On Null 3-Hypergraphs

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Abstract

Given a 3-uniform hypergraph H consisting of a set V of vertices, and $T \subseteq \binom{V}{3}$ triples, a *null labelling* is an assignment of ± 1 to the triples such that each vertex is contained in an equal number of triples labelled $+1$ and -1. Thus, the signed degree of each vertex is zero. A necessary condition for a null labelling is that the degree of every vertex of H is even. The Null Labelling Problem is to determine whether H has a null labelling. It is proved that this problem is NP-complete. Computer enumerations suggest that most hypergraphs which satisfy the necessary condition do have a null labelling. Some constructions are given which produce hypergraphs satisfying the necessary condition, but which do not have a null labelling. A self complementary 3-hypergraph with this property is also constructed.

Keywords: 3-hypergraph, null labelling, null hypergraph, NP-complete, self-complementary

1. Introduction

The problem of characterizing hypergraphs from their degree sequences is a longstanding challenging problem. Many necessary and a few sufficient conditions are present in the literature, and they rely mainly on a result by Dewdney [6] who established a necessary and sufficient condition for a sequence π to be k-graphic, i.e., the degree sequence of a k-uniform hypergraph, briefly k-hypergraph. The condition is based on a recursive decomposition of π that does not result in an efficient algorithm for the construction of a corresponding hypergraph. For example, inspired by this study, Behrens et al. [1] proposed a sufficient and polynomially testable condition for a degree sequence to be k-graphic; their result still does not provide any information about the associated k-hypergraphs. Soon after, Brlek and Frosini in [3] overcome this problem by designing a polynomial time algorithm to reconstruct one of the k-hypergraphs associated with a given instance satisfying the condition in [1].

Recently, Deza et al. [7] proved that, for any fixed integer $k \geq 3$, deciding the k-graphicality of a sequence is an NP-complete problem.

However, some relevant related questions remain open. In [12], the notion of null hypergraph has been introduced to study all 3-hypergraphs with a given degree sequence. The present research links this notion with that of intersection graphs, showing some sufficient conditions for the existence of a null labelling.

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More precisely, in the last part of this section we introduce the notion of null-labelling for graphs and hypergraphs and we provide some preliminary properties. The second section focuses on 3-uniform hypergraphs: a necessary condition is given for the existence of a null labelling and some examples of its non-sufficiency.

In Section 3 we prove the NP-completeness of the existence of a null labelling for 3-hypergraphs. A final section including conclusions and some research perspectives is also provided.

Let $V = \{1, 2, \ldots, n\}$, where $n \geq 3$, be a set of *vertices*. A graph G with vertex set V is considered to be a collection of pairs of vertices, called edges, namely, $G \subseteq \binom{V}{2}$. A 3-hypergraph H with vertex set V is a collection of triples of vertices, namely $H \subseteq \binom{V}{3}$. Given a graph G or 3-hypergraph H, we can assign $+1$ or -1 to each edge or triple, resulting in positive edges or triples, and negative edges or triples. The positive degree of a vertex v is $d^+(v)$, the number of positive edges or triples containing v. The negative degree is $d^-(v)$, the number of negative edges or triples containing v. The signed degree of each vertex v is $d(v) = d^+(v) - d^-(v)$. The unsigned degree is $\deg(v) = d^+(v) + d^-(v)$.

Definition 1. *An assignment of ± 1 to the edges or triples of a graph G or 3-hypergraph H is a* null labelling *if $d(v) = 0$, for all vertices v. A graph or hypergraph with a null labelling is said to be a* null graph *or* null hypergraph.

It is easy to characterize graphs G with a null labelling. Each vertex must have even degree, so that each connected component of G is Eulerian. The number of positive and negative edges in each connected component must be equal, so that the number of edges must be even. We state this as a lemma.

Lemma 1. *A graph G has a null labelling iff every connected component is an Eulerian graph with an even number of edges.*

Proof. The necessity is clear from the comments preceding the lemma. To prove sufficiency, consider a connected component which is an Eulerian graph with an even number of edges. By following an Euler tour, assigning $+1$ and -1 to alternate edges, a null labelling is obtained. □

This lemma also characterizes graphs G with even degrees and an even number of edges that do not have a null labelling — they must be disconnected graphs such that at least two connected components have an odd number of edges. The smallest graph with a null labelling is a cycle on four vertices, which we denote by C_4.

Graphs with null labellings arise when two graphs with equal degree sequences are considered. Let G_1 and G_2 be graphs with the same vertex set and the same degrees $d_1, d_2, \ldots, d_n$. Assign $+1$ to the edges of G_1, and -1 to the edges of G_2. The exclusive or (XOR) or symmetric difference of G_1 and G_2 is $G_1 \oplus G_2$. It is a graph whose edges have been assigned ± 1, such that each vertex v satisfies $d(v) = 0$, i.e., $G_1 \oplus G_2$ has a null labelling. A result of Havel and Hakimi [11, 10] states that G_1 can always be transformed into G_2 using a sequence of edge interchanges, that is, using a sequence of graphs isomorphic to C_4 with a null labelling, such that each intermediate graph is a simple graph. In symbols this can be written

$$G_1 = G_2 \oplus M_1 \oplus M_2 \oplus \ldots \oplus M_k$$

where each M_i is isomorphic to C_4.

