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Reconstructing quantum theory from its possibilistic operational formalism

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Abstract

We develop a possibilistic semantic formalism for quantum phenomena from an operational perspective. This semantic system is based on a Chu duality between preparation processes and yes/no tests, the target space being a three-valued set equipped with an informational interpretation. A basic set of axioms is introduced for the space of states. This basic set of axioms suffices to constrain the space of states to be a projective domain. The subset of pure states is then characterized within this domain structure.

After having specified the notions of properties and measurements, we explore the notion of compatibility between measurements and of minimally disturbing measurements.

We achieve the characterization of the domain structure on the space of states by requiring the existence of a scheme of discriminating yes/no tests, necessary condition for the construction of an orthogonality relation on the space of states. This last requirement about the space of states constrain the corresponding projective domain to be ortho-complemented. An orthogonality relation is then defined on the space of states and its properties are studied. Equipped with this relation, the orthoposet of ortho-closed subsets of pure states inherits naturally a structure of Hilbert lattice.

Finally, the symmetries of the system are characterized as a general subclass of Chu morphisms. We prove that these Chu symmetries preserve the class of minimally disturbing measurements and the orthogonality relation between states. These symmetries lead naturally to the ortho-morphisms of Hilbert lattice defined on the set of ortho-closed subsets of pure states.

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1 Introduction

The basic description of an 'experimental act' relies generally on (i) a description of a given *preparation setting* that produces *samples* of a given physical object, through some well-established procedures, and (ii) a particular set of *operations/tests* that can be realized by the *observer* on these prepared samples. Each prepared sample is associated with a set of *information*, checked throughout the preparation process and recorded on devices, and to a set of *instructions*, processed by a computer or monitored by the human operator. These information/instructions are naturally attached to macroscopically observable events that occur during the preparation process and to macroscopically distinguishable modes of the preparation apparatuses used throughout this preparation process. Analogously, the operations/tests, realized on the prepared samples, must be attached unambiguously to some *knowledge/belief* of the observer about the outcomes of this experimental step and to a new *state of facts* concerning the outcomes of the whole experiment, in order for these operations/tests to give sense to any 'sorting' of these outcomes. If these operations/tests do not 'destroy' or 'alter irremediably' the samples under study, we can then consider the whole experimental protocol^[1] as a completed preparation procedure for forthcoming experiments. The information characterizing the chosen operation, and the information retrieved by the observer during these tests, should then be recorded as new information/instructions in this completed preparation procedure.^[2] This description of preparation procedures and operations/tests is a fundamental ingredient of the physical description. It is a *sine qua non condition* for the experimental protocol to satisfy a basic requirement: *the reproducibility*. As noted by Kraus [44], there exist macroscopic devices undergoing macroscopic changes when interacting with micro-systems, and the observation of micro-objects always requires this inter-mediation. This fundamental empirical fact justifies the attempt to establish such an *operational description* for quantum experiments as well. Explicitly, the different procedures designed to prepare collection of similar quantum micro-systems may combine measurements and filtering operations (associated with the unambiguously measured properties) in order to produce collections of samples, that may be subjected to subsequent measurements.

Obviously, different preparation procedures may be used to produce distinct samples, to which the observer would nevertheless attach the same *informational content*. This is the case, in particular, if this observer does not know of any experiments that could be realized conjointly on these differently prepared samples and that would produce 'unambiguously-incompatible' logical conclusions. A physical description (of the objects subjected to experiment) is an attempt to establish a *semantic perspective* adapted to previous descriptions of the process of preparation/measurement.^[3] The notion of *physical state* occupies a central position in this semantic construction. The physical states are abstract names for the different possible realizations of the object under study. Adopting this ontological perspective, the observer may associate an element of the *space of states* with any preparation process, which will *a priori* characterize any particular sample 'prepared' through this process. The logical truth of a proposition about the 'similarity' of two given samples should then be directly linked to the logical truth of the proposition regarding the identity of the associated states. However, the ontological notion of state has to be faced with its epistemological counterpart. From an ontological perspective, we consider that a given physical system is necessarily in a particular realization, but from an epistemological perspective, the observer should test the hypothesis that this system may indeed be described by this state.^[4] Adopting this perspective, it seems that a given physical state could also be meant as a *denotation* for the set of

¹i.e., the initial preparation followed by the operation/test step as a global preparation process for subsequent tests

²We note that the description of the preparation/measurement process should then exploit some tools of recursion theory.

³Here, and in the following, we will adopt the following basic definition of the word 'semantic' recalled by Reichenbach: "Modern logic distinguishes between object language and metalanguage; the first speaks about physical objects, the second about statements, which in turn are referred to objects. The first part of the meta-language, syntax, concerns only statements, without dealing with physical objects; this part formulates the structure of statements. The second part of the metalanguage, semantics, refers to both statements and physical objects. This part formulates, in particular, the rules concerning truth and meaning of statements, since these rules include a reference to physical objects. The third part of the meta-language, pragmatics, includes a reference to persons who use the object language." [63]

⁴Note that, in practice, the observer is rather led to infer a 'mixture' on the basis of his limited knowledge about this sample.

preparations that the observer is led to identify empirically. From a strictly operational point of view, the observer will always establish the equivalence between different preparation procedures, by testing conjointly the corresponding prepared samples through a 'well-chosen' collection of control tests. In other words, the state may be defined by the set of common facts that could be established by realizing these control-tests on the corresponding samples. Adopting another perspective, we could also consider that the state should encode the determined aspects relative to the possible results of forthcoming experiments. There should be no problem with such a 'versatile' perspective in the operational description of classical experiments, as long as the properties, established as 'actual' during the preparation process, characterize the sample in a way that will be questioned neither by any control test realized on it, nor by any future non-destructive experiments. The operational description of quantum experiments is in fact significantly more intricate, due to some inconvenient features of the measurement operation. Indeed, it is a fundamental fact of quantum experiments that, whichever set of properties has been checked as 'actual' by the preparation process, the outcomes of an irreducible part of the measurements that can be made on these prepared samples, remain completely indeterminate. More than that, if some of these measurements are realized in order to establish some new properties, it generically occurs that the measured samples no longer exhibit, afterwards, some of the properties that had been previously established on the prepared samples.

Despite this indeterministic character of quantum theory, it is an empirical fact that the distinct outcomes of these measurements, operated on a large collection of samples, prepared according to the same experimental procedure, have reproducible relative frequencies. This fundamental fact has led physicists to consider large collections of statistically independent experimental sequences as the basic objects of physical description, rather than a single experiment on a singular realization of the object under study (see [53] for a reference book). According to *Generalized Probabilistic Theory* (GPT)^[5], a physical state (corresponding to a class of operationally equivalent preparation procedures) is defined by a vector of probabilities associated with the outcomes of a maximal and irredundant set of fiducial tests that can be effectuated on collections of samples produced by any of these preparation procedures.^[6] In other words, two distinct collections of prepared samples will be considered as operationally equivalent if they lead to the same probabilities for the outcomes of any test on them. The physical description consists, therefore, in a set of prescriptions that allows sophisticated constructs to be defined from elementary ones. In particular, combination rules are defined for the concrete mixtures of states and for the allowed operations/tests. The different attempts to reconstruct quantum mechanics along this path ([36, 37, 38][19][51][13]) proceed by the determination of a minimal set of plausible constraints, imposed on the space of states, sufficient to 'derive' the usual Von Neumann axiomatic quantum theory. Although this probabilistic approach is now accepted as a standard conceptual framework for the reconstruction of quantum theory, the adopted perspective appears puzzling for different reasons. First of all, the observer contributes fundamentally to give an intuitive meaning to the notions of preparation, operation and measurement on physical systems. However, the concrete process of 'acquisition of information' (by the observer / on the system) has no real place in this description. Secondly, the definition of the state has definitively lost its meaning for a singular prepared sample, and the physical state is now intrinsically attached to large collections of similarly prepared samples. This point has concentrated many critics since the original article of Einstein, Podolsky and Rosen [27], although the empirical testing of quantum theory in EPR experiments has led physicists to definitively accept the traditional probabilistic interpretation. The GPT approach adopts the probabilistic description of quantum phenomena without any discussion or attempt to explain why it is necessary. Thirdly, in order to clarify the requirements of the basic set of fiducial tests necessary and sufficient to define the space of states, this approach must proceed along a technical analysis which fundamentally limits this description to 'finite dimensional' systems (finite dimensional Hilbert spaces of states). Lastly, the axioms chosen to characterize quantum

⁵This formalism has its origins in the pioneering works of Mackey [48], Ludwig [45, 46, 47] and Kraus [44]. See [42] for a recent review.

⁶The description of quantum theory in this framework then must deal with the problem of defining the notions of consistency, completeness and irredundancy for the set of control tests that define an element of the quantum space of states.

theory, among other theories encompassed by the GPT formalism, must exhibit a 'naturalness' that - from our point of view - is still missing in the existing proposals.

Alternative research programs have tried to overcome some of these conceptual problems. In particular, they try to put the emphasis on the 'informational' relation emerging between the observer and the system, through the concrete set of 'yes/no tests' that can be addressed by this observer to this system, and to characterize quantum theory in these terms through a small number of basic semantic requirements. It must be noted that, although these programs try to clarify the central notion of "information" in the quantum description an observer can develop about the system under study, they basically adopt the 'probabilistic' interpretation of measurements. The fundamental limitation of the information that an observer can retrieve from a given quantum system through yes-no experiments, has been taken by different physicists as a central principle for the reconstruction of quantum theory (see reference paper [64, Chap III], and see [71] for another perspective on this basic principle) :

Information Principle *'Information is a discrete quantity: there is a minimum amount of information exchangeable (a single bit, or the information that distinguishes between just two alternatives).[...] Since information is discrete, any process of acquisition of information can be decomposed into acquisitions of elementary bits of information.'* [64, p.1655].

We intend to adopt our own version of this fundamental pre-requisite to build the quantum space of states.

To be more explicit, according to C.Rovelli, quantum mechanics appears to be governed by two seemingly incompatible principles (Postulate 1 and 2 of [64, Chap III]). According to the first, the amount of independent information that can be retrieved from a 'bounded' quantum system is fundamentally 'finite'. According to the second, however, the test of any observable property that has not been stated beforehand as 'actual' for a given state^[7], remains fundamentally indeterminate (of course, this test will establish an actual value for this observable property, valid after the measurement). Nevertheless, this 'new information' established through this measurement operation will have been compensated by the restoration of an indeterminacy in some of the properties that may have been established as actual beforehand. Another interesting analysis regarding this 'balance' principle concerning the knowledge of the observer about the quantum system is given in [66]. It must be noted that the combinatorics of the 'incompatibilities' between different measurements can be exploited to explore the algebraic structures behind quantum theory and to proceed to the 'reconstruction' of this theory [39, 40]. Nevertheless, these reconstruction programs stay technically imprisoned by a finite dimensional analysis. A third postulate in Rovelli's axiomatic proposal prescribes the nature of automorphisms acting on the state space (the continuous unitary transformations corresponding, in particular, to Schrodinger's dynamics). However, the form of this last postulate is not entirely satisfactory, as it imposes some intricate relations on the probabilities associated with transitions between states.

Adopting another perspective, the *operational quantum logic* approach tries to avoid the introduction of probabilities and explores the relevant categorical structures underlying the space of states and the set of properties of a quantum system. In this description, probabilities appear only as a derived concept.^[8] Following G. Birkhoff and J. Von Neumann [15] and G. W. Mackey [48], this approach focuses on the structured space of 'testable properties' of a physical system.^[9] Mackey identifies axioms on the set of yes/no questions sufficient to relate this set to the set of closed subspaces of a complex Hilbert space.

⁷In Von Neumann's formalism, this point means: the quantum state of the system is not an eigenstate of the associated operator.

⁸For the sake of usual quantum formalism, quantum probabilities can be recovered from this formalism, under the assumption of the existence of some well-behaved measurements, using Gleason's theorem [34] (see [54] for a historical and pedagogical presentation of these elements).

⁹In Von Neumann's quantum mechanics, each entity is associated with a complex Hilbert space H . A state ψ of this entity is defined by a ray $v(\psi)$ in H , and an observable is defined by a self-adjoint operator on H . In particular, a yes/no test t is represented by an orthogonal projector Π_t or equivalently by the closed subspace $\mathfrak{A}_t$ defined as the range of Π_t . The answer "yes" or "no" is obtained with certainty for the yes/no test t , if and only if the state ψ is such that $v(\psi)$ is included in $\mathfrak{A}_t$ or in the

Later, C. Piron proposed a set of axioms that (almost) lead back to the general framework of quantum mechanics (see [21] for a historical perspective of the abundant literature inherited from Piron's original works [54, 55]). Piron's framework has been developed into a full operational approach and the categories underlying this approach were analyzed (see [50, 49] for a detailed account of this categorical perspective). It must be noted that these constructions are established in reference to some general results of projective geometry [28] and are not restricted to a finite-dimensional perspective. Despite some beautiful results (in particular the restriction of the division ring associated with Piron's reconstruction of the Hilbert space from Piron's propositional lattices [41][8]) and the attractiveness of a completely categorical approach (see [67] for an analysis of the main results on propositional systems), this approach has encountered many problems. Among these problems, we may cite the difficulty of building a consistent description of compound systems due to no-go results related to the existence of a tensor product of Piron's propositional systems [62, 29][6, 7]. These works have cast doubts on the adequacy of Piron's choice of an "orthomodular complete lattice" structure for the set of properties of the system. D. Foulis, C. Piron and C. Randall [30] produced a jointly refined version of their respective approaches in order to rule out these problems (see also [69]). This description emphasizes the centrality of the treatment of the incompatibilities between measurements.^[10] Our work will emphasize the necessity of replacing the lattice structures, introduced to describe the set of propositions about the quantum system, with domains. Another central problem with the logico-algebraic approach was its inability to describe the dynamic aspect of the measurement operation. The operational quantum logic approach has then been developed later on in different categorical perspectives in order to clarify the links between quantum logic, and modal/dynamic [9, 10, 11][20, 22, 24][72, 14], or linear [23][32, 33] logic.

Other categorical formalisms, adapted to the axiomatic study of quantum theory, have been developed more recently [3] and their relation with the 'operational approach' has been partly explored [1, 2, 4]. In [1, Theorem 3.15], S. Abramsky makes explicit the fact that the *Projective quantum symmetry groupoid* $PSymmH$ ^[11] is fully and faithfully represented by the category $bmChu_{[0,1]}$, i.e., by the sub-category of the category of bi-extensional Chu spaces associated with the evaluation set $[0, 1]$ obtained by restricting it to Chu morphisms (f_*, f^*) for which f_* is injective. This result suggests that Chu categories could have a central role in the construction of axiomatic quantum mechanics^[12], as they provide a natural characterization of the automorphisms of the theory. More surprisingly, and interestingly for us, in [1] S. Abramsky shows that the previously mentioned representation of $PSymmH$ is 'already' full and faithful if we replace the evaluation space of the Chu category by a three-element set, where the three values represent "definitely yes", "definitely no" and "maybe" [1, Theorem 4.4] :

'The results on reduction to finite value sets are also intriguing. Not only is the bare Chu condition on morphisms sufficient to whittle them down to the semiunitaries, this is even the case when the discriminations on which the condition is based are reduced to three values. The general case for two values remains open, but we have shown that the two standard possibilistic reductions both fail to preserve fullness. A negative answer for two-valued semantics in general would suggest an unexpected rôle for three-valued logic in the

orthogonal of $\mathfrak{A}_t$, respectively. Birkhoff and von Neumann proposed to focus not on the structure of the Hilbert space itself, but on the structure of the set of closed subspaces of H . The mathematical structure associated with the set of quantum propositions defined by the closed subspaces of H is not a Boolean algebra (contrary to the case encountered in classical mechanics). By shifting the attention to the set of closed subspaces of H instead of H , the possibility is open to build an operational approach to quantum mechanics, because the basic elements of this description are yes/no experiments.

¹⁰'It is our contention that the realistic view implicit in classical physics need not be abandoned to accommodate the contemporary conceptions of quantum physics. All that must be abandoned is the presumption that each set of experiments possesses a common refinement (that is, the experiments are compatible). As we shall argue, this in no way excludes the notion of physical systems existing exterior to an observer, nor does it imply that the properties of such systems depend on the knowledge of the observer.' [30, p.813]

¹¹The objects of this category are the natural space of states in quantum mechanics, i.e., the Hilbert spaces of dimension greater than two, and the morphisms are the orbits under the $U(1)$ group action on semi-unitary maps (i.e. unitary or anti-unitary), which are the relevant symmetries of Hilbert spaces from the point of view of quantum mechanics.

¹²The centrality of the notion of Chu spaces for quantum foundations had been noted already by V. Pratt [59].

The introduction of a three-valued logic in the foundational studies of quantum mechanics goes back to H. Reichenbach's work [63]. H. Reichenbach's formalism has been introduced as an alternative to the quantum logic approaches developed to continue the seminal work of G. Birkhoff and J. von Neumann [15].^[13] This extension of standard logic was designed as a means of dealing with certain conceptual tensions arising in the quantum mechanical description of the world. Briefly, the third truth value, called 'indeterminate', was introduced to capture 'meaningless' statements associated with unmeasured entities during the experimental process. It is important to note that H. Reichenbach did not clarify how this three-valued logic may lead to results 'comparable' to those obtained using the usual probabilistic formalism, and why this extension of classical logic could be the right setting for quantum theory.

S. Abramsky's work adopts a radically different perspective. His study begins with continuous-valued Chu space. He shows that quantum symmetries are elegantly captured by Chu morphisms and proves finally that three values are actually sufficient for this characterization of quantum symmetries. Despite the suggestive similarities pointed to by S. Abramsky between such a three-valued Chu space description and the Geneva school's formalism, no bridge has been built between these two formalisms until now. Thus, S. Abramsky did not affirm that these three values are sufficient to found a complete axiomatic quantum theory, close to Piron's program or alternative to it, and allowing a complete reconstruction of the usual Hilbert formalism. It is the purpose of the present paper to achieve this goal. We intend to present the basic elements of this 'possibilistic'^[14] semantic formalism, and to give the precise axiomatics that leads to a reconstruction of a generalized Hilbert space structure on the space of states.

Our formalism is based on a Chu duality between preparation processes and quantum tests. This Chu duality refers to a three-valued target space. This three-valued target space is equipped with a 'possibilist' semantic formalism which leads to an 'informational' interpretation of the set of preparations. In the first part of our study, we formulate a precise semantic description of the space of states. The 'Information Principle' introduced by C. Rovelli plays a central role in this formalism. After having introduced a basic set of axioms about the space of states, it will be shown that the space of states is a projective domain. The space of pure states will then be characterized.

In a second part of the study, we clarify successively the notion of 'property' and the notion of a 'measurement' associated with a given property of the system. We explore the consequences of the incompatibilities existing between measurements.

An orthogonality relation is then defined on the space of states and its properties are studied using the domain structure obtained on the space of states. The axiomatics on the space of states is achieved by the addition of some conditions relative to the existence of the orthogonality relation. Equipped with this relation, the ortho-poset of ortho-closed subsets of pure states inherits a structure of Hilbert lattice. This result is the first part of our reconstruction theorem.

In the third part of this paper, we build the set of symmetries of the system as a particular sub-algebra of Chu morphisms. These symmetries appear to leave this subset of minimally disturbing measurement operations stable and preserves the orthogonality relation between states. Endly, it is shown that these symmetries lead naturally to the ortho-morphisms of Hilbert lattice defined on the set of ortho-closed subsets of pure states.

Throughout the paper we clearly distinguish (1) elements formulated in the 'material mode of speech' and concerning the structure of the language in which the experimental setting can be described operationally (these elements are designated as 'Notions' throughout the text), although these notions are the occasion to introduce the corresponding mathematical elements, from (2) purely mathematical definitions (these elements are classically designated as 'Definitions').

¹³In order to clarify the fundamental difference in nature between Reichenbach Quantum Logic and Mainstream Quantum Logic the reader is invited to consult [35].

¹⁴In the rest of this paper we refer to this construction, based on a three-valued Chu space, as a 'possibilistic' approach to distinguish it from the 'probabilistic' one.

We intentionally emphasize the different requirements of our reconstruction program. Every requirement is introduced accompanied with an analysis of its motivation and summarized under the term 'Axiom'; the analysis of the mathematical consequences of each axiom is declined along Lemmas and Theorems.

2 Preparations and states

2.1 Operational formalism

Adopting the operational perspective on quantum experiments, we will introduce the following definitions :

Notion 1. A *preparation process* is an objectively defined, and thus 'repeatable', experimental sequence that allows singular samples of a certain physical system to be produced, in such a way that we are able to submit them to tests. We will denote by $\mathfrak{P}$ the set of preparation processes (each element of $\mathfrak{P}$ can be equivalently considered as the collection of samples produced through this preparation procedure). The information corresponding to macroscopic events/operations describing the procedure depend on an observer O . If this dependence has to be made explicit, we will adopt the notation $\mathfrak{P}^{(O)}$ to denote the set of preparation processes defined by the observer O .

Notion 2. For each *property*, that the observer aims to test macroscopically on *any particular sample* of the considered micro-system, it will be assumed that the observer is able to define (i) some detailed 'procedure', in reference to the modes of use of some experimental apparatuses chosen to perform the operation/test, and (ii) a 'rule' allowing the answer 'yes' to be extracted if the macroscopic outcome of the experiment conforms with the expectation of the observer, when the test is performed on any input sample (as soon as this experimental procedure can be opportunely applied to this particular sample). These operations/tests, designed to determine the occurrence of a given property for a given sample, will be called *yes/no tests associated with this property* (also called a *definite experimental project* in [56]). If a yes/no test, associated with a given property, is effectuated according to the established procedure, and if a positive result is actually obtained for a given sample, we will say that *this property has been measured* for this sample. The set of 'yes/no tests' at the disposal of the observer will be denoted by $\mathfrak{T}$. If the dependence with respect to the observer O has to be made explicit, we will adopt the notation $\mathfrak{T}^{(O)}$ to denote the set of tests defined by the observer O .

We are essentially interested in the information gathered by the observer through the implementation of some yes/no tests, designated by elements in $\mathfrak{T}$, on finite collections of samples prepared similarly through any of the preparation procedures, given as elements of $\mathfrak{P}$. With this perspective, we have to abandon any reference to the probabilistic interpretation^[15]. Nevertheless, the observer is still able to distinguish the situations where one can pronounce a statement with 'certainty', from the situations where one can judge the result as 'indeterminate', on the basis of the knowledge gathered beforehand.

Notion 3. A yes/no test $t \in \mathfrak{T}$ will be said to be *positive with certainty* (resp. *negative with certainty*) relatively to a preparation process $p \in \mathfrak{P}$ iff the observer is led to affirm that the result of this test, realized on any of the particular samples that could be prepared according to this preparation process, would be 'positive with certainty' (resp. would be 'negative with certainty'), 'should' this test be effectuated. If the yes/no test can not be stated as 'certain', this yes/no test will be said to be *indeterminate*. Concretely, the observer can

¹⁵The finite character of the tested collection of prepared samples renders any notion of relative frequency of the outcomes 'meaningless'.

establish the 'certainty' of the result of a given yes/no test on any given sample issued from a given preparation procedure, by running the same test on a sufficiently large (but finite) collection of samples issued from this same preparation process: if the outcome is always the same, the observer will be led to claim that similarly prepared 'new' samples would also produce the same result, if the experiment was effectuated.

To summarize, for any preparation process p and any yes/no test t , the element $e(p, t) \in \mathfrak{B} := \{\perp, \mathbf{Y}, \mathbf{N}\}$ will be defined to be $\perp$ (alternatively, $\mathbf{Y}$ or $\mathbf{N}$) if the outcome of the yes/no test t on any sample prepared according to the preparation procedure p is judged as 'indeterminate' ('positive with certainty' or 'negative with certainty', respectively) by the observer.

$$\begin{aligned} e : \mathfrak{P} \times \mathfrak{T} &\longrightarrow \mathfrak{B} := \{\perp, \mathbf{Y}, \mathbf{N}\} \\ (p, t) &\mapsto e(p, t). \end{aligned} \quad (1)$$

Several remarks should be made regarding the above definitions.

Remark 1. It is essential to note the *counterfactual* aspect of these definitions: in the 'determinate' case, the observer is asked to predict the result of this test *before the test* and *regardless of whether the test is effectuated*. Of course, any 'determinate statement' (positive or negative) produced by the observer, about the result of any forthcoming yes/no test relative to a given preparation process, is a strictly falsifiable statement: it may be proved to be false after some test realized on a *finite collection* of new, similarly prepared samples^[16].

Remark 2. The 'certainty' of the observer about the occurrence of the considered 'property' is intrinsically attached to *any singular sample* prepared through this preparation process and can be falsified as a property of this sample. In other words, it is not necessary to consider *a statistical ensemble* of similarly prepared samples to give a meaning to these notions and to the logical perspective adopted to confront these statements with the measurable state of facts.

Remark 3. When the determinacy of a yes/no test is established for an observer, we can consider that this observer possesses some elementary 'information' about the state of the system, whereas, in the 'indeterminate case', the observer has none (relatively to the occurrence of the considered property).

Notion 4. The set $\mathfrak{B}$ will be equipped with the following poset structure, characterizing the 'information' gathered by the observer:

$$\forall u, v \in \mathfrak{B}, \quad (u \leq v) :\Leftrightarrow (u = \perp \text{ or } u = v). \quad (2)$$

$(\mathfrak{B}, \leq)$ will be called a *flat boolean domain* in the rest of this paper.

Notion 5. $(\mathfrak{B}, \leq)$ is equipped with the following involution map :

$$\overline{\perp} := \perp \quad \overline{\mathbf{Y}} := \mathbf{N} \quad \overline{\mathbf{N}} := \mathbf{Y}. \quad (3)$$

The conjugate of a yes/no test $t \in \mathfrak{T}$ is the yes/no test denoted $\bar{t}$ and defined from t by exchanging the roles of $\mathbf{Y}$ and $\mathbf{N}$ in every result obtained by applying t to any given input sample. In other words,

$$\forall t \in \mathfrak{T}, \forall p \in \mathfrak{P}, \quad e(p, \bar{t}) := \overline{e(p, t)}. \quad (4)$$

¹⁶If the observer is certain of the positive result after having performed a given yes/no test on a finite number of similarly prepared samples, a negative result obtained for any newly tested sample will lead the observer to revise that prediction and to consider this yes/no test as being 'indeterminate' for this preparation.

Notion 6. For any yes/no test t , the set of preparation processes p for which this test is established as *actual*, i.e., 'positive with certainty', will be denoted $\mathfrak{A}_t$.

$$\forall t \in \mathfrak{T}, \quad \mathfrak{A}_t := \{p \in \mathfrak{P} \mid \epsilon(p, t) = \mathbf{Y}\}. \quad (5)$$

Notion 7. For a given yes/no test t , we define the subset $\mathfrak{Q}_t$ of preparation processes that are known by the observer to produce collections of samples leading to positive results to the yes/no test t . Regarding these prepared samples, the observer is then asked to pronounce a statement about any future result of this test on similarly prepared new samples: 'positive' or 'indeterminate'. The collections of samples, resulting from these preparation processes, may then be filtered in order to select collections of samples that are known by the observer to have 'passed the yes/no test t positively'. If a preparation process p is in $\mathfrak{Q}_t$, we will say that the property associated with the yes/no test t is *potential* for the samples produced through p (or p is *questionable* by t). The subset $\mathfrak{Q}_t$ is given by

$$\mathfrak{Q}_t := \{p \in \mathfrak{P} \mid \epsilon(p, t) \leq \mathbf{Y}\} \subseteq \mathfrak{P}. \quad (6)$$

The evaluation map ϵ defines a particular 'duality' between the spaces $\mathfrak{P}$ and $\mathfrak{T}$. Formally, $(\mathfrak{P}, \mathfrak{T}, \epsilon)$ defines a Chu space^{[17][18]}. The set of preparations $\mathfrak{P}$ (or the set of yes/no tests $\mathfrak{T}$) will be *a priori* interpreted as the set of *points* (the set of *opens*) of this Chu space.^[19] Indeed, the preparation processes are naturally considered as 'coexisting entities' distinguished by the properties they posses, whereas the yes/no tests are naturally interpreted as 'alternative predicates' relative to the properties attached to the prepared samples.

According to the perspective adopted by [53], we will define the *states* of the physical system as follows:

Notion 8. An equivalence relation, denoted $\sim_{\mathfrak{P}}$, is defined on the set of preparations $\mathfrak{P}$:

Two preparation processes are identified iff the statements established by the observer about the corresponding prepared samples are identical.

A *state* of the physical system is an equivalence class of preparation processes corresponding to the same informational content, i.e., a class of preparation processes that are not distinguished by the statements established by the observer in reference to the tests realized on finite collections of samples produced through these preparation processes.

