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Abstract. The null controllability of the heat equation is known for decades [21, 25, 34]. The finite time stabilizability of the one dimensional heat equation was proved by Coron–Nguyên [15], while the same question for high dimensional spaces remained widely open. Inspired by Coron–Trélat [16] we find explicit stationary feedback laws that quantitatively exponentially stabilize the heat equation with decay rate λ and $Ce^{C\sqrt{\lambda}}$ estimates, where Lebeau–Robbiano's spectral inequality [34] is naturally used. Then a piecewise controlling argument leads to null controllability with optimal cost $Ce^{C/T}$, as well as finite time stabilization.

1. Introduction

Let Ω be an open domain in $\mathbb{R}^d$ with smooth boundary and $\omega \subset \Omega$ an open subset. We are interested in the stabilization and controllability of the internal controlled heat equation,

$$y_t = \Delta y + 1_\omega f \quad \text{in } \Omega, \tag{1.1}$$

$$y = 0 \quad \text{on } \partial\Omega. \tag{1.2}$$

1.1. **Stabilization problems.** It is well known that in the 90's the null controllability of the above system was simultaneously discovered by Lebeau–Robbiano and Fursikov–Imanuvilov via different approaches [25, 34], rely on Lions' H.U.M. [36], Russell's idea on controllability and observability [41], and most importantly Carleman estimates [9, 26]. See [32] for a complete and pedagogical introduction on these different but somehow complementary methods. Later on many people have contributed in the related controllability problems [16, 20, 22, 23, 24].

Though the study on the controllability of the heat equation is nearly complete, less is known concerning stabilization. Generally speaking, exponential stabilization for the heat equation even for evolution equations with operators generating analytic semi-groups should be easier than null controllability problems. Indeed from a spectrum point of view one needs to stabilize finitely many unstable modes, which in some sense is easier than unique continuation problems, while null controllability corresponds to observability inequality. The controlled wave equation is probably the best example to see this difference, Hörmander–Tataru–Robbiano–Zuily [27, 30, 40, 42] proved the unique continuation for arbitrary control domain, while observability requires the control domain satisfying G.C.C. [4, 8] according to Bardos–Lebeau–Rauch. Several methods have been introduced for exponential stabilization problems on partial differential equations, among which the most commonly used should be the so-called Riccati method motivated from finite dimensional optimal control theory (see for example [2, 3, 5, 35, 36, 39]). Modulo some systematic arguments, in the end of the day it suffices to solve some non-linear algebraic Riccati equation in order to define a stabilizing feedback law. Though powerful the feedback law is not explicit, because it is an implicit solution of the Riccati equation, for which even the numerical computation is demanding. Thus it is always asked to design simper and more efficient exponentially stabilizing feedback laws that provide quantitative stabilizing estimates.

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Finite time stabilization can be regarded as one of the ultimate questions to be asked for control theory, which is definitely much more involved than null controllability problems. In fact, even for the one dimensional heat equation the finite time stabilization problem was solved quite recently by Coron–Nguyên [15], the controllability of which was known for nearly half century [21]. We refer to the paper by Coron and the author [18, Introduction] for a detailed review on this problem. The crucial point for [15] is the exponential stabilization by stationary feedback laws for decay rate λ with quantitative estimates $e^{C\sqrt{\lambda}}$ via backstepping method. The backstepping method, first introduced by Krstic and his collaborators [29], corresponds to moving the spectrum with the help of some feedback laws. It has been improved in [12, 14], which turned out to be quite efficient for many one dimensional models [13, 47, 48]. From a spectrum point of view, this method is different from any other stabilizing techniques concentrating on finite dimensional low frequency terms, as a result it can be applied to hyperbolic type systems. However, it is a challenging problem to introduce backstepping method for models in high dimensional spaces. Because the other stabilization methods are rather abstract and the backstepping method provides satisfying $e^{C\sqrt{\lambda}}$ estimates, it was believed that the generalization of backstepping should appear before the proof of finite time stabilization of the heat equation.

1.2. **The main results.** In this paper, we solve the finite time stabilization problem of the heat equation in any dimensional space, and provide a quantitative exponential stabilization method. Instead of using Riccati methods, or of generalizing backstepping to high dimensional spaces, we use a straightforward Lyapunov functional method. It turns out that the exponential stabilization of the heat equation with arbitrary decay rate λ can be achieved via simple and explicit feedback laws. Surprisingly, the spectral estimates found by Lebeau–Robbiano is naturally and elegantly used to provide an $e^{C\sqrt{\lambda}}$ stabilizing estimate. Thanks to this powerful estimate, by applying a standard piecewise controlling argument we further prove the null controllability via stabilization approach, and more importantly, solve the finite time stabilization problem with arbitrarily small time $T > 0$ (hence small-time stabilization).

Before stating the detailed theorems, we briefly explain some terminologies used for finite time stabilization. A *time-varying feedback law* U is an application

$$\begin{cases} U : \mathbb{R} \times L^2(\Omega) & \to & L^2(\Omega) \\ \qquad (t; y) & \mapsto & U(t; y). \end{cases}$$

A *stationary* feedback law is such an application only depends on $L^2(\Omega)$, and a T-*periodic* feedback law is a time-varying feedback law such that $U(t+T; y) = U(t; y)$. The *closed-loop system* associated to a feedback law U is the evolution equation

$$\begin{cases} y_t = \Delta y + 1_\omega U(t; y), \ (t, x) \in (s, +\infty) \times \Omega, \\ y(t, x) = 0, \ (t, x) \in (s, +\infty) \times \partial\Omega. \end{cases} \tag{1.3}$$

Eventually we are interested in T-*periodic proper* feedback laws. Heuristically speaking, a feedback law U is called *proper* if the Cauchy problem associated to the closed-loop system (1.3) admits a unique solution for every $s \in \mathbb{R}$ and for every initial data $y_0 \in L^2(\Omega)$ at time s. Therefore, formally we are allowed to define a "*flow*", $\Phi(t, s; y_0)$, as the state at time t of the solution of (1.3) with initial state $y(s, x) = y_0(x)$, where $y_0 \in L^2(\Omega)$ and $t \geq s$. Please follow Section 4.1 for precise definitions on solutions of closed-loop systems, proper feedback laws, "flow" with respect to systems with proper feedback laws, as well as finite time stabilization.

Successively we are able to prove the following theorems concerning rapid stabilization, null controllability, and finite time stabilization in Section 2, Section 3, and Section 4 respectively. In

Section 2 we also comment on the stabilization of nonlinear systems and the stabilization on higher regularity spaces.

Theorem 1.1 (Quantitative rapid stabilization)**.** *There exists an effectively computable constant $C > 0$ such that for any $\lambda > 0$ we construct an explicit stationary feedback law $\mathcal{G}_\lambda : L^2(\Omega) \to L^2(\Omega)$, such that the closed-loop system*

$$\begin{aligned} y_t &= \Delta y + 1_\omega \mathcal{G}_\lambda y \quad \text{in } \Omega, \\ y &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

is exponentially stable:

$$||\Phi(t,s;y_0)||_{L^2(\Omega)} + ||1_\omega \mathcal{G}_\lambda \Phi(t,s;y_0)||_{L^2(\omega)} \leq C e^{C\sqrt{\lambda}} e^{-\frac{\lambda}{2}(t-s)} ||y_0||_{L^2(\Omega)}, \ \forall\, s \in \mathbb{R}, \forall\, t \in [s, +\infty).$$

Theorem 1.2 (Null controllability with optimal cost)**.** *There exists an effectively computable constant $C > 0$ such that, for any $T \in (0,1)$, for any $y_0 \in L^2(\Omega)$, we find an explicit control $f|_{[0,T]}(t,x)$ for the control system (1.1)–(1.2) such that the unique solution verifies*

$$y(0,x) = y_0(x) \text{ and } y(T,x) = 0,$$

moreover,

$$||1_\omega f(t,x)||_{L^\infty(0,T;L^2(\Omega))} \leq e^{C/T} ||y_0||_{L^2(\Omega)}.$$

Theorem 1.3 (Semi-global finite time stabilization with explicit feedback laws)**.** *For any $\Lambda \geq 1$, for any $T > 0$, we construct an explicit T-periodic proper feedback law U satisfying*

$$||1_\omega U(t;y)||_{L^2(\Omega)} \leq C||y||_{L^2(\Omega)} + 2||y||^{1/2}_{L^2(\Omega)}, \ \forall\, y \in L^2(\Omega), \ \forall\, t \in \mathbb{R},$$

with some C effectively computable, that stabilizes system (1.3) in finite time:

(i) *($2T$ stabilization) $\Phi(2T + t, t; y_0) = 0, \ \ \forall\, t \in \mathbb{R}, \ \forall\, ||y_0||_{L^2(\Omega)} \leq \Lambda$.*

(ii) *(Uniform stability) For every $\delta > 0$ there exists an effectively computable $\eta > 0$ such that*

$$\left(||(y_0||_{L^2(\Omega)} \leq \eta \right) \Rightarrow \left(||\Phi(t,t';y_0)||_{L^2(\Omega)} \leq \delta, \ \forall\, t' \in \mathbb{R}, \ \forall\, t \in (t', +\infty) \right).$$

Remark 1.4. *Let us emphasize that the "uniform stability" condition is one of the essential differences between null controllability and finite time stabilization. Indeed, this condition is crucial for stabilization problems as in reality systems may have errors and exist perturbations, thus the stabilizing system are required to overcome these difficulties. Another main difficulty for closed-loop stabilization compared to open-loop control is that the feedback only depends on current states, while control may depend on backward states.*

Statement on notations: for readers convenience we summarize some notations and constants that will be defined and used later on. Moreover, once a constant is defined, from then on we will use it directly. Notations (τ_i, e_i) and $N(\lambda)$ about eigenvalues defined in Section 2.2; orthogonal projection P_N, P_N^T defined after equation (2.12); truncated operator $\mathcal{K}_r$ in (2.3); γ_λ and μ_λ in (2.13); feedback law $\mathcal{F}_\lambda$ in (2.14); r_λ in (2.15); the partition T_n, λ_n, and I_n by (3.1). All the following constants are independent of $\lambda > 0$: C_1 defined in Proposition 2.5; C_2 in equation (2.15); Γ by (3.5) and C_3 by (3.7).

