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# Reconstructing quantum theory from its possibilistic operational formalism

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**Abstract**

We develop a possibilistic semantic formalism for quantum phenomena from an operational perspective. This semantic system is based on a Chu duality between preparation processes and yes/no tests, the target space being a three-valued set equipped with an informational interpretation.
A basic set of axioms is introduced for the space of states; it suffices to constrain this space to be a locally boolean qualitative domain. The subset of pure states is then characterized within this domain structure.
After having specified the notions of properties and measurements, we explore the notion of compatibility between measurements and of minimally disturbing measurements. The general existence of such measurements is then emphasized as the last key axiom to achieve the characterization of the domain structure on the space of states. This domain structure appears to be a new interesting generalization of 'concrete domains'.
An orthogonality relation is then defined on the space of states and its properties are studied. Equipped with this relation, the ortho-poset of ortho-closed subsets of pure states inherits naturally a structure of Hilbert lattice.
Finally, the symmetries of the system are characterized as a general subclass of Chu morphisms. We prove that these Chu symmetries preserve the class of minimally disturbing measurements and the orthogonality relation between states. These symmetries lead naturally to the ortho-morphisms of Hilbert lattice defined on the set of ortho-closed subsets of pure states.

2020 *Mathematics subject classification.* 81P10, 18C50, 18B35

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# 1 Introduction

The basic description of an 'experimental act' relies generally on (i) a description of a given *preparation setting* that produces *samples* of a given physical object, through some well-established procedures, and (ii) a particular set of *operations/tests* that can be realized by the *observer* on these prepared samples. Each prepared sample is associated with a set of *information*, checked throughout the preparation process and recorded on devices, and to a set of *instructions*, processed by a computer or monitored by the human operator. These information/instructions are naturally attached to macroscopically observable events that occur during the preparation process and to macroscopically distinguishable modes of the preparation apparatuses used throughout this preparation process. Analogously, the operations/tests, realized on the prepared samples, must be attached unambiguously to some *knowledge/belief* of the observer about the outcomes of this experimental step and to a new *state of facts* concerning the outcomes of the whole experiment, in order for these operations/tests to give sense to any 'sorting' of these outcomes. If these operations/tests do not 'destroy' or 'alter irremediably' the samples under study, we can then consider the whole experimental protocol[1] as a completed preparation procedure for forthcoming experiments. The information characterizing the chosen operation, and the information retrieved by the observer during these tests, should then be recorded as new information/instructions in this completed preparation procedure.[2] This description of preparation procedures and operations/tests is a fundamental ingredient of the physical description. It is a *sine qua non condition* for the experimental protocol to satisfy a basic requirement: *the reproducibility*. As noted by Kraus [46], there exist macroscopic devices undergoing macroscopic changes when interacting with micro-systems, and the observation of micro-objects always requires this inter-mediation. This fundamental empirical fact justifies the attempt to establish such an *operational description* for quantum experiments as well. Explicitly, the different procedures designed to prepare collection of similar quantum micro-systems may combine measurements and filtering operations (associated with the unambiguously measured properties) in order to produce collections of samples, that may be subjected to subsequent measurements.

Obviously, different preparation procedures may be used to produce distinct samples, to which the observer would nevertheless attach the same *informational content*. This is the case, in particular, if this observer does not know of any experiments that could be realized conjointly on these differently prepared samples and that would produce 'unambiguously-incompatible' logical conclusions. A physical description (of the objects subjected to experiment) is an attempt to establish *a semantic perspective* adapted to previous descriptions of the process of preparation/measurement. [3] The notion of *physical state* occupies a central position in this semantic construction. The physical states are abstract names for the different possible realizations of the object under study. Adopting this ontological perspective, the observer may associate an element of the *space of states* with any preparation process, which will *a priori* characterize any particular sample 'prepared' through this process. The logical truth of a proposition about the 'similarity' of two given samples should then be directly linked to the logical truth of the proposition regarding the identity of the associated states. However, the ontological notion of state has to be faced with its epistemological counterpart. From an ontological perspective, we consider that a given physical system is necessarily in a particular realization, but from an epistemological perspective, the observer should test the hypothesis that this system may indeed be described by this state.[4] Adopting this perspective, it seems that a given physical state could also be meant as a *denotation* for the set of

[1] i.e., the initial preparation followed by the operation/test step as a global preparation process for subsequent tests

[2] We note that the description of the preparation/measurement process should then exploit some tools of recursion theory.

[3] Here, and in the following, we will adopt the following basic definition of the word 'semantic' recalled by Reichenbach: "Modern logic distinguishes between object language and metalanguage; the first speaks about physical objects, the second about statements, which in turn are referred to objects. The first part of the meta-language, syntax, concerns only statements, without dealing with physical objects; this part formulates the structure of statements. The second part of the metalanguage, semantics, refers to both statements and physical objects. This part formulates, in particular, the rules concerning truth and meaning of statements, since these rules include a reference to physical objects. The third part of the meta-language, pragmatics, includes a reference to persons who use the object language."[69]

[4] Note that, in practice, the observer is rather led to infer a 'mixture' on the basis of his limited knowledge about this sample.

preparations that the observer is led to identify empirically. From a strictly operational point of view, the observer will always establish the equivalence between different preparation procedures, by testing conjointly the corresponding prepared samples through a 'well-chosen' collection of control tests. In other words, the state may be defined by the set of common facts that could be established by realizing these control-tests on the corresponding samples. Adopting another perspective, we could also consider that the state should encode the determined aspects relative to the possible results of forthcoming experiments. There should be no problem with such a 'versatile' perspective in the operational description of classical experiments, as long as the properties, established as 'actual' during the preparation process, characterize the sample in a way that will be questioned neither by any control test realized on it, nor by any future non-destructive experiments. The operational description of quantum experiments is in fact significantly more intricate, due to some inconvenient features of the measurement operation. Indeed, it is a fundamental fact of quantum experiments that, whichever set of properties has been checked as 'actual' by the preparation process, the outcomes of an irreducible part of the measurements that can be made on these prepared samples, remain completely indeterminate. More than that, if some of these measurements are realized in order to establish some new properties, it generically occurs that the measured samples no longer exhibit, afterwards, some of the properties that had been previously established on the prepared samples.

Despite this indeterministic character of quantum theory, it is an empirical fact that the distinct outcomes of these measurements, operated on a large collection of samples, prepared according to the same experimental procedure, have reproducible relative frequencies. This fundamental fact has led physicists to consider large collections of statistically independent experimental sequences as the basic objects of physical description, rather than a single experiment on a singular realization of the object under study (see [59] for a reference book). According to *Generalized Probabilistic Theory* (GPT)[5], a physical state (corresponding to a class of operationally equivalent preparation procedures) is defined by a vector of probabilities associated with the outcomes of a maximal and irredundant set of fiducial tests that can be effectuated on collections of samples produced by any of these preparation procedures.[6] In other words, two distinct collections of prepared samples will be considered as operationally equivalent if they lead to the same probabilities for the outcomes of any test on them. The physical description consists, therefore, in a set of prescriptions that allows sophisticated constructs to be defined from elementary ones. In particular, combination rules are defined for the concrete mixtures of states and for the allowed operations/tests. The different attempts to reconstruct quantum mechanics along this path ([36, 37, 38][18][57][13]) proceed by the determination of a minimal set of plausible constraints, imposed on the space of states, sufficient to 'derive' the usual Von Neumann axiomatic quantum theory. Although this probabilistic approach is now accepted as a standard conceptual framework for the reconstruction of quantum theory, the adopted perspective appears puzzling for different reasons. First of all, the observer contributes fundamentally to give an intuitive meaning to the notions of preparation, operation and measurement on physical systems. However, the concrete process of 'acquisition of information' (by the observer / on the system) has no real place in this description. Secondly, the definition of the state has definitively lost its meaning for a singular prepared sample, and the physical state is now intrinsically attached to large collections of similarly prepared samples. This point has concentrated many critics since the original article of Einstein, Podolsky and Rosen [26], although the empirical testing of quantum theory in EPR experiments has led physicists to definitively accept the traditional probabilistic interpretation. The GPT approach adopts the probabilistic description of quantum phenomena without any discussion or attempt to explain why it is necessary. Thirdly, in order to clarify the requirements of the basic set of fiducial tests necessary and sufficient to define the space of states, this approach must proceed along a technical analysis which fundamentally limits this description to 'finite dimensional' systems (finite dimensional Hilbert spaces of states). Lastly, the axioms chosen to characterize quantum

[5]This formalism has its origins in the pioneering works of Mackey [52], Ludwig [49, 50, 51] and Kraus [46]. See [42] for a recent review.

[6]The description of quantum theory in this framework then must deal with the problem of defining the notions of consistency, completeness and irredundancy for the set of control tests that define an element of the quantum space of states.

theory, among other theories encompassed by the GPT formalism, must exhibit a 'naturality' that - from our point of view - is still missing in the existing proposals.

Alternative research programs have tried to overcome some of these conceptual problems. In particular, they try to put the emphasis on the 'informational' relation emerging between the observer and the system, through the concrete set of 'yes/no tests' that can be addressed by this observer to this system, and to characterize quantum theory in these terms through a small number of basic semantic requirements. It must be noted that, although these programs try to clarify the central notion of "information" in the quantum description an observer can develop about the system under study, they basically adopt the 'probabilistic' interpretation of measurements. The fundamental limitation of the information that an observer can retrieve from a given quantum system through yes-no experiments, has been taken by different physicists as a central principle for the reconstruction of quantum theory (see reference paper [70, Chap III], and see [79] for another perspective on this basic principle) :

> **Information Principle** *'Information is a discrete quantity: there is a minimum amount of information exchangeable (a single bit, or the information that distinguishes between just two alternatives).[...] Since information is discrete, any process of acquisition of information can be decomposed into acquisitions of elementary bits of information.'* [70, p.1655].

We intend to adopt our own version of this fundamental pre-requisite to build the quantum space of states.
To be more explicit, according to C.Rovelli, quantum mechanics appears to be governed by two seemingly incompatible principles (Postulate 1 and 2 of [70, Chap III]). According to the first, the amount of independent information that can be retrieved from a 'bounded' quantum system is fundamentally 'finite'. According to the second, however, the test of any observable property that has not been stated beforehand as 'actual' for a given state[7], remains fundamentally indeterminate (of course, this test will establish an actual value for this observable property, valid after the measurement). Nevertheless, this 'new information' established through this measurement operation will have been compensated by the restoration of an indeterminacy in some of the properties that may have been established as actual beforehand. Another interesting analysis regarding this 'balance' principle concerning the knowledge of the observer about the quantum system is given in [72]. It must be noted that the combinatorics of the 'incompatibilities' between different measurements can be exploited to explore the algebraic structures behind quantum theory and to proceed to the 'reconstruction' of this theory [39, 40]. Nevertheless, these reconstruction programs stay technically imprisoned by a finite dimensional analysis. A third postulate in Rovelli's axiomatic proposal prescribes the nature of automorphisms acting on the state space (the continuous unitary transformations corresponding, in particular, to Schrodinger's dynamics). However, the form of this last postulate is not entirely satisfactory, as it imposes some intricate relations on the probabilities associated with transitions between states.

Adopting another perspective, the *operational quantum logic* approach tries to avoid the introduction of probabilities and explores the relevant categorical structures underlying the space of states and the set of properties of a quantum system. In this description, probabilities appear only as a derived concept.[8] Following G. Birkhoff and J. Von Neumann [15] and G. W. Mackey [52], this approach focuses on the structured space of 'testable properties' of a physical system.[9] Mackey identifies axioms on the set of yes/no questions sufficient to relate this set to the set of closed subspaces of a complex Hilbert space.

[7] In Von Neumann's formalism, this point means: the quantum state of the system is not an eigenstate of the associated operator.

[8] For the sake of usual quantum formalism, quantum probabilities can be recovered from this formalism, under the assumption of the existence of some well-behaved measurements, using Gleason's theorem [33] (see [60] for a historical and pedagogical presentation of these elements).

[9] In Von Neumann's quantum mechanics, each entity is associated with a complex Hilbert space $H$. A state $\psi$ of this entity is defined by a ray $v(\psi)$ in $H$, and an observable is defined by a self-adjoint operator on $H$. In particular, a yes/no test $\mathfrak{t}$ is represented by an orthogonal projector $\Pi_{\mathfrak{t}}$ or equivalently by the closed subspace $\mathfrak{A}_{\mathfrak{t}}$ defined as the range of $\Pi_{\mathfrak{t}}$. The answer "yes" or "no" is obtained with certainty for the yes/no test $\mathfrak{t}$, if and only if the state $\psi$ is such that $v(\psi)$ is included in $\mathfrak{A}_{\mathfrak{t}}$ or in the

Later, C. Piron proposed a set of axioms that (almost) lead back to the general framework of quantum mechanics (see [21] for a historical perspective of the abundant literature inherited from Piron's original works [60, 61].). Piron's framework has been developed into a full operational approach and the categories underlying this approach were analyzed (see [56, 55] for a detailed account of this categorical perspective). It must be noted that these constructions are established in reference to some general results of projective geometry [27] and are not restricted to a finite-dimensional perspective. Despite some beautiful results (in particular the restriction of the division ring associated with Piron's reconstruction of the Hilbert space from Piron's propositional lattices [41][8]) and the attractiveness of a completely categorical approach (see [73] for an analysis of the main results on propositional systems), this approach has encountered many problems. Among these problems, we may cite the difficulty of building a consistent description of compound systems due to no-go results related to the existence of a tensor product of Piron's propositional systems [68, 28][6, 7]. These works have cast doubts on the adequacy of Piron's choice of an "orthomodular complete lattice" structure for the set of properties of the system. D. Foulis, C. Piron and C. Randall [29] produced a jointly refined version of their respective approaches in order to rule out these problems (see also [76]). This description emphasizes the centrality of the treatment of the incompatibilities between measurements.⌊10⌋ Our work will emphasize the necessity of replacing the lattice structures, introduced to describe the set of propositions about the quantum system, with domains. Another central problem with the logico-algebraic approach was its inability to describe the dynamic aspect of the measurement operation. The operational quantum logic approach has then been developed later on in different categorical perspectives in order to clarify the links between quantum logic, and modal/dynamic [9, 10, 11][20, 22, 24][81, 14], or linear [23][31, 32] logic.

Other categorical formalisms, adapted to the axiomatic study of quantum theory, have been developed more recently [3] and their relation with the 'operational approach' has been partly explored [1, 2, 4]. In [1, Theorem 3.15], S. Abramsky makes explicit the fact that the *Projective quantum symmetry groupoid PSymmH*⌊11⌋ is fully and faithfully represented by the category $bmChu_{[0,1]}$, i.e., by the sub-category of the category of bi-extensional Chu spaces associated with the evaluation set $[0,1]$ obtained by restricting it to Chu morphisms $(f_*, f^*)$ for which $f_*$ is injective. This result suggests that Chu categories could have a central role in the construction of axiomatic quantum mechanics⌊12⌋, as they provide a natural characterization of the automorphisms of the theory. More surprisingly, and interestingly for us, in [1] S. Abramsky shows that the previously mentioned representation of *PSymmH* is 'already' full and faithful if we replace the evaluation space of the Chu category by a three-element set, where the three values represent "definitely yes", "definitely no" and "maybe" [1, Theorem 4.4] :

> *'The results on reduction to finite value sets are also intriguing. Not only is the bare Chu condition on morphisms sufficient to whittle them down to the semiunitaries, this is even the case when the discriminations on which the condition is based are reduced to three values. The general case for two values remains open, but we have shown that the two standard possibilistic reductions both* fail to preserve fullness. *A negative answer for two-valued semantics in general would suggest an unexpected rôle for three-valued logic in the*

orthogonal of $\mathfrak{A}_t$, respectively. Birkhoff and von Neumann proposed to focus not on the structure of the Hilbert space itself, but on the structure of the set of closed subspaces of $H$. The mathematical structure associated with the set of quantum propositions defined by the closed subspaces of $H$ is not a Boolean algebra (contrary to the case encountered in classical mechanics). By shifting the attention to the set of closed subspaces of $H$ instead of $H$, the possibility is open to build an operational approach to quantum mechanics, because the basic elements of this description are yes/no experiments.

[10] *'It is our contention that the realistic view implicit in classical physics need not be abandoned to accommodate the contemporary conceptions of quantum physics. All that must be abandoned is the presumption that each set of experiments possesses a common refinement (that is, the experiments are compatible). As we shall argue, this in no way excludes the notion of physical systems existing exterior to an observer, nor does it imply that the properties of such systems depend on the knowledge of the observer.'* [29, p.813]

[11] The objects of this category are the natural space of states in quantum mechanics, i.e., the Hilbert spaces of dimension greater than two, and the morphisms are the orbits under the $U(1)$ group action on semi-unitary maps (i.e. unitary or anti-unitary), which are the relevant symmetries of Hilbert spaces from the point of view of quantum mechanics.

[12] The centrality of the notion of Chu spaces for quantum foundations had been noted already by V. Pratt [65].

*foundations of Quantum Mechanics.'*

The introduction of a three-valued logic in the foundational studies of quantum mechanics goes back to H. Reichenbach's work [69]. H. Reichenbach's formalism has been introduced as an alternative to the quantum logic approaches developed to continue the seminal work of G. Birkhoff and J. von Neumann [15].⌊[13]⌋ This extension of standard logic was designed as a means of dealing with certain conceptual tensions arising in the quantum mechanical description of the world. Briefly, the third truth value, called 'indeterminate', was introduced to capture 'meaningless' statements associated with unmeasured entities during the experimental process. It is important to note that H. Reichenbach did not clarify how this three-valued logic may lead to results 'comparable' to those obtained using the usual probabilistic formalism, and why this extension of classical logic could be the right setting for quantum theory.
S. Abramsky's work adopts a radically different perspective. His study begins with continuous-valued Chu space. He shows that quantum symmetries are elegantly captured by Chu morphisms and proves finally that three values are actually sufficient for this characterization of quantum symmetries. Despite the suggestive similarities pointed to by S. Abramsky between such a three-valued Chu space description and the Geneva school's formalism, no bridge has been built between these two formalisms until now. Thus, S. Abramsky did not affirm that these three values are sufficient to found a complete axiomatic quantum theory, close to Piron's program or alternative to it, and allowing a complete reconstruction of the usual Hilbert formalism. It is the purpose of the present paper to achieve this goal. We intend to present the basic elements of this 'possibilistic'⌊[14]⌋ semantic formalism, and to give the precise axiomatics that leads to a reconstruction of a generalized Hilbert space structure on the space of states.

Our formalism is based on a Chu duality between preparation processes and quantum tests. This Chu duality refers to a three-valued target space. This three-valued target space is equipped with a 'possibilist' semantic formalism which leads to an 'informational' interpretation of the set of preparations. In the first part of our study, we formulate a precise semantic description of the space of states. The 'Information Principle' introduced by C. Rovelli plays a central role in this formalism. After having introduced a basic set of axioms about the space of states, we will show that the space of states is necessarily equipped with a structure of locally boolean qualitative domain. The technical details concerning the properties of this domain structure are given in an appendix to this paper. The space of pure states will then be characterized.
In a second part of the study, we clarify successively the notion of 'property' and the notion of a 'measurement' associated with a given property of the system. We explore the consequences of the incompatibilities existing between measurements. We clarify in particular the general conditions of existence of minimally disturbing measurements of the properties of the system. This new requirement closes the axiomatic characterization of the space of states. The domain structure on the space of states appears to be an interesting extension of the 'concrete domains' studied by Kahn G. and Plotkin G. [44].
An orthogonality relation is then defined on the space of states and its properties are studied using the domain structure obtained on the space of states. Equipped with this relation, the ortho-poset of ortho-closed subsets of pure states inherits a structure of Hilbert lattice. This result is the first part of our reconstruction theorem.
In the third part of this paper, we build the set of symmetries of the system as a particular sub-algebra of Chu morphisms. These symmetries appear to leave this subset of minimally disturbing measurement operations stable and preserves the orthogonality relation between states. Endly, it is shown that these symmetries lead naturally to the ortho-morphisms of Hilbert lattice defined on the set of ortho-closed subsets of pure states.

Throughout the paper we clearly distinguish (1) elements formulated in the 'material mode of speech'

[13] In order to clarify the fundamental difference in nature between Reichenbach Quantum Logic and Mainstream Quantum Logic the reader is invited to consult [35].

[14] In the rest of this paper we refer to this construction, based on a three-valued Chu space, as a 'possibilistic' approach to distinguish it from the 'probabilistic' one.

and concerning the structure of the language in which the experimental setting can be described operationally (these elements are designated as 'Notions' throughout the text), although these notions are the occasion to introduce the corresponding mathematical elements, from (2) purely mathematical definitions (these elements are classically designated as 'Definitions').
We intentionally emphasize the different requirements of our reconstruction program. Every requirement is introduced accompanied with an analysis of its motivation and summarized under the term 'Axiom'; the analysis of the mathematical consequences of each axiom is declined along Lemmas and Theorems.

Regarding the basic notations, notions and results relative to order theory used in the present paper, we refer the reader to section 7.

# 2 Preparations and states

## 2.1 Operational formalism

Adopting the operational perspective on quantum experiments, we will introduce the following definitions :

**Notion 1.** A *preparation process* is an objectively defined, and thus 'repeatable', experimental sequence that allows singular samples of a certain physical system to be produced, in such a way that we are able to submit them to tests. We will denote by $\mathfrak{P}$ the set of preparation processes (each element of $\mathfrak{P}$ can be equivalently considered as the collection of samples produced through this preparation procedure). The information corresponding to macroscopic events/operations describing the procedure depend on an observer $O$. If this dependence has to be made explicit, we will adopt the notation $\mathfrak{P}^{(O)}$ to denote the set of preparation processes defined by the observer $O$.

**Notion 2.** For each *property*, that the observer aims to test macroscopically on *any particular sample* of the considered micro-system, it will be assumed that the observer is able to define (i) some detailed 'procedure', in reference to the modes of use of some experimental apparatuses chosen to perform the operation/test, and (ii) a 'rule' allowing the answer 'yes' to be extracted if the macroscopic outcome of the experiment conforms with the expectation of the observer, when the test is performed on any input sample (as soon as this experimental procedure can be opportunely applied to this particular sample). These operations/tests, designed to determine the occurrence of a given property for a given sample, will be called *yes/no tests associated with this property* (also called a *definite experimental project* in [62]). If a yes/no test, associated with a given property, is effectuated according to the established procedure, and if a positive result is actually obtained for a given sample, we will say that *this property has been measured* for this sample. The set of 'yes/no tests' at the disposal of the observer will be denoted by $\mathfrak{T}$. If the dependence with respect to the observer $O$ has to be made explicit, we will adopt the notation $\mathfrak{T}^{(O)}$ to denote the set of tests defined by the observer $O$.

We are essentially interested in the information gathered by the observer through the implementation of some yes/no tests, designated by elements in $\mathfrak{T}$, on finite collections of samples prepared similarly through any of the preparation procedures, given as elements of $\mathfrak{P}$. With this perspective, we have to abandon any reference to the probabilistic interpretation[15]. Nevertheless, the observer is still able to distinguish the situations where one can pronounce a statement with 'certainty', from the situations where one can judge the result as 'indeterminate', on the basis of the knowledge gathered beforehand.

[15]The finite character of the tested collection of prepared samples renders any notion of relative frequency of the outcomes 'meaningless'.

**Notion 3.** A yes/no test $\mathfrak{t} \in \mathfrak{T}$ will be said to be *positive with certainty* (resp. *negative with certainty*) relatively to a preparation process $\mathfrak{p} \in \mathfrak{P}$ iff the observer is led to affirm that the result of this test, realized on any of the particular samples that could be prepared according to this preparation process, would be 'positive with certainty' (resp. would be 'negative with certainty'), 'should' this test be effectuated. If the yes/no test can not be stated as 'certain', this yes/no test will be said to be *indeterminate*. Concretely, the observer can establish the 'certainty' of the result of a given yes/no test on any given sample issued from a given preparation procedure, by running the same test on a sufficiently large (but finite) collection of samples issued from this same preparation process: if the outcome is always the same, the observer will be led to claim that similarly prepared 'new' samples would also produce the same result, if the experiment was effectuated.
To summarize, for any preparation process $\mathfrak{p}$ and any yes/no test $\mathfrak{t}$, the element $\mathfrak{e}(\mathfrak{p},\mathfrak{t}) \in \mathfrak{B} := \{\perp, \mathbf{Y}, \mathbf{N}\}$ will be defined to be $\perp$ (alternatively, $\mathbf{Y}$ or $\mathbf{N}$) if the outcome of the yes/no test $\mathfrak{t}$ on any sample prepared according to the preparation procedure $\mathfrak{p}$ is judged as 'indeterminate' ('positive with certainty' or 'negative with certainty', respectively) by the observer.

$$\begin{array}{rcccl} \mathfrak{e} & : & \mathfrak{P} \times \mathfrak{T} & \longrightarrow & \mathfrak{B} := \{\perp, \mathbf{Y}, \mathbf{N}\} \\ & & (\mathfrak{p},\mathfrak{t}) & \mapsto & \mathfrak{e}(\mathfrak{p},\mathfrak{t}). \end{array} \tag{1}$$

Several remarks should be made regarding the above definitions.

*Remark* 1. It is essential to note the *counterfactual* aspect of these definitions: in the 'determinate' case, the observer is asked to predict the result of this test *before the test* and *regardless of whether the test is effectuated*. Of course, any 'determinate statement' (positive or negative) produced by the observer, about the result of any forthcoming yes/no test relative to a given preparation process, is a strictly falsifiable statement: it may be proved to be false after some test realized on a *finite collection* of new, similarly prepared samples[16].

*Remark* 2. The 'certainty' of the observer about the occurrence of the considered 'property' is intrinsically attached to *any singular sample* prepared through this preparation process and can be falsified as a property of this sample. In other words, it is not necessary to consider *a statistical ensemble* of similarly prepared samples to give a meaning to these notions and to the logical perspective adopted to confront these statements with the measurable state of facts.

*Remark* 3. When the determinacy of a yes/no test is established for an observer, we can consider that this observer possesses some elementary 'information' about the state of the system, whereas, in the 'indeterminate case', the observer has none (relatively to the occurrence of the considered property).

**Notion 4.** The set $\mathfrak{B}$ will be equipped with the following poset structure, characterizing the 'information' gathered by the observer:

$$\forall u, v \in \mathfrak{B}, \qquad (u \leq v) \;:\Leftrightarrow\; (u = \perp \text{ or } u = v). \tag{2}$$

$(\mathfrak{B}, \leq)$ will be called a *flat boolean domain* in the rest of this paper.

**Notion 5.** $(\mathfrak{B}, \leq)$ is equipped with the following involution map :

$$\overline{\perp} := \perp \qquad \overline{\mathbf{Y}} := \mathbf{N} \qquad \overline{\mathbf{N}} := \mathbf{Y}. \tag{3}$$

[16] If the observer is certain of the positive result after having performed a given yes/no test on a finite number of similarly prepared samples, a negative result obtained for any newly tested sample will lead the observer to revise that prediction and to consider this yes/no test as being 'indeterminate' for this preparation.

The conjugate of a yes/no test $\mathfrak{t} \in \mathfrak{T}$ is the yes/no test denoted $\bar{\mathfrak{t}}$ and defined from $\mathfrak{t}$ by exchanging the roles of **Y** and **N** in every result obtained by applying $\mathfrak{t}$ to any given input sample. In other words,

$$\forall \mathfrak{t} \in \mathfrak{T}, \forall \mathfrak{p} \in \mathfrak{P}, \qquad \mathfrak{e}(\mathfrak{p}, \bar{\mathfrak{t}}) := \overline{\mathfrak{e}(\mathfrak{p}, \mathfrak{t})}. \tag{4}$$

**Notion 6.** For any yes/no test $\mathfrak{t}$, the set of preparation processes $\mathfrak{p}$ for which this test is established as *actual*, i.e., 'positive with certainty', will be denoted $\mathfrak{A}_{\mathfrak{t}}$.

$$\forall \mathfrak{t} \in \mathfrak{T}, \qquad \mathfrak{A}_{\mathfrak{t}} := \{\, \mathfrak{p} \in \mathfrak{P} \mid \mathfrak{e}(\mathfrak{p}, \mathfrak{t}) = \mathbf{Y} \,\}. \tag{5}$$

**Notion 7.** For a given yes/no test $\mathfrak{t}$, we define the subset $\mathfrak{Q}_{\mathfrak{t}}$ of preparation processes that are known by the observer to produce collections of samples leading to positive results to the yes/no test $\mathfrak{t}$. Regarding these prepared samples, the observer is then asked to pronounce a statement about any future result of this test on similarly prepared new samples: 'positive' or 'indeterminate'. The collections of samples, resulting from these preparation processes, may then be filtered in order to select collections of samples that are known by the observer to have 'passed the yes/no test $\mathfrak{t}$ positively'. If a preparation process $\mathfrak{p}$ is in $\mathfrak{Q}_{\mathfrak{t}}$, we will say that the property associated with the yes/no test $\mathfrak{t}$ is *potential* for the samples produced through $\mathfrak{p}$ (or $\mathfrak{p}$ is *questionable* by $\mathfrak{t}$). The subset $\mathfrak{Q}_{\mathfrak{t}}$ is given by

$$\mathfrak{Q}_{\mathfrak{t}} := \{\, \mathfrak{p} \in \mathfrak{P} \mid \mathfrak{e}(\mathfrak{p}, \mathfrak{t}) \leq \mathbf{Y} \,\} \subseteq \mathfrak{P}. \tag{6}$$

The evaluation map $\mathfrak{e}$ defines a particular 'duality' between the spaces $\mathfrak{P}$ and $\mathfrak{T}$. Formally, $(\mathfrak{P}, \mathfrak{T}, \mathfrak{e})$ defines a Chu space⌊[17]⌋⌊[18]⌋. The set of preparations $\mathfrak{P}$ (or the set of yes/no tests $\mathfrak{T}$) will be *a priori* interpreted as the set of *points* (the set of *opens*) of this Chu space.⌊[19]⌋ Indeed, the preparation processes are naturally considered as 'coexisting entities' distinguished by the properties they posses, whereas the yes/no tests are naturally interpreted as 'alternative predicates' relative to the properties attached to the prepared samples.

According to the perspective adopted by [59], we will define the *states* of the physical system as follows:

**Notion 8.** An equivalence relation, denoted $\sim_{\mathfrak{P}}$, is defined on the set of preparations $\mathfrak{P}$:

Two preparation processes are identified iff the statements established by the observer about the corresponding prepared samples are identical.

A *state* of the physical system is an equivalence class of preparation processes corresponding to the same informational content, i.e., a class of preparation processes that are not distinguished by the statements established by the observer in reference to the tests realized on finite collections of samples produced through these preparation processes.

The set of equivalence classes, modulo the relation $\sim_{\mathfrak{P}}$, will be denoted $\mathfrak{S}$. In other words,

$$\forall \mathfrak{p}_1, \mathfrak{p}_2 \in \mathfrak{P}, \ (\mathfrak{p}_1 \sim_{\mathfrak{P}} \mathfrak{p}_2) \quad :\Leftrightarrow \quad (\, \forall \mathfrak{t} \in \mathfrak{T}, \ \mathfrak{e}(\mathfrak{p}_1, \mathfrak{t}) = \mathfrak{e}(\mathfrak{p}_2, \mathfrak{t}) \,), \tag{7}$$

$$\sim_{\mathfrak{P}} \ \text{ is an equivalence relation}, \tag{8}$$

$$\lceil \mathfrak{p} \rceil \quad := \quad \{\, \mathfrak{p}' \in \mathfrak{P} \mid \mathfrak{p}' \sim_{\mathfrak{P}} \mathfrak{p} \,\}, \tag{9}$$

$$\mathfrak{S} \quad := \quad \{\, \lceil \mathfrak{p} \rceil \mid \mathfrak{p} \in \mathfrak{P} \,\}. \tag{10}$$

[17] See [12] for a reference paper and [66][67] for a basic presentation.