The set of equivalence classes, modulo the relation $\sim_{\mathfrak{P}}$, will be denoted $\mathfrak{S}$. In other words,

$$\forall p_1, p_2 \in \mathfrak{P}, \quad (p_1 \sim_{\mathfrak{P}} p_2) \quad :\Leftrightarrow \quad (\forall t \in \mathfrak{T}, \epsilon(p_1, t) = \epsilon(p_2, t)), \quad (7)$$

$$\sim_{\mathfrak{P}} \text{ is an equivalence relation,} \quad (8)$$

$$[p] := \{p' \in \mathfrak{P} \mid p' \sim_{\mathfrak{P}} p\}, \quad (9)$$

$$\mathfrak{S} := \{[p] \mid p \in \mathfrak{P}\}. \quad (10)$$

Remark 4. It must be noticed that a given yes/no test t can be applied separately on the two distinct collections of samples prepared through the two distinct preparation procedures p_1 and p_2 . The corresponding counterfactual statements $\epsilon(p_1, t)$ and $\epsilon(p_2, t)$, established by the observer about p_1 and p_2 , are then formulated 'consistently' after these two independent experimental sequences.

We will derive a map $\tilde{\epsilon}$ from the evaluation map ϵ according to the following definition :

$$\begin{aligned} \tilde{\epsilon} : \mathfrak{T} &\rightarrow \mathfrak{B}^{\mathfrak{S}} \\ t &\mapsto \tilde{\epsilon}_t \mid \tilde{\epsilon}_t([p]) := \epsilon(p, t), \forall p \in \mathfrak{P}. \end{aligned} \quad (11)$$

¹⁷See [12] for a reference paper and [60][61] for a basic presentation.

¹⁸In this section, we are concerned with the duality aspect and the situation of Chu morphisms will be treated later.

¹⁹These designations are reminiscent of the basic fact that Chu spaces are generalizations of topological spaces. However this distinction is largely obsolete, as soon as the Chu space construction establishes a duality between these two sets.

As a result of this quotient operation on the space of preparation processes, it appears that we have the following natural property of our Chu space.

Lemma 1. The Chu space $(\mathfrak{S}, \mathfrak{T}, \tilde{\epsilon})$ is *separated*. Different preparation procedures are indeed 'identified' by the observer as soon as this observer attributes the same statements to the differently prepared samples. In other words,

$$\forall \sigma_1, \sigma_2 \in \mathfrak{S}, \quad (\forall t \in \mathfrak{T}, \tilde{\epsilon}_t(\sigma_1) = \tilde{\epsilon}_t(\sigma_2)) \Rightarrow (\sigma_1 = \sigma_2). \quad (12)$$

■

2.2 First axioms for the space of states

A pre-order relation can be defined on the set $\mathfrak{P}$ of preparation processes.

Notion 9. A preparation process $p_2 \in \mathfrak{P}$ is said to be *sharper* than another preparation process $p_1 \in \mathfrak{P}$ (this fact will be denoted $p_1 \sqsubseteq_{\mathfrak{P}} p_2$) iff any yes/no test $t \in \mathfrak{T}$ that is 'determinate' for the samples prepared through p_1 is also necessarily 'determinate' with the same value for the samples prepared through p_2 , i.e.,

$$\forall p_1, p_2 \in \mathfrak{P}, \quad (p_1 \sqsubseteq_{\mathfrak{P}} p_2) :\Leftrightarrow (\forall t \in \mathfrak{T}, \epsilon(p_1, t) \leq \epsilon(p_2, t)), \quad (13)$$

If $p_1 \sqsubseteq_{\mathfrak{P}} p_2$ (i.e., p_2 is 'sharper' than p_1), p_1 is said to be 'coarser' than p_2 .

Lemma 2. $(\mathfrak{P}, \sqsubseteq_{\mathfrak{P}})$ is a pre-ordered set.

■

The reflexivity and transitivity properties of the binary relation $\sqsubseteq_{\mathfrak{P}} \in \mathfrak{P} \times \mathfrak{P}$ are trivial to check.

The equivalence relation defined in **Notion 8** derives from this pre-order:

$$\forall p_1, p_2 \in \mathfrak{P}, \quad (p_1 \sqsubseteq_{\mathfrak{P}} p_2 \text{ and } p_1 \supseteq_{\mathfrak{P}} p_2) \Rightarrow (p_1 \sim_{\mathfrak{P}} p_2). \quad (14)$$

We will define an operation of mixtures on the set of preparations.

Axiom 1. If we consider a collection of preparation processes $P \subseteq \mathfrak{P}$, we can define a new preparation procedure, called *mixture* and denoted $\bigcap_{\mathfrak{P}} P$, as follows:

the samples produced from the preparation procedure $\bigcap_{\mathfrak{P}} P$ are obtained by a random mixing of the samples issued from the preparation processes of the collection P indiscriminately.

As a consequence, the statements that the observer can establish after a sequence of tests $t \in \mathfrak{T}$ on these samples produced through the procedure $\bigcap_{\mathfrak{P}} P$ is given as the infimum of the statements that the observer can establish for the elements of P separately. In other words,

$$\forall P \subseteq \mathfrak{P}, \exists! \bigcap_{\mathfrak{P}} P \in \mathfrak{P} \mid (\forall t \in \mathfrak{T}, \epsilon(\bigcap_{\mathfrak{P}} P, t) = \bigwedge_{p \in P} \epsilon(p, t)), \quad (15)$$

where $\bigwedge$ denotes the infimum of a collection of elements in the poset $(\mathfrak{B}, \leq)$. We will adopt the following notation $p_1 \bigcap_{\mathfrak{P}} p_2 = \bigcap_{\mathfrak{P}} \{p_1, p_2\}$.

We note the following obvious properties deduced from the literal definitions of the random mixing operation $\bigcap_{\mathfrak{P}}$ and the equivalence relation $\sim_{\mathfrak{P}}$ defining the space of states.

Lemma 3. For any $p_1, p_2, p_3 \in \mathfrak{P}$, we have

$$p_1 \sqcap_{\mathfrak{P}} (p_2 \sqcap_{\mathfrak{P}} p_3) \sim_{\mathfrak{P}} (p_1 \sqcap_{\mathfrak{P}} p_2) \sqcap_{\mathfrak{P}} p_3 \quad (16)$$

$$(p_1 \sqcap_{\mathfrak{P}} p_1) \sim_{\mathfrak{P}} p_1, \quad (17)$$

$$(p_2 \sqcap_{\mathfrak{P}} p_1) \sim_{\mathfrak{P}} (p_1 \sqcap_{\mathfrak{P}} p_2) \quad (18)$$

■

The properties of the equivalence relation $\sim_{\mathfrak{P}}$ with respect to the pre-order $\sqsubseteq_{\mathfrak{P}}$ and the random-mixing binary operation $\sqcap_{\mathfrak{P}}$ leads to the following properties:

Lemma 4. The binary operation $\sqcap_{\mathfrak{P}}$ being literally designed to satisfy properties (16), (17) and (18), the binary relation $\sqsubseteq_{\mathfrak{P}}$ is then equivalently defined by:

$$\forall p_1, p_2 \in \mathfrak{P}, \quad (p_1 \sqsubseteq_{\mathfrak{P}} p_2) \Leftrightarrow (p_1 \sim_{\mathfrak{P}} (p_1 \sqcap_{\mathfrak{P}} p_2)). \quad (19)$$

The following properties of the pre-order $\sqsubseteq_{\mathfrak{P}}$ are direct consequences of this fact:

$$\forall p, p_1, p_2 \in \mathfrak{P}, \quad (p_1 \sqcap_{\mathfrak{P}} p_2) \sqsubseteq_{\mathfrak{P}} p_1, \quad (20)$$

$$(p \sqsubseteq_{\mathfrak{P}} p_1 \text{ and } p \sqsubseteq_{\mathfrak{P}} p_2) \Rightarrow (p \sqsubseteq_{\mathfrak{P}} (p_1 \sqcap_{\mathfrak{P}} p_2)). \quad (21)$$

■

Lemma 5. The space of states $\mathfrak{S}$ is partially ordered

$$\forall \sigma_1, \sigma_2 \in \mathfrak{S}, \quad (\sigma_1 \sqsubseteq_{\mathfrak{S}} \sigma_2) :\Leftrightarrow (\forall p_1, p_2 \in \mathfrak{P}, (\sigma_1 = \lceil p_1 \rceil, \sigma_2 = \lceil p_2 \rceil) \Rightarrow (p_1 \sqsubseteq_{\mathfrak{P}} p_2)), \quad (22)$$

$$(\mathfrak{S}, \sqsubseteq_{\mathfrak{S}}) \text{ is a partial order.} \quad (23)$$

■

Moreover, the existence of the 'mixed' preparations satisfying property (15) leads to the following definition.

Notion 10.

$$\forall P \subseteq \mathfrak{P}, \quad \bigcap_{p \in P} \lceil p \rceil := \lceil \bigcap_{\mathfrak{P}} P \rceil. \quad (24)$$

Quite naturally, we will assume the existence and uniqueness of a bottom element in $\mathfrak{P}$:

Axiom 2. There exists a unique preparation process, that can be interpreted as a 'randomly-selected' collection of 'un-prepared samples'. This element leads to complete indeterminacy for any yes/no test realized on it. In other words, the following axiom will be imposed

$$\exists! p_{\perp} \in \mathfrak{P} \mid (\forall t \in \mathfrak{T}, \quad \mathfrak{c}(p_{\perp}, t) = \perp). \quad (25)$$

Lemma 6. $(\mathfrak{S}, \sqsubseteq_{\mathfrak{S}})$ admits a bottom element, denoted $\perp_{\mathfrak{S}}$:

$$\perp_{\mathfrak{S}} := \lceil p_{\perp} \rceil \quad \text{is the bottom element of } (\mathfrak{S}, \sqsubseteq_{\mathfrak{S}}). \quad (26)$$

■

Notion 11. A collection of preparation processes $P \subseteq \mathfrak{P}$ will be said to be *consistent* (this fact will be denoted $\widehat{P}^{\mathfrak{P}}$) iff the different elements can be considered as different incomplete preparations of the same targeted collection of prepared samples, i.e., iff there exists a preparation process $p \in \mathfrak{P}$ which is simultaneously sharper than any p' in P (i.e., which is a common upper-bound in $\mathfrak{P}$). In other words,

$$\forall P \subseteq \mathfrak{P}, \widehat{P}^{\mathfrak{P}} :\Leftrightarrow (\exists p \in \mathfrak{P} \mid p' \sqsubseteq_{\mathfrak{P}} p, \forall p' \in P). \quad (27)$$

We will also denote $\widehat{p_1 p_2}^{\mathfrak{P}} := \widehat{\{p_1, p_2\}}^{\mathfrak{P}}$. This consistency relation is obviously reflexive and symmetric.

Due to the following relation $\forall p_1, p_2, p_3 \in \mathfrak{P}, (\widehat{p_1 p_2}^{\mathfrak{P}} \text{ and } p_2 \sim_{\mathfrak{P}} p_3) \Rightarrow \widehat{p_1 p_3}^{\mathfrak{P}}$, the consistency relation can be defined between states as follows.

$$\forall \sigma_1, \sigma_2 \in \mathfrak{S}, \widehat{\sigma_1 \sigma_2}^{\mathfrak{S}} :\Leftrightarrow (\forall p_1, p_2 \in \mathfrak{P}, \sigma_1 = \lceil p_1 \rceil, \sigma_2 = \lceil p_2 \rceil \Rightarrow \widehat{p_1 p_2}^{\mathfrak{P}}). \quad (28)$$

For any $\sigma_1, \sigma_2 \in \mathfrak{S}$, we will denote $\widetilde{\sigma_1 \sigma_2}^{\mathfrak{S}}$ the property $\neg \widehat{\sigma_1 \sigma_2}^{\mathfrak{S}}$.

Theorem 1. $(\mathfrak{S}, \sqcap_{\mathfrak{S}})$ is bounded complete. In other words,

$$\forall S \subseteq \mathfrak{S} \mid \widehat{S}^{\mathfrak{S}}, \quad (\bigsqcup_{\mathfrak{S}} S) \text{ exists in } \mathfrak{S}. \quad (29)$$

■

The previous construction of the space of states, albeit usual, appears a bit strange from an operational perspective. Indeed, concretely, the observer is never confronted with a given 'state' (i.e., to a generically-infinite class of preparation processes, indistinguishable by the generically-infinite set of tests that can be realized on them) in order to decide if it is consistent (or not) to affirm with 'certainty' the occurrence of a given 'property' for a given sample corresponding to this state. The observer is rather confronted with a restricted set of preparation processes, enabling mixtures to be produced, which generically lead to undetermined results when they are confronted with a family of tests.

In order to produce 'determinacy', relative to the occurrence of a given property for a given state of the system, the observer extracts (from the selected family of preparation processes available, that are detected to produce samples corresponding more or less to the chosen state) some sub-families that concretely realize a *sharpening* of the parameters that define the preparation setting/procedure, according to a given set of prerequisites concerning the samples that will be submitted to the test. Through each 'sharpening' of its preparation procedures, the observer intends to fix 'unambiguously', but 'inductively', a 'state' of the system. This limit process is understood in terms of the limit taken for every statement that can be made about the selected samples (i.e., the samples prepared according to any of the preparation processes that are elements of the chosen sharpening family).

Notion 12. A family $\Omega \subseteq \mathfrak{P}$ is a *sharpening family of preparation processes* (denoted $\Omega \subseteq_{Chain} \mathfrak{P}$) iff every pair of elements of Ω are ordered by $\sqsubseteq_{\mathfrak{P}}$, i.e., for any p_1 and p_2 in Ω , we have necessarily $p_1 \sqsubseteq_{\mathfrak{P}} p_2$ or $p_2 \sqsubseteq_{\mathfrak{P}} p_1$.

Axiom 3. For any family Ω in $\mathfrak{P}$ defining a 'sharpening', there exists a state σ in $\mathfrak{S}$ which is the supremum of the chain of states corresponding to the elements of Ω .

$$\forall \Omega \subseteq_{Chain} \mathfrak{P}, \quad \text{the supremum } (\bigsqcup_{\mathfrak{S}} \lceil \Omega \rceil) \text{ exists in the partially ordered set } \mathfrak{S} \quad (30)$$

In other words, $\mathfrak{S}$ will be required to be a *chain-complete partial order*.

Remark 5. $\mathfrak{S}$ is then also a *directed-complete partial order*.

Let us then fix a yes/no test t . If the observer intends to designate the corresponding property as an 'element of reality' attached to the system itself, and not as a datum depending on the explicit operational requirements used to define the state, the following condition must be satisfied.

Axiom 4. The observer is authorized to formulate a 'determinate' statement, about the occurrence of a given property, for the 'limit state' induced from a given sharpening family of preparation processes, iff it is possible to formulate this same statement for another preparation process that is an element of the chosen sharpening family (and thus also for any sharper preparation process). In other words,

$$\forall t \in \mathfrak{T}, \forall \mathfrak{C} \subseteq_{\text{Chain}} \mathfrak{S}, \quad ((\bigsqcup_{\mathfrak{S}} \mathfrak{C}) \in [\mathfrak{A}_t]) \Rightarrow (\exists \sigma \in \mathfrak{C} \mid \sigma \in [\mathfrak{A}_t]). \quad (31)$$

The set of states $[\mathfrak{A}_t]$, for which the property tested by t is actual, must be a 'Scott-open filter' in the directed-complete partial order $(\mathfrak{S}, \sqsubseteq_{\mathfrak{S}})$.

We can reformulate the above requirement in terms of a continuity property of the evaluation map with respect to the sharpening process.

Lemma 7. For any $t \in \mathfrak{T}$, the map $\tilde{e}_t$ is chain-continuous, i.e., continuous with respect to the Scott topology on $\mathfrak{S}$ and $\mathfrak{B}$. In other words,

$$\forall t \in \mathfrak{T}, \forall \mathfrak{C} \subseteq_{\text{Chain}} \mathfrak{S}, \quad \bigvee_{\sigma \in \mathfrak{C}} \tilde{e}_t(\sigma) = \tilde{e}_t(\bigsqcup_{\mathfrak{S}} \mathfrak{C}). \quad (32)$$

■

An interesting consequence of the continuity property (32) formalizing Lemma 7 is the following property of $(\mathfrak{S}, \sqsubseteq_{\mathfrak{S}})$.

Lemma 8. The chain-complete Inf semi-lattice $(\mathfrak{S}, \sqsubseteq_{\mathfrak{S}})$ is meet-continuous, i.e.,

$$\forall \mathfrak{C} \subseteq_{\text{Chain}} \mathfrak{S}, \forall \sigma \in \mathfrak{S}, \quad \sigma \sqcap_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} \mathfrak{C}) = \bigsqcup_{\sigma' \in \mathfrak{C}} (\sigma \sqcap_{\mathfrak{S}} \sigma'). \quad (33)$$

■

Proof. This fact is easily established using the continuity of map $\tilde{e}_t$ and the meet-continuity of dcpo $\mathfrak{B}$. For any $\mathfrak{C} \subseteq_{\text{Chain}} \mathfrak{S}$ and any $\sigma \in \mathfrak{S}$, $(\sigma \sqcap_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} \mathfrak{C}))$ and $\bigsqcup_{\sigma' \in \mathfrak{C}} (\sigma \sqcap_{\mathfrak{S}} \sigma')$ exist as elements of $\mathfrak{S}$. Moreover,

$$\begin{aligned} \forall t \in \mathfrak{T}, \tilde{e}_t(\sigma \sqcap_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} \mathfrak{C})) &= \tilde{e}_t(\sigma) \wedge \tilde{e}_t(\bigsqcup_{\sigma' \in \mathfrak{C}} \sigma') \\ &= \tilde{e}_t(\sigma) \wedge \bigvee_{\sigma' \in \mathfrak{C}} \tilde{e}_t(\sigma') \\ &= \bigvee_{\sigma' \in \mathfrak{C}} (\tilde{e}_t(\sigma) \wedge \tilde{e}_t(\sigma')) \\ &= \bigvee_{\sigma' \in \mathfrak{C}} \tilde{e}_t(\sigma \sqcap_{\mathfrak{S}} \sigma') \\ &= \tilde{e}_t(\bigsqcup_{\sigma' \in \mathfrak{C}} (\sigma \sqcap_{\mathfrak{S}} \sigma')). \end{aligned}$$

We finally use the separation of the Chu space $(\mathfrak{B}, \mathfrak{T}, \epsilon)$ (i.e., Lemma 1) to conclude that $(\sigma \sqcap_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} \mathfrak{C})) = \bigsqcup_{\sigma' \in \mathfrak{C}} (\sigma \sqcap_{\mathfrak{S}} \sigma')$. $\square$

Axioms 1, 2, 3, 4 shall be complemented by several new axioms that are closely related to the specific character of quantum systems. Indeed, C.Rovelli aims to reconstruct quantum mechanics from the following conceptual proposal [64, Chap.III]:

'Information is a discrete quantity: there is a minimum amount of information exchangeable (a single bit, or the information that distinguishes between just two alternatives). [...] Since information is discrete, any process of acquisition of information can be decomposed into acquisitions of elementary bits of information.' [64, p.1655].

We will translate Rovelli's conceptual proposal as follows.

Notion 13. A state σ_2 is said to *contain one more bit of information than* another state σ_1 (this fact will be denoted $\sigma_1 \sqsubseteq_{\mathfrak{S}} \sigma_2$), iff (i) σ_2 is strictly sharper than σ_1 , and (ii) there is no state strictly separating σ_1 and σ_2 . In other words,

$$\forall \sigma_1, \sigma_2 \in \mathfrak{S}, (\sigma_1 \sqsubseteq_{\mathfrak{S}} \sigma_2) :\Leftrightarrow \left\{ \begin{array}{l} (\sigma_1 \sqsubset_{\mathfrak{S}} \sigma_2) \\ (\forall \sigma \in \mathfrak{S}, (\sigma_1 \sqsubseteq_{\mathfrak{S}} \sigma \sqsubseteq_{\mathfrak{S}} \sigma_2) \Rightarrow (\sigma_1 = \sigma \text{ or } \sigma_2 = \sigma)) \end{array} \right\}. \quad (34)$$

The discreteness of the informational content, encoded in the partial order $\sqsubseteq_{\mathfrak{S}}$ defined on the space of states $\mathfrak{S}$, is translated into the following topological condition.

Axiom 5. For any state σ_1 admitting a sharper state σ_2 , there exists another state σ_3 which contains one more 'bit of information' than σ_1 and coarser than σ_2 . In other words, the partial-order $(\mathfrak{S}, \sqsubseteq_{\mathfrak{S}})$ is strongly atomic, i.e.

$$\forall \sigma, \sigma' \in \mathfrak{S} \mid \sigma \sqsubset \sigma' \Rightarrow (\exists \sigma'' \in \mathfrak{S}, \sigma \sqsubseteq_{\mathfrak{S}} \sigma'' \sqsubseteq_{\mathfrak{S}} \sigma'). \quad (35)$$

However, this requirement is not sufficient to capture all the aspects of our conceptual proposal. The exchangeable information comprises a collection of distinguishable bits of information and the acquisition of the informational content is reduced to the acquisition of these bits of information.

Axiom 6. For any state $\sigma \in \mathfrak{S}$ and any other $\sigma' \in \mathfrak{S}$ containing more information (i.e., $\sigma \sqsubset_{\mathfrak{S}} \sigma'$), the pair (σ, σ') 'encodes' an unambiguous piece of information entering into the 'decomposition' of σ' .

More explicitly, for any $\sigma_2 \in \mathfrak{S}$, any sharper state $\sigma'_2 \in \mathfrak{S}$, and any $\sigma_1 \in \mathfrak{S}$ 'coarser' than σ_2 , there exists a state σ'_1 such that (i) σ'_2 is the coarsest state sharper than σ'_1 and σ_2 , (ii) σ_1 is the sharpest state coarser than σ'_1 and σ_2 . Moreover, (i) if (σ_2, σ'_2) encodes for a 'bit' of information, this is also the case for (σ_1, σ'_1) , and (ii) if (σ_1, σ_2) encodes for a 'bit' of information, this is also the case for (σ'_1, σ'_2) . In other words,

$$\forall \sigma_1, \sigma_2, \sigma'_2 \in \mathfrak{S}, (\sigma_1 \sqsubseteq_{\mathfrak{S}} \sigma_2 \sqsubseteq_{\mathfrak{S}} \sigma'_2) \Rightarrow (\exists \sigma'_1 \in \mathfrak{S}, | (\sigma_1 = \sigma'_1 \sqcap_{\mathfrak{S}} \sigma_2 \text{ and } \sigma'_2 = \sigma'_1 \sqcup_{\mathfrak{S}} \sigma_2)), \quad (36)$$

and

$$\forall \sigma'_1, \sigma_2, \sigma'_2 \in \mathfrak{S}, (\sigma'_1 \sqsubseteq_{\mathfrak{S}} \sigma'_2, \sigma_2 \sqsubseteq_{\mathfrak{S}} \sigma'_2) \Rightarrow ((\sigma'_1 \sqcap_{\mathfrak{S}} \sigma_2) \sqsubseteq_{\mathfrak{S}} \sigma'_1), \quad (37)$$

$$\forall \sigma'_2, \sigma'_1, \sigma_2, \sigma_1 \in \mathfrak{S}, (\sigma'_2 \sqsupseteq_{\mathfrak{S}} \sigma'_1 \sqsupseteq_{\mathfrak{S}} \sigma_1, \sigma'_2 \sqsupseteq_{\mathfrak{S}} \sigma_2 \sqsupseteq_{\mathfrak{S}} \sigma_1) \Rightarrow ((\sigma'_1 \sqcup_{\mathfrak{S}} \sigma_2) \sqsupseteq_{\mathfrak{S}} \sigma'_1). \quad (38)$$

As a summary, $\mathfrak{S}$ is relatively-complemented (property (36)) and satisfies lower semi-modularity (property (37)) and conditional upper semi-modularity (property (38)).

Remark 6. Note that, in (37), $(\sigma'_1 \sqsubseteq_{\mathfrak{S}} \sigma'_2, \sigma_2 \sqsubseteq_{\mathfrak{S}} \sigma'_2)$ implies trivially $\sigma'_2 = (\sigma'_1 \sqcup_{\mathfrak{S}} \sigma_2)$, and, in (38), $(\sigma'_1 \sqsupseteq_{\mathfrak{S}} \sigma_1, \sigma_2 \sqsupseteq_{\mathfrak{S}} \sigma_1)$ implies trivially $\sigma_1 = (\sigma'_1 \sqcap_{\mathfrak{S}} \sigma_2)$.

2.3 First properties of the space of states as a domain

We now exploit the above requirements to characterize the structure of the space of states. Let us firstly summarize the collected information.

$(\mathfrak{S}, \sqsubseteq_{\mathfrak{S}})$ is a partial order (property (23)). Due to **Axiom 2**, this partial order is pointed (property (26)). Due to **Axiom 1**, $(\mathfrak{S}, \sqsubseteq_{\mathfrak{S}})$ is also a complete Inf semi-lattice (it is then also a bounded-complete partial order, see Theorem 1). Moreover, due to **Axiom 3**, this partial order is a directed complete partial order (property (30)). Due to Lemma 8, deriving from **Axiom 4**, this partial order is meet-continuous. Due to **Axiom 5**, the space of states $(\mathfrak{S}, \sqsubseteq_{\mathfrak{S}})$ satisfies the strong atomicity property (35). Lastly, due to **Axiom 6**, the space of states $(\mathfrak{S}, \sqsubseteq_{\mathfrak{S}})$ is relatively complemented (property (36)) and satisfies lower semi-modularity (property (37)) and conditional upper semi-modularity (property (38)).

Lemma 9. [25, Theorem 3.6 p.24] Due to the meet-continuity property (33), the strong-atomicity property (35), the lower semi-modularity (property (37)) and the conditional upper semi-modularity (property (38)), we deduce that $\mathfrak{S}$ is conditionally-modular. In other words,

$$\forall \sigma_1, \sigma_2, \sigma_3 \in \mathfrak{S} \mid \widehat{\sigma_2 \sigma_3}^{\mathfrak{S}} \quad (\sigma_1 \sqsupseteq_{\mathfrak{S}} \sigma_2) \Rightarrow \sigma_1 \sqcap_{\mathfrak{S}} (\sigma_2 \sqcup_{\mathfrak{S}} \sigma_3) = \sigma_2 \sqcup_{\mathfrak{S}} (\sigma_1 \sqcap_{\mathfrak{S}} \sigma_3), \quad (39)$$

or, equivalently,

$$\forall \sigma_1, \sigma_2, \sigma_3 \in \mathfrak{S} \mid \widehat{\sigma_1 \sigma_3}^{\mathfrak{S}}, (\sigma_1 \sqsupseteq_{\mathfrak{S}} \sigma_2), \quad (\sigma_1 \sqcap_{\mathfrak{S}} \sigma_3 = \sigma_2 \sqcap_{\mathfrak{S}} \sigma_3 \text{ and } \sigma_1 \sqcup_{\mathfrak{S}} \sigma_3 = \sigma_2 \sqcup_{\mathfrak{S}} \sigma_3) \Rightarrow (\sigma_1 = \sigma_2). \quad (40)$$

■

Lemma 10. [25, Theorem 4.3 (3) $\Rightarrow$ (4)] Due to the meet-continuity property (33), the strong-atomicity property (35), the bounded-completeness (Theorem 1) and the relative complement property (property (36)), we deduce that $\mathfrak{S}$ is atomistic.

$$\forall \sigma \in \mathfrak{S}, \quad \sigma = \bigsqcup_{\mathfrak{S}} \{a \mid \sigma \sqsupseteq_{\mathfrak{S}} a \sqsupseteq_{\mathfrak{S}} \perp_{\mathfrak{S}}\}. \quad (41)$$

■

Lemma 11. Due to the meet-continuity property (33), the atoms are compact. Then, using Lemma 10, we deduce that the domain $\mathfrak{S}$ is algebraic (i.e. compactly generated).

$$\forall \sigma \in \mathfrak{S}, \quad \sigma = \bigsqcup_{\mathfrak{S}} \{c \mid c \sqsubseteq_{\mathfrak{S}} \sigma, c \in \mathfrak{S}_c\}. \quad (42)$$

■

Let us recall that a lattice is said to be a *projective lattice* iff it is complete, atomistic, meet-continuous (or algebraic), complemented and modular [28, Definition 2.4.7 p.38]. Moreover, every complemented modular lattice is relatively-complemented [25, Theorem 4.2 p.31]. We introduce the definition :

Definition 1. A directed complete, bounded complete, atomistic, meet-continuous (or algebraic), relatively-complemented and conditionally-modular poset will be called *projective domain*.

As a consequence of previous lemmas, we have

Theorem 2. The space of states is a projective domain. ■

Conjecture 1. As a consequence of relative-complement property and meet-continuity property we suggest that $\mathfrak{S}$ is join-continuous. In other words,

$$\forall \sigma \in \mathfrak{S}, \forall \mathfrak{C} \subseteq_{Chain} \mathfrak{S} \mid \exists \sigma' \in \mathfrak{C}, \widehat{\sigma \sigma'}^{\mathfrak{S}}, \quad \sigma \sqcup_{\mathfrak{S}} \bigsqcap_{\mathfrak{S}} \mathfrak{C} = \bigsqcap_{\sigma'' \in \mathfrak{C}, \sigma' \sqsupseteq_{\mathfrak{S}} \sigma''} (\sigma \sqcup_{\mathfrak{S}} \sigma''). \quad (43)$$

Remark 7. see [16, Corollary 6] for a key intermediate step in the proof.

