



HAL
open science

Multi-period Multi-item Lot sizing under Random Yield and Stochastic Capacity

Mehrnoosh Mohammadi, Simon Thevenin, Alexandre Dolgui

► **To cite this version:**

Mehrnoosh Mohammadi, Simon Thevenin, Alexandre Dolgui. Multi-period Multi-item Lot sizing under Random Yield and Stochastic Capacity. 13th International Conference on Modeling, Optimization and Simulation - MOSIM'20, Nov 2020, Agadir (en ligne), Morocco. hal-03052909

HAL Id: hal-03052909

<https://hal.science/hal-03052909v1>

Submitted on 10 Dec 2020

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Multi-period Multi-item Lot sizing under Random Yield and Stochastic Capacity

Mehrnoosh MOHAMMADI, Simon THEVENIN, Alexandre DOLGUI

IMT Atlantique

Nantes, France

m.mohammadi94@ut.ac.ir, simon.thevenin@imt-atlantique.fr, alexandre.dolgui@imt-atlantique.fr

ABSTRACT: *This study addresses a multi-product multi-period lot-sizing problem with stochastic yield and capacity. Typically, random yield is the result of defective items, whereas the capacity is uncertain due to machine breakdowns. Considering both stochastic yield and capacity leads to a complex problem, since these two uncertainty types are in conflict. Indeed, the planners tend to produce large batches to overcome random yields, but the resulting plan might not be feasible due to random capacity. We provide an analytical expression for the service level function. A dynamic programming algorithm is also proposed to find the optimal released quantities. We validate the proposed methodology with a numerical example.*

KEYWORDS: *Lot-sizing, Uncertainties, Random yield, Random capacity, Stochastic production planning.*

1 INTRODUCTION

To respond to today's competitive business world, companies tend to increase the size of their product assortments, launch new products frequently, and adjust their production processes constantly (acquire new equipment). In these conditions, production systems can become very difficult to manage due to the lack of regularity and consistency in the production process, making it very difficult for manufacturers to ensure product-process quality. To face these challenges, inventory control systems must be able to cope with uncertainties from both process (e.g., equipment failure) and product quality. The goal of this work is to provide a methodology to size the production lots in a multi-item multi-period production line subject to break-down and defective items. In inventory control, machine breakdowns leads to stochastic resource capacity, whereas bad product quality leads to random yield.

One of the first studies on inventory control with imperfect item quality and its effect on production run time and economic production quantity (EPQ) was (Karlin et al. 1958). However, despite the large amount of works following this seminal work, there is a lake of study considering both random yield and capacity. The joint consideration of these uncertainties is critical since they are conflicting. Indeed, companies tends to produce large lot-size to hedge against yield uncertainty. This large lot sizes lead to a tight usage of the production capacity, which is not desirable when the capacity is unknown.

The only works considering jointly these two types

of uncertainties are limited to single period models (Schemeleva et al. 2018). To avoid nervousness, companies manage their production plan with a frozen horizon. In this context, companies multi-periods problem, where the uncertainties in a period results in over and under productions that affects future periods.

This study, we consider a single machine production line which operates with a lot for lot policy to satisfy the demand of each item in each period. Multi-item production planning requires the known sequence of products; there are some works in literature that propose a method for schedule the production sequence (e.g., Dolgui et al. 2005; Schemeleva et al. 2018, 2012), regarding to focus on the lot sizing in this paper, we assume that the sequence is predefined and known. Based on our reviews, lot sizing under yield and capacity uncertainties is consider as a single period problem; it means that at the end of the period there are maybe backlog or inventory which are ignored totally in the probable next period. In this regard, in this study we consider backlog/inventory in all periods. In other word, if there is a shortage in a period we add it to the next period demand and if there is inventory we reduce the next period demand. The goal of the study is to maximize the total service level which is probability of meet the demand of all item in all period at the end of the horizon. We also assume that in each stage of planning the released quantity is frozen and the next stage is planned based on non-changeable decided released quantities of previous stages.

The contributions of this paper include a mathemati-

cal analysis of the multi-item lot-sizing under stochastic yield and capacity. This analysis leads to the definition of the service level function, that incorporates the probabilistic backlog/inventory from one period to the next. In addition, we propose a dynamic programming algorithm that find the optimal released quantities, along with some efficient speed up strategies. We evaluate the impact of the multi-period model compare to single period. Finally, we provide some managerial insights.

The rest of this study is as follows: Section2 reviews the literature, section3 introduces the considered problem and gives the proposed mathematical model, section5 presents the employed optimization approach, section6 presents the empirical validation of proposed methodologies and section7 discusses about conclusion of work.

2 LITERATURE REVIEW

Inventory control plays a crucial role in the management of a production system since it impacts the customer satisfaction and the supply chain costs. Indeed, an improper inventory policy may lead to shortage of products or needless stocks, and both are costly for companies. Therefore, there is a significant motivation for developing efficient and cost-effective inventory policy/methods.

There is a vast literature on lot-sizing problem under uncertainty, and various uncertain parameters have been considered such as demand, yield, lead time or capacity. Review papers on this topic include (Aloulou et al. 2014; Dolgui and Prodhon 2007; Yano and Lee 1995). In the recent review of (Aloulou et al. 2014) on non-deterministic lot-sizing models, the authors classified such problems based on: the number of periods, the number of products, the number of machines, the uncertain parameters, and the modelling approaches. The present work focuses on multi-period, multi product, single machine, under uncertain yield and capacity. Therefore, the review below focuses on these two types of uncertainties.

Since the seminal paper of (E. Silver 1976) where the traditional EOQ model is extended with yield uncertainty, an extensive amount of research has been done on lot sizing problems with random yields (e.g., Gerchak and Grosfeld-Nir 1998; Gerchak et al. 1994; Inderfurth 2009; Khan et al. 2010; Yano and Lee 1995). There exists three approaches to represent random yield in a production context. A first stream of research (e.g., Agnihotri et al. 2000; Dolgui and Proth 2010; Haji et al. 2008; Li et al. 2008; Papachristos and Konstantaras 2006; Wang and Gerchak 2000) considers that the fraction of defective items is known, and the problem is deterministic. A second stream of research (e.g., Singh et al. 1988; Teunter and Flapper

2003) represents random yield as Bernoulli process, where a certain fraction of goods is defective, and the number of defective items depends on the lot size. The third research stream (e.g., Maddah and M. Y. Jaber 2008; Salameh and M. Jaber 2000) models random yield with a geometric distribution, where the manufacturing system becomes out of control with a given probability, and all produced items are defective. However, yield uncertainty does not necessarily result from bad product quality, but it can also arise due to shortage from the suppliers (e.g., Güllü et al. 1999; Moon et al. 2012).

