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Input-output linearization of nonlinear time-varying delay systems: the single-input single-output case

Ihab Haidar, Florentina Nicolau, Jean-Pierre Barbot,
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Abstract

This paper deals with the input-output linearization of nonlinear time-varying delay systems. We introduce an extension of the Lie derivative for time-varying delay systems and derive sufficient conditions for the existence of a causal and bounded nonlinear feedback linearizing the input-output behavior of the system. Sufficient conditions ensuring the internal stability after output stabilization are also presented. Finally, several examples illustrating our main results are discussed.

Keywords: Nonlinear control systems, delay systems, input-output linearization, Lie derivative.

1 Introduction

The problem of static linearization is an important tool in nonlinear control theory and consists in transforming the nonlinear system dynamics into a linear one such that linear control laws can be used in order to achieve the desired control properties. Two types of feedback linearization can be distinguished: input-state linearization and input-output linearization. In the first case, we need a state transformation (playing the role of a change of coordinates) and a static feedback transformation (that changes coordinates in the control space in a way which is state dependent) that bring the original system into an equivalent linear one. In the case of input-output linearization, the output is connected to the control only indirectly through the state and to achieve input-output linearization, a direct relation between the input and the output of the system must be found and, after the application of a suitable feedback transformation, the input-output map of the feedback modified systems has to be linear.

The problems of input-state and input-output linearization are well known for nonlinear control systems without delays (see, e.g., [12, 15], for input-state linearization, and [9, 3, 14], for input-output decoupling and linearization). Various aspects of those problems have been studied in the literature using different approaches and some of those approaches have been extended to encompass nonlinear control systems with constant delays (see, e.g., [1, 2], for the algebraic approach, and [8, 19, 20], for the geometric one).

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However in the particular case of nonlinear time-varying delay control systems, which is the subject of this paper, the problem is still largely open. Our goal is thus to propose a solution for the problem of input-output linearization for single-input single-output nonlinear time-varying delay systems, to discuss how our solution differs and how it reminds that for the constant-delay case and finally, to understand the problems that may arise when constructing a feedback transformation for delay systems. The multi-input multi-output case is more involved and will be considered in a future paper (see [18] for the case of two-input two-output time-varying delay systems). Preliminary results leading to this paper appeared in [10, 11].

Like for systems without delays, the input-output linearization of time-delay systems can be accomplished by successive differentiation of the output until the input appears in the resulting derivative equation (the number of times that we need to differentiate will be called relative degree) and, then, by applying a feedback transformation for which the input-output map of the feedback modified systems is linear. The particularity of time-delay systems resides in the resulting input-output relation which is a recursive equation involving (in addition to the current and past values of the state) not only the input at the current time, but also its past values. Thus, the main difficulty is the construction of a causal and bounded linearizing feedback. We give sufficient conditions ensuring those properties, explain how to verify them, and compare them with those presented in the literature for the constant-delay case [8, 7, 19, 20]. The interest of the paper is also to understand the role played by the time-varying delay. An extension of the Lie derivative to the time-varying case has to be defined. The particularity is that the consecutive derivatives of the considered time-varying delay appear explicitly in the Lie derivatives as well as in the expression of the linearizing feedback.

Input-output linearization is useful to study the asymptotic stability of time-varying delay systems. But, unlike for systems without delay, even if a full input-output linearization has been achieved (i.e., the relative degree equals the state dimension), the internal stability is not guaranteed after output stabilization. This problem is deeply studied in [7] in the case of constant-delay systems. In that paper, the authors prove that, in the case of fully linearization and under some assumptions on the internal state dynamics, when the output is asymptotically driven to zero, both the state and the control tend to zero asymptotically. Here, we develop sufficient conditions ensuring the internal stability in both cases (of a complete, respectively, of a partial input-output linearization).

The paper is organized as follows. In Section 2, we present some notations and recall the definition of the Lie derivative for time-varying delay systems (introduced by the authors in [11]). In Section 3, we present our main results and compare them to the existing ones for the constant-delay case. Several examples are discussed in Section 4. The proofs are given in Section 5.

2 Notations, definitions and problem statement

Throughout, \mathbb{R}^n denotes the n -dimensional Euclidean space with norm $\|\cdot\|$ and \mathbb{R}_+ the set of non-negative real numbers. Given $\bar{\theta} > 0$ and $q \in \mathbb{N}$, we denote by $\mathcal{C} = (\mathcal{C}([-q\bar{\theta}, 0], \mathbb{R}^n), \|\cdot\|_\infty)$ the Banach space of continuous functions from

$[-q\bar{\theta}, 0]$ into \mathbb{R}^n , with the norm $\|\cdot\|_\infty$ defined by

$$\|\varphi\|_\infty = \sup_{\theta \in [-q\bar{\theta}, 0]} \|\varphi(\theta)\|, \quad \varphi \in \mathcal{C}.$$

We denote by $\mathcal{U} = \mathcal{U}([-\bar{\theta}, +\infty), \mathbb{R})$ the set of measurable inputs from $[-\bar{\theta}, +\infty)$ into \mathbb{R} .

Definition 1 (δ -operators and their properties). *Let $\bar{\theta} > 0$ and $\theta : \mathbb{R} \mapsto (0, \bar{\theta}]$ be a sufficiently smooth function which is supposed to be known¹ and satisfying $\frac{d\theta}{dt} \leq 1$. over \mathbb{R}_+ and satisfies $\frac{d\theta}{dt} < 1$. Consider the recursive relation*

$$\tau_{i+1}(t) = \tau_i(t) - \theta \circ \tau_i(t), \quad \text{for } i \geq 0,$$

where $\tau_0(t) = t$. We denote by δ^i the time delay operator that shifts the time from t to $\tau_i(t)$ and which is defined as

$$\delta^0 \sigma(t) = \sigma(t) \quad \text{and} \quad \delta^i \sigma(t) = \sigma(\tau_i(t)), \quad \text{for } i \geq 0,$$

where σ is a function defined on an interval containing $[t - i\bar{\theta}, t]$. The application of δ^i on a composed function is given by

$$\delta^i \varphi(t, \sigma(t)) = \varphi(\tau_i, \delta^i \sigma(t)) = \varphi(\tau_i, \sigma(\tau_i)), \quad \text{for } i \geq 0. \quad (1)$$

Applied on the product of two functions, this delay operator acts as the following

$$\delta^i \varphi(t) \cdot \sigma(t) = (\delta^i \varphi(t)) \cdot (\delta^i \sigma(t)), \quad \text{for } i \geq 0, \quad (2)$$

i.e., the delay spreads to the right. If brackets are present, i.e., we have $(\delta^i \varphi(t))\sigma(t)$, then the delay affects only the first function (here φ).

We introduce the δ and $\delta^{\geq i}$ operators which are defined, respectively, by

$$\begin{aligned} \delta \sigma(t) &= (\sigma(t), \delta^1 \sigma(t), \dots, \delta^q \sigma(t)), \\ \delta^{\geq i} \sigma(t) &= (\delta^i \sigma(t), \dots, \delta^q \sigma(t)), \end{aligned} \quad (3)$$

where $0 \leq i < q$ and q is the maximal order of the delay operator acting on σ .

Finally, we introduce the advance operator denoted by δ^{-i} that shifts the time from t to $\tau_i^{-1}(t)$ and which is defined by

$$\delta^{-i} \sigma(t) = \sigma(\tau_i^{-1}(t)), \quad \text{for } i \geq 0.$$

Similarly to the δ -operator, we define

$$\delta^- \sigma(t) = (\delta^{-1} \sigma(t), \dots, \delta^{-q} \sigma(t)). \quad (4)$$

Remark 1. *If the delay operator δ^i is applied on the composition of two functions of one variable, relation (1) writes*

$$\delta^i \varphi \circ \sigma(t) = \delta^i \varphi(\sigma(t)) = \varphi(\delta^i \sigma(t)), \quad \text{for } i \geq 0.$$

¹This assumption could be weakened by supposing that the delay function has an observable dynamics.

Remark 2. *The condition $\frac{d\theta}{dt} < 1$ on the derivative of the delay function is important for causality reasons. Without that assumption, when dealing with time-varying delay systems, we would obtain a contradiction with the fact that the information available at t is not available at $t + \epsilon$. Usually, in the literature, this condition is stated as $\frac{d\theta}{dt} \leq 1$. In this paper, the case $\frac{d\theta}{dt} = 1$ is also excluded. Condition $\frac{d\theta}{dt} < 1$, together with the fact that the delay function is completely known, allows to define the inverse function τ_i^{-1} of τ_i over \mathbb{R}_+ , for $i \geq 0$. This is a "future function" which can be interpreted as follows: $\tau_i^{-1}(t_1)$, for some fixed $t_1 \geq 0$, is the future instant t_2 verifying $\tau_i(t_2) = t_1$ (and we clearly have $t_2 > t_1$).*

In this paper, we study input-output linearization of single-input single-output nonlinear time-varying delay systems of the form

$$\begin{aligned} \dot{x}(t) &= f(\delta x(t), t) + g_0(\delta x(t), t)\delta^0 u(t) + g_1(\delta x(t), t)\delta^1 u(t), \quad \forall t \geq 0, \\ y(t) &= h(\delta x(t), t), \\ x(s) &= x_0(s), \quad \forall s \in [-q\bar{\theta}, 0], \\ u(s) &= u_0(s), \quad \forall s \in [-\bar{\theta}, 0], \end{aligned} \tag{5}$$

where $x(t) \in \mathbb{R}^n$, the vector fields $f, g_0, g_1 : \mathbb{R}^{n(q+1)} \times \mathbb{R}_+ \rightarrow \mathbb{R}^n$ and the function $h : \mathbb{R}^{n(q+1)} \times \mathbb{R}_+ \rightarrow \mathbb{R}$ are sufficiently smooth. The integer q corresponds to the maximal delay order explicitly involved in f, g_0, g_1 and h (but it does not mean that f, g_0, g_1 and h have necessarily the same delay orders). According to (3), $\delta x(t) = (\delta^0 x(t), \dots, \delta^q x(t))$ denotes the δ -operator associated to a sufficiently smooth time-varying delay function $\theta : \mathbb{R} \rightarrow (0, \bar{\theta}]$ satisfying $\frac{d}{dt}\theta(t) < 1$, for all $t \in \mathbb{R}$, and where $\bar{\theta}$ is a positive real number. The initial condition x_0 belongs to $\mathcal{C}([-q\bar{\theta}, 0], \mathbb{R}^n)$ and the input $u : [-\bar{\theta}, +\infty) \rightarrow \mathbb{R}$ is a Lebesgue measurable function. We also assume that system (5) is forward complete; this assumption guarantees the existence of solutions on $[0, +\infty)$ for each fixed determined u .