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2. Rapid stabilization

2.1. **Well-posedness results.** In this section we quickly review the well-posedness results for the following Cauchy problem

$$\begin{cases} y_t = \Delta y + f(t,x),\ (t,x) \in (t_1,t_2)\times\Omega, \\ y(t,x) = 0,\ \ (t,x) \in (t_1,t_2)\times\partial\Omega, \\ y(t_1,x) = y_0(x), \end{cases} \tag{2.1}$$

as well as the related closed-loop systems with stationary feedback laws *i.e.* $f(t,x) = \mathcal{L}y$, where $\mathcal{L}$ is a bounded operator on $L^2(\Omega)$.

The well-posedness for both open-loop systems and closed-loop systems with stationary feedback laws are well-known, here we adapt the definition of the solution in the transposition sense, for which the well-poseness results are derived from classical Hille–Yosida semi-group theory. Transposition sense solution is introduced by Lions [36], for those who are not familiar with those definitions, we refer to the monograph by Coron [11, Chapter 1–2] for an excellent introduction on this subject.

Definition 2.1. *Let $t_1, t_2 \in \mathbb{R}$ be such that $t_1 < t_2$. Let $y_0 \in L^2(\Omega)$ and $f(t,x) \in L^2(t_1,t_2;L^2(\Omega))$. A solution to the Cauchy problem (2.1) is a function $y \in C^0([t_1,t_2];L^2(\Omega)) \cap L^2(t_1,t_2;H_0^1(\Omega))$ such that, for every $\tau \in [t_1,t_2]$ and for every $\phi \in C^0([t_1,\tau];H^1(\Omega))$ such that*

$$\phi_t \in L^2((t_1,t_2)\times\Omega),\ \Delta\phi \in L^2((0,T)\times\Omega),\ \text{and } \phi(t,\cdot) \in H_0^1(\Omega)\ \forall\, t \in [t_1,\tau],$$

one has

$$\int_\Omega y(\tau,x)\phi(\tau,x)dx - \int_\Omega y_0(x)\phi(t_1,x)dx - \int_{t_1}^{\tau}\int_\Omega f\phi dxdt - \int_{t_1}^{\tau}\int_\Omega (\phi_t + \Delta\phi)ydxdt = 0.$$

Theorem 2.2. *For any $T \in (0,1]$, for any $y_0 \in L^2(\Omega)$, and for any $f \in L^2(0,T;L^2(\Omega))$, the Cauchy problem (2.1) has a unique solution. Moreover, this solution satisfies*

$$\begin{aligned} ||y(t)||_{L^2(\Omega)} &\le ||y_0||_{L^2(\Omega)} + 2||f||_{L^2(0,t;L^2(\Omega))},\ \forall\, t \in (0,T], \\ ||\nabla y||_{L^2(0,t;L^2(\Omega))} &\le ||y_0||_{L^2(\Omega)} + 2||f||_{L^2(0,t;L^2(\Omega))},\ \forall\, t \in (0,T]. \end{aligned}$$

We do not recall the solution definition to closed-loop systems with stationary feedback laws as classical. Besides it can be covered by the more general definition of solutions for time-varying feedback systems that will be presented in Section 4.1. Concerning closed-loop systems with stationary feedback laws we have the following well-posedness results.

Theorem 2.3. *Let $\varphi_i \in L^2(\Omega), 1 \le i \le n$ be given functions. Let $l_i : L^2(\Omega) \to \mathbb{R}, 1 \le i \le n$ be given bounded linear operators. For any $y_0 \in L^2(\Omega)$ the Cauchy problem*

$$\begin{cases} y_t = \Delta y + 1_\omega\Big(\sum_{i=1}^n l_i(y)\varphi_i\Big),\ (t,x) \in (0,T)\times\Omega, \\ y(t,x) = 0,\ (t,x) \in (0,T)\times\partial\Omega, \\ y(0,x) = y_0(x), \end{cases}$$

has a unique solution.

Similar results exist for non-linear Lipschitz stationary feedback laws, the proof of which is a simple modification based on fixed point arguments and *a priori* estimates. For $r \in (0,1/2]$ we introduce the cutoff function $\chi_r \in C^\infty(\mathbb{R};[0,1])$ and the Lipschitz operator $\mathcal{K}_r : L^2(\Omega) \to L^2(\Omega)$ satisfying

$$\chi_r(x) = 1 \text{ for } x \in [0,r],\ \ \chi_r(x) = 0 \text{ for } x \in [2r,+\infty), \tag{2.2}$$

$$\mathcal{K}_r(y) = y\cdot\chi_r\left(||y||_{L^2(\Omega)}\right),\ \forall\, y \in L^2(\Omega). \tag{2.3}$$

Theorem 2.4. *Let $T \in (0,1]$. Let $r \in (0,1/2]$. Let $\varphi_i \in L^2(\Omega), 1 \leq i \leq n$ be given functions. Let $l_i : L^2(\Omega) \to \mathbb{R}, 1 \leq i \leq n$ be given bounded linear operators. For any $y_0 \in L^2(\Omega)$ the Cauchy problem*

$$\begin{cases} y_t = \Delta y + 1_\omega \mathcal{K}_r\Big(\sum_{i=1}^n l_i(y)\varphi_i\Big), \ (t,x) \in (0,T)\times\Omega, \\ y(t,x) = 0, \ (t,x) \in (0,T)\times\partial\Omega, \\ y(0,x) = y_0(x), \end{cases}$$

has a unique solution.

Proof. It suffices to prove the local (in time) existence of a unique solution. For ease of notations we simply denote the feedback law $1_\omega \mathcal{K}_r\Big(\sum_{i=1}^n l_i(y)\varphi_i\Big)$ as $\mathcal{H}(y)$, whose value is bounded by $C||y||_{L^2(\Omega)}$. The feedback law $\mathcal{H}$ is Lipschitz in $L^2(\Omega)$ with Lipschitz constant L.

For $||y_0||_{L^2(\Omega)} = R$, we select some $\widetilde{T} \leq \min\{(4C)^{-2}, (4L)^{-2}\}$ and denote $\mathcal{X}_{\widetilde{T}}$ by the Banach space $C^0([0,\widetilde{T}]; L^2(\Omega))$ with the corresponding norm given by the $C^0([0,\widetilde{T}]; L^2(\Omega))$-norm. We further define $\mathcal{X}_{\widetilde{T}}(2R)$ as the $2R$-radius closed ball in $\mathcal{X}_{\widetilde{T}}$. Next, for any $y \in \mathcal{X}_{\widetilde{T}}$ we denote $\mathcal{S}(y)$ as the unique solution of the Cauchy problem (2.1) on $[0,\widetilde{T}]$ with the initial state y_0 and the source term $\mathcal{H}(y)$. Thanks to Theorem 2.2, this solution is also in $L^2(0,\widetilde{T}; H_0^1(\Omega))$. By applying Theorem 2.2 one easily verifies that

$$||\mathcal{S}(y)||_{\mathcal{X}_{\widetilde{T}}} \leq 2R, \ \forall\, y \in \mathcal{X}_{\widetilde{T}}(2R),$$
$$||\mathcal{S}(y_1) - \mathcal{S}(y_2)||_{\mathcal{X}_{\widetilde{T}}} \leq \frac{1}{2}||y_1 - y_2||_{\mathcal{X}_{\widetilde{T}}}, \ \forall\, y_1, y_2 \in \mathcal{X}_{\widetilde{T}}(2R),$$

which, together with Banach fixed point theorem, yields the existence of the unique solution. □

2.2. **Spectral estimates.** Let us consider the Laplace operator with Dirichlet boundary condition $\Delta : H^2(\Omega) \cap H_0^1(\Omega) \to L^2(\Omega)$, there is a Hilbert orthogonal basis of $L^2(\Omega)$:

$$0 < \tau_1 \leq \tau_2 \leq \tau_3 \leq ... \leq \tau_n \leq ...,$$
$$-\Delta e_i = \tau_i e_i \text{ with } e_i|_{\partial\Omega} = 0.$$

Different τ_n may coincident, but every eigenvalue only have finite algebraic multiplicity. For any given positive number $\lambda > 0$, we define $N(\lambda)$ the number of eigenvalues (counting multiplicity) that are not strictly bigger than λ, *i.e.* $\tau_{N(\lambda)} \leq \lambda < \tau_{N(\lambda)+1}$. Moreover, the distribution of $\{\tau_k\}_{k=1}^\infty$ obeys Weyl's law: $N(\lambda) \sim (2\pi)^{-d}\omega_d \,\mathrm{vol}(\Omega)\lambda^{d/2}$, where ω_d is the volume of the unit ball. For ease of notations, in the following, if there is no confusion sometimes we simply denote N_λ by N.