Random capacity is a practical issue in different types of industries such as high-tech industries like semiconductor industry (Geng and Jiang 2007). (Wang and Gerchak 1996a) illustrate the possible impacts of variable capacity on the optimal solution in the continuous lot sizing problem. The uncertainty on the capacity has various causes, such as unexpected breakdowns and unplanned maintenance, material shortages; an uncertain duration of repair, even when the repair is planned and strikes (Ciarallo et al. 1994; Wang and Gerchak 1996a). Among these causes, machine breakdowns attracted the most attention in the literature (e.g., Chakraborty et al. 2009; Chiu 2010; Giri and Dohi 2004; Lin and Gong 2006). The mathematical modelling of machine breakdown is usually based on two factors, namely, the mean time to/between failure (MTTF/MTBF) and mean time to repair. For instance, (Lin and Gong 2006) considered the EPQ model with random breakdowns and constant repair times. Similarly, (Giri and Dohi 2004; Giri and Yun 2005) examined the optimal lot-sizing problem with at most two failures during the planning period (both failures and repair times follow a general (arbitrary) distribution).

Very few papers (Dolgui et al. 2005; Jain and E. A. Silver 1995; Schemelewa et al. 2018, 2012; Wang and Gerchak 1996b) consider simultaneously random yields and capacity. (Wang and Gerchak 1996b) proposed a stochastic dynamic programming for the single item case, to minimize the expected total cost (i.e., production, holding, and shortage cost). (Jain and E. A. Silver 1995) investigated the case where suppliers allow to reserve a dedicated capacity level by paying a premium charge. These assumptions illustrated the independence of the optimal order size and the random capacity. (Dolgui et al. 2005) study a sequencing and lot sizing problem for a multi-item single-period production system under uncertainties. The goal of their problem is maximizing service level, and they use dynamic programming for finding the solution of lot sizing part of the model. This work was extended in (Schemelewa et al. 2012) and (Schemelewa et al. 2018), where the author propose a memetic algorithm for lot sizing (production planning) part and a heuristic method without decomposition the two

sub-problems.

To conclude this literature review, despite its relevance in practice, very few studies considered random yield and capacity simultaneously. As these two uncertainty types are in conflicts, it is crucial to manage them simultaneously. Indeed, companies tend to produce more than required to overcome random yield issues, but a random capacity should not be used tightly. The few studies on lot-sizing with random yield and capacity are limited to one period. To the best of our knowledge, the multi-period multi-item lot sizing problem with random yield and capacity uncertainties has never been studied in the literature.

3 PROBLEM STATEMENT

This section gives a formulation of the considered problem. Table 1 summarizes the notations used in the rest of the paper. Note that we adopt the following convention. $M_{1\dots m,1\dots n}$ denotes a $m \times n$ matrix, $M_{i,1\dots n}$ (resp. $M_{1\dots m,i}$) denotes the i^{th} lines (resp. column) denotes the i^{th} lines, and $M_{i,j}$ denotes an element of the matrix. Finally, $M_{i,1\rightarrow n}$ denotes the sum of the components in the vector $M_{i,1\dots n}$.

We consider a multi-item multi-period lot sizing problem with stochastic yield and stochastic capacity. Given the demand D_{it} for each item i in period t of the planning horizon, the considered problem is to determine the production quantity x_{it} to be released for each item i in each period t , in order to maximize the service level.

The production must respect the capacity of the bottleneck resource. More precisely, there is C time capacity in each period, and the production of one unit of item i consumes k_i time of the capacity. However, the machine can breakdown during operation, and it becomes unusable during a random repair time. Similarly to (Dolgui et al. 2005), we assume that the failure and repair incidents follow independent exponential distributions, with mean time between failures $1/U$ and mean time to repair $1/\bar{U}$.

In addition, the machine can produce defective items. More precisely, we assume that each item i is of good quality with probability p_i . Therefore, the actual produced quantities y_{it} is lower than or equal to the released quantity x_{it} .

Because of these uncertainties, the produced quantity might not meet the demand, and there can be backlog and inventory. In other words, the unmet demand in period t can be met in period $t + 1$.

The decision framework assumes that the production quantities are decided for the entire horizon, and they are frozen.

$i = 1, \dots, n$	Index for items
$t = 1, \dots, T$	Index for period
p_i	Probability to produce a good item of type i
k_i	Capacity consumption per unit of item i
C	Production capacity per period
D_{it}	Demand of item i in period t
$1/U$	Mean time between failures
$1/\bar{U}$	Mean time to repair
$\gamma_{1\dots n}$	Vector of the items sequence
$R_{1\dots t}$	Repair time from period 1 to t
$x_{1\dots n,1\dots T}$	Matrix of released quantities for all items and all periods
$y_{1\dots n,1\dots T}$	Matrix of processed quantities for all items and all periods
$z_{1\dots n,1\dots T}$	Matrix of produced quantities for all items and all periods
$P_{it}^b(\tilde{D}_{it} x_{1\dots i,1\dots t-1})$	Probability to have a total demand \tilde{D}_{it} (including backlog and inventory) in period t given the released quantities.
$p(z_{it} \geq \tilde{D}_{it} x_{1\dots i,1\dots t})$	Probability to meet the demand given the released quantities.
$p_{it}^{y=} (z_{it} y_{it})$	Probability to yield exactly z_{it} good quality items, given the processed quantities
$p_{it}^{y>} (z_{it} y_{it})$	Probability to yield at least z_{it} good quality items, given the processed quantities
$p_{it}^{c=} (y_{it} x_{1\dots it})$	Probability to process exactly y_{it} item i in period t given the released quantities.
$p_{it}^{c>} (y_{it} x_{1\dots it})$	Probability to process at least y_{it} item i in period t given the released quantities.
$p_{it}^{cy=} (z_{it} x_{it})$	Probability to yield exactly z_{it} good quality items, given the released quantities
$p_{it}^{cy>} (z_{it} x_{it})$	Probability to yield at least z_{it} good quality items, given the released quantities
$p^r(R \leq \bar{K} K)$	Probability to have a repair time R lower than \bar{K} , with a required operation time K .