To achieve input-output linearization (the problem that we are studying here), we must find a direct relation between the input and the output of the system by successive differentiation of the output until the input appears in the resulting derivative equations. An important tool when differentiating is the Lie derivative. Let us next define the Lie derivative for time-varying delay systems which is a generalization of that presented in [5, 19] for constant-delay systems.

Definition 2 (Lie derivative). *Let $f : \mathbb{R}^{n(q+1)} \times \mathbb{R}_+ \rightarrow \mathbb{R}^n$ be a smooth vector field whose components are functions of $(\delta x(t), t)$, and $h : \mathbb{R}^{n(q+1)} \times \mathbb{R}_+ \rightarrow \mathbb{R}$ a real valued function of $(\delta x(t), t)$. The Lie derivative of h along f at $(\delta x(t), t)$ is defined as*

$$L_f h(\delta x(t), t) = \sum_{i=0}^q \frac{\partial h}{\partial \delta^i x} \dot{\tau}_i \delta^i f(\delta x(t), t) + \frac{\partial h}{\partial t}(\delta x(t), t). \tag{6}$$

Notice that (6) is also an extension of the Lie-Backlund derivative introduced in [4] in a control theory context. The difference resides in our consideration of time-varying delays which leads to multiplicative coefficients (the time-derivative of the recursive delay function τ_i) affecting the operator δ^i .

Observe that, by taking the Lie derivative of h along f , we introduce new delays (via the term $\delta^i f$). In the above definition q denotes the maximal delay order explicitly involved in f or h , but the number of new delays introduced in $L_f h$ is related to the maximal delay order appearing in h only. Since $L_f h$ is a real-valued function with delays, the above operation can be recursively repeated for higher order as

$$L_f^k h(\delta x(t), t) = \sum_{i=0}^{kq} \frac{\partial L_f^{k-1} h}{\partial \delta^i x} \dot{\tau}_i \delta^i f(\delta x(t), t) + \frac{\partial L_f^{k-1} h}{\partial t}(\delta x(t), t), \text{ for } k \geq 2.$$

We also need to define a reduced Lie derivative.

Definition 3 (Reduced Lie derivative). *Let $g : \mathbb{R}^{n(q+1)} \times \mathbb{R}_+ \rightarrow \mathbb{R}^n$ be a smooth vector field and $h : \mathbb{R}^{n(q+1)} \times \mathbb{R}_+ \rightarrow \mathbb{R}$ a real valued function. The reduced Lie derivative of h along g is defined as*

$$\bar{L}_g h(\delta x(t), t) = \sum_{i=0}^q \frac{\partial h}{\partial \delta^i x} \dot{\tau}_i \delta^i g(\delta x(t), t). \quad (7)$$

Contrary to (6), the partial derivative of h with respect to t is not present in (7). If h does not depend explicitly on t , then $\bar{L}_g h$ coincide with $L_g h$. Notice also that, in the case of time-varying delay systems, the Lie derivatives $L_f h$ and $\bar{L}_g h$ will always depend explicitly on t through the terms $\dot{\tau}_i$.

As indicated by the notations, in our study the reduced Lie derivative (7) will always be computed for the control vector fields g_0 and g_1 , while the Lie derivative (6) will always be associated to the drift f .

Similarly to control systems without delays, the time-derivative of $L_f^k h$, for $k \geq 0$, can be written as follows:

$$\frac{dL_f^k h}{dt} = \sum_{i=0}^m \frac{\partial L_f^k h}{\partial \delta^i x} \frac{d\delta^i x}{dt} + \frac{\partial L_f^k h}{\partial t} = L_f^{k+1} h + \bar{L}_{g_0} L_f^k h u + \bar{L}_{g_1} L_f^k h \delta^1 u, \quad (8)$$

where m is the highest delay order of $L_f^k h$ (in this case $m = kq$). Note that, in order to be able to compute the time-derivatives of $\delta^i x$, for $0 \leq i \leq m$, equation (8) should be defined only for $t \geq \tau_m^{-1}(0)$.

Recall that the operator δ spreads to the right (see equation (2)), thus $\bar{L}_{g_0} L_f^k h u$ involves as many delayed controls as the delays present in the function $L_f^k h$. Indeed, according to the definition of the reduced Lie derivative, we have:

$$\bar{L}_{g_0} L_f^k h u = \sum_{i=0}^m \frac{\partial L_f^k h}{\partial \delta^i x} \dot{\tau}_i(\delta^i g_0) \cdot (\delta^i u).$$

A similar remark can be made for $\bar{L}_{g_1} L_f^k h \delta^1 u$, but now the number of delayed controls is no longer m , but $m + 1$ (because $\bar{L}_{g_1} L_f^k h$ acts on a control that is already delayed):

$$\bar{L}_{g_1} L_f^k h \delta^1 u = \sum_{i=0}^m \frac{\partial L_f^k h}{\partial \delta^i x} \dot{\tau}_i(\delta^i g_1) \cdot (\delta^{i+1} u).$$

Therefore, the reduced Lie derivatives $\bar{L}_{g_0}L_f^k h$ and $\bar{L}_{g_1}L_f^k h$ associated to the control vector fields g_0 and g_1 , respectively, can be seen as δ -polynomials and can be developed with respect to the operator δ^i as follows:

$$\bar{L}_{g_0}L_f^k h(\delta x(t), t) = \sum_{i=0}^m a_0^i(\delta x(t), t)\delta^i, \quad \bar{L}_{g_1}L_f^k h = \sum_{i=1}^{m+1} a_1^i(\delta x(t), t)\delta^i,$$

where

$$a_0^i(\delta x(t), t) = \frac{\partial L_f^k h}{\partial \delta^i x} \dot{\tau}_i(\delta^i g_0(\delta x(t), t)), \quad 0 \leq i \leq m,$$

and

$$a_1^i(\delta x(t), t) = \frac{\partial L_f^k h}{\partial \delta^{i-1} x} \dot{\tau}_{i-1}(\delta^{i-1} g_1(\delta x(t), t)), \quad 1 \leq i \leq m+1.$$

Combining these two δ -polynomials, we define

$$a[\delta] = a^0(\delta x(t), t)\delta^0 + \dots + a^{m+1}(\delta x(t), t)\delta^{m+1}, \quad (9)$$

where $a^0 = a_0^0$, $a^i = a_0^i + a_1^i$, for $1 \leq i \leq m$, and $a^{m+1} = a_1^{m+1}$. The δ -polynomial $a[\delta]$ is an element of the ring of differential time-varying delay operators (see [21] for more details about the algebraic properties of that ring). Equation (8) can then be rewritten as follows:

$$\frac{dL_f^k h}{dt} = L_f^{k+1}(\delta x(t), t) + a[\delta]u(t), \quad t \geq \tau_m^{-1}(0). \quad (10)$$

Before formalizing the problem of input-output linearization, a definition of the relative degree in the case of time-varying delay systems is needed. This can be done in the same way as for the finite dimensional case (see, e.g., [13]): the relative degree of an output h corresponds to the number of times that we have to differentiate h with respect to time t before the input or the delayed input appears.

In this paper, we do not deal with singularities and we make the following assumption:

Remark 3. *When we say that a function does not vanish, we mean that it is nonzero for any $t \geq 0$.*

Definition 4 (Relative degree). *The relative degree of the output h of system (5) is the positive integer ρ defined as follows:*

$$\begin{cases} \bar{L}_{g_0}L_f^k h(\delta x(t), t) = \bar{L}_{g_1}L_f^k h(\delta x(t), t) = 0, & 0 \leq k < \rho - 1, \\ (\bar{L}_{g_0}L_f^{\rho-1} h, \bar{L}_{g_1}L_f^{\rho-1} h)(\delta x(t), t) \neq (0, 0), \end{cases}$$

for any $(\delta x(t), t) \in \mathbb{R}^n \times \mathbb{R}^{nq(\rho+1)} \times \mathbb{R}_+$.

Since, in the time-varying delay case, the Lie derivatives depend explicitly on time, the relative degree may also be time-dependent. Here, due to Remark 3, this cannot occur (i.e., the relative degree is uniform with respect to t).

From now on, we denote by $a[\delta]$ the δ -polynomial associated to the ρ -th time-derivative of h , that is

$$h^{(\rho)} = L_f^\rho h + a[\delta]u,$$

where

$$a[\delta] = \bar{L}_{g_0}L_f^{\rho-1} h + \bar{L}_{g_1}L_f^{\rho-1} h.$$

Definition 5 (Minimal and maximal degree). *We call minimal degree of $a[\delta]$ the order of its first coefficient non identically zero, i.e., the integer j such that*

$$\begin{aligned} \alpha^j(\delta x(t), t) &\neq 0, \\ \alpha^i(\delta x(t), t) &\equiv 0, \quad \forall i < j. \end{aligned} \quad (11)$$

The highest delay order of $a[\delta]$ is called the maximal degree and is denoted by $m + 1$ (where the integer m can be expressed in function of ρ and the highest delay order q involved in the system: $m = q\rho$).