Definition 2. *An* even *graph, or* even *3-hypergraph, is a graph or 3-hypergraph with an even number of edges or triples, such that all vertices have even degree.*

The situation with 3-hypergraphs is similar. Let H_1 and H_2 be two 3-hypergraphs with the same vertex set, and same degree sequence $d_1, d_2, \ldots, d_n$. Assign $+1$ to the triples of H_1 and -1 to the triples of H_2, and construct $H_1 \oplus H_2$. It is a 3-hypergraph with a null labelling. It was proved by Kocay and Li [12] that H_1 can always be transformed into H_2

using a sequence of null 3-hypergraphs isomorphic to N_5 or N_6. Here N_5 and N_6 are the two null 3-hypergraphs depicted in Figure 1, each with four triples. The triples are represented by triangles either shaded (+1) or unshaded (−1). This raises the question of whether there is an Eulerian result for null-labelled 3-hypergraphs similar to Lemma 1, i.e., a characterization of null 3-hypergraphs. This result was extended to k-hypergraphs by Behrens et al [1]. In a section following, it is shown that the problem of finding a null labelling for 3-hypergraphs is NP-complete.

Question: Let H be a connected, even 3-hypergraph. When can ± 1 be assigned to the triples of H to produce a null-labelled 3-hypergraph?

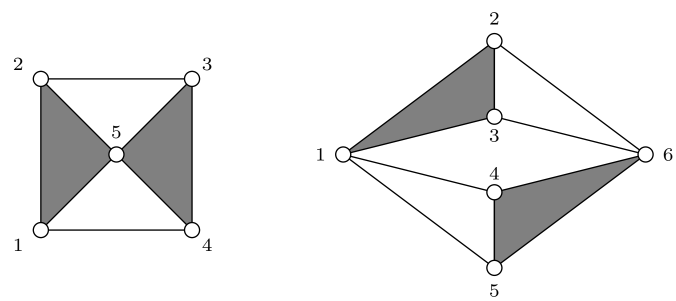


Figure 1: N_5 and N_6, null 3-hypergraphs on 5 and 6 vertices and 4 triples.

2. Null 3-Hypergraphs

In this section we construct an infinite family of connected, even 3-hypergraphs, such that the triples cannot be assigned ± 1 so as to create a null labelling.

Definition 3. *Given a 3-hypergraph H, and a vertex v of H, the triples containing v determine a collection of pairs H_v obtained by deleting v from the triples, called the* derived *graph of v in H.*

Let G be a graph. We can add a new vertex v, and using the edges of G, create the triples of a 3-hypergraph H, by simply adding v to each edge of G. Then $H_v = G$.

Lemma 2. *Let G be an even graph. Construct H such that $H_v = G$, where v is a new vertex. Then H is a connected, even 3-hypergraph. It has a null labelling iff G has a null labelling.*

This provides a simple means of constructing connected, even 3-hypergraphs with no null labelling. Consider any even graph G with at least two connected components each of which has an odd number of edges. Then G is an even graph, so that H is even and connected, but nevertheless, H does not have a null labelling. This is considered to be a degenerate example of a connected, even 3-hypergraph with no null labelling.

The smallest even, connected 3-hypergraph with no null labelling is shown in Figure 2. It has six vertices, and four triples, which are all shaded in the diagram. Its incidence graph is also shown, such that the vertices which represent triples are shaded gray. It is easy to see by inspection that it does not have a null labelling.

In the example of Figure 2, every vertex of H has degree two. In the incidence graph, we can replace each vertex of H and its two incident edges with a single edge, thereby obtaining a 3-regular graph whose vertices are the triples of H.

Definition 4. *The* intersection graph *of a 3-hypergraph H is denoted $I(H)$. Its vertices are the triples of H. Two triples are adjacent if their intersection is non-empty.*

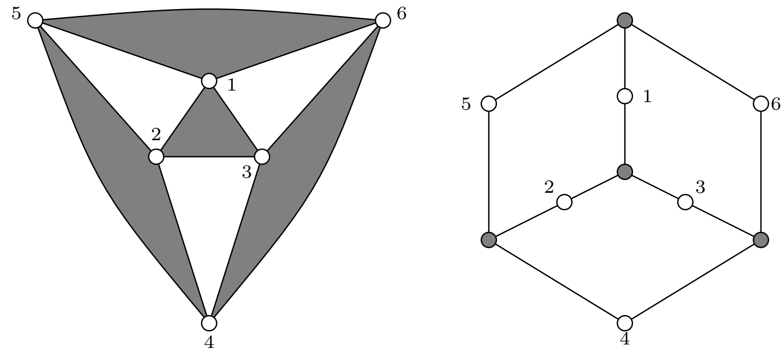


Figure 2: $H_{6,4}$, the smallest connected, even 3-hypergraph with no null-labelling. 6 vertices, 4 triples.

This is an extension of the idea of a line graph to 3-hypergraphs. In the example of Figure 2, K_4 is obtained as the intersection graph.