Table 1 – Notations

In the rest of this paper, we make a distinction between the quantity released, processed and produced. The *released* quantities ($x_{1\dots n,1\dots t}$) is the amount of item the planner asked to produced. The quantity *processed* ($y_{1\dots n,1\dots t}$) is the quantity that can be process by the machine with its actual capacity. The quantity *produced* ($z_{1\dots n,1\dots t}$) is the amount of item process that are of good quality.

The objective is to maximize the service level S , which is defined as the probability to meet the demand for all items in all period. That is

$$S = \prod_{i=1}^n \prod_{t=1}^T p(z_{i1\rightarrow t} \geq D_{i1\rightarrow t} | x_{1\dots i,1\dots t}), \quad (1)$$

where $p(z_{i1\rightarrow t} \geq D_{i1\rightarrow t} | x_{1\dots i,1\dots t})$ is the probability that the total produce quantities $z_{i1\rightarrow t}$ from period 1 to t is larger than the total demand of these periods.

4 MATHEMATICAL ANALYSIS

The service level can be decomposed based on three stochastic parameters

$$p(z_{i1\rightarrow t} \geq D_{i1\rightarrow t} | x_{1\dots i,1\dots t}) = \sum_{\tilde{D}_{it}=D_{i1\rightarrow t}-x_{i1\rightarrow t-1}}^{D_{i1\rightarrow t}} (p_{it}^b(\tilde{D}_{it} | x_{1\dots i,1\dots t-1}) \times p_{it}^{y>}(\tilde{D}_{it} | y_{it}) \times p_{it}^{c=} (y_{it} | x_{1\dots it})), \quad (2)$$

where $p(z_{it} \geq \tilde{D}_{it} | y_{it})$ is the probability to produce at least \tilde{D}_{it} good items of type i given that y_{it} items of type i are processed in period t , $p_{it}^{c=} (y_{it} | x_{1\dots it})$ is the probability to process y_{it} given the quantity released in period t , and $p_{it}^b(\tilde{D}_{it} | x_{1\dots i,1\dots t-1})$ is the probability to have a total demand \tilde{D}_{it} (including backlog and inventory) in period t .

This function tries to maximize the probability of meeting demand of item i from period one to the period t when all $i-1$ released quantities are planned. The total demand \tilde{D}_{it} can vary from subtraction of sum of the demands of item i from first period to period t ($D_{i1\rightarrow t}$) from the total released quantity of item i until end of period $t-1$ ($x_{i1\rightarrow t-1}$), which means all released items have good quality and there maybe inventory from the previous periods, to sum of the demands of item i from first period to period t , which indicates a large backlog. The probability of this value depends on the previous periods released quantities, then base on this value, the function calculate the probability of the meeting this demand with the pre-assumed released quantity of the item i in period t .

It should be noted that for the first period, there is no variation for the total demand and it is fixed as

the initial demands of each item, so we have to delete the external summation and replace \tilde{D}_{it} with D_{it} .

Basically, the larger are the sizes of the lots, the larger is the value of $p_{it}^y(z_{it} \geq \tilde{D}_{it} | y_{it})$, and the smaller is $p_{it}^{c=} (y_{it} | x_{it})$ since there may be less capacity to produce the last lots.

We detail below the computation of these three probability functions.

4.1 Yield uncertainty

The probability to $p_{it}^y(z_{it} | y_{it})$ to yield exactly z_{it} good quality items when y_{it} items i are processed can be computed with Bernoulli's formula:

$$p_{it}^{y=} (z_{it} | y_{it}) = C_{y_{it}}^{z_{it}} (1 - p_i)^{(y_{it}-z_{it})} p_i^{z_{it}}, \quad (3)$$

where p_i is the given perfection probability.

The probability to yield more than z_{it} item i in period t if y_{it} items are processes is

$$p_{it}^{y>} (z_{it} | y_{it}) = \sum_{a=z_{it}}^{y_{it}} C_{y_{it}}^a (1 - p_i)^{(y_{it}-a)} p_i^a \quad (4)$$

And the probability to yield more than z_{it} item i in period t if $x_{1\dots it}$ items are released is

$$p_{it}^{cy>} (z_{it} | x_{1\dots it}) = \sum_{y_{it}=0}^{x_{it}} p_{it}^{y>} (z_{it} | y_{it}) \times p_{it}^{c=} (y_{it} | x_{1\dots it}) \quad (5)$$

Similarly,

$$p_{it}^{cy=} (z_{it} | x_{1\dots it}) = \sum_{y_{it}=0}^{x_{it}} p_{it}^{y=} (z_{it} | y_{it}) \times p_{it}^{c=} (y_{it} | x_{1\dots it}) \quad (6)$$

4.2 Capacity uncertainty

The processing of items i starts only once the first $i-1$ lots are completed. More precisely, y_{it} items i are processed in period t if the repairing time R_t respects the capacity constraint

$$\sum_{j=1}^{j=i-1} k_j x_{jt} + y_{it} k_i + R_t \leq C \quad (7)$$

where k_i is the capacity consumption per unit of item i , and C is the capacity per period.

For sake of clarity, we define:

$$K = K(x_{1\dots i-1,t}, y_{it}) = \sum_{j \in 1\dots i-1} k_j x_{jt} + k_i y_{it} \quad (8)$$

and

$$\bar{K} = \bar{K}(x_{1\dots i-1t}, y_{it}) = C - K(x_{1\dots i-1t}, y_{it}) \quad (9)$$

The probability $p_{it}^{c>}(y_{it}|x_{1\dots it})$ to process more than y_{it} items i in periods t coincides with the probability $p^r(R_t < \bar{K}|K)$ to have less than \bar{K} units of repair time in period t if the required processing time is K . That is,

$$p_{it}^{c>}(y_{it}|x_{1\dots it}) = p^r(R_t < \bar{K}|K) \quad (10)$$

If the time to failure and repair time follow an exponential distribution, (Dolgui et al. 2005) showed that such a repair time probability can be computed as follows:

$$p^r(R_t \leq \bar{K}|K) = 1 - e^{-(KU + \bar{K}U)} \sum_{\nu=1}^{\infty} \left[\frac{(KU)^\nu}{\nu!} \sum_{j=0}^{\nu-1} \frac{(\bar{K}U)^j}{j!} \right] \quad (11)$$

where $\frac{1}{U}$ is the mean time between failures, and $\frac{1}{\bar{U}}$ is the mean time to repair. As mentioned in (Dolgui 2002), the infinite sum in (9) can be approximated with a large enough number, since the function converge quickly.