Observe that $h^{(\rho)}$ is actually defined for $t \geq \tau_m^{-1}(0)$. Throughout, we will use the notation $t_0 = \tau_m^{-1}(0)$.

Definition 6 (Input-output linearization). *We say that the problem of input-output linearization is solvable for system (5) if the output h admits a finite relative degree ρ and if there exists a causal and bounded feedback u verifying*

$$a[\delta]u(t) = -L_f^\rho h(\delta x(t), t) + \delta^j v(t), \quad \forall t \geq \tau_j^{-1}(t_0), \quad (12)$$

where j is the minimal degree of $a[\delta]$, and v is the new control (assigned with respect to the properties that we want to achieve). If such u exists, then the feedback modified system satisfies:

$$y^{(\rho)}(t) = \delta^j v(t), \quad \forall t \geq t_0.$$

Moreover, the system is said input-output linearizable with delay if $j > 0$ (respectively, without delay if $j = 0$).

The input-output linearization usually leads to the existence of an unobservable part (called *zero-dynamics*) of system (5). Thus, a condition on the zero-dynamics is needed in order to guarantee the internal stability of the system. We stress that even in the case of a complete linearization (i.e., $\rho = n$), we may also have an implicit relation between u and $\delta^i u$, for $j \leq i \leq m$, (relation which is called internal input dynamics in the literature, see [8]). By consequence, two main problems may arise when constructing a feedback u from an equation of form (12). The first problem is the boundedness of u which comes from the zero-dynamics or the internal input dynamics or both at once. The second one is the causality of u that stems from the fact that the drift may involve more delays than the control vector fields or vice versa. In the next section we give sufficient conditions allowing to solve these problems. Under those conditions, we design new coordinates allowing the transformation of (a part of) system (5) into the ρ -th order linear input-output system

$$\begin{aligned} \dot{z}_i(t) &= z_{i+1}(t), \quad 1 \leq i \leq \rho - 1, \\ \dot{z}_\rho(t) &= \delta^j v(t), \end{aligned} \quad (13)$$

where $z_i = L_f^{i-1} h$, for $1 \leq i \leq \rho$. The fact that in (12) we consider $\delta^j v$ (instead of v , which would also lead to a linear input-output map, but without delay) will be justified in the sequel. If we replace $\delta^j v$ by v in (12), i.e., u should be computed from

$$a[\delta]u(t) = -L_f^\rho h(\delta x(t), t) + v(t), \quad (14)$$

then the control would act instantaneously on the system and no predictor would be needed for the construction of a feedback v for the z -system (for instance, if

the aim is to design a stabilizing feedback for (13)), but when designing a feedback u from equation (14), a j -order predictor (i.e., a predictor over $[t_0, \tau_j^{-1}(t_0)]$) would have to be implemented for the x -system. Here, we simply chose to make the predictor over the linear z -system (rather than on the nonlinear x -system). In that case, we suppose that the system is in free evolution over $[t_0, \tau_j^{-1}(t_0)]$ and that the feedback v (which has to be computed for $t \geq t_0$) affects the system starting from the instant $\tau_j^{-1}(t_0)$ only. This remark explains two points: first, the reason for which (12) is defined for $t \geq \tau_j^{-1}(t_0)$, and second, why condition (11) is needed starting from $\tau_j^{-1}(t_0)$ only. Remark that, even if we place the predictor over the z -system, the computation of a feedback u may request a 1-order predictor (over $[t_0, \tau_1^{-1}(t_0)]$) on the x -system (this may happen, for instance, when $g_1 \neq 0$). Finally, notice that if trajectory planning is intended, the artificial delay in the control v does not generate any technical problem because, in that case, we dispose entirely of the desired trajectory.

3 Main results

3.1 Input-output linearization

In this section, we give the main result of the paper: we design new coordinates and a causal and bounded feedback that linearizes the input-output behavior of the original system. To that end, we introduce the following conditions, that will be discussed in details after stating our main theorem:

(C1) - Relative degree: The output h has a finite relative degree $\rho \leq n$.

(C2) - New coordinates: There exists a smallest integer $0 \leq \ell \leq m$ such that the matrix $\left(\frac{\partial L_f^k h}{\partial \delta^\ell x_i}\right)$, for $0 \leq k \leq \rho - 1$ and $1 \leq i \leq n$, is of full rank (equal to ρ) at any $(\delta x(t), t) \in \mathbb{R}^n \times \mathbb{R}^{nq(\rho+1)} \times \mathbb{R}_+$.

Introduce the z -variables $z_i(t) = L_f^{i-1} h(\delta x(t), t)$, for $t \geq t_0$ and $1 \leq i \leq \rho$, and complete them (which is always possible) by $n - \rho$ functions $\xi_i(t) = \phi_i(\delta x(t), t)$, for $\rho + 1 \leq i \leq n$, such that the Jacobian matrix $\left(\frac{\partial \Phi}{\partial \delta^\ell x}\right)$, where $\Phi = (h, \dots, L_f^{\rho-1} h, \phi_{\rho+1}, \dots, \phi_n)$, is of full rank (equal to n) at any $(\delta x(t), t) \in \mathbb{R}^n \times \mathbb{R}^{nq(\rho+1)} \times \mathbb{R}_+$.

(C3) - Causal and bounded inversion: There exist ξ -coordinates, a continuous function Ψ such that

$$\delta^\ell x(t) = \Psi(\delta z(t), \delta^- z(t), \delta \xi(t), \delta^- \xi(t), \delta^{\geq \ell+1} x(t), t), \quad \forall t \geq t_0, \quad (15)$$

and a constant $\mu > 0$ such that if $\|\delta^{\geq \ell+1} x(t)\| < \mu$, then

$$\|\delta^\ell x(t)\| \leq \alpha_1 \|(\delta z(t), \delta^- z(t))\| + \alpha_2 \|(\delta \xi(t), \delta^- \xi(t))\| + \alpha_3 \|\delta^{\geq \ell+1} x(t)\|, \quad \forall t \geq t_0, \quad (16)$$

where $\alpha_1 > 0$, $\alpha_2 > 0$ and $0 < \alpha_3 < 1/(q(\rho + 1) - \ell)$. Recall that δ^- and $\delta^{\geq \ell+1}$ were defined in Section 2.

(C4) - Causal feedback: The Lie derivative $L_f^\rho h$ and the coefficients of the δ -polynomial $a(\delta)$ verify

$$\frac{\partial L_f^\rho h}{\partial \delta^i x} \equiv 0 \text{ and } \frac{\partial a^k}{\partial \delta^i x} \equiv 0, \quad (17)$$

for $j \leq k \leq m+1$ and $0 \leq i \leq j-1$ if $g_0 \neq 0$ (respectively, $0 \leq i \leq j-2$ if $g_0 \equiv 0$).

(C5) - Bounded feedback: Let $v : [\tau_j(t_0), +\infty) \rightarrow \mathbb{R}$ be such that

$$\frac{-L_f^\rho h(\delta x(t), t) + \delta^j v(t)}{a^j(\delta x(t), t)} \text{ is bounded over } [\tau_j^{-1}(t_0), +\infty) \quad (18)$$

and u be a solution of (12) for that v . Suppose that u is bounded over $[\tau_{m+1-j}(t_0), t_0]$ and that there exists a constant $c > 1$ such that

$$\sup_{t \geq \tau_j^{-1}(t_0)} \left\| \frac{a^i(\delta x(t), t)}{a^j(\delta x(t), t)} \right\| \leq \frac{1}{c(m+1-j)}, \quad \forall i > j. \quad (19)$$

Theorem 1. *Consider system (5) and suppose that conditions (C1)-(C5) are satisfied. Then, system (5) is input-output linearizable via the causal and bounded feedback transformation (12) and (a part of) it can be transformed into the ρ -th order linear input-output system (13). Moreover, if $\rho = n$, then the system is fully input-output linearizable with delay and if, in addition, $j = 0$, then the system is fully input-output linearizable without delay.*

Theorem 1 gives only sufficient conditions. We do not claim that our conditions are also necessary, see Section 4, where we give several examples that do not satisfy the conditions of Theorem 1.

Conditions discussion

We next discuss the role and the importance of each condition. Condition (C1) requires that the relative degree of h is smaller or equal to n . It is clear that if this is not the case, system (5) is not input-output linearizable (in fact, a part of the system is not controllable).

Contrary to the cases of control systems without delays and constant-delay control systems (see Section 3.3, where we give a comparison of our results with the constant-delay case), condition (C1) may not be sufficient to define a local change of coordinates and additional conditions (given by (C2) and (C3)) may be needed.

It is important to note that by a change of coordinates (or a coordinate transformation), we do not mean (like for systems without delays) a diffeomorphism between x and the new state (z, ξ) , see condition (C2). In fact the coordinate transformation map $\Phi : \mathbb{R}^n \times \mathbb{R}^{nq(\rho+1)} \times \mathbb{R}_+ \mapsto \mathbb{R}^n$ of (C2), that allows us to construct the new variables z transforming a part of system (5) into the ρ -th order linear system (13), is never a diffeomorphism. However, condition (C2), together with (C3), enables us to come back from the (z, ξ) -state space into the x -state space by expressing the delayed states $\delta^\ell x_i$, for a certain delay order $0 \leq \ell \leq m$, as functions of the new state, their delays and advances, and possible delayed state $\delta^p x$, where $p \geq \ell + 1$ (hence the name "causal inversion") and

to preserve (in the x -state space) properties (like boundedness) that hold in the (z, ξ) -state space. As we will see below, conditions **(C2)** and **(C3)** play a crucial role in the verification of **(C5)** ensuring the boundedness of the input-output linearizing feedback.

