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SMALL-TIME LOCAL STABILIZATION OF THE TWO DIMENSIONAL INCOMPRESSIBLE NAVIER–STOKES EQUATIONS

SHENGQUAN XIANG

ABSTRACT. We provide explicit time-varying feedback laws that locally stabilize the two dimensional internal controlled incompressible Navier–Stokes equations in arbitrarily small time. We also obtain quantitative rapid stabilization via stationary feedback laws, as well as quantitative null controllability with explicit controls having $e^{C/T}$ costs.

1. INTRODUCTION

Let Ω be a bounded connected open set in \mathbb{R}^2 with smooth boundary. Let the controlled domain $\omega \subset \Omega$ be a nonempty open subset. We are interested in the stabilization and null controllability of the two dimensional incompressible Navier–Stokes system with internal control,

$$(1.1) \quad \begin{cases} y_t - \Delta y + (y \cdot \nabla) y + \nabla p = 1_\omega f & \text{in } \Omega, \\ \operatorname{div} y = 0 & \text{in } \Omega, \\ y = 0 & \text{on } \partial\Omega, \end{cases}$$

where, the state $y(t, \cdot)$ and the control term $f(t, \cdot)$ are in \mathcal{H} . We adapt the standard fluid mechanics framework,

$$\begin{aligned} \mathcal{H} &:= \{y \in L^2(\Omega)^2 : \operatorname{div} y = 0 \text{ in } \Omega, y \cdot n = 0 \text{ on } \partial\Omega\}, \\ \mathcal{V}_\sigma &:= \{y \in H_0^1(\Omega)^2 : \operatorname{div} y = 0 \text{ in } \Omega\} \quad \text{and} \quad \mathcal{V} := \{y \in H_0^1(\Omega)^2\}, \end{aligned}$$

with $\mathcal{V}_\sigma \hookrightarrow \mathcal{V} \hookrightarrow \mathcal{H} \hookrightarrow L^2(\Omega)^2 \hookrightarrow \mathcal{V}' \hookrightarrow \mathcal{V}'_\sigma$. We have taken the viscosity coefficient as 1 to simplify the presentation.

When dealing with stabilization problems the control term f is regarded as a feedback control governed by some “feedback application” that depends on current states and time, $U(t; y)$:

$$(1.2) \quad f(t, x) := U(t; y(t, x)),$$

where the application U is the so called *time-varying feedback law*,

$$(1.3) \quad \begin{cases} U : \mathbb{R} \times \mathcal{H} & \rightarrow & \mathcal{H} \\ (t; y) & \mapsto & U(t; y). \end{cases}$$

The *closed-loop system* associated to the preceding feedback law U is the evolution system (1.1)–(1.3). A *stationary* feedback law is such an application only depends on \mathcal{H} . A *T-periodic* feedback law is a time-varying feedback law that is periodic with respect to time, *i.e.* $U(T + t; y) = U(t; y)$. A *proper* feedback law U , roughly speaking, is some time-varying feedback law such that, for every $s \in \mathbb{R}$, and for every $y_0 \in \mathcal{H}$ as initial state at time s , *i.e.* $y(s, x) = y_0(x)$, the closed-loop system (1.1)–(1.3) admits a unique solution. For the closed-loop system with proper feedback law we can define the “*flow*”, $\Phi(t, s; y_0)$, as the state at time t of the solution of (1.1)–(1.3) with initial state $y(s, x) = y_0(x)$.

The local controllability and stabilization of Navier–Stokes equations have been extensively studied in the literature. Based on global Carleman estimates introduced by Fursikov–Imanuvilov [25]

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a nearly complete local exact controllability result is obtained in [21], other works include but not limited to [12, 20, 22, 24, 31, 34]. Eventually one can even control the system locally via some reduced control terms [13]. The global controllability of Navier–Stokes equations with controlling on boundary (namely Lions’ problem), which, different from the cases on Riemannian manifolds [1, 11], is far away from been answered due to boundary layer difficulties, by far the best results are given by [14, 15].

The study on local exponential stabilization around 0 and other trajectories of Navier–Stokes equations both with internal controls or with boundary controls is fruitful, notably based on Riccati type methods by optimal control theory. For example, [2, 5] for local exponential stabilization with finite dimensional internal control (feedback laws); exponential stabilization by boundary feedback laws [3, 23, 32]; stabilization around trajectories or unstable steady states [4, 7, 33], *etc.* To the best of our knowledge, result on quantitative rapid stabilization or even finite time stabilization of Navier–Stokes equations is extremely limited, we refer to [19] for a detailed review on these questions.

Recently, the author has introduced a method to stabilize the multi-dimensional heat equations in finite time [37], which is based on quantitative rapid stabilization relying on *spectral inequalities* and *Lyapunov functionals*, as well as *piecewise feedback laws*. Methodologically speaking, the technical spectral inequalities is achieved via local Carleman estimates on elliptic operators up to boundaries (as always fulfilling Hörmander’s pseudoconvex condition), starting from the seminal paper [27, 28] these results can be regarded as standard, at least compared to wave type operators; the Lyapunov functions [18] aim at finding artfully chosen energy and multiplier to characterize the variation of the energy from a global point of view without knowing any microlocal information, which have been extensively developed in the study of hyperbolic systems of conservation laws as well as scattering theory [6, 26, 30, 38]; the piecewise (in time) feedback law is formulated by [16] to stabilize the one dimensional heat equation in finite time together with the backstepping method, instead of using stationary feedback laws. This method shares several advantages:

- The designed feedback laws are simple and explicit to be compared with some other stabilization techniques, for instance the powerful Riccati method requires on solving some algebraic nonlinear Riccati equation.
- The quantitative rapid stabilization combined with the piecewise continuous feedback law argument leads to finite time stabilization. It also provides a constructive approach to null controllability without applying Lions’ fundamental H.U.M. [30] sharing explicit (and probably optimal) controlling costs.
- The feedback laws are stable under perturbation. As a direct consequence, the same feedback law can be used to stabilize (rapidly or even in finite time) nonlinear models with satisfying costs.

Inspired by [37] we prove the following theorems concerning quantitative rapid stabilization, local null controllability with cost estimates, and finite time stabilization for the two dimensional incompressible internal controlled Navier–Stokes equations, the proofs of which are presented in Section 3, Section 4, and Section 5 respectively. In Section 2 we introduce some preliminary results concerning well-posedness of Navier–Stokes equations, spectral inequalities of Stokes operators, as well as the related control problems.

Theorem 1.1 (Quantitative rapid stabilization). *There exists an effectively computable constant $C_2 > 0$ such that for any $\lambda > 0$ we can construct an explicit stationary feedback law $\mathcal{F}_\lambda : \mathcal{H} \rightarrow \mathcal{H}$, such that the closed-loop system (1.1)–(1.2) with the feedback law $U(t; y) := \mathcal{F}_\lambda y$ is locally exponentially stable:*

$$\|\Phi(t, s; y_0)\|_{\mathcal{H}} + \|\mathcal{F}_\lambda \Phi(t, s; y_0)\|_{\mathcal{H}} \leq 2C_2 e^{C_2 \sqrt{\lambda}} e^{-\frac{\lambda}{4}(t-s)} \|y_0\|_{\mathcal{H}}, \quad \forall s \in \mathbb{R}, \forall t \in [s, +\infty),$$

for any $\|y_0\|_{\mathcal{H}} \leq C_2^{-1}e^{-C_2\sqrt{\lambda}}$.

Theorem 1.2 (Quantitative null controllability with cost estimates). *There exists an effectively computable constant $C_3 > 0$ such that, for any $T \in (0, 1)$, and for any $\|y_0\|_{\mathcal{H}} \leq e^{-\frac{C_3}{T}}$ we can find an explicit control $f|_{[0,T]}(t, x)$ satisfying*

$$\|f(t, x)\|_{L^\infty(0,T;\mathcal{H})} \leq e^{\frac{C_3}{T}} \|y_0\|_{\mathcal{H}},$$

such that the unique solution of the controlled system (1.1) with initial state $y(0, x) = y_0(x)$ and the control $f|_{[0,T]}$ verifies $y(T, x) = 0$.

Theorem 1.3 (Small-time local stabilization with explicit feedback laws). *For any $T > 0$, we find an effectively computable constant Λ_T and construct an explicit T -periodic proper feedback law U satisfying*

$$\|U(t; y)\|_{\mathcal{H}} \leq \min\{1, 2\|y\|_{\mathcal{H}}^{1/2}\}, \quad \forall y \in \mathcal{H}, \quad \forall t \in \mathbb{R},$$

that stabilizes system (1.1)–(1.3) in finite time:

- (i) (*2T stabilization*) $\Phi(2T + t, t; y_0) = 0, \quad \forall t \in \mathbb{R}, \quad \forall \|y_0\|_{\mathcal{H}} \leq \Lambda_T.$
- (ii) (*Uniform stability*) For every $\delta > 0$ there exists an effectively computable $\eta > 0$ such that

$$(\|y_0\|_{\mathcal{H}} \leq \eta) \Rightarrow (\|\Phi(t, t'; y_0)\|_{\mathcal{H}} \leq \delta, \quad \forall t' \in \mathbb{R}, \quad \forall t \in (t', +\infty)).$$

2. PRELIMINARY

2.1. Functional framework. We refer to the book by Chemin [9] for the functional analysis framework and well-posedness results concerning Navier–Stokes equations, and the book by Coron [10] for introduction on the related control theory. In the context if there is no confusing sometimes we simply denote $L^2(\Omega)^2$ by $L^2(\Omega)$ or L^2 .

(1) *Leray projection and spectral decomposition.*

According to Helmholtz decomposition, for any $u \in L^2(\Omega)^2$ there exist unique $v \in \mathcal{H}$ and $\nabla p \in L^2(\Omega)^2$ such that $u = v + \nabla p$, which defines the (orthogonal) Leray projection \mathbb{P} on $L^2(\Omega)^2$:

$$\begin{cases} \mathbb{P} : L^2(\Omega)^2 & \rightarrow & \mathcal{H} \\ u & \mapsto & \mathbb{P}u := u - \nabla p. \end{cases}$$

Notice that for any $f \in \mathcal{H}$,

$$\|\mathbb{P}(1_\omega f)\|_{\mathcal{H}} \leq \|1_\omega f\|_{L^2(\Omega)^2} \leq \|f\|_{L^2(\Omega)^2} = \|f\|_{\mathcal{H}},$$

which allows us to estimate the control term via $\|f\|_{L^2}$ (or equivalently $\|f\|_{\mathcal{H}}$).