Finally, the probability $p_{it}^{c=}(y_{it}|x_{1\dots it})$ to process precisely y_{it} items i in period t . can be inferred from $p_{it}^{c>}(y_{it}|x_{1\dots it})$ as follows:

$$p_{it}^{c=}(y_{it}|x_{1\dots it}) = p_{it}^{c>}(y_{it}|x_{1\dots it}) - p_{it}^{c>}(y_{it} + 1|x_{1\dots it}) \quad (12)$$

In the case of $y_{it} = x_{it}$, if $p_{it}^{c>}(y_{it}|x_{1\dots it})$ would be equal to 1, we set $p_{it}^{c=}(y_{it}|x_{1\dots it})$ for $y_{it} = x_{it}$ equal to 1 and for all $y_{it} \leq x_{it}$ the value of the probability will ignore and consider it as 0.

4.3 Backlog and inventory uncertainty

The backlog/inventory level in period t can be computed from the backlog/inventory level in period $t-1$. More precisely, the flow conservation constraint can be expressed as follows

$$\tilde{D}_{it} = \tilde{D}_{it-1} - z_{it-1} + D_{it} \quad (13)$$

Equation (14) gives a recursive formula to compute the actual requirements (including backlog/inventory) in period t , where s_1 refers to the production in period $t-1$, and s_2 refers to the actual requirements in period $t-1$.

$$p_{it}^b(\tilde{D}_{it}|x_{1\dots i1\dots t-1}) = \sum_{s_1, s_2 | s_2 - s_1 + D_{it} = \tilde{D}_{it}} p_{it-1}^{cy=}(s_1|x_{1\dots it-1}) p_{it-1}^b(s_2|x_{1\dots i1\dots t-2}) \quad (14)$$

For the first period equation 14 must be changed as equations 15 and 16.

$$p_{i1}^b(\tilde{D}_{i1} = D_{i1}|\emptyset) = 1 \quad (15)$$

$$p_{i1}^b(\tilde{D}_{i1} \neq D_{i1}|\emptyset) = 0 \quad (16)$$

The computation can be simplified as follows.

$$p_{it}^b(\tilde{D}_{it}|x_{1\dots i1\dots t-1}) = \sum_{s_1=x_{it-1}}^{s_1=x_{it-1}} p_{it-1}^{cy=}(s_1|x_{1\dots it-1}) p_{it-1}^b(\tilde{D}_{it} - D_{it} + s_1|x_{1\dots i1\dots t-2}) \quad (17)$$

The values of x_{it} are limited to predefined intervals $[\underline{x}_{it}, \bar{x}_{it}]$ which are computed based on a user defined minimum service level β_i .

$$x_{it}^+ = \min\{\bar{x}_{it}, \lfloor \underline{x}_{it} + (C - \sum_{j=1}^n k_j x_{j,t})/k_i \rfloor\} \quad (18)$$

where \underline{x}_{it} is the minimum lot size of item i such that $p_{it}^{cy>}(z_{it} \geq D_{it}|x_{it}) \geq \beta_i$, and \bar{x}_{it} is the minimum lot size of item i such that $p_{it}^{cy>}(z_{it} \geq D_{it}|x_{it}) \geq 1 - \epsilon$ for a sufficiently small ϵ .

5 OPTIMIZATION APPROACH

We propose below a dynamic programming algorithm to compute the released quantities. In each stage, the algorithm computes the cost $H_i(V_{1\dots T}^i)$ of the optimal released quantities to complete the first i lots before the vector of times $V_{1\dots T}^i$ in each period.

$$H_i(V_{1\dots T}^i) = \max \left\{ \prod_{i=1}^n p(z_{i1 \rightarrow T} \geq D_{i1 \rightarrow T} | x_{1\dots i1\dots T}) \mid \sum_{j=1}^i k_j x_{j\tau} = V_\tau, x_{j\tau} \in [\underline{x}_{j\tau}, \bar{x}_{j\tau}], \tau \in 1 \dots t \right\} \quad (19)$$

with

$$p(z_{i1 \rightarrow T} \geq D_{i1 \rightarrow T} | x_{1\dots i1\dots T}) = \prod_{t=1}^T p(z_{i1 \rightarrow t} \geq D_{i1 \rightarrow t} | x_{1\dots i1\dots t}) \quad (20)$$

The cost $H_i(V_{1\dots T}^i)$ of the optimal lots sizes up to item i with completion times vector $V_{1\dots T}^i$, can be computed from the optimal cost of the previous stage

$H_{(i-1)}(V_{1...T}^{(i-1)'}) :$

$$\begin{aligned}
H_i(V_{1...T}^i) = \max\{ & H_{(i-1)}(V_{1...T}^{(i-1)'}) \times \\
& p(z_{i1 \rightarrow t} \geq D_{i1 \rightarrow t} | x_{1...i1...t}) \\
& | k_j x_{j\tau} = (V_{j\tau} - V_{(j-1)\tau}'), \\
& x_{j\tau} \in [\underline{x}_{j\tau}, \bar{x}_{j\tau}], \\
& V_{(i-1)\tau}' \in [\underline{V}_{(i-1)\tau}', \bar{V}_{(i-1)\tau}'], \\
& (V_{j\tau} - V_{(j-1)\tau}') \in [\underline{w}_{j\tau}, \bar{w}_{j\tau}], \\
& \tau \in 1 \dots T, \\
& j \in 1 \dots i\}
\end{aligned} \tag{21}$$

where

$$\underline{V}_{it} = \sum_{j=1}^i \underline{w}_{jt} \tag{22}$$

$$\bar{V}_{it} = \min\{T_t - \sum_{j=i+1}^n \underline{w}_{jt}, \sum_{j=1}^i \bar{w}_{jt}\} \tag{23}$$

$$\underline{w}_{jt} = k_j \underline{x}_{jt} \tag{24}$$

$$\bar{w}_{jt} = k_j x_{jt}^+ \tag{25}$$

The initial state $V_{1...T}^0 = 0 \dots 0$, and $H_0(V_{1...T}^0) = 1$.