A less restrictive condition than **(C2)** would be to assume the existence of delay orders ℓ_1, \dots, ℓ_n such that $0 \leq \ell_i \leq m$ and the rank of the matrix $\left(\frac{\partial \phi_k}{\partial \delta^{\ell_i} x_i}\right)$, for $1 \leq k \leq \rho$, $1 \leq i \leq n$, equals ρ , and for each $\delta^{\ell_i} x_i$ we have a causal inversion (i.e., in (15)-(16), we should take $\delta^{\geq L+1} x$, where $L = \max_{1 \leq i \leq n} \ell_i$ instead of $\delta^{\ell} x$). This would considerably complicate the notations and in order to simplify the understanding of the paper, we assume that the more restrictive condition **(C2)** holds.

We stress that condition **(C3)** (in particular relation (15) which should hold for all $t \geq t_0$) supposes indirectly that z and ξ are completely known over $[\tau_{q(\rho+1)}(t_0), t_0]$ and that their maximum delay order cannot exceed $q(\rho+1)$. This does not necessarily mean that all their components should be known over $[\tau_{q(\rho+1)}(t_0), t_0]$. Throughout this paper, we suppose that the maximum delay-advance order in (z, ξ) cannot exceed $q(\rho+1)$.

Conditions **(C4)** and **(C5)** deal with the construction of a causal and bounded feedback. **(C4)** should be understood as follows: at an instant $t \geq t_0$, we have to assign to system (5) either $u(t)$, if $g_0 \neq 0$, or $\delta^1 u(t)$, if $g_0 \equiv 0$, and, in both cases, the construction of u is obtained from (12) by applying the advance operator δ^{-j} and thus shifting the coefficients of the δ -polynomial as well as $L_f^j h$ of order j in the future. In the case when $u(t)$ acts on the system (i.e., $g_0 \neq 0$), the causality condition (17) guarantees that the value of $u(t)$ depends only on the current and past values of the state; thus no causality problem can be produced when introducing in (5) the control u so obtained. In the case when only the delayed input appears (i.e., $g_0 \equiv 0$ and, of course, $g_1 \neq 0$), the causality condition provides that the value of $u(t)$, computed from the feedback equation, depends on the past, current and 1-advance values of the state. Again, when introducing in (5) the associated $\delta^1 u(t)$, no causality problem appears. Nevertheless, in this later case, a predictor is needed in order to compute the control as a function of the future values of the state. Finally, observe that when $j = 0$, there is no causality problem (and thus there is no condition to be checked).

We will see, when proving Theorem 1, that condition **(C5)** guarantees that the feedback u stays bounded when property (18) holds. Thus an important issue is how to verify this property and next we explain how to do it. In fact, (18) can be verified in the (z, ξ) -coordinates. This does not necessarily implies its validity in the x -coordinates (recall that, in the time-varying delay context, the change of coordinates map does not define a diffeomorphism). Nevertheless, thanks to property (16) given by **(C3)**, under some additional assumptions (see Proposition 1 below), the boundedness in (z, ξ) -coordinates implies the boundedness in x -coordinates. In the following proposition, we use the notation

$$\varepsilon = \frac{1 - \alpha_3(q(\rho+1) - \ell)}{(2q(\rho+1) + 1) \max\{\alpha_1, \alpha_2\}} \mu, \quad (20)$$

where the parameters $\ell, \mu, \alpha_1, \alpha_2$, and α_3 are given by **(C2)**-**(C3)**.

Proposition 1. *Suppose that conditions **(C1)**-**(C3)** hold. If*

$$\sup_{t \geq \tau_{q(\rho+1)}(t_0)} \{\|z(t)\|, \|\xi(t)\|\} < \varepsilon \quad \text{and} \quad \sup_{\tau_{q(\rho+1)}(t_0) \leq t \leq t_0} \|x(t)\| < \mu,$$

then $x(\cdot)$ stays bounded by μ over $[t_0, +\infty)$.

3.2 Stability of the closed loop system

In this section, we study the stability of the closed loop system. In addition to **(C1)**-**(C5)**, we suppose that the following conditions are satisfied:

(C6) - Equilibrium: System (5) satisfies

$$f(0, t) = 0, \quad \text{and} \quad h(0, t) = 0, \quad \forall t \geq 0.$$

where the 0-argument of f and h corresponds to the $(q + 1)$ -tuple $(0, \dots, 0)$.

(C7) - Total causal inversion: We suppose that **(C3)** holds, with the ξ -coordinates and the function Ψ such that

$$\delta^\ell x(t) = \Psi(\delta z(t), \delta^- z(t), \delta \xi(t), \delta^- \xi(t), t), \quad \forall t \geq t_0, \quad (21)$$

and that the dynamics of ξ do not contain advances in the ξ -states, i.e., $\dot{\xi}$ is of the form $\dot{\xi}(t) = G(\delta z(t), \delta^- z(t), \delta \xi(t), \delta v(t), t)$.

Condition **(C7)** is quite restrictive: it supposes that all components of the state $\delta^\ell x(t)$ are explicit functions of $\delta z(t), \delta^- z(t), \delta \xi(t), \delta^- \xi(t), t$, for all $t \geq t_0$. This condition is important when dealing with the stability of the zero dynamics, but it is not necessary in the sense that only the x -components present in the functions $\phi_i(\delta x(t), t)$, for $\rho + 1 \leq i \leq n$, defining the ξ -variables (see condition **(C2)**) have to be explicit functions of $\delta z(t), \delta^- z(t), \delta \xi(t), \delta^- \xi(t), t$, for all $t \geq t_0$ (see Section 4, for an example illustrating the nonnecessity of **(C7)**).

In the case when $\rho < n$, by applying feedback (12) and introducing the new (z, ξ) -coordinates, system (5) is decomposed into two parts: a ρ -th order linear z -subsystem and a nonlinear ξ -subsystem (zero-dynamics) which may contain delayed and advanced states. Under assumptions **(C1)**-**(C7)**, system (5) can be transformed, for $t \geq t_0$, as follows:

$$\begin{cases} \dot{z}(t) = Az(t) + B\delta^j v(t), & \forall t \geq t_0 \\ \dot{\xi}(t) = G(\delta z(t), \delta^- z(t), \delta \xi(t), \delta v(t), t), & \forall t \geq t_0 \\ y(t) = Cz(t), \end{cases} \quad (22)$$

where the matrices A , B and C are associated to form (13). Note that the initial condition of (22) is defined through that of the initial problem (5) (see the remark explaining condition **(C3)** after the statement of Theorem 1).

Let

$$\delta^j v(t) = \begin{cases} k_1 z_1(t) + \dots + k_\rho z_\rho(t), & t \geq \tau_j^{-1}(t_0), \\ 0 & t_0 \leq t \leq \tau_j^{-1}(t_0), \end{cases} \quad (23)$$

where k_i , for $i = 1, \dots, \rho$, are chosen in such a way that $A + BK$ has ρ different eigenvalues with strictly negative real parts, where $K = (k_1, \dots, k_\rho)$. In this case, the resulting closed loop subsystem

$$\dot{z}(t) = (A + BK)z(t), \quad (24)$$

is asymptotically stable. Therefore, under this choice of v , the state $z(t)$ tends asymptotically to zero when t tends to $+\infty$. Even if the input-output behavior can be stabilized by a feedback, the internal dynamics may be unstable and the global system cannot be stabilized.

In order to characterize the internal stability of the closed loop system, we have to recall the definition of input-to-state stability for time-delay systems (see, e.g., [6, 16, 22]). We will also need the following function classes: a function $\gamma : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is said to be of class \mathcal{K}_∞ if it is continuous, strictly increasing and $\gamma(0) = 0$. A continuous function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is said to be of class \mathcal{KL} if $\beta(\cdot, t)$ is of class \mathcal{K}_∞ for each $t \geq 0$, $\beta(s, \cdot)$ is decreasing to zero for each $s > 0$ and $\beta(s, t) \rightarrow 0$ as $t \rightarrow +\infty$.

Definition 7. Consider the time-delay system

$$\dot{\xi}(t) = G(\delta\xi(t), w(t)), \quad \forall t \geq 0, \quad (25)$$

where $\xi(t) \in \mathbb{R}^n$ is the state of the system, $w(t) \in \mathbb{R}^m$ is an exogenous input, the dynamics G is continuously differentiable and such that $G(0, 0) = 0$. System (25) is said to be uniformly input-to-state stable if there exist a \mathcal{KL} function β and a \mathcal{K}_∞ function γ such that, for any initial time $t_0 \geq 0$, any initial state $\xi_{t_0} = \varphi \in \mathcal{C}([-q\bar{\theta}, 0], \mathbb{R}^n)$ and any measurable, locally essentially bounded input w , the solution $\xi(t)$ exists for all $t \geq t_0$ and furthermore, it satisfies

$$\|\xi(t)\| \leq \beta(\|\xi_{t_0}\|_\infty, t - t_0) + \gamma(\|w_{[t_0, t]}\|_\infty), \quad \forall t \geq t_0.$$

In our case, we apply the above definition to the zero-dynamics $\dot{\xi}(t) = G(\delta z(t), \delta^- z(t), \delta\xi(t), \delta v(t))$ of (22), for which the exogenous input is $w(t) = (\delta z(t), \delta^- z(t), \delta v(t))$. Notice, however, that $w(t)$ is not exogenous for the global system.

Proposition 2. Consider system (22). Let v , given by equation (23), be such that the matrix $A + BK$ in (24) is Hurwitz. If the zero-dynamics is uniformly input-to-state stable then system (22) is locally asymptotically stable around the origin.