[18] In this section, we are concerned with the duality aspect and the situation of Chu morphisms will be treated later.

[19] These designations are reminiscent of the basic fact that Chu spaces are generalizations of topological spaces. However this distinction is largely obsolete, as soon as the Chu space construction establishes a duality between these two sets.

*Remark* 4. It must be noticed that a given yes/no test $\mathfrak{t}$ can be applied separately on the two distinct collections of samples prepared through the two distinct preparation procedures $\mathfrak{p}_1$ and $\mathfrak{p}_2$. The corresponding counterfactual statements $\mathfrak{e}(\mathfrak{p}_1,\mathfrak{t})$ and $\mathfrak{e}(\mathfrak{p}_2,\mathfrak{t})$, established by the observer about $\mathfrak{p}_1$ and $\mathfrak{p}_2$, are then formulated 'consistently' after these two independent experimental sequences.

We will derive a map $\widetilde{\mathfrak{e}}$ from the evaluation map $\mathfrak{e}$ according to the following definition :

$$\begin{array}{rcl} \widetilde{\mathfrak{e}} \ : \ \mathfrak{T} & \to & \mathfrak{B}^{\mathfrak{S}} \\ \mathfrak{t} & \mapsto & \widetilde{\mathfrak{e}}_{\mathfrak{t}} \ \mid \ \widetilde{\mathfrak{e}}_{\mathfrak{t}}(\lceil \mathfrak{p} \rceil) := \mathfrak{e}(\mathfrak{p},\mathfrak{t}),\ \forall \mathfrak{p} \in \mathfrak{P}. \end{array} \tag{11}$$

As a result of this quotient operation on the space of preparation processes, it appears that we have the following natural property of our Chu space.

**Lemma 1.** The Chu space $(\mathfrak{S},\mathfrak{T},\widetilde{\mathfrak{e}})$ is *separated*. Different preparation procedures are indeed 'identified' by the observer as soon as this observer attributes the same statements to the differently prepared samples. In other words,

$$\forall \sigma_1,\sigma_2 \in \mathfrak{S}, \qquad (\forall \mathfrak{t} \in \mathfrak{T},\ \widetilde{\mathfrak{e}}_{\mathfrak{t}}(\sigma_1) = \widetilde{\mathfrak{e}}_{\mathfrak{t}}(\sigma_2)) \ \Rightarrow \ (\sigma_1 = \sigma_2). \tag{12}$$

■

## 2.2 Mixtures

We will define a binary operation on the set of preparations.

**Axiom 1.** If we consider two preparation processes $\mathfrak{p}_1$ and $\mathfrak{p}_2$ in $\mathfrak{P}$, we can define a new preparation procedure, called a *mixture* and denoted $\mathfrak{p}_1 \sqcap_{\mathfrak{P}} \mathfrak{p}_2$, as follows:

the samples produced from the preparation procedure $\mathfrak{p}_1 \sqcap_{\mathfrak{P}} \mathfrak{p}_2$ are obtained by a random mixing of the samples issued from the preparation processes $\mathfrak{p}_1$ and $\mathfrak{p}_2$ indiscriminately.
As a consequence, the statements that the observer can establish after a sequence of tests $\mathfrak{t} \in \mathfrak{T}$ on these samples produced through the procedure $\mathfrak{p}_1 \sqcap_{\mathfrak{P}} \mathfrak{p}_2$ is given as the infimum of the statements that the observer can establish for $\mathfrak{p}_1$ and $\mathfrak{p}_2$ separately. In other words,

$$\forall \mathfrak{p}_1,\mathfrak{p}_2 \in \mathfrak{P},\ \exists!\, (\mathfrak{p}_1 \sqcap_{\mathfrak{P}} \mathfrak{p}_2) \in \mathfrak{P} \ \mid \ (\forall \mathfrak{t} \in \mathfrak{T},\ \mathfrak{e}((\mathfrak{p}_1 \sqcap_{\mathfrak{P}} \mathfrak{p}_2),\mathfrak{t}) = \mathfrak{e}(\mathfrak{p}_1,\mathfrak{t}) \wedge \mathfrak{e}(\mathfrak{p}_2,\mathfrak{t})), \tag{13}$$

where $\wedge$ denotes the infimum of a pair of elements in the poset $(\mathfrak{B},\leq)$.

We note the following obvious properties deduced from the literal definitions of the random mixing operation $\sqcap_{\mathfrak{P}}$ and the equivalence relation $\sim_{\mathfrak{P}}$ defining the space of states.

**Lemma 2.** For any $\mathfrak{p}_1,\mathfrak{p}_2,\mathfrak{p}_3 \in \mathfrak{P}$, we have

$$\mathfrak{p}_1 \sqcap_{\mathfrak{P}} (\mathfrak{p}_2 \sqcap_{\mathfrak{P}} \mathfrak{p}_3) \sim_{\mathfrak{P}} (\mathfrak{p}_1 \sqcap_{\mathfrak{P}} \mathfrak{p}_2) \sqcap_{\mathfrak{P}} \mathfrak{p}_3 \tag{14}$$

$$(\mathfrak{p}_1 \sqcap_{\mathfrak{P}} \mathfrak{p}_1) \sim_{\mathfrak{P}} \mathfrak{p}_1, \tag{15}$$

$$(\mathfrak{p}_2 \sqcap_{\mathfrak{P}} \mathfrak{p}_1) \sim_{\mathfrak{P}} (\mathfrak{p}_1 \sqcap_{\mathfrak{P}} \mathfrak{p}_2) \tag{16}$$

■

More generally, a pre-order relation can be defined on the set $\mathfrak{P}$ of preparation processes.

**Notion 9.** A preparation process $\mathfrak{p}_2 \in \mathfrak{P}$ is said to be *sharper* than another preparation process $\mathfrak{p}_1 \in \mathfrak{P}$ (this fact will be denoted $\mathfrak{p}_1 \sqsubseteq_{\mathfrak{P}} \mathfrak{p}_2$) iff any yes/no test $\mathfrak{t} \in \mathfrak{T}$ that is 'determinate' for the samples prepared through $\mathfrak{p}_1$ is also necessarily 'determinate' with the same value for the samples prepared through $\mathfrak{p}_2$, i.e.,

$$\forall \mathfrak{p}_1,\mathfrak{p}_2 \in \mathfrak{P},\ (\ \mathfrak{p}_1 \sqsubseteq_{\mathfrak{P}} \mathfrak{p}_2\ ) \ :\Leftrightarrow \ (\ \forall \mathfrak{t} \in \mathfrak{T},\ \mathfrak{e}(\mathfrak{p}_1,\mathfrak{t}) \leq \mathfrak{e}(\mathfrak{p}_2,\mathfrak{t})\ ), \tag{17}$$

If $\mathfrak{p}_1 \sqsubseteq_{\mathfrak{P}} \mathfrak{p}_2$ (i.e., $\mathfrak{p}_2$ is 'sharper' than $\mathfrak{p}_1$), $\mathfrak{p}_1$ is said to be 'coarser' than $\mathfrak{p}_2$.

**Lemma 3.** $(\mathfrak{P}, \sqsubseteq_{\mathfrak{P}})$ is a pre-ordered set. ■

The equivalence relation defined in **Notion 8** derives from this pre-order:

$$\forall \mathfrak{p}_1, \mathfrak{p}_2 \in \mathfrak{P}, \qquad (\mathfrak{p}_1 \sqsubseteq_{\mathfrak{P}} \mathfrak{p}_2 \text{ and } \mathfrak{p}_1 \sqsupseteq_{\mathfrak{P}} \mathfrak{p}_2) \Rightarrow (\mathfrak{p}_1 \sim_{\mathfrak{P}} \mathfrak{p}_2). \tag{18}$$

The properties of the equivalence relation $\sim_{\mathfrak{P}}$ with respect to the pre-order $\sqsubseteq_{\mathfrak{P}}$ and the random-mixing binary operation $\sqcap_{\mathfrak{P}}$ leads to the following properties:

**Lemma 4.** The binary operation $\sqcap_{\mathfrak{P}}$ being literally designed to satisfy properties (14), (15) and (16), the binary relation $\sqsubseteq_{\mathfrak{P}}$ is then equivalently defined by:

$$\forall \mathfrak{p}_1, \mathfrak{p}_2 \in \mathfrak{P}, \qquad (\mathfrak{p}_1 \sqsubseteq_{\mathfrak{P}} \mathfrak{p}_2) \Leftrightarrow (\mathfrak{p}_1 \sim_{\mathfrak{P}} (\mathfrak{p}_1 \sqcap_{\mathfrak{P}} \mathfrak{p}_2)). \tag{19}$$

The following properties of the pre-order $\sqsubseteq_{\mathfrak{P}}$ are direct consequences of this fact:

$$\forall \mathfrak{p}, \mathfrak{p}_1, \mathfrak{p}_2 \in \mathfrak{P}, \qquad (\mathfrak{p}_1 \sqcap_{\mathfrak{P}} \mathfrak{p}_2) \sqsubseteq_{\mathfrak{P}} \mathfrak{p}_1, \tag{20}$$

$$(\mathfrak{p} \sqsubseteq_{\mathfrak{P}} \mathfrak{p}_1 \text{ and } \mathfrak{p} \sqsubseteq_{\mathfrak{P}} \mathfrak{p}_2) \Rightarrow (\mathfrak{p} \sqsubseteq_{\mathfrak{P}} (\mathfrak{p}_1 \sqcap_{\mathfrak{P}} \mathfrak{p}_2)). \tag{21}$$

■

Quite naturally, we will assume the existence and uniqueness of a bottom element in $\mathfrak{P}$:

**Axiom 2.** There exists a unique preparation process, that can be interpreted as a 'randomly-selected' collection of 'un-prepared samples'. This element leads to complete indeterminacy for any yes/no test realized on it. In other words, the following axiom will be imposed

$$\exists! \, \mathfrak{p}_\perp \in \mathfrak{P} \mid (\forall \mathfrak{t} \in \mathfrak{T}, \ \mathfrak{e}(\mathfrak{p}_\perp, \mathfrak{t}) = \perp). \tag{22}$$

**Lemma 5.** The space of states $\mathfrak{S}$ is partially ordered

$$\forall \sigma_1, \sigma_2 \in \mathfrak{S}, \qquad (\sigma_1 \sqsubseteq_{\mathfrak{S}} \sigma_2) :\Leftrightarrow (\forall \mathfrak{p}_1, \mathfrak{p}_2 \in \mathfrak{P}, \ (\sigma_1 = \lceil \mathfrak{p}_1 \rceil, \sigma_2 = \lceil \mathfrak{p}_2 \rceil) \Rightarrow (\mathfrak{p}_1 \sqsubseteq_{\mathfrak{P}} \mathfrak{p}_2)), \tag{23}$$

$$(\mathfrak{S}, \sqsubseteq_{\mathfrak{S}}) \text{ is a partial order.} \tag{24}$$

■

Moreover, the existence of the preparation procedure $\mathfrak{p}_1 \sqcap_{\mathfrak{P}} \mathfrak{p}_2$ satisfying property (13) leads to the following definition.

**Notion 10.**

$$\forall \mathfrak{p}_1, \mathfrak{p}_2 \in \mathfrak{P}, \qquad \lceil \mathfrak{p}_1 \rceil \sqcap_{\mathfrak{S}} \lceil \mathfrak{p}_2 \rceil := \lceil \mathfrak{p}_1 \sqcap_{\mathfrak{P}} \mathfrak{p}_2 \rceil. \tag{25}$$

**Lemma 6.** $(\mathfrak{S}, \sqcap_{\mathfrak{S}})$ is an Inf semi-lattice. ■

**Lemma 7.** $(\mathfrak{S}, \sqsubseteq_{\mathfrak{S}})$ admits a bottom element, denoted $\perp_{\mathfrak{S}}$:

$$\perp_{\mathfrak{S}} := \lceil \mathfrak{p}_\perp \rceil \qquad \text{is the bottom element of } (\mathfrak{S}, \sqsubseteq_{\mathfrak{S}}). \tag{26}$$

■

**Notion 11.** Two preparation processes $\mathfrak{p}_1, \mathfrak{p}_2 \in \mathfrak{P}$ will be said to be *consistent* (this fact will be denoted $\widehat{\mathfrak{p}_1\mathfrak{p}_2}^{\mathfrak{P}}$) iff they can be considered as two different incomplete preparations of the same targeted collection of prepared samples, i.e., iff there exists a preparation process $\mathfrak{p} \in \mathfrak{P}$ which is simultaneously sharper than $\mathfrak{p}_1$ and $\mathfrak{p}_2$ (i.e., which is a common upper-bound in $\mathfrak{P}$). In other words,

$$\forall \mathfrak{p}_1, \mathfrak{p}_2 \in \mathfrak{P},\ \widehat{\mathfrak{p}_1\mathfrak{p}_2}^{\mathfrak{P}} \quad :\Leftrightarrow \quad (\exists \mathfrak{p} \in \mathfrak{P} \mid \mathfrak{p}_1 \sqsubseteq_{\mathfrak{P}} \mathfrak{p},\ \mathfrak{p}_2 \sqsubseteq_{\mathfrak{P}} \mathfrak{p}). \tag{27}$$

The consistency relation is obviously reflexive and symmetric.
Due to the following relation $\forall \mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3 \in \mathfrak{P},\ (\widehat{\mathfrak{p}_1\mathfrak{p}_2}^{\mathfrak{P}} \text{ and } \mathfrak{p}_2 \sim_{\mathfrak{P}} \mathfrak{p}_3) \Rightarrow \widehat{\mathfrak{p}_1\mathfrak{p}_3}^{\mathfrak{P}}$, the consistency relation can be defined between states as follows.

$$\forall \sigma_1, \sigma_2 \in \mathfrak{S},\ \widehat{\sigma_1\sigma_2}^{\mathfrak{S}} \quad :\Leftrightarrow \quad (\forall \mathfrak{p}_1, \mathfrak{p}_2 \in \mathfrak{P},\ \sigma_1 = \lceil \mathfrak{p}_1 \rceil, \sigma_2 = \lceil \mathfrak{p}_2 \rceil \ \Rightarrow\ \widehat{\mathfrak{p}_1\mathfrak{p}_2}^{\mathfrak{P}}). \tag{28}$$

For any $\sigma_1, \sigma_2 \in \mathfrak{S}$, we will denote $\widetilde{\sigma_1\sigma_2}^{\mathfrak{S}}$ the property $\neg\ \widehat{\sigma_1\sigma_2}^{\mathfrak{S}}$.

The previous construction of the space of states, albeit usual, appears a bit strange from an operational perspective. Indeed, concretely, the observer is never confronted with a given 'state' (i.e., to a generically-infinite class of preparation processes, indistinguishable by the generically-infinite set of tests that can be realized on them) in order to decide if it is consistent (or not) to affirm with 'certainty' the occurrence of a given 'property' for a given sample corresponding to this state. The observer is rather confronted with a restricted set of preparation processes, enabling mixtures to be produced, which generically lead to undetermined results when they are confronted with a family of tests.

In order to produce 'determinacy', relative to the occurrence of a given property for a given state of the system, the observer extracts (from the selected family of preparation processes available that are detected to produce samples corresponding more or less to the chosen state) some sub-families that concretely realize *a sharpening* of the parameters that define the preparation setting/procedure, according to a given set of prerequisites concerning the samples that will be submitted to the test. Through each 'sharpening' of its preparation procedures, the observer intends to fix 'unambiguously', but 'inductively', a 'state' of the system. This limit process is understood in terms of the limit taken for every statement that can be made about the selected samples (i.e., the samples prepared according to any of the preparation processes that are elements of the chosen sharpening family).

**Notion 12.** A family $\mathfrak{Q} \subseteq \mathfrak{P}$ is *a sharpening family of preparation processes* (denoted $\mathfrak{Q} \subseteq_{Chain} \mathfrak{P}$) iff every pair of elements of $\mathfrak{Q}$ are ordered by $\sqsubseteq_{\mathfrak{P}}$, i.e., for any $\mathfrak{p}_1$ and $\mathfrak{p}_2$ in $\mathfrak{Q}$, we have necessarily $\mathfrak{p}_1 \sqsubseteq_{\mathfrak{P}} \mathfrak{p}_2$ or $\mathfrak{p}_2 \sqsubseteq_{\mathfrak{P}} \mathfrak{p}_1$.

**Axiom 3.** For any family $\mathfrak{Q}$ in $\mathfrak{P}$ defining a 'sharpening', there exists a state $\sigma$ in $\mathfrak{S}$ which is the supremum of the chain of states corresponding to the elements of $\mathfrak{Q}$.

$$\forall \mathfrak{Q} \subseteq_{Chain} \mathfrak{P}, \qquad \text{the supremum } \left(\bigsqcup\nolimits_{\mathfrak{S}} \lceil \mathfrak{Q} \rceil\right) \text{ exists in the partially ordered set } \mathfrak{S} \tag{29}$$

In other words, $\mathfrak{S}$ will be required to be *a chain-complete partial order*.

*Remark* 5. $\mathfrak{S}$ is then also *a directed-complete partial order*.

Let us then fix a yes/no test $\mathfrak{t}$. If the observer intends to designate the corresponding property as an 'element of reality' attached to the system itself, and not as a datum depending on the explicit operational requirements used to define the state, the following condition must be satisfied.

**Axiom 4.** The observer is authorized to formulate a 'determinate' statement, about the occurrence of a given property, for the 'limit state' induced from a given sharpening family of preparation processes, iff it is possible to formulate this same statement for another preparation process that is an element of the chosen sharpening family (and thus also for any sharper preparation process). In other words,

$$\forall \mathfrak{t} \in \mathfrak{T},\ \forall \mathfrak{C} \subseteq_{Chain} \mathfrak{S}, \qquad \left( (\bigsqcup_{\mathfrak{S}} \mathfrak{C}) \in \lceil \mathfrak{A}_{\mathfrak{t}} \rceil \right) \ \Rightarrow\ \left( \exists \sigma \in \mathfrak{C} \mid \sigma \in \lceil \mathfrak{A}_{\mathfrak{t}} \rceil \right). \tag{30}$$

The set of states $\lceil \mathfrak{A}_{\mathfrak{t}} \rceil$, for which the property tested by $\mathfrak{t}$ is actual, must be a 'Scott-open filter' in the directed-complete partial order $(\mathfrak{S}, \sqsubseteq_{\mathfrak{S}})$.

We can reformulate the above requirement in terms of a continuity property of the evaluation map with respect to the sharpening process.

**Lemma 8.** For any $\mathfrak{t} \in \mathfrak{T}$, the map $\widetilde{\mathfrak{e}}_{\mathfrak{t}}$ is chain-continuous, i.e., continuous with respect to the Scott topology on $\mathfrak{S}$ and $\mathfrak{B}$. In other words,

$$\forall \mathfrak{t} \in \mathfrak{T},\ \forall \mathfrak{C} \subseteq_{Chain} \mathfrak{S}, \qquad \bigvee_{\sigma \in \mathfrak{C}} \widetilde{\mathfrak{e}}_{\mathfrak{t}}(\sigma) = \widetilde{\mathfrak{e}}_{\mathfrak{t}}(\bigsqcup_{\mathfrak{S}} \mathfrak{C}). \tag{31}$$

■

**Axioms 1, 2, 3, 4** shall be complemented by another one that is closely related to the specific character of quantum systems. C.Rovelli aims to reconstruct quantum mechanics from the following conceptual proposal [70, Chap.III]:

> *'Information is a discrete quantity: there is a minimum amount of information exchangeable (a single bit, or the information that distinguishes between just two alternatives). [...] Since information is discrete, any process of acquisition of information can be decomposed into acquisitions of elementary bits of information.'* [70, p.1655].

We will translate Rovelli's conceptual proposal as follows.

**Notion 13.** A preparation process $\mathfrak{p}_2$ is said to *contain one more bit of information than* another preparation process $\mathfrak{p}_1$ (this fact will be denoted $\mathfrak{p}_1 \,\bar{\sqsubset}_{\mathfrak{P}} \mathfrak{p}_2$), iff (i) $\mathfrak{p}_2$ is strictly sharper than $\mathfrak{p}_1$, and (ii) there is no preparation process strictly separating $\mathfrak{p}_1$ and $\mathfrak{p}_2$. In other words,

$$\forall \mathfrak{p}_1, \mathfrak{p}_2 \in \mathfrak{P},\ \left( \mathfrak{p}_1 \,\bar{\sqsubset}_{\mathfrak{P}} \mathfrak{p}_2 \right) :\Leftrightarrow \left\{ \begin{array}{l} \left( \mathfrak{p}_1 \sqsubset_{\mathfrak{P}} \mathfrak{p}_2 \right) \\ \left( \forall \mathfrak{p} \in \mathfrak{P},\ \left( \mathfrak{p}_1 \sqsubseteq_{\mathfrak{P}} \mathfrak{p} \sqsubseteq_{\mathfrak{P}} \mathfrak{p}_2 \right) \Rightarrow \left( \mathfrak{p}_1 \sim_{\mathfrak{P}} \mathfrak{p} \ \text{or}\ \mathfrak{p}_2 \sim_{\mathfrak{P}} \mathfrak{p} \right) \right). \end{array} \right. \tag{32}$$

The discreteness of the informational content, encoded in the pre-order $\sqsubseteq_{\mathfrak{P}}$ defined on the set of preparation processes $\mathfrak{P}$, is translated into the following topological condition.

**Axiom 5.** For any preparation process admitting some sharper preparation processes, there exists another preparation process which contains one more 'bit of information' than this preparation process. In other words, the pre-order $(\mathfrak{P}, \sqsubseteq_{\mathfrak{P}})$ satisfies

$$\forall \mathfrak{p}_1 \in \mathfrak{P} \smallsetminus Max(\mathfrak{P}),\ \left( \exists \mathfrak{p}_2 \in \mathfrak{P} \mid \mathfrak{p}_1 \,\bar{\sqsubset}_{\mathfrak{P}} \mathfrak{p}_2 \right). \tag{33}$$

However, this requirement is not sufficient to capture all the aspects of Rovelli's proposal. In Rovelli's terms, the exchangeable information comprises a collection of distinguishable 'bits' and the acquisition of the informational content is reduced to the acquisition of these bits of information.

**Axiom 6.** For any preparation process $\mathfrak{p}_2 \in \mathfrak{P}$ and any other $\mathfrak{p}_2' \in \mathfrak{P}$ containing one more bit of information (i.e., $\mathfrak{p}_2 \,\overline{\sqsubset}_{\mathfrak{P}}\, \mathfrak{p}_2'$), the pair $(\mathfrak{p}_2, \mathfrak{p}_2')$ *'defines' a bit of information entering into the 'decomposition' of* $\mathfrak{p}_2'$. More explicitly, for any $\mathfrak{p}_1 \in \mathfrak{P}$ 'coarser' than $\mathfrak{p}_2$, there exists a preparation process, $\mathfrak{p}_1'$, unique from the point of view of the statements that can be made about it, that (i) contains one more bit of information than $\mathfrak{p}_1$, (ii) is not coarser than $\mathfrak{p}_2$, and (iii) is coarser than $\mathfrak{p}_2'$. In other words,

$$\forall \mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_2' \in \mathfrak{P}, \qquad (\mathfrak{p}_1 \sqsubseteq_{\mathfrak{P}} \mathfrak{p}_2 \,\overline{\sqsubset}_{\mathfrak{P}}\, \mathfrak{p}_2') \;\Rightarrow\; \begin{cases} \exists \mathfrak{p}_1' \,|\, (\mathfrak{p}_1' \not\sqsubseteq_{\mathfrak{P}} \mathfrak{p}_2 \text{ and } \mathfrak{p}_1 \,\overline{\sqsubset}_{\mathfrak{P}}\, \mathfrak{p}_1' \sqsubseteq_{\mathfrak{P}} \mathfrak{p}_2') \\ \text{and } \mathfrak{p}_1' \text{ is unique up to } \sim_{\mathfrak{P}}. \end{cases} \tag{34}$$

Due to **Axiom 5** and **Axiom 6**, the space of states $(\mathfrak{S}, \sqsubseteq_{\mathfrak{S}})$ satisfies the following properties:

$$\forall \sigma \in \mathfrak{S}, \quad (\sigma \notin Max(\mathfrak{S})) \;\Rightarrow\; (\exists \sigma' \in \mathfrak{S},\ \sigma \,\overline{\sqsubset}\, \sigma'). \tag{35}$$

$$\forall \sigma_1, \sigma_2, \sigma_2' \in \mathfrak{S}, \quad (\sigma_1 \sqsubseteq \sigma_2 \,\overline{\sqsubset}\, \sigma_2',) \;\Rightarrow\; (\exists!\, \sigma_1' \in \mathfrak{S},\ |\, (\sigma_1 \,\overline{\sqsubset}\, \sigma_1' \sqsubseteq \sigma_2' \text{ and } \sigma_1' \not\sqsubseteq \sigma_2)). \tag{36}$$

## 2.3 First properties of the space of states as a domain

We now exploit the above requirements to characterize the structure of the space of states. Let us firstly summarize the collected information.
$(\mathfrak{S}, \sqsubseteq_{\mathfrak{S}})$ is a partial order (property (24)). Due to **Axiom 2**, this partial order is pointed (property (26)). Due to **Axiom 1**, $(\mathfrak{S}, \sqsubseteq_{\mathfrak{S}})$ is also an Inf semi-lattice (Lemma 6). Moreover, due to **Axiom 3**, this partial order is chain-complete (property (29)). Due to **Axiom 5**, the space of states $(\mathfrak{S}, \sqsubseteq_{\mathfrak{S}})$ satisfies the semi-artinian property (35). Lastly, due to **Axiom 6**, the space of states $(\mathfrak{S}, \sqsubseteq_{\mathfrak{S}})$ satisfies the polarization property (36).

An interesting consequence of the continuity property (31) formalizing Lemma 8 is the following property of $(\mathfrak{S}, \sqsubseteq_{\mathfrak{S}})$.

**Lemma 9.** The chain-complete Inf semi-lattice $(\mathfrak{S}, \sqsubseteq_{\mathfrak{S}})$ is meet-continuous, i.e.,

$$\forall \mathfrak{C} \subseteq_{Chain} \mathfrak{S},\ \forall \sigma \in \mathfrak{S}, \qquad \sigma \sqcap_{\mathfrak{S}} (\textstyle\bigsqcup_{\mathfrak{S}} \mathfrak{C}) = \bigsqcup_{\sigma' \in \mathfrak{C}} (\sigma \sqcap_{\mathfrak{S}} \sigma'). \tag{37}$$

■

*Proof.* This fact is easily established using the continuity of map $\widetilde{\mathfrak{e}}_{\mathfrak{t}}$ and the meet-continuity of dcpo $\mathfrak{B}$. For any $\mathfrak{C} \subseteq_{Chain} \mathfrak{S}$ and any $\sigma \in \mathfrak{S}$, $(\sigma \sqcap_{\mathfrak{S}} (\bigsqcup \mathfrak{C}))$ and $\bigsqcup_{\sigma' \in \mathfrak{C}} (\sigma \sqcap_{\mathfrak{S}} \sigma')$ exist as elements of $\mathfrak{S}$. Moreover,

$$\begin{aligned} \forall \mathfrak{t} \in \mathfrak{T},\ \widetilde{\mathfrak{e}}_{\mathfrak{t}}(\sigma \sqcap_{\mathfrak{S}} (\textstyle\bigsqcup_{\mathfrak{S}} \mathfrak{C})) &= \widetilde{\mathfrak{e}}_{\mathfrak{t}}(\sigma) \wedge \widetilde{\mathfrak{e}}_{\mathfrak{t}}((\textstyle\bigsqcup_{\sigma' \in \mathfrak{C}} \sigma')) \\ &= \widetilde{\mathfrak{e}}_{\mathfrak{t}}(\sigma) \wedge \textstyle\bigvee_{\sigma' \in \mathfrak{C}} \widetilde{\mathfrak{e}}_{\mathfrak{t}}(\sigma') \\ &= \textstyle\bigvee_{\sigma' \in \mathfrak{C}} (\widetilde{\mathfrak{e}}_{\mathfrak{t}}(\sigma) \wedge \widetilde{\mathfrak{e}}_{\mathfrak{t}}(\sigma')) \\ &= \textstyle\bigvee_{\sigma' \in \mathfrak{C}} \widetilde{\mathfrak{e}}_{\mathfrak{t}}(\sigma \sqcap_{\mathfrak{S}} \sigma') \\ &= \widetilde{\mathfrak{e}}_{\mathfrak{t}}(\textstyle\bigsqcup_{\sigma' \in \mathfrak{C}} (\sigma \sqcap_{\mathfrak{S}} \sigma')). \end{aligned}$$

We finally use the separation of the Chu space $(\mathfrak{P}, \mathfrak{T}, \mathfrak{e})$ (i.e., Lemma 1) to conclude that $(\sigma \sqcap_{\mathfrak{S}} (\bigsqcup \mathfrak{C})) = \bigsqcup_{\sigma' \in \mathfrak{C}} (\sigma \sqcap_{\mathfrak{S}} \sigma')$. □

As a consequence of the above properties, the space of states can be characterized as follows:

**Theorem 1.** $(\mathfrak{S}, \sqsubseteq_{\mathfrak{S}})$ is a selection structure (see Definition 4). ■

*Proof.* Note that, for technical reasons, the presentation of selection structures, given in Definition 4, adopts a more elementary form than that of the space of states. However, we can easily show the strict equivalence of the two presentations and thus the quoted result. More precisely, we can adopt an equivalent set of axioms for selection structures, by replacing the axioms of strong-atomicity (property 224) and weak-continuity (property 229) with the axioms of meet-continuity (property 37) and the semi-artinian property (35), and by replacing property (225) with the Inf semi-lattice axiom.
Firstly, we note that the semi-artinian property (35), is clearly a weakening of the strong-atomicity property (224).
Secondly, the weak-continuity property (229) is a weakening of the meet-continuity property (37)). Indeed, if the meet-continuity property (37) is satisfied by the selection structure $\mathfrak{S}$, we have for any $s \in \mathfrak{S}$, $(s \sqcap_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} C)) = (\bigsqcup_{\mathfrak{S}} \{s \sqcap_{\mathfrak{S}} c \mid c \in C\}) = (\bigsqcup_{\mathfrak{S}} \{(s \sqcap_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} C)) \sqcap_{\mathfrak{S}} c \mid c \in C\})$. Hence, for any $u \sqsubseteq_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} C)$ (i.e., any $u$ that can be rewritten $u = u \sqcap (\bigsqcup_{\mathfrak{S}} C)$), we have $u = (\bigsqcup_{\mathfrak{S}} \{u \sqcap_{\mathfrak{S}} c \mid c \in C\})$. Now, we obtain property (229) as a special case of this reformulation of the meet-continuity property, where $u$ is only able to satisfy $u \overline{\sqsubset}_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} C)$ and not $u \sqsubseteq_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} C)$ in general.
Thirdly, a meet-continuous semi-artinian poset is necessarily strongly atomic (i.e., properties (35) and (37)) together imply property (224)). More precisely, we aim to check the strong atomicity of $\mathfrak{S}$ (i.e., property (224)) using exclusively axioms (223)(225)(226)(227) and (37)(35).
To begin, note that if $]x,y[ = \varnothing$, then $x \overline{\sqsubset}_{\mathfrak{S}} y$ and the result is trivially established. We will then only be interested in the non-trivial case, $]x,y[ \neq \varnothing$. Let us begin with some considerations about $Max(\mathfrak{S})$.
We will denote for any $s \in \mathfrak{S}$, $\hat{s} := Max(\mathfrak{S}) \cap [s\rangle$.
First of all, for any $v \in ]x,y[$, we have necessarily $\hat{x} \supseteq \hat{v} \supseteq \hat{y} \neq \varnothing$. Now, we can distinguish two cases: either $(\exists v \in ]x,y], \hat{x} = \hat{v})$ or $(\forall v \in ]x,y], \hat{v} \subset \hat{x})$.
In the first case, we have necessarily $]x\rangle = ]x,v] \cup ]v\rangle$, and then, $\mathfrak{S}$ being semi-artinian (equation 35), $\exists t \in \mathfrak{S}, x \overline{\sqsubset}_{\mathfrak{S}} t \sqsubseteq_{\mathfrak{S}} v \sqsubseteq_{\mathfrak{S}} y$ which is the announced result.
In the second case, let us choose any $v \in ]x,y]$, such that $\hat{v} \subset \hat{x}$, and choose $u \in \hat{v}$. We have obviously $x \sqsubset_{\mathfrak{S}} v \sqsubseteq_{\mathfrak{S}} u$ and $x \sqsubset_{\mathfrak{S}} v \sqsubseteq_{\mathfrak{S}} (u \sqcap_{\mathfrak{S}} y)$.
Let us consider the subset $Q$ of $P$ given by $Q := \{s \in \mathfrak{S} \mid s \sqsubseteq_{\mathfrak{S}} u \text{ and } y \sqcap_{\mathfrak{S}} s = x\}$. $Q$ is not empty because $x$ is an element of $Q$. Let us consider $C \subseteq_{chain} Q$. We remark that, due to the definition of supremum, we have $(\bigsqcup_{\mathfrak{S}} C) \sqsubseteq_{\mathfrak{S}} u$ and, due to the meet-continuity of $\mathfrak{S}$ (equation 37), we have $y \sqcap_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} C) = \bigsqcup_{\mathfrak{S}} \{(y \sqcap_{\mathfrak{S}} c) \mid c \in C\} = x$. As a result, $(\bigsqcup_{\mathfrak{S}} C) \in Q$. From Zorn's lemma, we deduce that $Q$ has some maximal elements. Let us choose $z \in Max(Q)$. We first note that $z \sqsubseteq_{\mathfrak{S}} u$ by definition of $Q$, but the equality $z = u$ is excluded because $(u \sqcap_{\mathfrak{S}} y) \neq x$. Hence, our element $u \in \mathfrak{S}$ satisfies $z \sqsubset_{\mathfrak{S}} u$. In other words, $z \notin Max(\mathfrak{S})$. $\mathfrak{S}$ being a semi-artinian (equation 35), there thus exists $w \in \mathfrak{S}$, such that $z \overline{\sqsubset}_{\mathfrak{S}} w$. For any choice of $w$ and $y$, we can define the element $t := w \sqcap_{\mathfrak{S}} y$ that satisfies $t \sqsubseteq_{\mathfrak{S}} w$ and $t \sqsubseteq_{\mathfrak{S}} y$. We note that $z \sqcap_{\mathfrak{S}} t = z \sqcap_{\mathfrak{S}} (w \sqcap_{\mathfrak{S}} y) = (z \sqcap_{\mathfrak{S}} y) \sqcap_{\mathfrak{S}} w = x \sqcap_{\mathfrak{S}} w$. The lower-covering property (Lemma 54) has been shown without using the strong-atomicity of $\mathfrak{S}$, i.e., equation (224), and property $z \overline{\sqsubset}_{\mathfrak{S}} w$ implies that $(z \sqcap_{\mathfrak{S}} y) \overline{\sqsubset}_{\mathfrak{S}} (w \sqcap_{\mathfrak{S}} y)$ or $(z \sqcap_{\mathfrak{S}} y) = (w \sqcap_{\mathfrak{S}} y)$, i.e. $x \overline{\sqsubset}_{\mathfrak{S}} (w \sqcap_{\mathfrak{S}} y)$ or $x = (w \sqcap_{\mathfrak{S}} y)$. However, the former equality is forbidden by the maximality condition imposed on $z$ (i.e., $z \in Max(Q)$ implies $z \overline{\sqsubset}_{\mathfrak{S}} w \notin Q$).
As a result, we deduce that $x \overline{\sqsubset}_{\mathfrak{S}} t \sqsubseteq_{\mathfrak{S}} y$. This concludes the proof of the strong-atomicity of $\mathfrak{S}$.
Finally, the replacement of property (225) by the Inf semi-lattice axiom is allowed by Theorem 23 in Section 6. □

In the following, we will extensively use the definitions and results relative to selection structures above and detailed in the first appendix of this paper. In particular, we recall the following fundamental result about the space of states :

**Corollary 1. [The space of states is a locally-boolean qualitative domain]**
The space of states $(\mathfrak{S}, \sqsubseteq_{\mathfrak{S}})$ is a 'qualitative-domain' (i.e., a conditionally-distributive and finitary Scott domain in which the join-prime elements are the atoms of $\mathfrak{S}$) such that, for any finite and therefore compact element $\sigma$, the subset $(\downarrow_{\mathfrak{S}} \sigma)$ is a finite boolean lattice constituted of finite elements. ■

*Proof.* This conclusion has been obtained from Theorem 29 complemented by Remark 30. □

*Remark* 6. The condition on the space of states given by Theorem 1 (or equivalently by its corollary 1) is only a **necessary** condition that has to satisfy a space of states. We will impose further constraints (**Axiom 7**, **Axiom 8** and **Axiom 9**) on these structures in order to select those that can be retained to characterize a space of states.

**Definition 1. Sub-selection structures**
Let $(\mathfrak{S}, \sqsubseteq_{\mathfrak{S}})$ be a selection structure and $(\mathfrak{S}', \sqsubseteq_{\mathfrak{S}})$ a sub-poset of $(\mathfrak{S}, \sqsubseteq_{\mathfrak{S}})$.
$(\mathfrak{S}', \sqsubseteq_{\mathfrak{S}})$ will be said to be *a sub-selection structure of* $(\mathfrak{S}, \sqsubseteq_{\mathfrak{S}})$ iff : (i) $\mathfrak{S}'$ is a downset of $(\mathfrak{S}, \sqsubseteq_{\mathfrak{S}})$, and (ii) $(\mathfrak{S}', \sqsubseteq_{\mathfrak{S}})$ is a selection structure in itself. The set of sub-selection structures of a given selection structure $(\mathfrak{S}, \sqsubseteq_{\mathfrak{S}})$ will be denoted $\mathscr{O}_{\mathfrak{S}}$.

**Lemma 10.** $\lceil \mathfrak{Q}_{\mathfrak{l}} \rceil$ is a sub-selection structure of $\mathfrak{S}$. ■

*Proof.* $\lceil \mathfrak{Q}_{\mathfrak{l}} \rceil$ is a Scott-closed subset in $\mathfrak{S}$, as the reverse image of the Scott-closed subset $\{\bot, \mathbf{Y}\}$ by the Scott-continuous map $\widetilde{\mathfrak{e}}_{\mathfrak{l}}$. Using Lemma 70, we conclude that $\lceil \mathfrak{Q}_{\mathfrak{l}} \rceil$ is a sub-selection structure of $\mathfrak{S}$. □

## 2.4 Pure states

**Notion 14.** A state is said to be *a pure state* if and only if it cannot be built as a mixture of other states (the set of pure states will be denoted $\mathfrak{S}_{pure}$). More explicitly,

$$\sigma \in \mathfrak{S}_{pure} \quad :\Leftrightarrow \quad (\forall S \subseteq^{\neq \varnothing} \mathfrak{S},\ (\sigma = \bigsqcap\nolimits_{\mathfrak{S}} S) \ \Rightarrow\ (\sigma \in S)). \tag{38}$$

In other words, pure states are associated with complete meet-irreducible elements in the selection structure $\mathfrak{S}$.

*Remark* 7. Complete meet-irreducibility implies meet-irreducibility. In other words,

$$\sigma \in \mathfrak{S}_{pure} \quad \Rightarrow \quad \forall \sigma_1, \sigma_2 \in \mathfrak{S},\ (\sigma = \sigma_1 \sqcap_{\mathfrak{S}} \sigma_2) \ \Rightarrow\ (\sigma = \sigma_1 \text{ or } \sigma = \sigma_2). \tag{39}$$

A simple characterization of completely meet-irreducible elements within posets is adopted in [30, Definition I-4.21]. This characterization is equivalent to the above one for a bounded-complete inf semi-lattice ($\mathfrak{S}$ is bounded complete due to Remark 28). We have explicitly

**Theorem 2. [Characterization of pure states]**

$$\sigma \in \mathfrak{S}_{pure} \quad \Leftrightarrow \quad \begin{cases} \sigma \in Max(\mathfrak{S}) & \text{(Type 1)} \\ \text{or} & \\ (\uparrow^{\mathfrak{S}} \sigma) \smallsetminus \{\sigma\} \text{ admits a minimum element} & \text{(Type 2)} \end{cases} \tag{40}$$

■

It is clear that 'type 2' pure states have no physical meaning. Indeed, for any 'type 2' pure states, it exists a unique 'type 1' pure state sharper than it (and, then, containing more information than it). The existence of 'type 2' pure states in the space of state leads then to a redundant description of the system. We are then led to exclude certain locally boolean qualitative domains by a first simple condition in order to describe a consistent space of states.

**Axiom 7.** $\mathfrak{S}_{pure}$ admits no element of type 2.

**Theorem 3.** Every state can be written as a mixture of pure states. In other words,

$$\forall \sigma \in \mathfrak{S} \qquad \sigma = \bigsqcap_{\mathfrak{S}} (\mathfrak{S}_{pure} \cap (\uparrow^{\mathfrak{S}} \sigma)). \tag{41}$$

Moreover, $\mathfrak{S}_{pure}$ is the unique smallest subset of states generating any state by mixture. In other words, $\mathfrak{S}_{pure}$ is the unique smallest order-generating subset in $\mathfrak{S}$ (i.e., the unique smallest subset of $\mathfrak{S}$ satisfying property (41)). ■

*Proof.* $\mathfrak{S}$ being, in particular, a bounded-complete algebraic domain, this result is a direct consequence of [30, Theorem I-4.26]. □

**Notion 15.** We will introduce the following subset of pure states associated with any state :

$$\forall S \subseteq \mathfrak{S}^*, \qquad \underline{S} := (\mathfrak{S}_{pure} \cap (\uparrow^{\mathfrak{S}} S)). \tag{42}$$

# 3 Properties and measurements

## 3.1 Properties and States

Let us now focus on the set of yes/no tests. Adopting our perspective on the Chu duality between $\mathfrak{P}$ and $\mathfrak{T}$, it is natural to introduce the following equivalence relation on $\mathfrak{T}$.

**Notion 16.** An equivalence relation, denoted $\sim_{\mathfrak{T}}$, is defined on the set of yes/no tests $\mathfrak{T}$ :

Two yes/no tests are identified iff the corresponding statements established by the observer about any given preparation process are the same.

A *property* of the physical system is an equivalence class of yes/no tests, i.e., a class of yes/no tests that are not distinguished from the point of view of the statements that the observer can produce by using these yes/no tests on finite collections of samples.

The set of equivalence classes of yes/no tests, modulo the relation $\sim_{\mathfrak{T}}$, will be denoted $\mathscr{L}$. In other words,

$$\forall \mathfrak{t}_1, \mathfrak{t}_2 \in \mathfrak{T}, \ (\mathfrak{t}_1 \sim_{\mathfrak{T}} \mathfrak{t}_2) \ :\Leftrightarrow \ (\forall \mathfrak{p} \in \mathfrak{P}, \ \mathfrak{e}(\mathfrak{p}, \mathfrak{t}_1) = \mathfrak{e}(\mathfrak{p}, \mathfrak{t}_2)), \tag{43}$$

$$\sim_{\mathfrak{T}} \text{ is an equivalence relation}, \tag{44}$$

$$\lfloor \mathfrak{t} \rfloor := \{ \mathfrak{t}' \in \mathfrak{T} \mid \mathfrak{t}' \sim_{\mathfrak{T}} \mathfrak{t} \}, \tag{45}$$

$$\mathscr{L} := \{ \lfloor \mathfrak{t} \rfloor \mid \mathfrak{t} \in \mathfrak{T} \}. \tag{46}$$

The following equivalence justifies the use of the notion of 'property' in the literal definitions of 'potentiality' and 'actuality' :

$$\forall \mathfrak{t}_1, t_2 \in \mathfrak{T}, \qquad (\mathfrak{t}_1 \sim_{\mathfrak{T}} \mathfrak{t}_2) \ \Leftrightarrow \ (\mathfrak{Q}_{\mathfrak{t}_1} = \mathfrak{Q}_{\mathfrak{t}_2} \text{ and } \mathfrak{A}_{\mathfrak{t}_1} = \mathfrak{A}_{\mathfrak{t}_2}). \tag{47}$$

Hence,

**Notion 17.** for any $\mathfrak{l} \in \mathscr{L}$, we will now denote by $\mathfrak{Q}_{\mathfrak{l}}$ (resp. $\mathfrak{A}_{\mathfrak{l}}$) the set $\mathfrak{Q}_{\mathfrak{t}}$ (resp. $\mathfrak{A}_{\mathfrak{t}}$) taken for any $\mathfrak{t}$ such that $\mathfrak{l} = \lfloor \mathfrak{t} \rfloor$.

Moreover,

**Notion 18.** for any $\mathfrak{l} \in \mathscr{L}$, we will now denote by $\widetilde{\mathfrak{e}}_{\mathfrak{l}}$ the evaluation map defined on $\mathfrak{S}$ and defined by $\widetilde{\mathfrak{e}}_{\mathfrak{l}} := \widetilde{\mathfrak{e}}_{\mathfrak{t}}$ for any $\mathfrak{t}$ such that $\mathfrak{l} = \lfloor \mathfrak{t} \rfloor$.

**Notion 19.** A property $\mathfrak{l} \in \mathscr{L}$ will be said to be *testable* iff it can be revealed as 'actual' at least for some collections of prepared samples. In other words,

$$\mathfrak{l} \text{ is 'testable'} \quad :\Leftrightarrow \quad \mathfrak{A}_{\mathfrak{l}} \text{ is a non-empty subset of } \mathfrak{S}. \tag{48}$$

We check immediately that $\forall \mathfrak{t}_1, \mathfrak{t}_2 \in \mathfrak{T},\ (\ \mathfrak{t}_1 \sim_{\mathfrak{T}} \mathfrak{t}_2\ ) \ \Leftrightarrow\ (\ \overline{\mathfrak{t}_2} \sim_{\mathfrak{T}} \overline{\mathfrak{t}_1}\ )$. As a consequence, the bar involution will be defined on the space of properties simply by requiring

**Notion 20.**

$$\forall \mathfrak{l} \in \mathscr{L}, \qquad \overline{\mathfrak{l}} := \{\, \overline{\mathfrak{t}} \mid \mathfrak{l} = \lfloor \mathfrak{t} \rfloor \,\}. \tag{49}$$

$(\mathfrak{S}, \sqsubseteq_{\mathfrak{S}})$ being a selection structure, we know that $(\mathfrak{S}, \sqsubseteq_{\mathfrak{S}})$ is a bounded-complete Inf semi-lattice, and in particular is closed under arbitrary infima. In other words,

**Lemma 11.**

$$\forall \mathfrak{Q} \subseteq^{\neq \varnothing} \mathfrak{P}, \qquad \text{the infimum } (\textstyle\bigsqcap_{\mathfrak{S}} \lceil \mathfrak{Q} \rceil) \text{ exists in } \mathfrak{S}, \tag{50}$$

■

Moreover, we inherit from Lemma 8 the following continuity property :

**Lemma 12.**

$$\forall \mathfrak{R} \subseteq^{\neq \varnothing} \mathfrak{S},\ \forall \mathfrak{t} \in \mathfrak{T}, \qquad \widetilde{\mathfrak{e}}_{\mathfrak{t}}(\textstyle\bigsqcap_{\mathfrak{S}} \mathfrak{R}) \ = \ \bigwedge_{\sigma \in \mathfrak{R}} \widetilde{\mathfrak{e}}_{\mathfrak{t}}(\sigma). \tag{51}$$

■

*Proof.* $\forall \mathfrak{R} \subseteq^{\neq \varnothing} \mathfrak{S}$, we define $\mathfrak{M}_{\mathfrak{R}} := \{\sigma \in \mathfrak{S} \mid \sigma \sqsubseteq_{\mathfrak{S}} \mathfrak{R}\}$. $\mathfrak{M}_{\mathfrak{R}}$ is obviously directed and $\bigsqcap_{\mathfrak{S}} \mathfrak{R} = \bigsqcup_{\mathfrak{S}}^{\overline{\wedge}} \mathfrak{M}_{\mathfrak{R}}$. Now, using the Scott continuity of $\widetilde{\mathfrak{e}}_{\mathfrak{t}}$, we have $\widetilde{\mathfrak{e}}_{\mathfrak{t}}(\bigsqcap_{\mathfrak{S}} \mathfrak{R}) = \bigvee_{\sigma \in \mathfrak{M}_{\mathfrak{R}}}^{\overline{\wedge}} \widetilde{\mathfrak{e}}_{\mathfrak{t}}(\sigma)$. The monotonicity of $\widetilde{\mathfrak{e}}_{\mathfrak{t}}$ and the fact that the target space of $\widetilde{\mathfrak{e}}_{\mathfrak{t}}$ is the boolean domain $\mathfrak{B}$ implies moreover that $\bigvee_{\sigma \in \mathfrak{M}_{\mathfrak{R}}}^{\overline{\wedge}} \widetilde{\mathfrak{e}}_{\mathfrak{t}}(\sigma) = \bigwedge_{\sigma \in \mathfrak{R}} \widetilde{\mathfrak{e}}_{\mathfrak{t}}(\sigma)$. □

**Theorem 4.** For any property $\mathfrak{l} \in \mathscr{L}$, the evaluation map $\widetilde{\mathfrak{e}}_{\mathfrak{l}}$ is order preserving and continuous with respect to the Lawson topologies on $\mathfrak{S}$ and $\mathfrak{B}$. ■

*Proof.* From previous lemma we have that, for any $\mathfrak{t} \in \mathfrak{T}$, the map $\widetilde{\mathfrak{e}}_{\mathfrak{t}}$ is continuous with respect to the lower topologies on $\mathfrak{P}$ and $\mathfrak{B}$.
Due to Lemma 8 and property (51), and using [30, Theorem III-1.8 p.213], we then prove the announced continuity property. □

We can then deduce the form of the subsets $\lceil \mathfrak{A}_{\mathfrak{t}} \rceil$ determining the sub-space of states for which the property tested by $\mathfrak{t}$ is actual.

**Theorem 5. [Property-state]**
For any $\mathfrak{t}$ in $\mathfrak{T}$, corresponding to a testable property $\lfloor \mathfrak{t} \rfloor$, there exists an element $\Sigma_{\mathfrak{t}} \in \mathfrak{S}$ (in fact, a compact element in the algebraic domain $\mathfrak{S}$), such that the Scott-open filter$\lceil \mathfrak{A}_{\mathfrak{t}} \rceil$ is the principal filter associated with $\Sigma_{\mathfrak{t}}$ :

$$\forall \mathfrak{t} \in \mathfrak{T} \mid \lfloor \mathfrak{t} \rfloor \text{ is testable},\ \exists \Sigma_{\mathfrak{t}} \in \mathfrak{S}_c \quad | \quad \lceil \mathfrak{A}_{\mathfrak{t}} \rceil = (\uparrow^{\mathfrak{S}} \Sigma_{\mathfrak{t}}). \tag{52}$$

In particular, we have

$$\forall \mathfrak{t} \in \mathfrak{T}, \forall \sigma \in \mathfrak{S}, \qquad \widetilde{\mathfrak{e}}_{\mathfrak{t}}(\sigma) = \mathbf{Y} \ \Leftrightarrow\ \Sigma_{\mathfrak{t}} \sqsubseteq_{\mathfrak{S}} \sigma. \tag{53}$$

If the conjugate test $\underline{\mathfrak{t}}$ corresponds to a testable property as well, there exists an element an element $\Sigma_{\underline{\mathfrak{t}}}$ such that

$$\Sigma_{\overline{\mathfrak{t}}} = \bigsqcap_{\mathfrak{S}} \widetilde{\mathfrak{e}}_{\mathfrak{t}}^{-1}(\mathbf{N}) \tag{54}$$

$$\mathfrak{Q}_{\mathfrak{t}} = \mathfrak{S} \smallsetminus (\uparrow^{\mathfrak{S}} \Sigma_{\overline{\mathfrak{t}}}) \tag{55}$$

■

*Proof.* We have already remarked that $\lceil \mathfrak{A}_{\mathfrak{t}} \rceil$ is a Scott-open filter. Using the fact that $\mathfrak{S}$ is a complete Inf semi-lattice as well as property (51), we also note that the element $\Sigma_{\mathfrak{t}} := \bigsqcap_{\mathfrak{S}} \lceil \mathfrak{A}_{\mathfrak{t}} \rceil$ obeys $\widetilde{\mathfrak{e}}_{\mathfrak{t}}(\Sigma_{\mathfrak{t}}) = \bigwedge_{\sigma \in \lceil \mathfrak{A}_{\mathfrak{t}} \rceil} \widetilde{\mathfrak{e}}_{\mathfrak{t}}(\sigma) = \mathbf{Y}$. Then, $\Sigma_{\mathfrak{t}}$ is the 'minimum' of $\lceil \mathfrak{A}_{\mathfrak{t}} \rceil$. As a consequence, the filter $\lceil \mathfrak{A}_{\mathfrak{t}} \rceil$ is revealed to be the principal filter $(\uparrow^{\mathfrak{S}} \Sigma_{\mathfrak{t}})$. From [30, Remark I-4.24], we deduce that $\Sigma_{\mathfrak{t}}$ is a compact element in $\mathfrak{S}$. The properties associated to $\Sigma_{\overline{\mathfrak{t}}}$ are derived along the same way. □

**Notion 21.** For any yes/no test $\mathfrak{t} \in \mathfrak{T}$, corresponding to the testable property $\lfloor \mathfrak{t} \rfloor$, the state $\Sigma_{\mathfrak{t}}$ is defined to be the minimal element of the principal filter $\lceil \mathfrak{A}_{\mathfrak{t}} \rceil$ in $(\mathfrak{S}, \sqsubseteq_{\mathfrak{S}})$.

$$\forall \mathfrak{t} \in \mathfrak{T} \mid \lfloor \mathfrak{t} \rfloor \text{ is testable}, \qquad \Sigma_{\mathfrak{t}} := \bigsqcap_{\mathfrak{S}} \lceil \mathfrak{A}_{\mathfrak{t}} \rceil = \bigsqcap_{\mathfrak{S}} \widetilde{\mathfrak{e}}_{\mathfrak{t}}^{-1}(\mathbf{Y}). \tag{56}$$

The state $\Sigma_{\mathfrak{t}}$ depends only on the testable property $\lfloor \mathfrak{t} \rfloor$ associated with $\mathfrak{t}$. This state will then be called the *property-state* associated to $\mathfrak{t}$ and we will henceforth adopt the following abuse of notation $\Sigma_{\lfloor \mathfrak{t} \rfloor} := \Sigma_{\mathfrak{t}}$.

## 3.2 Tests and measurements

The measurement process offers another perspective on the relation between the spaces $\mathfrak{P}$ and $\mathfrak{T}$ (the first being a duality relation): this new perspective emphasizes the 'recursive' aspect of the preparation process. Indeed, a given yes/no test $\mathfrak{t} \in \mathfrak{T}$ can be used to complete a given preparation procedure $\mathfrak{p} \in \mathfrak{P}$ in order to produce a new preparation procedure, as the 'filtering operation' associated with $\mathfrak{t}$ actually operates on a collection of produced samples that can exhibit the desired property associated with $\mathfrak{t}$.

**Notion 22.** For any yes/no test $\mathfrak{t} \in \mathfrak{T}$ and any preparation procedure $\mathfrak{p} \in \mathfrak{Q}_{\mathfrak{t}}$, we can define a preparation procedure denoted $\mathfrak{p}.\mathfrak{t}$ and defined as follows :

the samples, previously prepared through procedure $\mathfrak{p}$, are actually submitted to the measurement operation defined according to the yes/no test $\mathfrak{t}$, the resulting (or 'outcoming') samples of the 'whole preparation process' (i.e., the initial preparation $\mathfrak{p}$ followed by the filtering operation defined by $\mathfrak{t}$), denoted $(\mathfrak{p}.\mathfrak{t})$, are the samples 'actually measured as positive' through the yes/no test $\mathfrak{t}$.

For any yes/no test $\mathfrak{t} \in \mathfrak{T}$, we will then associate it with the partial map denoted $.\mathfrak{t}$ and defined by (where its domain will be denoted $Dom_{.\mathfrak{t}}^{\mathfrak{P}} \subseteq \mathfrak{P}$, and its range $Im_{.\mathfrak{t}}^{\mathfrak{P}}$):

$$\begin{array}{rcccl} .\mathfrak{t} & : & \mathfrak{P} & \dashrightarrow & \mathfrak{P} \\ & & \mathfrak{p} & \mapsto & \mathfrak{p}.\mathfrak{t} \end{array} \qquad Dom_{.\mathfrak{t}}^{\mathfrak{P}} := \mathfrak{Q}_{\mathfrak{t}} \tag{57}$$

These maps define the measurement operation associated with a given property. The map $.\mathfrak{t}$ associated with a yes/no test $\mathfrak{t}$ is called *the measurement map* associated with $\mathfrak{t}$.

To begin, let us clarify the notion of 'succession' of measurements.

**Notion 23.** For any yes/no tests $\mathfrak{t}_1, \mathfrak{t}_2 \in \mathfrak{T}$, we can build a new yes/no test denoted $(\mathfrak{t}_1.\mathfrak{t}_2) \in \mathfrak{T}$ and called *the succession of* $\mathfrak{t}_1$ *by* $\mathfrak{t}_2$. It is defined as follows :

to begin, the incoming sample is tested through the yes/no test $\mathfrak{t}_1$; if the result of this test is negative, then the whole yes/no test $(\mathfrak{t}_1 \centerdot \mathfrak{t}_2)$ is declared 'negative', but if the result is positive, the outcoming sample having been positively measured by $\mathfrak{t}_1$ is submitted to the yes/no test $\mathfrak{t}_2$; the result of this test is then attributed to the whole yes/no test $(\mathfrak{t}_1 \centerdot \mathfrak{t}_2)$ for the given prepared sample.
In other words,

$$\forall \mathfrak{t}_1, \mathfrak{t}_2 \in \mathfrak{T}, \exists (\mathfrak{t}_1 \centerdot \mathfrak{t}_2) \in \mathfrak{T} \quad |$$

$$\mathfrak{A}_{\mathfrak{t}_1 \centerdot \mathfrak{t}_2} := \{\, \mathfrak{p} \in \mathfrak{A}_{\mathfrak{t}_1} \mid (\mathfrak{p} \centerdot \mathfrak{t}_1) \in \mathfrak{A}_{\mathfrak{t}_2} \,\} \tag{58}$$

$$\mathfrak{Q}_{\mathfrak{t}_1 \centerdot \mathfrak{t}_2} := \{\, \mathfrak{p} \in \mathfrak{Q}_{\mathfrak{t}_1} \mid (\mathfrak{p} \centerdot \mathfrak{t}_1) \in \mathfrak{Q}_{\mathfrak{t}_2} \,\} \tag{59}$$

$$\forall \mathfrak{p} \in Dom^{\mathfrak{P}}_{\centerdot(\mathfrak{t}_1 \centerdot \mathfrak{t}_2)} = \mathfrak{Q}_{\mathfrak{t}_1 \centerdot \mathfrak{t}_2}, \;\; \mathfrak{p} \centerdot (\mathfrak{t}_1 \centerdot \mathfrak{t}_2) := (\mathfrak{p} \centerdot \mathfrak{t}_1) \centerdot \mathfrak{t}_2. \tag{60}$$

The 'succession law' satisfies the following associativity properties:

$$\forall \mathfrak{p} \in \mathfrak{P}, \forall \mathfrak{t}_1, \mathfrak{t}_2, \in \mathfrak{T}, \qquad \mathfrak{p} \centerdot (\mathfrak{t}_1 \centerdot \mathfrak{t}_2) = (\mathfrak{p} \centerdot \mathfrak{t}_1) \centerdot \mathfrak{t}_2, \tag{61}$$

$$\forall \mathfrak{t}_1, \mathfrak{t}_2, \mathfrak{t}_3 \in \mathfrak{T}, \qquad (\mathfrak{t}_1 \centerdot \mathfrak{t}_2) \centerdot \mathfrak{t}_3 = \mathfrak{t}_1 \centerdot (\mathfrak{t}_2 \centerdot \mathfrak{t}_3). \tag{62}$$

We note the following natural relations.

**Lemma 13.**

$$\forall \mathfrak{t}_1, \mathfrak{t}_2 \in \mathfrak{T}, \qquad \forall \mathfrak{p} \in \mathfrak{Q}_{\mathfrak{t}_1}, \;\; \mathfrak{e}(\mathfrak{p} \centerdot \mathfrak{t}_1, \mathfrak{t}_2) \geq \mathfrak{e}(\mathfrak{p}, \mathfrak{t}_1 \centerdot \mathfrak{t}_2), \tag{63}$$

$$\overline{\mathfrak{t}_1 \centerdot \mathfrak{t}_2} = \overline{\mathfrak{t}_1} \centerdot \overline{\mathfrak{t}_2} \tag{64}$$

■

As a basic requirement of measurement maps, we will impose that they are monotone maps on their domain. The following simple analysis justifies this requirement.

**Theorem 6.** A measurement operation associated with any yes/no test, $\mathfrak{t}$, respects the ordering of information established by the observer about the collection of samples on which it is realized.

$$\forall \mathfrak{t} \in \mathfrak{T}, \;\; \forall \mathfrak{p}_1, \mathfrak{p}_2 \in \mathfrak{Q}_{\mathfrak{t}}, \qquad (\mathfrak{p}_1 \sqsubseteq_{\mathfrak{P}} \mathfrak{p}_2) \;\Rightarrow\; ((\mathfrak{p}_1 \centerdot \mathfrak{t}) \sqsubseteq_{\mathfrak{P}} (\mathfrak{p}_2 \centerdot \mathfrak{t})). \tag{65}$$

The measurement map $(\centerdot \mathfrak{t})$ is an order-preserving map on $(\mathfrak{P}, \sqsubseteq_{\mathfrak{P}})$. ■

*Proof.* If the preparation processes $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are ordered by $\sqsubseteq_{\mathfrak{P}}$ ($\mathfrak{p}_2$ being sharper than $\mathfrak{p}_1$), every statement made by the observer about $\mathfrak{p}_1$ is also necessarily made about $\mathfrak{p}_2$, i.e., $\forall \mathfrak{u} \in \mathfrak{T}, \mathfrak{e}(\mathfrak{p}_1, \mathfrak{u}) \leq \mathfrak{e}(\mathfrak{p}_2, \mathfrak{u})$. This is true in particular for the statements that can be made by the observer about the corresponding collections of samples after having separately measured the property associated with a given yes/no test, $\mathfrak{t}$, on each collection of samples beforehand. More precisely, we must then have $\forall \mathfrak{v} \in \mathfrak{T}, \mathfrak{e}(\mathfrak{p}_1 \centerdot \mathfrak{t}, \mathfrak{v}) \leq \mathfrak{e}(\mathfrak{p}_2 \centerdot \mathfrak{t}, \mathfrak{v})$. □

**Corollary 2.** As a consequence, the measurement operation $(\centerdot \mathfrak{t})$ associated with a given yes/no test, $\mathfrak{t}$, cannot distinguish different collections of samples on which it acts, when these collections of samples correspond to the same state of the system, i.e.,

$$\forall \mathfrak{t} \in \mathfrak{T}, \;\; \forall \mathfrak{p}_1, \mathfrak{p}_2 \in \mathfrak{Q}_{\mathfrak{t}}, \qquad (\mathfrak{p}_1 \sim_{\mathfrak{P}} \mathfrak{p}_2) \;\Rightarrow\; ((\mathfrak{p}_1 \centerdot \mathfrak{t}) \sim_{\mathfrak{P}} (\mathfrak{p}_2 \centerdot \mathfrak{t})). \tag{66}$$

■

**Notion 24.** The measurement operation will then be defined to act on states as follows:

$$\forall \mathfrak{t} \in \mathfrak{T},\ \forall \mathfrak{p} \in \mathfrak{Q}_{\mathfrak{t}}, \qquad \lceil \mathfrak{p} \rceil . \mathfrak{t} := \lceil \mathfrak{p} . \mathfrak{t} \rceil \tag{67}$$

We will adopt the following notations $Dom^{\mathfrak{S}}_{.\mathfrak{t}} := \lceil Dom^{\mathfrak{P}}_{.\mathfrak{t}} \rceil$ and $Im^{\mathfrak{S}}_{.\mathfrak{t}} := \lceil Im^{\mathfrak{P}}_{.\mathfrak{t}} \rceil$.