5.1 Speedup strategy

This section provide the complexity of the proposed dynamic program, and it shows that the number of states grows exponentially with the number of period. To circumvent this issue, we propose to prune the states based on a lower bound and dominance properties.

A lower bound on the service level: Such a lower bound can be computed with a heuristic. To do so, we use the method proposed in ... for a the single period lot-sizing with random yield and capacity. This method is applied successively on each period, and this yield a set of released quantity $x_{1...n,1...T}^h$. The service level associated with $x_{1...n,1...T}^h$ is computed with equation (6). The resulting service level is sub-optimal, but this lower bound allows to prune a large number of node.

Dominated states: A state $V_{1...T}^i$ dominates the state $W_{i1...T}$, if $V_{1...T}^i \leq W_{i1...T}$ and $H_i(V_{1...T}^i) \geq H_i(W_{i1...T})$.

Algorithm 1 Dynamic programming algorithms

```

Set  $V_{1...T}^0 = 0 \dots 0$ , and  $H_0(V_{1...T}^0) = 1$ 
for all items  $i \in \{1 \dots n\}$  do
  for all  $V_{1...T}^i$  such that  $V_{it} \in [\underline{V}_{it}, \bar{V}_{it}]$  do
     $H_i(V_{1...T}^i) = 0$ 
    for all  $x_{i1...T}$  such that  $x_{it} \in [\underline{x}_{it}, \bar{x}_{it}]$  do
      for all states  $V_{1...T}^{(i-1)}$  such that  $V_{1...T}^{(i-1)} +$ 
       $K(x_{i1...T}) = V_{1...T}^i$  do
        if  $H_{(i-1)}(V_{1...T}^{(i-1)}) \times p(z_{i1 \rightarrow T} \geq$ 
         $D_{i1 \rightarrow T} | x_{1...i1...T}) \geq H_i(V_{1...T}^i)$  then
           $H_i(V_{1...T}^i) = H_{(i-1)}(V_{1...T}^{(i-1)}) \times$ 
           $p(z_{i1 \rightarrow T} \geq D_{i1 \rightarrow T} | x_{1...i1...T})$ 
        end if
      end for
    end for
  end for
  Remove states based on the lower bound and
  dominance properties;
end for

```

6 NUMERICAL EXAMPLE AND ANALYSIS

6.1 INSTANCE GENERATION

The optimization procedure is applied on the following small example.

We assume a single-machine line that produces two types of item (n=2) during two periods (T=2). Each period duration is equal to 1.2 hours ($C = 1.2$). The other feature of the problem is also known as follows (the times are given in hours).

- The initial demand of the items (D_{it})
 $D_{11} = 2, D_{12} = 1, D_{21} = 1, D_{22} = 3$
- The processing time for each type of items
 $k_1 = 0.17, k_2 = 0.09$
- The mean time between failures (MTBF)
 $U = 0.6667$
- The mean time to repair (MTTR)
 $\bar{U} = 4$
- Probability of defectiveness for each type of items
 $p_1 = 0.85, p_2 = 0.83$
- Sequence of producing lots
 $\gamma = \{\gamma_1 = 1, \gamma_2 = 2\}$

It is considered that probability of minimum service for all lots in all periods is equal to 0.93 ($\beta_1 = \beta_2 = \beta_3 = \beta_4 = 0.93$). As it is obvious to maximize the service level we should obtain at least the produced quantities (z_{it}) as same as demand quantities for each item at the end of each period.

The objective is to determine the optimal lot sizes x_{it} when we know the sequence in order to maximize the probability of equation 13.

In the optimization algorithm, the items appears as the stages of dynamic programming and number of stages varies by the number of items. The number of periods has impact on the number of states in each stage and also the complexity of computation of each loop in the algorithm.

6.2 EXPERIMENTAL RESULTS

With considering probability of minimum service level as 0.93 for items and using equation (18) we can obtain $x_{it} \in [x_{it}^-, x_{it}^+]$ interval as:

$$x_{11} \in [3, 6], x_{12} \in [2, 4], x_{21} \in [2, 5], x_{22} \in [5, 8]$$

To be able to apply optimization procedure the components of dynamic programming (equation 19 to 25) are calculated in the table 2.

This instance has two stages. The first stage is producing item 1 and the second is producing item 2. Below, all possible released quantities and states of each stage based on the x_{it} and V_{it} bounds are shown.

Stage 1:

$$x_{11...2} \in \{[3, 2], [3, 3], [3, 4], [4, 2], [4, 3], [4, 4], [5, 2], [5, 3], [6, 2], [6, 3], [6, 4]\}$$

$$V_{1...2}^1 \in \{[0.51, 0.34], [0.51, 0.51], [0.51, 0.68], [0.68, 0.34], [0.68, 0.51], [0.68, 0.68], [0.85, 0.34], [0.85, 0.51], [0.85, 0.68], [1.02, 0.34], [1.02, 0.51], [1.02, 0.68]\}$$

stage 2:

$$x_{21...2} \in \{[2, 5], [2, 6], [2, 7], [2, 8], [3, 5], [3, 6], [3, 7], [3, 8], [4, 5], [4, 6], [4, 7], [4, 8], [5, 5], [5, 6], [5, 7], [5, 8]\}$$

$$V_{1...2}^2 \in \{[0.69, 0.79], [0.69, 0.88], [0.69, 0.97], [0.69, 1.06], [0.69, 1.05], [0.69, 1.14], [0.69, 1.13], [0.78, 0.79], [0.78, 0.88], [0.78, 0.97], [0.78, 1.06], [0.78, 0.96], [0.78, 1.05], [0.78, 1.14], [0.78, 1.13], [0.87, 0.79], [0.87, 0.88], [0.87, 0.97], [0.87, 1.06], [0.87, 0.96], [0.87, 1.05], [0.87, 1.14], [0.87, 1.13], [0.96, 0.79], [0.96, 0.88], [0.96, 0.97], [0.96, 1.06], [0.96, 0.96], [0.96, 1.05], [0.96, 1.14], [0.96, 1.13], [0.86, 0.79], [0.86, 0.88], [0.86, 0.97], [0.86, 0.96], [0.86, 1.05], [0.86, 1.14], [0.86, 1.13], [0.95, 0.79], [0.95, 0.88], [0.95, 0.97], [0.95, 1.06], [0.95, 0.96], [0.95, 1.05], [0.95, 1.14], [0.95, 1.13], [1.04, 0.79], [1.04, 0.88], [1.04, 0.97], [1.04, 1.06], [1.04, 0.96], [1.04, 1.05], [1.04, 1.14], [1.04, 1.13], [1.13, 0.79], [1.13, 0.88], [1.13, 0.97], [1.13, 1.06], [1.13, 0.96], [1.13, 1.05], [1.13, 1.14], [1.13, 1.13], [1.03, 0.79], [1.03, 0.88], [1.03, 0.97], [1.03, 1.06], [1.03, 0.96], [1.03, 1.05], [1.03, 1.14], [1.03, 1.13], [1.12, 0.79], [1.12, 0.88], [1.12, 0.97], [1.12, 1.06], [1.12, 0.96], [1.12, 1.05], [1.12, 1.14], [1.12, 1.13], [1.2, 0.79], [1.2, 0.88], [1.2, 0.97], [1.2, 1.06], [1.2, 0.96], [1.2, 1.05], [1.2, 1.14], [1.2, 1.13]\}$$