Let $\{e_i\}_{i=1}^\infty \subset \mathcal{V}_\sigma$ be the orthonormal basis of \mathcal{H} given by the eigenvectors of the the Stokes operator

$$\begin{cases} -\Delta e_i + \nabla p_i = \tau_i e_i & \text{in } \Omega, \\ \operatorname{div} e_i = 0 & \text{in } \Omega, \\ e_i = 0 & \text{on } \partial\Omega, \end{cases}$$

with $0 < \tau_1 \leq \tau_2 \leq \tau_3 \leq \dots \leq \tau_n \leq \dots$ and $\lim_{i \rightarrow \infty} \tau_i = +\infty$. Let \mathcal{H}_N be the low frequency subspace of \mathcal{H} , and \mathbb{P}_N be its orthogonal projection,

$$\mathcal{H}_N := \operatorname{Vect}\{e_i\}_{i=1}^N \subset \mathcal{V}_\sigma.$$

In terms of the above eigenvectors Leray projection can be extended to \mathcal{V}' ,

$$\begin{cases} \mathbb{P} : \mathcal{V}' & \rightarrow & \mathcal{V}' \\ u & \mapsto & \mathbb{P}u := u - \nabla p, \end{cases}$$

where $p \in L^2_{loc}(\Omega)$, and ∇p belongs to \mathcal{V}_σ^0 as polar space of \mathcal{V}_σ ,

$$\mathcal{V}_\sigma^0 := \{f \in \mathcal{V}' : \langle f, v \rangle_{\mathcal{V}' \times \mathcal{V}} = 0, \quad \forall v \in \mathcal{V}_\sigma\}.$$

More precisely,

$$\begin{aligned}\mathbb{P}u &:= \sum_{i=1}^{\infty} \langle u, e_i \rangle_{\mathcal{V}' \times \mathcal{V}} e_i \in \mathcal{V}' \quad \text{for } u \in \mathcal{V}', \\ \mathbb{P}_N u &:= \sum_{i=1}^N \langle u, e_i \rangle_{\mathcal{V}' \times \mathcal{V}} e_i \in \mathcal{V}_\sigma \quad \text{for } u \in \mathcal{V}', \\ \mathbb{P}_N^\perp u &:= \sum_{i=N+1}^{\infty} \langle u, e_i \rangle_{\mathcal{V}' \times \mathcal{V}} e_i \in \mathcal{V}' \quad \text{for } u \in \mathcal{V}', \\ \mathbb{P}u &:= \sum_{i=1}^{\infty} (u, e_i)_{L^2(\Omega)^2} e_i \in \mathcal{H} \quad \text{for } u \in L^2(\Omega)^2,\end{aligned}$$

which satisfies,

$$\langle u, v \rangle_{\mathcal{V}' \times \mathcal{V}} = \langle u, v \rangle_{\mathcal{V}'_\sigma \times \mathcal{V}_\sigma} = \langle \mathbb{P}u, v \rangle_{\mathcal{V}'_\sigma \times \mathcal{V}_\sigma}, \quad \forall u \in \mathcal{V}', \quad \forall v \in \mathcal{V}_\sigma.$$

Furthermore, the related \mathcal{H} -norm, \mathcal{V}_σ -norm, and \mathcal{V}'_σ -norm can be characterized by

$$\begin{aligned}\sum_{i=1}^{\infty} |(u, e_i)_{L^2(\Omega)^2}|^2 &= \|\mathbb{P}u\|_{\mathcal{H}}^2 \leq \|u\|_{L^2}^2 \quad \text{for } u \in L^2(\Omega)^2, \\ \sum_{i=1}^{\infty} |(u, e_i)_{L^2(\Omega)^2}|^2 \tau_i &= \|\mathbb{P}u\|_{\mathcal{V}'_\sigma}^2 \leq \|u\|_{\mathcal{V}}^2 \quad \text{for } u \in \mathcal{V}, \\ \sum_{i=1}^{\infty} |\langle u, e_i \rangle_{\mathcal{V} \times \mathcal{V}'}|^2 \tau_i^{-1} &= \|\mathbb{P}u\|_{\mathcal{V}'_\sigma}^2 = \|u\|_{\mathcal{V}'_\sigma}^2 \leq \|u\|_{\mathcal{V}'}^2 \quad \text{for } u \in \mathcal{V}'.\end{aligned}$$

(2) *Spectral inequalities.*

For any $\lambda > 0$, we denote by $N(\lambda)$ the number of the eigenvalues that are smaller than or equal to λ , i.e. $\tau_{N(\lambda)} \leq \lambda < \tau_{N(\lambda)+1}$, and define the following symmetric matrix $J_{N(\lambda)}$,

$$(2.1) \quad J_{N(\lambda)} := \left((e_i, e_j)_{L^2(\omega)^2} \right)_{i,j=1}^{N(\lambda)}.$$

Proposition 2.1 (Spectral inequalities). *There exists an effectively computable constant $C_1 \geq 1$ only depends on (Ω, ω) that is independent of $\lambda > 0$ such that, for any $\lambda > 0$ and for any $E_{N(\lambda)} = (a_1, a_2, \dots, a_{N(\lambda)}) \in \mathbb{R}^{N(\lambda)}$ the following inequality holds,*

$$E_{N(\lambda)}^T J_{N(\lambda)} E_{N(\lambda)} \geq C_1^{-1} e^{-C_1 \sqrt{\lambda}} \|E_{N(\lambda)}\|_2^2.$$

Proof. This is a Lebeau–Robbiano type spectral inequality [28]. By letting N be representing N_λ , we get

$$E_N^T J_N E_N = \sum_{1 \leq i, j \leq N} a_i (e_i, e_j)_{L^2(\omega)^2} a_j = \left\| \sum_{i=1}^N a_i e_i \right\|_{L^2(\omega)^2}^2 \geq C_1^{-1} e^{-C_1 \sqrt{\lambda}} \|E_N\|_2^2,$$

where we have recalled the spectral inequality concerning the Stokes operator from the recent paper [8, Theorem 3.1],

$$C_1 e^{C_1 \sqrt{\lambda}} \int_{\omega} \left(\sum_{\tau_i \leq \lambda} a_i e_i(x) \right)^2 dx \geq \sum_{\tau_i \leq \lambda} a_i^2.$$

Or equivalently, the preceding positive quadratic form can be read as

$$C_1 e^{C_1 \sqrt{\lambda}} \int_{\omega} (\mathbb{P}_{N(\lambda)} y)^2 dx \geq \int_{\Omega} (\mathbb{P}_{N(\lambda)} y)^2 dx, \quad \forall y \in \mathcal{H}.$$

□

(3) *Nonlinear terms.*

Next, we define the following bilinear map \mathcal{Q} as well as trilinear functional \mathcal{B} ,

$$\begin{cases} \mathcal{Q} : \mathcal{V} \times \mathcal{V} & \rightarrow & \mathcal{V}' \\ (u, v) & \mapsto & \operatorname{div} (u \otimes v) \end{cases},$$

$$\mathcal{B}(u, v, w) := \langle \mathcal{Q}(u, v), w \rangle_{\mathcal{V}' \times \mathcal{V}}, \quad \forall u, v, w \in \mathcal{V}.$$

Proposition 2.2 (Nonlinearity estimates). *There exists a constant c_0 such that for any u, v and w in \mathcal{V} , we have the following estimates,*

$$\begin{aligned} \mathcal{B}(u, u, w) &= \langle (u \cdot \nabla)u, w \rangle_{\mathcal{V}' \times \mathcal{V}} \quad \text{if } u \in \mathcal{V}_\sigma, \\ \mathcal{B}(u, v, w) + \mathcal{B}(u, w, v) &= 0 \quad \text{if } u \in \mathcal{V}_\sigma, \\ |\mathcal{B}(u, v, w)| &\leq c_0 \|u\|_{L^2}^{\frac{1}{2}} \|v\|_{L^2}^{\frac{1}{2}} \|\nabla u\|_{L^2}^{\frac{1}{2}} \|\nabla v\|_{L^2}^{\frac{1}{2}} \|\nabla w\|_{L^2}. \end{aligned}$$

2.2. Open loop controlled (inhomogeneous) Navier–Stokes systems. The open loop controlled equation is indeed an inhomogeneous equation with a force term located in the controlled domain. A general inhomogeneous equation (without any restriction on force terms) is presented by,

$$(2.2) \quad \begin{cases} y_t - \Delta y + (y \cdot \nabla) y + \nabla p = f(t, x), & (t, x) \in (t_1, t_2) \times \Omega, \\ \operatorname{div} y(t, x) = 0, & (t, x) \in (t_1, t_2) \times \Omega, \\ y(t, x) = 0, & (t, x) \in (t_1, t_2) \times \partial\Omega, \\ y(t_1, x) = y_0(x), & x \in \Omega, \end{cases}$$

where t_2 can be taken as $+\infty$. We are interested in the solutions under Leray’s weak solution sense [29]: for any $y_0 \in \mathcal{H}$ and any $f \in L^2_{loc}(t_1, t_2; \mathcal{V}')$, the solution of equation (2.2) is some $y \in C([t_1, t_2]; \mathcal{H}) \cap L^2_{loc}(t_1, t_2; \mathcal{V}_\sigma)$ such that, for any test function ϕ in $C^1([t_1, t_2]; \mathcal{V}_\sigma)$, the vector field y satisfies the following condition:

$$(2.3) \quad \begin{aligned} (y(t), \phi(t))_{\mathcal{H}} &= (y_0, \phi(0))_{\mathcal{H}} + \int_{t_1}^t \langle \Delta \phi(s) + \partial_t \phi(s), y(s) \rangle_{\mathcal{V}'_\sigma \times \mathcal{V}_\sigma} ds \\ &\quad + \int_{t_1}^t (y(s) \otimes y(s), \nabla \phi(s))_{L^2(\Omega)^2} ds + \int_{t_1}^t \langle f(s), \phi(s) \rangle_{\mathcal{V}'_\sigma \times \mathcal{V}_\sigma} ds, \end{aligned}$$

for every $t \in [t_1, t_2]$.