**Corollary 3.** The measurement map $(.\mathfrak{t})$ is an order-preserving map on $(\mathfrak{S}, \sqsubseteq_{\mathfrak{S}})$

$$\forall \mathfrak{t} \in \mathfrak{T}, \forall \sigma_1, \sigma_2 \in \mathfrak{S}_{\mathfrak{t}}, \qquad (\sigma_1 \sqsubseteq_{\mathfrak{S}} \sigma_2) \ \Rightarrow\ ((\sigma_1 . \mathfrak{t}) \sqsubseteq_{\mathfrak{S}} (\sigma_2 . \mathfrak{t})). \tag{68}$$

■

We then complement the above property of monotonicity with a requirement concerning random mixing.

**Theorem 7.** The operation of measurement 'commutes' with the operation of random mixing on the measured samples. We will more generally have

$$\forall \mathfrak{t} \in \mathfrak{T}, \forall Q \subseteq \mathfrak{Q}_{\mathfrak{t}}, \qquad (\bigsqcap\nolimits_{\mathfrak{P}} Q) . \mathfrak{t} \sim_{\mathfrak{P}} \bigsqcap\nolimits_{\mathfrak{p} \in Q} (\mathfrak{p} . \mathfrak{t}). \tag{69}$$

■

*Proof.* The statement of the observer being realized after the measurement has been realized separately on each 'individual' sample the proposal is tautological. More precisely, the collection of samples produced by the random mixing of two collections of samples obtained by separately measuring by $(.\mathfrak{t})$ the collections of samples prepared through preparation processes $\mathfrak{p}_1$ and $\mathfrak{p}_2$, which can potentially exhibit property $\lfloor \mathfrak{t} \rfloor$, cannot be distinguished by the statements of the observer from the collection of samples produced by performing the measurement operation $(.\mathfrak{t})$ on the collection of samples obtained by the random mixing of the collections prepared through $\mathfrak{p}_1$ and $\mathfrak{p}_2$. In other words,

$$\forall \mathfrak{t} \in \mathfrak{T}, \forall \mathfrak{p}_1, \mathfrak{p}_2 \in \mathfrak{Q}_{\mathfrak{t}}, \qquad (\mathfrak{p}_1 \sqcap_{\mathfrak{P}} \mathfrak{p}_2) . \mathfrak{t} \sim_{\mathfrak{P}} (\mathfrak{p}_1 . \mathfrak{t}) \sqcap_{\mathfrak{P}} (\mathfrak{p}_2 . \mathfrak{t}). \tag{70}$$

More generally, we have the property (69). □

**Corollary 4.** As an immediate consequence of the above requirement, for any yes/no test, $\mathfrak{t}$, the measurement map $(.\mathfrak{t})$ is a partial map preserving arbitrary infima on the space of states. In other words,

$$\forall \mathfrak{t} \in \mathfrak{T}, \forall S \subseteq \lceil \mathfrak{Q}_{\mathfrak{t}} \rceil, \qquad (\bigsqcap\nolimits_{\mathfrak{S}} S) . \mathfrak{t} = \bigsqcap\nolimits_{\sigma \in S} (\sigma . \mathfrak{t}). \tag{71}$$

■

Endly, we will observe the following property.

**Theorem 8.** The operation of measurement respects the induction process of a limit state from any sharpening family. In other words, for any yes/no test, $\mathfrak{t}$, the measurement map $(.\mathfrak{t})$ is a Scott-continuous partial map.

$$\forall \mathfrak{t} \in \mathfrak{T}, \forall \mathfrak{C} \subseteq_{Chain} \lceil \mathfrak{Q}_{\mathfrak{t}} \rceil, \qquad \bigsqcup\nolimits_{\sigma \in \mathfrak{C}} (\sigma . \mathfrak{t}) = (\bigsqcup\nolimits_{\mathfrak{S}} \mathfrak{C}) . \mathfrak{t}. \tag{72}$$

■

*Proof.* Let us consider a sharpening family of preparation processes $\mathfrak{Q}$. The existence of the limit state $(\bigsqcup_{\mathfrak{S}} \lceil \mathfrak{Q} \rceil)$ is ensured by the collection of properties $\bigvee_{\sigma \in \lceil \mathfrak{Q} \rceil} \widetilde{\mathfrak{e}}_{\mathfrak{u}}(\sigma) = \widetilde{\mathfrak{e}}_{\mathfrak{u}}(\bigsqcup_{\mathfrak{S}} \lceil \mathfrak{Q} \rceil)$ considered for every yes/no test $\mathfrak{u} \in \mathfrak{T}$. If we consider, in particular, the subset of statements associated with any yes/no test, $\mathfrak{t}_2$, made by the observer concerning the states outcoming from the measurement associated with a yes/no test, $\mathfrak{t}_1$, we deduce $\bigvee_{\sigma \in \lceil \mathfrak{Q} \rceil} \widetilde{\mathfrak{e}}_{\mathfrak{t}_2}(\sigma . \mathfrak{t}_1) = \widetilde{\mathfrak{e}}_{\mathfrak{t}_2}((\bigsqcup_{\mathfrak{S}} \lceil \mathfrak{Q} \rceil) . \mathfrak{t}_1)$. And thus, for any $\mathfrak{t}_1, \mathfrak{t}_2 \in \mathfrak{T}$, we obtain $\widetilde{\mathfrak{e}}_{\mathfrak{t}_2}(\bigsqcup_{\sigma \in \lceil \mathfrak{Q} \rceil} (\sigma . \mathfrak{t}_1)) = \widetilde{\mathfrak{e}}_{\mathfrak{t}_2}((\bigsqcup_{\mathfrak{S}} \lceil \mathfrak{Q} \rceil) . \mathfrak{t}_1)$. □

**Definition 2.** The set of partial maps defined from the selection structure $\mathfrak{S}_1$ (the domain of the partial map has to be a Scott-closed subset of $\mathfrak{S}_1$) to the selection structure $\mathfrak{S}_2$, which are order preserving, Scott continuous and preserve arbitrary meets will be denoted $[\mathfrak{S}_1 \dashrightarrow \mathfrak{S}_2]^{\overline{\wedge}}_{\sqcap}$. The notation $[\mathfrak{S}_1 \dashrightarrow \mathfrak{S}_2]_{\sqcap}$ will be reserved for the set of order-preserving partial maps which preserve arbitrary meets (we omit the requirement of Scott continuity).

To summarize Theorem 6, Theorem 7 and Theorem 8 , we will have :

$$\forall \mathfrak{t} \in \mathfrak{T}, \qquad .\mathfrak{t} \in [\mathfrak{S} \dashrightarrow \mathfrak{S}]^{\overline{\wedge}}_{\sqcap} \mid Dom^{\mathfrak{S}}_{.\mathfrak{t}} = \lceil \mathfrak{Q}_{\mathfrak{t}} \rceil. \tag{73}$$

### 3.3 Minimally disturbing measurements

As analyzed above, the 'certainty' of the observer about the occurrence (or not) of a given 'property' for a given state, is formulated as a counterfactual statement ('actuality' or 'impossibility') about the tests that 'could' be realized on any sample corresponding to this state (and this certainty has been produced after having tested this property on similarly prepared samples). *Stricto sensu* this statement is then formulated without disturbing in any way the considered new sample, according to the definition of the 'elements of reality' for the system given in the celebrated paper of A. Einstein, B. Podolsky, and N. Rosen :

> *'If, without in any way disturbing a system, we can predict with certainty (i.e., with probability equal to unity) the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity.'* [26].

Nevertheless, in order to establish an interpretation of 'properties' as elements of reality, the observer must be able to establish and confirm 'conjointly' statements about the different properties that are considered 'simultaneously' actual for a given collection of similarly prepared samples. It is thus necessary to restrict the measurement operations that will be used by the observer, with regard to the disturbance they cause in the measured sample. These measurement operations should guarantee that the state of the system after measurement be characterized by the properties established as actual beforehand, through 'successive' measurement operations. The possibility (and necessity) to characterize minimally disturbing measurements exists in the classical and in the quantum situation as well. However, although the existence of such measurements does not pose any conceptual problem in the classical situation, things are more complex in the quantum situation as soon as the measurement process irreducibly 'alters' the state of the measured system.
Despite the indeterministic character of quantum measurements, we note that the realization of a 'careful' yes/no test does allow the observer to make some statements about the state 'after' the measurement, although it appears risky to extend these conclusions to the state of the system 'before' the measurement (due to the irreducible alteration of the state during the measurement operation). At least, a 'careful' measurement of a given property on a given sample[20] may guarantee the actuality of this property just after the measurement. As a consequence, the immediate repetition of this test should produce with certainty the same 'answer'. These sort of careful measurements do exist in the classical and in the quantum situation as well, and have been called *first-kind measurements* by W. Pauli :

[20] Here we mean that the test is effectuated on the considered sample, according to the procedure defined by a yes/no test associated with this property, and the 'answer' received by the observer is 'positive'.

*'The method of measurement [...] has the property that a repetition of measurement gives the same value for the quantity measured as in the first measurement. In other words, if the result of using the measuring apparatus is not known, but only the fact of its use is known [...], the probability that the quantity measured has a certain value is the same, both before and after the measurement. We shall call such measurements the measurements of the first kind.'* [58, p.75][21]

We will adopt the following formal definition for this type of measurement.

**Notion 25.** A yes/no test $\mathfrak{t} \in \mathfrak{T}$ is said to lead to a *first-kind measurement* associated with the corresponding testable property $\lfloor \mathfrak{t} \rfloor$ iff (i) a positive result of the test $\mathfrak{t}$ realized on any input sample is necessarily confirmed by an immediate repetition of this test realized on the samples outcoming from the first test, and (ii) the observer cannot establish if this new 'check' has been performed or not on the basis of the new tests that could be performed on the outcoming samples of the experiment. In other words, the testable property $\lfloor \mathfrak{t} \rfloor$ can be considered as 'actual' after the measurement by $\mathfrak{t}$, and the actuality of the property $\lfloor \mathfrak{t} \rfloor$ can be 'confirmed' through any repetition of the measurement by $\mathfrak{t}$, because this repeated measurement leaves the state of the system unchanged.
The subset of yes/no tests leading to first-kind measurements will be denoted $\mathfrak{T}_{FKM}$. In other words,

$$\forall \mathfrak{t} \in \mathfrak{T}, \qquad \mathfrak{t} \in \mathfrak{T}_{FKM} \;:\Leftrightarrow\; \begin{cases} \text{(i)} & \forall \mathfrak{p} \in \mathfrak{Q}_{\mathfrak{t}},\ (\mathfrak{p}\boldsymbol{.}\mathfrak{t}) \in \mathfrak{A}_{\mathfrak{t}} \\ \text{(ii)} & \forall \mathfrak{p} \in \mathfrak{A}_{\mathfrak{t}},\ (\mathfrak{p}\boldsymbol{.}\mathfrak{t}) \sim_{\mathfrak{P}} \mathfrak{p} \end{cases} \tag{74}$$

**Lemma 14.** In terms of the action on the space of states, we then have (using **Notion 24**) :

$$\forall \mathfrak{t} \in \mathfrak{T}_{FKM}, \qquad \begin{cases} \text{(i)} & \forall \sigma \in \lceil \mathfrak{Q}_{\mathfrak{t}} \rceil,\ (\sigma\boldsymbol{.}\mathfrak{t}) \in \lceil \mathfrak{A}_{\mathfrak{t}} \rceil \\ \text{(ii)} & \forall \sigma \in \lceil \mathfrak{A}_{\mathfrak{t}} \rceil,\ (\sigma\boldsymbol{.}\mathfrak{t}) = \sigma \end{cases} \tag{75}$$

■

As a remark, we also note the following trivial lemmas.

**Lemma 15.**

$$\forall \mathfrak{t} \in \mathfrak{T}_{FKM}, \qquad (\mathfrak{p}_{\perp}\boldsymbol{.}\mathfrak{t}) \in \mathfrak{A}_{\mathfrak{t}} \neq \varnothing \tag{76}$$

$$\forall \mathfrak{t} \in \mathfrak{T}_{FKM}, \forall \mathfrak{p} \in \mathfrak{Q}_{\mathfrak{t}}, \qquad ((\mathfrak{p}\boldsymbol{.}\mathfrak{t})\boldsymbol{.}\mathfrak{t}) \sim_{\mathfrak{P}} (\mathfrak{p}\boldsymbol{.}\mathfrak{t}) \tag{77}$$

$$\forall \mathfrak{t} \in \mathfrak{T}_{FKM}, \forall \mathfrak{p} \in \mathfrak{Q}_{\mathfrak{t}}, \qquad \big( (\mathfrak{p}\boldsymbol{.}\mathfrak{t}) \sim_{\mathfrak{P}} \mathfrak{p} \big) \;\Rightarrow\; \mathfrak{p} \in \mathfrak{A}_{\mathfrak{t}} \tag{78}$$

$$\forall \mathfrak{t} \in \mathfrak{T}_{FKM}, \qquad \lceil \mathfrak{A}_{\mathfrak{t}} \rceil = Im^{\mathfrak{S}}_{\boldsymbol{.}\mathfrak{t}} = \{ \sigma \in \mathfrak{S} \mid \sigma = (\sigma\boldsymbol{.}\mathfrak{t}) \}. \tag{79}$$

■

[21] Note the distinction made by W.Pauli between measurements of the first and second kind : *'On the other hand it can also happen that the system is changed but in a controllable fashion by the measurement — even when, in the state before the measurement, the quantity measured had with certainty a definite value. In this method, the result of a repeated measurement is not the same as that of the first measurement. But still it may be that from the result of this measurement, an un-ambiguous conclusion can be drawn regarding the quantity being measured for the concerned system before the measurement. Such measurements, we call the measurements of the second kind.'* [58, p.75] To be concrete: (i) the determination of the position of a particle by a test of the presence of the particle in a given box appears to be a measurement of the first kind, (ii) the determination of the momentum of a particle by the evaluation of the 'impact' of this particle on a given detector appears to be a measurement of the second kind.

**Notion 26.** The subset of testable properties that can be tested through first-kind measurements will be denoted $\mathscr{L}_{FKM}$. The definition of this subset of $\mathscr{L}$ is then summarized as follows

$$\mathfrak{l} \in \mathscr{L}_{FKM} \quad :\Leftrightarrow \quad \exists(\centerdot\mathfrak{t}) \in [\mathfrak{S} \dashrightarrow \mathfrak{S}]^{\bar{\wedge}}_{\sqcap} \ \mid \ \begin{cases} Dom^{\mathfrak{S}}_{\centerdot\mathfrak{t}} = \lceil \mathfrak{Q}_{\mathfrak{l}} \rceil \\ Im^{\mathfrak{S}}_{\centerdot\mathfrak{t}} = \lceil \mathfrak{A}_{\mathfrak{l}} \rceil = (\uparrow^{\mathfrak{S}} \Sigma_{\mathfrak{l}}) \\ \forall \sigma \in Dom^{\mathfrak{S}}_{\centerdot\mathfrak{t}}, \ (\sigma\centerdot\mathfrak{t})\centerdot\mathfrak{t} = \sigma\centerdot\mathfrak{t} \end{cases} \tag{80}$$

In order to pronounce synthetic statements concerning the actuality of a 'collection' of properties for a given state, it appears necessary to clarify how 'successive' measurements of different properties can be used to pronounce the actuality of these properties 'conjointly' for a given sample. Indeed, an inconvenient aspect of quantum measurements emerges when different properties are tested successively on a given sample: it generically happens that the effective measurements associated with different properties 'interfere', prohibiting the actuality of two such 'incompatible' properties to be affirmed simultaneously for a given preparation process (in this context, the definition of first-kind measurements appears to be insufficient for our purpose, as this subset of measurement maps is not closed under the succession of distinct measurement operations).
This limitation of effective measurement leads to no-go theorems in the pathway to the construction of a classical logic for the description of the properties of the system, as noted originally by E.Specker :

> *'Is it possible to extend the description of a quantum mechanical system through the introduction of supplementary — fictitious — propositions in such a way that in the extended domain the classical propositional logic holds (whereby, of course, for simultaneously decidable proposition negation, conjunction and disjunction must retain their meaning)? The answer to this question is negative, except in the case of Hilbert spaces of dimension 1 and 2.'*[71]

The origins of this puzzling fact can be presented in terms of concrete measurements performed on the system. Let us consider any preparation procedure $\mathfrak{p} \in \mathfrak{P}$ guaranteeing the actuality of a given property $\mathfrak{l} \in \mathscr{L}$ for its outcoming samples (i.e., $\mathfrak{p} \in \mathfrak{A}_{\mathfrak{l}}$). The actuality of this property may then be checked by applying any yes/no test $\mathfrak{t} \in \mathfrak{T}$ associated with this property (i.e., $\mathfrak{l} = \lfloor \mathfrak{t} \rfloor$) on these prepared samples before any other experiment as long as this yes/no test leads to a first-kind measurement, i.e., $(\mathfrak{p}\centerdot\mathfrak{t}) \in \mathfrak{A}_{\mathfrak{l}}$. However, there generically exists another property $\mathfrak{l}' \in \mathscr{L}$ (and a yes/no test $\mathfrak{t}' \in \mathfrak{T}$ associated with it, i.e., $\mathfrak{l}' = \lfloor \mathfrak{t}' \rfloor$) such that: (i) the result of any yes/no test associated with this second property on these prepared samples is fundamentally indeterminate (i.e., $\mathfrak{e}(\mathfrak{p}, \mathfrak{t}') = \perp$), (ii) if we select among the outcomes of these yes/no tests the samples exhibiting this second property (i.e., the samples produced through the preparation procedure $(\mathfrak{p}\centerdot\mathfrak{t}') \in \mathfrak{A}_{\mathfrak{l}'}$), then any subsequent measurement relative to the first property on these selected samples will exhibit an indeterminacy (i.e., $\mathfrak{e}((\mathfrak{p}\centerdot\mathfrak{t}'), \mathfrak{t}) = \perp$). In other words, the actuality of the first property can no longer be affirmed (i.e., $(\mathfrak{p}\centerdot\mathfrak{t}') \notin \mathfrak{A}_{\mathfrak{l}}$) after the actuality of the second property has been effectively established by a measurement, even if the actuality of the first property had been established beforehand on these same prepared samples! To summarize, contrary to the context of classical measurements, for $\mathfrak{p} \in \mathfrak{P}$ and $\mathfrak{t} \in \mathfrak{T}$, we can not affirm that $\mathfrak{p} \sqsubseteq_{\mathfrak{P}} (\mathfrak{p}\centerdot\mathfrak{t})$.
Despite this severe limitation on the determination process of the actual properties of a quantum system, it is however possible to singularize some measurements, chosen for their ability to 'minimally perturb the system'. C.Piron summarizes the proposal for these ideal measurements as follows :

> *'In general if we test a property $\mathfrak{a}$ by performing $\alpha$, one of the corresponding questions, we disturb completely the given physical system even if $\mathfrak{a}$ is actually true. We will say that a question $\alpha$ is an ideal measurement if, when we perform it, we can assert that (i) If the answer is "yes", then the corresponding proposition $\mathfrak{a}$ is true afterwards, and (ii) If the answer is "yes" and if a property $\mathfrak{b}$ is true and compatible with $\mathfrak{a}$, then $\mathfrak{b}$ is still true afterwards.'*[63]

Nevertheless, the definitions of 'compatible properties' and 'ideal first-kind measurements' seem to be trapped in a vicious circle: two properties are *compatible* as soon as they can be 'simultaneously' stated as actual using successive ideal first-kind measurements, and measurements are defined to be ideal as soon as they respect the actuality of the properties that are compatible with the measured property![22] In order to establish a consistent description, it appears necessary to clarify these notions in our vocabulary.

**Notion 27.** A family of testable properties $\mathfrak{L} = (\mathfrak{l}_i)_{i\in I} \subseteq \mathscr{L}$ will be said to be *a compatible family of properties* (this fact will be denoted $\overbrace{\mathfrak{L}}$), iff there exists at least one preparation process $\mathfrak{p} \in \mathfrak{P}$ producing collections of samples exhibiting all of these properties as 'actual' (the statements about the occurrence of the properties $(\mathfrak{l}_i)_{i\in I}$ for the samples prepared through $\mathfrak{p}$ will all be simultaneously 'positive with certainty'). In other words,

$$\overbrace{\mathfrak{L}} \quad :\Leftrightarrow \quad \bigcap_{\mathfrak{l}\in\mathfrak{L}} \mathfrak{A}_{\mathfrak{l}} \neq \varnothing \tag{81}$$

In particular, a property $\mathfrak{l}_1$ is said to be compatible with a property $\mathfrak{l}_2$ iff $\overbrace{\mathfrak{l}_1\mathfrak{l}_2}$. This defines a binary relation called the *compatibility relation* on $\mathscr{L}$.
The compatibility relation is a reflexive and symmetric relation. Moreover, $\mathfrak{t}_1 \sqsubseteq_{\mathfrak{P}} \mathfrak{t}_2$ implies $\overbrace{\lfloor\mathfrak{t}_1\rfloor\lfloor\mathfrak{t}_2\rfloor}$.

The *ideal measurements* will be characterized as follows.

**Notion 28.** A yes/no test $\mathfrak{t}_1 \in \mathfrak{T}$ is said to *lead to an ideal measurement* of the corresponding testable property $\lfloor\mathfrak{t}_1\rfloor$ (this fact is denoted $\mathfrak{t}_1 \in \mathfrak{T}_{Ideal}$) iff, for any property $\lfloor\mathfrak{t}_2\rfloor$ compatible with $\lfloor\mathfrak{t}_1\rfloor$, the statement pronounced by the observer beforehand concerning the 'actuality' of the property $\lfloor\mathfrak{t}_2\rfloor$ is conserved after the measurement operation associated with $\mathfrak{t}_1$ has been realized, i.e.,

$$\forall\mathfrak{t}_1 \in \mathfrak{T},\quad \mathfrak{t}_1 \in \mathfrak{T}_{Ideal} \quad :\Leftrightarrow \quad (\forall\mathfrak{t}_2 \in \mathfrak{T} \mid \overbrace{\lfloor\mathfrak{t}_1\rfloor\lfloor\mathfrak{t}_2\rfloor}, \forall\mathfrak{p} \in \mathfrak{Q}_{\mathfrak{t}_1},\ \mathfrak{e}(\mathfrak{p},\mathfrak{t}_2) = \mathbf{Y} \Rightarrow \mathfrak{e}(\mathfrak{p}\centerdot\mathfrak{t}_1,\mathfrak{t}_2) = \mathbf{Y}) \tag{82}$$

$$\Leftrightarrow \quad (\forall\mathfrak{t}_2 \in \mathfrak{T},\ \overbrace{\lfloor\mathfrak{t}_1\rfloor\lfloor\mathfrak{t}_2\rfloor} \Rightarrow (\mathfrak{Q}_{\mathfrak{t}_1} \cap \mathfrak{A}_{\mathfrak{t}_2})\centerdot\mathfrak{t}_1 \subseteq \mathfrak{A}_{\mathfrak{t}_2}) \tag{83}$$

We will adopt the following notation

$$\mathfrak{T}_{IFKM} \quad := \quad \mathfrak{T}_{Ideal} \cap \mathfrak{T}_{FKM}. \tag{84}$$

*Remark* 8. When an ideal measurement operation associated with a yes/no test $\mathfrak{t}_1$ is performed on a given collection of samples, similarly prepared in such a way that property $\lfloor\mathfrak{t}_2\rfloor$ was actual before this measurement operation, the outcoming samples are such that the observer cannot distinguish if property $\lfloor\mathfrak{t}_2\rfloor$ was tested (or not) after this measurement. In effect,

$$\forall\mathfrak{t}_1,\mathfrak{t}_2 \in \mathfrak{T}_{IFKM} \mid \overbrace{\lfloor\mathfrak{t}_1\rfloor\lfloor\mathfrak{t}_2\rfloor} \qquad \forall\mathfrak{p} \in \mathfrak{A}_{\mathfrak{t}_2} \cap \mathfrak{Q}_{\mathfrak{t}_1}, \qquad \mathfrak{p}\centerdot\mathfrak{t}_1\centerdot\mathfrak{t}_2 \sim_{\mathfrak{P}} \mathfrak{p}\centerdot\mathfrak{t}_1. \tag{85}$$

[22] A basic solution to this problem has been formalized by C. Piron [63]. This construction relies on an orthomodular lattice structure introduced on the space of properties (Note: we do not expect any such a construction in our perspective). In Piron's vocabulary,

> two properties are compatible as soon as they form a boolean sub-algebra in the orthomodular lattice of properties (this requirement about the sub- boolean structure is a remnant of the particular structure on the space of properties in the classical situation).[60, p.295]

**Notion 29.** The subset of testable properties that can be tested through ideal first-kind measurements will be denoted $\mathscr{L}_{IFKM}$. The definition of this subset of $\mathscr{L}$ is then summarized as follows

$$\mathfrak{l} \in \mathscr{L}_{IFKM} \quad :\Leftrightarrow \quad \exists(\boldsymbol{.}\mathfrak{t}) \in [\mathfrak{S} \dashrightarrow \mathfrak{S}]^{\bar{\wedge}}_{\sqcap} \;\mid\; \begin{cases} Dom^{\mathfrak{S}}_{\boldsymbol{.}\mathfrak{t}} = \lceil \mathfrak{Q}_{\mathfrak{l}} \rceil \\ Im^{\mathfrak{S}}_{\boldsymbol{.}\mathfrak{t}} = \lceil \mathfrak{A}_{\mathfrak{l}} \rceil = (\uparrow^{\mathfrak{S}} \Sigma_{\mathfrak{l}}) \\ \forall \boldsymbol{\sigma} \in Dom^{\mathfrak{S}}_{\boldsymbol{.}\mathfrak{t}},\ (\boldsymbol{\sigma}\boldsymbol{.}\mathfrak{t})\boldsymbol{.}\mathfrak{t} = \boldsymbol{\sigma}\boldsymbol{.}\mathfrak{t} \\ \forall \boldsymbol{\sigma} \in \mathfrak{S} \mid \widehat{\boldsymbol{\sigma}\Sigma_{\mathfrak{l}}}^{\mathfrak{S}},\ (Dom^{\mathfrak{S}}_{\boldsymbol{.}\mathfrak{t}} \cap (\uparrow^{\mathfrak{S}} \boldsymbol{\sigma}))\boldsymbol{.}\mathfrak{t} \subseteq ((\uparrow^{\mathfrak{S}} \boldsymbol{\sigma}) \cap \lceil \mathfrak{A}_{\mathfrak{l}} \rceil) \end{cases} \tag{86}$$

The motivation to introduce 'ideal first-kind measurements' is very clear from an operational point of view.[23] If such measurement operations exist for a basic set of compatible properties, they can be used to build preparation processes designed to produce collections of samples for which these properties will be found to be 'conjointly actual'. It is also completely clear that ideal first-kind measurements exist in concrete quantum mechanical experiments [64] and it is then natural to impose their existence at the center of a quantum axiomatics. Nevertheless, nothing guaranties that such a measurement operation exists for any given property $\mathfrak{l}$ (or, equivalently, that a measurement map satisfying property (86) exists for the property $\mathfrak{l}$). Our aim, by the end of the present subsection, will be to prove that an ideal first-kind measurement map existsif the corresponding 'property' satisfies a quasi-classicality criterion.

**Lemma 16.** For any $\mathfrak{t}_1, \mathfrak{t}_2 \in \mathfrak{T}_{IFKM}$ two yes/no tests leading to ideal first-kind measurements such that $\mathfrak{t}_1$ is compatible with $\mathfrak{t}_2$, the yes/no test $(\mathfrak{t}_1\boldsymbol{.}\mathfrak{t}_2)$ leads to ideal first-kind measurements associated with the conjunction of the properties $\lfloor \mathfrak{t}_1 \rfloor$ and $\lfloor \mathfrak{t}_2 \rfloor$. In other words,

$$\forall \mathfrak{t}_1, \mathfrak{t}_2 \in \mathfrak{T}_{IFKM} \mid \overbrace{\lfloor \mathfrak{t}_1 \rfloor \lfloor \mathfrak{t}_2 \rfloor}, \qquad (\mathfrak{t}_1\boldsymbol{.}\mathfrak{t}_2) \in \mathfrak{T}_{IFKM} \tag{87}$$

$$\mathfrak{A}_{\mathfrak{t}_1\boldsymbol{.}\mathfrak{t}_2} = (\mathfrak{A}_{\mathfrak{t}_1} \cap \mathfrak{A}_{\mathfrak{t}_2}). \tag{88}$$

■

*Proof.* Using equations (58) and (74 (ii)), we deduce (88). Let us now prove that $(\mathfrak{t}_1\boldsymbol{.}\mathfrak{t}_2) \in \mathfrak{T}_{IFKM}$. For any $\mathfrak{p} \in \mathfrak{Q}_{(\mathfrak{t}_1\boldsymbol{.}\mathfrak{t}_2)}$, we know from equation (59) that $(\mathfrak{p}\boldsymbol{.}\mathfrak{t}_1) \in \mathfrak{A}_{\mathfrak{t}_1} \cap \mathfrak{Q}_{\mathfrak{t}_2}$. Using equations (74 (i)) and (83), we then deduce that $((\mathfrak{p}\boldsymbol{.}\mathfrak{t}_1)\boldsymbol{.}\mathfrak{t}_2) \in \mathfrak{A}_{\mathfrak{t}_1} \cap \mathfrak{A}_{\mathfrak{t}_2} = \mathfrak{A}_{(\mathfrak{t}_1\boldsymbol{.}\mathfrak{t}_2)}$ (using (88)), i.e., equation (74 (i)) applied to the yes/no test $(\mathfrak{t}_1\boldsymbol{.}\mathfrak{t}_2)$. For any $\mathfrak{p} \in \mathfrak{A}_{(\mathfrak{t}_1\boldsymbol{.}\mathfrak{t}_2)} = (\mathfrak{A}_{\mathfrak{t}_1} \cap \mathfrak{A}_{\mathfrak{t}_2})$ (using (88)), we deduce from equations (74 (ii)) and (60) that $(\mathfrak{p}\boldsymbol{.}(\mathfrak{t}_1\boldsymbol{.}\mathfrak{t}_2)) = ((\mathfrak{p}\boldsymbol{.}\mathfrak{t}_1)\boldsymbol{.}\mathfrak{t}_2) \sim_{\mathfrak{P}} (\mathfrak{p}\boldsymbol{.}\mathfrak{t}_2) \sim_{\mathfrak{P}} \mathfrak{p}$, i.e., equation (74 (ii)) applied to the yes/no test $(\mathfrak{t}_1\boldsymbol{.}\mathfrak{t}_2)$. Let us now consider a yes/no test $\mathfrak{t} \in \mathfrak{T}$ compatible with the yes/no test $(\mathfrak{t}_1\boldsymbol{.}\mathfrak{t}_2)$. In other words, we have $\mathfrak{A}_{\mathfrak{t}} \cap \mathfrak{A}_{(\mathfrak{t}_1\boldsymbol{.}\mathfrak{t}_2)} \neq \varnothing$. As a result, we have then $\mathfrak{A}_{\mathfrak{t}} \cap \mathfrak{A}_{\mathfrak{t}_1} \neq \varnothing$ and $\mathfrak{A}_{\mathfrak{t}} \cap \mathfrak{A}_{\mathfrak{t}_2} \neq \varnothing$, i.e., $\overbrace{\lfloor \mathfrak{t} \rfloor \lfloor \mathfrak{t}_1 \rfloor}$ and $\overbrace{\lfloor \mathfrak{t} \rfloor \lfloor \mathfrak{t}_2 \rfloor}$. Using successively (60), (59), and the compatibility relations $\overbrace{\lfloor \mathfrak{t} \rfloor \lfloor \mathfrak{t}_1 \rfloor}$ and $\overbrace{\lfloor \mathfrak{t} \rfloor \lfloor \mathfrak{t}_2 \rfloor}$ coupled with property (83), we obtain $(\mathfrak{Q}_{(\mathfrak{t}_1\boldsymbol{.}\mathfrak{t}_2)} \cap \mathfrak{A}_{\mathfrak{t}})\boldsymbol{.}(\mathfrak{t}_1\boldsymbol{.}\mathfrak{t}_2) \subseteq ((\mathfrak{Q}_{\mathfrak{t}_2} \cap \mathfrak{A}_{\mathfrak{t}})\boldsymbol{.}\mathfrak{t}_2) \subseteq \mathfrak{A}_{\mathfrak{t}}$, i.e., property (83) for the compatibility property $\overbrace{\lfloor \mathfrak{t} \rfloor \lfloor (\mathfrak{t}_1\boldsymbol{.}\mathfrak{t}_2) \rfloor}$. □

**Notion 30.** A preparation process $\mathfrak{p} \in \mathfrak{P}$ is said to be *consistent with the actuality of a given testable property* $\mathfrak{l} \in \mathscr{L}$ iff there exists a preparation process $\mathfrak{p}' \in \mathfrak{P}$, sharper than $\mathfrak{p}$, and for which the property $\mathfrak{l}$ is actual. We denote by $\mathfrak{K}_{\mathfrak{l}}$ the set of preparation processes consistent with the actuality of the testable property $\mathfrak{l}$, i.e.,

$$\forall \mathfrak{l} \in \mathscr{L}, \qquad \mathfrak{K}_{\mathfrak{l}} := \{\, \mathfrak{p} \in \mathfrak{P} \mid \exists \mathfrak{p}' \in \mathfrak{A}_{\mathfrak{l}},\ \mathfrak{p} \sqsubseteq_{\mathfrak{P}} \mathfrak{p}' \,\}. \tag{89}$$

[23] These measurements played a fundamental role in Mackey's traditional axiomatic approach to quantum theory [17].