2.4 Pure states

Notion 14. A state is said to be a *pure state* if and only if it cannot be built as a mixture of other states (the set of pure states will be denoted $\mathfrak{S}_{\text{pure}}$). More explicitly,

$$\sigma \in \mathfrak{S}_{\text{pure}} \quad :\Leftrightarrow \quad (\forall S \subseteq \neq \emptyset \mathfrak{S}, (\sigma = \bigcap_{\mathfrak{S}} S) \Rightarrow (\sigma \in S)). \quad (44)$$

In other words, pure states are associated with complete meet-irreducible elements in the selection structure $\mathfrak{S}$.

Remark 8. Complete meet-irreducibility implies meet-irreducibility. In other words,

$$\sigma \in \mathfrak{S}_{\text{pure}} \Rightarrow \forall \sigma_1, \sigma_2 \in \mathfrak{S}, (\sigma = \sigma_1 \sqcap_{\mathfrak{S}} \sigma_2) \Rightarrow (\sigma = \sigma_1 \text{ or } \sigma = \sigma_2). \quad (45)$$

A simple characterization of completely meet-irreducible elements within posets is adopted in [31, Definition I-4.21]. This characterization is equivalent to the above one for a bounded-complete inf semi-lattice. We have explicitly

Theorem 3. [Characterization of pure states]

$$\sigma \in \mathfrak{S}_{\text{pure}} \Leftrightarrow \begin{cases} \sigma \in \text{Max}(\mathfrak{S}) & \text{(Type 1)} \\ \text{or} \\ (\uparrow^{\mathfrak{S}} \sigma) \setminus \{\sigma\} \text{ admits a minimum element} & \text{(Type 2)} \end{cases} \quad (46)$$

■

It is clear that 'type 2' pure states have no physical meaning. Indeed, for any 'type 2' pure states, it exists a unique 'type 1' pure state sharper than it (and, then, containing more information than it). The existence of 'type 2' pure states in the space of state leads then to a redundant description of the system.

Axiom 7. $\mathfrak{S}_{\text{pure}}$ admits no element of type 2.

Theorem 4. Every state can be written as a mixture of pure states. In other words,

$$\forall \sigma \in \mathfrak{S} \quad \sigma = \bigcap_{\mathfrak{S}} (\mathfrak{S}_{\text{pure}} \cap (\uparrow^{\mathfrak{S}} \sigma)). \quad (47)$$

Moreover, $\mathfrak{S}_{\text{pure}}$ is the unique smallest subset of states generating any state by mixture. In other words, $\mathfrak{S}_{\text{pure}}$ is the unique smallest order-generating subset in $\mathfrak{S}$ (i.e., the unique smallest subset of $\mathfrak{S}$ satisfying property (47)). ■

Proof. $\mathfrak{S}$ being, in particular, a bounded-complete algebraic domain, this result is a direct consequence of [31, Theorem I-4.26]. □

Notion 15. We will introduce the following subset of pure states associated with any state :

$$\forall S \subseteq \mathfrak{S}^*, \quad \underline{S} := (\mathfrak{S}_{\text{pure}} \cap (\uparrow^{\mathfrak{S}} S)). \quad (48)$$

3 Properties and measurements

3.1 Properties and States

Let us now focus on the set of yes/no tests. Adopting our perspective on the Chu duality between $\mathfrak{P}$ and $\mathfrak{T}$, it is natural to introduce the following equivalence relation on $\mathfrak{T}$.

Notion 16. An equivalence relation, denoted $\sim_{\mathfrak{T}}$, is defined on the set of yes/no tests $\mathfrak{T}$:

Two yes/no tests are identified iff the corresponding statements established by the observer about any given preparation process are the same.

A *property* of the physical system is an equivalence class of yes/no tests, i.e., a class of yes/no tests that are not distinguished from the point of view of the statements that the observer can produce by using these yes/no tests on finite collections of samples.

The set of equivalence classes of yes/no tests, modulo the relation $\sim_{\mathfrak{T}}$, will be denoted $\mathcal{L}$. In other words,

$$\forall t_1, t_2 \in \mathfrak{T}, (t_1 \sim_{\mathfrak{T}} t_2) :\Leftrightarrow (\forall p \in \mathfrak{P}, \mathfrak{e}(p, t_1) = \mathfrak{e}(p, t_2)), \quad (49)$$

$$\sim_{\mathfrak{T}} \text{ is an equivalence relation,} \quad (50)$$

$$[t] := \{t' \in \mathfrak{T} \mid t' \sim_{\mathfrak{T}} t\}, \quad (51)$$

$$\mathcal{L} := \{[t] \mid t \in \mathfrak{T}\}. \quad (52)$$

The following equivalence justifies the use of the notion of 'property' in the literal definitions of 'potentiality' and 'actuality' :

$$\forall t_1, t_2 \in \mathfrak{T}, (t_1 \sim_{\mathfrak{T}} t_2) \Leftrightarrow (\mathfrak{Q}_{t_1} = \mathfrak{Q}_{t_2} \text{ and } \mathfrak{A}_{t_1} = \mathfrak{A}_{t_2}). \quad (53)$$

Hence,

Notion 17. for any $l \in \mathcal{L}$, we will now denote by $\mathfrak{Q}_l$ (resp. $\mathfrak{A}_l$) the set $\mathfrak{Q}_t$ (resp. $\mathfrak{A}_t$) taken for any t such that $l = [t]$.

Moreover,

Notion 18. for any $l \in \mathcal{L}$, we will now denote by $\tilde{\mathfrak{e}}_l$ the evaluation map defined on $\mathfrak{S}$ and defined by $\tilde{\mathfrak{e}}_l := \tilde{\mathfrak{e}}_t$ for any t such that $l = [t]$.

Notion 19. A property $l \in \mathcal{L}$ will be said to be *testable* iff it can be revealed as 'actual' at least for some collections of prepared samples. In other words,

$$l \text{ is 'testable'} :\Leftrightarrow \mathfrak{A}_l \text{ is a non-empty subset of } \mathfrak{S}. \quad (54)$$

We check immediately that $\forall t_1, t_2 \in \mathfrak{T}, (t_1 \sim_{\mathfrak{T}} t_2) \Leftrightarrow (\overline{t_2} \sim_{\mathfrak{T}} \overline{t_1})$. As a consequence, the bar involution will be defined on the space of properties simply by requiring

Notion 20.

$$\forall l \in \mathcal{L}, \quad \bar{l} := \{\bar{t} \mid l = [t]\}. \quad (55)$$

($\mathfrak{S}, \sqsubseteq_{\mathfrak{S}}$) being a selection structure, we know that ($\mathfrak{S}, \sqsubseteq_{\mathfrak{S}}$) is a bounded-complete Inf semi-lattice, and in particular is closed under arbitrary infima. In other words,

Lemma 12.

$$\forall \mathfrak{Q} \subseteq^{\neq \emptyset} \mathfrak{P}, \quad \text{the infimum } (\bigwedge_{\mathfrak{S}} [\mathfrak{Q}]) \text{ exists in } \mathfrak{S}, \quad (56)$$

■

Moreover, we inherit from Lemma 7 the following continuity property :

Lemma 13.

$$\forall \mathfrak{R} \subseteq \neq \emptyset \mathfrak{S}, \forall t \in \mathfrak{T}, \quad \tilde{e}_t(\bigcap_{\mathfrak{S}} \mathfrak{R}) = \bigwedge_{\sigma \in \mathfrak{R}} \tilde{e}_t(\sigma). \quad (57)$$

■

Proof. $\forall \mathfrak{R} \subseteq \neq \emptyset \mathfrak{S}$, we define $\mathfrak{M}_{\mathfrak{R}} := \{\sigma \in \mathfrak{S} \mid \sigma \sqsubseteq_{\mathfrak{S}} \mathfrak{R}\}$. $\mathfrak{M}_{\mathfrak{R}}$ is obviously directed and $\bigcap_{\mathfrak{S}} \mathfrak{R} = \bigcap_{\mathfrak{S}} \mathfrak{M}_{\mathfrak{R}}$. Now, using the Scott continuity of $\tilde{e}_t$, we have $\tilde{e}_t(\bigcap_{\mathfrak{S}} \mathfrak{R}) = \bigvee_{\sigma \in \mathfrak{M}_{\mathfrak{R}}} \tilde{e}_t(\sigma)$. The monotonicity of $\tilde{e}_t$ and the fact that the target space of $\tilde{e}_t$ is the boolean domain $\mathfrak{B}$ implies moreover that $\bigvee_{\sigma \in \mathfrak{M}_{\mathfrak{R}}} \tilde{e}_t(\sigma) = \bigwedge_{\sigma \in \mathfrak{R}} \tilde{e}_t(\sigma)$. $\square$

Theorem 5. For any property $l \in \mathcal{L}$, the evaluation map $\tilde{e}_l$ is order preserving and continuous with respect to the Lawson topologies on $\mathfrak{S}$ and $\mathfrak{B}$. $\blacksquare$

Proof. From previous lemma we have that, for any $t \in \mathfrak{T}$, the map $\tilde{e}_t$ is continuous with respect to the lower topologies on $\mathfrak{P}$ and $\mathfrak{B}$.

Due to Lemma 7 and property (57), and using [31, Theorem III-1.8 p.213], we then prove the announced continuity property. $\square$

We can then deduce the form of the subsets $[\mathfrak{A}_t]$ determining the sub-space of states for which the property tested by t is actual.

Theorem 6. [Property-state]

For any t in $\mathfrak{T}$, corresponding to a testable property $[t]$, there exists an element $\Sigma_t \in \mathfrak{S}$ (in fact, a compact element in the algebraic domain $\mathfrak{S}$), such that the Scott-open filter $[\mathfrak{A}_t]$ is the principal filter associated with Σ_t :

$$\forall t \in \mathfrak{T} \mid [t] \text{ is testable}, \exists \Sigma_t \in \mathfrak{S}_c \mid [\mathfrak{A}_t] = (\uparrow^{\mathfrak{S}} \Sigma_t). \quad (58)$$

In particular, we have

$$\forall t \in \mathfrak{T}, \forall \sigma \in \mathfrak{S}, \quad \tilde{e}_t(\sigma) = \mathbf{Y} \Leftrightarrow \Sigma_t \sqsubseteq_{\mathfrak{S}} \sigma. \quad (59)$$

If the conjugate test $\bar{t}$ corresponds to a testable property as well, there exists an element $\Sigma_{\bar{t}}$ such that

$$\Sigma_{\bar{t}} = \bigcap_{\mathfrak{S}} \tilde{e}_t^{-1}(\mathbf{N}) \quad (60)$$

$$\Omega_t = \mathfrak{S} \setminus (\uparrow^{\mathfrak{S}} \Sigma_{\bar{t}}) \quad (61)$$

■

Proof. We have already remarked that $[\mathfrak{A}_t]$ is a Scott-open filter. Using the fact that $\mathfrak{S}$ is a complete Inf semi-lattice as well as property (57), we also note that the element $\Sigma_t := \bigcap_{\mathfrak{S}} [\mathfrak{A}_t]$ obeys $\tilde{e}_t(\Sigma_t) = \bigwedge_{\sigma \in [\mathfrak{A}_t]} \tilde{e}_t(\sigma) = \mathbf{Y}$. Then, Σ_t is the 'minimum' of $[\mathfrak{A}_t]$. As a consequence, the filter $[\mathfrak{A}_t]$ is revealed to be the principal filter $(\uparrow^{\mathfrak{S}} \Sigma_t)$. From [31, Remark I-4.24], we deduce that Σ_t is a compact element in $\mathfrak{S}$.

The properties associated to $\Sigma_{\bar{t}}$ are derived along the same way. $\square$

Notion 21. For any yes/no test $t \in \mathfrak{T}$, corresponding to the testable property $[t]$, the state Σ_t is defined to be the minimal element of the principal filter $[\mathfrak{A}_t]$ in $(\mathfrak{S}, \sqsubseteq_{\mathfrak{S}})$.

$$\forall t \in \mathfrak{T} \mid [t] \text{ is testable}, \quad \Sigma_t := \bigcap_{\mathfrak{S}} [\mathfrak{A}_t] = \bigcap_{\mathfrak{S}} \tilde{e}_t^{-1}(\mathbf{Y}). \quad (62)$$

The state Σ_t depends only on the testable property $[t]$ associated with t . This state will then be called the *property-state* associated to t and we will henceforth adopt the following abuse of notation $\Sigma_{[t]} := \Sigma_t$.

3.2 Tests and measurements

The measurement process offers another perspective on the relation between the spaces $\mathfrak{P}$ and $\mathfrak{T}$ (the first being a duality relation): this new perspective emphasizes the 'recursive' aspect of the preparation process. Indeed, a given yes/no test $t \in \mathfrak{T}$ can be used to complete a given preparation procedure $p \in \mathfrak{P}$ in order to produce a new preparation procedure, as the 'filtering operation' associated with t actually operates on a collection of produced samples that can exhibit the desired property.

Notion 22. For any yes/no test $t \in \mathfrak{T}$ and any preparation procedure $p \in \Omega_t$, we can define a preparation procedure denoted $p.t$ and defined as follows :

the samples, previously prepared through procedure p , are actually submitted to the measurement operation defined according to the yes/no test t , the resulting (or 'outcoming') samples of the 'whole preparation process' (i.e., the initial preparation p followed by the filtering operation defined by t), denoted $(p.t)$, are the samples 'actually measured as positive' through the yes/no test t .

For any yes/no test $t \in \mathfrak{T}$, we will then associate it with the partial map denoted $.t$ and defined by (where its domain will be denoted $Dom_t^{\mathfrak{P}} \subseteq \mathfrak{P}$, and its range $Im_t^{\mathfrak{P}}$):

$$\begin{array}{ccc} .t & : & \mathfrak{P} \dashrightarrow \mathfrak{P} \\ p & \mapsto & p.t \end{array} \quad Dom_t^{\mathfrak{P}} := \Omega_t \quad (63)$$

These maps define the measurement operation associated with a given property. The map $.t$ associated with a yes/no test t is called *the measurement map* associated with t .

Notion 23. For any yes/no tests $t_1, t_2 \in \mathfrak{T}$, we can build a new yes/no test denoted $(t_1.t_2) \in \mathfrak{T}$ and called *the succession of t_1 by t_2* . It is defined as follows :

to begin, the incoming sample is tested through the yes/no test t_1 ; if the result of this test is negative, then the whole yes/no test $(t_1.t_2)$ is declared 'negative', but if the result is positive, the outcoming sample having been positively measured by t_1 is submitted to the yes/no test t_2 ; the result of this test is then attributed to the whole yes/no test $(t_1.t_2)$ for the given prepared sample.

In other words,

$$\forall t_1, t_2 \in \mathfrak{T}, \exists (t_1.t_2) \in \mathfrak{T} \quad | \quad \mathfrak{A}_{t_1.t_2} := \{ p \in \mathfrak{A}_{t_1} \mid (p.t_1) \in \mathfrak{A}_{t_2} \} \quad (64)$$

$$\Omega_{t_1.t_2} := \{ p \in \Omega_{t_1} \mid (p.t_1) \in \Omega_{t_2} \} \quad (65)$$

$$\mathfrak{A}_{\overline{t_1.t_2}} := \mathfrak{A}_{\overline{t_1}} \cup \{ p \in \Omega_{t_1} \mid (p.t_1) \in \mathfrak{A}_{\overline{t_2}} \} \quad (66)$$

$$\forall p \in Dom_{(t_1.t_2)}^{\mathfrak{P}} = \Omega_{t_1.t_2}, \quad p.(t_1.t_2) := (p.t_1).t_2. \quad (67)$$

The 'succession law' satisfies the following associativity properties:

$$\forall p \in \mathfrak{P}, \forall t_1, t_2 \in \mathfrak{T}, \quad p.(t_1.t_2) = (p.t_1).t_2, \quad (68)$$

$$\forall t_1, t_2, t_3 \in \mathfrak{T}, \quad (t_1.t_2).t_3 = t_1.(t_2.t_3). \quad (69)$$

We note the following natural relations.

Lemma 14.

$$\forall t_1, t_2 \in \mathcal{T}, \quad \forall p \in \Omega_{t_1}, \quad c(p \cdot t_1, t_2) \geq c(p, t_1 \cdot t_2), \quad (70)$$

$$\overline{t_1 \cdot t_2} = \overline{t_1} \cdot \overline{t_2} \quad (71)$$

■

As a basic requirement of measurement maps, we will impose that they are monotone maps on their domain. The following simple analysis justifies this requirement.

Theorem 7. A measurement operation associated with any yes/no test, t , respects the ordering of information established by the observer about the collection of samples on which it is realized.

$$\forall t \in \mathcal{T}, \quad \forall p_1, p_2 \in \Omega_t, \quad (p_1 \sqsubseteq_{\mathfrak{P}} p_2) \Rightarrow ((p_1 \cdot t) \sqsubseteq_{\mathfrak{P}} (p_2 \cdot t)). \quad (72)$$

The measurement map $(\cdot t)$ is an order-preserving map on $(\mathfrak{P}, \sqsubseteq_{\mathfrak{P}})$. ■

Proof. If the preparation processes p_1 and p_2 are ordered by $\sqsubseteq_{\mathfrak{P}}$ (p_2 being sharper than p_1), every statement made by the observer about p_1 is also necessarily made about p_2 , i.e., $\forall u \in \mathcal{T}, c(p_1, u) \leq c(p_2, u)$. This is true in particular for the statements that can be made by the observer about the corresponding collections of samples after having separately measured the property associated with a given yes/no test, t , on each collection of samples beforehand. More precisely, we must then have $\forall v \in \mathcal{T}, c(p_1 \cdot t, v) \leq c(p_2 \cdot t, v)$. □

Corollary 1. As a consequence, the measurement operation $(\cdot t)$ associated with a given yes/no test, t , cannot distinguish different collections of samples on which it acts, when these collections of samples correspond to the same state of the system, i.e.,

$$\forall t \in \mathcal{T}, \quad \forall p_1, p_2 \in \Omega_t, \quad (p_1 \sim_{\mathfrak{P}} p_2) \Rightarrow ((p_1 \cdot t) \sim_{\mathfrak{P}} (p_2 \cdot t)). \quad (73)$$

■

Notion 24. The measurement operation will then be defined to act on states as follows:

$$\forall t \in \mathcal{T}, \quad \forall p \in \Omega_t, \quad [p] \cdot t := [p \cdot t] \quad (74)$$

We will adopt the following notations $Dom_{\cdot t}^{\mathfrak{S}} := [Dom_{\cdot t}^{\mathfrak{P}}]$ and $Im_{\cdot t}^{\mathfrak{S}} := [Im_{\cdot t}^{\mathfrak{P}}]$.

Corollary 2. The measurement map $(\cdot t)$ is an order-preserving map on $(\mathfrak{S}, \sqsubseteq_{\mathfrak{S}})$

$$\forall t \in \mathcal{T}, \quad \forall \sigma_1, \sigma_2 \in \mathfrak{S}_t, \quad (\sigma_1 \sqsubseteq_{\mathfrak{S}} \sigma_2) \Rightarrow ((\sigma_1 \cdot t) \sqsubseteq_{\mathfrak{S}} (\sigma_2 \cdot t)). \quad (75)$$

■

We then complement the above property of monotonicity with a requirement concerning random mixing.

Endly, we will observe the following property.

Theorem 8. The operation of measurement respects the induction process of a limit state from any sharpening family. In other words, for any yes/no test, t , the measurement map $(\cdot t)$ is a Scott-continuous partial map.

$$\forall t \in \mathcal{T}, \quad \forall \mathcal{C} \subseteq_{Chain} [\Omega_t], \quad \bigsqcup_{\sigma \in \mathcal{C}} (\sigma \cdot t) = (\bigsqcup_{\sigma \in \mathcal{C}} \sigma) \cdot t. \quad (76)$$

■

Proof. Let us consider a sharpening family of preparation processes Ω . The existence of the limit state $(\sqcup_{\mathfrak{S}} [\Omega])$ is ensured by the collection of properties $\bigvee_{\sigma \in [\Omega]} \tilde{e}_u(\sigma) = \tilde{e}_u(\sqcup_{\mathfrak{S}} [\Omega])$ considered for every yes/no test $u \in \mathfrak{T}$. If we consider, in particular, the subset of statements associated with any yes/no test, t_2 , made by the observer concerning the states outcoming from the measurement associated with a yes/no test, t_1 , we deduce $\bigvee_{\sigma \in [\Omega]} \tilde{e}_{t_2}(\sigma.t_1) = \tilde{e}_{t_2}((\sqcup_{\mathfrak{S}} [\Omega]).t_1)$. And thus, for any $t_1, t_2 \in \mathfrak{T}$, we obtain $\tilde{e}_{t_2}(\sqcup_{\sigma \in [\Omega]} (\sigma.t_1)) = \tilde{e}_{t_2}((\sqcup_{\mathfrak{S}} [\Omega]).t_1)$. $\square$

The set of partial maps defined from $\mathfrak{S}_1$ (the domain of the partial map has to be a Scott-closed subset of $\mathfrak{S}_1$) to $\mathfrak{S}_2$, which are order preserving and Scott continuous will be denoted $[\mathfrak{S}_1 \dashrightarrow \mathfrak{S}_2]^{\wedge}$.

To summarize Theorem 7 and Theorem 8, we will have :

$$\forall t \in \mathfrak{T}, \quad .t \in [\mathfrak{S} \dashrightarrow \mathfrak{S}]^{\wedge} \mid \text{Dom}_{.t}^{\mathfrak{S}} = [\Omega_t]. \quad (77)$$

3.3 Minimally disturbing measurements

As analyzed above, the 'certainty' of the observer about the occurrence (or not) of a given 'property' for a given state, is formulated as a counterfactual statement ('actuality' or 'impossibility') about the tests that 'could' be realized on any sample corresponding to this state (and this certainty has been produced after having tested this property on similarly prepared samples). *Stricto sensu* this statement is then formulated without disturbing in any way the considered new sample, according to the definition of the 'elements of reality' for the system given in the celebrated paper of A. Einstein, B. Podolsky, and N. Rosen :

'If, without in any way disturbing a system, we can predict with certainty (i.e., with probability equal to unity) the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity.' [27].

Nevertheless, in order to establish an interpretation of 'properties' as elements of reality, the observer must be able to establish and confirm 'conjunctly' statements about the different properties that are considered 'simultaneously' actual for a given collection of similarly prepared samples. It is thus necessary to restrict the measurement operations that will be used by the observer, with regard to the disturbance they cause in the measured sample. These measurement operations should guarantee that the state of the system after measurement be characterized by the properties established as actual beforehand, through 'successive' measurement operations. The possibility (and necessity) to characterize minimally disturbing measurements exists in the classical and in the quantum situation as well. However, although the existence of such measurements does not pose any conceptual problem in the classical situation, things are more complex in the quantum situation as soon as the measurement process irreducibly 'alters' the state of the measured system.

Despite the indeterministic character of quantum measurements, we note that the realization of a 'careful' yes/no test does allow the observer to make some statements about the state 'after' the measurement, although it appears risky to extend these conclusions to the state of the system 'before' the measurement (due to the irreducible alteration of the state during the measurement operation). At least, a 'careful' measurement of a given property on a given sample^[20] may guarantee the actuality of this property just after the measurement. As a consequence, the immediate repetition of this test should produce with certainty the same 'answer'. These sort of careful measurements do exist in the classical and in the quantum situation as well, and have been called *first-kind measurements* by W. Pauli :

'The method of measurement [...] has the property that a repetition of measurement gives the same value for the quantity measured as in the first measurement. In other words, if the

²⁰Here we mean that the test is effectuated on the considered sample, according to the procedure defined by a yes/no test associated with this property, and the 'answer' received by the observer is 'positive'.

result of using the measuring apparatus is not known, but only the fact of its use is known [...], the probability that the quantity measured has a certain value is the same, both before and after the measurement. We shall call such measurements the measurements of the first kind.' [52, p.75]^[21]

We will adopt the following formal definition for this type of measurement.

Notion 25. A yes/no test $t \in \mathcal{T}$ is said to lead to a *first-kind measurement* associated with the corresponding testable property $[t]$ iff (i) a positive result of the test t realized on any input sample is necessarily confirmed by an immediate repetition of this test realized on the samples outcoming from the first test, and (ii) the observer cannot establish if this new 'check' has been performed or not on the basis of the new tests that could be performed on the outcoming samples of the experiment. In other words, the testable property $[t]$ can be considered as 'actual' after the measurement by t , and the actuality of the property $[t]$ can be 'confirmed' through any repetition of the measurement by t , because this repeated measurement leaves the state of the system unchanged.

The subset of yes/no tests leading to first-kind measurements will be denoted $\mathcal{T}_{FKM}$. In other words,

$$\forall t \in \mathcal{T}, \quad t \in \mathcal{T}_{FKM} \quad :\Leftrightarrow \quad \begin{cases} \text{(i)} & \forall p \in \mathcal{Q}_t, (p.t) \in \mathcal{A}_t \\ \text{(ii)} & \forall p \in \mathcal{A}_t, (p.t) \sim_{\mathfrak{p}} p \end{cases} \quad (78)$$

Lemma 15. In terms of the action on the space of states, we then have (using **Notion 24**) :

$$\forall t \in \mathcal{T}_{FKM}, \quad \begin{cases} \text{(i)} & \forall \sigma \in [\mathcal{Q}_t], (\sigma.t) \in [\mathcal{A}_t] \\ \text{(ii)} & \forall \sigma \in [\mathcal{A}_t], (\sigma.t) = \sigma \end{cases} \quad (79)$$

■

As a remark, we also note the following trivial lemmas.

Lemma 16.

$$\forall t \in \mathcal{T}_{FKM}, \quad (p \perp .t) \in \mathcal{A}_t \neq \emptyset \quad (80)$$

$$\forall t \in \mathcal{T}_{FKM}, \forall p \in \mathcal{Q}_t, \quad ((p.t).t) \sim_{\mathfrak{p}} (p.t) \quad (81)$$

$$\forall t \in \mathcal{T}_{FKM}, \forall p \in \mathcal{Q}_t, \quad ((p.t) \sim_{\mathfrak{p}} p) \Rightarrow p \in \mathcal{A}_t \quad (82)$$

$$\forall t \in \mathcal{T}_{FKM}, \quad [\mathcal{A}_t] = \text{Im}_{.t}^{\mathfrak{S}} = \{ \sigma \in \mathfrak{S} \mid \sigma = (\sigma.t) \}. \quad (83)$$

■

Notion 26. The subset of testable properties that can be tested through first-kind measurements will be denoted $\mathcal{L}_{FKM}$. The definition of this subset of $\mathcal{L}$ is then summarized as

²¹Note the distinction made by W.Pauli between measurements of the first and second kind : 'On the other hand it can also happen that the system is changed but in a controllable fashion by the measurement — even when, in the state before the measurement, the quantity measured had with certainty a definite value. In this method, the result of a repeated measurement is not the same as that of the first measurement. But still it may be that from the result of this measurement, an un-ambiguous conclusion can be drawn regarding the quantity being measured for the concerned system before the measurement. Such measurements, we call the measurements of the second kind.' [52, p.75] To be concrete: (i) the determination of the position of a particle by a test of the presence of the particle in a given box appears to be a measurement of the first kind, (ii) the determination of the momentum of a particle by the evaluation of the 'impact' of this particle on a given detector appears to be a measurement of the second kind.

follows

$$l \in \mathcal{L}_{FKM} \quad :\Leftrightarrow \quad \exists (\cdot t) \in [\mathfrak{S} \dashrightarrow \mathfrak{S}]^{\wedge} \mid \begin{cases} Dom_{\cdot t}^{\mathfrak{S}} = [\mathfrak{Q}_l] \\ Im_{\cdot t}^{\mathfrak{S}} = [\mathfrak{A}_l] = (\uparrow^{\mathfrak{S}} \Sigma_l) \\ \forall \sigma \in Dom_{\cdot t}^{\mathfrak{S}}, (\sigma \cdot t) \cdot t = \sigma \cdot t \end{cases} \quad (84)$$

In order to pronounce synthetic statements concerning the actuality of a 'collection' of properties for a given state, it appears necessary to clarify how 'successive' measurements of different properties can be used to pronounce the actuality of these properties 'conjointly' for a given sample. Indeed, an inconvenient aspect of quantum measurements emerges when different properties are tested successively on a given sample: it generically happens that the effective measurements associated with different properties 'interfere', prohibiting the actuality of two such 'incompatible' properties to be affirmed simultaneously for a given preparation process (in this context, the definition of first-kind measurements appears to be insufficient for our purpose, as this subset of measurement maps is not closed under the succession of distinct measurement operations).