Theorem 2.3 (Leray theorem on well-posedness and stability of the solutions). *For any $y_0 \in \mathcal{H}$ and any $f \in L^2_{loc}(t_1, t_2; L^2(\Omega)^2)$, the Cauchy problem (2.2) admits a unique solution. This unique solution is also in $H^1_{loc}(t_1, t_2; \mathcal{V}'_\sigma)$. Moreover, there exists some constant C_0 independent of t_1 and t_2 such that this unique solution satisfies,*

$$(2.4) \quad \frac{1}{2} \|y(t, x)\|_{\mathcal{H}}^2 + \int_{t_1}^t \|\nabla y(s, x)\|_{L^2}^2 ds = \frac{1}{2} \|y_0\|_{\mathcal{H}}^2 + \int_{t_1}^t \langle f(s), y(s) \rangle_{\mathcal{V}'_\sigma \times \mathcal{V}_\sigma} ds,$$

$$(2.5) \quad \|y(t, x)\|_{\mathcal{H}}^2 + \int_{t_1}^t \|\nabla y(s, x)\|_{L^2}^2 ds \leq \|y_0\|_{\mathcal{H}}^2 + C_0 \int_{t_1}^t \|f(s)\|_{L^2}^2 ds,$$

for any $t \in [t_1, t_2]$.

Furthermore, the Leray solutions are stable in the following sense. Let y (resp. z) be the Leray solution associated with y_0 (resp. z_0) in \mathcal{H} and f (resp. g) in the space $L^2_{loc}(t_1, +\infty; L^2(\Omega)^2)$, then

for $w := y - z$ and for any $t \in (t_1, +\infty)$ we have

$$\begin{aligned} \|w(t)\|_{\mathcal{H}}^2 + \int_{t_1}^t \|\nabla w(s, x)\|_{L^2}^2 ds &\leq \left(\|w_0\|_{\mathcal{H}}^2 + C_0 \int_{t_1}^t \|(f - g)(s)\|_{L^2}^2 ds \right) \exp(C_s E^2(t)), \\ E(t) &:= \min \left\{ \|y_0\|_{\mathcal{H}}^2 + C_0 \int_{t_1}^t \|f(s)\|_{L^2}^2 ds, \|z_0\|_{\mathcal{H}}^2 + C_0 \int_{t_1}^t \|g(s)\|_{L^2}^2 ds \right\}. \end{aligned}$$

Actually Theorem 2.3 holds for f in $L^2_{loc}(t_1, +\infty; \mathcal{V}')$, for which the related inequalities are governed by the $L^2(\mathcal{V}'_\sigma)$ -norm of f and the constant C_0 can be taken as 1.

2.3. Time-varying feedback laws, closed-loop systems, and finite time stabilization. In this section we recall the precise definition of time-varying feedback laws as well as the related closed-loop solutions.

Definition 2.4 (Closed-loop systems). *Let $s_1 \in \mathbb{R}$ and $s_2 \in \mathbb{R}$ be given such that $s_1 < s_2$. Let the time-varying feedback law on interval $[s_1, s_2]$ be an application*

$$(2.6) \quad \begin{cases} U : [s_1, s_2] \times \mathcal{H} &\rightarrow \mathcal{H} \\ (t; y) &\mapsto U(t; y). \end{cases}$$

Let $t_1 \in [s_1, s_2]$, $t_2 \in (t_1, s_2]$, and $y_0 \in \mathcal{H}$. A solution on $[t_1, t_2]$ to the Cauchy problem associated to the closed-loop system (1.1)–(1.2) with (2.6) for initial data y_0 at time t_1 is some $y : [t_1, t_2] \rightarrow \mathcal{H}$ such that

$$t \in (t_1, t_2) \mapsto f(t, x) := 1_\omega U(t; y(t)) \in L^2(t_1, t_2; L^2(\Omega)^2),$$

y is a Leray solution of (2.2) with initial data y_0 at time t_1 and the above force term $f(t, x)$.

Definition 2.5 (Proper feedback laws). *Let $s_1 \in \mathbb{R}$ and $s_2 \in \mathbb{R}$ be given such that $s_1 < s_2$. A proper feedback law on $[s_1, s_2]$ is an application U of type (2.6) such that, for every $t_1 \in [s_1, s_2]$, for every $t_2 \in (t_1, s_2]$, and for every $y_0 \in \mathcal{H}$, there exists a unique solution on $[t_1, t_2]$ to the Cauchy problem associated to the closed-loop system (1.1)–(1.2) with (2.6) for initial data y_0 at time t_1 according to Definition 2.4.*

A proper feedback law is an application U of type (1.3) such that, for every $s_1 \in \mathbb{R}$ and for every $s_2 \in \mathbb{R}$ satisfying $s_1 < s_2$, the feedback law restricted to $[s_1, s_2] \times \mathcal{H}$ is a proper feedback law on $[s_1, s_2]$.

For a proper feedback law, one can define the flow $\Phi : \Delta \times \mathcal{H} \rightarrow \mathcal{H}$ associated to this feedback law, with $\Delta := \{(t, s); t > s\}$: $\Phi(t, s; y_0)$ is the value at time t of the solution y to the closed-loop system (1.1)–(1.3) which is equal to y_0 at time s .

Definition 2.6 (Finite time local stabilization of Navier-Stokes equations). *Let $T > 0$. A T -periodic proper feedback law U locally stabilizes the two dimensional Navier-Stokes equation in finite time, if for some $\varepsilon > 0$ the flow Φ of the closed-loop system (1.1)–(1.3) verifies,*

- (i) (2T stabilization) $\Phi(2T + t, t; y_0) = 0, \quad \forall t \in \mathbb{R}, \forall \|y_0\|_{\mathcal{H}} \leq \varepsilon,$
- (ii) (Uniform stability) For every $\delta > 0$, there exists $\eta > 0$ such that

$$(\|y_0\|_{\mathcal{H}} \leq \eta) \Rightarrow (\|\Phi(t, t'; y_0)\|_{\mathcal{H}} \leq \delta, \quad \forall t' \in \mathbb{R}, \forall t \in (t', +\infty)).$$

2.4. Well-posedness of closed-loop systems. Finally we present well-posedness results concerning closed-loop systems with stationary Lipschitz feedback laws. Concerning linear feedback laws one has the following well-posedness result.

Theorem 2.7. *Let $T > 0$. Let given vector functions $\{\varphi_i\}_{i=1}^n \in \mathcal{H}$ and bounded linear operators $\{l_i\}_{i=1}^n : \mathcal{H} \rightarrow \mathbb{R}$. For any $y_0 \in \mathcal{H}$ the Cauchy problem*

$$\begin{cases} y_t - \Delta y + (y \cdot \nabla) y + \nabla p = 1_\omega \left(\sum_{i=1}^n l_i(y) \varphi_i \right), & (t, x) \in (0, T) \times \Omega, \\ \operatorname{div} y = 0, & (t, x) \in (0, T) \times \Omega, \\ y(t, x) = 0, & (t, x) \in (0, T) \times \partial\Omega, \\ y(0, x) = y_0(x), & x \in \Omega, \end{cases}$$

admits a unique solution.

As will consider finite time stabilization problems we also introduce ‘‘cutoff’’ type feedback laws. For any $r \in (0, 1/2]$ we find some smooth *cutoff* function $\chi_r \in C^\infty(\mathbb{R}^+; [0, 1])$ satisfying

$$(2.7) \quad \chi_r(x) = \begin{cases} 1 & \text{if } x \in [0, r], \\ 0 & \text{if } x \in [2r, +\infty), \end{cases}$$

and further define the following related *Lipschitz* operator $\mathcal{K}_r : \mathcal{H} \rightarrow \mathcal{H}$,

$$(2.8) \quad \mathcal{K}_r(y) := y \cdot \chi_r(\|y\|_{\mathcal{H}}), \quad \forall y \in \mathcal{H},$$

satisfying, for some constant L_r depending on r ,

$$\begin{aligned} \|\mathcal{K}_r(y)\|_{\mathcal{H}} &\leq \min\{1, \|y\|_{\mathcal{H}}\}, \\ \|\mathcal{K}_r(y) - \mathcal{K}_r(z)\|_{\mathcal{H}} &\leq L_r \|y - z\|_{\mathcal{H}}, \quad \forall y, z \in \mathcal{H}. \end{aligned}$$

Theorem 2.8. *Let $T > 0$. Let $r \in (0, 1/2]$. Let given vector functions $\{\varphi_i\}_{i=1}^n \in \mathcal{H}$ and bounded linear operators $\{l_i\}_{i=1}^n : \mathcal{H} \rightarrow \mathbb{R}$. For any $y_0 \in \mathcal{H}$ the Cauchy problem*

$$\begin{cases} y_t - \Delta y + (y \cdot \nabla) y + \nabla p = 1_\omega \mathcal{K}_r \left(\sum_{i=1}^n l_i(y) \varphi_i \right), & (t, x) \in (0, T) \times \Omega, \\ \operatorname{div} y = 0, & (t, x) \in (0, T) \times \Omega, \\ y(t, x) = 0, & (t, x) \in (0, T) \times \partial\Omega, \\ y(0, x) = y_0(x), & x \in \Omega, \end{cases}$$

admits a unique solution.