Lemma 3. *If $I(H)$ is bipartite, then each vertex of H is contained in at most two triples.*

Proof. If some vertex v were contained in more than two triples, then these triples would all be adjacent in $I(H)$, so that $I(H)$ is not bipartite. $\square$

Theorem 1. *Let H be a connected, even 3-hypergraph, in which every vertex has degree two. Then H has a null labelling if and only if $I(H)$ is bipartite.*

Proof. If H has a null labelling, each vertex of H must be contained in a triple labelled $+1$, and a triple labelled -1. As each vertex has degree two, the null-labelling is unique, up to interchanging ± 1. If $I(H)$ contains an edge between two triples T_1, T_2 with the same label, the vertices of $T_1 \cap T_2$ have both neighbours with the same label, which is impossible. Therefore $I(B)$ is bipartite.

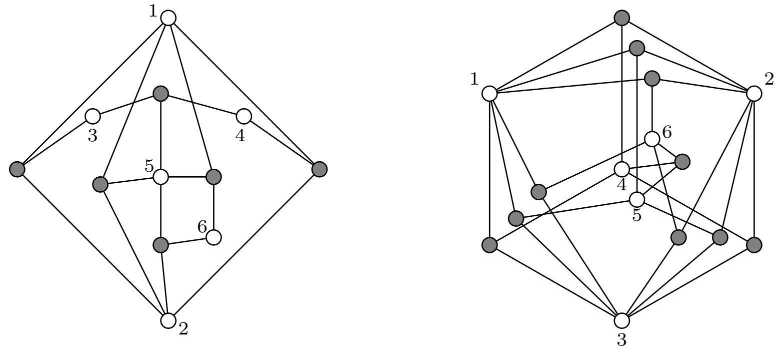
Conversely, if $I(H)$ is bipartite, with bipartition (V_1, V_2), assign $+1$ to V_1 and -1 to V_2. $I(B)$ is connected because H is connected. As every vertex of H has degree two, each vertex corresponds to an edge of $I(H)$, and so has one incident triple $+1$ and one -1. The bipartition defines a null labelling of H. $\square$

It is relatively straightforward to generate the distinct isomorphism types of connected, even 3-hypergraphs on five and six vertices, by computer, and to determine by exhaustive search whether they have null labellings. It is more difficult on seven or more vertices. The method used is described below. Every even 3-hypergraph on five vertices has a null-labelling, and there are three on six vertices which do not have a null labelling. The smallest with six vertices has four triples, shown in Figure 2. The other two have six triples and 10 triples, and are shown in Figure 3 as incidence graphs, as it does not seem to be feasible to draw the triples as triangles.

Lemma 4. *The connected, even 3-hypergraph $H_{6,10}$ of Figure 3 has no null labelling.*

Proof. Observe first that the vertices of H have degrees six and four, and that H is connected. If all three triples containing $\{1, 2\}$ have sign $+1$, then all triples containing $\{1, 3\}$ and all triples containing $\{2, 3\}$ must have sign -1. But then $d^-(3) = 6$, which is impossible. A similar conclusion follows if all three triples containing $\{1, 2\}$ have sign -1.

Otherwise, without loss of generality, two of the triples containing $\{1, 2\}$ have sign $+1$, and one has sign -1. Then two of the triples containing $\{1, 3\}$ and two of the triples containing $\{2, 3\}$ have sign -1, so that $d^-(3) = 4$, which is impossible. $\square$



$H_{6,6}$, 6 triples $\qquad$ $H_{6,10}$, 10 triples

Figure 3: Even 3-hypergraphs on 6 vertices with no null labelling.

The distinct isomorphism types of all connected, even 3-hypergraphs on $n = 5, 6, 7$ vertices were found by using a computer to generate them, and then to test if they have null labellings. Given the number of vertices n, the search first generates all connected 3-hypergraphs on n vertices, storing them in a file. They are tested for isomorphism using graph isomorphism software. The software of [13] was used. Each hypergraph is represented as a bipartite graph, with one side of the bipartition distinguished. Isomorphic copies are rejected. The connected, even 3-hypergraphs are then extracted and copied to another file and tested for a null labelling. The 3-hypergraphs on n vertices were generated in order of τ, the number of triples. The process was started by constructing the 3-hypergraphs with $\tau = 2$ by hand. Those with $\tau \geq 3$ triples can be constructed from those on $\tau - 1$ triples by adding a triple in all possible ways. All 3-hypergraphs were constructed for $n = 5, 6, 7$, and some with $n = 8$ were constructed. In generating the hypergraphs, one has to be careful not to generate each one numerous times. An overcount of $n!$ must be avoided. A more complete description of generating hypergraphs avoiding overcounting can be found in [14].

The number of 3-hypergraphs on eight vertices is much, much greater than on seven. In general, the number of isomorphism types of 3-hypergraphs on n vertices and τ triples is at least $\binom{\binom{n}{3}}{\tau}/n!$ (see [14]). We have generated and tested only some of them on eight vertices. The following table summarizes the numbers, where n is the number of vertices, and τ is the number of triples. Note that 3-hypergraphs with $\tau > \frac{1}{2}\binom{n}{3}$ are the complements of those with $\tau < \frac{1}{2}\binom{n}{3}$. The table only shows the numbers with τ even, and $\tau \leq \frac{1}{2}\binom{n}{3}$. Those with $\tau > \frac{1}{2}\binom{n}{3}$ and $n \leq 7$ all have a null labelling.