Proof. The proof derives straightforwardly from Definition 7. \square

Remark 4. The uniform input-to-state stability of the zero dynamics is not necessary if only the boundedness of the input is requested. For example, if the unstable part of the zero dynamics does not affect (or at least not significantly) the feedback control, which linearizes the system, then the input-output behavior is stable. Such a remark relaxes the conditions of Proposition 2, but this is far from the scope of the paper.

The following proposition gives sufficient conditions guaranteeing the local asymptotic stability of the equilibrium point in x -coordinates.

Proposition 3. Consider system (5) and suppose that conditions (C1)-(C7) are satisfied. Let u verifying equation (12) with v , given by (23), and such that the matrix $A + BK$ in (24) is Hurwitz. Suppose that the zero-dynamics is uniformly input-to-state stable. If $\|z(t)\| < \varepsilon_1$ over $[\tau_{q(\rho+1)}(t_0), \tau_j^{-1}(t_0)]$, and $\|\xi(t)\| < \varepsilon_1$ and $\|x(t)\| < \mu$ over $[\tau_{q(\rho+1)}(t_0), t_0]$, for a suitable constant ε_1 , then $\lim_{t \rightarrow +\infty} x(t) = 0$.

The constant ε_1 of the above proposition depends on the parameters defined by conditions (C2) and (C3), see the proof of Proposition 3.

3.3 Constant-delay case: comparison with the existent literature

A first attempt to extend the geometric tools of Lie derivative and Lie bracket to constant-delay systems has been given in [19, 20], where sufficient input-output linearizing conditions have been developed for single-input single-output constant-delay systems. Some of those conditions have been relaxed in [7, 8], in the case of fully linearizable time-delay systems.

It may be interesting to give a comparison of our conditions (simplified for the constant-delay case and for autonomous vector fields f, g_0, g_1 and output h) with the results of the above cited papers. We will next briefly recall their results, but let us first underline that in [7, 8, 19, 20], only the case of non-delayed input (i.e., $g_1 \equiv 0$) has been considered.

In addition to the existence of a relative degree $\rho \leq n$, conditions like

$$\frac{\partial L_f^{\rho-1} h}{\partial x} g_0(\delta x(t)) \neq 0, \quad \text{and} \quad \frac{\partial L_f^{\rho-1} h}{\partial \delta^i x} \delta^i g_0(\delta x(t)) = 0, \quad \forall i \geq 1, \quad (26)$$

for any $\delta x(t) \in \mathbb{R}^{nq}$ are required in [19, 20]. It is evident that under (26), the boundedness and causality properties of the control u (discussed after Definition 6) are no longer problematic. Indeed, if (26) holds, then (12) gives

$$u(t) = \frac{-L_f^\rho h(\delta x(t), t) + v(t)}{a^0(\delta x(t))}, \quad t \geq t_0, \quad (27)$$

which is clearly causal, and bounded when the right hand side of (27) is bounded. Condition (26) has been relaxed in [8], where only the less-restrictive relation

$$\frac{\partial L_f^{\rho-1} h}{\partial x} g_0(\delta x(t)) \neq 0, \quad (28)$$

for any $\delta x(t) \in \mathbb{R}^{nq}$, has been required. In this case, the control u is given by a recursive equation like (12) with $j = 0$. Thus, the boundedness (but not the causality) of the control u becomes questionable. Under condition (28) and in the case of fully linearization (i.e., $\rho = n$), the internal stability (i.e., the state and input stability), is then deeply studied and solved in [8].

Unlike for delay-free systems, even if a fully input-output linearization has been achieved, the internal stability is not guaranteed after output stabilization. Sufficient conditions ensuring that property (after output stabilization) are given in [7, 8]. These conditions are similar to what is requested by the statement of Proposition 3, but without any additional condition on the boundedness of the input. In fact, in [8], the authors show that, in the case of fully linearization and under some assumptions on the internal state dynamics, when the output is asymptotically driven to zero, both the state and the control tend to zero asymptotically. The proof of this result lies particularly in the fact that the delay is constant. The possibility of extension to the time-varying delay case is not so evident and will be treated elsewhere.

We will next explain why our conditions are less-restrictive comparing to what is done in [8, 7, 19, 20]. Condition **(C1)** demands that either $L_{g_0} L_f^{\rho-1} h \neq 0$ or $L_{g_1} L_f^{\rho-1} h \neq 0$, that is to only have (compare with (26)-(28)):

$$\frac{\partial L_f^{\rho-1} h}{\partial \delta^i x} \delta^i g_j(\delta x(t)) \neq 0, \quad \text{for some } i \geq 0 \text{ and } 0 \leq j \leq 1.$$

Of course, in addition to **(C1)**, further conditions like **(C3)**-**(C5)** are needed. Indeed, as explained in Section 2, in general, condition **(C1)** alone does not allow the construction of a bounded and causal feedback u from (12) that input-output linearizes the system. Conditions **(C4)** and **(C5)** guaranty the boundedness (additional condition on the zero-dynamics is needed in the case of partial linearization) and causality properties of u , while **(C2)** and **(C3)** are used in order to verify **(C5)**. Another important point is that, in the case of constant delays, conditions like **(C2)** are not needed since, in that case, **(C1)** is sufficient for the existence of a local coordinate transformation. In fact, it is easy to prove that in that case and for autonomous vector fields f, g_0, g_1 and output h , condition **(C2)** is a straightforward consequence of **(C1)**, see [19, 20] for a related result. For time-varying delays, the fact that the Lie derivatives are not autonomous (due to the presence of the terms $\dot{\tau}_i$) complicates the proof.

4 Examples

In this section, we discuss several examples illustrating Theorem 1. The presented examples are separated into two categories: the first one showing that our conditions are sufficient (Examples 1, 2 and 3) and the second one proving that they are not necessary (Example 4).

Example 1 (satisfying Theorem 1). *Consider the nonlinear time-varying delay system*

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) + \frac{1}{3}\delta^1 x_2(t) \\ \dot{x}_2(t) &= \delta^1 x_3^2(t) + \delta^1 u(t) \\ \dot{x}_3(t) &= -x_3(t) - x_1(t) \\ h(t) &= \delta^1 x_1(t), \quad \forall t \geq 0, \end{aligned} \tag{29}$$

with initial condition $x_0 \in \mathcal{C}([-\bar{\theta}, 0], \mathbb{R}^3)$ and $u \in \mathcal{U}$. The delay function $\theta(\cdot)$ is chosen in such a way that $|\dot{\theta}| < \frac{1}{4}$. Here, we have

$$g_0 = (0, 0, 0)^T, \quad g_1 = (0, 1, 0)^T, \quad \text{and} \quad f = (x_2 + \frac{1}{3}\delta^1 x_2, \delta^1 x_3^2, -x_3 - x_1)^T,$$

and the maximum delay order is $q = 1$. We can easily verify that all assumptions of Theorem 1 are all fulfilled. Indeed:

Condition (C1): A straightforward computation gives

$$\bar{L}_{g_1} h = 0, \quad L_f h = \dot{\tau}_1 \delta^1 x_2 + \frac{1}{3} \dot{\tau}_1 \delta^2 x_2, \quad \bar{L}_{g_1} L_f h = \dot{\tau}_1^2 \delta^2 + \frac{1}{3} \dot{\tau}_1 \dot{\tau}_2 \delta^3$$

and

$$L_f^2 h = \ddot{\tau}_1 \delta^1 x_2 + \frac{1}{3} \ddot{\tau}_1 \delta^2 x_2 + \dot{\tau}_1^2 \delta^2 x_3^2 + \frac{1}{3} \dot{\tau}_1 \dot{\tau}_2 \delta^3 x_3^2.$$

Thus, condition **(C1)** is satisfied and the relative degree is $\rho = 2$.

Condition (C2): By introducing

$$z_1 = \delta^1 x_1, \quad z_2 = \dot{\tau}_1 \delta^1 x_2 + \frac{1}{3} \dot{\tau}_1 \delta^2 x_2, \tag{30}$$

and choosing

$$\xi = \delta^1 x_3, \tag{31}$$

the rank of $\frac{\partial \Phi}{\partial \delta^1 x}$ is equal to 3 at any $(\delta x(t), t) \in \mathbb{R}^9 \times \mathbb{R}_+$, thus **(C2)** holds with $\ell = 1$.

Condition (C3): From (30)-(31), we obtain

$$\delta^1 x_1 = z_1, \quad \delta^1 x_2 = \frac{1}{\dot{\tau}_1} z_2 - \frac{1}{3\dot{\tau}_1} \delta^2 x_2, \quad \delta^1 x_3 = \xi, \quad \forall t \geq t_0, \quad (32)$$

with $t_0 = \tau_2^{-1}(0)$. In addition, for any $\mu > 0$, we have

$$\|\delta^1 x(t)\| \leq \alpha_1 \|z(t)\| + \alpha_2 \|\xi(t)\| + \alpha_3 \|\delta^2 x(t)\|, \quad \forall t \geq t_0,$$

with $\alpha_1 = 3, \alpha_2 = 1$ and $\alpha_3 = \frac{4}{9}$, and **(C3)** is valid.

Condition (C4): The feedback equation is given by

$$a(\delta]u = \dot{\tau}_1^2 \delta^2 u + k \dot{\tau}_1 \dot{\tau}_2 \delta^3 u = -L_f^2 h + \delta^2 v(t), \quad t \geq \tau_2^{-1}(t_0).$$

We are in the case where $j = 2$ and $g_0 \equiv 0$. Since the coefficients of the δ -polynomial do not depend on the state and that $\frac{\partial L_f^2 h}{\partial x} \equiv 0$, it follows that condition **(C4)** is valid.