Both the closed-loop systems with linear feedback laws and the closed-loop systems with Lipschitz nonlinear feedback laws are well-posed correspond to Theorem 2.7 and Theorem 2.8, the detailed proofs of which we omit. Indeed, local (in time) existence and uniqueness of solutions are based on Leray’s theorem 2.3 concerning energy estimates and stability of the solutions, and Banach fixed point theorem. Let the Lipschitz constant of the feedback law be L , let the feedback of y is always bounded by $C\|y\|_{\mathcal{H}}$, and let $\|y_0\|_{\mathcal{H}} = M$. For some \tilde{T} small enough to be fixed later on we consider the Banach space,

$$\begin{aligned} \mathcal{X}_{\tilde{T}} &:= C([0, \tilde{T}]; \mathcal{H}) \cap L^2(0, \tilde{T}; \mathcal{V}_\sigma), \\ \mathcal{X}_{\tilde{T}}(2M) &:= \left\{ y \in \mathcal{X}_{\tilde{T}} : \|y\|_{\mathcal{X}_{\tilde{T}}}^2 = \|y\|_{C([0, \tilde{T}]; \mathcal{H})}^2 + \|\nabla y\|_{L^2(0, \tilde{T}; L^2)}^2 \leq 4M^2 \right\}, \end{aligned}$$

as well as the following application,

$$\begin{cases} \mathcal{S} : \mathcal{X}_{\tilde{T}}(2M) & \rightarrow \mathcal{X}_{\tilde{T}} \\ y & \mapsto \mathcal{S}(y), \end{cases}$$

where $\mathcal{S}(y)$ is the solution of Cauchy problem (2.2) with the initial state y_0 and force (control) term $f = 1_\omega \mathcal{K}_r \left(\sum_{i=1}^n l_i(y) \varphi_i \right)$. One can check that, thanks to Theorem 2.3, the preceding application is a contraction map on $\mathcal{X}_{\tilde{T}}(2M)$ for \tilde{T} sufficiently small, *e.g.*

$$\tilde{T} \leq \min \left\{ (4C_0 C^2)^{-1}, (4C_0 L^2 \exp(4C_s M^4))^{-1} \right\},$$

thus admits a fixed point \tilde{y} which is the unique solution of the closed-loop system. Moreover, since $\tilde{y} = \mathcal{S}(\tilde{y})$ is the solution of the Cauchy problem (2.2) with control $f = 1_\omega \mathcal{K}_r(\sum_{i=1}^n l_i(\tilde{y})\varphi_i)$, thanks to Theorem 2.3, this solution also belongs to the space $H^1(0, T; \mathcal{V}'_\sigma)$. In the end, some *a priori* energy estimates lead to global (in time) solutions.

We also emphasize that the Lipschitz condition is crucial in order to guarantee the uniqueness. Otherwise one may need to use other compactness arguments to prove existence of solutions, see for example [17].

2.5. On the choice of constants. In this section we conclude the values of the constants that will be used later on, which also correspond to the constants that appeared in Theorem 1.1 and Theorem 1.2.

- For any given $\lambda > 0$, we define

$$(2.9) \quad \gamma_\lambda := C_1 e^{C_1 \sqrt{\lambda}} \lambda \quad \text{and} \quad \mu_\lambda := \frac{\gamma_\lambda^2}{\lambda^2} = C_1^2 e^{2C_1 \sqrt{\lambda}} > 1.$$

- By recalling the definition of C_1 in Proposition 2.1, we further select some $C_2 \in [3C_1, +\infty)$ such that for all $\lambda > 0$,

$$(2.10) \quad (1 + \lambda C_1) e^{C_1 \sqrt{\lambda}}, 8(1 + \lambda) C_1^2 e^{2C_1 \sqrt{\lambda}}, 9c_0 C_1^3 e^{3C_1 \sqrt{\lambda}} \leq C_2 e^{C_2 \sqrt{\lambda}},$$

and define

$$(2.11) \quad r_\lambda := \left(C_2 e^{C_2 \sqrt{\lambda}} \right)^{-1}.$$

- Then we choose some constant $Q > 0$ satisfying

$$(2.12) \quad C_1 e^{C_1 Q^m}, C_2 e^{C_2 Q^m} \leq e^{\frac{Q^2}{64} m}, \quad \forall m \geq 1,$$

and select

$$(2.13) \quad C_3 := \frac{Q^2}{32}.$$

3. QUANTITATIVE RAPID STABILIZATION

Inspired by the recent work [37] on the stabilization of the heat equations, we directly define the following stationary feedback law

$$(3.1) \quad \mathcal{F}_\lambda y := -\gamma_\lambda \mathbb{P}_{N(\lambda)} y, \quad \forall y \in \mathcal{H},$$

and consider the following closed-loop system,

$$(3.2) \quad \begin{cases} y_t = \Delta y - (y \cdot \nabla) y - \nabla p - \gamma_\lambda 1_\omega \mathbb{P}_N y & \text{in } \Omega, \\ \operatorname{div} y = 0 & \text{in } \Omega, \\ y = 0 & \text{on } \partial\Omega, \end{cases}$$

where, and from now on, we simply denote $N(\lambda)$ by N . Furthermore, the low frequency system satisfies

$$(3.3) \quad \frac{d}{dt} (\mathbb{P}_N y) = \mathbb{P}_N (\Delta y) - \mathbb{P}_N ((y \cdot \nabla) y) - \gamma_\lambda \mathbb{P}_N (1_\omega \mathbb{P}_N y).$$

Because y lives in \mathcal{H} , we can decompose

$$y(t, x) = \mathbb{P}y(t, x) = \sum_{i=1}^{\infty} y_i(t) e_i,$$

$$\mathbb{P}(1_\omega e_i) = \sum_{j=1}^{\infty} (1_\omega e_i, e_j)_{L^2(\Omega)^2} e_j = \sum_{j=1}^{\infty} (e_i, e_j)_{L^2(\omega)^2} e_j,$$

which further implies

$$\mathbb{P}_N(1_\omega \mathbb{P}_N y) = \mathbb{P}_N\left(1_\omega \sum_{i=1}^N y_i(t) e_i\right) = \sum_{i=1}^N \sum_{j=1}^N y_i(t) (e_i, e_j)_{L^2(\omega)^2} e_j.$$

Furthermore, for y in \mathcal{V}_σ the nonlinear term $(y \cdot \nabla)y$ (which is equivalent to $\operatorname{div}(y \otimes y)$) belongs to the space \mathcal{V}' , thus

$$\mathbb{P}_N((y \cdot \nabla)y) = \sum_{i=1}^N \langle (y \cdot \nabla)y, e_i \rangle_{\mathcal{V}' \times \mathcal{V}} e_i,$$

moreover

$$\mathbb{P}_N(\Delta y) = - \sum_{i=1}^N \tau_i y_i e_i.$$

By defining

$$(3.4) \quad X_N(t) := \begin{pmatrix} y_1(t) \\ y_2(t) \\ \dots \\ y_N(t) \end{pmatrix}, Y_N(t) := \begin{pmatrix} -\langle (y \cdot \nabla)y, e_1 \rangle_{\mathcal{V}' \times \mathcal{V}}(t) \\ -\langle (y \cdot \nabla)y, e_2 \rangle_{\mathcal{V}' \times \mathcal{V}}(t) \\ \dots \\ -\langle (y \cdot \nabla)y, e_N \rangle_{\mathcal{V}' \times \mathcal{V}}(t) \end{pmatrix}, A_N := \begin{pmatrix} -\tau_1 & & & \\ & -\tau_2 & & \\ & & \dots & \\ & & & -\tau_N \end{pmatrix},$$

we know that the finite dimensional system $X_N(t)$ satisfies the following ordinary differential equation,

$$(3.5) \quad \dot{X}_N(t) = A_N X_N(t) - \gamma_\lambda J_N X_N(t) + Y_N(t).$$

Let us consider the following Lyapunov functional on \mathcal{H} : for every $y \in \mathcal{H}$,

$$(3.6) \quad V(y) := \mu_\lambda (\mathbb{P}_N y, \mathbb{P}_N y)_{L^2(\Omega)^2} + \left(\mathbb{P}_N^\perp y, \mathbb{P}_N^\perp y \right)_{L^2(\Omega)^2} = \mu_\lambda \|X_N\|_2^2 + \left(\mathbb{P}_N^\perp y, \mathbb{P}_N^\perp y \right)_{L^2(\Omega)^2},$$

satisfying

$$\|y\|_{L^2(\Omega)^2}^2 \leq V(y) \leq \mu_\lambda \|y\|_{L^2(\Omega)^2}^2, \quad \forall y \in \mathcal{H},$$

where

$$\|X_N\|_2^2 := \sum_{i=1}^N y_i^2 = (\mathbb{P}_N y, \mathbb{P}_N y)_{L^2(\Omega)^2}.$$

Let $T > 0$. Concerning the variation of the value of the above Lyapunov function, at least when the solution is regular enough, for example $y \in C^1([0, T]; \mathcal{V}'_\sigma) \cap C^0([0, T]; \mathcal{V}_\sigma)$ and (thus) $\mathbb{P}_N y \in C^1([0, T]; \mathcal{H}_N)$, one has

$$\begin{aligned} \frac{d}{dt} V(y(t)) &= \mu_\lambda \frac{d}{dt} \|X_N\|_2^2 + \frac{d}{dt} \left(\mathbb{P}_N^\perp y, \mathbb{P}_N^\perp y \right)_{L^2(\Omega)^2}, \\ &= 2\mu_\lambda X_N^T \dot{X}_N + 2 \left\langle \mathbb{P}_N^\perp y, \frac{d}{dt} \left(\mathbb{P}_N^\perp y \right) \right\rangle_{\mathcal{V}_\sigma \times \mathcal{V}'_\sigma}, \\ &= 2\mu_\lambda X_N^T \left(A_N X_N - \gamma_\lambda J_N X_N + Y_N \right) + 2 \left\langle \mathbb{P}_N^\perp y, \frac{d}{dt} y \right\rangle_{\mathcal{V}_\sigma \times \mathcal{V}'_\sigma}, \end{aligned}$$

where the value of $-X_N^T Y_N$ is given by

$$\left\langle \mathbb{P}_N((y \cdot \nabla)y), \mathbb{P}_N y \right\rangle_{\mathcal{V}'_\sigma \times \mathcal{V}_\sigma} = \left\langle \mathbb{P}((y \cdot \nabla)y), \mathbb{P}_N y \right\rangle_{\mathcal{V}'_\sigma \times \mathcal{V}_\sigma} = \langle (y \cdot \nabla)y, \mathbb{P}_N y \rangle_{\mathcal{V}' \times \mathcal{V}} = \mathcal{B}(y, y, \mathbb{P}_N y).$$

Eventually, according to Theorem 2.3 and Theorem 2.7, the solution y lives indeed in the space $H^1(0, T; \mathcal{V}'_\sigma) \cap C^0([0, T]; \mathcal{H}) \cap L^2(0, T; \mathcal{V}_\sigma)$. This further implies that Δy and $(y \cdot \nabla)y$ belong to $L^2(0, T; \mathcal{V}')$, the finite dimensional projection Y_N is in $L^2(0, T; \mathbb{R}^N)$, $\frac{d}{dt}(X_N)$ is in $L^2(0, T; \mathbb{R}^N)$, X_N is in $C^0([0, T]; \mathbb{R}^N)$, $\frac{d}{dt}(\mathbb{P}_N y)$ is in $L^2(0, T; \mathcal{H}_N)$, $\frac{d}{dt}(\mathbb{P}_N^\perp y)$ and $\frac{d}{dt}y$ live in $L^2(0, T; \mathcal{V}'_\sigma)$. Consequently, the preceding equations hold in the distribution sense in $L^1(0, T)$.