n	τ	non-null	connected,even	# hypergraphs
6	4	1	1	21
6	6	1	1	94
6	8	0	14	249
6	10	1	19	352
7	4	0	0	38
7	6	5	12	509
7	8	15	104	5,557
7	10	31	705	39,433
7	12	9	2882	172,933
7	14	4	7828	473,827
7	16	1	13355	824,410
8	6	5	9	1,413
8	8	117	328	41,868
8	10	> 500	7297	936,130
8	12	?	?	$\geq 13,848,793$

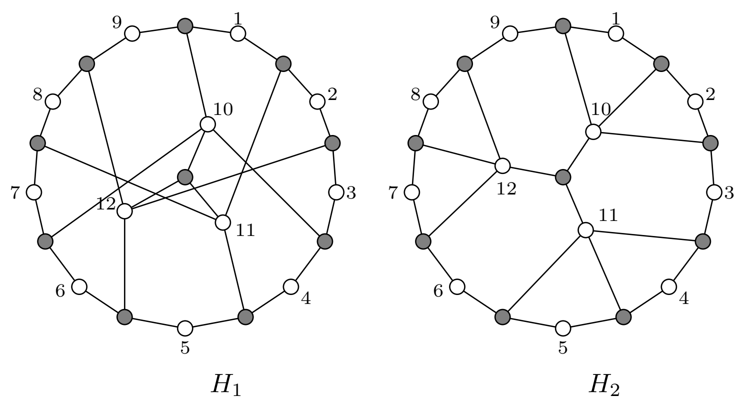


Figure 4: Two connected, even 3-hypergraphs H_1 and H_2 on 12 vertices, with no null labelling.

Two even 3-hypergraphs on 12 vertices are represented by their incidence graphs in Figure 4. The vertices are labelled $\{1, 2, \ldots, 12\}$. The triples are represented by shaded vertices. It is easy to see that they have no null-labellings, as the "outer" ring has nine vertices of degree two, and nine triples, an odd number. There are several ways in which this example can be generalized to an infinite family of even 3-hypergraphs with no null-labelling.

Theorem 2. *Let $n \geq 1$. There are at least two connected even 3-hypergraphs H_1 and H_2 on $12n$ vertices with no null labelling.*

Proof. Let $V = \{u_1, u_2, \ldots, u_{9n}\} \cup \{v_1, v_2, \ldots, v_{3n}\}$ be the vertices of of either H_1 or H_2. The triples of H_1 are the following, where subscripts m of u_m are reduced to the range $1 \ldots 9n$, and subscripts m of v_m are reduced to the range $1 \ldots 3n$:

$$T_i = \{u_i, u_{i+1}, v_{i+1}\}, \quad \text{where } i = 1, 2, \ldots, 9n$$

$$T_i' = \{v_i, v_{i+n}, v_{i+2n}\}, \quad \text{where } i = 1, 2, \ldots, n$$

Then $\deg(u_i, H_1) = 2$ and $\deg(v_i, H_1) = 4$. If n is odd, then it is not possible to assign the triples T_i values ± 1 such that $\deg(u_i) = 0$. If n is even, then without loss of generality, the triples $T_1, T_3, \ldots$

must be assigned $+1$, and $T_2, T_4, \ldots$ must be assigned -1. But then v_1 is contained in three triples of sign $+1$, which is impossible in a null labelling.

The triples of H_2 are the following.

$$T_i = \{u_i, u_{i+1}, v_{1+\lfloor i/3 \rfloor}\}, \quad \text{where } i = 1, 2, \ldots, 9n$$

$$T_i' = \{v_i, v_{i+1}, v_{i+2}\}, \quad \text{where } i = 1, 4, 7, \ldots, 3n-2$$

A similar proof shows that H_2 has no null labelling. □

It is clear that various other families of 3-hypergraphs without null labellings, similar to the examples of Figure 4, can also be constructed. The proof of Theorem 2 is based on the degrees of the vertices u_i being two, so that a null labelling is forced once T_1 has been given its sign. Examples where all degrees are at least four also exist. One such is $H_{6,10}$, shown in Figure 3, which is a non-null 3-hypergraph with six vertices and 10 triples.

A family of connected, even 3-hypergraphs without a null labelling can be constructed based on this model.

Definition 5. *Let $n \geq 1$. Denote by $H(3n)$ the 3-hypergraph with vertices $\{u_1, u_2, \ldots, u_{3n}\} \cup \{v_1, v_2, \ldots, v_{3n}\}$ with the following triples. Here subscripts larger than $3n$ are reduced modulo $3n$ to the range $1 \ldots 3n$.*

$$\{u_i, u_{i+1}, v_i\}, \{u_i, u_{i+1}, v_{i+n}\}, \{u_i, u_{i+1}, v_{i+2n}\}, \text{where } i = 1, 2, \ldots, 3n$$

$$\{v_i, v_{i+n}, v_{i+2n}\}, \text{where } i = 1, 2, \ldots, n$$

Theorem 3. *Let $n \geq 1$ be odd. Then $H(3n)$ has no null labelling.*

Proof. For every i, vertex u_i is contained in six triples. There are three triples containing u_i and u_{i+1}. Start with $i = 1$. If all three triples containing u_1 and u_2 are positive, then all three containing u_2 and u_3 are negative. Continuing like this, we find that all triples containing u_m and u_{m+1} are positive when m is odd, and negative when m is even. But $3n$ is odd, a contradiction.