This limitation of effective measurement leads to no-go theorems in the pathway to the construction of a classical logic for the description of the properties of the system, as noted originally by E. Specker :

'Is it possible to extend the description of a quantum mechanical system through the introduction of supplementary — fictitious — propositions in such a way that in the extended domain the classical propositional logic holds (whereby, of course, for simultaneously decidable proposition negation, conjunction and disjunction must retain their meaning)? The answer to this question is negative, except in the case of Hilbert spaces of dimension 1 and 2.'[65]

The origins of this puzzling fact can be presented in terms of concrete measurements performed on the system. Let us consider any preparation procedure $p \in \mathfrak{P}$ guaranteeing the actuality of a given property $l \in \mathcal{L}$ for its outcoming samples (i.e., $p \in \mathfrak{A}_l$). The actuality of this property may then be checked by applying any yes/no test $t \in \mathfrak{T}$ associated with this property (i.e., $l = [t]$) on these prepared samples before any other experiment as long as this yes/no test leads to a first-kind measurement, i.e., $(p \cdot t) \in \mathfrak{A}_l$. However, there generically exists another property $l' \in \mathcal{L}$ (and a yes/no test $t' \in \mathfrak{T}$ associated with it, i.e., $l' = [t']$) such that: (i) the result of any yes/no test associated with this second property on these prepared samples is fundamentally indeterminate (i.e., $c(p, t') = \perp$), (ii) if we select among the outcomes of these yes/no tests the samples exhibiting this second property (i.e., the samples produced through the preparation procedure $(p \cdot t') \in \mathfrak{A}_{l'}$), then any subsequent measurement relative to the first property on these selected samples will exhibit an indeterminacy (i.e., $c((p \cdot t'), t) = \perp$). In other words, the actuality of the first property can no longer be affirmed (i.e., $(p \cdot t') \notin \mathfrak{A}_l$) after the actuality of the second property has been effectively established by a measurement, even if the actuality of the first property had been established beforehand on these same prepared samples! To summarize, contrary to the context of classical measurements, for $p \in \mathfrak{P}$ and $t \in \mathfrak{T}$, we can not affirm that $p \sqsubseteq_{\mathfrak{P}} (p \cdot t)$.

Despite this severe limitation on the determination process of the actual properties of a quantum system, it is however possible to singularize some measurements, chosen for their ability to 'minimally perturb the system'. C. Piron summarizes the proposal for these ideal measurements as follows :

'In general if we test a property a by performing α , one of the corresponding questions, we disturb completely the given physical system even if a is actually true. We will say that a question α is an ideal measurement if, when we perform it, we can assert that (i) If the answer is "yes", then the corresponding proposition a is true afterwards, and (ii) If the answer is "yes" and if a property b is true and compatible with a , then b is still true afterwards.'[57]

Nevertheless, the definitions of 'compatible properties' and 'ideal first-kind measurements' seem to be trapped in a vicious circle: two properties are *compatible* as soon as they can be 'simultaneously' stated as actual using successive ideal first-kind measurements, and measurements are defined to be ideal as

soon as they respect the actuality of the properties that are compatible with the measured property!^[22] In order to establish a consistent description, it appears necessary to clarify these notions in our vocabulary.

Notion 27. A family of testable properties $\mathfrak{L} = (\iota_i)_{i \in I} \subseteq \mathcal{L}$ will be said to be a *compatible family of properties* (this fact will be denoted $\widehat{\mathfrak{L}}$), iff there exists at least one preparation process $p \in \mathfrak{P}$ producing collections of samples exhibiting all of these properties as 'actual' (the statements about the occurrence of the properties $(\iota_i)_{i \in I}$ for the samples prepared through p will all be simultaneously 'positive with certainty'). In other words,

$$\widehat{\mathfrak{L}} \quad :\Leftrightarrow \quad \bigcap_{i \in I} \mathfrak{A}_{\iota_i} \neq \emptyset \quad (85)$$

In particular, a property ι_1 is said to be compatible with a property ι_2 iff $\widehat{\iota_1 \iota_2}$. This defines a binary relation called the *compatibility relation* on $\mathcal{L}$.

The compatibility relation is a reflexive and symmetric relation. Moreover, $t_1 \sqsubseteq_{\mathfrak{P}} t_2$ implies $\widehat{[t_1][t_2]}$.

The *ideal measurements* will be characterized as follows.

Notion 28. A yes/no test $t_1 \in \mathfrak{T}$ is said to *lead to an ideal measurement* of the corresponding testable property $[t_1]$ (this fact is denoted $t_1 \in \mathfrak{T}_{ideal}$) iff, for any property $[t_2]$ compatible with $[t_1]$, the statement pronounced by the observer beforehand concerning the 'actuality' of the property $[t_2]$ is conserved after the measurement operation associated with t_1 has been realized, i.e.,

$$\forall t_1 \in \mathfrak{T}, \quad t_1 \in \mathfrak{T}_{ideal} \quad :\Leftrightarrow \quad (\forall t_2 \in \mathfrak{T} \mid \widehat{[t_1][t_2]}, \forall p \in \Omega_{t_1}, \epsilon(p, t_2) = Y \Rightarrow \epsilon(p \cdot t_1, t_2) = Y) \quad (86)$$

$$\Leftrightarrow \quad (\forall t_2 \in \mathfrak{T}, \quad \widehat{[t_1][t_2]} \Rightarrow (\Omega_{t_1} \cap \mathfrak{A}_{t_2}) \cdot t_1 \subseteq \mathfrak{A}_{t_2}) \quad (87)$$

We will adopt the following notation

$$\mathfrak{T}_{IFKM} := \mathfrak{T}_{ideal} \cap \mathfrak{T}_{FKM}. \quad (88)$$

Remark 9. When an ideal measurement operation associated with a yes/no test t_1 is performed on a given collection of samples, similarly prepared in such a way that property $[t_2]$ was actual before this measurement operation, the outcoming samples are such that the observer cannot distinguish if property $[t_2]$ was tested (or not) after this measurement. In effect,

$$\forall t_1, t_2 \in \mathfrak{T}_{IFKM} \mid \widehat{[t_1][t_2]} \quad \forall p \in \mathfrak{A}_{t_2} \cap \Omega_{t_1}, \quad p \cdot t_1 \cdot t_2 \sim_{\mathfrak{P}} p \cdot t_1. \quad (89)$$

Notion 29. The subset of testable properties that can be tested through ideal first-kind measurements will be denoted $\mathcal{L}_{IFKM}$. The definition of this subset of $\mathcal{L}$ is then summarized as

²²A basic solution to this problem has been formalized by C. Piron [57]. This construction relies on an orthomodular lattice structure introduced on the space of properties (Note: we do not expect any such a construction in our perspective). In Piron's vocabulary,

two properties are compatible as soon as they form a boolean sub-algebra in the orthomodular lattice of properties (this requirement about the sub-boolean structure is a remnant of the particular structure on the space of properties in the classical situation).[54, p.295]

follows

$$l \in \mathcal{L}_{IFKM} \quad :\Leftrightarrow \quad \exists (\cdot, t) \in [\mathfrak{S} \dashrightarrow \mathfrak{S}]^{\bar{\wedge}} \mid \begin{cases} \text{Dom}_{\cdot, t}^{\mathfrak{S}} = [\mathfrak{Q}_l] \\ \text{Im}_{\cdot, t}^{\mathfrak{S}} = [\mathfrak{A}_l] = (\uparrow^{\mathfrak{S}} \Sigma_l) \\ \forall \sigma \in \text{Dom}_{\cdot, t}^{\mathfrak{S}}, (\sigma \cdot t) \cdot t = \sigma \cdot t \\ \forall \sigma \in \mathfrak{S} \mid \overline{\sigma \Sigma_l^{\mathfrak{S}}}, (\text{Dom}_{\cdot, t}^{\mathfrak{S}} \cap (\uparrow^{\mathfrak{S}} \sigma)) \cdot t \subseteq ((\uparrow^{\mathfrak{S}} \sigma) \cap [\mathfrak{A}_l]) \end{cases} \quad (90)$$

The motivation to introduce 'ideal first-kind measurements' is very clear from an operational point of view.^[23] If such measurement operations exist for a basic set of compatible properties, they can be used to build preparation processes designed to produce collections of samples for which these properties will be found to be 'conjointly actual'. It is also completely clear that ideal first-kind measurements exist in concrete quantum mechanical experiments [58] and it is then natural to impose their existence at the center of a quantum axiomatics. Nevertheless, nothing guaranties that such a measurement operation exists for any given property l (or, equivalently, that a measurement map satisfying property (90) exists for the property l). Our aim, by the end of the present subsection, will be to prove that an ideal first-kind measurement map exists if the corresponding 'property' satisfies a quasi-classicality criterion.

Lemma 17. For any $t_1, t_2 \in \mathfrak{T}_{IFKM}$ two yes/no tests leading to ideal first-kind measurements such that t_1 is compatible with t_2 , the yes/no test $(t_1 \cdot t_2)$ leads to ideal first-kind measurements associated with the conjunction of the properties $[t_1]$ and $[t_2]$. In other words,

$$\forall t_1, t_2 \in \mathfrak{T}_{IFKM} \mid \overbrace{[t_1][t_2]}, \quad (t_1 \cdot t_2) \in \mathfrak{T}_{IFKM} \quad (91)$$

$$\mathfrak{A}_{t_1 \cdot t_2} = (\mathfrak{A}_{t_1} \cap \mathfrak{A}_{t_2}), \quad (92)$$

$$\text{Dom}_{\cdot, (t_1 \cdot t_2)}^{\mathfrak{S}} = \mathfrak{Q}_{t_1 \cdot t_2} = \mathfrak{Q}_{t_1} \cap (\cdot t_1)^{-1}(\mathfrak{Q}_{t_2} \cap \mathfrak{A}_{t_1}), \quad (93)$$

$$\mathfrak{A}_{\overline{t_1 \cdot t_2}} = \mathfrak{A}_{\overline{t_1}} \cup (\cdot t_1)^{-1}(\mathfrak{A}_{\overline{t_2}} \cap \mathfrak{A}_{t_1}) = \mathfrak{S} \setminus \mathfrak{Q}_{t_1 \cdot t_2} \quad (94)$$

■

Proof. Using equations (64) and (78 (ii)), we deduce (92). Let us now prove that $(t_1 \cdot t_2) \in \mathfrak{T}_{IFKM}$. For any $p \in \mathfrak{Q}_{(t_1 \cdot t_2)}$, we know from equation (65) that $(p \cdot t_1) \in \mathfrak{A}_{t_1} \cap \mathfrak{Q}_{t_2}$. Using equations (78 (i)) and (87), we then deduce that $((p \cdot t_1) \cdot t_2) \in \mathfrak{A}_{t_1} \cap \mathfrak{A}_{t_2} = \mathfrak{A}_{(t_1 \cdot t_2)}$ (using (92)), i.e., equation (78 (i)) applied to the yes/no test $(t_1 \cdot t_2)$. For any $p \in \mathfrak{A}_{(t_1 \cdot t_2)} = (\mathfrak{A}_{t_1} \cap \mathfrak{A}_{t_2})$ (using (92)), we deduce from equations (78 (ii)) and (67) that $(p \cdot (t_1 \cdot t_2)) = ((p \cdot t_1) \cdot t_2) \sim_{\mathfrak{P}} (p \cdot t_2) \sim_{\mathfrak{P}} p$, i.e., equation (78 (ii)) applied to the yes/no test $(t_1 \cdot t_2)$. Let us now consider a yes/no test $t \in \mathfrak{T}$ compatible with the yes/no test $(t_1 \cdot t_2)$. In other words, we have $\mathfrak{A}_t \cap \mathfrak{A}_{(t_1 \cdot t_2)} \neq \emptyset$. As a result, we have then $\mathfrak{A}_t \cap \mathfrak{A}_{t_1} \neq \emptyset$ and $\mathfrak{A}_t \cap \mathfrak{A}_{t_2} \neq \emptyset$, i.e., $\overbrace{[t][t_1]}$ and $\overbrace{[t][t_2]}$.

Using successively (67), (65), and the compatibility relations $\overbrace{[t][t_1]}$ and $\overbrace{[t][t_2]}$ coupled with property (87), we obtain $(\mathfrak{Q}_{(t_1 \cdot t_2)} \cap \mathfrak{A}_t) \cdot (t_1 \cdot t_2) \subseteq ((\mathfrak{Q}_{t_2} \cap \mathfrak{A}_{t_1}) \cdot t_2) \subseteq \mathfrak{A}_t$, i.e. property (87) for the compatibility property $\overbrace{[t][t_1][t_2]}$.

Properties (93) and (94) are explicit rewriting of properties (65) and (66). □

Notion 30. A preparation process $p \in \mathfrak{P}$ is said to be *consistent with the actuality of a given testable property* $l \in \mathcal{L}$ iff there exists a preparation process $p' \in \mathfrak{P}$, sharper than p , and for which the property l is actual. We denote by $\mathfrak{K}_l$ the set of preparation processes consistent with the actuality of the testable property l , i.e.,

$$\forall l \in \mathcal{L}, \quad \mathfrak{K}_l := \{p \in \mathfrak{P} \mid \exists p' \in \mathfrak{A}_l, p \sqsubseteq_{\mathfrak{P}} p'\}. \quad (95)$$

²³These measurements played a fundamental role in Mackey's traditional axiomatic approach to quantum theory [18].

$\lceil \mathfrak{K}_l \rceil$ will be called *the consistency domain of the property l*. We have

$$\forall l \in \mathcal{L}, \quad \lceil \mathfrak{K}_l \rceil = \downarrow_{\mathfrak{S}} \lceil \mathfrak{A}_l \rceil = \{ \sigma' \in \mathfrak{S} \mid \widehat{\Sigma_l \sigma'}^{\mathfrak{S}} \}. \quad (96)$$

Lemma 18. For any testable property $l \in \mathcal{L}$, the consistency domain $\lceil \mathfrak{K}_l \rceil$ is a Scott-closed subset of $\lceil \mathfrak{Q}_l \rceil$. $\blacksquare$

Proof. First, for any $\sigma' \in \lceil \mathfrak{K}_l \rceil$ the property $\widehat{\Sigma_l \sigma'}^{\mathfrak{S}}$ implies $\exists \sigma'' \in \lceil \mathfrak{A}_l \rceil$ with $\sigma' \sqsubseteq_{\mathfrak{S}} \sigma''$, and therefore $\tilde{\mathfrak{e}}_l(\sigma') \leq \tilde{\mathfrak{e}}_l(\sigma'') = \mathbf{Y}$ for any t such that $l = \lfloor t \rfloor$. As a consequence, $\lceil \mathfrak{K}_l \rceil \subseteq \lceil \mathfrak{Q}_l \rceil$. Moreover, $\lceil \mathfrak{K}_l \rceil$ is obviously a downset. Lastly, let us consider $\mathfrak{C} \subseteq_{Chain} \mathfrak{S}$ such that $(\forall c \in \mathfrak{C}, \widehat{\Sigma_l c}^{\mathfrak{S}})$. For any $c \in \mathfrak{C}$, we can define the element $(\Sigma_l \sqcup_{\mathfrak{S}} c)$. The chain $\mathfrak{C}' := \{ \Sigma_l \sqcup_{\mathfrak{S}} c \mid c \in \mathfrak{C} \}$ admits a supremum in $\mathfrak{S}$ satisfying (i) $\forall c' \in \mathfrak{C}', \Sigma_l \sqsubseteq_{\mathfrak{S}} c'$ and thus $\Sigma_l \sqsubseteq_{\mathfrak{S}} \sqcup_{\mathfrak{S}} \mathfrak{C}'$, and (ii) $\forall c \in \mathfrak{C}, \exists c' \in \mathfrak{C}' \mid c \sqsubseteq_{\mathfrak{S}} c'$ and thus $c \sqsubseteq_{\mathfrak{S}} \sqcup_{\mathfrak{S}} \mathfrak{C}'$ and therefore $\sqcup_{\mathfrak{S}} \mathfrak{C} \sqsubseteq_{\mathfrak{S}} \sqcup_{\mathfrak{S}} \mathfrak{C}'$. As a result, we have obtained $(\widehat{\Sigma_l \sqcup_{\mathfrak{S}} \mathfrak{C}'}^{\mathfrak{S}})$ and thus $(\sqcup_{\mathfrak{S}} \mathfrak{C}) \in \lceil \mathfrak{K}_l \rceil$. This chain-completeness property implies the directed completeness of $\lceil \mathfrak{K}_l \rceil$. $\square$

Lemma 19. For any testable property $l \in \mathcal{L}$, the map

$$\begin{aligned} \theta_l : \lceil \mathfrak{K}_l \rceil &\rightarrow \lceil \mathfrak{A}_l \rceil \\ \sigma &\mapsto \theta_l(\sigma) := \Sigma_l \sqcup_{\mathfrak{S}} \sigma. \end{aligned} \quad (97)$$

is an idempotent, order-preserving, Scott-continuous partial map. It preserves also filtered infima and existing suprema. $\blacksquare$

Proof. Firstly, from the basic properties of $\sqcup_{\mathfrak{S}}$, we know that θ_l is idempotent and order preserving. Secondly, if we denote by ι the inclusion of $\lceil \mathfrak{A}_l \rceil$ in $\lceil \mathfrak{K}_l \rceil$, we have $id \sqsubseteq \iota \circ \theta_l$ and $\theta_l \circ \iota = id$. As a result, the right adjoint θ_l of this Galois connection preserves existing suprema. In particular, θ_l is Scott continuous.

Thirdly, from the join-continuity property satisfied in $\mathfrak{S}$ (Conjecture 1), we deduce that θ_l preserves filtered infima.

The idempotency is a direct consequence of the properties of $\sqcup_{\mathfrak{S}}$. $\square$

Notion 31. A yes/no test $t \in \mathfrak{T}$ is said to *lead to a minimally disturbing measurement* of the corresponding testable property $\lfloor t \rfloor$ (this fact is denoted $t \in \mathfrak{T}_{min}$) iff (i) this measurement is first-kind, and (ii) for any preparation p 'consistent with the actuality of $\lfloor t \rfloor$ ', the observer is able to pronounce statements about the measured state $\lceil p \cdot t \rceil$ that are the minimal statements simultaneously finer than the statements pronounced separately about $\lceil p \rceil$ and about $\Sigma_{\lfloor t \rfloor}$ before the measurement. In other words,

$$\forall t \in \mathfrak{T}, \quad t \in \mathfrak{T}_{min} : \Leftrightarrow \begin{cases} Dom_{\cdot t}^{\mathfrak{P}} = \mathfrak{Q}_{\lfloor t \rfloor} \\ t \in \mathfrak{T}_{FKM} \\ \forall p \in \mathfrak{K}_{\lfloor t \rfloor}, \lceil p \cdot t \rceil = \lceil p \rceil \sqcup_{\mathfrak{S}} \Sigma_{\lfloor t \rfloor}. \end{cases} \quad (98)$$

Notion 32. The subset of properties that can be tested through minimally disturbing measurements will be denoted $\mathcal{L}_{min}$. In other words,

$$\forall l \in \mathcal{L}, \quad l \in \mathcal{L}_{min} : \Leftrightarrow \exists (\cdot t) \in [\mathfrak{S} \dashrightarrow \mathfrak{S}]^{\bar{\cdot}} \mid \begin{cases} Dom_{\cdot t}^{\mathfrak{S}} = \lceil \mathfrak{Q}_l \rceil \\ Im_{\cdot t}^{\mathfrak{S}} = \lceil \mathfrak{A}_l \rceil = (\uparrow^{\mathfrak{S}} \Sigma_l) \\ \forall \sigma \in Dom_{\cdot t}^{\mathfrak{S}}, (\sigma \cdot t) \cdot t = \sigma \cdot t \\ \forall \sigma \in \lceil \mathfrak{K}_l \rceil, \sigma \cdot t = \sigma \sqcup_{\mathfrak{S}} \Sigma_l. \end{cases} \quad (99)$$

Remark 10. We note that $\sigma \in [\mathfrak{K}_t]$ means $\widehat{\sigma \sqcup_{\mathfrak{S}} \Sigma_t}^{\mathfrak{S}}$ and then the supremum $(\sigma \sqcup_{\mathfrak{S}} \Sigma_t)$ exists, due to the consistent completeness of $\mathfrak{S}$.

Remark 11. Note the following implicit property of minimally disturbing measurement maps :

$$t \in \mathfrak{T}_{min} \Rightarrow Im_{\cdot, t}^{\mathfrak{P}} = \mathfrak{A}_{[t]} \quad (100)$$

Let us distinguish two sub-cases.

As a first sub-case, let us suppose that $p \in (\downarrow_{\mathfrak{P}} \mathfrak{A}_t)$. Applying property (98), we deduce immediately that $[p \cdot t] \sqsubseteq_{\mathfrak{S}} \Sigma_t$, i.e. $[p \cdot t] \in (\uparrow^{\mathfrak{S}} \Sigma_t) = [\mathfrak{A}_{[t]}]$, and therefore $(p \cdot t) \in \mathfrak{A}_t$.

As a second sub-case, we now suppose $p \in \Omega_t \setminus (\downarrow_{\mathfrak{P}} \mathfrak{A}_t)$. Let us denote by p'' an element of $(\downarrow_{\mathfrak{P}} p) \cap \mathfrak{K}_{[t]}$ which is non-empty (it contains $\perp_{\mathfrak{P}}$). Firstly, we have $p'' \sqsubseteq_{\mathfrak{P}} p$. We note in particular that p'' is then an element of Ω_t , because Ω_t is a downset. Moreover, the monotonicity of the measurement map associated with t implies $(p'' \cdot t) \sqsubseteq_{\mathfrak{P}} (p \cdot t)$. Secondly, we have $p'' \in (\downarrow_{\mathfrak{P}} \mathfrak{A}_t) = \mathfrak{K}_t$, and thus $(p'' \cdot t) \in \mathfrak{A}_t$ as proved in the first sub-case. We now use the fact that $\mathfrak{A}_t$ is an upper-set to conclude that $(p \cdot t) \in \mathfrak{A}_t$.

Lemma 20. We will assume the following preservation property

$$\forall \mathfrak{F} \subseteq_{Fil} [\Omega_t], \quad (\bigsqcap_{\mathfrak{S}}^{\vee} \mathfrak{F}) \cdot t = \bigsqcap_{\sigma \in \mathfrak{F}}^{\vee} (\sigma \cdot t). \quad (101)$$

We then have

$$t \in \mathfrak{T}_{IFKM} \Rightarrow t \in \mathfrak{T}_{min}. \quad (102)$$

■

Proof. Let us consider $t \in \mathfrak{T}_{IFKM}$ and $\sigma \in [\mathfrak{K}_{[t]}]$. Using the fact that the measurement map $(\cdot, t)$ preserves filtered-infima, we deduce

$$\begin{aligned} (\sigma \cdot t) &= (\bigsqcap_{\mathfrak{S}} (\uparrow^{\mathfrak{S}} \sigma)) \cdot t \\ &= \bigsqcap_{\mathfrak{S}} ((Dom_{\cdot, t}^{\mathfrak{S}} \cap (\uparrow^{\mathfrak{S}} \sigma)) \cdot t) \\ &\sqsubseteq_{\mathfrak{S}} \bigsqcap_{\mathfrak{S}} ((\uparrow^{\mathfrak{S}} \sigma) \cap [\mathfrak{A}_{[t]}]) = (\sigma \sqcup_{\mathfrak{S}} \Sigma_{[t]}) \end{aligned}$$

Using $\sigma \sqsubseteq_{\mathfrak{S}} (\sigma \sqcup_{\mathfrak{S}} \Sigma_{[t]})$ and the monotonicity of the measurement map, along with property (78 (ii)), we also obtain, for any $\sigma \in [\mathfrak{K}_{[t]}]$, the property $\sigma \cdot t \sqsubseteq_{\mathfrak{S}} (\sigma \sqcup_{\mathfrak{S}} \Sigma_{[t]}) \cdot t = \sigma \sqcup_{\mathfrak{S}} \Sigma_{[t]}$. As a result,

$$\forall \sigma \in [\mathfrak{K}_{[t]}], \quad (\sigma \cdot t) = (\sigma \sqcup_{\mathfrak{S}} \Sigma_{[t]}). \quad (103)$$

□

Lemma 21. Let l be a testable property and let t be a yes/no test leading to a minimally disturbing measurement of the property $l = [t]$ (i.e., $l = [t]$ and $t \in \mathfrak{T}_{min}$), then necessarily t leads to ideal first-kind measurements of the property l . As a conclusion,

$$t \in \mathfrak{T}_{min} \Rightarrow t \in \mathfrak{T}_{IFKM}. \quad (104)$$

■

Proof. First, t being in $\mathfrak{T}_{min}$, we know that $[t]$ is in $\mathcal{L}_{min}$ and therefore in $\mathcal{L}_{q-cl}$ using Lemma 25. We now intend to prove that the yes/no tests t satisfying the properties given in (98) also satisfy the properties (78 (i)), (78 (ii)) and (87), as long as the property $[t]$ is quasi-classical.

We have proved property (78 (i)) in Remark 11.

Using property (98) for the particular case $p \in \mathfrak{A}_t$, we obtain $[p \cdot t] = [p]$, because by definition $\Sigma_t =$

$\sqcap_{\mathfrak{S}} \mathfrak{A}_t$. We have then proved property (78 (ii)).

Let us now consider a second yes/no test $t' \in \mathfrak{T}$ such that $\overbrace{[t][t']}$, i.e., $\mathfrak{A}_t \cap \mathfrak{A}_{t'} \neq \emptyset$. We note in particular that this compatibility property implies $\tilde{e}_{t'}(\Sigma_t) \leq Y$. Let us also consider $p \in \mathfrak{Q}_t$ such that $e(p, t') = Y$, i.e., $p \in \mathfrak{A}_{t'}$. We will distinguish two sub-cases as before.

As a first sub-case, let us suppose that $p \in (\downarrow_{\mathfrak{P}} \mathfrak{A}_t)$. Applying property (98), we have $e(p \cdot t, t') \geq e(p, t') = Y$ using the hypotheses.

As a second sub-case, we now suppose $p \in \mathfrak{Q}_t \setminus (\downarrow_{\mathfrak{P}} \mathfrak{A}_t)$. Let us consider once again any preparation process $p'' \in \mathfrak{P}$ whose state corresponds to the supremum $\sqcup_{\mathfrak{S}} ((\downarrow_{\mathfrak{S}} [p]) \cap (\downarrow_{\mathfrak{S}} [\mathfrak{A}_t]))$. We conclude as before that $p'' \in \mathfrak{Q}_t$ and $(p'' \cdot t) \sqsubseteq_{\mathfrak{P}} (p \cdot t)$. In particular $e(p'' \cdot t, t') \leq e(p \cdot t, t')$.

We know also that $p'' \in (\downarrow_{\mathfrak{P}} \mathfrak{A}_t)$ and therefore, using property (98), $e(p'' \cdot t, t') \geq e(p'', t')$.

Lastly, we know that $\Sigma_{t'} \sqsubseteq_{\mathfrak{S}} [p''] = \sqcup_{\mathfrak{S}} ((\downarrow_{\mathfrak{S}} [p]) \cap (\downarrow_{\mathfrak{S}} [\mathfrak{A}_t]))$ because (i) $p \in \mathfrak{A}_{t'}$ implies $\Sigma_{t'} \sqsubseteq_{\mathfrak{S}} [p]$, which implies $\Sigma_{t'} \in (\downarrow_{\mathfrak{S}} [p])$, and (ii) $\mathfrak{A}_t \cap \mathfrak{A}_{t'} \neq \emptyset$ implies $\Sigma_{t'} \in (\downarrow_{\mathfrak{S}} [\mathfrak{A}_t])$. But $\Sigma_{t'} \sqsubseteq_{\mathfrak{S}} [p'']$ is equivalently rewritten as $e(p'', t') = Y$. We therefore deduce that $e(p'' \cdot t, t') = Y$.

Using the two intermediary results, we conclude this analysis of the second sub-case by $e(p \cdot t, t') = Y$.

As a conclusion, we have for any $p \in \mathfrak{Q}_t$, $e(p \cdot t, t') = Y$. We have thus demonstrated property (87). $\square$

Notion 33. A testable property $l \in \mathcal{L}$ is said to be *quasi-classical* (this fact is denoted $l \in \mathcal{L}_{q-cl}$) iff the consistency-domain $[\mathfrak{K}_l]$ is a continuous retract of the domain $[\mathfrak{Q}_l]$, i.e.,

$$\forall l \in \mathcal{L}, \quad l \in \mathcal{L}_{q-cl} \quad :\Leftrightarrow \quad \exists \pi_l : [\mathfrak{Q}_l] \longrightarrow [\mathfrak{K}_l] \quad | \quad \pi_l \text{ is Scott continuous} \quad (105)$$

$$\forall \sigma \in [\mathfrak{K}_l], \quad \pi_l(\sigma) = \sigma \quad (106)$$

$$\forall \sigma' \in [\mathfrak{Q}_l], \quad \pi_l(\sigma') \sqsubseteq_{\mathfrak{S}} \sigma'. \quad (107)$$

Let $\mathfrak{S}'$ be a Scott-closed subset of $\mathfrak{S}$. $\mathfrak{S}'$ is said to be a *Scott ideal* of $\mathfrak{S}$ iff

$$\forall S \subseteq_{fin} \mathfrak{S}', \quad \hat{S}^{\mathfrak{S}} \Rightarrow \hat{S}^{\mathfrak{S}'}. \quad (108)$$

Lemma 22. A Scott-closed subset $\mathfrak{S}'$ of a domain $\mathfrak{S}$ is a Scott-ideal iff it satisfies the property:

$$\forall s \in \mathfrak{S}, \quad \sqcup_{\mathfrak{S}} ((\downarrow_{\mathfrak{S}} s) \cap_{\mathfrak{S}} \mathfrak{S}') \in \mathfrak{S}'. \quad (109)$$

■

Proof. Let us consider $S \subseteq_{fin} \mathfrak{S}'$ such that it exists $s \in \mathfrak{S}$ which is a common upper-bound of elements of S . We have then $S \subseteq ((\downarrow_{\mathfrak{S}} s) \cap_{\mathfrak{S}} \mathfrak{S}')$. Using property (109), the supremum $\sqcup_{\mathfrak{S}} ((\downarrow_{\mathfrak{S}} s) \cap_{\mathfrak{S}} \mathfrak{S}')$ exists as an element of $\mathfrak{S}'$. It is then a common upper-bound of elements of S that belongs to $\mathfrak{S}'$. Hence, property (109) implies that $\mathfrak{S}'$ is a Scott-ideal of $\mathfrak{S}$.