The dynamic programming algorithm calculation begins with setting $V_{1...2}^0 = [0, 0]$, and $H_0(V_{1...2}^0) = 1$; Then it continues by considering first stage or item, i.e $i = 1$. The first selected state of first stage in the set of possible state is $V_{1...2}^1 = [0.51, 0.34]$ and $H_1([0.51, 0.34]) = 0$ is set for this state.

Procedure is started with $x_{11...2} = [3, 2]$, since this is the first stage of dynamic programming there is just one state for the $V_{1...2}^0$ which is equal to $[0, 0]$, so it should be checked if $K([3, 2] = [0.51, 0.34])$ with using equation 8. $K[3, 2] = 0.17 * [3, 2] = [0.51, 0.34]$ which is equal to $V_{1...2}^1 = [0.51, 0.34]$.

The next step is to check if $H_0([0, 0]) \times p(z_{11 \rightarrow 2} \geq D_{11 \rightarrow 2} | [3, 2]) \geq H_1([0.51, 0.34])$ is satisfied or not; where $H_0([0, 0])$ is equal to 1. According to equation 20, $p(z_{11 \rightarrow 2} \geq D_{11 \rightarrow 2} | [3, 2])$ has 2 component as follows:

$$p(z_{11 \rightarrow 2} \geq D_{11 \rightarrow 2} | [3, 2]) = \prod_{t=1}^2 p(z_{11 \rightarrow t} \geq D_{11 \rightarrow t} | x_{1...11...t})$$

where for $t=1$, $p(z_{11 \rightarrow 1} \geq D_{11} | [3])$ should be calculated.

$$p(z_{11 \rightarrow 1} \geq D_{11} | [3]) = p_{11}^b(D_{11} | \emptyset) \times \left(\sum_{y_{11}=0}^3 p_{11}^{y>}(D_{11} | y_{11}) \times p_{11}^{c=}(y_{11} | [3]) \right)$$

Refer to the equation 15, $p_{11}^b(D_{11} = 2 | \emptyset) = 1$.

Calculation of $p_{11}^{y>}(D_{11} | y_{11})$ needs using equation 4 as well, and the result is $p_{11}^{y>}(2 | 0) = 0, p_{11}^{y>}(2 | 1) = 0, p_{11}^{y>}(2 | 2) = 0.7225, p_{11}^{y>}(2 | 3) = 0.9393$.

Use of equations 10, 11 and 12 to calculate $p_{11}^{c=}(y_{11} | [3])$ results in $p_{11}^{c=}(0 | [3]) = 0, p_{11}^{c=}(1 | [3]) = 0, p_{11}^{c=}(2 | [3]) = 0, p_{11}^{c=}(3 | [3]) = 1$.

At last:

$$p(z_{11 \rightarrow 1} \geq D_{11 \rightarrow 1} | [3]) = 1 \times (0 \times 0 + 0 \times 0 + 0.7225 \times 0 + 0.9393 \times 1) = 0.9393$$

The next is calculation of $p(z_{11 \rightarrow 2} \geq D_{11 \rightarrow 2} | [3, 2])$ value.

$$p(z_{11 \rightarrow 2} \geq D_{11 \rightarrow 2} | [3, 2]) = \sum_{\tilde{D}_{12}=3-3}^3 \left(p_{12}^b(\tilde{D}_{12} | [3]) \left(\sum_{y_{12}=0}^2 p_{12}^{y>}(\tilde{D}_{12} | y_{12}) \times p_{12}^{c=}(y_{12} | [2]) \right) \right)$$

To compute the probabilities of actual requirements $(p_{12}^b(\tilde{D}_{12} | [3]))$ of the period to equation 15, 16 and 17 are used. So

$$p(z_{11 \rightarrow 2} \geq D_{11 \rightarrow 2} | [3, 2]) = 0.6141 \times (1 \times 0 + 1 \times 0 + 1 \times 1) + 0.3251 \times (0 \times 0 + 0.85 \times 0 + 0.9775 \times 1) + 0.0574 \times (0 \times 0 + 0 \times 0 + 0.7225 \times 1) + 0.0034 \times (1 \times 0 + 1 \times 0 + 0 \times 1) = 0.9734$$

i	t	\underline{w}_{it}	\bar{w}_{it}	$\sum_{j=1}^i \bar{w}_{jt}$	$C - \sum_{j=i+1}^2 \underline{w}_{jt}$	\underline{V}_{it}	\bar{V}_{it}
1	1	0.51	1.02	1.02	1.02	0.51	1.02
	2	0.34	0.68	0.68	0.75	0.34	0.68
2	1	0.18	0.45	1.47	1.2	0.69	1.2
	2	0.45	0.71	1.39	1.2	0.79	1.2

Table 2 – Calculation of the components of dynamic programming

Finally, These calculation result in:

$$p(z_{11 \rightarrow 2} \geq D_{11 \rightarrow 2} | [3, 2]) = p(z_{11 \rightarrow 1} \geq D_{11 \rightarrow 1} | [3]) \times p(z_{11 \rightarrow 2} \geq D_{11 \rightarrow 2} | [3, 2]) = 0.9393 \times 0.9734 = 0.9143$$

Which results in $H_1([0.51, 0.34]) = 0.9943$.