Otherwise, for some u_i there are two positive triples containing u_i and u_{i+1}, and one negative triple containing u_i and u_{i+1}. Without loss of generality, it is u_1. Then there are two negative triples and one positive triple containing both u_2 and u_3, etc. In general, there are two positive triples containing u_m and u_{m+1}, and one negative triple, whenever m is odd. But as $3n$ is odd, we again have a contradiction. □

The proof depends on n being odd. Note that the role played by the vertices v_i is basically to ensure that the 3-hypergraph is even, as the proof depends mainly on the degrees of the u_i. When n is even, $H(3n)$ always has a null labelling. We would like to thank the referee for providing a proof of this fact. The triples can be partitioned according to the following table, for odd i, with signs reversed when i is even. Here $1 \leq i \leq n$. Each u_i has degree six, and each v_i has degree four.

$u_i u_{i+1} v_i$	$-$	$u_{i+n} u_{i+n+1} v_i$	$+$	$u_{i+2n} u_{i+2n+1} v_i$	$+$	$v_i v_{i+n} v_{i+2n}$	$-$
$u_i u_{i+1} v_{i+n}$	$+$	$u_{i+n} u_{i+n+1} v_{i+n}$	$-$	$u_{i+2n} u_{i+2n+1} v_{i+n}$	$+$		
$u_i u_{i+1} v_{i+2n}$	$+$	$u_{i+n} u_{i+n+1} v_{i+2n}$	$+$	$u_{i+2n} u_{i+2n+1} v_{i+2n}$	$-$		

An additional property of $H_{6,10}$ is that it is self-complementary:

Remark 1. *The 3-hypergraph $H_{6,10}$ of Figure 3 is self-complementary.*

Proof. Note that $H_{6,10}$ has 10 triples out of a possible $\binom{6}{3} = 20$. It is easy to see that the permutation $(1, 4, 2, 5, 3, 6)$ maps the 10 triples of $H_{6,10}$ to the triples $\{1, 4, 5\}, \{2, 4, 5\}, \{3, 4, 5\}, \{1, 4, 6\}, \{2, 4, 6\}, \{3, 4, 6\}, \{2, 4, 5\}, \{1, 5, 6\}, \{2, 5, 6\}, \{3, 5, 6\}$, which is the complement of $H_{6,10}$. □

Lemma 5. *An even self-complementary 3-hypergraph on n vertices must have either n even, or $n \equiv 1 \pmod 8$.*

Proof. An even self-complementary 3-hypergraph on n vertices must have $\frac{1}{2}\binom{n}{3} = n(n-1)(n-2)/12$ triples, which must be an even number. $\square$

By constructing all the connected, even 3-hypergraphs on six vertices, we find that this is the only connected even 3-hypergraph with 6 vertices and 10 triples which does not have a null labelling, and that all those with more than 10 triples do have a null labelling. Similarly, on 7 vertices, the number with no null-labellling at first increases as the number of triples increases, and then decreases. All connected even 3-hypergraphs with 7 vertices and more than 17 triples do have null labellings. On the basis of this evidence, we make the following conjecture.

Conjecture: All connected, even 3-hypergraphs on n vertices with more than $\frac{1}{2}\binom{n}{3}$ triples have null labellings.

Related to this conjecture, we can formulate the following decision problem. It will be proved to be NP-complete in the next section.

Null Labelling Problem (NLP):

Instance: a connected, even 3-hypergraph H.

Question: is there an assignment ± 1 to the triples of H so as to produce a null labelling of H?

3. The NP-Completeness of the Null Labelling Problem

In this section it is shown that it is NP-complete to determine whether an arbitrary connected, even 3-hypergraph has a null labelling. We reduce from the problem

3-Partition (SP16 in [9]):

Instance: a set of $n = 3m$ positive integers $A = \{a_1, a_2, \ldots, a_n\}$, with sum $S = \Sigma a_i$ divisible by m, and such that $S = mB$.

Question: Can A can be partitioned into m triples $\{a_{i_1}, a_{i_2}, a_{i_3}\}, \{a_{i_4}, a_{i_5}, a_{i_6}\}, \ldots$ such that each triple sums to B?

Note that it is usual to require that each a_i satisfies $B/4 < a_i < B/2$. This has the effect of forcing each set of the partition which sums to B to have exactly three elements.

Given an instance A of 3-Partition, we construct a 3-hypergraph H_A such that there is a solution of the 3-Partition instance if and only if H_A has a null labelling. This will imply that NLP is NP-complete. Given the set A with sum $S = mB$, the 3-hypergraph is constructed in steps. The first step is to multiply each a_i by 3 to obtain $a'_i = 3a_i$ and $B' = 3B$. It is straightforward that the solutions of the instance $A' = \{a'_1, a'_2, \ldots a'_n\}$ are in 1-1 correspondence with those of the initial instance A.

Representing the elements of A'

Define a *gadget*, denoted $\mathcal{A}(a'_i)$, to represent each integer a'_i in A'. The gadget is a 3-hypergraph. We describe it through its incidence graph (see Figure 5).