Proposition 2.5. *The eigenfunctions $\{e_i\}_{i=1}^\infty$ satisfy*

1) *Orthonormal basis: $(e_i, e_j)_{L^2(\Omega)} = \delta_{ij}$.*
2) *(Unique continuation) The symmetric matrix J_N given below is invertible,*

$$J_N := \big((e_i, e_j)_{L^2(\omega)}\big)_{i,j=1}^N. \tag{2.4}$$

3) *(Tunneling estimate) There exist $C_0 > 0$ that is independent of τ_n such that*

$$||e_n||_{L^2(\omega)}^2 \geq C_0^{-1} e^{-C_0\sqrt{\tau_n}}.$$

4) *(Spectral estimate) There exist $C_1 \geq 1$ that is independent of $\lambda > 0$ such that*

$$||\sum_{i=1}^{N(\lambda)} a_i e_i||_{L^2(\omega)}^2 \geq C_1^{-1} e^{-C_1\sqrt{\lambda}} \sum_{i=1}^{N(\lambda)} a_i^2.$$

Proof. 1) This is a well-known result upon self-adjoint operators with compact resolvent.
2) This is a consequence of the unique continuation of the Dirichlet operator. One can see Barbu–Triggiani [3] for more general results.
3) First proved by Donnelly–Fefferman in [19] for compact Riemannian manifolds, the latest related result is given by Léautaud–Laurent [31] for hypoelliptic equations.
4) This highly non-trivial observation is found by Lebeau–Robbiano [34] via Carleman estimates, which is essentially the core of their proof on null controllability of the heat equation. Indeed the form $e^{\sqrt{\lambda}}$ is optimal once $\overline{\omega} \neq \Omega$, as illustrated in [32]. However, the optimality of the constant C_1, which clearly depends on the geometry of (Ω, ω), is still open. This kind of spectral inequalities has been extensively studied in the literature, for example, [28] for nodal sets of Laplace operator, [33] for bi-Laplace operators, [1, 44] for Laplace operators on measurable sets, [7] for degenerate one dimensional elliptic operators, [10] for Stokes operators, *etc.* □

As a direct consequence of property 4) of the preceding proposition, we have a quantitative estimate of J_N as quadratic form.

Lemma 2.6. *For $Y_{N(\lambda)} = (a_1, a_2, ..., a_{N(\lambda)})$, we have*

$$Y_{N(\lambda)}^T J_{N(\lambda)} Y_{N(\lambda)} \geq C_1^{-1} e^{-C_1\sqrt{\lambda}} ||Y_{N(\lambda)}||_2^2.$$

Actually, letting N be presenting N_λ, we get

$$Y_N^T J_N Y_N = \sum_{1\leq i,j\leq N} a_i \left(e_i, e_j\right)_{L^2(\omega)} a_j = \left(\sum_{i=1}^N a_i e_i, \sum_{j=1}^N a_j e_j\right)_{L^2(\omega)} = ||\sum_{i=1}^N a_i e_i||_{L^2(\omega)}^2 \geq C_1^{-1} e^{-C_1\sqrt{\lambda}} ||Y_N||_2^2.$$

2.3. **Rapid stabilization via Lyapunov function approach and explicit feedback law.** The following rapid stabilization result is inspired by the Lyapunov function idea introduced by Coron–Trélat [16], where it was used as an intermediate step for their proof of global controllability of steady states of non-linear parabolic equations in one dimensional space. This idea has been adapted to various models, for example [17] on global controllability of one dimensional wave equations and [43] for others. However, though relatively efficient and effectively calculable, no attempt on quantitative estimates has been made. Probably this is because in the proof some general theories as Kalman's rank condition and stabilization matrix are used.

Instead of using abstract stabilizing matrix arguments, here we construct precise Lyapunov functionals and quite surprisingly the spectral estimates by Lebeau–Robbiano are naturally used. That is the reason we get a quantitative rapid stabilization result with $Ce^{C\sqrt{\lambda}}$ estimates.

For any given $\lambda > 0$, we suggest control terms in forms of $\sum_i^{N(\lambda)} e_i|_\omega u_i(t)$ with $u_i(t) \in \mathbb{R}$, thus consider the following controlled problem:

$$y_t = \Delta y + 1_\omega \left(\sum_{i=1}^{N(\lambda)} e_i u_i(t)\right) \quad \text{in } \Omega, \tag{2.5}$$

$$y = 0 \quad \text{on } \partial\Omega. \tag{2.6}$$

In the rest part of this section, we simply denote N_λ by N. By decomposing

$$y(t,x) = \sum_{i=1}^\infty \left(y(t), e_i\right)_{L^2(\Omega)} e_i = \sum_{i=1}^\infty y_i(t) e_i, \tag{2.7}$$

$$1_\omega e_j = \sum_{i=1}^\infty \left(1_\omega e_j, e_i\right)_{L^2(\Omega)} e_i = \sum_{i=1}^\infty \left(e_i, e_j\right)_{L^2(\omega)} e_i, \tag{2.8}$$

and by defining

$$X_N(t) := \begin{pmatrix} y_1(t) \\ y_2(t) \\ ... \\ y_N(t) \end{pmatrix}, \quad U_N(t) := \begin{pmatrix} u_1(t) \\ u_2(t) \\ ... \\ u_N(t) \end{pmatrix}, \quad A_N := \begin{pmatrix} -\tau_1 & & & \\ & -\tau_2 & & \\ & & ... & \\ & & & -\tau_N \end{pmatrix}, \tag{2.9}$$

we know, thanks to the definition of J_N in (2.4), that the finite dimensional system $X_N(t)$ satisfies

$$\dot{X}_N(t) = A_N X_N(t) + J_N U_N(t). \tag{2.10}$$

For any given λ (thus N is given), for $\gamma_\lambda, \mu_\lambda > 0$ that will be fixed later on, we suggest the feedback law

$$U_N(y(t)) := -\gamma_\lambda X_N(t), \tag{2.11}$$

as well as the Lyapunov function:

$$V(y) := \mu_\lambda ||X_N||_2^2 + \left(P_N^\perp y, P_N^\perp y\right)_{L^2(\Omega)}, \ \forall y \in L^2(\Omega), \tag{2.12}$$

where $||X_N||_2^2$ is given by $X_N^T X_N = \sum_{i=1}^N y_i^2 = ||P_N y||_{L^2(\Omega)}^2$, P_N is the projection on $\text{Vect}\{e_i\}_{i=1}^N$, and $P_N^\perp$ be its co-projection. Thanks to Theorem 2.3, the closed-loop system (2.5)–(2.11) is well-posed. According to the preceding feedback law, $y(t)$ and $X_N(t)$ verify

$$\begin{gathered} \dot{X}_N(t) = A_N X_N(t) - \gamma_\lambda J_N X_N(t), \\ y_t = \Delta y - \gamma_\lambda 1_\omega \left(\sum_{i=1}^N e_i y_i(t) \right) \quad \text{in } \Omega, \\ y = 0 \quad \text{on } \partial\Omega. \end{gathered}$$

As we know from Theorem 2.4 that the solution of the above closed-loop system y is indeed in $C^0([0,T]; L^2(\Omega)) \cap L^2(0,T; H_0^1(\Omega))$, which further implies that X_N and $\frac{d}{dt}(X_N)$ in $C^0([0,T]; \mathbb{R}^N)$, $\frac{d}{dt} y$ and $\frac{d}{dt}(P_N^\perp y)$ in $L^2(0,T; H^{-1}(\Omega))$. Consequently

$$\frac{d}{dt} V(y(t)) = \mu_\lambda \frac{d}{dt} ||X_N||_2^2 + \frac{d}{dt} \left(P_N^\perp y, P_N^\perp y\right)_{L^2(\Omega)} = \mu_\lambda \frac{d}{dt} ||X_N||_2^2 + 2 \left\langle P_N^\perp y, \frac{d}{dt} y \right\rangle_{H_0^1(\Omega) \times H^{-1}(\Omega)},$$

holds in the distribution sense in $L^1(0,T)$.
On the one hand we know that

$$\mu_\lambda \frac{d}{dt} ||X_N||_2^2 = \mu_\lambda X_N^T \left(A_N^T + A_N - 2\gamma_\lambda J_N\right) X_N = 2\mu_\lambda X_N^T \left(A_N - \gamma_\lambda J_N\right) X_N \leq -2\mu_\lambda \gamma_\lambda C_1^{-1} e^{-C_1\sqrt{\lambda}} ||X_N||_2^2.$$

On the other hand we have

$$\begin{aligned} 2 \left\langle P_N^\perp y, \frac{d}{dt} y \right\rangle_{H_0^1(\Omega) \times H^{-1}(\Omega)} &= 2 \left\langle P_N^\perp y, \Delta y \right\rangle_{H_0^1(\Omega) \times H^{-1}(\Omega)} - 2\gamma_\lambda \left(P_N^\perp y, 1_\omega \left(\sum_{i=1}^N e_i y_i(t) \right) \right)_{L^2(\Omega)}, \\ &= -2 \sum_{i=N+1}^\infty \tau_i y_i^2 - 2\gamma_\lambda \left(P_N^\perp y, 1_\omega (P_N y)\right)_{L^2(\Omega)}, \\ &\leq -2\lambda ||P_N^\perp y||_{L^2(\Omega)}^2 + 2\gamma_\lambda ||P_N^\perp y||_{L^2(\Omega)} ||P_N y||_{L^2(\Omega)}, \\ &\leq -2\lambda ||P_N^\perp y||_{L^2(\Omega)}^2 + \lambda ||P_N^\perp y||_{L^2(\Omega)}^2 + \frac{\gamma_\lambda^2}{\lambda} ||P_N y||_{L^2(\Omega)}^2, \\ &= -\lambda ||P_N^\perp y||_{L^2(\Omega)}^2 + \frac{\gamma_\lambda^2}{\lambda} ||X_N||_2^2. \end{aligned}$$