Condition (C5): One can easily verify that the $a(\delta]$ -polynomial satisfies condition (19). Thus, it remains to check the boundedness of

$$\frac{-L_f^2 h + \delta^2 v(t)}{\dot{\tau}_1^2} \quad \forall t \geq \tau_2^{-1}(t_0). \quad (33)$$

As mentioned in Section 3, the boundedness will be verified in the (z, ξ) -coordinates. By consequence, we have to check if

$$\frac{-\frac{\dot{\tau}_1}{\tau_1} z_2 - \dot{\tau}_1^2 \delta^1 \xi^2 - k \dot{\tau}_1 \dot{\tau}_2 \delta^2 \xi^2 + \delta^2 v(t)}{\dot{\tau}_1^2},$$

stays bounded for every $t \geq \tau_2^{-1}(t_0)$. System (29) can be written in the (z, ξ) -coordinates as follows

$$\begin{cases} \dot{z}_1(t) = z_2(t) \\ \dot{z}_2(t) = \delta^2 v(t) \\ \dot{\xi}(t) = -\dot{\tau}_1(t)\xi(t) - \dot{\tau}_1(t)z_1(t) \\ y(t) = z_1(t), \quad \forall t \geq t_0. \end{cases}$$

Let v be as in equation (23), thus guaranteeing that the state $(z_1(t), z_2(t))$ converges asymptotically to zero when t tends to $+\infty$. Since $\dot{\tau}_1$ is bounded over $[0, +\infty)$, it follows that $w(t) = -\dot{\tau}_1(t)z_1(t)$ (which is seen as an external exogenous input of the zero-dynamics $\dot{\xi}(t) = -\dot{\tau}_1(t)\delta\xi(t) - w(t)$, for $t \geq t_0$) tends to 0 when t tends to $+\infty$. Knowing that the zero-dynamics is uniformly input-to-state stable (because the unforced zero-dynamics $\dot{\xi}(t) = -\dot{\tau}_1(t)\xi(t)$ has a globally exponentially stable equilibrium point at $\xi = 0$ and $G(\xi, z_1, t) = -\dot{\tau}_1(t)\xi - \dot{\tau}_1(t)z_1$ is continuously differentiable and globally Lipschitz, uniformly in t , see [17]), then $\xi(t)$ tends to 0 when t tends to $+\infty$. Since **(C3)** is valid for any arbitrary value of μ , by Proposition 1, we have that x stays bounded for $t \geq t_0$. It follows that (33) is bounded in the x -coordinates as well and, finally, that **(C5)** holds.

Example 2 (not satisfying the causality condition **(C4)**). Consider the following time-varying delay system

$$\begin{aligned}\dot{x}_1(t) &= x_2^2(t)u(t) \\ \dot{x}_2(t) &= u(t) \\ \dot{x}_3(t) &= x_1(t) \\ h(t) &= x_3(t) + \delta^4 x_2(t),\end{aligned}\tag{34}$$

for which $x_0 \in \mathcal{C}([-4\bar{\theta}, 0], \mathbb{R}^3)$ and $u_0 \in \mathcal{U}$ are given. We will show that (34) is not input-output linearizable via a causal and bounded feedback and that it is not in conformity with Theorem 1. We have $g_0 = (x_2^2, 1, 0)^T$, $g_1 \equiv 0$, $f = (0, 0, x_1)^T$ and the δ -polynomial is given by

$$a[\delta] = \dot{\tau}_4 \delta^4.\tag{35}$$

Thus the feedback u has to verify the following relation

$$x_1(t) + \dot{\tau}_4(t)\delta^4 u(t) = \delta^4 v(t), \quad t \geq \tau_4^{-1}(0).\tag{36}$$

It is clear that condition **(C4)** is violated since $\partial L_f h / \partial x_1 = 1 \neq 0$ and the causality problem becomes now clear. Indeed, the computation of $u(t)$ from (36) depends on the future of x_1 . However, from the first equation of (34), we see that u acts instantaneously on x_1 which prevents us to assign $u(t)$ computed by (36) to (34).

Example 3 (not satisfying the bounded feedback condition **(C5)**). This example shows that if inequality (19) of condition **(C5)** is not satisfied, then even for a very simple choice of a bounded v , there is no bounded feedback u satisfying (12). For simplicity, we suppose that the delay is constant and equal to 1. Consider

$$\begin{aligned}\dot{x}_1(t) &= x_3(t) - 2x_2(t) \\ \dot{x}_2(t) &= (1 + x_1^2(t))\delta^1 u(t) \\ \dot{x}_3(t) &= (1 + x_1^2(t))u(t) \\ h(t) &= x_1(t),\end{aligned}\tag{37}$$

with $x(0) = 0$ and $u(t) = 1$, for $t \in [-\bar{\theta}, 0]$. Notice that in this example only the control is delayed. We have

$$a[\delta]u(t) = (1 + x_1^2(t))u(t) - 2(1 + x_1^2(t))\delta^1 u(t).\tag{38}$$

Remark from the latter equation, that (19) is not satisfied. Now, suppose that we want to stay at $x_1(t) = 0$ for all $t \geq 0$. Hence u has to verify

$$u(t) - 2\delta^1 u(t) = 0, \quad t \geq 0,\tag{39}$$

i.e., in this case $v = 0$. It follows from (39) that, over $[0, \bar{\theta}]$, the control u should be equal to 2. Similarly, we should have $u(t) = 4$ over $[\bar{\theta}, 2\bar{\theta}]$. By repeating this reasoning, we find recursively that over the interval $[n\bar{\theta}, (n+1)\bar{\theta}]$, the control u should be equal to 2^{n+1} . Then, there is no bounded u satisfying (39).

Example 4 (Non-necessity of conditions **(C3)** and **(C7)**). The non-necessity of condition **(C7)** can be easily seen through Example 1. Indeed, even if we cannot express $\delta^1 x$ as a pure function of (z, ξ) , like in expression (21) required

by **(C7)**, we can always write the (z, ξ) -dynamics. In fact, condition **(C7)** can be relaxed by supposing that only the x -states which appear in the dynamics of ξ should respect (21). Concerning condition **(C3)** we give the following example. For simplicity, we consider a constant-delay nonlinear system:

$$\begin{aligned}\dot{x}_1(t) &= 2\delta^1 x_2(t) + \delta^1 u(t) \\ \dot{x}_2(t) &= \arctan(x_1(t)) - u(t) \\ y(t) &= x_1(t) + \delta^1 x_2(t), \quad \forall t \geq 0,\end{aligned}$$

with initial conditions $x_0 \in \mathcal{C}([-\bar{\theta}, 0], \mathbb{R}^2)$ and $u_0 \in \mathcal{U}$. We have $g_0 = (0, -1)^T$, $g_1 = (1, 0)^T$, $f = (2\delta^1 x_2, \arctan(x_1))^T$. A straightforward computation shows that $\bar{L}_{g_0} h = \bar{L}_{g_1} h = 0$, $L_f h = \arctan(\delta^1 x_1) + 2\delta^1 x_2$, $a[\delta] = \bar{L}_{g_0} L_f h + \bar{L}_{g_1} L_f h = -2\delta^1 + \frac{1}{1+(\delta^1 x_1)^2} \delta^2$ and $L_f^2 h = 2 \arctan(\delta^1 x_1) + 2 \frac{\delta^2 x_2}{1+(\delta^1 x_1)^2}$. Condition **(C1)** is satisfied, with $\rho = 2$, the feedback u has to verify the following relation

$$-2\delta^1 u(t) + \frac{1}{1+(\delta^1 x_1)^2} \delta^2 u(t) = -2 \arctan(\delta^1 x_1) - 2 \frac{\delta^2 x_2}{1+(\delta^1 x_1)^2} + \delta^1 v(t), \quad t \geq 3\bar{\theta}, \quad (40)$$

and it is immediate that the causality condition **(C4)** is valid. The new coordinates

$$z_1 = x_1 + \delta^1 x_2, \quad z_2 = \arctan(\delta^1 x_1) + 2\delta^1 x_2, \quad \forall t \geq t_0,$$

satisfies condition **(C2)** with $\ell = 1$. Remark that from the above two equations, we can never express $\delta^1 x$ in function of $\delta z(t)$, $\delta^- z(t)$ and $\delta^{\geq 2} x(t)$, thus condition **(C3)** is not satisfied. Finally, notice that even if **(C3)** does not hold, the boundedness condition **(C5)** is satisfied for this example and its verification is obvious.

5 Proofs

5.1 Proof of Theorem 1

We assume that system (5) satisfies conditions **(C1)**-**(C5)**. Introduce, as explained when stated **(C2)**, the new variables $z_i = L_f^{i-1} h$, for $1 \leq i \leq \rho$. For $t \geq t_0$, we obtain:

$$\begin{aligned}\dot{z}_i &= z_{i+1}, \quad 1 \leq i \leq \rho - 1, \\ \dot{z}_\rho &= L_f^\rho h + \bar{L}_{g_0} L_f^{\rho-1} h u + \bar{L}_{g_1} L_f^{\rho-1} h \delta^1 u = L_f^\rho h + a[\delta] u(t),\end{aligned}$$

System (5) is input-output linearizable if the feedback u verifying

$$a[\delta] u(t) = -L_f^\rho h(\delta x(t), t) + \delta^j v(t), \quad t \geq \tau_j^{-1}(t_0),$$

is bounded and causal. Those properties are guaranteed by the two lemmas stated and proven below. According to Lemma 1, if condition **(C5)** is verified, then the feedback u stays bounded when property (18) is satisfied. Lemma 2 states that **(C4)** is actually a sufficient condition for the causality of u . Therefore, by applying (12), the input-output map becomes $z^{(\rho)} = h^{(\rho)} = \delta^j v(t)$ and systems (5) is thus input-output linearizable by a causal and bounded feedback. If $\rho = n$, then the system is fully input-output linearizable. If, moreover, $j = 0$, it is fully input-output linearizable without delay (with delay, otherwise).