Moreover, on the one hand, thanks to Proposition 2.1, Proposition 2.2, as well as the choice of γ_λ and μ_λ , we have

$$\begin{aligned}
& 2\mu_\lambda X_N^T \dot{X}_N, \\
& = 2\mu_\lambda X_N^T (A_N - \gamma_\lambda J_N) X_N + 2\mu_\lambda X_N^T Y_N, \\
& \leq -2\mu_\lambda \gamma_\lambda \left(C_1 e^{C_1 \sqrt{\lambda}} \right)^{-1} \|X_N\|_2^2 - 2\mu_\lambda \|\nabla \mathbb{P}_N y\|_{L^2(\Omega)}^2 + 2\mu_\lambda |\mathcal{B}(y, y, \mathbb{P}_N y)|, \\
& \leq -2\mu_\lambda \lambda \|X_N\|_2^2 - 2\mu_\lambda \|\nabla \mathbb{P}_N y\|_{L^2(\Omega)}^2 + 2\mu_\lambda c_0 \|y\|_{L^2(\Omega)} \|\nabla y\|_{L^2(\Omega)} \|\nabla \mathbb{P}_N y\|_{L^2(\Omega)}, \\
& \leq -2\mu_\lambda \lambda \|X_N\|_2^2 - 2\mu_\lambda \|\nabla \mathbb{P}_N y\|_{L^2(\Omega)}^2 + 2\mu_\lambda c_0 \|y\|_{L^2(\Omega)} \|\nabla y\|_{L^2(\Omega)}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
& 2 \left\langle \mathbb{P}_N^\perp y, \frac{d}{dt} y \right\rangle_{\mathcal{V}_\sigma \times \mathcal{V}'_\sigma}, \\
& = 2 \left\langle \mathbb{P}_N^\perp y, \Delta y - (y \cdot \nabla) y - \gamma_\lambda 1_\omega \mathbb{P}_N y - \nabla p \right\rangle_{\mathcal{V}_\sigma \times \mathcal{V}'_\sigma}, \\
& = -2 \left(\mathbb{P}_N^\perp y, y \right)_{\mathcal{V}_\sigma} - 2\gamma_\lambda \left(\mathbb{P}_N^\perp y, 1_\omega \mathbb{P}_N y \right)_{L^2(\Omega)^2} - 2 \left\langle (y \cdot \nabla) y, \mathbb{P}_N^\perp y \right\rangle_{\mathcal{V}'_\sigma \times \mathcal{V}_\sigma}, \\
& = -2 \sum_{i=N+1}^{\infty} \tau_i y_i^2 - 2\gamma_\lambda \left(\mathbb{P}_N^\perp y, 1_\omega \mathbb{P}_N y \right)_{L^2(\Omega)^2} - 2\mathcal{B}(y, y, \mathbb{P}_N^\perp y), \\
& \leq -2 \sum_{i=N+1}^{\infty} \tau_i y_i^2 + 2\gamma_\lambda \|\mathbb{P}_N^\perp y\|_{L^2(\Omega)} \|1_\omega \mathbb{P}_N y\|_{L^2(\Omega)} + 2\mathcal{B}(y, y, \mathbb{P}_N y), \\
& \leq -\frac{3}{2} \lambda \|\mathbb{P}_N^\perp y\|_{L^2(\Omega)}^2 - \frac{1}{2} \|\nabla \mathbb{P}_N^\perp y\|_{L^2(\Omega)}^2 + \lambda \|\mathbb{P}_N^\perp y\|_{L^2(\Omega)}^2 + \frac{\gamma_\lambda^2}{\lambda} \|X_N\|_2^2 + 2c_0 \|y\|_{L^2(\Omega)} \|\nabla y\|_{L^2(\Omega)}^2, \\
& \leq -\frac{1}{2} \lambda \|\mathbb{P}_N^\perp y\|_{L^2(\Omega)}^2 - \frac{1}{2} \|\nabla \mathbb{P}_N^\perp y\|_{L^2(\Omega)}^2 + \mu_\lambda \lambda \|X_N\|_2^2 + 2c_0 \|y\|_{L^2(\Omega)} \|\nabla y\|_{L^2(\Omega)}.
\end{aligned}$$

Combining the preceding three inequalities, we further derive that

$$\begin{aligned}
(3.7) \quad \frac{d}{dt} (V(y(t))) & \leq -2\mu_\lambda \lambda \|X_N\|_2^2 - 2\mu_\lambda \|\nabla \mathbb{P}_N y\|_{L^2(\Omega)}^2 + 2\mu_\lambda c_0 \|y\|_{L^2(\Omega)} \|\nabla y\|_{L^2(\Omega)}^2 \\
& \quad - \frac{1}{2} \lambda \|\mathbb{P}_N^\perp y\|_{L^2(\Omega)}^2 - \frac{1}{2} \|\nabla \mathbb{P}_N^\perp y\|_{L^2(\Omega)}^2 + \mu_\lambda \lambda \|X_N\|_2^2 + 2c_0 \|y\|_{L^2(\Omega)} \|\nabla y\|_{L^2(\Omega)}^2, \\
& \leq -\mu_\lambda \lambda \|X_N\|_2^2 - \frac{1}{2} \lambda \|\mathbb{P}_N^\perp y\|_{L^2(\Omega)}^2 - \frac{1}{2} \|\nabla y\|_{L^2(\Omega)}^2 + 4\mu_\lambda c_0 \|y\|_{L^2(\Omega)} \|\nabla y\|_{L^2(\Omega)}^2, \\
& \leq \left(-\frac{\lambda}{2} \right) V(y(t)) - \frac{1}{2} \|\nabla y\|_{L^2(\Omega)}^2 + 4\mu_\lambda c_0 \|y\|_{L^2(\Omega)} \|\nabla y\|_{L^2(\Omega)}^2, \\
& \leq \left(-\frac{\lambda}{2} \right) V(y(t)) - \|\nabla y\|_{L^2(\Omega)}^2 \left(\frac{1}{2} - 4\mu_\lambda c_0 V^{\frac{1}{2}}(y(t)) \right).
\end{aligned}$$

Keep in mind that the continuous function $V(y(t)) \in W^{1,1}(0, T)$ and that inequality (3.7) holds almost everywhere with both hand sides being $L^1(0, T)$. Inspired by the above formula, at first by ignoring the first term in the right hand side of (3.7), we know that the value of $\min\{V(y(t)), (8\mu_\lambda c_0)^{-2}\}$ decreases with respect to time. Therefore, if $V(y(0)) < (8\mu_\lambda c_0)^{-2}$ then the value of $V(y(t))$ is always strictly smaller than $(8\mu_\lambda c_0)^{-2}$. As a consequence in the preceding inequality one can next ignore the second term involving ∇y , which results in the fact that the Lyapunov functional $V(y(t))$ decay exponentially with decay rate $\lambda/2$.

More precisely, by the choice of r_λ from (2.10)–(2.11), for any initial data $y_0 \in \mathcal{H}$ satisfying $\|y_0\|_{L^2(\Omega)} \leq r_\lambda$ we have

$$V(y_0) \leq \mu_\lambda \|y_0\|_{L^2(\Omega)}^2 \leq \mu_\lambda r_\lambda^2 \leq (9\mu_\lambda c_0)^{-2} < (8\mu_\lambda c_0)^{-2},$$

which, combined with the fact due to inequality (3.7) that for almost every t in $(0, T)$,

$$\frac{d}{dt} \left(\min \{V(y(t)), (8\mu_\lambda c_0)^{-2}\} \right) \leq 0,$$

implies that the continuous function $\min \{V(y(t)), (8\mu_\lambda c_0)^{-2}\}$ is always smaller than $(9\mu_\lambda c_0)^{-2}$. Hence $V(y(t))$ is always smaller than $(9\mu_\lambda c_0)^{-2}$, which, combined with inequality (3.7), yields that for almost every t in $(0, T)$,

$$\frac{d}{dt} (V(y(t))) \leq \left(-\frac{\lambda}{2}\right) V(y(t)).$$

Therefore, the continuous function $V(y(t))$ verifies

$$V(y(t)) \leq e^{-\frac{\lambda}{2}t} V(y(0)), \quad \forall t \in [0, T].$$

Consequently

$$\|y(t)\|_{L^2(\Omega)}^2 \leq V(y(t)) \leq e^{-\frac{\lambda}{2}t} V(y(0)) \leq e^{-\frac{\lambda}{2}t} \mu_\lambda \|y(0)\|_{L^2(\Omega)}^2 \leq C_1^2 e^{2C_1\sqrt{\lambda}} e^{-\frac{\lambda}{2}t} \|y(0)\|_{L^2(\Omega)}^2.$$

Therefore, for any initial data $y_0 \in \mathcal{H}$ satisfying $\|y_0\|_{L^2(\Omega)} \leq r_\lambda$ the unique solution decays exponentially,

$$\|y(t)\|_{L^2(\Omega)} \leq C_1 e^{C_1\sqrt{\lambda}} e^{-\frac{\lambda}{4}t} \|y(0)\|_{L^2(\Omega)}, \quad \forall t \in [0, +\infty),$$

$$\|\mathcal{F}_\lambda y(t)\|_{L^2(\Omega)} \leq \gamma_\lambda \|y(t)\|_{L^2(\Omega)} \leq \lambda C_1^2 e^{2C_1\sqrt{\lambda}} e^{-\frac{\lambda}{4}t} \|y(0)\|_{L^2(\Omega)}, \quad \forall t \in [0, +\infty),$$

which can be quantified in the following theorem.