Therefore,

$$\frac{d}{dt}V(y(t)) \leq -2\mu_\lambda\gamma_\lambda C_1^{-1}e^{-C_1\sqrt{\lambda}}||X_N||_2^2 - \lambda||P_N^\perp y||^2_{L^2(\Omega)} + \frac{\gamma_\lambda^2}{\lambda}||X_N||_2^2.$$

Motivated from the above estimate, we choose

$$\gamma_\lambda := C_1e^{C_1\sqrt{\lambda}}\lambda, \;\; \mu_\lambda := \frac{\gamma_\lambda^2}{\lambda^2} = C_1^2e^{2C_1\sqrt{\lambda}}, \tag{2.13}$$

which further yields

$$\begin{aligned}\frac{d}{dt}V(y(t)) &\leq -2\mu_\lambda\lambda||X_N||_2^2 - \lambda||P_N^\perp y||^2_{L^2(\Omega)} + \mu_\lambda\lambda||X_N||_2^2,\\ &\leq -\mu_\lambda\lambda||X_N||_2^2 - \lambda||P_N^\perp y||^2_{L^2(\Omega)},\\ &= -\lambda\left(\mu_\lambda||X_N||_2^2 + ||P_N^\perp y||^2_{L^2(\Omega)}\right) = -\lambda V(y(t)).\end{aligned}$$

Since $\mu_\lambda \geq 1$ for $C_1 \geq 1$, we know that,

$$||y(t)||^2_{L^2(\Omega)} \leq V(y(t)) \leq e^{-\lambda t}V(y(0)) \leq e^{-\lambda t}\mu_\lambda||y(0)||^2_{L^2(\Omega)} \leq C_1^2e^{2C_1\sqrt{\lambda}}e^{-\lambda t}||y(0)||^2_{L^2(\Omega)},$$

thus

$$||y(t)||_{L^2(\Omega)} \leq C_1e^{C_1\sqrt{\lambda}}e^{-\frac{\lambda}{2}t}||y(0)||_{L^2(\Omega)}.$$

Moreover since the control (feedback) is given by

$$1_\omega f(t,x) = -\gamma_\lambda 1_\omega\left(\sum_{i=1}^N e_iX_i(t)\right),$$

we know that

$$||f(t,\cdot)||_{L^2(\Omega)} \leq \gamma_\lambda||X_{N(\lambda)}||_2 \leq \gamma_\lambda||y(t)||_{L^2(\Omega)} \leq \lambda C_1^2e^{2C_1\sqrt{\lambda}}e^{-\frac{\lambda}{2}t}||y(0)||_{L^2(\Omega)}.$$

By applying the above explicit feedback law, we get the following rapid stabilization result. For any $\lambda > 0$ we define an explicit stationary feedback law $\mathcal{F}_\lambda : L^2(\Omega) \to L^2(\Omega)$,

$$\mathcal{F}_\lambda y := -\gamma_\lambda\left(\sum_{i=1}^{N(\lambda)}\Big(y, e_i\Big)_{L^2(\Omega)}e_i\right) = -\gamma_\lambda P_{N(\lambda)}y \text{ with } \gamma_\lambda = C_1e^{C_1\sqrt{\lambda}}\lambda, \tag{2.14}$$

where $P_{N(\lambda)}$ is the projection on the sub-space spanned by $\{e_i\}_{i=1}^{N(\lambda)}$, and $N(\lambda)$ is the number of eigenvalues (counting multiplicity) that are not strictly bigger than λ. Clearly, there exists $C_2 \geq 2C_1$ such that for all $\lambda > 0$,

$$\lambda C_1^2e^{2C_1\sqrt{\lambda}},\; C_1^2e^{2C_1\sqrt{\lambda}},\; \lambda C_1e^{C_1\sqrt{\lambda}},\; C_1e^{C_1\sqrt{\lambda}} \leq C_2e^{C_2\sqrt{\lambda}} =: \frac{1}{r_\lambda}. \tag{2.15}$$

Theorem 2.7. *For any $\lambda > 0$ the closed-loop system*

$$\begin{aligned}y_t &= \Delta y + 1_\omega\mathcal{F}_\lambda y \;\; \text{in } \Omega,\\ y &= 0 \;\; \text{on } \partial\Omega,\end{aligned}$$

is exponentially stable. More precisely, for any $s \in \mathbb{R}$ the Cauchy problem

$$\begin{aligned}y_t &= \Delta y - \gamma_\lambda 1_\omega\left(\sum_{i=1}^{N(\lambda)}\Big(y(t), e_i\Big)_{L^2(\Omega)}e_i\right),\; \forall (t,x) \in [s,+\infty)\times\Omega,\\ y(t,x) &= 0,\; \forall (t,x) \in [s,+\infty)\times\partial\Omega,\\ y(s,x) &= y_0(x),\end{aligned}$$

has a unique solution in $C^0([s,+\infty);L^2(\Omega))\cap L^2_{loc}(s,+\infty;H^1_0(\Omega))$*, and this unique solution verifies*

$$||y(t)||_{L^2(\Omega)} \leq C_1 e^{C_1\sqrt{\lambda}} e^{-\frac{\lambda}{2}(t-s)} ||y_0||_{L^2(\Omega)},\ \forall\, t\in[s,+\infty), \tag{2.16}$$

$$||\mathcal{F}_\lambda y(t)||_{L^2(\Omega)} \leq C_2 e^{C_2\sqrt{\lambda}} e^{-\frac{\lambda}{2}(t-s)} ||y_0||_{L^2(\Omega)},\ \forall\, t\in[s,+\infty). \tag{2.17}$$

2.4. **Rapid stabilization for higher regularity.** Actually the feedback law $\mathcal{F}_\lambda$ presented in (2.14) and Theorem 2.7 also stabilizes the system in $H^1_0(\Omega)$ space, let us briefly comment on this issue without going into details. We refer to Brezis [6, Chapter 9–10] and Lions–Magenes [37] for related well-posedness results.

Let the Hilbert space $H^1_0(\Omega)$ be endowed with scalar product $\int_\Omega \nabla u\cdot\nabla v$. The eigenfunctions $\{e_i/\sqrt{\tau_i}\}_{i=1}^\infty$ form an orthonormal basis of $H^1_0(\Omega)$. For any $y_0\in H^1_0(\Omega)$ and any $f(t,x)\in L^2(0,T;L^2(\Omega))$, the Cauchy problem (2.1) admits a unique solution $y(t)$ in $C^0([0,T];H^1_0(\Omega))\cap L^2(0,T;H^2(\Omega))$, thanks to the *a priori* estimate,

$$||\Delta y||^2_{L^2(0,t;L^2(\Omega))} + ||\nabla y(t)||^2_{L^2(\Omega)} \leq ||\nabla y_0||^2_{L^2(\Omega)} + ||f||^2_{L^2(0,t;L^2(\Omega))},\ \forall\, t\in(0,T].$$

We further adapt the notations in the preceding section, and even the same choice of γ_λ. For any $\lambda>0$, we consider the Lyapunov functional on $H^1_0(\Omega)$:

$$V_1(y) := \widetilde{\mu}_\lambda ||X_{N(\lambda)}||_2^2 + ||P^T_{N(\lambda)}y||^2_{H^1_0(\Omega)},\ \forall\, y\in H^1_0(\Omega),$$

with $\widetilde{\mu}_\lambda := \gamma_\lambda^2/\lambda$. It is actually equivalent to the $H^1_0(\Omega)$ norm by

$$\frac{1}{1+\lambda}||y||^2_{H^1_0(\Omega)} \leq V_1(y) \leq \left(1+\frac{\widetilde{\mu}_\lambda}{\tau_1}\right)||y||^2_{H^1_0(\Omega)},\ \forall\, y\in H^1_0(\Omega).$$

Then, similar estimation implies that the solution $y(t)$ of the closed-loop system (2.5)–(2.11) with feedback law $\mathcal{F}_\lambda$ verifies, where N means $N(\lambda)$,

$$\begin{aligned}
\dot{V}_1(y(t)) &\leq -2\widetilde{\mu}_\lambda\gamma_\lambda C_1^{-1} e^{-C_1\sqrt{\lambda}}||X_N||_2^2 - 2\left(P_N^T\Delta y, \Delta y - \gamma_\lambda 1_\omega(P_N y)\right)_{L^2(\Omega)},\\
&\leq -2\lambda\widetilde{\mu}_\lambda||X_N||_2^2 - \lambda||P_N^T y||^2_{H^1_0(\Omega)} - ||P_N^T\Delta y||^2_{L^2(\Omega)} + ||P_N^T\Delta y||^2_{L^2(\Omega)} + \gamma_\lambda^2||P_N y||^2_{L^2(\Omega)},\\
&\leq -2\lambda\widetilde{\mu}_\lambda||X_N||_2^2 - \lambda||P_N^T y||^2_{H^1_0(\Omega)} + \lambda\widetilde{\mu}_\lambda||X_N||_2^2,\\
&\leq -\lambda V_1(y(t)).
\end{aligned}$$

The stabilization on $H^1_0(\Omega)$ space becomes more important when it is combined with the Sobolev embedding $H^1(\Omega)\subseteq L^p(\Omega)$ with $p=\frac{2d}{d-2}$. As for stabilization for even higher regularities, $H^2(\Omega)\cap H^1_0(\Omega)$ for example, probably one needs to replace the control setting $1_\omega f$ by $\chi_\omega f$ with some smooth truncated function $\chi_\omega(x)$ that is supported in ω and equals to 1 in an open subset $\omega_1\subset\omega$.