Lemma 1. *Suppose that condition (C5) holds. Then, for every $t \geq t_0$, we have*

$$\|u(t)\| \leq \frac{1}{c-1} \sup_{\tau_{m+1-j}(t_0) \leq s \leq t_0} \|u(s)\| + \frac{c}{c-1} \sup_{s \geq \tau_j^{-1}(t_0)} \left\| \frac{-L_f^\rho h(\delta x(s), s) + \delta^j v(s)}{a^j(\delta x(s), s)} \right\|.$$

Proof. Throughout the proof, by abuse of notation, we write $a^i(t)$ (respectively $L_f^\rho(h)(t)$) instead of $a^i(\delta x(t), t)$, for $0 \leq i \leq m+1$, (respectively $L_f^\rho h(\delta x(t), t)$). If the minimal degree j of the δ -polynomial $a[\delta]$ is such that $j \geq 1$, we start by introducing a new time-scale \tilde{t} that allows us to shift the time in such a way that $\delta^j u(t)$ becomes $u(\tilde{t})$. In order to do that, remark that $\tau_i(\cdot)$ is a strictly increasing function of t , for $1 \leq i \leq m+1$. Indeed, it is easy to verify that

$$\dot{\tau}_{i+1}(t) = \dot{\tau}_i(t)(1 - \theta(\tau_i(t))), \quad \text{for } 1 \leq i \leq m+1,$$

from which we can deduce, by induction, that $\dot{\tau}_{i+1}(t) > 0$, for every $t \geq 0$. This allows us to define a new time-scale $\tilde{t} = \tau_j(t)$, for $t \geq 0$.

By introducing $\tilde{t} = \tau_j(t)$, equation (12) reads

$$\tilde{a}^j(\tilde{t})u(\tilde{t}) + \cdots + \tilde{a}^m(\tilde{t})\delta^{m+1-j}u(\tilde{t}) = -\tilde{L}_f^\rho h(\tilde{t}) + v(\tilde{t}), \quad \forall \tilde{t} \geq t_0, \quad (41)$$

where $\tilde{L}_f^\rho h(\tilde{t}) = L_f^\rho h(\tau_j^{-1}(\tilde{t}))$ and $\tilde{a}^i(\tilde{t}) = a^i(\tau_j^{-1}(\tilde{t}))$, for $j \leq i \leq m+1$, and condition (19) becomes

$$\sup_{\tilde{t} \geq t_0} \left\| \frac{\tilde{a}^i(\tilde{t})}{\tilde{a}^j(\tilde{t})} \right\| \leq \frac{1}{c(m+1-j)}, \quad j < i \leq m+1. \quad (42)$$

From the assumptions made on the delay function $\theta(\cdot)$ (in particular, $\theta(t) > 0$ and $\dot{\theta}(t) < 1$, for every $t \geq 0$), one can easily prove (the proof is left to the reader) the existence of a strictly increasing sequence $(T_k)_{k \geq 0}$ such that

$$T_0 = t_0, \quad T_k = T_{k+1} - \theta(T_{k+1}), \quad \forall k \geq 0,$$

and $T_k \rightarrow +\infty$ with k .

The idea of the proof of Lemma 1 is to give a bound for $u(\tilde{t})$, for \tilde{t} belonging to the sequence of intervals $[T_k, T_{k+1}]$, and, show that, when k goes to infinity, this bound is finite.

Denote

$$M = \sup_{\tilde{t} \geq t_0} \left\| \frac{-\tilde{L}_f^\rho h(\tilde{t}) + v(\tilde{t})}{\tilde{a}^j(\tilde{t})} \right\|.$$

Since τ_i , $1 \leq i \leq m+1$, is an increasing function (as we have just proven), over $[T_0, T_1]$, we have (recall that $T_0 = T_1 - \theta(T_1) = \tau_1(T_1)$):

$$\tau_1(\tilde{t}) \in [\tau_1(T_0), \tau_1(T_1)] = [\tau_1(T_0), T_0],$$

and we can easily verify that

$$\tau_i(\tilde{t}) \in [\tau_i(T_0), \tau_i(T_1)] \subset [\tau_i(T_0), T_0], \quad \forall 1 \leq i \leq m+1-j.$$

Then, from relation (41), it follows that, for every $\tilde{t} \in [T_0, T_1]$, we have

$$\begin{aligned} \|u(\tilde{t})\| &\leq \left\| \frac{\tilde{a}^{j+1}}{\tilde{a}^j} u(\tau_1(\tilde{t})) \right\| + \cdots + \left\| \frac{\tilde{a}^{m+1}}{\tilde{a}^j} u(\tau_{m+1-j}(\tilde{t})) \right\| + M \\ &\leq \left(\left\| \frac{\tilde{a}^{j+1}}{\tilde{a}^j} \right\| + \cdots + \left\| \frac{\tilde{a}^{m+1}}{\tilde{a}^j} \right\| \right) \sup_{\tau_{m+1-j}(T_0) \leq s \leq T_0} \|u(s)\| + M. \end{aligned}$$

Now, recall that according to our assumptions, there exists a constant $c > 1$ such that relation (42) holds for all $\tilde{t} \geq t_0$, thus for $\tilde{t} \in [T_0, T_1]$ also. Hence,

$$\|u(\tilde{t})\| \leq \frac{1}{c} \sup_{\tau_{m+1-j}(T_0) \leq s \leq T_0} \|u(s)\| + M.$$

The latter inequality allows to estimate $u(\tilde{t})$ over $[T_0, T_1]$. As above, over $[T_1, T_2]$, we have $\tau_1(\tilde{t}) \in [\tau_1(T_1), \tau_1(T_2)] = [\tau_1(T_1), T_1]$, and, in the general case, $\tau_i(\tilde{t}) \in [\tau_i(T_1), \tau_i(T_2)] \subset [\tau_i(T_1), T_1]$, for all $1 \leq i \leq m+1-j$. Then, for every $\tilde{t} \in [T_1, T_2]$, we obtain

$$\begin{aligned} \|u(\tilde{t})\| &\leq \left(\left\| \frac{\tilde{a}^{j+1}}{\tilde{a}^j} \right\| + \dots + \left\| \frac{\tilde{a}^{m+1}}{\tilde{a}^j} \right\| \right) \sup_{\tau_{m+1-j}(T_1) \leq s \leq T_1} \|u(s)\| + M \\ &\leq \frac{1}{c} \sup_{\tau_{m+1-j}(T_0) \leq s \leq T_1} \|u(s)\| + M \\ &\leq \frac{1}{c} \left(\sup_{\tau_{m+1-j}(T_0) \leq s \leq T_0} \|u(s)\| + \sup_{T_0 \leq s \leq T_1} \|u(s)\| \right) + M \\ &\leq \frac{1}{c} \left(1 + \frac{1}{c} \right) \sup_{\tau_{m+1-j}(T_0) \leq s \leq T_0} \|u(s)\| + \left(1 + \frac{1}{c} \right) M. \end{aligned}$$

By an induction argument, for $k \geq 2$, we deduce that over $[T_k, T_{k+1}]$, we have

$$\|u(\tilde{t})\| \leq \frac{1}{c} \left(\sum_{i=0}^{k-1} \left(\frac{1}{c} \right)^i \right) \sup_{\tau_{m+1-j}(T_0) \leq s \leq T_0} \|u(s)\| + M \sum_{i=0}^{k-1} \left(\frac{1}{c} \right)^i.$$

Recall that the constant c is such that $c > 1$. Thus, for all $\tilde{t} \geq t_0$ we have

$$\|u(\tilde{t})\| \leq \frac{1}{c-1} \sup_{\tau_{m+1-j}(T_0) \leq s \leq T_0} \|u(s)\| + \frac{c}{c-1} M.$$

□

Lemma 2. *If condition (C4) holds then the feedback u , solution of (12), is causal.*

Proof. Let us first suppose that $g_0 \neq 0$. Applying the operator δ^{-j} on both sides of (12), gives

$$(\delta^{-j} a^j(t))u(t) + \dots + (\delta^{-j} a^m(t))\delta^{m+1-j}u(t) = -\delta^{-j} L_f^p h(t) + v(t), \quad \forall t \geq t_0. \quad (43)$$

As for the proof of Lemma 1, for simplicity of notation, we omit the state dependency of the coefficients a^i as well as of $L_f^p h$. From equation (43), the input that we have to assign to system (5) is given by

$$u(t) = \frac{-\delta^{-j} L_f^p h(t) + v(t)}{\delta^{-j} a^j(t)} - \sum_{k=j+1}^{m+1} \left(\frac{\delta^{-j} a^k(t)}{\delta^{-j} a^j(t)} \right) \delta^{k-j} u(t), \quad \forall t \geq t_0.$$

We see clearly, that condition (17) avoids the existence of advanced states in the later equation, and then no causality problem can appear in this case.