Conversely, let us suppose $\mathfrak{S}'$ is a Scott-ideal of $\mathfrak{S}$ and let us fix any $s \in \mathfrak{S}$. The set $((\downarrow_{\mathfrak{S}} s) \cap_{\mathfrak{S}} \mathfrak{S}')$ is upper-bounded by s . Using property (108), for any $F \subseteq_{fin} ((\downarrow_{\mathfrak{S}} s) \cap_{\mathfrak{S}} \mathfrak{S}')$, it exists $M_F \in \mathfrak{S}'$ upper-bound of F . Hence, $\sqcup_{\mathfrak{S}'} F$ exists and is in $\mathfrak{S}'$, and $\sqcup_{\mathfrak{S}} F = \sqcup_{\mathfrak{S}'} F$. As an example, for $F \subseteq_{fin} \mathbb{A}(s) \cap_{\mathfrak{S}} \mathfrak{S}'$ We can build the following subset: $\{\sqcup_{\mathfrak{S}} F \mid F \subseteq_{fin} \mathbb{A}(s) \cap_{\mathfrak{S}} \mathfrak{S}'\} \subseteq ((\downarrow_{\mathfrak{S}} s) \cap_{\mathfrak{S}} \mathfrak{S}')$. It is a directed subset, by construction. It then admits a directed-supremum in $\mathfrak{S}$ which is in $\mathfrak{S}'$, because $\mathfrak{S}'$ is Scott-closed : $\sqcup_{\mathfrak{S}}^{\wedge} \{\sqcup_{\mathfrak{S}} F \mid F \subseteq_{fin} \mathbb{A}(s) \cap_{\mathfrak{S}} \mathfrak{S}'\} \in \mathfrak{S}'$. Due to algebraicity of $\mathfrak{S}$ (Theorem 11), we then obtain $(\sqcup_{\mathfrak{S}} \{t \in \mathfrak{S}' \mid t \sqsubseteq_{\mathfrak{S}} s\}) \in \mathfrak{S}'$. This achieves the proof of property (109). $\square$

Lemma 23. Let l be a quasi-classical property. Then, $\mathfrak{K}_l$ is a Scott-ideal of $[\mathfrak{Q}_l]$. Conversely, if $[\mathfrak{K}_l]$ is a Scott-ideal of $[\mathfrak{Q}_l]$, then l is a quasi-classical property. Moreover, the retraction π_l is given by

$$\forall \sigma \in [\mathfrak{Q}_l], \quad \pi_l(\sigma) = \sqcup_{\mathfrak{S}} ([\mathfrak{K}_l] \cap (\downarrow_{\mathfrak{S}} \sigma)). \quad (110)$$

The retraction π_l is an idempotent, order-preserving and Scott-continuous map, which preserves infima. $\blacksquare$

Proof. Let us consider $\sigma_1, \sigma_2 \in [\mathcal{R}_l]$ and let us suppose that $\exists \sigma \in [\mathcal{Q}_l] \mid \sigma_1, \sigma_2 \sqsubseteq_{\mathcal{E}} \sigma$. The monotonicity of π_l implies $\pi_l(\sigma_1) \sqsubseteq_{\mathcal{E}} \pi_l(\sigma)$ and $\pi_l(\sigma_2) \sqsubseteq_{\mathcal{E}} \pi_l(\sigma)$. Secondly, property (106) implies $\sigma_1 = \pi_l(\sigma_1)$ and $\sigma_2 = \pi_l(\sigma_2)$. Thirdly, property (107) implies $\pi_l(\sigma) \sqsubseteq_{\mathcal{E}} \sigma$. As a conclusion, $\exists \sigma' = \pi_l(\sigma) \in [\mathcal{R}_l] \mid \sigma_1, \sigma_2 \sqsubseteq_{\mathcal{E}} \sigma'$. Furthermore, from Lemma 18, we know already that $[\mathcal{R}_l]$ is Scott-closed. As a result, we then conclude that $[\mathcal{R}_l]$ is a Scott ideal in $[\mathcal{Q}_l]$.

Conversely, if $[\mathcal{R}_l]$ is a Scott ideal in $[\mathcal{Q}_l]$, we can use Lemma 22 to conclude that

$$\forall \sigma \in [\mathcal{Q}_l], \pi(\sigma) := \bigsqcup_{\mathcal{E}} ([\mathcal{R}_l] \cap (\downarrow_{\mathcal{E}} \sigma)) \in [\mathcal{R}_l]. \quad (111)$$

π is a map defined from $[\mathcal{Q}_l]$ to $[\mathcal{R}_l]$, which trivially satisfies properties (105), (106) and (107). l is then a quasi-classical property and the expression for the retraction π_l is given by (110).

π_l is also the Galois right adjoint of the inclusion map from $[\mathcal{R}_l]$ to $[\mathcal{Q}_l]$. As an immediate consequence, π_l is a surjective, order-preserving map, which preserves infima. $\square$

Lemma 24.

$$\mathcal{L}_{q-cl} \subseteq \mathcal{L}_{min} \quad (112)$$

More precisely, if $l \in \mathcal{L}_{q-cl}$, the map $\bullet\Theta_l$ defined by

$$\begin{array}{ccc} \bullet\Theta_l & : & \mathcal{E} \dashrightarrow \mathcal{E} \\ [\mathcal{Q}_l] \ni \sigma & \mapsto & \sigma \bullet\Theta_l := \Sigma_l \sqcup_{\mathcal{E}} \pi_l(\sigma). \end{array} \quad (113)$$

is an idempotent, order-preserving and Scott-continuous partial map from $[\mathcal{Q}_l]$ to $[\mathcal{R}_l]$, which preserves filtered-infima, and satisfies $\forall \sigma \in [\mathcal{R}_l], \sigma \bullet\Theta_l = \sigma \sqcup_{\mathcal{E}} \Sigma_l$. $\blacksquare$

Proof. Let us consider $l \in \mathcal{L}_{q-cl}$. π_l is an idempotent, order-preserving and Scott-continuous partial map from $[\mathcal{Q}_l]$ to $[\mathcal{R}_l]$, which preserves arbitrary infima, as shown in Lemma 23.

As proved in Lemma 19, the map

$$\begin{array}{ccc} [\mathcal{R}_l] & \longrightarrow & [\mathcal{Q}_l] \\ \sigma & \mapsto & \Sigma_l \sqcup_{\mathcal{E}} \sigma \end{array} \quad (114)$$

is an idempotent, order-preserving and Scott-continuous partial map, which preserves filtered-infima and existing suprema. As a result, $\bullet\Theta_l$ is an order-preserving, Scott-continuous partial map, which preserves filtered-infima.

The idempotency of $\bullet\Theta_l$ is verified as follows. Firstly, $\pi_l(\Sigma_l \sqcup_{\mathcal{E}} \pi_l(\sigma)) = \Sigma_l \sqcup_{\mathcal{E}} \pi_l(\sigma)$ because $\Sigma_l \sqcup_{\mathcal{E}} \pi_l(\sigma) \in [\mathcal{R}_l]$.

Endly, we have $\Sigma_l \sqcup_{\mathcal{E}} \Sigma_l = \Sigma_l$.

Moreover, for any $\sigma \in [\mathcal{R}_l] = (\downarrow_{\mathcal{E}} [\mathcal{Q}_l])$, we have $\pi_l(\sigma) = \sigma$. Hence, we obtain $\forall \sigma \in (\downarrow_{\mathcal{E}} [\mathcal{Q}_l]), \sigma \bullet\Theta_l = \sigma \sqcup_{\mathcal{E}} \Sigma_l$.

As a result, $\bullet\Theta_l$ satisfies all properties mentioned in (99), i.e., the properties required to conclude that $l \in \mathcal{L}_{min}$. As a conclusion, $\mathcal{L}_{q-cl} \subseteq \mathcal{L}_{min}$. $\square$

Lemma 25.

$$\mathcal{L}_{q-cl} \supseteq \mathcal{L}_{min} \quad (115)$$

More precisely, if $l \in \mathcal{L}_{min}$ and $\bullet t$ is a measurement map defined to satisfy the minimality require-

ment (99), then the partial map ρ_t defined on $[\mathfrak{Q}_l]$ by

$$\begin{array}{ccc} \rho_t & : & \mathfrak{S} \dashrightarrow \mathfrak{S} \\ [\mathfrak{Q}_l] \ni \sigma & \mapsto & \rho_t(\sigma) := \sigma \sqcap_{\mathfrak{S}} (\sigma \cdot t). \end{array} \quad (116)$$

is a Scott-continuous retraction from $[\mathfrak{Q}_l]$ to $[\mathfrak{R}_l]$. $\blacksquare$

Proof. We first note the obvious property $\forall \sigma' \in [\mathfrak{Q}_l], \rho_t(\sigma') \sqsubseteq_{\mathfrak{S}} \sigma'$.

Secondly, we note that, due to the monotonicity requirement on the measurement map $(\cdot t)$, $(\sigma \cdot t)$ is in $[\mathfrak{Q}_l]$. We then deduce that the range of ρ_t is included in $(\downarrow_{\mathfrak{S}} [\mathfrak{Q}_l])$, i.e., included in $[\mathfrak{R}_l]$.

Thirdly, due to property (99) being satisfied by $\cdot t$, we know that $\forall \sigma \in [\mathfrak{R}_l], \rho_t(\sigma) = \sigma \sqcap_{\mathfrak{S}} (\sigma \sqcup_{\mathfrak{S}} \Sigma_l) = \sigma$. The Scott continuity of ρ_t is derived from: the Scott continuity of the measurement map $(\cdot t)$, the meet-continuity property in $\mathfrak{S}$ and the property given in [5, Proposition 2.1.12]:

$$\begin{aligned} \forall \mathfrak{C} \subseteq_{Chain} [\mathfrak{Q}_l], \quad \rho_t(\bigsqcup_{\mathfrak{S}} \mathfrak{C}) &= (\bigsqcup_{\mathfrak{S}} \mathfrak{C}) \sqcap_{\mathfrak{S}} ((\bigsqcup_{\mathfrak{S}} \mathfrak{C}) \cdot t) \\ &= (\bigsqcup_{\mathfrak{C} \in \mathfrak{C}} \mathfrak{C}) \sqcap_{\mathfrak{S}} (\bigsqcup_{\mathfrak{C}' \in \mathfrak{C}} (\mathfrak{C}' \cdot t)) \\ &= \bigsqcup_{\mathfrak{C} \in \mathfrak{C}} \bigsqcup_{\mathfrak{C}' \in \mathfrak{C}} (\mathfrak{C} \sqcap_{\mathfrak{S}} (\mathfrak{C}' \cdot t)) \\ &= \bigsqcup_{\mathfrak{C} \in \mathfrak{C}} (\mathfrak{C} \sqcap_{\mathfrak{S}} (\mathfrak{C} \cdot t)) = \bigsqcup_{\mathfrak{C} \in \mathfrak{C}} \rho_t(\mathfrak{C}). \end{aligned}$$

Finally, chain-continuity is equivalent to Scott continuity.

We have then checked properties (105), (106) and (107) for ρ_t . As a result, $l \in \mathcal{L}_{q-cl}$ and thus $\mathcal{L}_{q-cl} \supseteq \mathcal{L}_{min}$. $\square$

Theorem 9. [Characterization of minimally disturbing measurements]

A property l can be measured through minimally disturbing measurements iff l is a testable quasi-classical property (i.e., iff $[\mathfrak{R}_l]$ is a Scott-ideal in $[\mathfrak{Q}_l]$). In other words,

$$\mathcal{L}_{q-cl} = \mathcal{L}_{min} \subseteq \mathcal{L}_{IFKM}. \quad (117)$$

For any quasi-classical property $l \in \mathcal{L}_{q-cl}$, the map given by

$$\begin{array}{ccc} \cdot \Theta_l & : & \mathfrak{S} \dashrightarrow \mathfrak{S} \\ [\mathfrak{Q}_l] \ni \sigma & \mapsto & \sigma \cdot \Theta_l := \Sigma_l \sqcup_{\mathfrak{S}} \bigsqcup_{\mathfrak{S}} ([\mathfrak{R}_l] \cap (\downarrow_{\mathfrak{S}} \sigma)). \end{array} \quad (118)$$

defines an idempotent, order-preserving, Scott-continuous partial map from $[\mathfrak{Q}_l]$ to $[\mathfrak{R}_l]$, preserving filtered-infima and satisfying $\forall \sigma \in [\mathfrak{R}_l], \sigma \cdot \Theta_l = \sigma \sqcup_{\mathfrak{S}} \Sigma_l$. This is the explicit form of the minimally disturbing measurement map (which is also an ideal first-kind measurement map) associated with the property l . $\blacksquare$

Proof. Direct consequence of Lemmas 23, 24, 25, 21 and 20. $\square$

Theorem 10. [The Chu space $(\mathfrak{S}, \mathfrak{T}_{min}, \tilde{\mathfrak{e}})$ is bi-extensional]

$$\begin{array}{ccc} \mu & : & \mathfrak{T}_{min} \longrightarrow [\mathfrak{S} \rightarrow \mathfrak{B}]_{\cap}^{\wedge} \\ t & \mapsto & \tilde{\mathfrak{e}}_t \end{array} \quad \text{is injective} \quad (119)$$

As a result, the Chu space $(\mathfrak{S}, \mathfrak{T}_{min}, \tilde{\mathfrak{e}})$ is *extensional*, i.e.

$$\forall t_1, t_2 \in \mathfrak{T}_{min}, \quad (\forall \sigma \in \mathfrak{S}, \tilde{\mathfrak{e}}_{t_1}(\sigma) = \tilde{\mathfrak{e}}_{t_2}(\sigma)) \Rightarrow (t_1 = t_2) \quad (120)$$

As a consequence of Lemma 1, this Chu space is *bi-extensional*. $\blacksquare$

Proof. Once the evaluation map is given, we obtain unambiguously (1) the Scott-closed subset $[\Omega_t]$ of $\mathfrak{S}$ as the reverse image of the subset $\{\perp, \mathbf{Y}\}$ by the Scott-continuous map $\tilde{e}_t$, and (2) the element Σ_t as the infimum of the Scott-open filter determined as the non-empty reverse image of the subset $\{\mathbf{Y}\}$ by the order-preserving and infima-preserving map $\tilde{e}_t$.

Let us now consider two minimally disturbing tests $t_1, t_2 \in \mathfrak{T}_{min}$ such that $\Sigma_{t_1} = \Sigma_{t_2}$ and $[\Omega_{t_1}] = [\Omega_{t_2}]$. We then have, firstly :

$$Dom_{t_1}^{\mathfrak{S}} = [\Omega_{t_1}] = [\Omega_{t_2}] = Dom_{t_2}^{\mathfrak{S}}. \quad (121)$$

Secondly, we have for any σ in $Dom_{t_1}^{\mathfrak{S}} = Dom_{t_2}^{\mathfrak{S}}$

$$\sigma.t_1 = \Sigma_{t_1} \sqcup_{\mathfrak{S}} \bigsqcup_{\mathfrak{S}} ((\downarrow_{\mathfrak{S}} \uparrow^{\mathfrak{S}} \Sigma_{t_1}) \cap (\downarrow_{\mathfrak{S}} \sigma)) = \Sigma_{t_2} \sqcup_{\mathfrak{S}} \bigsqcup_{\mathfrak{S}} ((\downarrow_{\mathfrak{S}} \uparrow^{\mathfrak{S}} \Sigma_{t_2}) \cap (\downarrow_{\mathfrak{S}} \sigma)) = \sigma.t_2 \quad (122)$$

As a consequence, $t_1 = t_2$. As a result, the map μ is injective, and the Chu space $(\mathfrak{S}, \mathfrak{T}_{min}, \tilde{e})$ is therefore extensional. Then, using Lemma 1, we deduce that the Chu space $(\mathfrak{S}, \mathfrak{T}_{min}, \tilde{e})$ is bi-extensional. $\square$

Notion 34. We allow for a generalized definition of yes/no tests and of minimally disturbing yes/no tests. The corresponding set of *generalized yes/no tests* (resp. *generalized minimally disturbing yes/no tests*) will be denoted $\tilde{\mathfrak{T}}$ (resp. $\tilde{\mathfrak{T}}_{min}$). Let us consider $\Sigma, \Sigma' \in \mathfrak{S}$ (not necessarily compact) such that $\widetilde{\Sigma \Sigma'}^{\mathfrak{S}}$. We denote a generic element of $\tilde{\mathfrak{T}}$ as $t_{(\Sigma, \Sigma')}$ and define it according to

$$\mathfrak{A}_{t_{(\Sigma, \Sigma')}} := \uparrow^{\mathfrak{S}} \Sigma \quad (123)$$

$$\Omega_{t_{(\Sigma, \Sigma')}} := (\mathfrak{S} \setminus \uparrow^{\mathfrak{S}} \Sigma') \quad (124)$$

If $t_{(\Sigma, \Sigma')} \in \tilde{\mathfrak{T}}_{min}$, i.e. if Σ, Σ' are such that $(\downarrow_{\mathfrak{S}} \uparrow^{\mathfrak{S}} \Sigma)$ is an idealized sub-selection structure of $\Omega := (\mathfrak{S} \setminus \uparrow^{\mathfrak{S}} \Sigma')$, we will obviously have

$$\forall \sigma \in \Omega_{t_{(\Sigma, \Sigma')}}, \quad \sigma.t_{(\Sigma, \Sigma')} := \Sigma \sqcup_{\mathfrak{S}} \bigsqcup_{\mathfrak{S}} ((\downarrow_{\mathfrak{S}} \uparrow^{\mathfrak{S}} \Sigma) \cap (\downarrow_{\mathfrak{S}} \sigma)). \quad (125)$$

We define the corresponding *generalized property* $l_{(\Sigma, \Sigma')}$ straightforwardly and denote the set of generalized properties (resp. generalized minimally disturbing properties) by $\tilde{\mathcal{L}}$ (resp. $\tilde{\mathcal{L}}_{min}$).

Notion 35. A generalized property $l \in \tilde{\mathcal{L}}_{min}$ will be said to be a *perfect property* (this fact will be denoted $l \in \tilde{\mathcal{L}}_{perfect}$) if $\bar{l}$ is also a testable generalized minimally disturbing property. In other words, $\downarrow_{\mathfrak{S}} \uparrow^{\mathfrak{S}} \Sigma_l$ is an idealized sub-selection structure of $\Omega_l = (\mathfrak{S} \setminus \uparrow^{\mathfrak{S}} \Sigma_{\bar{l}})$ and $\downarrow_{\mathfrak{S}} \uparrow^{\mathfrak{S}} \Sigma_{\bar{l}}$ is an idealized sub-selection structure of $\Omega_{\bar{l}} = (\mathfrak{S} \setminus \uparrow^{\mathfrak{S}} \Sigma_l)$. As a consequence, there exist two conjugate generalized minimally disturbing yes/no tests, t and $\bar{t}$, leading to ideal first-kind measurements of the respective generalized properties, l and $\bar{l}$. The corresponding measurement $.t$ is said to be a *perfect measurement*. The yes/no test t will be said to be perfect as well (this fact will be denoted $t \in \tilde{\mathfrak{T}}_{perfect}$).

3.4 The space of 'Descriptions'

Notion 36. We denote by $\mathcal{D}$ the following subset of the powerset $\mathcal{P}(\mathcal{L}_{IFKM})$:

$$\mathcal{D} := \{ \mathfrak{L} \subseteq \mathcal{L}_{IFKM} \mid \widehat{\mathfrak{L}} \} \quad (126)$$

An element of this set corresponds to a family of properties that can be checked by 'conjoint minimally disturbing measurements' as being 'simultaneously actual' on a given sample. Hence, $\mathcal{D}$ will be called *the space of descriptions*.

An element $D \in \mathcal{D}$ will be eventually denoted $[[l_1 \cdots l_n]]$ rather than $\{l_1, \dots, l_n\}$ to emphasize the compatibility between the properties constituting the description.

Lemma 26. [“Specker’s principle”] Let us consider a finite family of testable properties $\mathcal{L} = \{l_1, \dots, l_N\}$ with $l_k \in \mathcal{L}_{IFKM}$ for any $k \in \{1, \dots, N\}$, such that the elements of this family are pairwise compatible, i.e., $\forall k, k' \in \{1, \dots, N\}, \widehat{l_k l_{k'}}$. The family $\mathcal{L}$ is then a compatible family. In other words,

$$\forall \mathcal{L} = \{l_1, \dots, l_N\} \mid l_k \in \mathcal{L}_{IFKM}, \quad \widehat{\mathcal{L}} \Leftrightarrow (\forall k, k' \in \{1, \dots, N\}, \widehat{l_k l_{k'}}). \quad (127)$$

■

Proof. Given the above requirement, we consider $T = (t_k)_{k \in \{1, \dots, N\}}$ with $t_k \in \mathfrak{T}_{IFKM}$ and $l_k = \lfloor t_k \rfloor$ for any $k = 1, \dots, N$.

First, using the compatibility relations $\widehat{l_i l_k}$ for $k = 1, \dots, i-1$ and Lemma 17, we prove (i) $\lfloor t_1 \cdots t_{i-1} \rfloor \in \mathcal{L}_{IFKM}$, (ii) the compatibility relation $\widehat{l_i \lfloor t_1 \cdots t_{i-1} \rfloor}$ and (iii) $\mathfrak{A}_{t_1, \dots, t_{i-1}} = \mathfrak{A}_{t_1} \cap \cdots \cap \mathfrak{A}_{t_{i-1}}$ for any $i = 1, \dots, N$.

Let us now consider $p \in \mathfrak{Q}_{t_1, (t_2, (\dots (t_{N-1}, (t_N)) \dots))}$ (we note that $\perp_{\mathfrak{S}} \in \mathfrak{Q}_{t_1, (t_2, (\dots (t_{N-1}, (t_N)) \dots))} \neq \emptyset$). We have then from equation (65), $(p \in \mathfrak{Q}_{t_1} \text{ and } (p \cdot t_1) \in \mathfrak{Q}_{t_2, (\dots (t_{N-1}, (t_N)) \dots))}$). Let us denote by (P_i) the statement: $((p \cdot t_1 \cdots t_{i-1}) \in \mathfrak{A}_{t_1} \cap \cdots \cap \mathfrak{A}_{t_{i-1}} \cap \mathfrak{Q}_{t_i} \text{ and } (p \cdot t_1 \cdots t_i) \in \mathfrak{Q}_{t_{i+1}, (\dots (t_{N-1}, (t_N)) \dots)})$. Let us suppose that (P_i) is satisfied.

Using the compatibility relation $\widehat{l_i \lfloor t_1 \cdots t_{i-1} \rfloor}$ and the fact that t_i leads to an ideal measurement and thus satisfies property (87), we deduce that $(p \cdot t_1 \cdots t_i) \in \mathfrak{A}_{t_1} \cap \cdots \cap \mathfrak{A}_{t_{i-1}}$. Using the fact that t_i leads to a first-kind measurement and therefore satisfies (78 (i)), we also have $(p \cdot t_1 \cdots t_i) \in \mathfrak{A}_{t_i}$. Now using expression (65), we conclude that property (P_{i+1}) is satisfied.

As a result, we conclude by recursion that $(p \cdot t_1 \cdots t_N) \in \mathfrak{A}_{t_1} \cap \cdots \cap \mathfrak{A}_{t_N}$. The set $\mathfrak{A}_{t_1} \cap \cdots \cap \mathfrak{A}_{t_N}$ is then non-empty, and the family $\mathcal{L}$ is therefore a compatible family of properties. $\square$

Remark 12. The history of this “principle” is recalled in [17] and some results are given in this paper that “suggest that the principle that pairwise decidable propositions are jointly decidable, together with Boole’s condition that the sum of probabilities of jointly exclusive propositions cannot be higher than one [...], which I will collectively call Specker’s principle, may explain quantum contextuality.”^[24]

Theorem 11. The space of descriptions $\mathcal{D}$ is a ‘coherence domain’ [33] associated with the ‘web’ $\mathcal{L}_{IFKM}$ and with the coherence relation $\widehat{\cdot}$. In other words, $\mathcal{D} \subseteq \mathcal{P}(\mathcal{L}_{IFKM})$ satisfies

$$\forall \mathcal{L}_1, \mathcal{L}_2 \in \mathcal{P}(\mathcal{L}_{IFKM}), \quad (\mathcal{L}_1 \subseteq \mathcal{L}_2 \text{ and } \mathcal{L}_2 \in \mathcal{D}) \Rightarrow (\mathcal{L}_1 \in \mathcal{D}) \quad (128)$$

$$\forall l \in \mathcal{L}_{IFKM}, \quad \{l\} \in \mathcal{D} \quad (129)$$

$$\forall L \in \mathcal{P}(\mathcal{D}) \mid (\forall \mathcal{L}_1, \mathcal{L}_2 \in L, \widehat{\mathcal{L}_1 \cup \mathcal{L}_2}), \quad \bigcup L \in \mathcal{D}. \quad (130)$$

■

Proof. The first property is a direct consequence of the definition of $\mathcal{D}$. Indeed, $\forall \mathcal{L}_1, \mathcal{L}_2 \in \mathcal{P}(\mathcal{L}_{IFKM})$, and the properties $\mathcal{L}_1 \subseteq \mathcal{L}_2$ and $\mathcal{L}_2 \in \mathcal{D}$ imply $\emptyset \neq \bigcap_{l \in \mathcal{L}_2} \mathfrak{A}_l \subseteq \bigcap_{l \in \mathcal{L}_1} \mathfrak{A}_l$.

The second property is a trivial consequence of the definition of consistency and the fact that we consider

²⁴See [43] for the original results on non-contextuality in quantum mechanics.

testable properties.

The property $(\forall M \in \mathcal{P}_{fin}(\mathcal{D}) \mid (\forall \mathfrak{L}_1, \mathfrak{L}_2 \in M, \overbrace{\mathfrak{L}_1 \cup \mathfrak{L}_2}^{\text{testable}}, \bigcup M \in \mathcal{D})$ is a direct consequence of Lemma 26. Hence, for any $L \in \mathcal{P}(\mathcal{D}) \mid (\forall \mathfrak{L}_1, \mathfrak{L}_2 \in L, \overbrace{\mathfrak{L}_1 \cup \mathfrak{L}_2}^{\text{testable}})$, we deduce that $N \in \mathcal{D}$ for any $N \subseteq_{fin} \bigcup L$, which allows the third property to be deduced. $\square$

Notion 37. For any description $D \in \mathcal{D}$, and as soon as $\Sigma_D := \bigsqcup_{l \in D} \Sigma_l$ and $\Sigma'_D := \prod_{l \in D} \Sigma_l$ satisfy $\widehat{\Sigma_D \Sigma'_D}^\mathfrak{S}$, we define a generalized yes/no test denoted t_D by $t_D := t_{(\Sigma_D, \Sigma'_D)}$. This generalized yes/no test characterizes the conjoint measurement of the compatible properties belonging to the description D .

3.5 From discriminating yes/no tests to the complete axiomatics on the space of states

Let us introduce a notion that will reveal to be fundamental in the next subsection when we will have to consider the definition of an orthogonality relation on the space of states.

Notion 38. A collection $\mathfrak{U}$ of generalized yes/no tests is said to be *complete* iff

$$\forall \Sigma \in \mathfrak{S}, \exists \Sigma' \in \mathfrak{S} \mid t_{(\Sigma, \Sigma')} \in \mathfrak{U}, \quad (131)$$

Notion 39. A collection $\mathfrak{U}$ of generalized yes/no tests is said to be *irredundant* iff

$$\forall t_1, t_2 \in \mathfrak{U} \quad \Sigma_{[t_1]} \sqsubseteq_\mathfrak{S} \Sigma_{[t_2]} \Leftrightarrow \Sigma_{[t_2]} \sqsubseteq_\mathfrak{S} \Sigma_{[t_1]} \quad (132)$$

Notion 40. A collection $\mathfrak{U}$ of generalized yes/no tests is said to be *closed* iff

$$\begin{aligned} & \forall t \in \mathfrak{U}, \quad \bar{t} \in \mathfrak{U} \quad (133) \\ & \forall t_{(\Sigma_1, \Sigma'_1)}, t_{(\Sigma_2, \Sigma'_2)} \in \mathfrak{U} \mid \widehat{\Sigma_1 \Sigma_2}^\mathfrak{S} \quad t_{(\Sigma_1 \sqcup_\mathfrak{S} \Sigma_2, \Sigma'_1 \cap_\mathfrak{S} \Sigma'_2)} \in \mathfrak{U} \end{aligned} \quad (134)$$

Remark 13. The property (134) implies more generally

$$\forall D \in \mathcal{D} \mid \forall l \in D, l \in [\mathfrak{U}], \quad t_D \in \mathfrak{U} \quad (135)$$

Notion 41. A *scheme of yes/no tests* is a complete, irredundant and closed collection of generalized yes/no tests.