2.5. **Rapid stabilization for nonlinear systems.** Indeed the same feedback $\mathcal{F}_\lambda$ also stabilizes some nonlinear systems provided the initial data is small, we also comment on this issue without going into details. Again we adapt the notations and the choice of γ_λ in Section 2.3.

Let us consider the two dimensional subcritical heat equation with $\omega\subset\Omega\subset\mathbb{R}^2$,

$$\begin{aligned}
y_t &= \Delta y + y^3 - \gamma_\lambda 1_\omega\left(P_{N(\lambda)}y\right) \quad \text{in } \Omega,\\
y &= 0 \quad \text{on } \partial\Omega.
\end{aligned}$$

By defining

$$Y_{N(\lambda)}(t) := \left((y^3(t),e_i)_{L^2(\Omega)}\right)_{i=1}^{N(\lambda)},$$

and by denoting N by $N(\lambda)$, we know that

$$\dot{X}_N(t) = A_N X_N(t) - \gamma_\lambda J_N X_N(t) + Y_N(t).$$

As a consequence

$$\mu_\lambda \frac{d}{dt}||X_N||_2^2 = \mu_\lambda X_N^T \left(A_N^T + A_N - 2\gamma_\lambda J_N\right) X_N + \mu_\lambda \left(Y_N^T X_N + X_N^T Y_N\right)$$
$$\leq -2\mu_\lambda \lambda ||X_N||_2^2 - 2\mu_\lambda ||\nabla P_N y||_{L^2(\Omega)}^2 + 2\mu_\lambda \int_\Omega y^3 \left(P_N y\right) dx,$$

and

$$\frac{d}{dt}\left(P_N^\perp y, P_N^\perp y\right)_{L^2(\Omega)} = 2\left\langle P_N^\perp y, \Delta y - \gamma_\lambda 1_\omega \left(P_N y\right) + y^3 \right\rangle_{H_0^1(\Omega)\times H^{-1}(\Omega)},$$
$$\leq -\frac{3}{2}\lambda ||P_N^\perp y||_{L^2(\Omega)}^2 - \frac{1}{2}||\nabla P_N^\perp y||_{L^2(\Omega)}^2 + \lambda ||P_N^\perp y||_{L^2(\Omega)}^2 + \frac{\gamma_\lambda^2}{\lambda}||X_N||_2^2 + 2\int_\Omega y^3 \left(P_N^\perp y\right) dx,$$
$$= -\frac{\lambda}{2}||P_N^\perp y||_{L^2(\Omega)}^2 + \mu_\lambda \lambda ||X_N||_2^2 - \frac{1}{2}||\nabla P_N^\perp y||_{L^2(\Omega)}^2 + 2\int_\Omega y^3 \left(P_N^\perp y\right) dx.$$

By recalling Ladyzhenskaya's inequality

$$||f||_{L^4(\Omega)} \leq \sqrt{2}||f||_{L^2(\Omega)}^{1/2}||\nabla f||_{L^2(\Omega)}^{1/2},\ \forall\ f \in H_0^1(\Omega),$$

we further get

$$\frac{d}{dt}V(y(t)) \leq \left(-\frac{\lambda}{2}\right) V(y(t)) - \frac{1}{2}||\nabla y||_{L^2(\Omega)}^2 + 2\mu_\lambda ||y||_{L^4(\Omega)}^3 ||P_N y||_{L^4(\Omega)} + 2||y||_{L^4(\Omega)}^3 ||P_N^\perp y||_{L^4(\Omega)},$$
$$\leq \left(-\frac{\lambda}{2}\right) V(y(t)) - \frac{1}{2}||\nabla y||_{L^2(\Omega)}^2 + 16\mu_\lambda ||y||_{L^2(\Omega)}^2 ||\nabla y||_{L^2(\Omega)}^2,$$
$$\leq \left(-\frac{\lambda}{2}\right) V(y(t)) - ||\nabla y||_{L^2(\Omega)}^2 \left(\frac{1}{2} - 16\mu_\lambda V(y(t))\right).$$

Let us define

$$R_\lambda := \frac{r_\lambda}{8} \leq \left(8\mu_\lambda\right)^{-1}. \tag{2.18}$$

Then for any initial data y_0 satisfying $||y_0||_{L^2(\Omega)} \leq R_\lambda$ we have

$$V(y(t)) \leq e^{-\frac{\lambda}{2}t} V(y(0)),\ \forall\ t \geq 0,$$

which further implies,

$$||y(t)||_{L^2(\Omega)}^2 \leq V(y(t)) \leq e^{-\frac{\lambda}{2}t} V(y(0)) \leq e^{-\frac{\lambda}{2}t}\mu_\lambda ||y(0)||_{L^2(\Omega)}^2 \leq C_1^2 e^{2C_1\sqrt{\lambda}} e^{-\frac{\lambda}{2}t}||y(0)||_{L^2(\Omega)}^2.$$

Therefore the unique solution of the closed-loop system decays exponentially as

$$||y(t)||_{L^2(\Omega)} \leq C_1 e^{C_1\sqrt{\lambda}} e^{-\frac{\lambda}{4}t}||y_0||_{L^2(\Omega)},\ \forall\ ||y_0||_{L^2(\Omega)} \leq R_\lambda,\ \forall\ t \in [0, +\infty).$$

In the end, let us remark that without the feedback law the solution may blow up.

3. Null controllability with optimal cost estimates

Armed with the $Ce^{C\sqrt{\lambda}}$ estimates (2.16)–(2.17), exactly the same procedure proposed in [15, 45, 46] by using piecewise stabilizing controls leads to the null controllability. In this section we construct similar feedback laws while keeping an extra attention on control costs. Two different kind of precise feedback laws (control) are considered with control costs $Ce^{\frac{C}{T^{1+\varepsilon}}}$ and $Ce^{\frac{C}{T}}$ respectively.

We mainly focus on the following weaker result, Theorem 3.1, for which the feedback law (control) is nice and the calculation is easy. After that easy modification leads to stronger cases, Corollary 3.2 and Theorem 3.3.

Theorem 3.1. *There exists $C_3 > 0$ such that, for any $T \in (0,1)$ and for any $y_0 \in L^2(\Omega)$, we find an explicit control $f(t,x)$ for the control system (1.1)–(1.2) such that the unique solution with initial data $y(0,x) = y_0(x)$ verifies $y(T) = 0$. Moreover, the controlling cost is given by,*

$$||1_\omega f(t,x)||_{L^\infty(0,T;L^2(\Omega))} \leq e^{C_3/T^2}||y_0||_{L^2(\Omega)}.$$

Proof of Theorem 3.1. We only treat the case $1/T$ be integer to simplify the presentation. Let us take some $\Gamma > 0$ independent of $T \in (0,1)$ that will be fixed later on.

Control design. *Let $T = \frac{1}{n_T}$ with $n_T \in \mathbb{N}^*$. We define,*

$$T_n := T - \frac{1}{n},\ I_n := [T_n, T_{n+1}),\ \lambda_n := \Gamma^2 n^4 \text{ for } \forall\, n \geq n_T := \frac{1}{T}, \tag{3.1}$$

for any $n \geq n_T$ we consider the control (feedback law) as $\mathcal{F}_{\lambda_n}$ on interval I_n.

More precisely, first on I_{n_T} we consider the closed-loop system (1.1)–(1.2) with feedback law $\mathcal{F}_{\lambda_{n_T}}$ and $y(0,x) = y_0(x)$. According to Theorem 2.7 this system has a unique solution $\widetilde{y}|_{\bar{I}_{n_T}}$. Next, we consider the closed-loop system with feedback law $\mathcal{F}_{\lambda_{n_T+1}}$ and $y(T_{n_T+1},x) := \widetilde{y}(T_{n_T+1},x)$ on I_{n_T+1}, which, again, admit a unique solution $\widetilde{y}|_{\bar{I}_{n_T+1}}$. We continue this procedure on $\{I_n\}_{n=n_T}^\infty$ to find eventually a function $\widetilde{y}|_{[0,T)} \in C^0([0,T);L^2(\Omega))$ such that $\widetilde{y}(T) := \lim_{t\to T^-}\widetilde{y}(t) = 0$, and that

$$\widetilde{y}|_{[0,T]} \text{ is the solution of the Cauchy problem (2.1) with control } 1_\omega f|_{I_n} = 1_\omega \mathcal{F}_{\lambda_n}\widetilde{y}|_{I_n}, \forall n \geq n_T.$$

We denote the above constructed solution by $y(t)$ which, by Theorem 2.7, verifies

$$||y(t)||_{L^2(\Omega)} \leq C_1 e^{C_1\Gamma n^2} e^{-\frac{\Gamma^2 n^4}{2}(t-T_n)}||y(T_n)||_{L^2(\Omega)},\ \forall t \in I_n,\ \forall n \geq n_T, \tag{3.2}$$

$$||\mathcal{F}_{\lambda_n} y(t)||_{L^2(\Omega)} \leq C_2 e^{C_2\Gamma n^2} e^{-\frac{\Gamma^2 n^4}{2}(t-T_n)}||y(T_n)||_{L^2(\Omega)},\ \forall t \in I_n,\ \forall n \geq n_T. \tag{3.3}$$

Therefore, for $n \geq n_T + 1$ the value of the solution on T_n is controlled by,

$$||y(T_n)||_{L^2(\Omega)} \leq \left(\prod_{k=n_T}^{n-1} C_1 e^{C_1\Gamma k^2} e^{-\frac{\Gamma^2 k^2}{4}}\right)||y_0||_{L^2(\Omega)}. \tag{3.4}$$