Let us now assume that $g_0 \equiv 0$ (and $g_1 \neq 0$). In this case, we have to assign $\delta^1 u(t)$ to system (5), for $t \geq t_0$. Thus, applying the operator δ^{-j+1} on both sides of (12), we obtain that

$$\delta^1 u(t) = \frac{-\delta^{-j+1} L_f^\rho h(t) + \delta^1 v(t)}{\delta^{-j+1} a^j(t)} - \sum_{k=j+1}^{m+1} \left(\frac{\delta^{-j+1} a^k(t)}{\delta^{-j+1} a^j(t)} \right) \delta^{k-j+1} u(t), \quad \forall t \geq \tau_1^{-1}(t_0).$$

Condition (17) avoids the existence of advanced states in the later equation. \square

5.2 Proof of Proposition 1

Let $(T_k)_{k \geq 0}$ be the time sequence introduced in the proof of Lemma 1. Since τ_i is an increasing function, for $i \geq 0$, then over $[T_0, T_1]$ we have

$$\tau_i(t) \in [\tau_i(T_0), \tau_i(T_1)] \subset [\tau_{q(\rho+1)}(T_0), T_0], \quad \forall \ell + 1 \leq i \leq q(\rho + 1),$$

where the integer ℓ is defined by **(C2)**. Knowing that over $[\tau_{q(\rho+1)}(T_0), T_0]$ we have $\|x(t)\| < \mu$, then, thanks to inequality (16) of **(C3)**, for every $t \in [T_0, T_1]$, we have:

$$\|\delta^\ell x(t)\| \leq \alpha_1 \|(\delta z(t), \delta^- z(t))\| + \alpha_2 \|\delta \xi(t), \delta^- \xi(t)\| + \alpha_3 \|\delta^{\geq \ell+1} x(t)\|.$$

From which we obtain, knowing that

$$\sup_{t \geq \tau_{q(\rho+1)}(T_0)} \{\|z(t)\|, \|\xi(t)\|\} < \varepsilon,$$

where ε is given by (20), together with the fact that the maximum delay-advance order in (z, ξ) cannot exceed $q(\rho + 1)$ (see the discussion of condition **(C3)** after the statement of Theorem 1), that

$$\|\delta^\ell x(t)\| < (2q(\rho + 1) + 1) \max\{\alpha_1, \alpha_2\} \varepsilon + \alpha_3 (q(\rho + 1) - \ell) \mu.$$

Hence,

$$\|\delta^\ell x(t)\| < \mu, \quad \forall t \in [T_0, T_1].$$

Using the same reasoning, we can straightforwardly prove that $\|\delta^\ell x(t)\| < \mu$ over $[T_k, T_{k+1}]$, for every $k \geq 1$. Therefore, $\|\delta^\ell x(t)\| = \|x(\tau_\ell(t))\| < \mu$, for $t \geq t_0$, and since $\tau_\ell(t) < t$, it follows immediately that the trajectories of system (5) lie inside the ball $B(0, \mu)$ for every $t \geq t_0$.

5.3 Proof of Proposition 3

The proof is inspired from [7]. Let $\lambda_1, \dots, \lambda_\rho$ be the eigenvalues of the matrix $A + BK$ associated to (24). Recall that we have supposed that all of them are different. In addition, assume, without any loss of generality, that λ_1 is the largest eigenvalue. Let P be a change of basis matrix diagonalizing $A + BK$ and denote $M = \|P\| \|P^{-1}\| + 1$. Let ε_1 be any positive constant such that

$$\max\{M\varepsilon_1, \beta(\varepsilon_1, 0) + \gamma(M\varepsilon_1)\} \leq \frac{1 - \alpha_3(q(\rho + 1) - \ell)}{(2q(\rho + 1) + 1) \max\{\alpha_1, \alpha_2\}} \mu,$$

where $l, \mu, \alpha_1, \alpha_2$ and α_3 are given by conditions **(C2)**-**(C3)** and $\beta(\cdot)$ and $\gamma(\cdot)$ are given by Definition 7 applied to the zero-dynamics (which is assumed to be uniformly input-to-state stable).

For $t \geq \tau_j^{-1}(t_0)$, the dynamics of the z -subsystem is given by $\dot{z} = (A + BK)z$ and is governed by a Hurwitz matrix. Thus, for $t \geq \tau_j^{-1}(t_0)$, we have

$$z(t) = e^{(A+BK)(t-\tau_j^{-1}(t_0))} z(\tau_j^{-1}(t_0)) = P e^{D(t-\tau_j^{-1}(t_0))} P^{-1} z(\tau_j^{-1}(t_0)),$$

where D is the diagonal matrix associated to $A + BK$, and we deduce

$$\|z(t)\| \leq (M-1)e^{\lambda_1(t-\tau_j^{-1}(t_0))} \|z(\tau_j^{-1}(t_0))\| \leq (M-1)\|z(\tau_j^{-1}(t_0))\|, \quad \forall t \geq \tau_j^{-1}(t_0).$$

Now recall (as explained at the end of Section 2) that the z -subsystem is in free evolution over $[t_0, \tau_j^{-1}(t_0)]$ and the feedback $v(t) = \sum_{i=1}^{\rho} k_i z_i(t)$ starts acting on the system from the instant $\tau_j^{-1}(t_0)$. Thus, for every $1 \leq i \leq q(\rho + 1)$, the delayed z -states verify the following inequality

$$\|\delta^i z(t)\| \leq \max \left\{ \sup_{[\tau_{q(\rho+1)}(t_0), \tau_j^{-1}(t_0)]} \|z(t)\|, (M-1)\|z(\tau_j^{-1}(t_0))\| \right\}, \quad \forall t \geq t_0.$$

Since, by hypothesis, $\|z(t)\| < \varepsilon_1$ over $[\tau_{q(\rho+1)}(t_0), \tau_j^{-1}(t_0)]$, then

$$\|\delta^i z(t)\| < M\varepsilon_1, \quad \forall t \geq t_0,$$

for all $-q(\rho + 1) \leq i \leq q(\rho + 1)$. By consequence, for every $t \geq t_0$, we have

$$\begin{aligned} \|(\delta z(t), \delta^- z(t))\| &= \|z(t), \delta^1 z(t), \dots, \delta^{q(\rho+1)} z(t), \delta^{-1} z(t), \dots, \delta^{-q(\rho+1)} z(t)\| \\ &\leq (2q(\rho + 1) + 1) \max_{-q(\rho+1) \leq i \leq q(\rho+1)} \|\delta^i z(t)\| \\ &\leq (2q(\rho + 1) + 1) M\varepsilon_1. \end{aligned} \tag{44}$$

On other hand, since the zero-dynamics is input-to-state stable for the (exogenous) input $w(t) = (\delta z, \delta^- z)$, according to Definition 7, for every $t \geq t_0$, we have

$$\begin{aligned} \|\xi(t)\| &\leq \beta(\|\xi_{t_0}\|_{\infty}, t - t_0) + \gamma(\|z_{[\tau_{q(\rho+1)}(t_0), t]}\|_{\infty}) \\ &\leq \beta(\varepsilon_1, 0) + \gamma(M\varepsilon_1). \end{aligned} \tag{45}$$

Thus, for every $1 \leq i \leq q(\rho + 1)$, the delayed ξ -states verify the following inequality for all $t \geq t_0$

$$\begin{aligned} \|\delta^i \xi(t)\| &\leq \max\{\|\xi_{t_0}\|_{\infty}, \beta(\|\xi_{t_0}\|_{\infty}, 0) + \gamma(\|z_{[\tau_{q(\rho+1)}(t_0), t]}\|_{\infty})\} \\ &\leq \max\{\varepsilon_1, \beta(\varepsilon_1, 0) + \gamma(M\varepsilon_1)\}. \end{aligned} \tag{46}$$

By consequence, for every $t \geq t_0$, we have

$$\begin{aligned} \|(\delta \xi(t), \delta^- \xi(t))\| &= \|\xi(t), \delta^1 \xi(t), \dots, \delta^{q(\rho+1)} \xi(t), \delta^{-1} \xi(t), \dots, \delta^{-q(\rho+1)} \xi(t)\| \\ &\leq (2q(\rho + 1) + 1) \max\{\varepsilon_1, \beta(\varepsilon_1, 0) + \gamma(M\varepsilon_1)\}. \end{aligned} \tag{47}$$

Consider the same time intervals $(T_k)_{k \geq 0}$ as in Lemma 1 and using the same reasoning as in the proof of Proposition 1, we have (thanks to condition **(C3)**)

that for every $t \in [T_0, T_1]$

$$\begin{aligned} \|\delta^\ell x(t)\| &\leq \alpha_1 \|(\delta z(t), \delta^- z(t))\| + \alpha_2 \|\delta \xi(t), \delta^- \xi(t)\| + \alpha_3 \|\delta^{\geq \ell+1} x(t)\| \\ &\leq (2q(\rho+1) + 1) \max\{\alpha_1, \alpha_2\} \max\{M\varepsilon_1, \beta(\varepsilon_1, 0) + \gamma(M\varepsilon_1)\} \\ &\quad + \alpha_3(q(\rho+1) - \ell)\mu \\ &< \mu. \end{aligned}$$

Using the same reasoning, we can straightforwardly prove that $\|\delta^\ell x(t)\| < \mu$ over $[T_k, T_{k+1}]$, for every $k \geq 1$. Thus, under the assumption given by (16), the trajectories of system (5) lie inside the ball $B(0, \mu)$ for every $t \geq \tau_\ell(t_0)$. Thus, $\overline{\lim}_{t \rightarrow +\infty} \|x(t)\| \leq \mu$. Knowing the exponential stability of the z -subsystem, we have $\lim_{t \rightarrow +\infty} z(t) = 0$. On the other hand, from the input-to-state stability of the zero dynamics, $\lim_{t \rightarrow +\infty} \xi(t) = 0$. Thus,

$$\overline{\lim}_{t \rightarrow +\infty} \|x(t)\| \leq \alpha_3 \overline{\lim}_{t \rightarrow +\infty} \|x(t)\|.$$

Using the fact that $\alpha_3 < 1$ (recall that α_3 is such that $0 < \alpha_3 < 1/(q(\rho+1) - \ell)$, see condition **(C3)**), we have $\lim_{t \rightarrow +\infty} x(t) = 0$.

6 Conclusion

In this paper, we have considered the problem of input-output linearization of single-input single-output nonlinear systems with time-varying delays appearing in the state, the input and the output. Sufficient conditions have been developed in order to guarantee the existence of a causal and bounded feedback linearizing the input-output map of the feedback modified system. We have discussed and presented sufficient conditions ensuring the stability of the feedback modified system. In a future work, the input-output linearization and decoupling of multi-input multi-output systems will be studied.

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