It means that the $H_1([0.51, 0.34])$ is updated to 0.9143. The steps of algorithm should be applied on all other $x_{11...2}$ for $V_{1..2}^1 = [0.51, 0.34]$ and also for all other $V_{1..2}^1$. The $H_1(V_{11..2})$ for all possible states of producing item 1 is in the vector below:

$$H_1(V_{1..2}^1) = [0.9143; 0.9337; 0.9381; 0.9822; 0.9868; 0.9878; 0.9966; 0.9975; 0.9977; 0.9994; 0.9996; 0.9996]$$

If we assume that periods are independent (single period model) the released quantities for lots are:

$$x_{11} = 5, x_{12} = 3, x_{21} = 3, x_{22} = 6$$

When we consider this amount as production quantities in multi-period model the service level will be: 0.9913. We set this value as a lower bound for $H_i(V_{i1..T})$ and delete the states with H lower than this bound for the next item in algorithm. So the new set of states of stage 1 and their related H value are:

$$V_{1..2}^1 \in \{[0.85, 0.34], [0.85, 0.51], [0.85, 0.68], [1.02, 0.34], [1.02, 0.51], [1.02, 0.68]\}$$

$$H_1(V_{1..2}^1) = [0.9966; 0.9975; 0.9977; 0.9994; 0.9996; 0.9996]$$

Since some of the $V_{1..2}^1$ are removed by the lower bound, the $V_{1..2}^2$ must be also updated.

$$V_{1..2}^2 \in \{[1.03, 0.79], [1.03, 0.88], [1.03, 0.97], [1.03, 1.06], [1.03, 0.96], [1.03, 1.05], [1.03, 1.14], [1.03, 1.13], [1.12, 0.79], [1.12, 0.88], [1.12, 0.97], [1.12, 1.06], [1.12, 0.96], [1.12, 1.05], [1.12, 1.14], [1.12, 1.13], [1.2, 0.79], [1.2, 0.88], [1.2, 0.97], [1.2, 1.06], [1.2, 0.96], [1.2, 1.05], [1.2, 1.14], [1.2, 1.13]\}$$

Then all steps of algorithm should be done for the next stage $i = 2$. The $H_2(V_{1..2}^2)$ values for all $V_{1..2}^2$ are in the vector below:

$$H_2(V_{1..2}^2) = [0.9494; 0.9629; 0.9665; 0.9675; 0.9504; 0.9638; 0; 0.9506; 0.9867; 0.9904; 0.9914; 0.9916; 0.9876; 0.9914; 0; 0.9878; 0.9521; 0.9656; 0.9692; 0.9702; 0.9523;$$

$$0.9658; 0; 0.9523]$$

The optimal answer is max of $H_2(V_{1..2}^2)$, so:

$$\text{customer service} = H_2([1.12, 1.06]) = 0.9916$$

And the released quantities are :

$$x_{11} = 5, x_{12} = 2, x_{21} = 3, x_{22} = 8$$

6.3 Analysis and comparison

In this study we develop a multi-item multi-period stochastic lot sizing under yield and capacity uncertainties, in literature the models proposed for lot sizing under yield and capacity uncertainties are single-period horizon such as (Dolgui et al. 2005; Schemeleva et al. 2018, 2012), regarding to the literature which indicate single-period method is popular in the problem, here, we compare our model which consider the backlog/inventory in multi-period horizon production with a single-period horizon one in which the conditions and assumptions are the same. For comparison we use the example which is given in section 6.1. The considered example include two periods. First we solve the problem as a multi-period horizon and then for single-period method we divided the horizon to the two independent period and ignore the backlog/inventory assumption.

The table 3 illustrates the results. In the table the multi period method refers to the method in this study assuming backlog/inventory and single period method refers to model which ignore this assumption.

The table 3 indicates that the results for the first period are precisely the same in the both methods, it is found out easily, the reason is that there is no backlog/inventory in the beginning of the first period. The difference is in the second period. In multi-period horizon, the probability of maximizing service level is higher than single period. It should be noted, since in the multi-period method the periods are depended it's not possible to compute the each period service level.

If we produce as amount as optimal quantities in the single-period horizon, in the multi-period horizon the

periods t items i	multi period method				single period method			
	1		2		1		2	
1	x_{11}	5	x_{12}	2	x_{11}	5	x_{12}	3
2	x_{21}	3	x_{22}	8	x_{21}	3	x_{22}	6
period service level	-		-		0/9926		0/9873	
total service level	0/9913				0/9800			

Table 3 – Results of comparison of two methods

probability of satisfying the demand, objective function value would be equal to 0.9913. The result shows that the objective value (service level) wouldn't be optimal.

As it is mentioned before, here backlog/inventory means that the production quantity is less or more than enough, in other words if we may obtain more good items than the demand quantity we keep remaining to the next period or if we may get less than the demand quantity we add the backlogs to the demand of the same item in the next period. That is why we could get higher probability of service level in comparison with single period horizon.

Overall, this work in comparison with the lot sizing problem in literature results in a higher probability of satisfying demands, but it needs more time to solve the problem. It worth to mention the instance in this study is very small, if this method is applied on a larger problem the efficiency will be more obvious.

7 CONCLUSIONS

In this research, a multi-item multi-period lot sizing problem for a single-machine production line considering random yield and random capacity was studied. In which the uncertainties rise from item imperfection and machine breakdowns. A predefined item sequences and backlog/inventory allowance are assumed in the production line. The problem is to estimate the optimal lot size of each item to maximize the probability of having a sufficient number of good quality items by the end of a given planning horizon, i.e. maximize the overall customer service level. Regarding the similarity of the problem to the well-known knapsack problem, dynamic programming was employed to estimate the optimal lot sizes to be produced. As we went deep into the arts this problem has not been studied in the literature sufficiently specially for the multi-period horizon.

In order to validate the model and the proposed optimization approach, a numerical example was illustrated. Later, a comparison between this study, multi-period stochastic lot sizing and single-period stochastic lot sizing was drawn. The obtained results demonstrate the better efficiency of considering

multi-period than single-period setting which is effect of considering backlog/inventory.

This research can be extended in some direction as follow: Extending the approach to solve similar problems considering other objective functions such as minimizing production cost and also Implementing other optimization approaches such as (meta)heuristics to empower the model to solve the large problems.