Theorem 3.1 (Local stabilization with linear feedback laws). *For any $\lambda > 0$, for any $\|y_0\|_{\mathcal{H}} \leq r_\lambda$, and for any $s \in \mathbb{R}$ the Cauchy problem*

$$(3.8) \quad \begin{cases} y_t - \Delta y + (y \cdot \nabla) y + \nabla p = -\gamma_\lambda 1_\omega \mathcal{F}_\lambda y, & (t, x) \in [s, +\infty) \times \Omega, \\ \operatorname{div} y = 0, & (t, x) \in [s, +\infty) \times \Omega, \\ y(t, x) = 0, & (t, x) \in [s, +\infty) \times \partial\Omega, \\ y(s, x) = y_0(x), & x \in \Omega, \end{cases}$$

has a unique solution in $C^0([s, +\infty); \mathcal{H}) \cap L_{loc}^2(s, +\infty; \mathcal{V}_\sigma)$. Moreover, this unique solution verifies

$$(3.9) \quad \|y(t)\|_{\mathcal{H}} \leq C_1 e^{C_1\sqrt{\lambda}} e^{-\frac{\lambda}{4}(t-s)} \|y_0\|_{\mathcal{H}}, \quad \forall t \in [s, +\infty),$$

$$(3.10) \quad \|\mathcal{F}_\lambda y(t)\|_{\mathcal{H}} \leq C_2 e^{C_2\sqrt{\lambda}} e^{-\frac{\lambda}{4}(t-s)} \|y_0\|_{\mathcal{H}}, \quad \forall t \in [s, +\infty).$$

Nonlinear feedback laws

Actually similar result also holds for nonlinear feedback laws $\mathcal{K}_{r_\lambda}(\mathcal{F}_\lambda y)$ provided by equations (2.7)–(2.8) and (3.1). From the preceding theorem we observe that for initial state $\|y_0\|_{\mathcal{H}} \leq r_\lambda^2$, the unique solution $y(t)$ of the Cauchy problem (3.8) satisfies

$$(3.11) \quad \|\mathcal{F}_\lambda y(t)\|_{L^2(\Omega)} \leq C_2 e^{C_2\sqrt{\lambda}} e^{-\frac{\lambda}{4}(t-s)} \|y_0\|_{L^2(\Omega)} \leq C_2 e^{C_2\sqrt{\lambda}} r_\lambda^2 = r_\lambda, \quad \forall t \in [s, +\infty).$$

Now, we replace the linear feedback law \mathcal{F}_λ by $\mathcal{K}_{r_\lambda}(\mathcal{F}_\lambda y)$ (see equation (2.8)), which satisfies

$$(3.12) \quad \|\mathcal{K}_{r_\lambda}(\mathcal{F}_\lambda y)\|_{L^2(\Omega)} \leq \min\{1, \sqrt{2\|y\|_{L^2(\Omega)}}\}.$$

Indeed, if $\|\mathcal{F}_\lambda y\|_{L^2(\Omega)} \leq 2r_\lambda$, then since the operator norm $\|\mathcal{F}_\lambda\| \leq \gamma_\lambda \leq r_\lambda^{-1}$, we have,

$$\|\mathcal{K}_{r_\lambda}(\mathcal{F}_\lambda y)\|_{L^2(\Omega)} \leq \|\mathcal{F}_\lambda y\|_{L^2(\Omega)} \leq \sqrt{2r_\lambda \|\mathcal{F}_\lambda y\|_{L^2(\Omega)}} \leq \sqrt{2r_\lambda \|\mathcal{F}_\lambda\| \|y\|_{L^2(\Omega)}} \leq \sqrt{2\|y\|_{L^2(\Omega)}}.$$

If $\|\mathcal{F}_\lambda y\|_{L^2(\Omega)} > 2r_\lambda$, then by the definition of \mathcal{K}_{r_λ} we know that $\mathcal{K}_{r_\lambda}(\mathcal{F}_\lambda y) = 0$, which completes the proof of the condition (3.12).

Finally, we show that for $\|y_0\|_{\mathcal{H}} \leq r_\lambda^2$ the solution of the closed-loop system with feedback law $\mathcal{K}_{r_\lambda}(\mathcal{F}_\lambda y)$ also decays exponentially. Indeed, it suffices to prove that the solution y verifies

$$\mathcal{K}_{r_\lambda}(\mathcal{F}_\lambda y(t)) = \mathcal{F}_\lambda y(t), \quad \forall t \in [s, +\infty),$$

which, by recalling the definition of \mathcal{K}_{r_λ} in (2.7)–(2.8), is true according to (3.11),

$$\|\mathcal{F}_\lambda y(t)\|_{L^2(\Omega)} \leq C_2 e^{C_2 \sqrt{\lambda}} \|y_0\|_{L^2(\Omega)} \leq C_2 e^{C_2 \sqrt{\lambda}} r_\lambda^2 = r_\lambda, \quad \forall t \in [s, +\infty).$$

Theorem 3.2 (Local stabilization with nonlinear Lipschitz feedback laws). *For any $\lambda > 0$, for any $\|y_0\|_{\mathcal{H}} \leq r_\lambda^2$, and for any $s \in \mathbb{R}$ the Cauchy problem*

$$(3.13) \quad \begin{cases} y_t - \Delta y + (y \cdot \nabla)y + \nabla p = -\gamma_\lambda 1_\omega \mathcal{K}_{r_\lambda}(\mathcal{F}_\lambda y), & (t, x) \in [s, +\infty) \times \Omega, \\ \operatorname{div} y = 0, & (t, x) \in [s, +\infty) \times \Omega, \\ y(t, x) = 0, & (t, x) \in [s, +\infty) \times \partial\Omega, \\ y(s, x) = y_0(x), & x \in \Omega, \end{cases}$$

has a unique solution in $C^0([s, +\infty); \mathcal{H}) \cap L^2_{loc}(s, +\infty; \mathcal{V}_\sigma)$. Moreover, this unique solution verifies

$$\begin{aligned} \|y(t)\|_{\mathcal{H}} &\leq C_1 e^{C_1 \sqrt{\lambda}} e^{-\frac{\lambda}{4}(t-s)} \|y_0\|_{\mathcal{H}}, \quad \forall t \in [s, +\infty), \\ \|\mathcal{F}_\lambda y(t)\|_{\mathcal{H}} &\leq C_2 e^{C_2 \sqrt{\lambda}} e^{-\frac{\lambda}{4}(t-s)} \|y_0\|_{\mathcal{H}}, \quad \forall t \in [s, +\infty). \end{aligned}$$

4. QUANTITATIVE NULL CONTROLLABILITY WITH COST ESTIMATES

In this section we construct feedback laws (controls) that yield the solutions decay to zero in finite time.

Theorem 4.1. *There exists $C_3 > 0$ such that, for any $T \in (0, 1)$, for any $y_0 \in \mathcal{H}$ satisfying $\|y_0\|_{\mathcal{H}} \leq e^{-\frac{C_3}{T}}$ we construct an explicit control $f(t, x)$ for the controlled system (1.1) such that the unique solution with initial data $y(0, x) = y_0(x)$ verifies $y(T, x) = 0$. Moreover,*

$$\|f(t, x)\|_{L^\infty(0, T; \mathcal{H})} \leq e^{\frac{C_3}{T}} \|y_0\|_{\mathcal{H}}.$$

Proof. For the ease of presentation, we only consider the case when $1/T = 2^{n_0}$ with $n_0 \in \mathbb{N}^*$. The other cases can be treated via time transition, *i.e.* if $T \in (2^{-m-1}, 2^{-m})$ then we simply let the feedback law $U(t; y) := 0$ on the time interval $[2^{-m-1}, T]$. More precisely, we consider the following partition of $[0, T]$ and piecewise feedback laws,

$$(4.1) \quad T_n := 2^{-n_0} \left(1 - \frac{1}{2^n}\right), \quad I_n := [T_n, T_{n+1}), \quad \lambda_n := Q^2 2^{2(n_0+n)} \text{ for any } n \geq 0,$$

$$(4.2) \quad \text{for any } n \geq 0 \text{ we consider the control (feedback law) as } \mathcal{F}_{\lambda_n} \text{ on interval } I_n.$$

Control design.

Step 1. Let the constant $R_T > 0$ be sufficiently small to be fixed later on. First, for $\|y_0\|_{\mathcal{H}} \leq R_T$, on the interval I_0 we consider the closed-loop system (1.1)–(1.2) with feedback law $U := \mathcal{F}_{\lambda_0}$ and initial data $y(0, x) = y_0(x)$. Assuming that $R_T \leq r_{\lambda_0}$, then according to Theorem 3.1 the closed-loop system has a unique solution $\tilde{y}|_{I_0}$ that decays exponentially with decay rate $\lambda_0/4$.

Step 2. Next, we consider the closed-loop system with feedback law \mathcal{F}_{λ_1} and $y(T_1, x) := \tilde{y}(T_1, x)$ on I_1 . Again we assume $\|y(T_1)\| \leq r_{\lambda_1}$ to find a unique solution $\tilde{y}|_{I_1}$ that is exponentially stable.

Step 3. By continuing this procedure on I_n and by always assuming $\|y(T_n)\| \leq r_{\lambda_n}$, we find a stable solution $\tilde{y}|_{I_n}$.

Step 4. We denote this constructed solution $\tilde{y}|_{[0, T]} \in C^0([0, T]; \mathcal{H})$ by $y|_{[0, T]}$.

Step 5. For some sufficiently small R_T we prove that $\|y(T_n)\|$ is indeed smaller than r_{λ_n} for every $n \in \mathbb{N}$, and show that the solution tends to zero as $y(T) := \lim_{t \rightarrow T^-} y(t) = 0$.

Step 6. Eventually, thanks to Step 5, $y|_{[0,T]}$ is the unique solution of the Cauchy problem (2.2) with the control term f given by $f|_{I_n} := \mathcal{F}_{\lambda_n} y|_{I_n}, \forall n \geq 0$, which satisfies $y(T) = 0$.

Step 7. We calculate precise cost estimates.