Inspired by the preceding estimates, we choose the constant $\Gamma > 0$ be such that

$$C_1 e^{C_1\Gamma n^2}, C_2 e^{C_2\Gamma n^2} \leq e^{\frac{\Gamma^2}{16}n^2},\ \forall n \in \mathbb{N}^*. \tag{3.5}$$

The above choice of Γ, combined with (3.4), lead to

$$||y(T_n)||_{L^2(\Omega)} \leq \left(\prod_{k=n_T}^{n-1} e^{-\frac{3\Gamma^2}{16}k^2}\right)||y_0||_{L^2(\Omega)},\ \forall\, n \geq n_T + 1. \tag{3.6}$$

Essentially, it already implies that $y(T_n)$ is strictly decaying to 0 at time T. Next, we concentrate on its cost, *i.e.* the norm of the control term. From (3.2), (3.3), (3.5), and (3.6) we know that for $n \geq n_T + 1$ and $t \in I_n$,

$$\begin{aligned}||y(t)||_{L^2(\Omega)}, ||\mathcal{F}_{\lambda_n} y(t)||_{L^2(\Omega)} &\leq e^{\frac{\Gamma^2}{16}n^2}\left(\prod_{k=n_T}^{n-1} e^{-\frac{3\Gamma^2}{16}k^2}\right)||y_0||_{L^2(\Omega)},\\ &\leq \exp\left(-\frac{\Gamma^2}{16}\Big(3\big(\sum_{k=n_T}^{n-1} k^2\big) - n^2\Big)\right)||y_0||_{L^2(\Omega)} \leq ||y_0||_{L^2(\Omega)}.\end{aligned}$$

On the other hand, for $n = n_T$ we know that

$$||y(t)||_{L^2(\Omega)}, ||\mathcal{F}_\lambda y(t)||_{L^2(\Omega)} \leq e^{\frac{\Gamma^2}{16}n_T^2}||y_0||_{L^2(\Omega)} = e^{\frac{\Gamma^2}{16T^2}}||y_0||_{L^2(\Omega)} = e^{\frac{C_3}{T^2}}||y_0||_{L^2(\Omega)}, \tag{3.7}$$

where $C_3 := \frac{\Gamma^2}{16}$.

In conclusion, the constructed solution $y(t,x)$ with control $1_\omega f(t,x)$ satisfies

$$||y(t)||_{L^2(\Omega)} \text{ and } ||1_\omega f(t,\cdot)||_{L^2(\Omega)} \longrightarrow 0^+, \text{ as } t \to T^-,$$

$$||y(t)||_{L^2(\Omega)} \text{ and } ||1_\omega f(t,\cdot)||_{L^2(\Omega)} \leq e^{\frac{\Gamma^2}{16T^2}}||y_0||_{L^2(\Omega)}, \ \forall\, t \in [0,T],$$

which completes the proof. □

Actually Theorem 3.1 can be easily improved to the following one via simple modification on the choice of T_n and λ_n.

Corollary 3.2. *For any $\varepsilon \in (0,1)$, there exists $C_3^\varepsilon > 0$ such that, for any $T \in (0,1)$, for any $y_0 \in L^2(\Omega)$, we can find an explicit control $f(t,x)$ for the control system (1.1)–(1.2) such that the unique solution with initial data $y(0,x) = y_0(x)$ verifies $y(T,x) = 0$. Moreover, the cost is controlled by*

$$||1_\omega f(t,x)||_{L^\infty(0,T;L^2(\Omega))} \leq e^{C_3^\varepsilon/T^{1+\varepsilon}}||y_0||_{L^2}.$$

Idea of the proof. Indeed, it suffices to take

$$T_{k,n} := T - \frac{1}{n^k}, \ \lambda_{k,n} := \Gamma_\varepsilon^2 n^{2(k+1)} \text{ for } \forall n \geq n_T := \frac{1}{T^{1/k}}, \tag{3.8}$$

for some $k \geq 1/\varepsilon$, and to find some suitable Γ_ε. We observe that the energy decay on interval $I_{k,n}$ is dominated by

$$C_1 e^{C_1\Gamma_\varepsilon n^{k+1}} e^{-c\Gamma_\varepsilon^2 n^{k+1}},$$

which allows us to find some Γ_ε satisfying (3.5) type estimates. □

However, $e^{C/T^{1+\varepsilon}}$ is the best estimate that we can achieve from partitions of type (3.8), which is slightly weaker than the optimal cost [38]: $e^{C/T}$. Eventually with another choice of partition we can also get the optimal cost from stabilization approach.

Theorem 3.3 (Optimal cost)**.** *There exists $C_3^0 > 0$ such that, for any $T \in (0,1)$ and for any $y_0 \in L^2(\Omega)$, we find an explicit control $f(t,x)$ for the control system (1.1)–(1.2) such that the unique solution with initial data $y(0,x) = y_0(x)$ verifies $y(T,x) = 0$. Moreover, the controlling cost is given by*

$$||1_\omega f(t,x)||_{L^\infty(0,T;L^2(\Omega))} \leq e^{C_3^0/T}||y_0||_{L^2}.$$

Proof. As illustrated above we adapt another type of construction to get this optimal result. For the ease of presentation, we only consider the case $1/T = 2^{n_0}$ with $n_0 \in N^*$. More precisely, we consider the following partition as well as the piecewise controlling method explained in the proof of Theorem 3.1 (see Control design),

$$T_n^0 := 2^{-n_0}\left(1 - \frac{1}{2^n}\right), \ I_n^0 := [T_n^0, T_{n+1}^0), \ \lambda_n^0 := Q^2 2^{2(n_0+n)} \text{ for } \forall\, n \geq 0,$$

where $Q > 0$ is a given constant satisfying

$$C_1 e^{C_1 Q m}, C_2 e^{C_2 Q m} \leq e^{\frac{Q^2}{16}m}, \ \forall\, m \geq 1.$$

Suppose that $y(t)$ is the unique solution satisfying the designed control, then for $n \geq 1$ we are able to estimate $y(T_n)$ by

$$\begin{aligned}
||y(T_n)||_{L^2(\Omega)} &\leq \left(\prod_{k=0}^{n-1} C_1 e^{C_1\sqrt{\lambda_k^0}} e^{-\frac{\lambda_k^0}{2}2^{-(n_0+k+1)}}\right)||y_0||_{L^2(\Omega)},\\
&\leq \left(\prod_{k=0}^{n-1} C_1 e^{C_1 Q 2^{n_0+k}} e^{-\frac{Q^2}{4}2^{n_0+k}}\right)||y_0||_{L^2(\Omega)},\\
&\leq \left(\prod_{k=0}^{n-1} e^{-\frac{Q^2}{8}2^{n_0+k}}\right)||y_0||_{L^2(\Omega)}.
\end{aligned}$$

For $n \geq 1$, the preceding estimate further implies that the control term on $t \in I_n^0$ satisfies,

$$||1_\omega \mathcal{F}_{\lambda_n^0} y(t)||_{L^2(\Omega)} \leq C_2 e^{C_2 Q 2^{n_0+n}}||y(T_n)||_{L^2(\Omega)} \leq e^{\frac{Q^2}{16}2^{n_0+n}}\left(\prod_{k=0}^{n-1} e^{-\frac{Q^2}{8}2^{n_0+k}}\right)||y_0||_{L^2(\Omega)} \leq ||y_0||_{L^2(\Omega)}.$$

Therefore, the $L^\infty(0,T;L^2(\Omega))$ norm of the control term $1_\omega f$ is dominated by its $L^\infty(T_0^0,T_1^0;L^2(\Omega))$ norm. As a consequence, we know that for any $t \in [0,T]$,

$$||1_\omega f(t,x)||_{L^2(\Omega)} \leq C_2 e^{C_2 Q 2^{n_0}}||y_0||_{L^2(\Omega)} \leq e^{\frac{C_3^0}{T}}||y_0||_{L^2(\Omega)} \text{ with } C_3^0 = \frac{Q^2}{16}.$$

□

Remark 3.4. *It is noteworthy that the $e^{C/T}$ type cost is optimal to many other systems, for example the Stokes system [10] where similar spectral estimates are proved.*

4. Finite time stabilization

In this section, we construct T-periodic proper feedback laws that stabilize system (1.1)–(1.2) in finite time: Theorem 4.4.

4.1. Time-varying feedback laws and finite time stabilization.

We are interested in time-varying feedback laws, more precisely proper feedback laws. The following definition of time-varying feedback laws that allows the closed-loop system admit a unique solution borrows directly from the paper [18].

First, we recall the closed-loop system associated to a time-varying feedback law U.

$$\begin{cases} y_t = \Delta y + 1_\omega U(t;y),\ \forall\ (t,x) \in (s,+\infty)\times\Omega,\\ y = 0 \ \text{ on } \partial\Omega. \end{cases} \tag{4.1}$$

Definition 4.1. *Let $s_1 \in \mathbb{R}$ and $s_2 \in \mathbb{R}$ be given such that $s_1 < s_2$. Let*

$$\begin{array}{rcl} U : [s_1,s_2]\times L^2(\Omega) & \to & L^2(\Omega)\\ (t;y) & \mapsto & U(t;y). \end{array}$$

Let $t_1 \in [s_1,s_2]$, $t_2 \in (t_1,s_2]$, and $y_0 \in L^2(\Omega)$. A solution on $[t_1,t_2]$ to the Cauchy problem associated to the closed-loop system (4.1) with initial data y_0 at time t_1 is some $y : [t_1,t_2] \to L^2(\Omega)$ such that

$$t \in (t_1,t_2) \mapsto f(t,x) := U(t;y(t)) \in L^2(t_1,t_2;L^2(\Omega)),$$

y is a solution (see Definition 2.1) of (2.1) with initial data y_0 at time t_1 and the above $1_\omega f(t,x)$.