$\lceil \mathfrak{K}_{\mathfrak{l}} \rceil$ will be called *the consistency domain of the property* $\mathfrak{l}$. We have

$$\forall \mathfrak{l} \in \mathscr{L}, \qquad \lceil \mathfrak{K}_{\mathfrak{l}} \rceil = \downarrow_{\mathfrak{S}} \lceil \mathfrak{A}_{\mathfrak{l}} \rceil = \{ \sigma' \in \mathfrak{S} \mid \widehat{\Sigma_{\mathfrak{l}} \sigma'}^{\mathfrak{S}} \}. \tag{90}$$

**Lemma 17.** For any testable property $\mathfrak{l} \in \mathscr{L}$, the consistency domain $\lceil \mathfrak{K}_{\mathfrak{l}} \rceil$ is a Scott-closed subset of $\lceil \mathfrak{Q}_{\mathfrak{l}} \rceil$ and thus equivalently a sub-selection structure of $\lceil \mathfrak{Q}_{\mathfrak{l}} \rceil$. ■

*Proof.* First, for any $\sigma' \in \lceil \mathfrak{K}_{\mathfrak{l}} \rceil$ the property $\widehat{\Sigma_{\mathfrak{l}} \sigma'}^{\mathfrak{S}}$ implies $\exists \sigma'' \in \lceil \mathfrak{A}_{\mathfrak{l}} \rceil$ with $\sigma' \sqsubseteq_{\mathfrak{S}} \sigma''$, and therefore $\widetilde{\mathfrak{e}}_{\mathfrak{t}}(\sigma') \leq \widetilde{\mathfrak{e}}_{\mathfrak{t}}(\sigma'') = \mathbf{Y}$ for any $\mathfrak{t}$ such that $\mathfrak{l} = \lfloor \mathfrak{t} \rfloor$. As a consequence, $\lceil \mathfrak{K}_{\mathfrak{l}} \rceil \subseteq \lceil \mathfrak{Q}_{\mathfrak{l}} \rceil$. Moreover, $\lceil \mathfrak{K}_{\mathfrak{l}} \rceil$ is obviously a downset. Lastly, let us consider $\mathfrak{C} \subseteq_{Chain} \mathfrak{S}$ such that $(\forall \mathfrak{c} \in \mathfrak{C}, \widehat{\Sigma_{\mathfrak{l}} \mathfrak{c}}^{\mathfrak{S}})$. For any $\mathfrak{c} \in \mathfrak{C}$, we can define the element $(\Sigma_{\mathfrak{l}} \sqcup_{\mathfrak{S}} \mathfrak{c})$. The chain $\mathfrak{C}' := \{ \Sigma_{\mathfrak{l}} \sqcup_{\mathfrak{S}} \mathfrak{c} \mid \mathfrak{c} \in \mathfrak{C} \}$ admits a supremum in $\mathfrak{S}$ satisfying (i) $\forall \mathfrak{c}' \in \mathfrak{C}'$, $\Sigma_{\mathfrak{l}} \sqsubseteq_{\mathfrak{S}} \mathfrak{c}'$ and thus $\Sigma_{\mathfrak{l}} \sqsubseteq_{\mathfrak{S}} \bigsqcup_{\mathfrak{S}} \mathfrak{C}'$, and (ii) $\forall \mathfrak{c} \in \mathfrak{C}$, $\exists \mathfrak{c}' \in \mathfrak{C}' \mid \mathfrak{c} \sqsubseteq_{\mathfrak{S}} \mathfrak{c}'$ and thus $\mathfrak{c} \sqsubseteq_{\mathfrak{S}} \bigsqcup_{\mathfrak{S}} \mathfrak{C}'$ and therefore $\bigsqcup_{\mathfrak{S}} \mathfrak{C} \sqsubseteq_{\mathfrak{S}} \bigsqcup_{\mathfrak{S}} \mathfrak{C}'$. As a result, we have obtained $(\widehat{\Sigma_{\mathfrak{l}} \bigsqcup_{\mathfrak{S}} \mathfrak{C}}^{\mathfrak{S}})$ and thus $(\bigsqcup_{\mathfrak{S}} \mathfrak{C}) \in \lceil \mathfrak{K}_{\mathfrak{l}} \rceil$. This chain-completeness property implies the directed completeness of $\lceil \mathfrak{K}_{\mathfrak{l}} \rceil$.
$\lceil \mathfrak{K}_{\mathfrak{l}} \rceil$ is then equivalently a sub-selection structure of $\lceil \mathfrak{Q}_{\mathfrak{l}} \rceil$, using Lemma 70. □

**Lemma 18.** For any testable property $\mathfrak{l} \in \mathscr{L}$, the map

$$\begin{array}{rcccl} \theta_{\mathfrak{l}} & : & \lceil \mathfrak{K}_{\mathfrak{l}} \rceil & \to & \lceil \mathfrak{A}_{\mathfrak{l}} \rceil \\ & & \sigma & \mapsto & \theta_{\mathfrak{l}}(\sigma) := \Sigma_{\mathfrak{l}} \sqcup_{\mathfrak{S}} \sigma. \end{array} \tag{91}$$

is an idempotent, order-preserving, Scott-continuous map. It preserves arbitrary infima and existing suprema. ■

*Proof.* Firstly, from the basic properties of $\sqcup_{\mathfrak{S}}$, we know that $\theta_{\mathfrak{l}}$ is idempotent and order preserving. Secondly, if we denote by $\iota$ the inclusion of $\lceil \mathfrak{A}_{\mathfrak{l}} \rceil$ in $\lceil \mathfrak{K}_{\mathfrak{l}} \rceil$, we have $id \sqsubseteq \iota \circ \theta_{\mathfrak{l}}$ and $\theta_{\mathfrak{l}} \circ \iota = id$. As a result, the right adjoint $\theta_{\mathfrak{l}}$ of this Galois connection preserves existing suprema. In particular, $\theta_{\mathfrak{l}}$ is Scott continuous.
Thirdly, from the conditional-distributivity property satisfied in $\mathfrak{S}$ (property (265)), we deduce that $\theta_{\mathfrak{l}}$ preserves finite meets. Furthermore, $\theta_{\mathfrak{l}}$ preserves filtered infima. Indeed, let us consider any filtered set $F \subseteq_{Fil} \lceil \mathfrak{K}_{\mathfrak{l}} \rceil$ and let us denote $\sigma \in \lceil \mathfrak{K}_{\mathfrak{l}} \rceil \mid F, \Sigma_{\mathfrak{l}} \sqsubseteq_{\mathfrak{S}} \sigma$, then we have, using the properties of complementation in $\mathfrak{S}$ (property (259)) and meet-continuity, $\theta_{\mathfrak{l}}(\bigsqcap^{\vee}_{\mathfrak{S}} F) = \Sigma_{\mathfrak{l}} \sqcup_{\mathfrak{S}} \bigsqcap^{\vee}_{\mathfrak{S}} F = \sigma \smallsetminus ((\sigma \smallsetminus \Sigma_{\mathfrak{l}}) \sqcap_{\mathfrak{S}} (\sigma \smallsetminus (\bigsqcap^{\vee}_{\mathfrak{S}} F))) = \sigma \smallsetminus ((\sigma \smallsetminus \Sigma_{\mathfrak{l}}) \sqcap_{\mathfrak{S}} \bigsqcup^{\overline{\wedge}}_{\sigma' \in F} (\sigma \smallsetminus \sigma'))) = \sigma \smallsetminus \bigsqcup^{\overline{\wedge}}_{\sigma' \in F} ((\sigma \smallsetminus \Sigma_{\mathfrak{l}}) \sqcap_{\mathfrak{S}} (\sigma \smallsetminus \sigma'))) = \bigsqcap^{\vee}_{\sigma' \in F} (\Sigma_{\mathfrak{l}} \sqcup_{\mathfrak{S}} \sigma')$. As a consequence of these two basic properties, we obtain that $\theta_{\mathfrak{l}}$ preserves arbitrary infima. □

**Notion 31.** A testable property $\mathfrak{l} \in \mathscr{L}$ is said to be *quasi-classical* (this fact is denoted $\mathfrak{l} \in \mathscr{L}_{q-cl}$) iff the consistency-domain $\lceil \mathfrak{K}_{\mathfrak{l}} \rceil$ is a continuous retract of the domain $\lceil \mathfrak{Q}_{\mathfrak{l}} \rceil$, i.e.,

$$\forall \mathfrak{l} \in \mathscr{L}, \ \ \mathfrak{l} \in \mathscr{L}_{q-cl} \ :\Leftrightarrow \ \exists \ \pi_{\mathfrak{l}} : \lceil \mathfrak{Q}_{\mathfrak{l}} \rceil \longrightarrow \lceil \mathfrak{K}_{\mathfrak{l}} \rceil \ \mid \qquad \pi_{\mathfrak{l}} \text{ is Scott continuous} \tag{92}$$

$$\forall \sigma \in \lceil \mathfrak{K}_{\mathfrak{l}} \rceil, \ \pi_{\mathfrak{l}}(\sigma) = \sigma \tag{93}$$

$$\forall \sigma' \in \lceil \mathfrak{Q}_{\mathfrak{l}} \rceil, \ \pi_{\mathfrak{l}}(\sigma') \sqsubseteq_{\mathfrak{S}} \sigma'. \tag{94}$$

**Definition 3. [Idealized sub-selection structures]** Let $(\mathfrak{S}, \sqsubseteq_{\mathfrak{S}})$ be a selection structure and $(\mathfrak{S}', \sqsubseteq_{\mathfrak{S}})$ be a sub-selection structure of $(\mathfrak{S}, \sqsubseteq_{\mathfrak{S}})$. $\mathfrak{S}' \in \mathscr{O}_{\mathfrak{S}}$ is said to be *idealized* iff

$$\forall S \subseteq_{fin} \mathfrak{S}', \qquad \widehat{S}^{\mathfrak{S}} \ \Rightarrow \ \widehat{S}^{\mathfrak{S}'}. \tag{95}$$

The set of idealized sub-selection structures of a given selection structure $(\mathfrak{S}, \sqsubseteq_{\mathfrak{S}})$ is denoted $\widehat{\mathscr{O}}_{\mathfrak{S}}$.

**Lemma 19.** Let $\mathfrak{l}$ be a quasi-classical property. Then,

$$\lceil\mathfrak{K}_{\mathfrak{l}}\rceil \text{ is an 'idealized sub-selection structure' in the 'sub-selection structure' } \lceil\mathfrak{Q}_{\mathfrak{l}}\rceil. \tag{96}$$

Conversely, if $\lceil\mathfrak{K}_{\mathfrak{l}}\rceil$ is an 'idealized sub-selection structure' in the 'sub-selection structure' $\lceil\mathfrak{Q}_{\mathfrak{l}}\rceil$, then $\mathfrak{l}$ is a quasi-classical property. Moreover, the retraction $\pi_{\mathfrak{l}}$ is given by

$$\forall\sigma \in \lceil\mathfrak{Q}_{\mathfrak{l}}\rceil,\ \ \pi_{\mathfrak{l}}(\sigma) = \bigsqcup\nolimits_{\mathfrak{S}}(\lceil\mathfrak{K}_{\mathfrak{l}}\rceil \cap (\downarrow_{\mathfrak{S}}\sigma)). \tag{97}$$

The retraction $\pi_{\mathfrak{l}}$ is a surjective, order-preserving map which preserves arbitrary infima and existing suprema. ■

*Remark* 9. $\forall\sigma \in \lceil\mathfrak{Q}_{\mathfrak{l}}\rceil,\ \ \bot_{\mathfrak{S}} \in (\lceil\mathfrak{K}_{\mathfrak{l}}\rceil \cap (\downarrow_{\mathfrak{S}}\sigma)) \neq \varnothing$

*Remark* 10. $\lceil\mathfrak{Q}_{\mathfrak{l}}\rceil$ is a downset containing $\lceil\mathfrak{A}_{\mathfrak{l}}\rceil$. Hence, $\lceil\mathfrak{Q}_{\mathfrak{l}}\rceil$ contains $\lceil\mathfrak{K}_{\mathfrak{l}}\rceil = \downarrow_{\mathfrak{S}}\lceil\mathfrak{A}_{\mathfrak{l}}\rceil$.

*Proof.* Let us consider $\sigma_1, \sigma_2 \in \lceil\mathfrak{K}_{\mathfrak{l}}\rceil$ and let us suppose that $\exists\sigma \in \lceil\mathfrak{Q}_{\mathfrak{l}}\rceil \mid \sigma_1, \sigma_2 \sqsubseteq_{\mathfrak{S}} \sigma$. The monotonicity of $\pi_{\mathfrak{l}}$ implies $\pi_{\mathfrak{l}}(\sigma_1) \sqsubseteq_{\mathfrak{S}} \pi_{\mathfrak{l}}(\sigma)$ and $\pi_{\mathfrak{l}}(\sigma_2) \sqsubseteq_{\mathfrak{S}} \pi_{\mathfrak{l}}(\sigma)$. Secondly, property (93) implies $\sigma_1 = \pi_{\mathfrak{l}}(\sigma_1)$ and $\sigma_2 = \pi_{\mathfrak{l}}(\sigma_2)$. Thirdly, property (94) implies $\pi_{\mathfrak{l}}(\sigma) \sqsubseteq_{\mathfrak{S}} \sigma$. As a conclusion, $\exists\sigma' = \pi_{\mathfrak{l}}(\sigma) \in \lceil\mathfrak{K}_{\mathfrak{l}}\rceil \mid \sigma_1, \sigma_2 \sqsubseteq_{\mathfrak{S}} \sigma'$. Furthermore, from Lemma 17, we know already that $\lceil\mathfrak{K}_{\mathfrak{l}}\rceil$ is Scott-closed. As a result, we then conclude that $\lceil\mathfrak{K}_{\mathfrak{l}}\rceil$ is a Scott ideal in $\lceil\mathfrak{Q}_{\mathfrak{l}}\rceil$, i.e., an idealized sub-selection structure of $\lceil\mathfrak{Q}_{\mathfrak{l}}\rceil$.
Conversely, if $\lceil\mathfrak{K}_{\mathfrak{l}}\rceil$ is a Scott ideal in $\lceil\mathfrak{Q}_{\mathfrak{l}}\rceil$, we can use Lemma 71 to conclude that

$$\forall\sigma \in \lceil\mathfrak{Q}_{\mathfrak{l}}\rceil,\ \ \pi(\sigma) := \bigsqcup\nolimits_{\mathfrak{S}}(\lceil\mathfrak{K}_{\mathfrak{l}}\rceil \cap (\downarrow_{\mathfrak{S}}\sigma)) \ \in \lceil\mathfrak{K}_{\mathfrak{l}}\rceil. \tag{98}$$

$\pi$ is a map defined from $\lceil\mathfrak{Q}_{\mathfrak{l}}\rceil$ to $\lceil\mathfrak{K}_{\mathfrak{l}}\rceil$, which trivially satisfies properties (92), (93) and(94). $\mathfrak{l}$ is then a quasi-classical property and the expression for the retraction $\pi_{\mathfrak{l}}$ is given by (97).
$\pi_{\mathfrak{l}}$ is also the Galois right adjoint of the inclusion map from $\lceil\mathfrak{K}_{\mathfrak{l}}\rceil$ to $\lceil\mathfrak{Q}_{\mathfrak{l}}\rceil$. As an immediate consequence, $\pi_{\mathfrak{l}}$ is a surjective, order-preserving map that preserves arbitrary infima.
Finally, the conditional-distributivity property satisfied in the sub-selection structure $\lceil\mathfrak{Q}_{\mathfrak{l}}\rceil$ leads to the preservation of existing finite joins by $\pi_{\mathfrak{l}}$. Indeed, let us consider $\sigma_1, \sigma_2 \in \lceil\mathfrak{Q}_{\mathfrak{l}}\rceil$ such that $(\sigma_1 \sqcup_{\mathfrak{S}} \sigma_2)$ exists. We remark that $\pi_{\mathfrak{l}}(\sigma_1) = \sigma_1 \sqcap_{\mathfrak{S}} \pi_{\mathfrak{l}}(\sigma_1 \sqcup_{\mathfrak{S}} \sigma_2)$. Indeed, (i) $\pi_{\mathfrak{l}}(\sigma_1) \sqsubseteq_{\mathfrak{S}} \sigma_1$ and $\pi_{\mathfrak{l}}(\sigma_1) = \pi_{\mathfrak{l}}(\sigma_1 \sqcup_{\mathfrak{S}} \sigma_2)$ implies $\pi_{\mathfrak{l}}(\sigma_1) \sqsubseteq_{\mathfrak{S}} \sigma_1 \sqcap_{\mathfrak{S}} \pi_{\mathfrak{l}}(\sigma_1 \sqcup_{\mathfrak{S}} \sigma_2)$, and (ii) $\sigma_1 \sqcap_{\mathfrak{S}} \pi_{\mathfrak{l}}(\sigma_1 \sqcup_{\mathfrak{S}} \sigma_2)$ is an element of $\lceil\mathfrak{K}_{\mathfrak{l}}\rceil$ below $\sigma_1$ and $\pi_{\mathfrak{l}}(\sigma_1)$ is the supremum of all these elements and therefore $\pi_{\mathfrak{l}}(\sigma_1) \sqsupseteq_{\mathfrak{S}} \sigma_1 \sqcap_{\mathfrak{S}} \pi_{\mathfrak{l}}(\sigma_1 \sqcup_{\mathfrak{S}} \sigma_2)$. With the same type of arguments, we show that $\pi_{\mathfrak{l}}(\sigma_2) = \sigma_2 \sqcap_{\mathfrak{S}} \pi_{\mathfrak{l}}(\sigma_1 \sqcup_{\mathfrak{S}} \sigma_2)$. We then obtain :

$$\begin{aligned}
\pi_{\mathfrak{l}}(\sigma_1) \sqcup_{\mathfrak{S}} \pi_{\mathfrak{l}}(\sigma_2) &= (\sigma_1 \sqcap_{\mathfrak{S}} \pi_{\mathfrak{l}}(\sigma_1 \sqcup_{\mathfrak{S}} \sigma_2)) \sqcup_{\mathfrak{S}} (\sigma_2 \sqcap_{\mathfrak{S}} \pi_{\mathfrak{l}}(\sigma_1 \sqcup_{\mathfrak{S}} \sigma_2)) \\
&= (\sigma_1 \sqcup_{\mathfrak{S}} \sigma_2) \sqcap_{\mathfrak{S}} \pi_{\mathfrak{l}}(\sigma_1 \sqcup_{\mathfrak{S}} \sigma_2) \qquad \textit{(conditional-distributivity)} \\
&= \pi_{\mathfrak{l}}(\sigma_1 \sqcup_{\mathfrak{S}} \sigma_2)
\end{aligned} \tag{99}$$

Any existing supremum being the directed supremum of the suprema of finite subsets of the considered family, and $\pi_{\mathfrak{l}}$ preserving directed suprema and finite joins, we obtain the preservation of any existing suprema. □

**Notion 32.** A yes/no test $\mathfrak{t} \in \mathfrak{T}$ is said to *lead to a minimally disturbing measurement* of the corresponding testable property $\lfloor\mathfrak{t}\rfloor$ (this fact is denoted $\mathfrak{t} \in \mathfrak{T}_{min}$) iff (i) this measurement is first-kind, and (ii) for any preparation $\mathfrak{p}$ 'consistent with the actuality of $\lfloor\mathfrak{t}\rfloor$', the observer is able to pronounce statements about the measured state $\lceil\mathfrak{p}.\mathfrak{t}\rceil$ that are the minimal statements simultaneously finer than the statements pronounced separately about $\lceil\mathfrak{p}\rceil$ and about $\Sigma_{\lfloor\mathfrak{t}\rfloor}$

before the measurement. In other words,

$$\forall \mathfrak{t} \in \mathfrak{T}, \qquad \mathfrak{t} \in \mathfrak{T}_{min} :\Leftrightarrow \begin{cases} Dom^{\mathfrak{P}}_{\bullet\mathfrak{t}} = \mathfrak{Q}_{\lfloor \mathfrak{t} \rfloor} \\ \mathfrak{t} \in \mathfrak{T}_{FKM} \\ \forall \mathfrak{p} \in \mathfrak{K}_{\lfloor \mathfrak{t} \rfloor},\ \lceil \mathfrak{p}\bullet\mathfrak{t} \rceil = \lceil \mathfrak{p} \rceil \sqcup_{\mathfrak{S}} \Sigma_{\lfloor \mathfrak{t} \rfloor}. \end{cases} \tag{100}$$

**Notion 33.** The subset of properties that can be tested through minimally disturbing measurements will be denoted $\mathscr{L}_{min}$. In other words,

$$\forall \mathfrak{l} \in \mathscr{L}, \qquad \mathfrak{l} \in \mathscr{L}_{min} :\Leftrightarrow \exists (\bullet\mathfrak{t}) \in [\mathfrak{S} \dashrightarrow \mathfrak{S}]^{\bar{\wedge}}_{\sqcap} \ \mid \begin{cases} Dom^{\mathfrak{S}}_{\bullet\mathfrak{t}} = \lceil \mathfrak{Q}_{\mathfrak{l}} \rceil \\ Im^{\mathfrak{S}}_{\bullet\mathfrak{t}} = \lceil \mathfrak{A}_{\mathfrak{l}} \rceil = (\uparrow^{\mathfrak{S}} \Sigma_{\mathfrak{l}}) \\ \forall \sigma \in Dom^{\mathfrak{S}}_{\bullet\mathfrak{t}},\ (\sigma\bullet\mathfrak{t})\bullet\mathfrak{t} = \sigma\bullet\mathfrak{t} \\ \forall \sigma \in \lceil \mathfrak{K}_{\mathfrak{l}} \rceil,\ \sigma\bullet\mathfrak{t} = \sigma \sqcup_{\mathfrak{S}} \Sigma_{\mathfrak{l}}. \end{cases} \tag{101}$$

*Remark* 11. We note that $\sigma \in \lceil \mathfrak{K}_{\mathfrak{l}} \rceil$ means $\widehat{\sigma\Sigma_{\mathfrak{l}}}^{\mathfrak{S}}$ and then the supremum $(\sigma \sqcup_{\mathfrak{S}} \Sigma_{\mathfrak{l}})$ exists, due to the consistent completeness of $\mathfrak{S}$.

*Remark* 12. Note the following implicit property of minimally disturbing measurement maps :

$$\mathfrak{t} \in \mathfrak{T}_{min} \quad \Rightarrow \quad Im^{\mathfrak{P}}_{\bullet\mathfrak{t}} = \mathfrak{A}_{\lfloor \mathfrak{t} \rfloor} \tag{102}$$

Let us distinguish two sub-cases.
As a first sub-case, let us suppose that $\mathfrak{p} \in (\downarrow_{\mathfrak{P}} \mathfrak{A}_{\mathfrak{t}})$. Applying property (100), we deduce immediately that $\lceil \mathfrak{p}\bullet\mathfrak{t} \rceil \sqsupseteq_{\mathfrak{S}} \Sigma_{\mathfrak{t}}$, i.e. $\lceil \mathfrak{p}\bullet\mathfrak{t} \rceil \in (\uparrow^{\mathfrak{S}} \Sigma_{\mathfrak{t}}) = \lceil \mathfrak{A}_{\mathfrak{t}} \rceil$, and therefore $(\mathfrak{p}\bullet\mathfrak{t}) \in \mathfrak{A}_{\mathfrak{t}}$.
As a second sub-case, we now suppose $\mathfrak{p} \in \mathfrak{Q}_{\mathfrak{t}} \smallsetminus (\downarrow_{\mathfrak{P}} \mathfrak{A}_{\mathfrak{t}})$. Let us denote by $\mathfrak{p}''$ an element of $(\downarrow_{\mathfrak{P}} \mathfrak{p}) \cap \mathfrak{K}_{\lfloor \mathfrak{t} \rfloor}$ which is non-empty (Remark 9). Firstly, we have $\mathfrak{p}'' \sqsubseteq_{\mathfrak{P}} \mathfrak{p}$. We note in particular that $\mathfrak{p}''$ is then an element of $\mathfrak{Q}_{\mathfrak{t}}$, because $\mathfrak{Q}_{\mathfrak{t}}$ is a downset. Moreover, the monotonicity of the measurement map associated with $\mathfrak{t}$ implies $(\mathfrak{p}''\bullet\mathfrak{t}) \sqsubseteq_{\mathfrak{P}} (\mathfrak{p}\bullet\mathfrak{t})$. Secondly, we have $\mathfrak{p}'' \in (\downarrow_{\mathfrak{P}} \mathfrak{A}_{\mathfrak{t}}) = \mathfrak{K}_{\mathfrak{t}}$, and thus $(\mathfrak{p}''\bullet\mathfrak{t}) \in \mathfrak{A}_{\mathfrak{t}}$ as proved in the first sub-case. We now use the fact that $\mathfrak{A}_{\mathfrak{t}}$ is an upper-set to conclude that $(\mathfrak{p}\bullet\mathfrak{t}) \in \mathfrak{A}_{\mathfrak{t}}$.