First we *assume* that for every I_n the value $\|y(T_n)\|_{L^2}$ is smaller than r_{λ_n} , which, together with Theorem 3.1, implies that the solution $y|_{I_n}$ verifies

$$(4.3) \quad \|y(t)\|_{L^2(\Omega)} \leq C_1 e^{C_1 Q 2^{n_0+n}} e^{-\frac{Q^2}{4} 2^{2(n_0+n)}(t-T_n)} \|y(T_n)\|_{L^2(\Omega)}, \quad \forall t \in I_n,$$

$$(4.4) \quad \|\mathcal{F}_{\lambda_n} y(t)\|_{L^2(\Omega)} \leq C_2 e^{C_2 Q 2^{n_0+n}} e^{-\frac{Q^2}{4} 2^{2(n_0+n)}(t-T_n)} \|y(T_n)\|_{L^2(\Omega)}, \quad \forall t \in I_n.$$

Consequently, for every $n \geq 1$ the value of $y(T_n)$ is dominated by

$$(4.5) \quad \begin{aligned} \|y(T_n)\|_{L^2(\Omega)} &\leq \left(\prod_{k=0}^{n-1} C_1 e^{C_1 \sqrt{\lambda_k}} e^{-\frac{\lambda_k}{4} 2^{-(n_0+k+1)}} \right) \|y_0\|_{L^2(\Omega)}, \\ &= \left(\prod_{k=0}^{n-1} C_1 e^{C_1 Q 2^{n_0+k}} e^{-\frac{Q^2}{8} 2^{n_0+k}} \right) \|y_0\|_{L^2(\Omega)}, \\ &\leq \left(\prod_{k=0}^{n-1} e^{\frac{Q^2}{64} 2^{n_0+k}} e^{-\frac{Q^2}{8} 2^{n_0+k}} \right) \|y_0\|_{L^2(\Omega)}, \\ &= \left(\prod_{k=0}^{n-1} e^{-\frac{7Q^2}{64} 2^{n_0+k}} \right) \|y_0\|_{L^2(\Omega)}, \\ &= \exp\left(-\frac{7Q^2}{64} 2^{n_0} (2^n - 1)\right) \|y_0\|_{L^2(\Omega)}. \end{aligned}$$

Observe that the above inequality also holds for $n = 0$. Furthermore, for any $n \geq 1$ and for any $t \in I_n$ the control term is bounded by

$$(4.6) \quad \|\mathcal{F}_{\lambda_n} y(t)\|_{L^2(\Omega)} \leq C_2 e^{C_2 Q 2^{n_0+n}} \|y(T_n)\|_{L^2(\Omega)} \leq \exp\left(-\frac{5Q^2}{64} 2^{n_0+n-1}\right) \|y_0\|_{L^2(\Omega)},$$

Clearly, the right hand side of the inequalities (4.5) and (4.6) tend to 0 as n tends to ∞ . Therefore, it suffices to prove the assumption $\|y(T_n)\|_{L^2} \leq r_{\lambda_n}$ to close the ‘‘bootstrap’’ and to conclude the null controllability. By recalling the definitions of λ_n , r_{λ_n} , and Q we know that

$$e^{-\frac{Q^2}{64} 2^{n_0+n}} \leq (C_2 e^{C_2 Q 2^{n_0+n}})^{-1} = (C_2 e^{C_2 \sqrt{\lambda_n}})^{-1} = r_{\lambda_n}, \quad \forall n \in \mathbb{N}.$$

Hence, it suffices to find some $R_T > 0$ such that

$$(4.7) \quad R_T \exp\left(-\frac{7Q^2}{64} 2^{n_0} (2^n - 1)\right) \leq e^{-\frac{Q^2}{64} 2^{n_0+n}} \leq r_{\lambda_n}, \quad \forall n \in \mathbb{N}.$$

Thus one can take

$$(4.8) \quad R_T := e^{-\frac{Q^2}{32} 2^{n_0}} = e^{-\frac{Q^2}{32T}} = e^{-\frac{C_3}{T}} \quad \text{where } C_3 = \frac{Q^2}{32}.$$

It only remains to estimate the controlling cost. Thanks to (4.6) we know that

$$\|f(t)\|_{L^2(\Omega)} \leq \|y_0\|_{L^2(\Omega)}, \quad \forall t \in [T_1, T].$$

As for $t \in [0, T_1)$ and the control $f|_{I_0}(t)$, we have

$$\|\mathcal{F}_{\lambda_0} y(t)\|_{L^2(\Omega)} \leq C_2 e^{C_2 Q 2^{n_0}} \|y_0\|_{L^2(\Omega)} \leq e^{\frac{Q^2}{64} 2^{n_0}} \|y_0\|_{L^2(\Omega)} \leq e^{\frac{C_3}{T}} \|y_0\|_{L^2(\Omega)}.$$

In conclusion, for any $\|y_0\|_{\mathcal{H}} \leq e^{-\frac{C_3}{T}}$, the constructed solution $y(t, x)$ and control $f(t, x)$ satisfy

$$\begin{aligned} & \|y(t, \cdot)\|_{L^2(\Omega)} \text{ and } \|f(t, \cdot)\|_{L^2(\Omega)} \longrightarrow 0^+, \text{ as } t \rightarrow T^-, \\ & \|y(t, \cdot)\|_{L^2(\Omega)} \text{ and } \|f(t, \cdot)\|_{L^2(\Omega)} \leq e^{\frac{C_3}{T}} \|y_0\|_{L^2(\Omega)}, \forall t \in [0, T]. \end{aligned}$$

□

Remark 4.2. *If we replace the linear feedback laws $\{\mathcal{F}_{\lambda_n} y\}_{n=1}^{\infty}$ by $\{\mathcal{K}_{r_{\lambda_n}}(\mathcal{F}_{\lambda_n} y)\}_{n=1}^{\infty}$ on interval I_n , then similar result holds. Indeed, according to Theorem 3.2 it suffices to find some initial state such that for every $n \in \mathbb{N}$ the value of $\|y(T_n)\|$ is smaller than $r_{\lambda_n}^2$. More precisely, instead of taking some $R_T > 0$ that verifies (4.7), one only needs to find $\tilde{R}_T := e^{-\frac{Q^2}{16T}} = e^{-\frac{2C_3}{T}}$ satisfying*

$$\tilde{R}_T \exp\left(-\frac{7Q^2}{64} 2^{n_0}(2^n - 1)\right) \leq e^{-\frac{Q^2}{32} 2^{n_0+n}} \leq r_{\lambda_n}^2, \forall n \in \mathbb{N},$$

to guarantee that for every $n \in \mathbb{N}$ we have $\|y(T_n)\|_{L^2} \leq r_{\lambda_n}^2$.

5. SMALL-TIME LOCAL STABILIZATION

As in the preceding section, we only focus on the case when $T = 1/2^{n_0}$ with n_0 be integer. We also adapt the same construction of T_n and λ_n given by (4.1) in Section 4.

Theorem 5.1 (Small-time local stabilization of Navier-Stokes equations). *Let $T = 1/2^{n_0}$ with $n_0 \in \mathbb{N}^*$. The following T -periodic feedback law $U(t; y) : \mathbb{R} \times \mathcal{H} \rightarrow \mathcal{H}$ satisfying (3.12),*

$$(5.1) \quad U|_{[0, T) \times \mathcal{H}}(t; y) := \mathcal{K}_{r_{\lambda_n}}(\mathcal{F}_{\lambda_n} y), \forall y \in \mathcal{H}, \forall t \in I_n, \forall n \in \mathbb{N},$$

is a proper feedback law for system (1.1)–(1.2). Moreover, for some effectively computable constant Λ_T this feedback law stabilizes system (1.1)–(1.2) in finite time:

- (i) (*2T stabilization*) $\Phi(2T + t, t; y_0) = 0, \forall t \in \mathbb{R}, \forall \|y_0\|_{\mathcal{H}} \leq \Lambda_T$.
- (ii) (*Uniform stability*) For every $\delta > 0$, there exists an effectively computable $\eta > 0$ such that

$$(\|y_0\|_{\mathcal{H}} \leq \eta) \Rightarrow (\|\Phi(t, t'; y_0)\|_{\mathcal{H}} \leq \delta, \forall t' \in \mathbb{R}, \forall t \in (t', +\infty)).$$

Proof of Theorem 5.1. We mimic the prove of the finite time stabilization of the heat equations [16, 37], as relatively standard, see also [19, 35, 36] for similar results. The proof is followed by five steps:

Step 1. The feedback law U is a proper feedback law, i.e. for any $y_0 \in \mathcal{H}$ and for any initial time $s \in \mathbb{R}$ there exists a unique global (in time) solution.

Step 2. Null controllability: $\Phi(T, 0; y_0) = 0$ for any y_0 satisfying $\|y_0\|_{\mathcal{H}} \leq \tilde{R}_T = e^{-\frac{2C_3}{T}}$. Moreover,

$$(5.2) \quad \|\Phi(t, 0; y_0)\|_{\mathcal{H}} \leq e^{\frac{C_3}{T}} \|y_0\|_{\mathcal{H}}, \forall \|y_0\|_{\mathcal{H}} \leq e^{-\frac{2C_3}{T}}, \forall t \in [0, T].$$

Step 3. For any $\tilde{\eta} > 0$, there exists some $\varepsilon(\tilde{\eta}) \in (0, \tilde{\eta})$ such that

$$(5.3) \quad \|\Phi(t, s; y_0)\|_{\mathcal{H}} \leq \tilde{\eta}, \forall \|y_0\|_{\mathcal{H}} \leq \varepsilon(\tilde{\eta}), \forall s \in [0, T), \forall t \in [s, T].$$

Step 4. 2T stabilization: $\Phi(2T, s; y_0) = 0$, for any $s \in [0, T)$, for any y_0 satisfying $\|y_0\|_{\mathcal{H}} \leq \varepsilon\left(e^{-\frac{2C_3}{T}}\right) =: \Lambda_T$.

Step 5. Uniform stability as direct consequence of Step 2–4.

Step 1. It suffices to prove that for any $s \in [0, T)$ the closed-loop system has a unique solution on $[s, T]$. Indeed, thanks to Theorem 2.8 there exists a unique solution on I_n for any I_n that intersects with $[s, T)$. Hence we find a unique solution y in $C^0([s, T]; \mathcal{H}) \cap L^2_{loc}(s, T; \mathcal{V}_\sigma)$. Observe that the control (provided by the related feedback law) is smaller than 1, i.e. $\|f(t, x)\|_{L^2(s, T; \mathcal{H})} \leq \sqrt{T}$.

Theorem 2.3 implies that the solution y is indeed in $C^0([s, T]; \mathcal{H}) \cap L^2(s, T; \mathcal{V}_\sigma)$. Finally, thanks to Theorem 2.3 again, the unique solution y never blow up,

$$\|y(t, x)\|_{\mathcal{H}}^2 + \|\nabla y(t, x)\|_{L^2(s, t; L^2)}^2 \leq \|y_0\|_{\mathcal{H}}^2 + C_0(t - s), \quad \forall t \in (s, +\infty).$$

Step 2. This step is a consequence of Theorem 4.1 and Remark 4.2.