The construction of $\mathcal{A}(a'_i)$ requires a central layer of $V = \{v_1, \ldots, v_{2a'_i}\}$ vertices of degree two, an upper layer $T = \{T_1, \ldots, T_{a'_i}\}$, and a lower layer $T' = \{T'_1, \ldots, T'_{a'_i}\}$ of vertices of degree three, representing triples, and two vertices X and Y of degree a'_i that are adjacent to T' and T, respectively. The vertices X and Y provide the external accesses to the gadget. The triples are the following, where subscripts of v_i are reduced to the range $1 \ldots 2a'_i$:

$$T_i = \{v_{2i}, v_{2i+1}, Y\} \quad \text{and} \quad T'_i = \{v_{2i-1}, v_{2i}, X\}, \quad \text{where} \quad i = 1, \ldots, a'_i$$

An example is shown in Figure 5, for the case $a'_i = 6$.

The edges containing the vertices in the central layer V of $\mathcal{A}(a'_i)$ and their connections throughout T and T' with X and Y lead to the following lemma, whose proof is straightforward

Lemma 6. *In any labelling of the triples T and T' of $\mathcal{A}(a'_i)$, such that the central layer of vertices V have signed degree 0, the triples T must all have the same sign $+1$, and the triples T' must all have the same opposite sign -1, or vice-versa.*

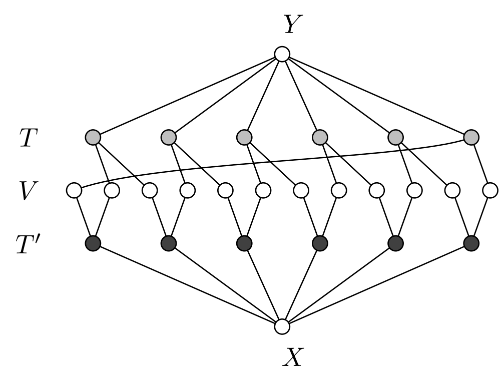


Figure 5: The gadget $\mathcal{A}(a'_i)$ for the case $a'_i = 6$. The white central layer of vertices represents V, and the grey vertices represent the upper layer T, and the black vertices represent the lower layer T'.

In the definition of the gadget $\mathcal{A}(a'_i)$, the element a'_i can be replaced with any other positive integer, so we will also consider $\mathcal{A}(a'_i(m-2))$ in the definition of H_A. They will be combined with the second type of gadget defined below to allow vertices X and Y to have a signed degree of 0.

Counting the incoming edges to a vertex

We slightly modify $\mathcal{A}(a'_i)$ to define a second gadget, denoted $\mathcal{C}(3B(m-2))$, and illustrated in Figure 6 when $B = 2$ and $m = 6$. The vertex Z will have $3B(m-2)$ incident edges to triples all of the same sign. So, Z will reach zero sum when connected with external edges whose signed degree is $-3B(m-2)$.

Let $t = 6B(m-2)$. Note that t is divisible by six. $\mathcal{C}(3B(m-2))$ consists of one central layer of $V = \{v_1, \ldots v_t\}$ vertices of degree two, an upper layer $T = \{T_1, \ldots, T_{\frac{t}{2}}\}$, and a lower layer $T' = \{T'_1, \ldots, T'_{\frac{t}{3}}\}$, of vertices of degree three (representing triples) and one vertex Z of degree $\frac{t}{2}$ that is adjacent to all vertices of T. The vertex Z provides external access to the gadget. The triples are the following:

$$T_i = \{v_{2i-1}, v_{2i}, Z\} \qquad \text{where} \quad i = 1, \dots, \frac{t}{2} \quad \text{and}$$

$$T' = \{\{v_1, v_2, v_3\}, \{v_{t-2}, v_{t-1}, v_t\}, \{v_{6i-2}, v_{6i-1}, v_{6i+1}\}, \{v_{6i}, v_{6i+2}, v_{6i+3}\}\},$$

where $i = 1, \dots, \frac{t}{6} - 1$. The construction of this counter is illustrated in Figure 6 for $m = 4$ and $3B = 6$. Note that all the indices of the triples are integers.

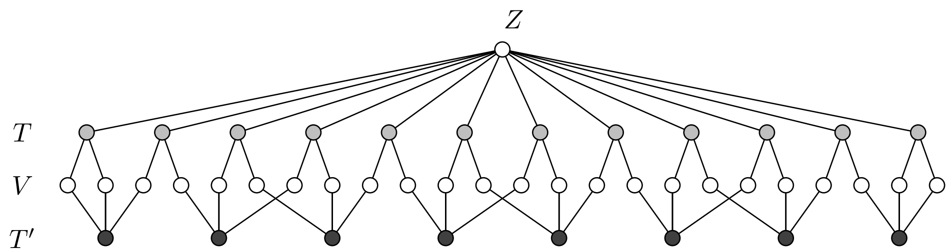


Figure 6: The counter $\mathcal{C}(3B(m-2))$, with $3B = 6$ and $m = 4$. The vertex Z has $3B(m-2) = 12$ incoming edges and all the vertices V have signed degree 0. The grey vertices represent the upper layer of triples T, and the black vertices the lower layer T'.