**Lemma 20.**

$$\mathscr{L}_{q-cl} \quad \subseteq \quad \mathscr{L}_{min} \tag{103}$$

More precisely, if $\mathfrak{l} \in \mathscr{L}_{q-cl}$, the map $\bullet\Theta_{\mathfrak{l}}$ defined by

$$\begin{array}{rcccl} \bullet\Theta_{\mathfrak{l}} & : & \mathfrak{S} & \dashrightarrow & \mathfrak{S} \\ & \lceil \mathfrak{Q}_{\mathfrak{l}} \rceil \ni & \sigma & \mapsto & \sigma\bullet\Theta_{\mathfrak{l}} := \Sigma_{\mathfrak{l}} \sqcup_{\mathfrak{S}} \pi_{\mathfrak{l}}(\sigma). \end{array} \tag{104}$$

is an idempotent, order-preserving partial map from $\lceil \mathfrak{Q}_{\mathfrak{l}} \rceil$ to $\lceil \mathfrak{K}_{\mathfrak{l}} \rceil$, which preserves arbitrary infima and existing suprema, and satisfies $\forall \sigma \in \lceil \mathfrak{K}_{\mathfrak{l}} \rceil$, $\sigma\bullet\Theta_{\mathfrak{l}} = \sigma \sqcup_{\mathfrak{S}} \Sigma_{\mathfrak{l}}$. ■

*Proof.* Let us consider $\mathfrak{l} \in \mathscr{L}_{q-cl}$. $\pi_{\mathfrak{l}}$ is an idempotent, order-preserving partial map from $\lceil \mathfrak{Q}_{\mathfrak{l}} \rceil$ to $\lceil \mathfrak{K}_{\mathfrak{l}} \rceil$, which preserves arbitrary infima and existing suprema as shown in Lemma 19.
As proved in Lemma 18, the map

$$\begin{array}{ccc} \lceil \mathfrak{K}_{\mathfrak{l}} \rceil & \longrightarrow & \lceil \mathfrak{A}_{\mathfrak{l}} \rceil \\ \sigma & \mapsto & \Sigma_{\mathfrak{l}} \sqcup_{\mathfrak{S}} \sigma \end{array} \tag{105}$$

is an idempotent, order-preserving partial map, which preserves arbitrary infima and existing suprema.
As a result, $\bullet\Theta_{\mathfrak{l}}$ is an idempotent, order-preserving partial map, which preserves arbitrary infima and existing suprema.
Moreover, for any $\sigma \in \lceil \mathfrak{K}_{\mathfrak{l}} \rceil = (\downarrow_{\mathfrak{S}} \lceil \mathfrak{A}_{\mathfrak{l}} \rceil)$, we have $\pi_{\mathfrak{l}}(\sigma) = \sigma$. Hence, we obtain $\forall \sigma \in (\downarrow_{\mathfrak{S}} \lceil \mathfrak{A}_{\mathfrak{l}} \rceil),\ \sigma\bullet\Theta_{\mathfrak{l}} = \sigma \sqcup_{\mathfrak{S}} \Sigma_{\mathfrak{l}}$.
As a result, $\bullet\Theta_{\mathfrak{l}}$ satisfies all properties mentioned in (101), i.e., the properties required to conclude that $\mathfrak{l} \in \mathscr{L}_{min}$. As a conclusion, $\mathscr{L}_{q-cl} \subseteq \mathscr{L}_{min}$. □

**Lemma 21.**

$$\mathscr{L}_{q-cl} \quad \supseteq \quad \mathscr{L}_{min} \tag{106}$$

More precisely, if $\mathfrak{l} \in \mathscr{L}_{min}$ and $\mathbf{.}\mathfrak{t}$ is a measurement map defined to satisfy the minimality requirement (101), then the partial map $\rho_\mathfrak{t}$ defined on $\lceil \mathfrak{Q}_\mathfrak{l} \rceil$ by

$$\begin{array}{rcccl} \rho_\mathfrak{t} & : & \mathfrak{S} & \dashrightarrow & \mathfrak{S} \\ & \lceil \mathfrak{Q}_\mathfrak{l} \rceil \ni & \sigma & \mapsto & \rho_\mathfrak{t}(\sigma) := \sigma \sqcap_\mathfrak{S} (\sigma\mathbf{.}\mathfrak{t}). \end{array} \tag{107}$$

is a Scott-continuous retraction from $\lceil \mathfrak{Q}_\mathfrak{l} \rceil$ to $\lceil \mathfrak{K}_\mathfrak{l} \rceil$. ■

*Proof.* We first note the obvious property $\forall \sigma' \in \lceil \mathfrak{Q}_\mathfrak{l} \rceil,\ \rho_\mathfrak{t}(\sigma') \sqsubseteq_\mathfrak{S} \sigma'$.
Secondly, we note that, due to the monotonicity requirement on the measurement map $(\mathbf{.}\mathfrak{t})$, $(\sigma\mathbf{.}\mathfrak{t})$ is in $\lceil \mathfrak{A}_\mathfrak{l} \rceil$. We then deduce that the range of $\rho_\mathfrak{t}$ is included in $(\downarrow_\mathfrak{S} \lceil \mathfrak{A}_\mathfrak{l} \rceil)$, i.e., included in $\lceil \mathfrak{K}_\mathfrak{l} \rceil$.
Thirdly, due to property (101) being satisfied by $\mathbf{.}\mathfrak{t}$, we know that $\forall \sigma \in \lceil \mathfrak{K}_\mathfrak{l} \rceil,\ \rho_\mathfrak{t}(\sigma) = \sigma \sqcap_\mathfrak{S} (\sigma \sqcup_\mathfrak{S} \Sigma_\mathfrak{l}) = \sigma$.
The Scott continuity of $\rho_\mathfrak{t}$ is derived from: the Scott continuity of the measurement map $(\mathbf{.}\mathfrak{t})$, the meet-continuity property in $\mathfrak{S}$ and the property given in [5, Proposition 2.1.12]:

$$\begin{array}{rcl} \forall \mathfrak{C} \subseteq_{Chain} \lceil \mathfrak{Q}_\mathfrak{l} \rceil, \qquad \rho_\mathfrak{t}(\bigsqcup_\mathfrak{S} \mathfrak{C}) & = & (\bigsqcup_\mathfrak{S} \mathfrak{C}) \sqcap_\mathfrak{S} ((\bigsqcup_\mathfrak{S} \mathfrak{C})\mathbf{.}\mathfrak{t}) \\ & = & (\bigsqcup_{\mathfrak{c}\in\mathfrak{C}} \mathfrak{c}) \sqcap_\mathfrak{S} (\bigsqcup_{\mathfrak{c}'\in\mathfrak{C}} (\mathfrak{c}'\mathbf{.}\mathfrak{t})) \\ & = & \bigsqcup_{\mathfrak{c}\in\mathfrak{C}} \bigsqcup_{\mathfrak{c}'\in\mathfrak{C}} (\mathfrak{c} \sqcap_\mathfrak{S} (\mathfrak{c}'\mathbf{.}\mathfrak{t})) \\ & = & \bigsqcup_{\mathfrak{c}\in\mathfrak{C}} (\mathfrak{c} \sqcap_\mathfrak{S} (\mathfrak{c}\mathbf{.}\mathfrak{t})) = \bigsqcup_{\mathfrak{c}\in\mathfrak{C}} \rho_\mathfrak{t}(\mathfrak{c}). \end{array}$$

Finally, chain-continuity is equivalent to Scott continuity.
We have then checked properties (92), (93) and (94) for $\rho_\mathfrak{t}$. As a result, $\mathfrak{l} \in \mathscr{L}_{q-cl}$ and thus $\mathscr{L}_{q-cl} \supseteq \mathscr{L}_{min}$. □

**Lemma 22.** Let $\mathfrak{l}$ be a testable quasi-classical property and let $\mathfrak{t}$ be a yes/no test leading to a minimally disturbing measurement of the property $\mathfrak{l} = \lfloor \mathfrak{t} \rfloor$ (i.e., $\mathfrak{l} = \lfloor \mathfrak{t} \rfloor$ and $\mathfrak{t} \in \mathfrak{T}_{min}$), then necessarily $\mathfrak{t}$ leads to ideal first-kind measurements of the property $\mathfrak{l}$. As a conclusion,

$$\mathfrak{t} \in \mathfrak{T}_{min} \quad \Rightarrow \quad \mathfrak{t} \in \mathfrak{T}_{IFKM}. \tag{108}$$

■

*Proof.* First, $\mathfrak{t}$ being in $\mathfrak{T}_{min}$, we know that $\lfloor \mathfrak{t} \rfloor$ is in $\mathscr{L}_{min}$ and therefore in $\mathscr{L}_{q-cl}$ using Lemma 21. We now intend to prove that the yes/no tests $\mathfrak{t}$ satisfying the properties given in (100) also satisfy the properties (74 (i)), (74 (ii)) and (83), as long as the property $\lfloor \mathfrak{t} \rfloor$ is quasi-classical.
We have proved property (74 (i)) in Remark 12.
Using property (100) for the particular case $\mathfrak{p} \in \mathfrak{A}_\mathfrak{t}$, we obtain $\lceil \mathfrak{p}\mathbf{.}\mathfrak{t} \rceil = \lceil \mathfrak{p} \rceil$, because by definition $\Sigma_\mathfrak{t} = \bigsqcap_\mathfrak{S} \mathfrak{A}_\mathfrak{t}$. We have then proved property (74 (ii)).
Let us now consider a second yes/no test $\mathfrak{t}' \in \mathfrak{T}$ such that $\widehat{\lfloor \mathfrak{t} \rfloor \lfloor \mathfrak{t}' \rfloor}$, i.e., $\mathfrak{A}_\mathfrak{t} \cap \mathfrak{A}_{\mathfrak{t}'} \neq \varnothing$. We note in particular that this compatibility property implies $\widetilde{\mathfrak{e}}_{\mathfrak{t}'}(\Sigma_\mathfrak{t}) \leq \mathbf{Y}$. Let us also consider $\mathfrak{p} \in \mathfrak{Q}_\mathfrak{t}$ such that $\mathfrak{e}(\mathfrak{p}, \mathfrak{t}') = \mathbf{Y}$, i.e., $\mathfrak{p} \in \mathfrak{A}_{\mathfrak{t}'}$. We will distinguish two sub-cases as before.
As a first sub-case, let us suppose that $\mathfrak{p} \in (\downarrow_\mathfrak{P} \mathfrak{A}_\mathfrak{t})$. Applying property (100), we have $\mathfrak{e}(\mathfrak{p}\mathbf{.}\mathfrak{t}, \mathfrak{t}') \geq \mathfrak{e}(\mathfrak{p}, \mathfrak{t}') = \mathbf{Y}$ using the hypotheses.
As a second sub-case, we now suppose $\mathfrak{p} \in \mathfrak{Q}_\mathfrak{t} \smallsetminus (\downarrow_\mathfrak{P} \mathfrak{A}_\mathfrak{t})$. Let us consider once again any preparation process $\mathfrak{p}'' \in \mathfrak{P}$ whose state corresponds to the supremum $\bigsqcup_\mathfrak{S} ((\downarrow_\mathfrak{S} \lceil \mathfrak{p} \rceil) \cap (\downarrow_\mathfrak{S} \lceil \mathfrak{A}_\mathfrak{t} \rceil))$. We conclude as before that $\mathfrak{p}'' \in \mathfrak{Q}_\mathfrak{t}$ and $(\mathfrak{p}''\mathbf{.}\mathfrak{t}) \sqsubseteq_\mathfrak{P} (\mathfrak{p}\mathbf{.}\mathfrak{t})$. In particular $\mathfrak{e}(\mathfrak{p}''\mathbf{.}\mathfrak{t}, \mathfrak{t}') \leq \mathfrak{e}(\mathfrak{p}\mathbf{.}\mathfrak{t}, \mathfrak{t}')$.

We know also that $\mathfrak{p}'' \in (\downarrow_{\mathfrak{P}} \mathfrak{A}_{\mathfrak{t}})$ and therefore, using property (100), $\mathfrak{e}(\mathfrak{p}''.\mathfrak{t}, \mathfrak{t}') \geq \mathfrak{e}(\mathfrak{p}'', \mathfrak{t}')$.
Lastly, we know that $\Sigma_{\mathfrak{t}'} \sqsubseteq_{\mathfrak{S}} \lceil \mathfrak{p}'' \rceil = \bigsqcup_{\mathfrak{S}} ((\downarrow_{\mathfrak{S}} \lceil \mathfrak{p} \rceil) \cap (\downarrow_{\mathfrak{S}} \lceil \mathfrak{A}_{\mathfrak{t}} \rceil))$ because (i) $\mathfrak{p} \in \mathfrak{A}_{\mathfrak{t}'}$ implies $\Sigma_{\mathfrak{t}'} \sqsubseteq_{\mathfrak{S}} \lceil \mathfrak{p} \rceil$, which implies $\Sigma_{\mathfrak{t}'} \in (\downarrow_{\mathfrak{S}} \lceil \mathfrak{p} \rceil)$, and (ii) $\mathfrak{A}_{\mathfrak{t}} \cap \mathfrak{A}_{\mathfrak{t}'} \neq \varnothing$ implies $\Sigma_{\mathfrak{t}'} \in (\downarrow_{\mathfrak{S}} \lceil \mathfrak{A}_{\mathfrak{t}} \rceil)$. But $\Sigma_{\mathfrak{t}'} \sqsubseteq_{\mathfrak{S}} \lceil \mathfrak{p}'' \rceil$ is equivalently rewritten as $\mathfrak{e}(\mathfrak{p}'', \mathfrak{t}') = \mathbf{Y}$. We therefore deduce that $\mathfrak{e}(\mathfrak{p}''.\mathfrak{t}, \mathfrak{t}') = \mathbf{Y}$.
Using the two intermediary results, we conclude this analysis of the second sub-case by $\mathfrak{e}(\mathfrak{p}.\mathfrak{t}, \mathfrak{t}') = \mathbf{Y}$.
As a conclusion, we have for any $\mathfrak{p} \in \mathfrak{Q}_{\mathfrak{t}}$, $\mathfrak{e}(\mathfrak{p}.\mathfrak{t}, \mathfrak{t}') = \mathbf{Y}$. We have thus demonstrated property (83). □

**Lemma 23.**

$$\forall \mathfrak{t} \in \mathfrak{T}_{IFKM}, \forall \sigma \in \lceil \mathfrak{K}_{\lfloor \mathfrak{t} \rfloor} \rceil, \qquad (\sigma.\mathfrak{t}) = (\sigma \sqcup_{\mathfrak{S}} \Sigma_{\lfloor \mathfrak{t} \rfloor}). \tag{109}$$

In other words,

$$\mathfrak{t} \in \mathfrak{T}_{IFKM} \quad \Rightarrow \quad \mathfrak{t} \in \mathfrak{T}_{min}. \tag{110}$$

■

*Proof.* Let us consider $\mathfrak{t} \in \mathfrak{T}_{IFKM}$ and $\sigma \in \{ \sigma' \mid \widehat{\sigma\Sigma_{\lfloor \mathfrak{t} \rfloor}}^{\mathfrak{S}} \} = \lceil \mathfrak{K}_{\lfloor \mathfrak{t} \rfloor} \rceil$. Using the fact that the measurement map $(.\mathfrak{t})$ preserves arbitrary infima, we deduce

$$\begin{aligned}
(\sigma.\mathfrak{t}) &= (\bigsqcap_{\mathfrak{S}} (\uparrow^{\mathfrak{S}} \sigma)).\mathfrak{t} \\
&= \bigsqcap_{\mathfrak{S}} ((Dom^{\mathfrak{S}}_{.\mathfrak{t}} \cap (\uparrow^{\mathfrak{S}} \sigma)).\mathfrak{t}) \\
&\sqsupseteq_{\mathfrak{S}} \bigsqcap_{\mathfrak{S}} ((\uparrow^{\mathfrak{S}} \sigma) \cap \lceil \mathfrak{A}_{\lfloor \mathfrak{t} \rfloor} \rceil) = (\sigma \sqcup_{\mathfrak{S}} \Sigma_{\lfloor \mathfrak{t} \rfloor})
\end{aligned}$$

Using $\sigma \sqsubseteq_{\mathfrak{S}} (\sigma \sqcup_{\mathfrak{S}} \Sigma_{\lfloor \mathfrak{t} \rfloor})$ and the monotonicity of the measurement map, along with property (74 (ii)), we also obtain, for any $\sigma \in \lceil \mathfrak{K}_{\lfloor \mathfrak{t} \rfloor} \rceil$, the property $\sigma.\mathfrak{t} \sqsubseteq_{\mathfrak{S}} (\sigma \sqcup_{\mathfrak{S}} \Sigma_{\lfloor \mathfrak{t} \rfloor}).\mathfrak{t} = \sigma \sqcup_{\mathfrak{S}} \Sigma_{\lfloor \mathfrak{t} \rfloor}$. □

**Theorem 9. [Characterization of minimally disturbing measurements]**
A property $\mathfrak{l}$ can be measured through ideal first-kind measurements iff $\mathfrak{l}$ is a testable quasi-classical property (i.e., iff $\lceil \mathfrak{K}_{\mathfrak{l}} \rceil$ is an idealized sub-selection structure in the selection structure $\lceil \mathfrak{Q}_{\mathfrak{l}} \rceil$). In other words,

$$\mathscr{L}_{q-cl} = \mathscr{L}_{min} = \mathscr{L}_{IFKM}. \tag{111}$$

For any quasi-classical property $\mathfrak{l} \in \mathscr{L}_{q-cl}$, the map given by

$$\begin{array}{rcccl}
.\Theta_{\mathfrak{l}} & : & \mathfrak{S} & \dashrightarrow & \mathfrak{S} \\
 & \lceil \mathfrak{Q}_{\mathfrak{l}} \rceil \ni & \sigma & \mapsto & \sigma.\Theta_{\mathfrak{l}} := \Sigma_{\mathfrak{l}} \sqcup_{\mathfrak{S}} \bigsqcup_{\mathfrak{S}} (\lceil \mathfrak{K}_{\mathfrak{l}} \rceil \cap (\downarrow_{\mathfrak{S}} \sigma)).
\end{array} \tag{112}$$

defines an idempotent, order-preserving, Scott-continuous partial map from $\lceil \mathfrak{Q}_{\mathfrak{l}} \rceil$ to $\lceil \mathfrak{K}_{\mathfrak{l}} \rceil$, preserving arbitrary infima and satisfying $\forall \sigma \in \lceil \mathfrak{K}_{\mathfrak{l}} \rceil$, $\sigma.\Theta_{\mathfrak{l}} = \sigma \sqcup_{\mathfrak{S}} \Sigma_{\mathfrak{l}}$. This is the explicit form of the minimally disturbing measurement map (it is also an ideal first-kind measurement map) associated with the property $\mathfrak{l}$.
Moreover, the measurement map $.\Theta_{\mathfrak{l}}$ satisfies the following 'linearity' property :

$$\forall \sigma_1, \sigma_2 \in \mathfrak{S} \mid (\sigma_1 \sqcup_{\mathfrak{S}} \sigma_2) \in \lceil \mathfrak{Q}_{\mathfrak{l}} \rceil, \qquad (\sigma_1 \sqcup_{\mathfrak{S}} \sigma_2).\Theta_{\mathfrak{l}} = (\sigma_1.\Theta_{\mathfrak{l}}) \sqcup_{\mathfrak{S}} (\sigma_2.\Theta_{\mathfrak{l}}). \tag{113}$$

■

*Proof.* Direct consequence of Lemmas 19, 20, 21, 22 and 23. □

*Remark* 13. We note also the following 'regularity' property

$$\forall \sigma_1, \sigma_2 \in \lceil \mathfrak{Q}_{\mathfrak{t}} \rceil, \qquad (\sigma_1 \,\overline{\sqsubseteq}_{\mathfrak{S}}\, \sigma_2) \;\Rightarrow\; ((\sigma_1 \centerdot \Theta_{\mathfrak{t}}) = (\sigma_2 \centerdot \Theta_{\mathfrak{t}}) \text{ or } (\sigma_1 \centerdot \Theta_{\mathfrak{t}}) \,\overline{\sqsubseteq}_{\mathfrak{S}}\, (\sigma_2 \centerdot \Theta_{\mathfrak{t}})). \tag{114}$$

Indeed, firstly, we note that $\sigma_2 = \sigma_1 \sqcup_{\mathfrak{S}} (\sigma_2 \smallsetminus \sigma_1)$. Secondly, $\sigma_2 \smallsetminus \sigma_1$ is an element in $\lceil \mathfrak{Q}_{\mathfrak{t}} \rceil$, and then $\pi_{\mathfrak{t}}(\sigma_2 \smallsetminus \sigma_1)$ is equal either to $\bot_{\mathfrak{S}}$ or to an element of $\lceil \mathfrak{K}_{\mathfrak{t}} \rceil$. Finally, we use property (113) to obtain $(\sigma_2 \centerdot \Theta_{\mathfrak{t}}) = (\sigma_1 \centerdot \Theta_{\mathfrak{t}}) \sqcup_{\mathfrak{S}} \pi_{\mathfrak{t}}(\sigma_2 \smallsetminus \sigma_1)$ and conclude.

**Theorem 10. [The Chu space $(\mathfrak{S}, \mathfrak{T}_{min}, \widetilde{\mathfrak{e}})$ is *bi-extensional*]**

$$\begin{array}{rccl} \mu & : & \mathfrak{T}_{min} & \longrightarrow & [\mathfrak{S} \to \mathfrak{B}]^{\overline{\wedge}}_{\sqcap} \\ & & \mathfrak{t} & \mapsto & \widetilde{\mathfrak{e}}_{\mathfrak{t}} \end{array} \qquad \text{is injective} \tag{115}$$

As a result, the Chu space $(\mathfrak{S}, \mathfrak{T}_{min}, \widetilde{\mathfrak{e}})$ is *extensional*, i.e.

$$\forall \mathfrak{t}_1, \mathfrak{t}_2 \in \mathfrak{T}_{min}, \qquad (\forall \sigma \in \mathfrak{S},\ \widetilde{\mathfrak{e}}_{\mathfrak{t}_1}(\sigma) = \widetilde{\mathfrak{e}}_{\mathfrak{t}_2}(\sigma)) \;\Rightarrow\; (\mathfrak{t}_1 = \mathfrak{t}_2) \tag{116}$$

As a consequence of Lemma 1, this Chu space is *bi-extensional*. ■

*Proof.* Once the evaluation map is given, we obtain unambiguously (1) the Scott-closed subset $\lceil \mathfrak{Q}_{\mathfrak{t}} \rceil$ of $\mathfrak{S}$ as the reverse image of the subset $\{\bot, \mathbf{Y}\}$ by the Scott-continuous map $\widetilde{\mathfrak{e}}_{\mathfrak{t}}$, and (2) the element $\Sigma_{\mathfrak{t}}$ as the infimum of the Scott-open filter determined as the non-empty reverse image of the subset $\{\mathbf{Y}\}$ by the order-preserving and infima-preserving map $\widetilde{\mathfrak{e}}_{\mathfrak{t}}$.
Let us now consider two minimally disturbing tests $\mathfrak{t}_1, \mathfrak{t}_2 \in \mathfrak{T}_{min}$ such that $\Sigma_{\mathfrak{t}_1} = \Sigma_{\mathfrak{t}_2}$ and $\lceil \mathfrak{Q}_{\mathfrak{t}_1} \rceil = \lceil \mathfrak{Q}_{\mathfrak{t}_2} \rceil$. We then have, firstly :

$$Dom^{\mathfrak{S}}_{\centerdot \mathfrak{t}_1} = \lceil \mathfrak{Q}_{\mathfrak{t}_1} \rceil = \lceil \mathfrak{Q}_{\mathfrak{t}_2} \rceil = Dom^{\mathfrak{S}}_{\centerdot \mathfrak{t}_2}. \tag{117}$$

Secondly, we have for any $\sigma$ in $Dom^{\mathfrak{S}}_{\centerdot \mathfrak{t}_1} = Dom^{\mathfrak{S}}_{\centerdot \mathfrak{t}_2}$

$$\sigma \centerdot \mathfrak{t}_1 = \Sigma_{\mathfrak{t}_1} \sqcup_{\mathfrak{S}} \bigsqcup_{\mathfrak{S}} \left( (\downarrow_{\mathfrak{S}} \uparrow^{\mathfrak{S}} \Sigma_{\mathfrak{t}_1}) \cap (\downarrow_{\mathfrak{S}} \sigma) \right) = \Sigma_{\mathfrak{t}_2} \sqcup_{\mathfrak{S}} \bigsqcup_{\mathfrak{S}} \left( (\downarrow_{\mathfrak{S}} \uparrow^{\mathfrak{S}} \Sigma_{\mathfrak{t}_2}) \cap (\downarrow_{\mathfrak{S}} \sigma) \right) = \sigma \centerdot \mathfrak{t}_2 \tag{118}$$

As a consequence, $\mathfrak{t}_1 = \mathfrak{t}_2$. As a result, the map $\mu$ is injective, and the Chu space $(\mathfrak{S}, \mathfrak{T}_{min}, \widetilde{\mathfrak{e}})$ is therefore extensional. Then, using Lemma 1, we deduce that the Chu space $(\mathfrak{S}, \mathfrak{T}_{min}, \widetilde{\mathfrak{e}})$ is bi-extensional. □

**Notion 34.** We allow for a generalized definition of minimally disturbing tests (the corresponding set of *generalized minimally disturbing yes/no tests* will be denoted $\widetilde{\mathfrak{T}}_{min}$), as follows. Let us consider $\Sigma \in \mathfrak{S}, \Sigma'$ (not necessarily compact) such that $(\downarrow_{\mathfrak{S}} \uparrow^{\mathfrak{S}} \Sigma)$ is an idealized sub- selection structure of $\mathfrak{Q} := (\mathfrak{S} \smallsetminus \uparrow^{\mathfrak{S}} \Sigma')$. We denote a generic element of $\widetilde{\mathfrak{T}}_{min}$ as $\mathfrak{t}_{(\Sigma,\Sigma')}$ and define it according to

$$\mathfrak{A}_{\mathfrak{t}_{(\Sigma,\Sigma')}} := \uparrow^{\mathfrak{S}} \Sigma \tag{119}$$

$$\mathfrak{Q}_{\mathfrak{t}_{(\Sigma,\Sigma')}} := (\mathfrak{S} \smallsetminus \uparrow^{\mathfrak{S}} \Sigma') \tag{120}$$

$$\forall \sigma \in \mathfrak{Q}_{\mathfrak{t}_{(\Sigma,\Sigma')}}, \quad \sigma \centerdot \mathfrak{t}_{(\Sigma,\Sigma')} := \Sigma \sqcup_{\mathfrak{S}} \bigsqcup_{\mathfrak{S}} \left( (\downarrow_{\mathfrak{S}} \uparrow^{\mathfrak{S}} \Sigma) \cap (\downarrow_{\mathfrak{S}} \sigma) \right). \tag{121}$$

We define the corresponding *generalized property* $\mathfrak{l}_{(\Sigma,\Sigma')}$ straightforwardly and denote the set of generalized properties by $\widetilde{\mathscr{L}}_{min}$.

**Notion 35.** A generalized property $\mathfrak{l} \in \widetilde{\mathscr{L}}_{min}$ will be said to be *a perfect property* (this fact will be denoted $\mathfrak{l} \in \widetilde{\mathscr{L}}_{perfect}$) if $\bar{\mathfrak{l}}$ is also a testable generalized property. In other words, $\downarrow_{\mathfrak{S}}\uparrow^{\mathfrak{S}} \Sigma_{\mathfrak{l}}$ is an idealized sub- selection structure of $\mathfrak{Q}_{\mathfrak{l}} = (\mathfrak{S} \setminus \uparrow^{\mathfrak{S}} \Sigma_{\bar{\mathfrak{l}}})$ and $\downarrow_{\mathfrak{S}}\uparrow^{\mathfrak{S}} \Sigma_{\bar{\mathfrak{l}}}$ is an idealized sub- selection structure of $\mathfrak{Q}_{\bar{\mathfrak{l}}} = (\mathfrak{S} \setminus \uparrow^{\mathfrak{S}} \Sigma_{\mathfrak{l}})$. As a consequence, there exist two conjugate generalized minimally disturbing yes/no tests, $\mathfrak{t}$ and $\bar{\mathfrak{t}}$, leading to ideal first-kind measurements of the respective generalized properties, $\mathfrak{l}$ and $\bar{\mathfrak{l}}$. The corresponding measurement $\mathbf{.}\mathfrak{t}$ is said to be *a perfect measurement*. The yes/no test $\mathfrak{t}$ will be said to be perfect as well (this fact will be denoted $\mathfrak{t} \in \widetilde{\mathfrak{T}}_{perfect}$).

### 3.4 The space of states as a 'Quantum domain'

From Theorem 9, it appears that some axiom is missing in order to guaranty that any property can be measured through ideal first-kind measurements.
A very naive proposal would be to require that every consistency domain of the form $\lceil \mathfrak{K}_{\mathfrak{l}} \rceil$ be an idealized sub- selection structure of the whole space of states $\mathfrak{S}$. In other words, we would require that, for any property $\mathfrak{l}$ and for any $\Sigma_1, \Sigma_2 \in \lceil \mathfrak{K}_{\mathfrak{l}} \rceil := (\downarrow_{\mathfrak{S}}\uparrow^{\mathfrak{S}} \Sigma_{\mathfrak{l}})$ such that $\widehat{\Sigma_1\Sigma_2}^{\mathfrak{S}}$, the supremum $(\Sigma_1 \sqcup_{\mathfrak{S}} \Sigma_2)$ be in $\lceil \mathfrak{K}_{\mathfrak{l}} \rceil$. We note that this condition is nothing else than the 'coherence condition' [44, Prop.1.3 p.3] imposed on the space of states

$$\forall \Sigma_1, \Sigma_2, \Sigma_3 \in \mathfrak{S} \mid \widehat{\Sigma_1\Sigma_2}^{\mathfrak{S}}, \widehat{\Sigma_1\Sigma_3}^{\mathfrak{S}}, \widehat{\Sigma_2\Sigma_3}^{\mathfrak{S}}, \qquad \widehat{(\Sigma_1 \sqcup_{\mathfrak{S}} \Sigma_2)\Sigma_3}^{\mathfrak{S}}. \tag{122}$$

Nevertheless, this requirement appears immediately too restrictive, because we want to have the testability condition $\mathfrak{A}_{\bar{\mathfrak{l}}} \neq \varnothing$.
We will rather impose a weaker condition than the 'coherence condition'. This new proposal is defined as follows.

**Notion 36.** A couple of states $(\Sigma, \Sigma') \in \mathfrak{S}^{\times 2}$ is said to be *quasi-consistent* (this fact is denoted $\Sigma \bowtie_{\mathfrak{S}} \Sigma'$) according to the following definition :

$$\Sigma \bowtie_{\mathfrak{S}} \Sigma' \quad :\Leftrightarrow \quad (\forall \Sigma'' \sqsubset_{\mathfrak{S}} \Sigma', \widehat{\Sigma\Sigma''}^{\mathfrak{S}} \text{ and } \forall \Sigma'' \sqsubset_{\mathfrak{S}} \Sigma, \widehat{\Sigma'\Sigma''}^{\mathfrak{S}}). \tag{123}$$

We have obviously $\widehat{\Sigma\Sigma'}^{\mathfrak{S}} \Rightarrow \Sigma \bowtie_{\mathfrak{S}} \Sigma'$.

**Axiom 8.** We will impose the following condition emphasizing the existence of a large set of perfect yes/no tests :

$$\forall (\Sigma, \Sigma') \in \mathfrak{S}^{\times 2} \mid (\Sigma \bowtie_{\mathfrak{S}} \Sigma' \text{ and } \widetilde{\Sigma\Sigma'}^{\mathfrak{S}}), \qquad \mathfrak{t}_{(\Sigma,\Sigma')} \in \widetilde{\mathfrak{T}}_{perfect} \tag{124}$$

**Lemma 24.** The constraint imposed in **Axiom 8** appears to be equivalent to a constraint on the space of states. This constraint is naturally called *quasi-coherence condition* :

$$\forall \Sigma_1, \Sigma_2, \Sigma_3 \in \mathfrak{S} \mid \widehat{\Sigma_1\Sigma_2}^{\mathfrak{S}}, \widehat{\Sigma_1\Sigma_3}^{\mathfrak{S}}, \widehat{\Sigma_2\Sigma_3}^{\mathfrak{S}}, \qquad (\Sigma_1 \sqcup_{\mathfrak{S}} \Sigma_2) \bowtie_{\mathfrak{S}} \Sigma_3, \tag{125}$$

■

*Proof.* Let us consider $(\Sigma_3, \Sigma_4) \in \mathfrak{S}^{\times 2}$ such that $\Sigma_3 \bowtie_{\mathfrak{S}} \Sigma_4$ and $\widetilde{\Sigma_3\Sigma_4}^{\mathfrak{S}}$. We then have $\mathfrak{t}_{(\Sigma_3,\Sigma_4)} \in \widetilde{\mathfrak{T}}_{perfect}$. It means in particular that $(\downarrow_{\mathfrak{S}}\uparrow^{\mathfrak{S}} \Sigma_3) \cap (\downarrow_{\mathfrak{S}} \Sigma_4)$ admits no supremum. More explicitely, we can consider $\Sigma_1$ and $\Sigma_2$ two elements of $(\downarrow_{\mathfrak{S}}\uparrow^{\mathfrak{S}} \Sigma_3)$ (i.e.$\widehat{\Sigma_1\Sigma_3}^{\mathfrak{S}}$, $\widehat{\Sigma_2\Sigma_3}^{\mathfrak{S}}$) such that $\widehat{\Sigma_1\Sigma_2}^{\mathfrak{S}}$. Two situations have to be

distinguished : $(\Sigma_1 \sqcup_{\mathfrak{S}} \Sigma_2) = \Sigma_4$ or $(\Sigma_1 \sqcup_{\mathfrak{S}} \Sigma_2) \in (\downarrow_{\mathfrak{S}} \uparrow^{\mathfrak{S}} \Sigma_3)$. In the first case, the condition $\Sigma_3 \bowtie_{\mathfrak{S}} \Sigma_4$ can be rewritten as $(\Sigma_1 \sqcup_{\mathfrak{S}} \Sigma_2) \bowtie_{\mathfrak{S}} \Sigma_3$. In the second case, the relation $(\Sigma_1 \sqcup_{\mathfrak{S}} \Sigma_2) \bowtie_{\mathfrak{S}} \Sigma_3$ is trivially implied by the property $\widehat{(\Sigma_1 \sqcup_{\mathfrak{S}} \Sigma_2)\Sigma_3}^{\mathfrak{S}}$. □

*Remark* 14. We have seen from Remark 27 that the condition C [44, p.23 section 3] is satisfied by the space of states. We have seen from Remark 24 that the condition I [44, p.13] is satisfied by the space of states. We have seen from Remark 23 that the condition R [44, Theorem 5.1 p.52] is satisfied by the space of states. We now observe that the 'coherence condition' [44, Prop.1.3 p.3], directly linked [44, Prop.5.6 p.54] to the last condition Q [44, p.40] defining concrete domains, is absent (or at least weakened) from our formalism. Our space of states is then definitely NOT a concrete domain in the sense of [44] but a generalization of them defined in the following Notion.

**Notion 37.** We will define the set of *discriminating yes/no tests* as follows.

$$\widetilde{\mathfrak{T}}_{disc} \quad := \quad \{\, \mathfrak{t}_{(\Sigma,\Sigma')} \in \widetilde{\mathfrak{T}}_{perfect} \mid \Sigma \bowtie_{\mathfrak{S}} \Sigma' \,\}. \tag{126}$$

The corresponding set of *discriminating properties* is denoted $\widetilde{\mathscr{L}}_{disc}$.

*Remark* 15. We note that, by construction of the perfect yes/no test $\mathfrak{t}_{(\Sigma,\Sigma')} \in \widetilde{\mathfrak{T}}_{perfect}$, we have also $\widetilde{\Sigma\Sigma'}^{\mathfrak{S}}$ and then consequently $\forall \Sigma'' \sqsupseteq_{\mathfrak{S}} \Sigma'$, $\widetilde{\Sigma''\Sigma}^{\mathfrak{S}}$ and $\forall \Sigma'' \sqsupseteq_{\mathfrak{S}} \Sigma$, $\widetilde{\Sigma''\Sigma'}^{\mathfrak{S}}$.