Step 3. Thanks to the fact that $\|f(t, x)\|_{\mathcal{H}} \leq 1$ and Theorem 2.3, there exists $\tilde{T} \in (0, T)$ such that

$$\|\Phi(t, s; y_0)\|_{\mathcal{H}} \leq \tilde{\eta}, \quad \forall \|y_0\|_{\mathcal{H}} \leq \tilde{\eta}/2, \quad \forall s \in [\tilde{T}, T], \quad \forall t \in [s, T].$$

Observe that the feedback law U on $[0, \tilde{T}]$ is composed by finitely many stationary feedback laws on intervals $\{I_n\}$, while, thanks to Theorem 3.2, on each of these intervals I_n the system is locally exponentially stable. Consequently, there exists some $\varepsilon = \varepsilon(\tilde{\eta}) \in (0, \tilde{\eta}/2)$ such that

$$\|\Phi(t, s; y_0)\|_{\mathcal{H}} \leq \tilde{\eta}/2, \quad \forall \|y_0\|_{\mathcal{H}} \leq \varepsilon, \quad \forall s \in [0, \tilde{T}], \quad \forall t \in [s, \tilde{T}].$$

Step 4 is a trivial combination of Step 2 and Step 3 by taking $\varepsilon \left(e^{-\frac{2C_3}{T}} \right)$.

Step 5 follows directly from Step 2–4. For instance, for $\delta > 0$, we can take

$$\delta_1 := \min\{\varepsilon(\delta), e^{-\frac{2C_3}{T}}\} < \delta \quad \text{and} \quad \eta := \varepsilon(\delta_1) < \delta_1 < \delta.$$

Indeed, for $s \in [0, T)$, and for $\|y_0\|_{\mathcal{H}} \leq \eta$, thanks to the choice of η as $\varepsilon(\delta_1)$, Step 3 yields,

$$\|y(t)\|_{\mathcal{H}} = \|\Phi(t, s; y_0)\|_{\mathcal{H}} \leq \delta_1, \quad \forall t \in [s, T].$$

Next, thanks to the choice of δ_1 , Step 3 and Step 2 lead to

$$\begin{aligned} \|\Phi(t, s; y_0)\|_{\mathcal{H}} &= \|\Phi(t, T; y(T))\|_{\mathcal{H}} \leq \delta, \quad \forall t \in [T, 2T], \\ \Phi(2T, s; y_0) &= \Phi(2T, T; y(T)) = 0. \end{aligned}$$

□

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REFERENCES

- [1] Andrey A. Agrachev and Andrey V. Sarychev. Navier-Stokes equations: controllability by means of low modes forcing. *J. Math. Fluid Mech.*, 7(1):108–152, 2005.
- [2] Mehdi Badra and Takéo Takahashi. Stabilization of parabolic nonlinear systems with finite dimensional feedback or dynamical controllers: application to the Navier-Stokes system. *SIAM J. Control Optim.*, 49(2):420–463, 2011.
- [3] Viorel Barbu, Irena Lasiecka, and Roberto Triggiani. Tangential boundary stabilization of Navier-Stokes equations. *Mem. Amer. Math. Soc.*, 181(852):x+128, 2006.
- [4] Viorel Barbu, Sérgio S. Rodrigues, and Armen Shirikyan. Internal exponential stabilization to a nonstationary solution for 3D Navier-Stokes equations. *SIAM J. Control Optim.*, 49(4):1454–1478, 2011.
- [5] Viorel Barbu and Roberto Triggiani. Internal stabilization of Navier-Stokes equations with finite-dimensional controllers. *Indiana Univ. Math. J.*, 53(5):1443–1494, 2004.
- [6] Georges Bastin and Jean-Michel Coron. *Stability and boundary stabilization of 1-D hyperbolic systems*, volume 88 of *Progress in Nonlinear Differential Equations and their Applications*. Birkhäuser/Springer, [Cham], 2016. Subseries in Control.
- [7] Tobias Breiten, Karl Kunisch, and Laurent Pfeiffer. Feedback stabilization of the two-dimensional Navier-Stokes equations by value function approximation. *Appl. Math. Optim.*, 80(3):599–641, 2019.
- [8] Felipe W. Chaves-Silva and Gilles Lebeau. Spectral inequality and optimal cost of controllability for the Stokes system. *ESAIM Control Optim. Calc. Var.*, 22(4):1137–1162, 2016.
- [9] Jean-Yves Chemin. *Perfect incompressible fluids*, volume 14 of *Oxford Lecture Series in Mathematics and its Applications*. The Clarendon Press, Oxford University Press, New York, 1998. Translated from the 1995 French original by Isabelle Gallagher and Dragos Iftimie.
- [10] Jean-Michel Coron. *Control and nonlinearity*, volume 136 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2007.

- [11] Jean-Michel Coron and Andrei V. Fursikov. Global exact controllability of the 2D Navier-Stokes equations on a manifold without boundary. *Russian J. Math. Phys.*, 4(4):429–448, 1996.
- [12] Jean-Michel Coron and Sergio Guerrero. Local null controllability of the two-dimensional Navier-Stokes system in the torus with a control force having a vanishing component. *J. Math. Pures Appl. (9)*, 92(5):528–545, 2009.
- [13] Jean-Michel Coron and Pierre Lissy. Local null controllability of the three-dimensional Navier-Stokes system with a distributed control having two vanishing components. *Invent. Math.*, 198(3):833–880, 2014.
- [14] Jean-Michel Coron, Frédéric Marbach, and Franck Sueur. Small-time global exact controllability of the Navier-Stokes equation with Navier slip-with-friction boundary conditions. *J. Eur. Math. Soc. (JEMS)*, 22(5):1625–1673, 2020.
- [15] Jean-Michel Coron, Frédéric Marbach, Franck Sueur, and Ping Zhang. Controllability of the Navier-Stokes equation in a rectangle with a little help of a distributed phantom force. *Ann. PDE*, 5(2):Paper No. 17, 49, 2019.
- [16] Jean-Michel Coron and Hoai-Minh Nguyen. Null controllability and finite time stabilization for the heat equations with variable coefficients in space in one dimension via backstepping approach. *Arch. Ration. Mech. Anal.*, 225(3):993–1023, 2017.
- [17] Jean-Michel Coron, Ivonne Rivas, and Shengquan Xiang. Local exponential stabilization for a class of Korteweg-de Vries equations by means of time-varying feedback laws. *Anal. PDE*, 10(5):1089–1122, 2017.
- [18] Jean-Michel Coron and Emmanuel Trélat. Global steady-state controllability of one-dimensional semilinear heat equations. *SIAM J. Control Optim.*, 43(2):549–569, 2004.
- [19] Jean-Michel Coron and Shengquan Xiang. Small-time global stabilization of the viscous Burgers equation with three scalar controls. *Preprint, hal-01723188*, 2018.
- [20] Sylvain Ervedoza, Olivier Glass, and Sergio Guerrero. Local exact controllability for the two- and three-dimensional compressible Navier-Stokes equations. *Comm. Partial Differential Equations*, 41(11):1660–1691, 2016.
- [21] E. Fernández-Cara, S. Guerrero, O. Yu. Imanuvilov, and J.-P. Puel. Local exact controllability of the Navier-Stokes system. *J. Math. Pures Appl. (9)*, 83(12):1501–1542, 2004.
- [22] Enrique Fernández-Cara, Sergio Guerrero, Oleg Yu. Imanuvilov, and Jean-Pierre Puel. Some controllability results for the N -dimensional Navier-Stokes and Boussinesq systems with $N - 1$ scalar controls. *SIAM J. Control Optim.*, 45(1):146–173, 2006.
- [23] A. V. Fursikov. Stabilizability of two-dimensional Navier-Stokes equations with help of a boundary feedback control. *J. Math. Fluid Mech.*, 3(3):259–301, 2001.
- [24] A. V. Fursikov and O. Yu. Émanuilov. Exact local controllability of two-dimensional Navier-Stokes equations. *Mat. Sb.*, 187(9):103–138, 1996.
- [25] Andrei V. Fursikov and Oleg Yu. Imanuvilov. *Controllability of evolution equations*, volume 34 of *Lecture Notes Series*. Seoul National University, Research Institute of Mathematics, Global Analysis Research Center, Seoul, 1996.
- [26] Amaury Hayat. Boundary stability of 1-D nonlinear inhomogeneous hyperbolic systems for the C^1 norm. *SIAM J. Control Optim.*, 57(6):3603–3638, 2019.
- [27] David Jerison and Gilles Lebeau. Nodal sets of sums of eigenfunctions. In *Harmonic analysis and partial differential equations (Chicago, IL, 1996)*, Chicago Lectures in Math., pages 223–239. Univ. Chicago Press, Chicago, IL, 1999.
- [28] Gilles Lebeau and Luc Robbiano. Contrôle exact de l'équation de la chaleur. *Comm. Partial Differential Equations*, 20(1-2):335–356, 1995.
- [29] Jean Leray. Sur le mouvement d'un liquide visqueux emplissant l'espace. *Acta Math.*, 63(1):193–248, 1934.
- [30] Jacques-Louis Lions. *Contrôlabilité exacte, perturbations et stabilisation de systèmes distribués. Tome 2*, volume 9 of *Recherches en Mathématiques Appliquées [Research in Applied Mathematics]*. Masson, Paris, 1988. Perturbations. [Perturbations].
- [31] Jacques-Louis Lions and Enrique Zuazua. Exact boundary controllability of Galerkin's approximations of Navier-Stokes equations. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 26(4):605–621, 1998.
- [32] J.-P. Raymond. Feedback boundary stabilization of the three-dimensional incompressible Navier-Stokes equations. *J. Math. Pures Appl. (9)*, 87(6):627–669, 2007.
- [33] Jean-Pierre Raymond. Feedback boundary stabilization of the two-dimensional Navier-Stokes equations. *SIAM J. Control Optim.*, 45(3):790–828, 2006.
- [34] Michael Schmidt and Emmanuel Trélat. Controllability of Couette flows. *Commun. Pure Appl. Anal.*, 5(1):201–211, 2006.
- [35] Shengquan Xiang. Small-time local stabilization for a Korteweg-de Vries equation. *Systems & Control Letters*, 111:64 – 69, 2018.
- [36] Shengquan Xiang. Null controllability of a linearized Korteweg-de Vries equation by backstepping approach. *SIAM J. Control Optim.*, 57(2):1493–1515, 2019.
- [37] Shengquan Xiang. Quantitative rapid and finite time stabilization of the heat equation. *Preprint*, 2020.

- [38] Christophe Zhang. Internal rapid stabilization of a 1-D linear transport equation with a scalar feedback. Preprint, October 2018.

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