The edges containing the vertices of the central layer V of $\mathcal{C}(3B(m-2))$ and their connections throughout T with Z lead to the following result, whose proof is straightforward

Property 1. *In any labelling of the triples T and T' of $\mathcal{C}(3B(m-2))$ such that the central layer of vertices V has signed degree 0, the triples T must all have the same sign $+1$, and the triples T' must all have the same opposite sign -1, or vice-versa. Furthermore, vertex Z has $3B(m-2)$ adjacent triples of the same sign.*

In the definition of the gadget $\mathcal{C}(3B(m-2))$ the element B can be replaced with any other positive integer, so we will also consider $\mathcal{C}(3mB(m-2))$ in the definition of H_A.

Putting gadgets together to define a 3-hypergraph

Let us consider the instance $A' = \{a'_1, \dots, a'_n\}$ of 3-Partition whose elements sum to $3mB$, where $n = 3m$ and $m > 1$. Based on the gadgets already defined, we construct a 3-hypergraph $H_{A'}$. The reader can follow the construction in Figure 7, where dashed links stand for multiple connections. We make m copies of the gadget $\mathcal{C}(3B(m-2))$, denoting the Z-vertex of the j^{th} $\mathcal{C}(3B(m-2))$ by $N_{1,j}$ where $j = 1, \dots, m$. They become the upper layer in Figure 7. We construct a gadget $\mathcal{C}(3mB(m-2))$, denoting its Z-vertex by $N_{3,1}$. This gadget becomes the lowermost layer in Figure 7. From now on, we will consider index $i = 1, \dots, n$ and index $j = 1, \dots, m$.

Now we define the vertices $N_{2,i}$: we make m copies of each $\mathcal{A}(a'_i)$, and denote by $X_{i,j}$ and $Y_{i,j}$ the corresponding X and Y nodes. Merge, for all j, all $X_{i,j}$ into a single vertex X_i, and then identify it with $N_{2,i}$. And merge, for all i, all $Y_{i,j}$ into a single vertex Y_j, and then merge it with $N_{1,j}$.

Finally, construct n gadgets $\mathcal{A}(a'_i(m-2))$, one for each i. Merge the Y vertex of the i^{the} gadget with $N_{2,i}$ and the X vertex with $N_{3,1}$. These gadgets form the second lowest layer in Figure 7.

The 3-sets in a solution to 3-Partition will be indicated by the vertices $N_{1,j}$. So, this will create an incidence graph as in Figure 7, with m equal blocks $\mathcal{A}(a'_i)$ adjacent to a common $N_{2,i}$, and the j^{th} copy of each $\mathcal{A}(a'_i)$, for $i = 1, \dots, n$, adjacent to a common $N_{1,j}$.

Theorem 4. *Given an instance A' of 3-Partition, the 3-hypergraph $H_{A'}$ has a null labelling if and only if the instance A' has a solution.*

Proof. Suppose there is a null labelling of $H_{A'}$. Property 1 states that all the edges of $N_{3,1}$ incoming from $\mathcal{C}(3mB(m-2))$ have the same sign. Since $\sum_{m}^{i=1} a_i'(m-2) = 3mB(m-2)$, then also all the edges incoming from $\mathcal{A}(a_i'(m-2))$ must have the same opposite sign, without loss of generality, choose $+1$.

A similar reasoning holds for each vertex $N_{2,i}$: the edges incoming from the $\mathcal{A}(a_i'(m-2))$ must all have sign -1, giving $a_i'(m-2)$ edges of sign -1 incoming to $N_{2,i}$. The number of edges incoming to $N_{2,i}$ from the m copies of $\mathcal{A}(a_i')$ is ma_i'. It must be that a_i' of them have sign -1, and $(m-1)a_i'$ of them have sign $+1$. Thus, in each group of m copies of $\mathcal{A}(a_i')$, *one* copy has a_i' edges of sign $+1$ to some $N_{1,j}$. The other $m-1$ copies have edges of sign -1 to their corresponding $N_{1,j}$ vertices.

In order for each $N_{1,j}$ to have zero sum, Property 1 assures that the signed degree of the edges incoming from the gadgets $\mathcal{A}(a_i')$, must be $\pm 3B(m-2)$. This counts all edges coming from n $(= 3m)$ gadgets $\mathcal{A}(a_i')$. This implies that $3B(m-1)$ edges reaching $N_{1,j}$ from $n-3$ gadgets $\mathcal{A}(a_i')$ have sign $+1$ and the $3B$ edges incoming from the three remaining gadgets $\mathcal{A}(a_i')$ have sign -1. Consequently, the $3B$ edges with sign $+1$ from these three $\mathcal{A}(a_i')$ can produce zero sum in only one $N_{1,j}$.

Since for every i, there is exactly one copy of $\mathcal{A}(a_i')$ that is contributing weight -1 to some $N_{1,j}$ and each of these $N_{1,j}$ can only have one such incident edge (because of the contributed weight from the $\mathcal{C}$ gadget above), then this defines a partition of the set $\{a_1', a_2', \ldots, a_n'\}$ into m parts. Furthermore, the labelling is a null labelling if and only if each part of the partition sums to $3B$.

It is clear that the converse holds, as the construction is reversible. □

Theorem 5. *The Null Labelling Problem is NP-complete.*

The result is a direct consequence of Theorem 4 since it is clear that the transformation relating the strongly NP-complete problem 3-Partition to NLP is a polynomial transformation.