Let us now introduce a class of generalized yes/no tests that will be fundamental to clarify the last axioms on the space of states.

Notion 42. A couple of states $(\sigma, \sigma') \in \mathfrak{S}^{\times 2}$ is said to be *quasi-consistent* (this fact is denoted $\sigma \bowtie_\mathfrak{S} \sigma'$) according to the following definition :

$$\sigma \bowtie_\mathfrak{S} \sigma' :\Leftrightarrow (\forall \sigma'' \sqsubseteq_\mathfrak{S} \sigma', \widehat{\sigma \sigma''}^\mathfrak{S} \text{ and } \forall \sigma'' \sqsubseteq_\mathfrak{S} \sigma, \widehat{\sigma' \sigma''}^\mathfrak{S}). \quad (136)$$

We have obviously $\widehat{\sigma \sigma'}^\mathfrak{S} \Rightarrow \sigma \bowtie_\mathfrak{S} \sigma'$.

We will denote $\sigma \widetilde{\bowtie}_\mathfrak{S} \sigma' :\Leftrightarrow (\sigma \bowtie_\mathfrak{S} \sigma' \text{ and } \widehat{\sigma \sigma'}^\mathfrak{S})$.

Notion 43. We will say that a generalized yes/no test $t_{(\Sigma, \Sigma')}$ is a *discriminating yes/no test* iff $\Sigma \bowtie_{\mathfrak{S}} \Sigma'$. In other words,

$$\widetilde{\mathfrak{T}}_{disc} := \{ t_{(\Sigma, \Sigma')} \in \widetilde{\mathfrak{T}} \mid \Sigma \bowtie_{\mathfrak{S}} \Sigma' \}. \quad (137)$$

The corresponding set of *discriminating properties* is denoted $\widetilde{\mathcal{L}}_{disc}$.

We will constrain further the space of states by requiring the existence of schemes of discriminating yes/no tests.

Axiom 8. The space of states is such that there exists a scheme of discriminating yes/no tests.

Now, we can make a summary of the axioms postulated for the space of states.

Theorem 12. The space of states is a projective domain, with no complete meet-irreducible elements except maximal elements (**Axiom 7**), such that a scheme of discriminating tests exists (**Axiom 8**). ■

Remark 14 clarifies the precise consequences of **Axiom 8** for the space of states.

Remark 14. A natural question to address to our program is about the general existence of examples of domains solution of the constraints given in Theorem 12.

In fact, it is easy to build a rather general class of examples of space of states. Let us consider a projective lattice equipped with an orthocomplementation (i.e. an involutive and order-reversing complementation). It will be denoted $(\mathfrak{L}, \sqcup_{\mathfrak{L}}, \sqcap_{\mathfrak{L}}, \star)$. We will require that the complete meet-irreducible elements be the co-atoms of $\mathfrak{L}$. Let us denote by $\perp_{\mathfrak{L}}$ (resp. $\top_{\mathfrak{L}}$) the bottom element (resp. top element) of $\mathfrak{L}$. We will assume that the join-irreducible elements of $\mathfrak{L}$ are the atoms of $\mathfrak{L}$.

Let us denote $\mathfrak{S} := \mathfrak{L} \setminus \{\top_{\mathfrak{L}}\}$. The laws $\sqcup_{\mathfrak{S}}$, $\sqcap_{\mathfrak{S}}$ and $\star$ are induced from the laws on $\mathfrak{L}$ and we have $\perp_{\mathfrak{S}} = \perp_{\mathfrak{L}}$.

$\mathfrak{L}$ being modular, $\mathfrak{S}$ is then conditionally-modular. $\mathfrak{L}$ being complete, $\mathfrak{S}$ is then bounded-complete. $\mathfrak{L}$ being atomistic, $\mathfrak{S}$ is then atomistic. $\mathfrak{L}$ being meet-continuous, $\mathfrak{S}$ is then meet-continuous. $\mathfrak{L}$ being relatively-complemented, $\mathfrak{S}$ is then relatively-complemented. Endly, it is easy to check that $\mathfrak{S}$ is directed-complete using the completeness of $\mathfrak{L}$. As a result, $\mathfrak{S}$ is a projective domain.

Due to our requirement on the complete meet-irreducible elements of $\mathfrak{L}$, **Axiom 7** is then immediately satisfied.

We now intent to consider the last axiom. But before that, let us establish a simple result about the discriminating relation.

We firstly note that

$$\forall u, x \in \mathfrak{S}, \quad \widetilde{ux}^{\mathfrak{S}} \Leftrightarrow (u \sqcup_{\mathfrak{L}} x = \top_{\mathfrak{L}}). \quad (138)$$

Let us consider $u, v, x \in \mathfrak{S}$ such that $u \sqsupseteq_{\mathfrak{L}} v$ and $u \parallel_{\mathfrak{L}} x$, we have the following case analysis :

- $x \sqsubseteq_{\mathfrak{L}} v$: impossible as it contradicts $x \parallel_{\mathfrak{L}} u$,
- $x \sqsupseteq_{\mathfrak{L}} v$: we have then $x \sqcup_{\mathfrak{L}} v = x$ and then $((x \sqcup_{\mathfrak{L}} u) \sqsupseteq_{\mathfrak{L}} x \sqsupseteq_{\mathfrak{L}} v, (x \sqcup_{\mathfrak{L}} u) \sqsupseteq_{\mathfrak{L}} u \sqsupseteq_{\mathfrak{L}} v)$ which implies $(x \sqcup_{\mathfrak{L}} u) \sqsupseteq_{\mathfrak{L}} x = (x \sqcup_{\mathfrak{L}} v)$ using upper semi-modularity property in $\mathfrak{L}$,
- $x \parallel_{\mathfrak{L}} v$: we distinguish three sub-cases

- $(x \sqcup_{\mathfrak{L}} v) \sqsupseteq_{\mathfrak{L}} u$: impossible as it implies $x \sqsupseteq_{\mathfrak{L}} u$ which contradicts $u \parallel_{\mathfrak{L}} x$,
- $(x \sqcup_{\mathfrak{L}} v) \parallel_{\mathfrak{L}} u$: we have then $((x \sqcup_{\mathfrak{L}} u) \sqsupseteq_{\mathfrak{L}} (x \sqcup_{\mathfrak{L}} v) \sqsupseteq_{\mathfrak{L}} v, (x \sqcup_{\mathfrak{L}} u) \sqsupseteq_{\mathfrak{L}} u \sqsupseteq_{\mathfrak{L}} v)$ which implies $(x \sqcup_{\mathfrak{L}} u) \sqsupseteq_{\mathfrak{L}} x = (x \sqcup_{\mathfrak{L}} v)$ using upper semi-modularity property in $\mathfrak{L}$,
- $(x \sqcup_{\mathfrak{L}} v) \sqsupseteq_{\mathfrak{L}} u$: this sub-case implies immediately $(x \sqcup_{\mathfrak{L}} v) = (x \sqcup_{\mathfrak{L}} u)$, however this configuration $(x \sqcup_{\mathfrak{L}} u) = (x \sqcup_{\mathfrak{L}} v)$ implies $u = v$ (which contradicts $u \sqsupseteq_{\mathfrak{L}} v$) as soon as $(x \sqcap_{\mathfrak{L}} u) = (x \sqcap_{\mathfrak{L}} v)$, due to modularity property (40).

As a conclusion, using the defining property

$$(u \bowtie_{\mathfrak{S}} x) \Leftrightarrow (\forall v \sqsubseteq_{\mathfrak{L}} u, (v \sqcup_{\mathfrak{L}} x) \sqsubseteq_{\mathfrak{L}} \top_{\mathfrak{L}} \text{ and } \forall y \sqsubseteq_{\mathfrak{L}} x, (u \sqcup_{\mathfrak{L}} y) \sqsubseteq_{\mathfrak{L}} \top_{\mathfrak{L}}), \quad (139)$$

we finally obtain the simple equivalence :

$$(u \bowtie_{\mathfrak{S}} x) \Leftrightarrow ((u \sqcup_{\mathfrak{L}} x) = \top_{\mathfrak{L}} \text{ and } (u \sqcap_{\mathfrak{L}} x) = \perp_{\mathfrak{L}}). \quad (140)$$

Let us now come back to our last axiom. Let us choose

$$\mathfrak{U} := \{t_{(\Sigma, \Sigma^*)} \mid \Sigma \in \mathfrak{S}\}. \quad (141)$$

The completeness condition for the scheme is satisfied as soon as

$$\forall \sigma_1 \in \mathfrak{S}^*, \exists \sigma'_1 := \sigma_1^* \in \mathfrak{S} \mid (\sigma_1 \bowtie_{\mathfrak{S}} \sigma'_1), \quad (142)$$

i.e., as soon as $\mathfrak{L}$ is ortho-complemented.

Secondly, the irredundancy condition for the scheme is satisfied as soon as

$$\forall \sigma_1, \sigma_2 \in \mathfrak{S}^*, \quad (\sigma_1 \sqsubseteq_{\mathfrak{S}} \sigma_2) \Rightarrow (\sigma_2^* \sqsubseteq_{\mathfrak{S}} \sigma_1^*), \quad (143)$$

i.e., as soon as the star operation satisfies the order reversing property.

Endly, the closedness property for the scheme is satisfied as soon as the property

$$\forall \sigma_1, \sigma_2 \in \mathfrak{S}^*, \quad (\widehat{\sigma_1 \sigma_2}^{\mathfrak{S}}) \Rightarrow ((\sigma_1 \sqcup_{\mathfrak{S}} \sigma_2) \bowtie_{\mathfrak{S}} (\sigma_1^* \sqcap_{\mathfrak{S}} \sigma_2^*)) \quad (144)$$

expresses the DeMorgan's law satisfied by the star operation (this law is also a simple consequence of the order reversing property and the involutive property).

As a result, $\mathfrak{S}$ is a well defined space of states.

The final conclusion of this remark is simple : **Axiom 8** expresses the necessity for $\mathfrak{S}$ to be equipped with an ortho-complementation.

Let us now formulate a conjecture relative to the set of discriminating yes/no tests.

Conjecture 2. Any discriminating yes/no test is a perfect yes/no test. In other words,

$$\widetilde{\mathfrak{T}}_{disc} \subseteq \widetilde{\mathfrak{T}}_{perfect}. \quad (145)$$

Remark 15. This is not the purpose of this paper to clarify the proof of this property and we prefer to postpone this analysis to a forthcoming paper. Nevertheless, we want to note that this result is trivial in a basic class of spaces of states. Indeed, if $(\mathfrak{L}, \sqcup_{\mathfrak{L}}, \sqcap_{\mathfrak{L}}, \star)$ is a finite boolean lattice and if we define $\mathfrak{S} := (\mathfrak{L} \setminus \{\top_{\mathfrak{L}}\})$, we obtain immediately $(\mathfrak{S} \setminus (\uparrow^{\mathfrak{S}} \Sigma^*)) = (\downarrow_{\mathfrak{S}} \uparrow^{\mathfrak{S}} \Sigma)$ for any $\Sigma \in \mathfrak{S}$, or, in other words, $[\mathfrak{Q}_l] = [\mathfrak{R}_l]$ for any discriminating property l . Hence, any discriminating property is a perfect property.

3.6 Orthogonality relation on the space of states

In the present subsection, $\mathfrak{U}$ is a fixed scheme of generalized yes/no tests. When the choice $\mathfrak{U} \subseteq \widetilde{\mathfrak{T}}_{disc}$ will have to be done, it will be mentioned explicitly.

Notion 44. Two states $\sigma_1, \sigma_2 \in \mathfrak{S}^*$ are said to be *orthogonal* (this fact will be denoted $\sigma_1 \perp \sigma_2$, the dependence with respect to $\mathfrak{U}$ is intentionally erased) iff they can be distinguished unambiguously by a statement associated with a perfect property. In other words,

$$\forall \sigma_1, \sigma_2 \in \mathfrak{S}^*, (\sigma_1 \perp \sigma_2) \Leftrightarrow (\exists l \in [\mathfrak{U}] \mid \sigma_1 \in [\mathfrak{A}_l] \text{ and } \sigma_2 \in [\mathfrak{A}_l]). \quad (146)$$

This orthogonality relation is obviously symmetric and anti-reflexive.
We denote as usual

$$\forall S \subseteq \mathfrak{S}^*, \quad S^\perp := \{ \sigma' \in \mathfrak{S} \mid \forall \sigma \in S, \sigma \perp \sigma' \}. \quad (147)$$

Remark 16. Moreover, we have obviously

$$\forall \sigma_1, \sigma_2 \in \mathfrak{S}^*, \quad \sigma_1 \perp \sigma_2 \Rightarrow \widehat{\sigma_1 \sigma_2}^\mathfrak{S}. \quad (148)$$

Indeed, following the definition of the consistency relation, we then note that, for any $\sigma_1, \sigma_2 \in \mathfrak{S}^*$, the property $\widehat{\sigma_1 \sigma_2}^\mathfrak{S}$ immediately implies $\forall t \in \mathfrak{U}, \mathfrak{e}(\sigma_1, t) \mathfrak{e}(\sigma_2, t)$. By negation, we obtain that $(\exists l \in [\mathfrak{U}] \mid \sigma_1 \in [\mathfrak{A}_l] \text{ and } \sigma_2 \in [\mathfrak{A}_l])$ implies $\widehat{\sigma_1 \sigma_2}^\mathfrak{S}$.

Lemma 27. For any $\sigma \in \mathfrak{S}^*$, $\{\sigma\}^\perp$ is a non-empty filter. ■

Proof. Firstly, let us consider $\sigma_1, \sigma_2 \in \mathfrak{S}^*$ such that $\sigma_1 \sqsubseteq_\mathfrak{S} \sigma_2$ and let us assume that $\sigma_1 \perp \sigma$. We have $(\exists l \in [\mathfrak{U}] \mid \sigma \in [\mathfrak{A}_l] \text{ and } \sigma_1 \in [\mathfrak{A}_l]) \Rightarrow (\exists l \in [\mathfrak{U}] \mid \sigma \in [\mathfrak{A}_l] \text{ and } \sigma_2 \in [\mathfrak{A}_l]) \Leftrightarrow \sigma \in \{\sigma_2\}^\perp$. Secondly, let us consider $S \subseteq \mathfrak{S}$ such that $\forall \sigma' \in S, \sigma \perp \sigma'$. There exists a family of generalized properties $(l_{\sigma'})_{\sigma' \in S} \subseteq [\mathfrak{U}]$ such that $\forall \sigma' \in S, \Sigma_{l_{\sigma'}} \sqsubseteq_\mathfrak{S} \sigma$ and $\Sigma_{l_{\sigma'}} \sqsubseteq_\mathfrak{S} \sigma'$. Using relation (135), we deduce that there exists a generalized property l defined by $\Sigma_l := \bigsqcup_{\sigma' \in S} \Sigma_{l_{\sigma'}} \sqsubseteq_\mathfrak{S} \sigma$ and $\Sigma_l := \bigcap_{\sigma' \in S} \Sigma_{l_{\sigma'}}$. And we have then $\Sigma_l \sqsubseteq_\mathfrak{S} \sigma$ and $\Sigma_l \sqsubseteq_\mathfrak{S} \sigma', \forall \sigma' \in S$. In other words, $\sigma \perp (\bigcap_\mathfrak{S} S)$. □

As a consequence, we will adopt the following definition

Notion 45. The space of states is equipped with a unary operation defined as follows

$$\forall \sigma \in \mathfrak{S}^*, \quad \sigma^* := \bigcap_\mathfrak{S} \{\sigma\}^\perp. \quad (149)$$

Lemma 28.

$$\forall \sigma \in \mathfrak{S}^*, \quad \{\sigma\}^\perp = \uparrow^\mathfrak{S} \sigma^*, \quad (150)$$

$$\forall \sigma_1, \sigma_2 \in \mathfrak{S}^*, \quad (\sigma_1 \perp \sigma_2) \Leftrightarrow (\sigma_1^* \sqsubseteq_\mathfrak{S} \sigma_2), \quad (151)$$

$$\forall \sigma_1, \sigma_2 \in \mathfrak{S}^*, \quad (\sigma_1 \sqsubseteq_\mathfrak{S} \sigma_2) \Rightarrow (\sigma_2^* \sqsubseteq_\mathfrak{S} \sigma_1^*). \quad (152)$$

$$\forall \sigma \in \mathfrak{S}^*, \forall l \in \mathcal{L}, \quad \tilde{\mathfrak{e}}_l(\sigma) = \tilde{\mathfrak{e}}_l(\sigma^*). \quad (153)$$

■

Proof. The properties (150) and (151) are trivial consequences of the defining property (149). Let us consider $\sigma_1, \sigma_2 \in \mathfrak{S}^*$ such that $\sigma_1 \sqsubseteq_\mathfrak{S} \sigma_2$. We have $\sigma \in \{\sigma_1\}^\perp \Leftrightarrow (\exists l \in [\mathfrak{U}] \mid \sigma_1 \in [\mathfrak{A}_l] \text{ and } \sigma \in [\mathfrak{A}_l]) \Rightarrow (\exists l \in [\mathfrak{U}] \mid \sigma_2 \in [\mathfrak{A}_l] \text{ and } \sigma \in [\mathfrak{A}_l]) \Leftrightarrow \sigma \in \{\sigma_2\}^\perp$. We then have

$$\sigma_1 \sqsubseteq_\mathfrak{S} \sigma_2 \Rightarrow \{\sigma_1\}^\perp \subseteq \{\sigma_2\}^\perp \Rightarrow (\bigcap_\mathfrak{S} \{\sigma_1\}^\perp) \sqsubseteq_\mathfrak{S} (\bigcap_\mathfrak{S} \{\sigma_2\}^\perp). \quad (154)$$

As a result, we obtain the order-reversing property (152) of the unary operation $\star$. □

Lemma 29.

$$\forall \sigma \in \mathfrak{S}, \exists ! l_\sigma \in [\mathcal{U}] \mid (\Sigma_{l_\sigma} = \sigma, \quad \Sigma_{\overline{l_\sigma}} = \sigma^*) \quad (155)$$

■

Proof. From $\sigma^* \in \{\sigma\}^\perp$ and the defining property (146), we deduce that there exists $l_\sigma \in [\mathcal{U}]$ such that $\Sigma_{l_\sigma} \sqsubseteq_{\mathfrak{S}} \sigma$ and $\Sigma_{\overline{l_\sigma}} \sqsubseteq_{\mathfrak{S}} \sigma^*$. From completeness property (131), we know that, for any $\sigma \in \mathfrak{S}$, there exists a $\sigma' \in \mathfrak{S}$ such that $l_{(\sigma, \sigma')} \in [\mathcal{U}]$. We note by the way that $\sigma' \in \{\sigma\}^\perp$ and then $\sigma' \sqsupseteq_{\mathfrak{S}} \bigcap_{\mathfrak{S}} \{\sigma\}^\perp = \sigma^*$. Using irredundancy property (132), we deduce that $\sigma^* \sqsubseteq_{\mathfrak{S}} \sigma' \sqsubseteq_{\mathfrak{S}} \Sigma_{\overline{l_\sigma}} \sqsubseteq_{\mathfrak{S}} \sigma^*$, and then $\Sigma_{\overline{l_\sigma}} = \sigma^*$. Using once again irredundancy property (132), we deduce that $\Sigma_{l_\sigma} = \sigma$. $\square$

Lemma 30.

$$\forall \sigma \in \mathfrak{S}^*, \quad \sigma^{**} = \sigma. \quad (156)$$

$$\forall S \subseteq_{fin}^{\neq \emptyset} \mathfrak{S}^* \mid \widehat{S}^{\mathfrak{S}}, \quad (\bigsqcup_{\mathfrak{S}} S)^* = \bigsqcap_{\sigma \in S} \sigma^* \quad (157)$$

■

Proof. Using Lemma 29 and property $(\forall l \in [\mathcal{U}], \quad l = \overline{\overline{l}})$, we deduce property (156). As usual, property (157) is a direct consequence of properties (156) and (152). To derive it directly, using Lemma 29, we introduce, for any $\sigma \in S$, the property l_σ such that $\Sigma_{l_\sigma} = \sigma$ and $\Sigma_{\overline{l_\sigma}} = \sigma^*$. We now deduce, from property (135), that there exists a generalized property $l \in [\mathcal{U}]$ such that $\Sigma_l = \bigsqcap_{\sigma \in S} \Sigma_{l_\sigma} = \bigsqcap_{\mathfrak{S}} S$ and $\Sigma_{\overline{l}} = \bigsqcap_{\sigma \in S} \Sigma_{\overline{l_\sigma}} = \bigsqcap_{\sigma \in S} \sigma^*$. Using Lemma 29, we deduce $\Sigma_l = \Sigma_l^*$, i.e. property (157). $\square$

Lemma 31. The closure operator associated with the orthogonality relation $\perp$ satisfies

$$\forall S \subseteq \mathfrak{S}^*, \quad S^{\perp\perp} = \uparrow^{\mathfrak{S}}(\bigsqcap_{\mathfrak{S}} S). \quad (158)$$

■

Proof. First of all, we have $S^\perp = \bigcap_{\sigma \in S} \{\sigma\}^\perp = \uparrow^{\mathfrak{S}} \bigsqcap_{\sigma \in S} \sigma^*$. As a consequence, we have $S^{\perp\perp} = (\uparrow^{\mathfrak{S}} \bigsqcap_{\sigma \in S} \sigma^*)^\perp = \{\bigsqcap_{\sigma \in S} \{\sigma^*\}^\perp\}^\perp = \uparrow^{\mathfrak{S}} (\bigsqcap_{\sigma \in S} \sigma^*)^*$. Now, using De Morgan's law (157), we deduce $S^{\perp\perp} = \uparrow^{\mathfrak{S}} (\bigsqcap_{\sigma \in S} \sigma^{**})$. We now use the involutive property (156) to deduce $S^{\perp\perp} = \uparrow^{\mathfrak{S}} (\bigsqcap_{\mathfrak{S}} S)$. $\square$

3.7 The space of ortho-closed subsets of pure states as a Hilbert lattice

In this subsection, we will impose $\mathcal{U} \subseteq \widetilde{\mathfrak{T}}_{disc}$.

Notion 46. The space of pure states $\mathfrak{S}_{pure}$ inherits an orthogonality relation (denoted $\perp$) from the orthogonality relation $\perp$ defined on the whole space of states $\mathfrak{S}$.

$$\forall \sigma_1, \sigma_2 \in \mathfrak{S}_{pure}, \quad (\sigma_1 \perp\!\!\!\perp \sigma_2) \quad :\Leftrightarrow \quad (\sigma_1 \perp \sigma_2), \quad (159)$$

$$\forall S \subseteq \mathfrak{S}_{pure}, \quad S^\perp = \{\sigma' \in \mathfrak{S}_{pure}, \mid \forall \sigma \in S, \sigma \perp \sigma'\} = \underline{S^\perp}. \quad (160)$$

Lemma 32.

$$\forall S \subseteq \mathfrak{S}_{pure}, \quad S^{\perp\perp} = \underline{\bigsqcap_{\mathfrak{S}} S}. \quad (161)$$

■

Notion 47. The set of ortho-closed subsets of the space of pure states equipped with the orthogonality relation $\perp$ is denoted $\mathcal{C}(\mathfrak{S}_{\text{pure}})$. The set $\mathcal{C}(\mathfrak{S}_{\text{pure}})$ is equipped with the following operations

$$\forall \mathfrak{c}_1, \mathfrak{c}_2 \in \mathcal{C}(\mathfrak{S}_{\text{pure}}), \quad \mathfrak{c}_1 \wedge \mathfrak{c}_2 := \mathfrak{c}_1 \cap \mathfrak{c}_2 \quad (162)$$

$$\mathfrak{c}_1 \vee \mathfrak{c}_2 := (\mathfrak{c}_1 \cup \mathfrak{c}_2)^{\perp\perp}. \quad (163)$$

and by the unary operation $\perp$.

Lemma 33. $\mathcal{C}(\mathfrak{S}_{\text{pure}})$ is a complete ortho-lattice. ■

Lemma 34. $\mathcal{C}(\mathfrak{S}_{\text{pure}})$ is atomic, i.e.

$$\forall \sigma \in \mathfrak{S}_{\text{pure}}, \{\sigma\}^{\perp\perp} = \{\sigma\}. \quad (164)$$

■

Proof. Due to **Axiom 7** (i.e. the absence of type 2's pure states), we have immediately

$$\forall \sigma \in \mathfrak{S}_{\text{pure}}, \underline{\{\sigma\}} = \{\sigma\}. \quad (165)$$

□

Lemma 35. The lattice $\mathcal{C}(\mathfrak{S}_{\text{pure}})$ is atomistic, i.e.

$$\forall \mathfrak{c} \in \mathcal{C}(\mathfrak{S}_{\text{pure}}), \quad \mathfrak{c} = \bigvee_{\sigma \in \mathfrak{c}} \{\sigma\}^{\perp\perp} = \bigvee_{\sigma \in \mathfrak{c}} \{\sigma\}. \quad (166)$$

■

Lemma 36. We have the following property

$$\forall S \subseteq_{\text{fin}}^{\neq \emptyset} \mathfrak{S}, \forall \sigma \in \mathfrak{S} \mid \sigma \notin S^{\perp\perp}, \quad \exists \sigma' \in S^{\perp} \mid (S \cup \{\sigma\})^{\perp\perp} = (S \cup \{\sigma'\})^{\perp\perp}. \quad (167)$$

We have then immediately

$$\forall S \subseteq_{\text{fin}}^{\neq \emptyset} \mathfrak{S}_{\text{pure}}, \forall \sigma \in \mathfrak{S}_{\text{pure}} \mid \sigma \notin S^{\perp\perp}, \quad \exists \sigma' \in S^{\perp} \mid (S \cup \{\sigma\})^{\perp\perp} = (S \cup \{\sigma'\})^{\perp\perp}. \quad (168)$$

■

Proof. First of all, from (158), we have $(S \cup \{\sigma\})^{\perp\perp} = \uparrow^{\mathfrak{S}}((\bigcap_{\mathfrak{S}} S) \sqcap_{\mathfrak{S}} \sigma)$. We have $((\bigcap_{\mathfrak{S}} S) \sqcap_{\mathfrak{S}} \sigma) \sqsubset_{\mathfrak{S}} (\bigcap_{\mathfrak{S}} S)$ because the condition $\sigma \notin S^{\perp\perp}$ means $\sigma \not\sqsubseteq_{\mathfrak{S}} (\bigcap_{\mathfrak{S}} S)$. Due to lemma 29, there exists a unique discriminating property $\mathfrak{l}$ such that $\Sigma_{\mathfrak{l}} = (\bigcap_{\mathfrak{S}} S)$ and $\Sigma_{\mathfrak{l}} = (\bigcap_{\mathfrak{S}} S)^*$. The discriminating character of the property $\mathfrak{l}$ implies that $\forall \Sigma'' \sqsubset_{\mathfrak{S}} \Sigma_{\mathfrak{l}}, \widehat{\Sigma_{\mathfrak{l}} \Sigma''}^{\mathfrak{S}}$. In particular, choosing $\Sigma'' = ((\bigcap_{\mathfrak{S}} S) \sqcap_{\mathfrak{S}} \sigma)$, we obtain $(\bigcap_{\mathfrak{S}} S)^* \widehat{((\bigcap_{\mathfrak{S}} S) \sqcap_{\mathfrak{S}} \sigma)}$. Explicitly, there exists an element $\sigma' \in S^{\perp} = \uparrow^{\mathfrak{S}}((\bigcap_{\mathfrak{S}} S)^*)$ such that $\sigma' \sqsupseteq_{\mathfrak{S}} (\sigma \sqcap_{\mathfrak{S}} (\bigcap_{\mathfrak{S}} S))$. In other words, there exists an element $\sigma' \in S^{\perp}$ such that $(S \cup \{\sigma\})^{\perp\perp} = (S \cup \{\sigma'\})^{\perp\perp}$. This proves the property (167).

We deduce (168) immediately from (167) using (161). □

Remark 17. The property (168) implies the following simple property :

$$\forall \sigma_1, \sigma_2 \in \mathfrak{S}_{\text{pure}} \mid \sigma_1 \neq \sigma_2, \quad \exists \sigma_3 \in \mathfrak{S}_{\text{pure}} \mid (\sigma_1 \perp \sigma_3 \text{ and } \sigma_2 \not\perp \sigma_3). \quad (169)$$

Indeed, we deduce from (168), for any $\sigma_1, \sigma_2 \in \mathfrak{S}_{pure} \mid \sigma_1 \neq \sigma_2$, the fact that $\exists \sigma_3 \perp \sigma_1$ such that $(\{\sigma_1\} \cup \{\sigma_2\})^{\perp\perp} = (\{\sigma_1\} \cup \{\sigma_3\})^{\perp\perp}$, and we cannot have $\sigma_2 \perp \sigma_3$ because this would mean $\sigma_2 \in \{\sigma_1\}^{\perp\perp} = \{\sigma_1\}$ which contradicts the assumption.