The so-called *proper* feedback laws is a time-varying feedback law such that the closed-loop system always admit a unique solution.

Definition 4.2. *Let $s_1 \in \mathbb{R}$ and $s_2 \in \mathbb{R}$ be given such that $s_1 < s_2$. A proper feedback law on $[s_1, s_2]$ is an application*

$$\begin{array}{rcl} U : [s_1, s_2] \times L^2(\Omega) & \to & L^2(\Omega) \\ (t; y) & \mapsto & U(t; y). \end{array}$$

such that, for every $t_1 \in [s_1, s_2]$, for every $t_2 \in (t_1, s_2]$, and for every $y_0 \in L^2(\Omega)$, there exists a unique solution on $[t_1, t_2]$ to the Cauchy problem associated to the closed-loop system (4.1) with initial data y_0 at time t_1 according to Definition 4.1.

A proper feedback law is an application U

$$\begin{array}{rcl} U : \mathbb{R} \times L^2(\Omega) & \to & L^2(\Omega) \\ (t; y) & \mapsto & U(t; y). \end{array}$$

such that, for every $s_1 \in \mathbb{R}$ and for every $s_2 \in \mathbb{R}$ satisfying $s_1 < s_2$, the feedback law restricted to $[s_1, s_2] \times L^2(\Omega)$ is a proper feedback law on $[s_1, s_2]$.

For a **proper** feedback law, one can define the **flow** $\Phi : \Delta \times L^2(\Omega) \to L^2(\Omega)$, with $\Delta := \{(t, s);\, t > s\}$ associated to this feedback law: $\Phi(t, s; y_0)$ is the value at time t of the solution y to the closed-loop system (4.1) which is equal to y_0 at time s.

Finally we state the exact definition of the finite time stabilization.

Definition 4.3 (Finite time stabilization of the heat equation)**.** *Let $T > 0$. A T-periodic proper feedback law U stabilizes the heat equation in finite time, if the flow Φ of the closed-loop system (4.1) verifies,*

(i) *(2T stabilization)* $\Phi(2T + t, t; y_0) = 0, \quad \forall t \in \mathbb{R},\ \forall\, y_0 \in L^2(\Omega)$.

(ii) *(Uniform stability) For every $\delta > 0$, there exists $\eta > 0$ such that*

$$\left(||(y_0||_{L^2(\Omega)} \leq \eta\right) \Rightarrow \left(||\Phi(t, t'; y_0)||_{L^2(\Omega)} \leq \delta,\ \forall\, t' \in \mathbb{R},\ \forall\, t \in (t', +\infty)\right).$$

4.2. **Finite time stabilization.** We only work on the case when $1/T$ is an integer, as the other cases can be trivially treated by time transition.

Different from null controllability we do not pay extra attention to the "stabilizing cost" with respect to T. Indeed, we directly apply the feedback law constructed in the preceding section as combination of stationary feedback laws $\mathcal{F}_{\lambda_n}$ on interval I_n. Then, $\mathcal{F}_{\lambda_n}$ can be regarded as "λ_n frequency" feedback, which is sensible with respect to the states for large n. For example, for some given $y_0 \in L^2(\Omega)$ we consider the Cauchy problem of the closed-loop system with $\mathcal{F}_{\lambda_n}$ and $y(T_n) = y_0$. Thanks to Theorem 2.7, $||y(t)||$ is uniformly bounded by $C_1 e^{C_1\sqrt{\lambda}}||y_0||$ on I_n. By letting n tends to ∞, we are not allowed to get "uniform stability", as commented in Remark 1.4. Therefore, we introduce some truncated operator on feedback laws, especially for high frequencies λ, to guarantee "uniform stability". However, in this case the natural *a priori* bound for the Cauchy problem that can be expected is $C_\varepsilon||y_0|| + \varepsilon, \forall \varepsilon > 0$. As a result the cost can not be bounded by $C||y_0||$, that explains why we do not characterize the stabilizing cost in details with respect to T. However, thanks to the precise construction of the feedback laws that will be presented in this section, for any given T an effectively computable stabilizing cost depending on "starting time" and "initial state" can be obtained.

Before stating the detailed stabilizing theorem, we first recall the following notations and facts:

truncated operator $\mathcal{K}_r : L^2(\Omega) \to L^2(\Omega)$ such that $||\mathcal{K}_r(y)||_{L^2(\Omega)} \leq \min\{1, ||y||_{L^2(\Omega)}\}$, defined in (2.3),

stationary feedback law $\mathcal{F}_\lambda : L^2(\Omega) \to L^2(\Omega)$ with $||\mathcal{F}_\lambda|| \leq C_2 e^{C_2\sqrt{\lambda}} = (r_\lambda)^{-1}$, given by (2.14),

$$T_n := T - \frac{1}{n}, \quad I_n := [T_n, T_{n+1}), \quad \lambda_n := \Gamma^2 n^4 \text{ for } \forall n \geq n_T := \frac{1}{T}, \text{ defined in (3.1).}$$

Next, we show that the feedback law, $\mathcal{K}_{r_{\lambda_n}}(\mathcal{F}_{\lambda_n}y)$, satisfies

$$||\mathcal{K}_{r_{\lambda_n}}(\mathcal{F}_{\lambda_n}y)||_{L^2(\Omega)} \leq \min\{1, \sqrt{2||y||_{L^2(\Omega)}}\}. \tag{4.2}$$

Indeed, by the choice of $\mathcal{K}_{r_{\lambda_n}}$ and $\mathcal{F}_{\lambda_n}$ we prove the preceding inequality by two steps.

If $||\mathcal{F}_{\lambda_n}y||_{L^2(\Omega)} \leq 2r_{\lambda_n}$, then

$$||\mathcal{K}_{r_{\lambda_n}}(\mathcal{F}_{\lambda_n}y)||_{L^2(\Omega)} \leq ||\mathcal{F}_{\lambda_n}y||_{L^2(\Omega)} \leq \sqrt{2r_{\lambda_n}||\mathcal{F}_{\lambda_n}y||_{L^2(\Omega)}} \leq \sqrt{2||y||_{L^2(\Omega)}},$$

moreover, it is also bounded by 1 as $2r_{\lambda_n} \leq 1$.

If $||\mathcal{F}_{\lambda_n}y||_{L^2(\Omega)} > 2r_{\lambda_n}$, then by the choice of $\mathcal{K}_{r_{\lambda_n}}$ we know that $\mathcal{K}_{r_{\lambda_n}}(\mathcal{F}_{\lambda_n}y) = 0$.

Theorem 4.4 (Semi-global finite time stabilization of the heat equation)**.** *Let $T = 1/n_T \in (0,1)$ with $n_T \in \mathbb{N}^*$. Let $\Lambda \geq 1$. For any integer $N_T > n_T$, the T-periodic feedback law $U(t;y) : \mathbb{R} \times L^2(\Omega) \to L^2(\Omega)$ given by*

$$U\big|_{[0,T)\times L^2(\Omega)}(t;y) := \begin{cases} \mathcal{F}_{\lambda_n}y, \quad \forall\, y \in L^2(\Omega), \forall\, t \in I_n, \forall\, n_T \leq n \leq N_T, \\ \mathcal{K}_{r_{\lambda_n}}(\mathcal{F}_{\lambda_n}y), \ \forall\, y \in L^2(\Omega), \forall\, t \in I_n, \forall\, n \geq N_T + 1, \end{cases} \tag{4.3}$$

is a proper feedback law for system (4.1).

Moreover, for an effectively computable large N_T the feedback law (4.3) stabilizes system (4.1) in finite time:

(i) *(2T stabilization)* $\Phi(2T + t, t; y_0) = 0, \ \forall\, t \in \mathbb{R}, \ \forall\, ||y_0||_{L^2(\Omega)} \leq \Lambda.$

(ii) *(Uniform stability) For every $\delta > 0$, there exists an effectively computable $\eta > 0$ such that*

$$\big(||(y_0||_{L^2(\Omega)} \leq \eta\big) \Rightarrow \big(||\Phi(t,t';y_0)||_{L^2(\Omega)} \leq \delta, \ \forall\, t' \in \mathbb{R}, \ \forall\, t \in (t', +\infty)\big)\,.$$

Proof of Theorem 4.3. Thanks to the $Ce^{C\sqrt{\lambda}}$ estimate, the proof of Theorem 4.4 is rather standard. Here we mimic the treatment for similar results on one dimensional parabolic equations [15]. The proof is followed by three steps: the feedback law is proper; condition (i); and condition (ii).