We point out that the case $m = 1$ is trivial and it is not considered in the construction, while in the case $m = 2$, the hypergraph $H_{A'}$ simplifies and the gadgets $\mathcal{C}$ and $\mathcal{A}(a_i'(m-2))$ vanish.

4. Conclusion and Further Perspective

The notion of null labelling is concerned with the possibility of relating all 3-hypergraphs sharing the same degree sequence. This can be determined by assigning $+1$ and -1 to the edges of two of them, say H_1 and H_2, and then constructing $H = H_1 \oplus H_2$. The resulting hypergraph H has a null labelling and it reveals the edges and the vertices that have to be considered in order to pass from H_1 to H_2.

In this paper we provide some necessary conditions for a 3-hypergraph to have a null labelling. The connected, even 3-hypergraphs with no null labelling are computed up to 7 vertices, and those on 6 vertices are illustrated. Only partial computations are performed on 8 vertices. Other examples are also provided to support the conjecture that all connected, even 3-hypergraphs on n vertices with more than $\frac{1}{2}\binom{n}{3}$ triples have a null labelling. Finally, the general problem of deciding if a given 3-hypergraph admits a null labelling is shown to be NP-complete.

However, many new open problems rise from the present study, among them we stress the following:

i) Can Theorem 1 be extended to 3-hypergraphs with some vertices of degree 2, but not all? Either a fixed maximum number c of vertices of degree 2, or a maximum proportion of vertices of degree 2 would be suitable.

ii) Characterize some suitable classes of connected, even 3-hypergraphs, which admit no null labelling.

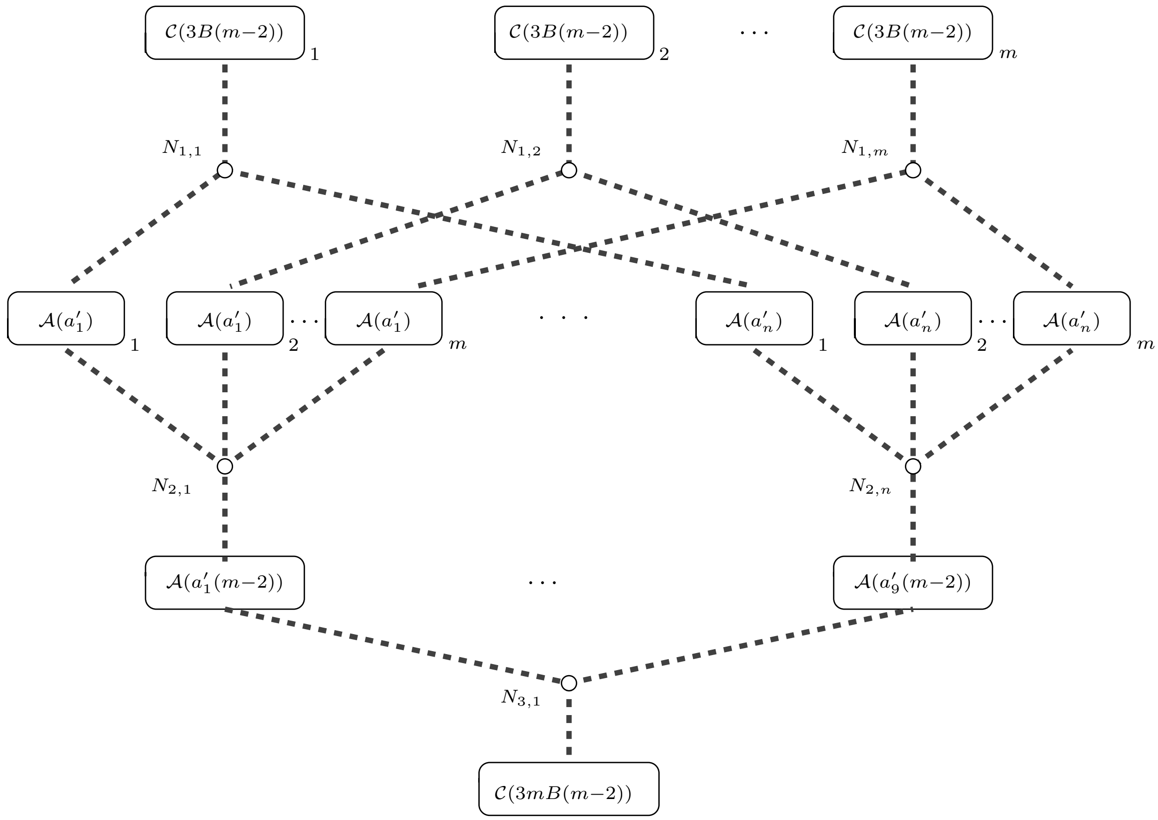


Figure 7: Schematic of the incidence graph of the 3-hypergraph for an instance of 3-Partition.

iii) Let H be a connected, even 3-hypergraph on n vertices, which does not have a null labelling. Let $\overline{H}$ be the complement of H. It is an even 3-hypergraph when $n \equiv 1, 2 \pmod 4$. Show that $\overline{H}$ has a null labelling, except when $\overline{H}$ and H are isomorphic.

iv) As self-complementary structures often have special properties, it would be interesting to characterize the self-complementary 3-hypergraphs that have a null labelling.

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