The property (169) is called 'Separation' axiom in [73, Definition 2.3].

Remark 18. For each $S \subseteq \mathfrak{S}_{pure}$, we can define the corresponding discriminating yes/no test $t_{(\prod S, (\prod S)^*)}$ and for any $\sigma \notin S^{\perp}$ define $\sigma_{\perp S} := \sigma \cdot t_{(\prod S, (\prod S)^*)}$. We have, for any $\sigma' \in S$, the equivalence $(\sigma' \perp \sigma) \Leftrightarrow (\sigma' \perp \sigma_{\perp S})$. This property is called 'Representation' axiom in [73, Definition 2.3]. Let us prove it in our context. We will suppose that S is not reduced to a single element, this particular case being in fact separately and trivially checked.

First of all, it is straightforward to prove that $(\sigma' \perp \sigma_{\perp S}) \Rightarrow (\sigma' \perp \sigma)$. Indeed, $(\sigma' \perp \sigma_{\perp S})$ can be rewritten $Y = \tilde{e}_{t_{(\sigma'^*, \sigma')}}(\sigma_{\perp S})$, i.e. $Y = \tilde{e}_{t_{(\sigma'^*, \sigma')}}(\sigma \cdot t_{(\prod S, (\prod S)^*)})$. The minimally disturbing yes/no tests $t_{(\sigma'^*, \sigma')}$ and $t_{(\prod S, (\prod S)^*)}$ are compatible. We then have $\tilde{e}_{t_{(\sigma'^*, \sigma')}}(\sigma \cdot t_{(\prod S, (\prod S)^*)}) \leq \tilde{e}_{t_{(\sigma'^*, \sigma')}}(\sigma)$, from (87). As a result, we obtain $Y = \tilde{e}_{t_{(\sigma'^*, \sigma')}}(\sigma)$, i.e. $(\sigma' \perp \sigma)$.

Let us now prove the implication $(\sigma' \perp \sigma) \Rightarrow (\sigma' \perp \sigma_{\perp S})$. Let us suppose that $\sigma \in \{\sigma'\}^{\perp} = \uparrow^{\mathfrak{S}} \sigma'^*$. We note that $\sigma'^* \sqsubseteq_{\mathfrak{S}} (\prod_{\mathfrak{S}} S)^*$. Moreover, we have $(\prod_{\mathfrak{S}} S)^* \bowtie_{\mathfrak{S}} \prod_{\mathfrak{S}} S$. As a consequence, we deduce that $(\uparrow^{\mathfrak{S}} \prod_{\mathfrak{S}} S) \cap (\uparrow^{\mathfrak{S}} \sigma'^*) \neq \emptyset$, i.e. $S \cap \{\sigma'\}^{\perp} \neq \emptyset$. Let us then consider $\sigma'' \in S \cap \{\sigma'\}^{\perp}$. We have obviously $(\sigma'' \sqsubseteq_{\mathfrak{S}} \sigma) \in (\downarrow_{\mathfrak{S}} \uparrow^{\mathfrak{S}} \prod_{\mathfrak{S}} S) \cap (\downarrow_{\mathfrak{S}} \sigma)$ and then $(\sigma'' \sqsubseteq_{\mathfrak{S}} \sigma) \sqsubseteq_{\mathfrak{S}} (\sigma_{\perp S} \sqsubseteq_{\mathfrak{S}} \sigma)$ because of the property (116). Moreover, as soon as (i) $(\downarrow_{\mathfrak{S}} \sigma)$ is a filter, (ii) $\sigma'' \in \{\sigma'\}^{\perp}$ by construction, and (iii) $\sigma \in \{\sigma'\}^{\perp}$ by assumption, we conclude that $(\sigma'' \sqsubseteq_{\mathfrak{S}} \sigma) \in \{\sigma'\}^{\perp}$. As a result, we deduce that $(\sigma_{\perp S} \sqsubseteq_{\mathfrak{S}} \sigma) \in \{\sigma'\}^{\perp}$, and in particular $\sigma_{\perp S} \in \{\sigma'\}^{\perp}$. This concludes the proof.

Lemma 37. The ortho-lattice $\mathcal{C}(\mathfrak{S}_{pure})$ is ortho-modular. ■

Proof. Let us firstly prove that, for any maximal orthogonal subset S of an ortho-closed set $A \in \mathcal{C}(\mathfrak{S}_{pure})$, we have $A = S^{\perp\perp}$. Let us suppose it exists $\sigma \in A \setminus S^{\perp\perp}$. Using previous lemma, we can identify $\sigma' \in S^{\perp}$ such that $(S \cup \{\sigma\})^{\perp\perp} = (S \cup \{\sigma'\})^{\perp\perp} \subseteq A$. This result contradicts the maximality of the orthogonal subset S in A . We have then necessarily $A = S^{\perp\perp}$. Using now [26, Corollary 2 p.7] (see also [70, Theorem 35]), we then conclude that the ortho-lattice $\mathcal{C}(\mathfrak{S}_{pure})$ is orthomodular. □

Lemma 38. The ortho-lattice $\mathcal{C}(\mathfrak{S}_{pure})$ satisfies the *covering property*, i.e.

$$\forall A \in \mathcal{C}(\mathfrak{S}_{pure}), \forall \sigma \in \mathfrak{S}_{pure} \mid \sigma \notin A, \quad \{\sigma\} \vee A \text{ covers } A. \quad (170)$$
■

Proof. Let $A \in \mathcal{C}(\mathfrak{S}_{pure})$ and $\sigma \notin A$. From Lemma 36, we know that there exists $\sigma' \in A^{\perp}$ such that $A \vee \{\sigma\} = (A \cup \{\sigma\})^{\perp\perp} = (A \cup \{\sigma'\})^{\perp\perp} = A \vee \{\sigma'\}$. Since σ' is an atom orthogonal to A , it follows from the orthomodularity of $\mathcal{C}(\mathfrak{S}_{pure})$ that $A \vee \{\sigma\}$ covers A . □

Theorem 13. The ortho-lattice $\mathcal{C}(\mathfrak{S}_{pure})$ forms a *Piron's propositional system* (also called *Hilbert lattice*) (see [67, Definition 5.9]). ■

Proof. Direct consequence of Lemma 33, Lemma 35, Lemma 37, Lemma 38, □

Notion 48. The orthogonality space $\mathfrak{S}_{pure}$ is said to be *reducible* iff $\mathfrak{S}_{pure}$ is the disjoint union of non-empty subsets $\mathfrak{S}_1, \mathfrak{S}_2 \subseteq \mathfrak{S}_{pure}$ such that $\sigma_1 \perp \sigma_2$ for any $\sigma_1 \in \mathfrak{S}_1$ and $\sigma_2 \in \mathfrak{S}_2$. Otherwise, $\mathfrak{S}_{pure}$ is said to be *irreducible*.

Axiom 9.

$$\forall \sigma_1, \sigma_2 \in \mathfrak{S}_{\text{pure}} \mid \sigma_1 \neq \sigma_2, \quad \exists \sigma_3 \in \mathfrak{S}_{\text{pure}} \mid (\sigma_3 \neq \sigma_1, \quad \sigma_3 \neq \sigma_2, \text{ and } \sigma_3 \in \{\sigma_1 \sqcap_{\mathfrak{S}} \sigma_2\}) \quad (171)$$

Remark 19. Previous axiom is nothing else than the 'Superposition' axiom in [73, Definition 2.3].

Remark 20. As a conclusion of remark 17, remark 18, and remark 19, we conclude that $(\mathfrak{S}_{\text{pure}}, \perp)$ is a 'quantum Kripke frame' as defined in [73, Definition 2.3].

Lemma 39. $\mathfrak{S}_{\text{pure}}$ is irreducible. ■

Proof. Let us assume that $\mathfrak{S}_{\text{pure}}$ is reducible, then $\mathfrak{S}_{\text{pure}}$ is the disjoint union of non-empty subsets $\mathfrak{S}_1, \mathfrak{S}_2 \subseteq \mathfrak{S}_{\text{pure}}$ such that $\sigma_1 \perp \sigma_2$ for any $\sigma_1 \in \mathfrak{S}_1$ and $\sigma_2 \in \mathfrak{S}_2$. Let us consider $\sigma_1 \in \mathfrak{S}_1$ and $\sigma_2 \in \mathfrak{S}_2$ and $\sigma_3 \in \{\sigma_1, \sigma_2\}^{\perp\perp}$. Necessarily $\sigma_3 \in \mathfrak{S}_1$ or $\sigma_3 \in \mathfrak{S}_2$, and then $\sigma_3 \perp \sigma_1$ or $\sigma_3 \perp \sigma_2$. If $\sigma_3 \perp \sigma_1$, we have $\{\sigma_3\} \subseteq \{\sigma_1, \sigma_2\}^{\perp\perp} \cap \{\sigma_1\}^{\perp} = (\{\sigma_1\} \vee \{\sigma_2\}) \cap \{\sigma_1\}^{\perp} = \{\sigma_2\}$. In the same way, if $\sigma_3 \perp \sigma_2$, then $\{\sigma_3\} = \{\sigma_1\}$. As a result, we conclude that $\{\sigma_1, \sigma_2\}^{\perp\perp} = \{\sigma_1, \sigma_2\}$.

This proof is given in the first part of the proof of [68, Lemma 2.9]. From **Axiom 9** we then deduce that $\mathfrak{S}_{\text{pure}}$ is irreducible. □

Corollary 3. $\mathcal{C}(\mathfrak{S}_{\text{pure}})$ is an irreducible ortho-lattice. ■

Proof. Let us suppose $\mathcal{C}(\mathfrak{S}_{\text{pure}})$ is reducible, then there exist a central element A distinct from the bottom element $\emptyset$ and the top element $\mathfrak{S}_{\text{pure}}$. Then any atom is either below A or below $A^{\perp}$, and then any $\sigma \in \mathfrak{S}_{\text{pure}}$ is either in A or in $A^{\perp}$. Hence $\mathfrak{S}_{\text{pure}}$ is reducible. □

Theorem 14. If the condition of the **Axiom 9** is satisfied, the ortho-lattice $\mathcal{C}(\mathfrak{S}_{\text{pure}})$ forms an irreducible Piron's propositional system. ■

Proof. Direct consequence of Theorem 13 and Corollary 3. □

Remark 21. It is important to note that the examples of spaces of states built in Remark 14 are in fact interesting to reconstruct a consistent quantum space of states, because they generically lead to **irreducible** $\mathcal{C}(\mathfrak{S}_{\text{pure}})$. This point can be easily verified. Indeed, if we suppose that it exists σ_1 and σ_2 in $\mathfrak{S}_{\text{pure}}$ such that $\exists \sigma_3 \in \mathfrak{S}_{\text{pure}} \mid (\sigma_3 \neq \sigma_1, \quad \sigma_3 \neq \sigma_2, \text{ and } \sigma_3 \in \{\sigma_1 \sqcap_{\mathfrak{S}} \sigma_2\})$, then the sub-graph of $\mathfrak{L}$ formed by the elements $\sigma_1, \sigma_2, \sigma_3, (\sigma_1 \sqcap_{\mathfrak{S}} \sigma_2), \top_{\mathfrak{L}}$ is a 'diamond sub-graph' of $\mathfrak{L}$, which is totally compatible with the modular character of $\mathfrak{L}$.

4 Symmetries

Let us consider two observers, O_1 and O_2 , who wish to formalize 'transactions' concerning their experimental results about the system.

Notion 49. Observer O_1 has chosen a preparation process, $p_1 \in \mathfrak{P}^{(O_1)}$, and intends to describe it to observer O_2 . Observer O_2 is able to interpret the macroscopic data defining p_1 in terms of the elements of $\mathfrak{P}^{(O_2)}$ using a map $f_{(12)} : \mathfrak{P}^{(O_1)} \rightarrow \mathfrak{P}^{(O_2)}$ (i.e., O_2 knows how to identify a preparation procedure $f_{(12)}(p_1)$ corresponding to any p_1).

Observer O_2 has chosen a yes/no test $t_2 \in \mathfrak{T}^{(O_2)}$ and intends to address the corresponding question to O_1 . Observer O_1 is able to interpret the macroscopic data defining t_2 in terms of the elements of $\mathfrak{T}^{(O_1)}$ using a map $f^{(21)} : \mathfrak{T}^{(O_2)} \rightarrow \mathfrak{T}^{(O_1)}$ (i.e., O_1 knows how to fix a test $f^{(21)}(t_2)$ corresponding to any t_2).

The pair of maps $(f_{(12)}, f^{(21)})$ where $f_{(12)} \in \mathfrak{P}^{(O_1)} \rightarrow \mathfrak{P}^{(O_2)}$ and $f^{(21)} : \mathfrak{T}^{(O_2)} \rightarrow \mathfrak{T}^{(O_1)}$ defines a

dictionary formalizing the transaction from O_1 to O_2 .

The first task these observers want to accomplish is to confront their knowledge, i.e., to compare their 'statements' about the system.

Notion 50. As soon as the transaction is formalized using a dictionary, the two observers can formulate their statements and each confront them with the statements of the other. First, observer O_1 can interpret the macroscopic data defining t_2 using the map $f^{(21)}$. Then, he produces the statement $\epsilon^{(o_1)}(p_1, f^{(21)}(t_2))$ concerning the results of this test on the chosen samples.

Secondly, observer O_2 can interpret the macroscopic data defining p_1 using the map $f_{(12)}$. Then, observer O_2 pronounces her statement $\epsilon^{(o_2)}(f_{(12)}(p_1), t_2)$ concerning the results of test t_2 on the correspondingly prepared samples.

The two observers, O_1 and O_2 , are said to *agree about their statements* iff

$$\forall p_1 \in \mathfrak{P}^{(o_1)}, \forall t_2 \in \mathfrak{T}^{(o_2)}, \quad \epsilon^{(o_2)}(f_{(12)}(p_1), t_2) = \epsilon^{(o_1)}(p_1, f^{(21)}(t_2)), \quad (172)$$

i.e., iff the adjoint pair $(f_{(12)}, f^{(21)})$ defines a morphism of Chu spaces from $(\mathfrak{P}^{(o_1)}, \mathfrak{T}^{(o_1)}, \epsilon^{(o_1)})$ to $(\mathfrak{P}^{(o_2)}, \mathfrak{T}^{(o_2)}, \epsilon^{(o_2)})$ [61].

Lemma 40. If the dictionary $(f_{(12)}, f^{(21)})$ satisfies property (172) (i.e., the adjoint pair defines a Chu morphism from $(\mathfrak{P}^{(o_1)}, \mathfrak{T}^{(o_1)}, \epsilon^{(o_1)})$ to $(\mathfrak{P}^{(o_2)}, \mathfrak{T}^{(o_2)}, \epsilon^{(o_2)})$) we have immediately

$$\forall p, p' \in \mathfrak{P}^{(o_1)}, \quad (p \sim_{\mathfrak{P}^{(o_1)}} p') \Rightarrow (f_{(12)}(p) \sim_{\mathfrak{P}^{(o_2)}} f_{(12)}(p')) \quad (173)$$

■

Proof. $p \sim_{\mathfrak{P}^{(o_1)}} p'$ implies, in particular, $\epsilon^{(o_1)}(p, f^{(21)}(t_2)) = \epsilon^{(o_1)}(p', f^{(21)}(t_2))$ for any $t_2 \in \mathfrak{T}^{(o_2)}$. Using property (172), we obtain $\epsilon^{(o_1)}(f_{(12)}(p), t_2) = \epsilon^{(o_1)}(f_{(12)}(p'), t_2)$ for any $t_2 \in \mathfrak{T}^{(o_2)}$, i.e., $f_{(12)}(p) \sim_{\mathfrak{P}^{(o_2)}} f_{(12)}(p')$. □

Notion 51. In order for observers O_1 and O_2 to be in *complete agreement* about the system (once they do agree about their statements), it is necessary for them to be unable to distinguish the outcomes of the control tests, realized to confirm (or not) their statements.

Firstly, the measurement realized by observer O_2 is given by $(f_{(12)}(p_1) \bullet t_2)$.

Secondly, observer O_2 interprets, using $f_{(12)}$, the measurement $(p_1 \bullet f^{(21)}(t_2))$ realized by observer O_1 .

In other words, it is necessary for the dictionary $(f_{(12)}, f^{(21)})$ also to satisfy the following property

$$\forall p_1 \in \mathfrak{P}^{(o_1)}, \forall t_2 \in \mathfrak{T}^{(o_2)}, \quad f_{(12)}(p_1) \bullet t_2 \sim_{\mathfrak{S}^{(o_2)}} f_{(12)}(p_1 \bullet f^{(21)}(t_2)), \quad (174)$$

Notion 52. If the dictionary $(f_{(12)}, f^{(21)})$ satisfies properties (172) and (174), as well as property (175) below

$$\forall t, t' \in \mathfrak{T}^{(o_2)}, \quad f^{(21)}(t \bullet t') = f^{(21)}(t) \bullet f^{(21)}(t'), \quad (175)$$

and properties (176) and (177) below

$$\forall \mathfrak{p}, \mathfrak{p}' \in \mathfrak{P}^{(O_1)}, (f_{(12)}(\mathfrak{p}) \sim_{\mathfrak{P}^{(O_2)}} f_{(12)}(\mathfrak{p}')) \Rightarrow (\mathfrak{p} \sim_{\mathfrak{P}^{(O_1)}} \mathfrak{p}'), \quad (176)$$

$$f^{(21)} \text{ surjective}, \quad (177)$$

then this dictionary is said to *relate by a symmetry* $(\mathfrak{P}^{(O_1)}, \mathfrak{T}^{(O_1)}, \mathfrak{e}^{(O_1)})$ to $(\mathfrak{P}^{(O_2)}, \mathfrak{T}^{(O_2)}, \mathfrak{e}^{(O_2)})$. This fact will be denoted

$$(f_{(12)}, f^{(21)}) \in \mathbf{Sym}[(\mathfrak{P}^{(O_1)}, \mathfrak{T}^{(O_1)}, \mathfrak{e}^{(O_1)}) \rightarrow (\mathfrak{P}^{(O_2)}, \mathfrak{T}^{(O_2)}, \mathfrak{e}^{(O_2)})]. \quad (178)$$

Remark 22. We note that the axiom (175) has been designed to preserve the associativity property of the succession rule. Indeed, for any $\mathfrak{p} \in \mathfrak{P}^{(O_1)}$, and any $\mathfrak{t}, \mathfrak{t}' \in \mathfrak{T}^{(O_2)}$, we have $f_{(12)}(\mathfrak{p}) \cdot (\mathfrak{t} \cdot \mathfrak{t}') = (f_{(12)}(\mathfrak{p}) \cdot \mathfrak{t}) \cdot \mathfrak{t}' = f_{(12)}(\mathfrak{p} \cdot f^{(21)}(\mathfrak{t})) \cdot \mathfrak{t}' = f_{(12)}((\mathfrak{p} \cdot f^{(21)}(\mathfrak{t})) \cdot f^{(21)}(\mathfrak{t}')) = f_{(12)}(\mathfrak{p} \cdot (f^{(21)}(\mathfrak{t}) \cdot f^{(21)}(\mathfrak{t}')) = f_{(12)}(\mathfrak{p} \cdot f^{(21)}(\mathfrak{t} \cdot \mathfrak{t}'))$.

Remark 23. Properties (176) and (177) have been introduced in order to be able to derive Theorem 16.

Theorem 15. [Composition of symmetries]

$$\left. \begin{aligned} (f_{(12)}, f^{(21)}) &\in \mathbf{Sym}[(\mathfrak{P}^{(O_1)}, \mathfrak{T}^{(O_1)}, \mathfrak{e}^{(O_1)}) \rightarrow (\mathfrak{P}^{(O_2)}, \mathfrak{T}^{(O_2)}, \mathfrak{e}^{(O_2)})] \\ (g_{(23)}, g^{(32)}) &\in \mathbf{Sym}[(\mathfrak{P}^{(O_2)}, \mathfrak{T}^{(O_2)}, \mathfrak{e}^{(O_2)}) \rightarrow (\mathfrak{P}^{(O_3)}, \mathfrak{T}^{(O_3)}, \mathfrak{e}^{(O_3)})] \end{aligned} \right\} \\ \Rightarrow (g_{(23)} \circ f_{(12)}, f^{(21)} \circ g^{(32)}) \in \mathbf{Sym}[(\mathfrak{P}^{(O_1)}, \mathfrak{T}^{(O_1)}, \mathfrak{e}^{(O_1)}) \rightarrow (\mathfrak{P}^{(O_3)}, \mathfrak{T}^{(O_3)}, \mathfrak{e}^{(O_3)})]. \quad (179)$$

■

Proof. We firstly note that

$$\forall \mathfrak{p}_1 \in \mathfrak{P}^{(O_1)}, \forall \mathfrak{t}_3 \in \mathfrak{T}^{(O_3)}, \mathfrak{e}^{(O_3)}(g_{(23)} \circ f_{(12)}(\mathfrak{p}_1), \mathfrak{t}_3) = \mathfrak{e}^{(O_2)}(f_{(12)}(\mathfrak{p}_1), g^{(32)}(\mathfrak{t}_3)) = \mathfrak{e}^{(O_1)}(\mathfrak{p}_1, f^{(21)} \circ g^{(32)}(\mathfrak{t}_3)). \quad (180)$$

Secondly, we have

$$\forall \mathfrak{p}_1 \in \mathfrak{P}^{(O_1)}, \forall \mathfrak{t}_3 \in \mathfrak{T}^{(O_3)}, (g_{(23)} \circ f_{(12)})(\mathfrak{p}_1) \cdot \mathfrak{t}_3 = g_{(23)}(f_{(12)}(\mathfrak{p}_1) \cdot g^{(32)}(\mathfrak{t}_3)) = (g_{(23)} \circ f_{(12)})(\mathfrak{p}_1) \cdot (f^{(21)} \circ g^{(32)})(\mathfrak{t}_3). \quad (181)$$

The properties (175), (176), and (177) are trivially preserved by composition. □

Lemma 41. The dictionary $(h_{(12)}, h^{(21)})$ defined by

$$\begin{aligned} h_{(12)} : \mathfrak{P} &\rightarrow \mathfrak{S} & h^{(21)} : \mathfrak{T} &\rightarrow \mathfrak{T} \\ \mathfrak{p} &\mapsto [\mathfrak{p}] & \mathfrak{t} &\mapsto \mathfrak{t} \end{aligned} \quad (182)$$

satisfies properties (172), (174), (175), (176) and (177). In other words, this dictionary relates by a symmetry $(\mathfrak{P}, \mathfrak{T}, \mathfrak{e})$ to $(\mathfrak{S}, \mathfrak{T}, \tilde{\mathfrak{e}})$.

$$(h_{(12)}, h^{(21)}) \in \mathbf{Sym}[(\mathfrak{P}, \mathfrak{T}, \mathfrak{e}) \rightarrow (\mathfrak{S}, \mathfrak{T}, \tilde{\mathfrak{e}})] \quad (183)$$

■

Proof. Property (172) is a direct consequence of the definition (11) of $\tilde{\mathfrak{e}}$.

Property (174) is a direct consequence of property (74).

Property (175) is tautologically verified.

Property (176) relies on the definition of the map $[\cdot]$.

Property (177) is trivial. □

As a consequence of Lemma 41 and Theorem 15, we can define the following notion.

Notion 53. For any dictionary $(f_{(12)}, f^{(21)}) \in \mathbf{Sym}[(\mathfrak{P}^{(o_1)}, \mathfrak{T}^{(o_1)}, \mathfrak{e}^{(o_1)}) \rightarrow (\mathfrak{P}^{(o_2)}, \mathfrak{T}^{(o_2)}, \mathfrak{e}^{(o_2)})]$, we will associate the dictionary $(\tilde{f}_{(12)}, f^{(21)})$ defined by

$$\forall \mathfrak{p} \in \mathfrak{P}^{(o_1)}, \quad \tilde{f}_{(12)}(\lceil \mathfrak{p} \rceil_1) := \lceil f_{(12)}(\mathfrak{p}) \rceil_2. \quad (184)$$

We have

$$(\tilde{f}_{(12)}, f^{(21)}) \in \mathbf{Sym}[(\mathfrak{S}^{(o_1)}, \mathfrak{T}^{(o_1)}, \tilde{\mathfrak{e}}^{(o_1)}) \rightarrow (\mathfrak{S}^{(o_2)}, \mathfrak{T}^{(o_2)}, \tilde{\mathfrak{e}}^{(o_2)})]. \quad (185)$$

Explicitly, $(\tilde{f}_{(12)}, f^{(21)})$ has to satisfy the following requirements :

$$\forall \sigma_1 \in \mathfrak{S}^{(o_1)}, \forall t_2 \in \mathfrak{T}^{(o_2)}, \quad \tilde{\mathfrak{e}}_{t_2}^{(o_2)}(\tilde{f}_{(12)}(\sigma_1)) = \tilde{\mathfrak{e}}_{f^{(21)}(t_2)}^{(o_1)}(\sigma_1), \quad (186)$$

$$\forall \sigma_1 \in \mathfrak{S}^{(o_1)}, \forall t_2 \in \mathfrak{T}^{(o_2)}, \quad \tilde{f}_{(12)}(\sigma_1) \cdot t_2 = \tilde{f}_{(12)}(\sigma_1 \cdot f^{(21)}(t_2)), \quad (187)$$

$$\forall t, t' \in \mathfrak{T}^{(o_2)}, \quad f^{(21)}(t \cdot t') = f^{(21)}(t) \cdot f^{(21)}(t'), \quad (188)$$

$$\tilde{f}_{(12)} \text{ injective} \quad (189)$$

$$f^{(21)} \text{ surjective} \quad (190)$$

Lemma 42. $\tilde{f}_{(12)}$ is as map satisfying

$$\tilde{f}_{(12)} \in [\mathfrak{S}^{(o_1)} \rightarrow \mathfrak{S}^{(o_2)}]_{\sqcap}^{\wedge} \quad \text{and} \quad \tilde{f}_{(12)}(\perp_{\mathfrak{S}^{(o_1)}}) = \perp_{\mathfrak{S}^{(o_2)}}. \quad (191)$$

In particular, $\tilde{f}_{(12)}$ is order-preserving. ■

Proof.

$$\forall t_2 \in \mathfrak{T}^{(o_2)}, \quad \tilde{\mathfrak{e}}_{t_2}^{(o_2)}(\tilde{f}_{(12)}(\perp_{\mathfrak{S}^{(o_1)}})) = \tilde{\mathfrak{e}}_{f^{(21)}(t_2)}^{(o_1)}(\perp_{\mathfrak{S}^{(o_1)}}) = \perp \quad \Rightarrow \quad \tilde{f}_{(12)}(\perp_{\mathfrak{S}^{(o_1)}}) = \perp_{\mathfrak{S}^{(o_2)}}. \quad (192)$$

Secondly, from Theorem 5, we know that $\forall t \in \mathfrak{T}^{(o_2)}, \tilde{\mathfrak{e}}_t^{(o_2)}$ is an order-preserving and Lawson-continuous map (i.e., $\tilde{\mathfrak{e}}_t^{(o_1)} \in [\mathfrak{S}^{(o_1)} \rightarrow \mathfrak{B}]_{\sqcap}^{\wedge}$). We can then deduce that $\tilde{f}_{(12)}$ is an order-preserving and Lawson-continuous map. Let us first check the Scott continuity of $\tilde{f}_{(12)}$:

$$\begin{aligned} \forall t \in \mathfrak{T}^{(o_2)}, \forall \mathfrak{C} \subseteq_{Chain} \mathfrak{S}^{(o_1)}, \quad \tilde{\mathfrak{e}}_t^{(o_2)}(\tilde{f}_{(12)}(\bigsqcup_{\mathfrak{S}^{(o_1)}} \mathfrak{C})) &= \tilde{\mathfrak{e}}_{f^{(21)}(t)}^{(o_1)}(\bigsqcup_{\mathfrak{S}^{(o_1)}} \mathfrak{C}) && \text{from eq. (186)} \\ &= \bigsqcup_{\sigma \in \mathfrak{C}} \tilde{\mathfrak{e}}_{f^{(21)}(t)}^{(o_1)}(\sigma) && \text{from eq. (32)} \\ &= \bigsqcup_{\sigma \in \mathfrak{C}} \tilde{\mathfrak{e}}_t^{(o_2)}(f_{(12)}(\sigma)) && \text{from eq. (186)} \\ &= \tilde{\mathfrak{e}}_t^{(o_2)}(\bigsqcup_{\sigma \in \mathfrak{C}} f_{(12)}(\sigma)) && \text{from eq. (32)}. \end{aligned} \quad (193)$$