Step 1. First, we show that the feedback law given by (4.3) is proper. Without loss of generality, we only need to prove that for any $s \in [0,T)$ and for any $y_0 \in L^2(\Omega)$ the Cauchy problem

$$\begin{cases} y_t = \Delta y + 1_\omega U(t;y), \ (t,x) \in (s,T) \times \Omega, \\ y(t,x) = 0, \ (t,x) \in (s,T) \times \partial\Omega, \\ y(s,x) = y_0(x), \end{cases}$$

has a unique solution y, and $\lim_{t\to T^-} y(t) \in L^2(\Omega)$. Actually, the existence of a unique solution on each interval I_n follows directly from Theorem 2.3 for $n \leq N_T$ and from Theorem 2.4 for $n > N_T$. Hence,

$$y|_{[s,T)}(t) \in C^0([s,T); L^2(\Omega)).$$

Moreover, $||y(t)||_{L^2(\Omega)}$ is uniformly bounded on $[s,T)$ thanks to Theorem 2.7 and Theorem 2.2. Therefore, the control term on time interval $[s,T)$ is uniformly bounded in $L^2(\Omega)$, thus by applying Theorem 2.2 again we know that

$$y|_{[s,T]}(t) \in C^0([s,T]; L^2(\Omega)) \cap L^2(s,T; H_0^1(\Omega)).$$

Or equivalently, $\lim_{t\to T^-} y(t) \in L^2(\Omega)$ can be proved by the Cauchy sequence argument suggested in [15, page 1018 for (4.42)]. Therefore, the flow $\Phi(s,t;y)$ is well-defined on $\Delta \times L^2(\Omega)$.

Step 2. Next, we need to find a suitable integer N_T such that the proper feedback law (4.3) stabilize system (4.1) in finite time, mainly focus on condition (i).

Lemma 4.5. *The following energy estimate concerning the flow of the closed-loop system holds,*

$$||\Phi(T,s;y_0)||_{L^2(\Omega)} \leq 2 + e^{\frac{\Gamma^2}{16}N_T^2}||y_0||_{L^2(\Omega)}, \ \forall\, ||y_0||_{L^2(\Omega)} \leq \Lambda, \ \forall\, s \in [0,T). \tag{4.4}$$

Proof of Lemma 4.5. For any given $||y_0||_{L^2(\Omega)} \leq \Lambda$.

If $s \in [T_{N_T+1}, T)$, then since the feedback is bounded by 1, Theorem 2.2 yields

$$||\Phi(T, s; y_0)||_{L^2(\Omega)} \leq ||y_0||_{L^2(\Omega)} + 2.$$

If $s \in [0, T_{N_T+1})$, then we estimate $||\Phi(T_{N_T+1}, s; y_0)||_{L^2(\Omega)}$. Suppose that $s \in I_n$ with $n_T \leq n \leq N_T$, then direct calculation shows that (recalling some estimates from Section 3, especially (3.2)–(3.6)),

$$\begin{aligned}
||\Phi(T_{N_T+1}, s; y_0)||_{L^2(\Omega)} &\leq C_1 e^{C_1\sqrt{\lambda_n}} \left(\prod_{k=n+1}^{N_T} C_1 e^{C_1\sqrt{\lambda_k}} e^{-\frac{\lambda_k}{2}(T_{k+1}-T_k)} \right) ||y_0||_{L^2(\Omega)}, \\
&\leq C_1 e^{C_1\Gamma n^2} \left(\prod_{k=n+1}^{N_T} C_1 e^{C_1\Gamma k^2} e^{-\frac{\Gamma^2}{4}k^2} \right) ||y_0||_{L^2(\Omega)}, \\
&\leq e^{\frac{\Gamma^2}{16}n^2} \left(\prod_{k=n+1}^{N_T} e^{-\frac{3\Gamma^2}{16}k^2} \right) ||y_0||_{L^2(\Omega)}, \\
&\leq e^{\frac{\Gamma^2}{16}N_T^2} ||y_0||_{L^2(\Omega)}.
\end{aligned}$$

Next, for $\widetilde{y}(T_{N_T+1}) := \Phi(T_{N_T+1}, s; y_0)$ we adapt the case that $s \in [T_{N_T+1}, T)$ to get the required result. □

By applying Lemma 4.5 we know that for any $s \in [0, T)$ and for any $||y_0||_{L^2(\Omega)} \leq \Lambda$,

$$||\Phi(T, s; y_0)||_{L^2(\Omega)} \leq 2 + e^{\frac{\Gamma^2}{16}N_T^2}\Lambda.$$

Let us define $\widetilde{y}(T) := \Phi(T, s; y_0)$. The next step is to show that $\Phi(2T, T; \widetilde{y}(T)) = 0$, which requires us to seek for suitable N_T such that for every $n \geq N_T + 1$ we have,

$$K_{r_{\lambda_n}}\left(\mathcal{F}_{\lambda_n}\Phi(t, T; \widetilde{y}(T))\right) = \mathcal{F}_{\lambda_n}\Phi(t, T; \widetilde{y}(T)), \forall\, t \in I_n + T. \tag{4.5}$$

For ease of notations we simply denote the unique solution of the closed-loop system by

$$\widetilde{y}(t) := \Phi(t, T; \widetilde{y}(T)), \forall\, t \in [T, 2T].$$

Thus, in order to prove (4.5) it suffices to show that

$$||\mathcal{F}_{\lambda_n}\widetilde{y}(t)||_{L^2(\Omega)} \leq r_{\lambda_n}, \forall\, t \in I_n + T,\ \forall\, n \geq N_T + 1.$$

As we know from the proof of Lemma 4.5, or from (3.6), that

$$||\widetilde{y}(T_{N_T+1})||_{L^2(\Omega)} \leq \left(\prod_{k=n_T}^{N_T} e^{-\frac{3\Gamma^2}{16}k^2} \right) ||\widetilde{y}(T)||_{L^2(\Omega)} \leq 2e^{\frac{\Gamma^2}{16}N_T^2}\Lambda \left(\prod_{k=n_T}^{N_T} e^{-\frac{3\Gamma^2}{16}k^2} \right),$$

and that for $n \geq N_T + 1$,

$$||\widetilde{y}(T_n)||_{L^2(\Omega)} \leq 2e^{\frac{\Gamma^2}{16}N_T^2}\Lambda \left(\prod_{k=n_T}^{N_T} e^{-\frac{3\Gamma^2}{16}k^2} \right) \left(\prod_{k=N_T+1}^{n-1} e^{-\frac{3\Gamma^2}{16}k^2} \right),$$

and that for $n \geq N_T + 1$ and $t \in I_n + T$,

$$||\mathcal{F}_{\lambda_n}\widetilde{y}(t)||_{L^2(\Omega)} \leq ||\widetilde{y}(T_n)||_{L^2(\Omega)} C_2 e^{C_2\Gamma n^2} \leq ||\widetilde{y}(T_n)||_{L^2(\Omega)} e^{\frac{\Gamma^2}{16}n^2}.$$

Therefore, it suffices to find N_T such that for every $n \geq N_T + 1$ we have

$$\left(2e^{\frac{\Gamma^2}{16}N_T^2}\Lambda \left(\prod_{k=n_T}^{N_T} e^{-\frac{3\Gamma^2}{16}k^2} \right) \right) \left(\prod_{k=N_T+1}^{n-1} e^{-\frac{3\Gamma^2}{16}k^2} \right) e^{\frac{\Gamma^2}{16}n^2} \leq e^{-\frac{\Gamma^2}{16}n^2}.$$

Thus one only needs to find the existence of N_T such that

$$2e^{\frac{\Gamma^2}{16}N_T^2}\Lambda\left(\prod_{k=n_T}^{N_T} e^{-\frac{3\Gamma^2}{16}k^2}\right)e^{\frac{\Gamma^2}{8}(N_T+1)^2} \leq 1,$$

which is obviously possible for any given $\Lambda > 1$.

Step 3. Finally, in order to complete the proof of finite time stabilization, it only remains to prove that the proper feedback law given by (4.3) satisfies condition (ii): uniform stability.

Thanks to Step 2 we know the existence of C such that

$$||\Phi(t,T;y_0)||_{L^2(\Omega)} \leq C||y_0||_{L^2(\Omega)}, \forall\ ||y_0||_{L^2(\Omega)} \leq 1, \forall\ t \in [T,+\infty).$$

As a consequence for any $\delta > 0$ there exists $\widetilde{\eta} \in (0,\delta)$ such that

$$||\Phi(t,T;y_0)||_{L^2(\Omega)} \leq \delta,\ \forall\ ||y_0||_{L^2(\Omega)} \leq \widetilde{\eta},\ \forall\ t \in [T,+\infty). \tag{4.6}$$

Moreover, there exists some $\varepsilon > 0$ such that

$$||\Phi(t,s;y_0)||_{L^2(\Omega)} \leq \widetilde{\eta},\ \forall\ ||y_0||_{L^2(\Omega)} \leq \varepsilon,\ \forall\ s \in [0,T),\ \forall\ t \in [s,T]. \tag{4.7}$$

Indeed, thanks to Theorem 2.2 and the fact that $||\mathcal{K}_{r_{\lambda_n}}(y)|| \leq 1$, there exists $\widetilde{T} \in (0,T)$ such that

$$||\Phi(t,s;y_0)||_{L^2(\Omega)} \leq \widetilde{\eta},\ \forall\ ||y_0||_{L^2(\Omega)} \leq \widetilde{\eta}/2,\ \forall\ s \in [\widetilde{T},T),\ \forall\ t \in [s,T]. \tag{4.8}$$

Because the time-varying feedback law U on $[0,\widetilde{T})$ is given by finitely many stationary feedback laws, there exists some $\varepsilon \in (0,\widetilde{\eta}/2)$ such that

$$||\Phi(t,s;y_0)||_{L^2(\Omega)} \leq \widetilde{\eta}/2,\ \forall\ ||y_0||_{L^2(\Omega)} \leq \varepsilon,\ \forall\ s \in [0,\widetilde{T}),\ \forall\ t \in [s,\widetilde{T}]. \tag{4.9}$$

In conclusion, inequalities (4.8)–(4.9) yields (4.7); then estimates (4.6)–(4.7), as well as the fact that $\Phi(2T,s;y_0) = 0$, imply the uniform stability condition (ii). □

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