References

- Agnihotri, S., J. S. Lee, and J. Kim (2000). "Lot sizing with random yields and tardiness costs". In: *Computers & Operations Research* 27.5, pp. 437–459.
- Aloulou, M. A., A. Dolgui, and M. Y. Kovalyov (2014). "A bibliography of non-deterministic lot-sizing models". In: *International Journal of Production Research* 52.8, pp. 2293–2310.
- Chakraborty, T., B. Giri, and K. Chaudhuri (2009). "Production lot sizing with process deterioration and machine breakdown under inspection schedule". In: *Omega* 37.2, pp. 257–271.
- Chiu, S. W. (2010). "Robust planning in optimization for production system subject to random machine breakdown and failure in rework". In: *Computers & Operations Research* 37.5, pp. 899–908.
- Ciarallo, F. W., R. Akella, and T. E. Morton (1994). "A periodic review, production planning model with uncertain capacity and uncertain demand—optimality of extended myopic policies". In: *Management Science* 40.3, pp. 320–332.
- Dolgui, A. (2002). "Performance analysis model for systems described by renewal process". In: *Engineering Simulation (Electronic Modeling)* 24.2, pp. 3–11.
- Dolgui, A., G. Levin, and M.-A. Louly (2005). "Decomposition approach for a problem of lot-sizing and sequencing under uncertainties". In: *International Journal of Computer Integrated Manufacturing* 18.5, pp. 376–385.
- Dolgui, A. and C. Prodhon (2007). "Supply planning under uncertainties in MRP environments:

- A state of the art". In: *Annual Reviews in Control* 31.2, pp. 269–279.
- Dolgui, A. and J.-M. Proth (2010). *Supply chain engineering: useful methods and techniques*. Springer Science & Business Media.
- Geng, N. and Z. Jiang (2007). "Capacity planning for semiconductor wafer fabrication with uncertain demand and capacity". In: *2007 IEEE International Conference on Automation Science and Engineering*. IEEE, pp. 100–105.
- Gerchak, Y. and A. Grosfeld-Nir (1998). "Multiple lot-sizing, and value of probabilistic information, in production to order of an uncertain size". In: *International Journal of Production Economics* 56, pp. 191–197.
- Gerchak, Y., Y. Wang, and C. A. Yano (1994). "Lot sizing in assembly systems with random component yields". In: *IIE transactions* 26.2, pp. 19–24.
- Giri, B. and T. Dohi (2004). "Optimal lot sizing for an unreliable production system based on net present value approach". In: *International Journal of Production Economics* 92.2, pp. 157–167.
- Giri, B. and W. Yun (2005). "Optimal lot sizing for an unreliable production system under partial backlogging and at most two failures in a production cycle". In: *International Journal of Production Economics* 95.2, pp. 229–243.
- Güllü, R., E. Öno, and N. Erkip (1999). "Analysis of an inventory system under supply uncertainty". In: *International Journal of Production Economics* 59.1-3, pp. 377–385.
- Haji, R., A. Haji, M. Sajadifar, and S. Zolfaghari (2008). "Lot sizing with non-zero setup times for rework". In: *Journal of Systems Science and Systems Engineering* 17.2, p. 230.
- Inderfurth, K. (2009). "How to protect against demand and yield risks in MRP systems". In: *International Journal of Production Economics* 121.2, pp. 474–481.
- Jain, K. and E. A. Silver (1995). "The single period procurement problem where dedicated supplier capacity can be reserved". In: *Naval Research Logistics (NRL)* 42.6, pp. 915–934.
- Karlin, S., H. Scarf, and K. J. Arrow (1958). *Studies in the mathematical theory of inventory and production*. Stanford University Press.
- Khan, M., M. Jaber, and M. Wahab (2010). "Economic order quantity model for items with imperfect quality with learning in inspection". In: *International journal of production economics* 124.1, pp. 87–96.
- Li, Q., H. Xu, and S. Zheng (2008). "Periodic-review inventory systems with random yield and demand: Bounds and heuristics". In: *IIE Transactions* 40.4, pp. 434–444.
- Lin, G. C. and D.-C. Gong (2006). "On a production-inventory system of deteriorating items subject to random machine breakdowns with a fixed repair time". In: *Mathematical and Computer Modelling* 43.7-8, pp. 920–932.
- Maddah, B. and M. Y. Jaber (2008). "Economic order quantity for items with imperfect quality: revisited". In: *International Journal of Production Economics* 112.2, pp. 808–815.
- Moon, I., B.-H. Ha, and J. Kim (2012). "Inventory systems with variable capacity". In: *European Journal of Industrial Engineering* 6.1, pp. 68–86.
- Papachristos, S. and I. Konstantaras (2006). "Economic ordering quantity models for items with imperfect quality". In: *International Journal of Production Economics* 100.1, pp. 148–154.
- Salameh, M. and M. Jaber (2000). "Economic production quantity model for items with imperfect quality". In: *International journal of production economics* 64.1-3, pp. 59–64.
- Schemeleva, K., X. Delorme, and A. Dolgui (2018). "Evaluation of solution approaches for a stochastic lot-sizing and sequencing problem". In: *International Journal of Production Economics* 199, pp. 179–192.
- Schemeleva, K., X. Delorme, A. Dolgui, and F. Grimaud (2012). "Multi-product sequencing and lot-sizing under uncertainties: A memetic algorithm". In: *Engineering Applications of Artificial Intelligence* 25.8, pp. 1598–1610.
- Silver, E. (1976). "Establishing the order quantity when the amount received is uncertain". In: *INFOR: Information Systems and Operational Research* 14.1, pp. 32–39.
- Singh, M., C. Abraham, and R. Akella (1988). "Planning for production of a set of components when yield is random". In: *Fifth IEEE/CHMT International Electronic Manufacturing Technology Symposium, 1988, Design-to-Manufacturing Transfer Cycle*. IEEE, pp. 196–200.
- Teunter, R. H. and S. D. P. Flapper (2003). "Lot-sizing for a single-stage single-product production system with rework of perishable production defectives". In: *Or Spectrum* 25.1, pp. 85–96.
- Wang, Y. and Y. Gerchak (1996a). "Continuous review inventory control when capacity is variable". In: *International Journal of Production Economics* 45.1-3, pp. 381–388.
- (1996b). "Periodic review production models with variable capacity, random yield, and uncertain demand". In: *Management science* 42.1, pp. 130–137.
- (2000). "Input control in a batch production system with lead times, due dates and random yields". In: *European Journal of Operational Research* 126.2, pp. 371–385.
- Yano, C. A. and H. L. Lee (1995). "Lot sizing with random yields: A review". In: *Operations Research* 43.2, pp. 311–334.