**Axiom 9.** The space of states $\mathfrak{S}$ is constrained to satisfy the following conditions :

$$\forall \Sigma \in \mathscr{A}^*_{\mathfrak{S}}, \exists ! \Sigma' \in \mathfrak{S}_{pure} \mid (\Sigma \bowtie_{\mathfrak{S}} \Sigma' \text{ and } \widetilde{\Sigma\Sigma'}^{\mathfrak{S}}), \tag{127}$$

$$\forall \Sigma' \in \mathfrak{S}_{pure}, \exists ! \Sigma \in \mathscr{A}^*_{\mathfrak{S}} \mid (\Sigma \bowtie_{\mathfrak{S}} \Sigma' \text{ and } \widetilde{\Sigma\Sigma'}^{\mathfrak{S}}). \tag{128}$$

**Notion 38.** We define, from **Axiom 9**, the map $\Delta_{\mathfrak{S}}$ from $\mathscr{A}^*_{\mathfrak{S}}$ to $\mathfrak{S}_{pure}$ as follows.

$$\begin{array}{rcl} \Delta_{\mathfrak{S}} \;:\; \mathscr{A}^*_{\mathfrak{S}} & \to & \mathfrak{S}_{pure} \\ \sigma & \mapsto & \Delta_{\mathfrak{S}}(\sigma) \quad \mid (\sigma \bowtie_{\mathfrak{S}} \Delta_{\mathfrak{S}}(\sigma) \text{ and } \widetilde{\sigma\Delta_{\mathfrak{S}}(\sigma)}^{\mathfrak{S}}). \end{array} \tag{129}$$

$\Delta_{\mathfrak{S}}$ is bijective.

**Notion 39.** A *quantum domain* is defined to be

- a locally boolean qualitative domain (Corollary 1),
- with no complete meet-irreducible elements except maximal elements (**Axiom 7**),
- satisfying the quasi-coherence condition (125)
- satisfying **Axiom 9**.

This new category of domains, generalizing the notion of 'concrete domain' introduced and studied by Kahn G. and Plotkin [44], will be studied in a forthcoming paper.

**Theorem 11.** The space of states is a quantum domain. ■

### 3.5 The space of 'Descriptions'

**Notion 40.** We denote by $\mathscr{D}$ the following subset of the powerset $\mathscr{P}(\mathscr{L}_{IFKM})$ :

$$\mathscr{D} := \{\mathfrak{L} \subseteq \mathscr{L}_{IFKM} \mid \overbrace{\mathfrak{L}} \} \tag{130}$$

An element of this set corresponds to a family of properties that can be checked by 'conjoint minimally disturbing measurements' as being 'simultaneously actual' on a given sample. Hence, $\mathscr{D}$ will be called *the space of descriptions*.
An element $D \in \mathscr{D}$ will be eventually denoted $[\![\mathfrak{l}_1 \cdots \mathfrak{l}_n]\!]$ rather than $\{\mathfrak{l}_1, \cdots, \mathfrak{l}_n\}$ to emphasize the compatibility between the properties constituting the description.

**Lemma 25. ["Specker's principle"]** Let us consider a finite family of testable properties $\mathfrak{L} = \{\mathfrak{l}_1, \cdots, \mathfrak{l}_N\}$ with $\mathfrak{l}_k \in \mathscr{L}_{IFKM}$ for any $k \in \{1, \cdots, N\}$, such that the elements of this family are pairwise compatible, i.e., $\forall k, k' \in \{1, \cdots, N\}$, $\overbrace{\mathfrak{l}_k \mathfrak{l}_{k'}}$. The family $\mathfrak{L}$ is then a compatible family. In other words,

$$\forall \mathfrak{L} = \{\mathfrak{l}_1, \cdots, \mathfrak{l}_N\} \mid \mathfrak{l}_k \in \mathscr{L}_{IFKM}, \quad \overbrace{\mathfrak{L}} \quad \Leftrightarrow \quad (\forall k, k' \in \{1, \cdots, N\}, \overbrace{\mathfrak{l}_k \mathfrak{l}_{k'}}). \tag{131}$$

■

*Proof.* Given the above requirement, we consider $T = (\mathfrak{t}_k)_{k \in \{1, \cdots, N\}}$ with $\mathfrak{t}_k \in \mathfrak{T}_{IFKM}$ and $\mathfrak{l}_k = \lfloor \mathfrak{t}_k \rfloor$ for any $k = 1, \cdots, N$.
First, using the compatibility relations $\overbrace{\mathfrak{l}_i \mathfrak{l}_k}$ for $k = 1, \cdots, i-1$ and Lemma 16, we prove (i) $\lfloor \mathfrak{t}_1 \centerdot \cdots \centerdot \mathfrak{t}_{i-1} \rfloor \in \mathscr{L}_{IFKM}$, (ii) the compatibility relation $\overbrace{\mathfrak{l}_i \lfloor \mathfrak{t}_1 \centerdot \cdots \centerdot \mathfrak{t}_{i-1} \rfloor}$ and (iii) $\mathfrak{A}_{\mathfrak{t}_1 \centerdot \cdots \centerdot \mathfrak{t}_{i-1}} = \mathfrak{A}_{\mathfrak{t}_1} \cap \cdots \cap \mathfrak{A}_{\mathfrak{t}_{i-1}}$ for any $i = 1, \cdots, N$.
Let us now consider $\mathfrak{p} \in \mathfrak{Q}_{\mathfrak{t}_1 \centerdot (\mathfrak{t}_2 \centerdot (\cdots (\mathfrak{t}_{N-1} \centerdot (\mathfrak{t}_N)) \cdots)}$ (we note that $\perp_{\mathfrak{S}} \in \mathfrak{Q}_{\mathfrak{t}_1 \centerdot (\mathfrak{t}_2 \centerdot (\cdots (\mathfrak{t}_{N-1} \centerdot (\mathfrak{t}_N)) \cdots)} \neq \varnothing$). We have then from equation (59), $(\mathfrak{p} \in \mathfrak{Q}_{\mathfrak{t}_1}$ and $(\mathfrak{p} \centerdot \mathfrak{t}_1) \in \mathfrak{Q}_{\mathfrak{t}_2 \centerdot (\cdots (\mathfrak{t}_{N-1} \centerdot (\mathfrak{t}_N)) \cdots)})$. Let us denote by $(P_i)$ the statement: $((\mathfrak{p} \centerdot \mathfrak{t}_1 \centerdot \cdots \centerdot \mathfrak{t}_{i-1}) \in \mathfrak{A}_{\mathfrak{t}_1} \cap \cdots \cap \mathfrak{A}_{\mathfrak{t}_{i-1}} \cap \mathfrak{Q}_{\mathfrak{t}_i}$ and $(\mathfrak{p} \centerdot \mathfrak{t}_1 \centerdot \cdots \centerdot \mathfrak{t}_i) \in \mathfrak{Q}_{\mathfrak{t}_{i+1} \centerdot (\cdots (\mathfrak{t}_{N-1} \centerdot (\mathfrak{t}_N)) \cdots)})$. Let us suppose that $(P_i)$ is satisfied.
Using the compatibility relation $\overbrace{\mathfrak{l}_i \lfloor \mathfrak{t}_1 \centerdot \cdots \centerdot \mathfrak{t}_{i-1} \rfloor}$ and the fact that $\mathfrak{t}_i$ leads to an ideal measurement and thus satisfies property (83), we deduce that $(\mathfrak{p} \centerdot \mathfrak{t}_1 \centerdot \cdots \centerdot \mathfrak{t}_i) \in \mathfrak{A}_{\mathfrak{t}_1} \cap \cdots \cap \mathfrak{A}_{\mathfrak{t}_{i-1}}$. Using the fact that $\mathfrak{t}_i$ leads to a first-kind measurement and therefore satisfies (74 (i)), we also have $(\mathfrak{p} \centerdot \mathfrak{t}_1 \centerdot \cdots \centerdot \mathfrak{t}_i) \in \mathfrak{A}_{\mathfrak{t}_i}$. Now using expression (59), we conclude that property $(P_{i+1})$ is satisfied.
As a result, we conclude by recursion that $(\mathfrak{p} \centerdot \mathfrak{t}_1 \centerdot \cdots \centerdot \mathfrak{t}_N) \in \mathfrak{A}_{\mathfrak{t}_1} \cap \cdots \cap \mathfrak{A}_{\mathfrak{t}_N}$. The set $\mathfrak{A}_{\mathfrak{t}_1} \cap \cdots \cap \mathfrak{A}_{\mathfrak{t}_N}$ is then non-empty, and the family $\mathfrak{L}$ is therefore a compatible family of properties. □

*Remark* 16. The history of this "principle" is recalled in [16] and some results are given in this paper that *"suggest that the principle that pairwise decidable propositions are jointly decidable, together with Boole's condition that the sum of probabilities of jointly exclusive propositions cannot be higher than one [$\cdots$], which I will collectively call Specker's principle, may explain quantum contextuality."* [24]

**Theorem 12.** The space of descriptions $\mathscr{D}$ is a 'coherence domain' [32] associated with the 'web'

[24] See [45] for the original results on non-contextuality in quantum mechanics.

$\mathscr{L}_{IFKM}$ and with the coherence relation $\overbrace{\cdot\ \cdot}$. In other words, $\mathscr{D} \subseteq \mathscr{P}(\mathscr{L}_{IFKM})$ satisfies

$$\forall \mathfrak{L}_1, \mathfrak{L}_2 \in \mathscr{P}(\mathscr{L}_{IFKM}), \quad (\mathfrak{L}_1 \subseteq \mathfrak{L}_2 \text{ and } \mathfrak{L}_2 \in \mathscr{D}) \Rightarrow (\mathfrak{L}_1 \in \mathscr{D}) \tag{132}$$

$$\forall \mathfrak{l} \in \mathscr{L}_{IFKM}, \quad \{\mathfrak{l}\} \in \mathscr{D} \tag{133}$$

$$\forall L \in \mathscr{P}(\mathscr{D}) \mid (\forall \mathfrak{L}_1, \mathfrak{L}_2 \in L, \overbrace{\mathfrak{L}_1 \cup \mathfrak{L}_2}), \quad \bigcup L \in \mathscr{D}. \tag{134}$$

■

*Proof.* The first property is a direct consequence of the definition of $\mathscr{D}$. Indeed, $\forall \mathfrak{L}_1, \mathfrak{L}_2 \in \mathscr{P}(\mathscr{L}_{IFKM})$, and the properties $\mathfrak{L}_1 \subseteq \mathfrak{L}_2$ and $\mathfrak{L}_2 \in \mathscr{D}$ imply $\varnothing \neq \bigcap_{\mathfrak{l} \in \mathfrak{L}_2} \mathfrak{A}_{\mathfrak{l}} \subseteq \bigcap_{\mathfrak{l} \in \mathfrak{L}_1} \mathfrak{A}_{\mathfrak{l}}$.
The second property is a trivial consequence of the definition of consistency and the fact that we consider testable properties.
The property $(\forall M \in \mathscr{P}_{fin}(\mathscr{D}) \mid (\forall \mathfrak{L}_1, \mathfrak{L}_2 \in M, \overbrace{\mathfrak{L}_1 \cup \mathfrak{L}_2}), \ \bigcup M \in \mathscr{D})$ is a direct consequence of Lemma 25. Hence, for any $L \in \mathscr{P}(\mathscr{D}) \mid (\forall \mathfrak{L}_1, \mathfrak{L}_2 \in L, \overbrace{\mathfrak{L}_1 \cup \mathfrak{L}_2})$, we deduce that $N \in \mathscr{D}$ for any $N \subseteq_{fin} \bigcup L$, which allows the third property to be deduced. □

**Notion 41.** For any description $D \subseteq_{fin}^{\neq \varnothing} \mathscr{D}$ such that $\forall \mathfrak{l} \in D$, $\mathfrak{l} \in \widetilde{\mathfrak{T}}_{min}$, we define a generalized minimally disturbing yes/no test denoted $\mathfrak{t}_D$ by

$$\mathfrak{t}_D := \mathfrak{t}_{(\Sigma_D, \Sigma'_D)} \quad \mid \quad \Sigma_D = \bigsqcup_{\mathfrak{l} \in D} \Sigma_{\mathfrak{l}}, \quad \Sigma'_D = \bigsqcap_{\mathfrak{l} \in D} \Sigma_{\overline{\mathfrak{l}}} \tag{135}$$

This yes/no test characterizes the conjoint measurement of the compatible properties belonging to the description $D$.

### 3.6 Scheme of discriminating yes/no tests

**Notion 42.** A collection $\mathfrak{U}$ of perfect yes/no tests is said to be *closed* iff

$$\forall D \subseteq_{fin}^{\neq \varnothing} \mathscr{D} \mid \forall \mathfrak{l} \in D, \ \mathfrak{l} \in \lfloor \mathfrak{U} \rfloor, \qquad \mathfrak{t}_D \in \mathfrak{U} \tag{136}$$

$$\forall \mathfrak{C}, \mathfrak{C}' \subseteq_{Chain} \mathfrak{S} \text{ and } \Sigma := \bigsqcup_{\mathfrak{S}} \mathfrak{C}, \Sigma' := \bigsqcap_{\mathfrak{S}} \mathfrak{C}' \qquad \mathfrak{t}_{(\Sigma, \Sigma')} \in \mathfrak{U} \tag{137}$$

$$\forall \mathfrak{t} \in \mathfrak{U}, \qquad \overline{\mathfrak{t}} \in \mathfrak{U} \tag{138}$$

**Notion 43.** A collection $\mathfrak{U}$ of perfect yes/no tests is said to be *complete* and *irredundant* iff

$$\forall \Sigma \in \mathfrak{S}, \exists \Sigma' \in \mathfrak{S} \mid \quad \mathfrak{t}_{(\Sigma, \Sigma')} \in \mathfrak{U}, \tag{139}$$

$$\forall \mathfrak{t}_1, \mathfrak{t}_2 \in \mathfrak{U} \qquad \Sigma_{\lfloor \mathfrak{t}_1 \rfloor} \sqsubseteq_{\mathfrak{S}} \Sigma_{\lfloor \mathfrak{t}_2 \rfloor} \ \Leftrightarrow \ \Sigma_{\overline{\lfloor \mathfrak{t}_2 \rfloor}} \sqsubseteq_{\mathfrak{S}} \Sigma_{\overline{\lfloor \mathfrak{t}_1 \rfloor}} \tag{140}$$

**Notion 44.** A *scheme of yes/no tests* is a closed, complete and irredundant collection of perfect yes/no tests.

Our aim, in the rest of this subsection, is to show that a scheme of discriminating yes/no tests can always been build.

**Lemma 26.**

$$\forall \Sigma \in \mathscr{A}_{\mathfrak{S}}^*, \exists ! \Sigma' = \Delta_{\mathfrak{S}}(\Sigma) \in \mathfrak{S}_{pure} \ \mid \ \mathfrak{t}_{(\Sigma, \Sigma')} \in \widetilde{\mathfrak{T}}_{disc}, \tag{141}$$

$$\forall \Sigma' \in \mathfrak{S}_{pure}, \exists ! \Sigma = \Delta_{\mathfrak{S}}^{-1}(\Sigma') \in \mathscr{A}_{\mathfrak{S}}^* \ \mid \ \mathfrak{t}_{(\Sigma, \Sigma')} \in \widetilde{\mathfrak{T}}_{disc}. \tag{142}$$

■

*Proof.* Direct consequence of **Axiom 9** and **Axiom 8**. □

**Lemma 27.** Let us consider $\Sigma_1, \Sigma_1', \Sigma_2, \Sigma_2' \in \mathfrak{S}$ with $\Sigma_1' \parallel \Sigma_2'$, ($\Sigma_1 \bowtie_{\mathfrak{S}} \Sigma_1'$ and $\widetilde{\Sigma_1\Sigma_1'}^{\mathfrak{S}}$) and ($\Sigma_2 \bowtie_{\mathfrak{S}} \Sigma_2'$ and $\widetilde{\Sigma_2\Sigma_2'}^{\mathfrak{S}}$). Let us then consider $\mathfrak{t}_{(\Sigma_1,\Sigma_1')}, \mathfrak{t}_{(\Sigma_2,\Sigma_2')}$ the corresponding discriminating yes/no tests. Let us moreover assume the compatibility condition $\overbrace{\lfloor \mathfrak{t}_{(\Sigma_1,\Sigma_1')} \rfloor \lfloor \mathfrak{t}_{(\Sigma_2,\Sigma_2')} \rfloor}$ ,i.e. $\widehat{\Sigma_1\Sigma_2}^{\mathfrak{S}}$. We have then

$$(\Sigma_1 \sqcup_{\mathfrak{S}} \Sigma_2)\widetilde{(\Sigma_1' \sqcap_{\mathfrak{S}} \Sigma_2')} \quad \text{and} \quad (\Sigma_1 \sqcup_{\mathfrak{S}} \Sigma_2) \bowtie_{\mathfrak{S}} (\Sigma_1' \sqcap_{\mathfrak{S}} \Sigma_2'). \tag{143}$$

In other words, $\mathfrak{t}_{(\Sigma_1 \sqcup_{\mathfrak{S}} \Sigma_2, \Sigma_1' \sqcap_{\mathfrak{S}} \Sigma_2')}$ is also a discriminating yes/no test. ■

*Proof.* Let us begin with the proof of $\forall \Sigma' \in \mathfrak{S} \mid \Sigma' \sqsubset_{\mathfrak{S}} (\Sigma_1' \sqcap_{\mathfrak{S}} \Sigma_2'), \widehat{(\Sigma_1 \sqcup_{\mathfrak{S}} \Sigma_2)\Sigma'}^{\mathfrak{S}}$.
Let us consider $\Sigma' \in \mathfrak{S}$ such that $\Sigma' \sqsubset_{\mathfrak{S}} (\Sigma_1' \sqcap_{\mathfrak{S}} \Sigma_2')$. We have $(\Sigma_1' \sqcap_{\mathfrak{S}} \Sigma_2') \sqsubset_{\mathfrak{S}} \Sigma_1'$ and $(\Sigma_1' \sqcap_{\mathfrak{S}} \Sigma_2') \sqsubset_{\mathfrak{S}} \Sigma_2'$ (because $\Sigma_1' \parallel \Sigma_2'$), and then, by assumption, $\widehat{(\Sigma_1' \sqcap_{\mathfrak{S}} \Sigma_2')\Sigma_1}^{\mathfrak{S}}$ and $\widehat{(\Sigma_1' \sqcap_{\mathfrak{S}} \Sigma_2')\Sigma_2}^{\mathfrak{S}}$. Using these two last consistency equations, $\Sigma' \sqsubset_{\mathfrak{S}} (\Sigma_1' \sqcap_{\mathfrak{S}} \Sigma_2')$ and $\widehat{\Sigma_1\Sigma_2}^{\mathfrak{S}}$, we deduce from the quasi coherence property (125) the equation $\widehat{(\Sigma_1 \sqcup_{\mathfrak{S}} \Sigma_2)\Sigma'}^{\mathfrak{S}}$.
Let us now continue with the proof of $\forall \Sigma \in \mathfrak{S} \mid \Sigma \sqsubset_{\mathfrak{S}} (\Sigma_1 \sqcup_{\mathfrak{S}} \Sigma_2), \widehat{\Sigma(\Sigma_1' \sqcap_{\mathfrak{S}} \Sigma_2')}^{\mathfrak{S}}$.
Let us consider $\Sigma \in \mathfrak{S}$ such that $\Sigma \sqsubset_{\mathfrak{S}} (\Sigma_1 \sqcup_{\mathfrak{S}} \Sigma_2)$. We know that $\forall \Sigma' \sqsubset_{\mathfrak{S}} \Sigma_1'$, $\widehat{\Sigma_1\Sigma'}^{\mathfrak{S}}$ and $\forall \Sigma' \sqsubset_{\mathfrak{S}} \Sigma_2'$, $\widehat{\Sigma_2\Sigma'}^{\mathfrak{S}}$. Combining these two results and choosing $\Sigma' := (\Sigma_1' \sqcap_{\mathfrak{S}} \Sigma_2')$ (this is possible because $\Sigma_1' \parallel \Sigma_2'$ and then $(\Sigma_1' \sqcap_{\mathfrak{S}} \Sigma_2') \sqsubset_{\mathfrak{S}} \Sigma_1', \Sigma_2'$), we obtain $\widehat{\Sigma_1(\Sigma_1' \sqcap_{\mathfrak{S}} \Sigma_2')}^{\mathfrak{S}}$ and $\widehat{\Sigma_2(\Sigma_1' \sqcap_{\mathfrak{S}} \Sigma_2')}^{\mathfrak{S}}$. Now, using the quasi-coherence property (125), we deduce that $\widehat{\Sigma(\Sigma_1' \sqcap_{\mathfrak{S}} \Sigma_2')}^{\mathfrak{S}}$ because $\Sigma \sqsubset_{\mathfrak{S}} (\Sigma_1 \sqcup_{\mathfrak{S}} \Sigma_2)$.
We now conclude using **Axiom 8** that $\mathfrak{t}_{(\Sigma_1 \sqcup_{\mathfrak{S}} \Sigma_2, \Sigma_1' \sqcap_{\mathfrak{S}} \Sigma_2')}$ is also a perfect yes/no test, and then a discriminating yes/no test. □

**Lemma 28.**

$$\begin{aligned} &\forall \Sigma \in \mathfrak{S}_c, \exists \Sigma' \in \mathfrak{S} \mid \mathfrak{t}_{(\Sigma,\Sigma')} \in \widetilde{\mathfrak{T}}_{disc}, \\ &\Sigma = \bigsqcup_{\sigma \in \mathscr{A}^*_{\mathfrak{S}} \mid \sigma \sqsubseteq_{\mathfrak{S}} \Sigma} \sigma, \qquad \Sigma' = \bigsqcap_{\sigma \in \mathscr{A}^*_{\mathfrak{S}} \mid \sigma \sqsubseteq_{\mathfrak{S}} \Sigma} \Delta_{\mathfrak{S}}(\sigma) \end{aligned} \tag{144}$$

■

*Proof.* Direct consequence of the previous lemma by recursion on the number of atoms of the compact element $\Sigma$. □

**Lemma 29.** Let us consider $(\mathfrak{c}_i)_{i \in I}$ a chain of elements of $\mathfrak{S}$ and let us denote $\Sigma := \bigsqcup_{i \in I} \mathfrak{c}_i$. Let us moreover consider a chain $(\mathfrak{c}_i')_{i \in I}$ such that $\mathfrak{t}_{(\mathfrak{c}_i,\mathfrak{c}_i')} \in \widetilde{\mathfrak{T}}_{disc}$, i.e. $\mathfrak{c}_i \bowtie \mathfrak{c}_i'$ and $\widetilde{\mathfrak{c}_i\mathfrak{c}_i'}$. Let us also denote $\Sigma' := \bigsqcap_{i \in I} \mathfrak{c}_i'$. Then we have $\mathfrak{t}_{(\Sigma,\Sigma')} \in \widetilde{\mathfrak{T}}_{disc}$, i.e. $\Sigma \bowtie_{\mathfrak{S}} \Sigma'$ and $\widetilde{\Sigma\Sigma'}^{\mathfrak{S}}$ ■

*Proof.* Firstly, $\forall \Sigma'' \sqsubset_{\mathfrak{S}} \Sigma'$ we have $\forall i \in I, \Sigma'' \sqsubset_{\mathfrak{S}} \mathfrak{c}_i'$ and then $\forall i \in I, \widehat{\mathfrak{c}_i\Sigma''}^{\mathfrak{S}}$. and then $\widehat{\Sigma\Sigma''}$. Secondly, $\forall \Sigma'' \sqsubset_{\mathfrak{S}} \Sigma$ there exists $j \in I$ such that $\Sigma'' \sqsubset_{\mathfrak{S}} \mathfrak{c}_k, \forall k \geq j$ and then $\widehat{\mathfrak{c}_k'\Sigma''}^{\mathfrak{S}}$. As a result, we have immediately $\widehat{\Sigma'\Sigma''}^{\mathfrak{S}}$. As a conclusion, $\Sigma \bowtie_{\mathfrak{S}} \Sigma'$. □

**Theorem 13.** The collection $\widetilde{\mathfrak{T}}_{disc}$ is a closed, complete and irredundant collection of perfect yes/no tests. Explicitely, we have

$$\forall \Sigma \in \mathfrak{S}, \exists! \Sigma' \in \mathfrak{S} \mid \mathfrak{t}_{(\Sigma,\Sigma')} \in \widetilde{\mathfrak{T}}_{disc},$$
$$\Sigma = \bigsqcup_{\sigma \in \mathscr{A}^*_{\mathfrak{S}} \mid \sigma \sqsubseteq_{\mathfrak{S}} \Sigma} \sigma, \qquad \Sigma' = \bigsqcap_{\sigma \in \mathscr{A}^*_{\mathfrak{S}} \mid \sigma \sqsubseteq_{\mathfrak{S}} \Sigma} \Delta_{\mathfrak{S}}(\sigma) \tag{145}$$

■

*Proof.* Using the two previous lemmas, we can compute the expression (145) for any $\Sigma$, due to the atomisticity of $\mathfrak{S}$. This result establishes the completeness result for $\widetilde{\mathfrak{T}}_{disc}$.
The closeness properties (136) and (137) are direct consequences of Lemma 27 and Lemma 29.
The irredundancy property (140) is a trivial consequence of the expression (145).
The last closeness property (138) is a trivial consequence of the definition (126). □

### 3.7 Orthogonality relation on the space of states

In the present subsection, $\mathfrak{U}$ is a fixed scheme of yes/no tests. When the choice $\mathfrak{U} = \widetilde{\mathfrak{T}}_{disc}$ will have to be done, it will be mentioned explicitly.

In a classical perspective, distinct prepared samples can be considered as corresponding to two 'orthogonal states' as soon as the observer can produce a statement that unambiguously distinguishes them. In other words, two states $\sigma_1, \sigma_2 \in \mathfrak{S}$ are said to be *orthogonal* (this fact will be denoted $\sigma_1 \perp \sigma_2$) iff there exists a property $\mathfrak{l} \in \mathscr{L}$ such that $\mathfrak{l}$ is simultaneously 'actual' on $\sigma_1$ and 'impossible' on $\sigma_2$.
Adopting an operational perspective, our definition will guarantee that the observer can prepare pairs of samples corresponding to pairs of orthogonal states associated with a given property.

**Notion 45.** Two states $\sigma_1, \sigma_2 \in \mathfrak{S}^*$ are said to be *orthogonal* (this fact will be denoted $\sigma_1 \perp \sigma_2$, the dependence with respect to $\mathfrak{U}$ is intentionally erased) iff

$$(\exists \mathfrak{t} \in \mathfrak{U}, \exists \sigma \in \lceil \mathfrak{Q}_{\mathfrak{t}} \rceil \cap \lceil \mathfrak{Q}_{\bar{\mathfrak{t}}} \rceil \mid (\sigma_1 \sqsupseteq_{\mathfrak{S}} \sigma.\mathfrak{t} \text{ and } \sigma_2 \sqsupseteq_{\mathfrak{S}} \sigma.\bar{\mathfrak{t}})). \tag{146}$$

We denote as usual

$$\forall S \subseteq \mathfrak{S}^*, \qquad S^{\perp} := \{\sigma' \in \mathfrak{S} \mid \forall \sigma \in S,\ \sigma \perp \sigma'\}. \tag{147}$$

**Lemma 30.** The relation $\perp$ is symmetric and anti-reflexive (i.e., $\sigma_1 \perp \sigma_2$ implies $\sigma_1 \neq \sigma_2$).
Two states $\sigma_1, \sigma_2 \in \mathfrak{S}^*$ are orthogonal iff they can be distinguished unambiguously by a statement associated with a perfect property. In other words,

$$\forall \sigma_1, \sigma_2 \in \mathfrak{S}^*,\ (\sigma_1 \perp \sigma_2) \quad \Leftrightarrow \quad (\exists \mathfrak{l} \in \lfloor \mathfrak{U} \rfloor \mid \sigma_1 \in \lceil \mathfrak{A}_{\mathfrak{l}} \rceil \text{ and } \sigma_2 \in \lceil \mathfrak{A}_{\bar{\mathfrak{l}}} \rceil). \tag{148}$$

Moreover, we have obviously

$$\forall \sigma_1, \sigma_2 \in \mathfrak{S}^*, \qquad \sigma_1 \perp \sigma_2 \ \Rightarrow\ \widetilde{\sigma_1 \sigma_2}^{\mathfrak{S}}. \tag{149}$$

■

*Proof.* The statement (148) is a direct consequence of property (74 (i)) and the equation (146) defining the orthogonality relation.
Following the definition of the consistency relation, we then note that, for any $\sigma_1, \sigma_2 \in \mathfrak{S}^*$, the property $\widehat{\sigma_1 \sigma_2}^{\mathfrak{S}}$ immediately implies $\forall \mathfrak{t} \in \mathfrak{U}, \widehat{\mathfrak{e}(\sigma_1, \mathfrak{t}) \mathfrak{e}(\sigma_2, \mathfrak{t})}^{\mathfrak{B}}$. By negation, we obtain that $(\exists \mathfrak{l} \in \lfloor \mathfrak{U} \rfloor \mid \sigma_1 \in \lceil \mathfrak{A}_{\mathfrak{l}} \rceil$ and $\sigma_2 \in \lceil \mathfrak{A}_{\bar{\mathfrak{l}}} \rceil)$ implies $\widetilde{\sigma_1 \sigma_2}^{\mathfrak{S}}$. □

**Lemma 31.** For any $\sigma \in \mathfrak{S}^*$, $\{\sigma\}^\perp$ is a non-empty filter. ■

*Proof.* Firstly, let us consider $\sigma_1, \sigma_2 \in \mathfrak{S}^*$ such that $\sigma_1 \sqsubseteq_{\mathfrak{S}} \sigma_2$ and let us assume that $\sigma_1 \perp \sigma$. We have $(\exists \mathfrak{l} \in \lfloor \mathfrak{U} \rfloor \mid \sigma \in \lceil \mathfrak{A}_{\mathfrak{l}} \rceil \text{ and } \sigma_1 \in \lceil \mathfrak{A}_{\bar{\mathfrak{l}}} \rceil) \Rightarrow (\exists \mathfrak{l} \in \lfloor \mathfrak{U} \rfloor \mid \sigma \in \lceil \mathfrak{A}_{\mathfrak{l}} \rceil \text{ and } \sigma_2 \in \lceil \mathfrak{A}_{\bar{\mathfrak{l}}} \rceil) \Leftrightarrow \sigma \in \{\sigma_2\}^\perp$.
Secondly, let us consider $S \subseteq \mathfrak{S}$ such that $\forall \sigma' \in S, \sigma \perp \sigma'$. There exists a family of generalized properties $(\mathfrak{l}_{\sigma''})_{\sigma'' \in S} \subseteq \lfloor \mathfrak{U} \rfloor$ such that $\forall \sigma' \in S$, $\Sigma_{\mathfrak{l}_{\sigma'}} \sqsubseteq_{\mathfrak{S}} \sigma$ and $\Sigma_{\overline{\mathfrak{l}_{\sigma'}}} \sqsubseteq_{\mathfrak{S}} \sigma'$. Using relation (136), we deduce that there exists a generalized property $\mathfrak{l}$ defined by $\Sigma_{\mathfrak{l}} := \bigsqcup_{\sigma' \in S} \Sigma_{\mathfrak{l}_{\sigma'}} \sqsubseteq_{\mathfrak{S}} \sigma$ and $\Sigma_{\bar{\mathfrak{l}}} := \bigsqcap_{\sigma' \in S} \Sigma_{\overline{\mathfrak{l}_{\sigma'}}}$. And we have then $\Sigma_{\mathfrak{l}} \sqsubseteq_{\mathfrak{S}} \sigma$ and $\Sigma_{\bar{\mathfrak{l}}} \sqsubseteq_{\mathfrak{S}} \sigma', \forall \sigma' \in S$. In other words, $\sigma \perp (\bigsqcap_{\mathfrak{S}} S)$. □