Then, using Lemma 1, we conclude that $\tilde{f}_{(12)}$ is Scott continuous (and in particular order preserving)

$$\forall \mathfrak{C} \subseteq_{Chain} \mathfrak{S}^{(o_1)}, \quad \tilde{f}_{(12)}(\bigsqcup_{\mathfrak{S}^{(o_1)}} \mathfrak{C}) = \bigsqcup_{\sigma \in \mathfrak{C}} \tilde{f}_{(12)}(\sigma). \quad (194)$$

We can prove property (195) below, using properties (57) and (186), along the same line of proof :

$$\forall \Omega \subseteq \mathfrak{S}^{(o_1)}, \quad \tilde{f}_{(12)}(\prod_{\mathfrak{S}^{(o_1)}} \Omega) = \prod_{\sigma \in \Omega} \tilde{f}_{(12)}(\sigma). \quad (195)$$

□

Theorem 16. [Preservation of the class of minimally disturbing measurements by symmetry]

Let $(\tilde{f}_{(12)}, f^{(21)})$ be a dictionary relating by symmetry $(\mathfrak{S}^{(O_1)}, \mathfrak{T}^{(O_1)}, \tilde{\mathfrak{e}}^{(O_1)})$ to $(\mathfrak{S}^{(O_2)}, \mathfrak{T}^{(O_2)}, \tilde{\mathfrak{e}}^{(O_2)})$, and let $t \in \mathfrak{T}_{IFKM}^{(O_2)}$ be a yes/no test leading to an ideal first-kind measurement, therefore $f^{(21)}(t)$ is a yes/no test that leads to an ideal first-kind measurement, i.e., $f^{(21)}(t) \in \mathfrak{T}_{IFKM}^{(O_1)}$. In other words,

$$\forall (\tilde{f}_{(12)}, f^{(21)}) \in \mathbf{Sym} [(\mathfrak{S}^{(O_1)}, \mathfrak{T}^{(O_1)}, \tilde{\mathfrak{e}}^{(O_1)}) \rightarrow (\mathfrak{S}^{(O_2)}, \mathfrak{T}^{(O_2)}, \tilde{\mathfrak{e}}^{(O_2)})], \quad t \in \mathfrak{T}_{IFKM}^{(O_2)} \Rightarrow f^{(21)}(t) \in \mathfrak{T}_{IFKM}^{(O_1)} \quad (196)$$

$$\text{i.e.} \quad (\tilde{f}_{(12)}, f^{(21)}) \in \mathbf{Sym} [(\mathfrak{S}^{(O_1)}, \mathfrak{T}_{min}^{(O_1)}, \tilde{\mathfrak{e}}^{(O_1)}) \rightarrow (\mathfrak{S}^{(O_2)}, \mathfrak{T}_{min}^{(O_2)}, \tilde{\mathfrak{e}}^{(O_2)})]. \quad (197)$$

■

Proof. Let us first prove that the symmetry preserves the Scott-continuity property of measurement maps.

$$\begin{aligned} \forall \mathfrak{C} \subseteq_{Chain} \mathfrak{Q}_{f^{(21)}(t)}^{(O_1)}, \quad \tilde{f}_{(12)}((\bigsqcup_{\mathfrak{S}^{(O_1)}} \mathfrak{C}) \cdot f^{(21)}(t)) &= \tilde{f}_{(12)}(\bigsqcup_{\mathfrak{S}^{(O_1)}} \mathfrak{C}) \cdot t && \text{from eq. (187)} \\ &= (\bigsqcup_{\sigma \in \mathfrak{C}} \tilde{f}_{(12)}(\sigma)) \cdot t && \text{from eq. (194)} \\ &= \bigsqcup_{\sigma \in \mathfrak{C}} (\tilde{f}_{(12)}(\sigma) \cdot t) && \text{from eq. (76)} \\ &= \bigsqcup_{\sigma \in \mathfrak{C}} \tilde{f}_{(12)}(\sigma \cdot f^{(21)}(t)) && \text{from eq. (187)} \\ &= \tilde{f}_{(12)}(\bigsqcup_{\sigma \in \mathfrak{C}} (\sigma \cdot f^{(21)}(t))) && \text{from eq. (194)} \end{aligned} \quad (198)$$

We now use the injectivity property (189) to confirm the preservation of Scott continuity of measurement maps by symmetries:

$$\forall \mathfrak{C} \subseteq_{Chain} \mathfrak{Q}_{f^{(21)}(t)}^{(O_1)}, \quad (\bigsqcup_{\mathfrak{S}^{(O_1)}} \mathfrak{C}) \cdot f^{(21)}(t) = \bigsqcup_{\sigma \in \mathfrak{C}} (\sigma \cdot f^{(21)}(t)) \quad (199)$$

The second continuity property is proved along the same line of proof, using properties (187), (195), (101), and (189)

$$\forall \mathfrak{F} \subseteq_{Fil} \mathfrak{Q}_{f^{(21)}(t)}^{(O_1)}, \quad (\bigcap_{\mathfrak{S}^{(O_1)}} \mathfrak{F}) \cdot f^{(21)}(t) = \bigcap_{\sigma \in \mathfrak{F}} (\sigma \cdot f^{(21)}(t)). \quad (200)$$

Secondly, let us prove that $f^{(21)}(t) \in \mathfrak{T}_{FKM}^{(O_1)}$. Let us consider any $\sigma \in \mathfrak{S}^{(O_1)}$, and $t \in \mathfrak{T}_{FKM}^{(O_2)}$. The preservation of equation (79 (i)) is proved as follows

$$\begin{aligned} \tilde{\mathfrak{e}}_{f^{(21)}(t)}^{(O_1)}(\sigma) \leq Y &\Rightarrow \tilde{\mathfrak{e}}_t^{(O_1)}(\tilde{f}_{(12)}(\sigma)) \leq Y && \text{from eq. (186)} \\ &\Rightarrow \tilde{\mathfrak{e}}_t^{(O_1)}(\tilde{f}_{(12)}(\sigma) \cdot t) = Y && \text{from } t \in \mathfrak{T}_{FKM}^{(O_2)} \\ &\Rightarrow \tilde{\mathfrak{e}}_t^{(O_1)}(\tilde{f}_{(12)}(\sigma \cdot f^{(21)}(t))) = Y && \text{from eq. (187)} \\ &\Rightarrow \tilde{\mathfrak{e}}_{f^{(21)}(t)}^{(O_1)}(\sigma \cdot f^{(21)}(t)) = Y && \text{from eq. (186)} \end{aligned} \quad (201)$$

The preservation of equation (79 (ii)) is proved as follows

$$\begin{aligned} \tilde{\mathfrak{e}}_{f^{(21)}(t)}^{(O_1)}(\sigma) = Y &\Rightarrow \tilde{\mathfrak{e}}_t^{(O_1)}(\tilde{f}_{(12)}(\sigma)) = Y && \text{from eq. (186)} \\ &\Rightarrow \tilde{f}_{(12)}(\sigma) \cdot t = \tilde{f}_{(12)}(\sigma) && \text{from } t \in \mathfrak{T}_{FKM}^{(O_2)} \\ &\Rightarrow \tilde{f}_{(12)}(\sigma \cdot f^{(21)}(t)) = \tilde{f}_{(12)}(\sigma) && \text{from eq. (187)} \\ &\Rightarrow \sigma \cdot f^{(21)}(t) = \sigma && \text{from eq. (189)} \end{aligned} \quad (202)$$

Thirdly, it remains to be shown that $f^{(21)}(t) \in \mathfrak{T}_{Ideal}^{(O_1)}$. Let us consider $u \in \mathfrak{T}^{(O_1)}$ such that $\overbrace{[f^{(21)}(t)] \sqcup [u]}$. The surjectivity of $f^{(21)}$ (equation (190)) implies that there exists $t' \in \mathfrak{T}^{(O_2)}$ such that $u = f^{(21)}(t')$.

The compatibility relation $\overbrace{[f^{(21)}(t)][f^{(21)}(t')]}^{\text{ }}$ implies the compatibility relation $\overbrace{[t][t']}. Indeed$

$$\begin{aligned} \overbrace{[f^{(21)}(t)][f^{(21)}(t')]} &\Leftrightarrow \exists \sigma \in \mathfrak{S} \mid \tilde{\mathfrak{e}}_{f^{(21)}(t)}^{(o_1)}(\sigma) = \tilde{\mathfrak{e}}_{f^{(21)}(t')}^{(o_1)}(\sigma) = Y \\ &\Rightarrow \exists \sigma' = \tilde{f}_{(12)}(\sigma) \in \mathfrak{S} \mid \tilde{\mathfrak{e}}_t^{(o_1)}(\sigma') = \tilde{\mathfrak{e}}_{t'}^{(o_1)}(\sigma') = Y \quad \text{from eq. (186)} \\ &\Rightarrow \overbrace{[t][t']}. \end{aligned} \quad (203)$$

Let us now consider any $\sigma \in \mathfrak{S}$ and $t' \in \mathfrak{T}^{(o_2)}$ such that $\overbrace{[f^{(21)}(t)][f^{(21)}(t')]}^{\text{ }}$, we then have

$$\begin{aligned} (\tilde{\mathfrak{e}}_{f^{(21)}(t)}^{(o_1)}(\sigma) \leq Y \text{ and } \tilde{\mathfrak{e}}_{f^{(21)}(t')}^{(o_1)}(\sigma) = Y) &\Rightarrow (\tilde{\mathfrak{e}}_t^{(o_1)}(\tilde{f}_{(12)}(\sigma)) \leq Y \text{ and } \tilde{\mathfrak{e}}_{t'}^{(o_1)}(\tilde{f}_{(12)}(\sigma)) = Y) \quad \text{from eq. (186)} \\ &\Rightarrow (\tilde{\mathfrak{e}}_{t'}^{(o_1)}(\tilde{f}_{(12)}(\sigma).t) = Y) \quad \text{from } t \in \mathfrak{T}_{Ideal}^{(o_2)} \text{ and } \widehat{tt'} \\ &\Rightarrow (\tilde{\mathfrak{e}}_{t'}^{(o_1)}(\tilde{f}_{(12)}(\sigma.f^{(21)}(t))) = Y) \quad \text{from eq. (187)} \\ &\Rightarrow (\tilde{\mathfrak{e}}_{f^{(21)}(t')}^{(o_1)}(\sigma.f^{(21)}(t)) = Y) \quad \text{from eq. (186)} \end{aligned} \quad (204)$$

□

Lemma 43. $\tilde{f}_{(12)}$ is surjective. ■

Proof. Let us introduce the following map on $\mathfrak{S}$:

$$\begin{aligned} f_{(21)}^\downarrow : \mathfrak{S}^{(o_2)} &\longrightarrow \mathfrak{S}^{(o_1)} \\ \Sigma &\mapsto f_{(21)}^\downarrow(\Sigma) := \bigcap_{\mathfrak{S}^{(o_1)}} \{ \sigma \mid \tilde{\mathfrak{e}}_{f^{(21)}(t_{(\Sigma, \Sigma^*)})}^{(o_1)}(\sigma) = Y \}, \end{aligned} \quad (205)$$

where $t_{(\Sigma, \Sigma^*)}$ designates the unique discriminating yes/no test in $\mathfrak{U}$ with $\Sigma_{t_{(\Sigma, \Sigma^*)}} = \Sigma$.

We have

$$\begin{aligned} \forall \Sigma \in \mathfrak{S}^{(o_2)}, \Sigma' \in \mathfrak{S}^{(o_1)}, \quad f_{(21)}^\downarrow(\Sigma) \sqsubseteq_{\mathfrak{S}^{(o_1)}} \Sigma' &\Leftrightarrow \tilde{\mathfrak{e}}_{f^{(21)}(t_{(\Sigma, \Sigma^*)})}^{(o_1)}(\Sigma') = Y \\ &\Leftrightarrow \tilde{\mathfrak{e}}_{t_{(\Sigma, \Sigma^*)}}^{(o_1)}(\tilde{f}_{(12)}(\Sigma')) = Y \\ &\Leftrightarrow \Sigma \sqsubseteq_{\mathfrak{S}^{(o_2)}} \tilde{f}_{(12)}(\Sigma') \end{aligned} \quad (206)$$

$\tilde{f}_{(12)}$ is then the right Galois adjunct of $f_{(21)}^\downarrow$. Then, $\tilde{f}_{(12)}$ is surjective and preserves infima (this last property was already proved as a part of Lemma 42). □

Lemma 44. For any $(\tilde{f}_{(12)}, f^{(21)}) \in \mathbf{Sym} \left[(\mathfrak{S}^{(o_1)}, \mathfrak{T}^{(o_1)}, \tilde{\mathfrak{e}}^{(o_1)}) \rightarrow (\mathfrak{S}^{(o_2)}, \mathfrak{T}^{(o_2)}, \tilde{\mathfrak{e}}^{(o_2)}) \right]$ and $t \in \mathfrak{T}^{(o_2)}$

$$f^{(21)}(\overline{t}) = \overline{f^{(21)}(t)}. \quad (207)$$

As a consequence,

$$\tilde{f}_{(12)}(\Sigma^*) = (\tilde{f}_{(12)}(\Sigma))^*. \quad (208)$$

■

Proof. Using successively two times properties (186) and (4), we obtain : $\forall \sigma \in \mathfrak{S}^{(O_1)}, \forall t \in \mathfrak{T}^{(O_2)}, \tilde{\mathfrak{e}}_{f^{(21)}(\bar{t})}^{(O_1)}(\sigma) = \tilde{\mathfrak{e}}_{\bar{t}}^{(O_2)}(\tilde{f}_{(12)}(\sigma)) = \overline{\tilde{\mathfrak{e}}_{f^{(21)}(t)}^{(O_1)}(\sigma)} = \tilde{\mathfrak{e}}_{f^{(21)}(t)}^{(O_1)}(\sigma)$.

Property (208) is a direct consequence of (153) and (207) : $\forall \sigma \in \mathfrak{S}^{(O_1)}, \forall t \in \mathfrak{T}^{(O_2)}, \tilde{\mathfrak{e}}_t^{(O_1)}(\tilde{f}_{(12)}(\sigma^*)) = \tilde{\mathfrak{e}}_{f^{(21)}(t)}^{(O_1)}(\sigma^*) = \tilde{\mathfrak{e}}_{f^{(21)}(\bar{t})}^{(O_1)}(\sigma) = \tilde{\mathfrak{e}}_{\bar{t}}^{(O_2)}(\tilde{f}_{(12)}(\sigma)) = \tilde{\mathfrak{e}}_t^{(O_1)}((\tilde{f}_{(12)}(\sigma))^*)$. $\square$

Before studying the preservation of orthogonality, we will firstly note some elementary results. First of all, we have

$$\forall t_{(\Sigma, \Sigma^*)} \in \tilde{\mathfrak{T}}_{disc}^{(O_2)}, \quad f^{(21)}(t_{(\Sigma, \Sigma^*)}) \in \tilde{\mathfrak{T}}_{disc}^{(O_1)}. \quad (209)$$

Indeed, let us introduce $\kappa, \kappa' \in \mathfrak{S}^{(O_1)}$ as follows

$$\kappa := \bigcap_{\mathfrak{S}} \{ \sigma \mid \tilde{\mathfrak{e}}_{f^{(21)}(t_{(\Sigma, \Sigma^*)})}^{(O_1)}(\sigma) = \mathbf{Y} \} \quad \text{and} \quad \kappa' := \bigcap_{\mathfrak{S}} \{ \sigma \mid \tilde{\mathfrak{e}}_{f^{(21)}(t_{(\Sigma, \Sigma^*)})}^{(O_1)}(\sigma) = \mathbf{N} \} \quad (210)$$

Using the surjectivity of $\tilde{f}_{(12)}$ (Lemma 43) and the fact that $\tilde{f}_{(12)}$ preserves infima (Lemma 42), we obtain

$$\tilde{f}_{(12)}(\kappa) = \Sigma \quad \text{and} \quad \tilde{f}_{(12)}(\kappa') = \Sigma^*. \quad (211)$$

$t_{(\Sigma, \Sigma^*)} \in \tilde{\mathfrak{T}}_{disc}^{(O_2)}$ means that $(\Sigma \bowtie_{\mathfrak{S}} \Sigma^*)$ and $\widetilde{\Sigma \Sigma'}^{\mathfrak{S}}$. Using the bijectivity of $\tilde{f}_{(12)}$ (Lemma 43 and property (189)) and its order-preserving property (Lemma 42), with the explicit definition of the relations $\cdot \bowtie_{\mathfrak{S}} \cdot$ and $\cdot \prec^{\mathfrak{S}} \cdot$, we deduce that $\kappa \bowtie_{\mathfrak{S}} \kappa'$ and $\widetilde{\kappa \kappa'}^{\mathfrak{S}}$. As a result, $f^{(21)}(t_{(\Sigma, \Sigma^*)}) \in \tilde{\mathfrak{T}}_{disc}^{(O_1)}$. Moreover,

$$\kappa = \tilde{f}_{(12)}^{-1}(\Sigma) \quad \text{and} \quad \kappa' = \tilde{f}_{(12)}^{-1}(\Sigma^*) = \kappa^*. \quad (212)$$

As a result, we have

$$f^{(21)}(t_{(\Sigma, \Sigma^*)}) = t_{(\tilde{f}_{(12)}^{-1}(\Sigma), \tilde{f}_{(12)}^{-1}(\Sigma^*))}. \quad (213)$$

In the following, we will impose the following property of preservation of schemes :

$$f^{(21)}(\mathfrak{U}^{(O_2)}) = \mathfrak{U}^{(O_1)}. \quad (214)$$

We now check the following fundamental property

Theorem 17. [Preservation of the orthogonality of states]

$$\forall (\tilde{f}_{(12)}, f^{(21)}) \in \mathbf{Sym} [(\mathfrak{S}^{(O_1)}, \mathfrak{T}^{(O_1)}, \tilde{\mathfrak{e}}^{(O_1)}) \rightarrow (\mathfrak{S}^{(O_2)}, \mathfrak{T}^{(O_2)}, \tilde{\mathfrak{e}}^{(O_2)})], \forall \sigma_1, \sigma_2 \in \mathfrak{S}^{(O_1)}, \quad \sigma_1 \perp \sigma_2 \Rightarrow \tilde{f}_{(12)}(\sigma_1) \perp \tilde{f}_{(12)}(\sigma_2). \quad (215)$$

■

Proof.

$$\begin{aligned} \sigma_1 \perp \sigma_2 &\Leftrightarrow \exists t \in \mathfrak{U}^{(O_1)} \mid (\tilde{\mathfrak{e}}_t(\sigma_1) = \mathbf{Y} \quad \text{and} \quad \tilde{\mathfrak{e}}_{\bar{t}}(\sigma_2) = \mathbf{Y}) \\ &\Leftrightarrow \exists u \in \mathfrak{U}^{(O_2)} \mid (\tilde{\mathfrak{e}}_{f^{(21)}(u)}(\sigma_1) = \mathbf{Y} \quad \text{and} \quad \tilde{\mathfrak{e}}_{\overline{f^{(21)}(u)}}(\sigma_2) = \mathbf{Y}) \quad \text{from eq. (190) and property (214)} \\ &\Leftrightarrow \exists u \in \mathfrak{U}^{(O_2)} \mid (\tilde{\mathfrak{e}}_u(\tilde{f}_{(12)}(\sigma_1)) = \mathbf{Y} \quad \text{and} \quad \tilde{\mathfrak{e}}_{\bar{u}}(\tilde{f}_{(12)}(\sigma_2)) = \mathbf{Y}) \quad \text{from eq. (186) and Lemma 44} \\ &\Leftrightarrow \tilde{f}_{(12)}(\sigma_1) \perp \tilde{f}_{(12)}(\sigma_2). \end{aligned} \quad (216)$$

□

Lemma 45.

$$\begin{aligned} \varphi_L : \mathcal{C}(\mathfrak{S}_{pure}) &\longrightarrow \mathfrak{S} & \varphi_D : \mathfrak{S} &\longrightarrow \mathcal{C}(\mathfrak{S}_{pure}) \\ \mathfrak{c} &\mapsto \bigcap_{\mathfrak{S}} \mathfrak{c} & \sigma &\mapsto \underline{\{\sigma\}} \end{aligned} \quad (217)$$

satisfy the following Galois adjunction relation

$$\forall \mathfrak{c} \in \mathcal{C}(\mathfrak{S}_{pure}), \forall \sigma \in \mathfrak{S}, \quad (\varphi_L(\mathfrak{c}) \sqsubseteq_{\mathfrak{S}} \sigma) \Leftrightarrow (\mathfrak{c} \subseteq \varphi_D(\sigma)), \quad (218)$$

and $\varphi_L \circ \varphi_D = id_{\mathfrak{S}}$. ■

Notion 54. Using the elements of a symmetry $(\tilde{f}_{(12)}, f^{(21)}) \in \mathbf{Sym}[(\mathfrak{S}^{(O_1)}, \mathfrak{T}^{(O_1)}, \tilde{\mathfrak{e}}^{(O_1)}) \rightarrow (\mathfrak{S}^{(O_2)}, \mathfrak{T}^{(O_2)}, \tilde{\mathfrak{e}}^{(O_2)})]$, we define the following maps

$$\begin{aligned} \underline{\tilde{f}_{(12)}} : \mathcal{C}(\mathfrak{S}_{pure}^{(O_1)}) &\longrightarrow \mathcal{C}(\mathfrak{S}_{pure}^{(O_2)}) & \underline{f^{(21)}} : \mathcal{C}(\mathfrak{S}_{pure}^{(O_2)}) &\longrightarrow \mathcal{C}(\mathfrak{S}_{pure}^{(O_1)}) \\ \mathfrak{c} &\mapsto \bigvee \{ \tilde{f}_{(12)}(\sigma) \mid \sigma \in \mathfrak{c} \} & \mathfrak{c} &\mapsto (\varphi_D^{(O_1)} \circ f_{(21)}^\downarrow \circ \varphi_L^{(O_2)})(\mathfrak{c}) \end{aligned} \quad (219)$$

Note that $f_{(21)}^\downarrow$ has been defined in equation (205).

Lemma 46. $(\underline{\tilde{f}_{(12)}}, \underline{f^{(21)}})$ forms a Galois connection. ■

Proof. For any $\mathfrak{c}_1 \in \mathcal{C}(\mathfrak{S}_{pure}^{(O_1)})$ and $\mathfrak{c}_2 \in \mathcal{C}(\mathfrak{S}_{pure}^{(O_2)})$, we have

$$\begin{aligned} (\underline{\tilde{f}_{(12)}}(\mathfrak{c}_1) \subseteq \mathfrak{c}_2) &\Leftrightarrow (\forall \sigma \in \mathfrak{c}_1, \varphi_L^{(O_2)}(\mathfrak{c}_2) \sqsubseteq_{\mathfrak{S}^{(O_2)}} \tilde{f}_{(12)}(\sigma)) \\ &\Leftrightarrow (\forall \sigma \in \mathfrak{c}_1, \tilde{\mathfrak{e}}_{(\varphi_L^{(O_2)}(\mathfrak{c}_2), (\varphi_L^{(O_2)}(\mathfrak{c}_2))^*)}^{(O_2)}(\tilde{f}_{(12)}(\sigma)) = \mathbf{Y}) \\ &\Leftrightarrow (\forall \sigma \in \mathfrak{c}_1, \tilde{\mathfrak{e}}_{(\varphi_L^{(O_2)}(\mathfrak{c}_2), (\varphi_L^{(O_2)}(\mathfrak{c}_2))^*)}^{(O_1)}(\sigma) = \mathbf{Y}) \\ &\Leftrightarrow (\forall \sigma \in \mathfrak{c}_1, \{\sigma\} \subseteq \left\{ \bigcap_{\mathfrak{S}^{(O_1)}} \{\sigma' \in \mathfrak{S}_{pure}^{(O_1)} \mid \tilde{\mathfrak{e}}_{(\varphi_L^{(O_2)}(\mathfrak{c}_2), (\varphi_L^{(O_2)}(\mathfrak{c}_2))^*)}^{(O_1)}(\sigma') = \mathbf{Y}\} \right\}) \\ &\Leftrightarrow (\forall \sigma \in \mathfrak{c}_1, \{\sigma\} \subseteq (\varphi_D^{(O_1)} \circ f_{(21)}^\downarrow \circ \varphi_L^{(O_2)})(\mathfrak{c}_2)) \\ &\Leftrightarrow (\mathfrak{c}_1 \subseteq \underline{f^{(21)}}(\mathfrak{c}_2)) \end{aligned} \quad (220)$$

□

Theorem 18. $\underline{\tilde{f}_{(12)}}$ is an injective map from $\mathcal{C}(\mathfrak{S}_{pure}^{(O_1)})$ to $\mathcal{C}(\mathfrak{S}_{pure}^{(O_2)})$ preserving suprema, mapping atoms to atoms and preserving orthogonality (i.e. $\forall \mathfrak{c} \in \mathcal{C}(\mathfrak{S}_{pure}^{(O_1)}), \underline{\tilde{f}_{(12)}}(\mathfrak{c}^\perp) = (\underline{\tilde{f}_{(12)}}(\mathfrak{c}))^\perp$).

It is then an ortho-morphism of Hilbert lattice (see [67, Definition 5.12 and Definition 5.17]). ■

Proof. $\underline{\tilde{f}_{(12)}}$ preserves atoms as an extension of $\tilde{f}_{(12)}$ from $\mathfrak{S}_{pure}^{(O_1)}$ to $\mathcal{C}(\mathfrak{S}_{pure}^{(O_1)})$. It is injective and preserves suprema as a left Galois adjunct. It preserves the orthogonality relation according to Theorem 17. □

5 Conclusion

We aim to develop a new axiomatic approach to quantum theory and this article is designed as a first decisive step for this axiomatic program. A precise semantic description of the space of preparations and of the associated 'mixed states' of the system was formulated. This semantic formalism is based on a Chu space construction involving the set of preparations, the set of yes/no tests and an evaluation map with a three-valued target space. The values taken by this map are associated with counterfactual statements of the observer for a given yes/no test and a given prepared sample. The three values are interpreted in a possibilistic perspective, i.e., as 'certainly yes', 'certainly no' and 'maybe'. This domain structure on the target space led to an 'informational' interpretation of the set of preparations. The space of preparations was equipped with a notion of 'mixtures', expressed in terms of the meet operation on this poset. From natural requirements about the inductive definition of states, it appeared that this Inf semi-lattice was also a pointed directed-complete partial order. Then, an 'Information Principle' was introduced in the form of two topological requirements on the space of states. Although new in its form, this principle is very standard in different quantum axiomatic programs. This basic set of axioms was shown to be sufficient to constrain the structure of the space of states to be a 'projective domain'. The space of pure states was then basically identified in terms of maximal elements of this domain.

Then, the relation between yes/no tests and states was studied from two perspectives: using the notion of Chu duality and using the notion of 'measurement'. Adopting the first perspective, the notion of 'properties' of the system was defined and a 'property-state' was identified for any property, as in Piron's construction. The second perspective emphasizes the recursive aspect of preparation processes. In order to identify a subclass of measurement operations corresponding to minimally disturbing measurements, it appeared necessary to clarify the notion of 'compatibility between measurements'. The compatibility between two measurements was defined in terms of the existence of preparations that simultaneously exhibited the two corresponding properties as actual. This notion was used to define 'ideal first-kind measurements' and to characterize them as 'minimally disturbing measurements'. We finally proved the existence of a 'bi-extensional Chu_3 duality' between the space of minimally disturbing yes/no tests and the space of states.

The simultaneous ideal first-kind measurements of compatible properties was then studied and 'Specker's principle' was proved. Using this result, we obtained a 'coherent domain' structure on the space of 'descriptions' formalizing the families of compatible properties used to define a state of the system. An orthogonality relation was then defined on the space of states in terms of the class of discriminating yes/no tests. We achieve the characterization of the domain structure on the space of states by requiring the existence of a scheme of discriminating yes/no tests necessary to the construction of an orthogonality relation on the space of states. In generic examples, this last constraint is equivalent to the existence of an ortho-complementation on the projective domain defining the space of states.

Using the properties of the domain structure established on the space of states, we deduced that the set of ortho-closed subsets of pure states, equipped with the induced orthogonality relation, inherits a structure of Hilbert lattice. This is the first part of our reconstruction theorem.

In Section 4, we explored the properties of Chu morphisms with respect to previous notions. As a central result, a sub-algebra of the algebra of Chu morphisms, corresponding to 'symmetries' of the system, was defined. These symmetries appear to preserve the class of minimally disturbing yes/no tests and the orthogonality of states. The link between these Chu symmetries and the morphisms of Hilbert lattice, defined on the space of ortho-closed subsets of pure states, is finally emphasized. This is the second part of our reconstruction theorem.

Despite its self-contained character and the fact that it achieves to produce a reconstruction theorem from very basic premises for quantum theory, this paper requires more developments to be plainly satisfactory. First of all, it appears necessary to study the ways to recover a probabilistic formalism from a possibilistic one. Secondly, it appears necessary to pursue the construction to the case of compound systems by studying the tensor products structures on the domains appearing in the present paper. Endly, it seems unavoidable to study extensively the category of 'projective domains' extracted from our set of axioms. These problems will be attacked in our forthcoming papers.

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