X_t)_{t \geq 0}$  solution to (1) in the regime  $h \rightarrow 0$ , when  $X_0 = x \in \Omega$ .

Under assumption **(A0)**, what is expected is the following: (i) let us consider the dynamics  $\dot{y} = -\nabla f(y)$  with initial condition  $y(0) = x$ , and let us assume that  $y(t)_{t \geq 0}$  remains in  $\Omega$  and reaches a local minimum  $x^*$  (recall that if  $y(t)$  leaves  $\Omega$  at a finite time, say at point  $y^*$ , then  $y^*$  will be the exit point in the small temperature regime, see Remark 7); (ii) let us then consider  $\lambda^* = \sup\{\lambda > f(x^*) \text{ s.t. the connected component of } \{f < \lambda\} \text{ which contains } x^* \text{ does not intersect } \partial\Omega\}$  and the associated connected component  $\mathcal{C}^*$  defined as the connected component of  $\{f < \lambda^*\}$  which contains  $x^*$  (notice that  $\mathcal{C}^* = \mathcal{C}(x^*)$ , as defined by (11)). Let us assume that  $\partial\mathcal{C}^* \cap \partial\Omega \neq \emptyset$ . Then, one expects that with probability one,  $X_t$  leaves  $\Omega$  through  $\partial\mathcal{C}^* \cap \partial\Omega$  (in other words, the law of the first exit point concentrates on  $\partial\mathcal{C}^* \cap \partial\Omega$ , according to Definition 1).

What we have proven in this work is that this property can indeed be proven (with explicit relative likelihoods of the different exit points in  $\partial\mathcal{C}^* \cap \partial\Omega$ ) if  $\arg \min\{f(x), x \in \mathcal{C}_{ext}^*\} = \arg \min\{f(x), x \in \mathcal{C}^*\}$ , and  $f$  is constant (equal to  $\lambda^*$ ) over the boundary of  $\mathcal{C}_{ext}^*$  where  $\mathcal{C}_{ext}^*$  is the connected component of  $\{f \leq \lambda^*\}$  which contains  $x^*$ . Indeed, this is exactly what is needed to apply Theorem 4 with  $\lambda = \lambda^*$ ,  $\mathcal{C}^* = \mathcal{C}_1$ , and  $\mathcal{C}_{ext}^* = \cup_{j=1}^m \bar{\mathcal{C}}_j$ . Notice in particular that the result holds if  $\bar{\mathcal{C}}^*$  is a connected component of  $\{f \leq \lambda^*\}$ .

In future works, we intend to study the same question while relaxing the assumption **(A0)**, by allowing  $f$  to have critical points on the boundary, see [30, 38] for preliminary works in that direction. This should also allow us to go beyond some of the restrictions above.

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