As a consequence, we will adopt the following definition

**Notion 46.** The space of states is equipped with a unary operation defined as follows

$$\forall \sigma \in \mathfrak{S}^*, \qquad \sigma^\star := \bigsqcap_{\mathfrak{S}} \{\sigma\}^\perp. \tag{150}$$

**Lemma 32.**

$$\forall \sigma \in \mathfrak{S}^*, \qquad \{\sigma\}^\perp = \uparrow^{\mathfrak{S}} \sigma^\star, \tag{151}$$

$$\forall \sigma_1, \sigma_2 \in \mathfrak{S}^*, \qquad (\sigma_1 \perp \sigma_2) \Leftrightarrow (\sigma_1^\star \sqsubseteq_{\mathfrak{S}} \sigma_2), \tag{152}$$

$$\forall \sigma_1, \sigma_2 \in \mathfrak{S}^*, \qquad (\sigma_1 \sqsubseteq_{\mathfrak{S}} \sigma_2) \Rightarrow (\sigma_2^\star \sqsubseteq_{\mathfrak{S}} \sigma_1^\star). \tag{153}$$

$$\forall \sigma \in \mathfrak{S}^*, \forall \mathfrak{l} \in \mathscr{L}, \qquad \widetilde{\mathfrak{e}}(\sigma, \bar{\mathfrak{l}}) = \widetilde{\mathfrak{e}}(\sigma^\star, \mathfrak{l}). \tag{154}$$

■

*Proof.* The properties (151) and (152) are trivial consequences of the defining property (150).
Let us consider $\sigma_1, \sigma_2 \in \mathfrak{S}^*$ such that $\sigma_1 \sqsubseteq_{\mathfrak{S}} \sigma_2$. We have $\sigma \in \{\sigma_1\}^\perp \Leftrightarrow (\exists \mathfrak{l} \in \lfloor \mathfrak{U} \rfloor \mid \sigma_1 \in \lceil \mathfrak{A}_{\mathfrak{l}} \rceil \text{ and } \sigma \in \lceil \mathfrak{A}_{\bar{\mathfrak{l}}} \rceil) \Rightarrow (\exists \mathfrak{l} \in \lfloor \mathfrak{U} \rfloor \mid \sigma_2 \in \lceil \mathfrak{A}_{\mathfrak{l}} \rceil \text{ and } \sigma \in \lceil \mathfrak{A}_{\bar{\mathfrak{l}}} \rceil) \Leftrightarrow \sigma \in \{\sigma_2\}^\perp$. We then have

$$\sigma_1 \sqsubseteq_{\mathfrak{S}} \sigma_2 \quad \Rightarrow \quad \{\sigma_1\}^\perp \subseteq \{\sigma_2\}^\perp \quad \Rightarrow \quad (\bigsqcap_{\mathfrak{S}} \{\sigma_1\}^\perp) \sqsubseteq_{\mathfrak{S}} (\bigsqcap_{\mathfrak{S}} \{\sigma_2\}^\perp). \tag{155}$$

As a result, we obtain the order-reversing property (153) of the unary operation $\star$. □

**Theorem 14.**

$$\forall \sigma \in \mathfrak{S}, \ \exists! \ \mathfrak{l}_\sigma \in \lfloor \mathfrak{U} \rfloor \mid (\Sigma_{\mathfrak{l}_\sigma} = \sigma, \quad \Sigma_{\overline{\mathfrak{l}_\sigma}} = \sigma^\star) \tag{156}$$

■

*Proof.* From $\sigma^\star \in \{\sigma\}^\perp$ and the defining property (148), we deduce that there exists $\mathfrak{l}_\sigma \in \lfloor \mathfrak{U} \rfloor$ such that $\Sigma_{\mathfrak{l}_\sigma} \sqsubseteq_{\mathfrak{S}} \sigma$ and $\Sigma_{\overline{\mathfrak{l}_\sigma}} \sqsubseteq_{\mathfrak{S}} \sigma^\star$. From equation (139), we know that, for any $\sigma \in \mathfrak{S}$, there exists a $\Sigma' \in \mathfrak{S}$ such that $\mathfrak{l}_{(\sigma, \Sigma')} \in \lfloor \mathfrak{U} \rfloor$. We note by the way that $\Sigma' \in \{\sigma\}^\perp$. Using equation (140), we deduce that $\Sigma' \sqsubseteq_{\mathfrak{S}} \Sigma_{\overline{\mathfrak{l}_\sigma}}$. Recalling that $\sigma^\star = \bigsqcap_{\mathfrak{S}} \{\sigma\}^\perp$, we then conclude that $\Sigma' = \Sigma_{\overline{\mathfrak{l}_\sigma}} = \sigma^\star$. □

*Remark* 17. We note that, using Theorem 14 for $\mathfrak{U} = \widetilde{\mathfrak{T}}_{disc}$ and equation 145 we obtain an explicit expression for the star operation

$$\Sigma^\star = \bigsqcap_{\sigma \in \mathscr{A}^*_{\mathfrak{S}} \mid \sigma \sqsubseteq_{\mathfrak{S}} \Sigma} \Delta_{\mathfrak{S}}(\sigma) \tag{157}$$

**Lemma 33.**

$$\forall S \subseteq^{\neq\varnothing}_{fin} \mathfrak{S}^* \mid \widehat{S}^{\mathfrak{S}}, \qquad (\bigsqcup\nolimits_{\mathfrak{S}} S)^\star = \bigsqcap\nolimits_{\sigma\in S} \sigma^\star \tag{158}$$

$$\forall \sigma \in \mathfrak{S}^*, \qquad \sigma^{\star\star} = \sigma. \tag{159}$$

■

*Proof.* Using Lemma 14, we introduce, for any $\sigma \in S$, the property $\mathfrak{l}_\sigma$ such that $\Sigma_{\mathfrak{l}_\sigma} = \sigma$ and $\Sigma_{\overline{\mathfrak{l}_\sigma}} = \sigma^\star$. We now deduce, from equation (136), that there exists a perfect property $\mathfrak{l} \in \lfloor\mathfrak{U}\rfloor$ such that $\Sigma_{\mathfrak{l}} = \bigsqcup_{\sigma\in S}\Sigma_{\mathfrak{l}_\sigma} = \bigsqcup_{\mathfrak{S}} S$ and $\Sigma'_{\mathfrak{l}} = \bigsqcap_{\sigma\in S}\Sigma_{\overline{\mathfrak{l}_\sigma}} = \bigsqcap_{\sigma\in S}\sigma^\star$. Using once again equation (140), we deduce $\Sigma'_{\mathfrak{l}} = \Sigma^\star_{\mathfrak{l}}$, i.e. the property (158).
Using Lemma 14, the property $(\ \forall \mathfrak{l} \in \lfloor\mathfrak{U}\rfloor,\ \mathfrak{l} = \overline{\overline{\mathfrak{l}}}\ )$ and equation (140), we deduce the property (159). □

**Lemma 34.** The closure operator associated with the orthogonality relation $\perp$ satisfies

$$\forall S \subseteq \mathfrak{S}^*, \qquad S^{\perp\perp} = \uparrow^{\mathfrak{S}}(\bigsqcap\nolimits_{\mathfrak{S}} S). \tag{160}$$

■

*Proof.* First of all, we have $S^\perp = \bigcap_{\sigma\in S}\{\sigma\}^\perp = \uparrow^{\mathfrak{S}} \bigsqcup_{\sigma\in S}\sigma^\star$. As a consequence, we have $S^{\perp\perp} = (\uparrow^{\mathfrak{S}} \bigsqcup_{\sigma\in S}\sigma^\star)^\perp = \{\bigsqcup_{\sigma\in S}\{\sigma\}^\star\}^\perp = \uparrow^{\mathfrak{S}} (\bigsqcup_{\sigma\in S}\sigma^\star)^\star$. Now, using the De Morgan's law (158), we deduce $S^{\perp\perp} = \uparrow^{\mathfrak{S}} (\bigsqcap_{\sigma\in S}\sigma^{\star\star})$. We now use the involutive property (159) to conclude $S^{\perp\perp} = \uparrow^{\mathfrak{S}}(\bigsqcap_{\mathfrak{S}} S)$. □

## 3.8 The space of ortho-closed subsets of pure states as a Hilbert lattice

In this subsection, we will impose $\mathfrak{U} = \widetilde{\mathfrak{T}}_{disc}$.

**Notion 47.** The space of pure states $\mathfrak{S}_{pure}$ inherits an orthogonality relation (denoted $\underline{\perp}$) from the orthogonality relation $\perp$ defined on the whole space of states $\mathfrak{S}$.

$$\forall \sigma_1, \sigma_2 \in \mathfrak{S}_{pure}, \quad (\sigma_1 \underline{\perp} \sigma_2) \quad :\Leftrightarrow \quad (\sigma_1 \perp \sigma_2), \tag{161}$$

$$\forall S \subseteq \mathfrak{S}_{pure}, \quad S^{\underline{\perp}} \quad = \quad \{\sigma' \in \mathfrak{S}_{pure}, \mid \forall \sigma \in S,\ \sigma \perp \sigma'\} = \underline{S^\perp}. \tag{162}$$

**Lemma 35.**

$$\forall S \subseteq \mathfrak{S}_{pure}, \quad S^{\underline{\perp}\underline{\perp}} \quad = \quad \underline{\{\bigsqcap\nolimits_{\mathfrak{S}} S\}}. \tag{163}$$

■

**Notion 48.** The set of ortho-closed subsets of the space of pure states equipped with the orthogonality relation $\underline{\perp}$ is denoted $\mathscr{C}(\mathfrak{S}_{pure})$. The set $\mathscr{C}(\mathfrak{S}_{pure})$ is equipped with the following operations

$$\forall \mathfrak{c}_1, \mathfrak{c}_2 \in \mathscr{C}(\mathfrak{S}_{pure}), \qquad \mathfrak{c}_1 \wedge \mathfrak{c}_2 \quad := \quad \mathfrak{c}_1 \cap \mathfrak{c}_2 \tag{164}$$

$$\mathfrak{c}_1 \vee \mathfrak{c}_2 \quad := \quad (\mathfrak{c}_1 \cup \mathfrak{c}_2)^{\underline{\perp}\underline{\perp}}. \tag{165}$$

and by the unary operation $\underline{\perp}$.

**Lemma 36.** $\mathscr{C}(\mathfrak{S}_{pure})$ is a complete ortho-lattice. ■

**Lemma 37.** $\mathscr{C}(\mathfrak{S}_{pure})$ is atomic, i.e.

$$\forall \sigma \in \mathfrak{S}_{pure}, \{\sigma\}^{\perp\perp} = \{\sigma\}. \tag{166}$$

■

*Proof.* Due to **Axiom 7** (i.e. the absence of type 2's pure states), we have immediately

$$\forall \sigma \in \mathfrak{S}_{pure}, \underline{\{\sigma\}} = \{\sigma\}. \tag{167}$$

□

**Lemma 38.** The lattice $\mathscr{C}(\mathfrak{S}_{pure})$ is atomistic, i.e.

$$\forall \mathfrak{c} \in \mathscr{C}(\mathfrak{S}_{pure}), \qquad \mathfrak{c} = \bigvee_{\sigma \in \mathfrak{c}} \{\sigma\}^{\perp\perp} = \bigvee_{\sigma \in \mathfrak{c}} \{\sigma\}. \tag{168}$$

■

**Lemma 39.** We have the following property

$$\forall S \subseteq_{fin}^{\neq \varnothing} \mathfrak{S}, \forall \sigma \in \mathfrak{S} \mid \sigma \notin S^{\perp\perp}, \qquad \exists \sigma' \in S^{\perp} \mid (S \cup \{\sigma\})^{\perp\perp} = (S \cup \{\sigma'\})^{\perp\perp}. \tag{169}$$

We have then immediately

$$\forall S \subseteq_{fin}^{\neq \varnothing} \mathfrak{S}_{pure}, \forall \sigma \in \mathfrak{S}_{pure} \mid \sigma \notin S^{\underline{\perp}\underline{\perp}}, \qquad \exists \sigma' \in S^{\underline{\perp}} \mid (S \cup \{\sigma\})^{\underline{\perp}\underline{\perp}} = (S \cup \{\sigma'\})^{\underline{\perp}\underline{\perp}}. \tag{170}$$

■

*Proof.* First of all, from (160), we have $(S \cup \{\sigma\})^{\perp\perp} = \uparrow^{\mathfrak{S}}((\bigsqcap_{\mathfrak{S}} S) \sqcap_{\mathfrak{S}} \sigma)$ and $(S \cup \{\sigma'\})^{\perp\perp} = \uparrow^{\mathfrak{S}}((\bigsqcap_{\mathfrak{S}} S) \sqcap_{\mathfrak{S}} \sigma')$. We have $((\bigsqcap_{\mathfrak{S}} S) \sqcap_{\mathfrak{S}} \sigma) \sqsubset_{\mathfrak{S}} (\bigsqcap_{\mathfrak{S}} S)$ because the condition $\sigma \notin S^{\perp\perp}$ means $\sigma \not\sqsupseteq_{\mathfrak{S}} (\bigsqcap_{\mathfrak{S}} S)$. Due to lemma 14, there exists a unique discriminating property $\mathfrak{l}$ such that $\Sigma_{\mathfrak{l}} = (\bigsqcap_{\mathfrak{S}} S)$ and $\Sigma_{\overline{\mathfrak{l}}} = (\bigsqcap_{\mathfrak{S}} S)^{\star}$. The discriminating character of the property $\mathfrak{l}$ implies that $\forall \Sigma'' \sqsubset_{\mathfrak{S}} \Sigma_{\mathfrak{l}}$, $\widehat{\Sigma_{\overline{\mathfrak{l}}} \Sigma''}^{\mathfrak{S}}$. In particular, choosing $\Sigma'' = ((\bigsqcap_{\mathfrak{S}} S) \sqcap_{\mathfrak{S}} \sigma)$, we obtain $\widehat{(\bigsqcap_{\mathfrak{S}} S)^{\star} ((\bigsqcap_{\mathfrak{S}} S) \sqcap_{\mathfrak{S}} \sigma)}$. Explicitely, there exists an element $\sigma' \in S^{\perp} = (\uparrow^{\mathfrak{S}} (\bigsqcap_{\mathfrak{S}} S)^{\star})$ such that $\sigma' \sqsupseteq_{\mathfrak{S}} (\sigma \sqcap_{\mathfrak{S}} (\bigsqcap_{\mathfrak{S}} S))$. In other words, there exists an element $\sigma' \in S^{\perp}$ such that $(S \cup \{\sigma'\})^{\perp\perp} = (S \cup \{\sigma\})^{\perp\perp}$.
We deduce (170) immediately from (169) using (163). □

**Lemma 40.** The ortho-lattice $\mathscr{C}(\mathfrak{S}_{pure})$ is ortho-modular. ■

*Proof.* Let us firstly prove that, for any maximal orthogonal subset $S$ of an ortho-closed set $A \in \mathscr{C}(\mathfrak{S}_{pure})$, we have $A = S^{\underline{\perp}\underline{\perp}}$. Let us suppose it exists $\sigma \in A \setminus S^{\underline{\perp}\underline{\perp}}$. Using previous lemma, we can identify $\sigma' \in S^{\underline{\perp}}$ such that $(S \cup \{\sigma\})^{\underline{\perp}\underline{\perp}} = (S \cup \{\sigma'\})^{\underline{\perp}\underline{\perp}} \subseteq A$. This result contradicts the maximality of the orthogonal subset $S$ in $A$. We have then necessarily $A = S^{\underline{\perp}\underline{\perp}}$. Using now [25] (see also [77, Theorem 35]), we then conclude that the ortho-lattice $\mathscr{C}(\mathfrak{S}_{pure})$ is orthomodular. □

**Lemma 41.** The ortho-lattice $\mathscr{C}(\mathfrak{S}_{pure})$ satisfies the *covering property*, i.e.

$$\forall A \in \mathscr{C}(\mathfrak{S}_{pure}), \forall \sigma \in \mathfrak{S}_{pure} \mid \sigma \notin A, \qquad \{\sigma\} \vee A \text{ covers } A. \tag{171}$$

■

*Proof.* Let $A \in \mathscr{C}(\mathfrak{S}_{pure})$ and $\sigma \notin A$. From Lemma 39, we know that there exists $\sigma' \in A^{\underline{\perp}}$ such that $A \vee \{\sigma\} = (A \cup \{\sigma\})^{\underline{\perp}\underline{\perp}} = (A \cup \{\sigma'\})^{\underline{\perp}\underline{\perp}} = A \vee \{\sigma'\}$. Since $\sigma'$ is an atom orthogonal to $A$, it follows from the orthomodularity of $\mathscr{C}(\mathfrak{S}_{pure})$ that $A \vee \{\sigma\}$ covers $A$. □

**Theorem 15.** The ortho-lattice $\mathscr{C}(\mathfrak{S}_{pure})$ forms a *Piron's propositional system* (also called *Hilbert lattice*) (see [73, Definition 5.9]). ■

*Proof.* Direct consequence of Lemma 36, Lemma 38, Lemma 40, Lemma 41, □

**Notion 49.** The orthogonality space $\mathfrak{S}_{pure}$ is said to be *reducible* iff $\mathfrak{S}_{pure}$ is the disjoint union of non-empty subsets $\mathfrak{S}_1, \mathfrak{S}_2 \subseteq \mathfrak{S}_{pure}$ such that $\sigma_1 \underline{\perp} \sigma_2$ for any $\sigma_1 \in \mathfrak{S}_1$ and $\sigma_2 \in \mathfrak{S}_2$. Otherwise, $\mathfrak{S}_{pure}$ is said to be *irreducible*.

**Axiom 10.**

$$\forall \sigma_1, \sigma_2 \in \mathfrak{S}_{pure} \mid \sigma_1 \neq \sigma_2, \quad \exists \sigma_3 \in \mathfrak{S}_{pure} \mid (\ \sigma_3 \neq \sigma_1, \quad \sigma_3 \neq \sigma_2, \text{ and } \sigma_3 \in \underline{\{\sigma_1 \sqcap_{\mathfrak{S}} \sigma_2\}}\ ) \tag{172}$$

**Lemma 42.** $\mathfrak{S}_{pure}$ is irreducible. ■

*Proof.* Let us assume that $\mathfrak{S}_{pure}$ is reducible, then $\mathfrak{S}_{pure}$ is the disjoint union of non-empty subsets $\mathfrak{S}_1, \mathfrak{S}_2 \subseteq \mathfrak{S}_{pure}$ such that $\sigma_1 \underline{\perp} \sigma_2$ for any $\sigma_1 \in \mathfrak{S}_1$ and $\sigma_2 \in \mathfrak{S}_2$. Let us consider $\sigma_1 \in \mathfrak{S}_1$ and $\sigma_2 \in \mathfrak{S}_2$ and $\sigma_3 \in \{\sigma_1, \sigma_2\}^{\underline{\perp}\underline{\perp}}$. Necessarily $\sigma_3 \in S_1$ or $\sigma_3 \in S_2$, and then $\sigma_3 \underline{\perp} \sigma_1$ or $\sigma_3 \underline{\perp} \sigma_2$. If $\sigma_3 \underline{\perp} \sigma_1$, we have $\{\sigma_3\} \subseteq \{\sigma_1, \sigma_2\}^{\underline{\perp}\underline{\perp}} \cap \{\sigma_1\}^{\underline{\perp}} = (\{\sigma_1\} \vee \{\sigma_2\}) \cap \{\sigma_1\}^{\underline{\perp}} = \{\sigma_2\}$. In the same way, if $\sigma_3 \underline{\perp} \sigma_2$, then $\{\sigma_3\} = \{\sigma_1\}$. As a result, we conclude that $\{\sigma_1, \sigma_2\}^{\underline{\perp}\underline{\perp}} = \{\sigma_1, \sigma_2\}$
From **Axiom 10** we then deduce that $\mathfrak{S}_{pure}$ is irreducible. □

**Corollary 5.** $\mathscr{C}(\mathfrak{S}_{pure})$ is an irreducible ortho-lattice. ■

*Proof.* Let us suppose $\mathscr{C}(\mathfrak{S}_{pure})$ is reducible, then there exist a central element $A$ distinct from the bottom element $\varnothing$ and the top element $\mathfrak{S}_{pure}$. Then any atom is either below $A$ or below $A^{\underline{\perp}}$, and then any $\sigma \in \mathfrak{S}_{pure}$ is either in $A$ or in $A^{\underline{\perp}}$. Hence $\mathfrak{S}_{pure}$ is reducible. □

**Theorem 16.** If the condition of the **Axiom 10** is satisfied, the ortho-lattice $\mathscr{C}(\mathfrak{S}_{pure})$ forms an irreducible Piron's propositional system. ■

*Proof.* Direct consequence of Theorem 15 and Corollary 5. □

# 4 Symmetries

Let us consider two observers, $O_1$ and $O_2$, who wish to formalize 'transactions' concerning their experimental results about the system.

**Notion 50.** Observer $O_1$ has chosen a preparation process, $\mathfrak{p}_1 \in \mathfrak{P}^{(O_1)}$, and intends to describe it to observer $O_2$. Observer $O_2$ is able to interpret the macroscopic data defining $\mathfrak{p}_1$ in terms of the elements of $\mathfrak{P}^{(O_2)}$ using a map $f_{(12)} : \mathfrak{P}^{(O_1)} \to \mathfrak{P}^{(O_2)}$ (i.e., $O_2$ knows how to identify a preparation procedure $f_{(12)}(\mathfrak{p}_1)$ corresponding to any $\mathfrak{p}_1$).

Observer $O_2$ has chosen a yes/no test $\mathfrak{t}_2 \in \mathfrak{T}^{(O_2)}$ and intends to address the corresponding question to $O_1$. Observer $O_1$ is able to interpret the macroscopic data defining $\mathfrak{t}_2$ in terms

of the elements of $\mathfrak{T}^{(O_1)}$ using a map $f^{(21)} : \mathfrak{T}^{(O_2)} \to \mathfrak{T}^{(O_1)}$ (i.e., $O_1$ knows how to fix a test $f^{(21)}(\mathfrak{t}_2)$ corresponding to any $\mathfrak{t}_2$).
The pair of maps $(f_{(12)}, f^{(21)})$ where $f_{(12)} \in \mathfrak{P}^{(O_1)} \to \mathfrak{P}^{(O_2)}$ and $f^{(21)} : \mathfrak{T}^{(O_2)} \to \mathfrak{T}^{(O_1)}$ defines *a dictionary* formalizing the transaction from $O_1$ to $O_2$.

The first task these observers want to accomplish is to confront their knowledge, i.e., to compare their 'statements' about the system.

**Notion 51.** As soon as the transaction is formalized using a dictionary, the two observers can formulate their statements and each confront them with the statements of the other.
First, observer $O_1$ can interpret the macroscopic data defining $\mathfrak{t}_2$ using the map $f^{(21)}$. Then, he produces the statement $\mathfrak{e}^{(O_1)}(\mathfrak{p}_1, f^{(21)}(\mathfrak{t}_2))$ concerning the results of this test on the chosen samples.
Secondly, observer $O_2$ can interpret the macroscopic data defining $\mathfrak{p}_1$ using the map $f_{(12)}$. Then, observer $O_2$ pronounces her statement $\mathfrak{e}^{(O_2)}(f_{(12)}(\mathfrak{p}_1), \mathfrak{t}_2)$ concerning the results of test $\mathfrak{t}_2$ on the correspondingly prepared samples.
The two observers, $O_1$ and $O_2$, are said to *agree about their statements* iff

$$\forall \mathfrak{p}_1 \in \mathfrak{P}^{(O_1)}, \forall \mathfrak{t}_2 \in \mathfrak{T}^{(O_2)}, \qquad \mathfrak{e}^{(O_2)}(f_{(12)}(\mathfrak{p}_1), \mathfrak{t}_2) = \mathfrak{e}^{(O_1)}(\mathfrak{p}_1, f^{(21)}(\mathfrak{t}_2)), \tag{173}$$

i.e., iff the adjoint pair $(f_{(12)}, f^{(21)})$ defines a morphism of Chu spaces from $(\mathfrak{P}^{(O_1)}, \mathfrak{T}^{(O_1)}, \mathfrak{e}^{(O_1)})$ to $(\mathfrak{P}^{(O_2)}, \mathfrak{T}^{(O_2)}, \mathfrak{e}^{(O_2)})$ [67].

**Lemma 43.** If the dictionary $(f_{(12)}, f^{(21)})$ satisfies property (173) (i.e., the adjoint pair defines a Chu morphism from $(\mathfrak{P}^{(O_1)}, \mathfrak{T}^{(O_1)}, \mathfrak{e}^{(O_1)})$ to $(\mathfrak{P}^{(O_2)}, \mathfrak{T}^{(O_2)}, \mathfrak{e}^{(O_2)})$) we have immediately

$$\forall \mathfrak{p}, \mathfrak{p}' \in \mathfrak{P}^{(O_1)}, \qquad (\mathfrak{p} \sim_{\mathfrak{P}^{(O_1)}} \mathfrak{p}') \Rightarrow (f_{(12)}(\mathfrak{p}) \sim_{\mathfrak{P}^{(O_2)}} f_{(12)}(\mathfrak{p}')) \tag{174}$$

■

*Proof.* $\mathfrak{p} \sim_{\mathfrak{P}^{(O_1)}} \mathfrak{p}'$ implies, in particular, $\mathfrak{e}^{(O_1)}(\mathfrak{p}, f^{(21)}(\mathfrak{t}_2)) = \mathfrak{e}^{(O_1)}(\mathfrak{p}', f^{(21)}(\mathfrak{t}_2))$ for any $\mathfrak{t}_2 \in \mathfrak{T}^{(O_2)}$. Using property (173), we obtain $\mathfrak{e}^{(O_1)}(f_{(12)}(\mathfrak{p}), \mathfrak{t}_2) = \mathfrak{e}^{(O_1)}(f_{(12)}(\mathfrak{p}'), \mathfrak{t}_2)$ for any $\mathfrak{t}_2 \in \mathfrak{T}^{(O_2)}$, i.e., $f_{(12)}(\mathfrak{p}) \sim_{\mathfrak{P}^{(O_2)}} f_{(12)}(\mathfrak{p}')$. □

**Notion 52.** In order for observers $O_1$ and $O_2$ to be in *complete agreement* about the system (once they do agree about their statements), it is necessary for them to be unable to distinguish the outcomes of the control tests, realized to confirm (or not) their statements.
Firstly, the measurement realized by observer $O_2$ is given by $(f_{(12)}(\mathfrak{p}_1).\mathfrak{t}_2)$.
Secondly, observer $O_2$ interprets, using $f_{(12)}$, the measurement $(\mathfrak{p}_1.f^{(21)}(\mathfrak{t}_2))$ realized by observer $O_1$.
In other words, it is necessary for the dictionary $(f_{(12)}, f^{(21)})$ also to satisfy the following property

$$\forall \mathfrak{p}_1 \in \mathfrak{P}^{(O_1)}, \forall \mathfrak{t}_2 \in \mathfrak{T}^{(O_2)}, \qquad f_{(12)}(\mathfrak{p}_1).\mathfrak{t}_2 \sim_{\mathfrak{S}^{(O_2)}} f_{(12)}(\mathfrak{p}_1.f^{(21)}(\mathfrak{t}_2)), \tag{175}$$

**Notion 53.** If the dictionary $(f_{(12)}, f^{(21)})$ satisfies properties (173) and (175), as well as property (176) below

$$\forall \mathfrak{t}, \mathfrak{t}' \in \mathfrak{T}^{(O_2)}, \qquad f^{(21)}(\mathfrak{t}.\mathfrak{t}') = f^{(21)}(\mathfrak{t}).f^{(21)}(\mathfrak{t}'), \tag{176}$$

and properties (177) and (178) below

$$\forall \mathfrak{p},\mathfrak{p}' \in \mathfrak{P}^{(O_1)},\ \ (f_{(12)}(\mathfrak{p}) \sim_{\mathfrak{P}^{(O_2)}} f_{(12)}(\mathfrak{p}')) \ \Rightarrow\ (\mathfrak{p} \sim_{\mathfrak{P}^{(O_1)}} \mathfrak{p}'), \tag{177}$$

$$f^{(21)} \text{ surjective}, \tag{178}$$

then this dictionary is said to *relate by a symmetry* $(\mathfrak{P}^{(O_1)},\mathfrak{T}^{(O_1)},\mathfrak{e}^{(O_1)})$ to $(\mathfrak{P}^{(O_2)},\mathfrak{T}^{(O_2)},\mathfrak{e}^{(O_2)})$. This fact will be denoted

$$(f_{(12)},f^{(21)}) \quad \in \quad \mathbf{Sym}\left[(\mathfrak{P}^{(O_1)},\mathfrak{T}^{(O_1)},\mathfrak{e}^{(O_1)}) \to (\mathfrak{P}^{(O_2)},\mathfrak{T}^{(O_2)},\mathfrak{e}^{(O_2)})\right]. \tag{179}$$

*Remark* 18. We note that the axiom (176) has been designed to preserve the associativity property of the succession rule. Indeed, for any $\mathfrak{p} \in \mathfrak{P}^{(O_1)}$, and any $\mathfrak{t},\mathfrak{t}' \in \mathfrak{T}^{(O_2)}$, we have $f_{(12)}(\mathfrak{p})\centerdot(\mathfrak{t}\centerdot\mathfrak{t}') = (f_{(12)}(\mathfrak{p})\centerdot\mathfrak{t})\centerdot\mathfrak{t}' = f_{(12)}(\mathfrak{p}\centerdot f^{(21)}(\mathfrak{t}))\centerdot\mathfrak{t}' = f_{(12)}((\mathfrak{p}\centerdot f^{(21)}(\mathfrak{t}))\centerdot f^{(21)}(\mathfrak{t}')) = f_{(12)}(\mathfrak{p}\centerdot(f^{(21)}(\mathfrak{t})\centerdot f^{(21)}(\mathfrak{t}'))) = f_{(12)}(\mathfrak{p}\centerdot f^{(21)}(\mathfrak{t}\centerdot\mathfrak{t}'))$.

*Remark* 19. Properties (177) and (178) have been introduced in order to be able to derive Theorem 18.

**Theorem 17. [Composition of symmetries]**

$$\left.\begin{array}{l}(f_{(12)},f^{(21)}) \in \mathbf{Sym}\left[(\mathfrak{P}^{(O_1)},\mathfrak{T}^{(O_1)},\mathfrak{e}^{(O_1)}) \to (\mathfrak{P}^{(O_2)},\mathfrak{T}^{(O_2)},\mathfrak{e}^{(O_2)})\right] \\ (g_{(23)},g^{(32)}) \in \mathbf{Sym}\left[(\mathfrak{P}^{(O_2)},\mathfrak{T}^{(O_2)},\mathfrak{e}^{(O_2)}) \to (\mathfrak{P}^{(O_3)},\mathfrak{T}^{(O_3)},\mathfrak{e}^{(O_3)})\right]\end{array}\right\}$$
$$\Rightarrow \quad (g_{(23)} \circ f_{(12)}, f^{(21)} \circ g^{(32)}) \in \mathbf{Sym}\left[(\mathfrak{P}^{(O_1)},\mathfrak{T}^{(O_1)},\mathfrak{e}^{(O_1)}) \to (\mathfrak{P}^{(O_3)},\mathfrak{T}^{(O_3)},\mathfrak{e}^{(O_3)})\right]. \tag{180}$$

■

*Proof.* We firstly note that

$$\forall \mathfrak{p}_1 \in \mathfrak{P}^{(O_1)}, \forall \mathfrak{t}_3 \in \mathfrak{T}^{(O_3)},\ \ \mathfrak{e}^{(O_3)}(g_{(23)} \circ f_{(12)}(\mathfrak{p}_1),\mathfrak{t}_3) = \mathfrak{e}^{(O_2)}(f_{(12)}(\mathfrak{p}_1),g^{(32)}(\mathfrak{t}_3)) = \mathfrak{e}^{(O_1)}(\mathfrak{p}_1, f^{(21)} \circ g^{(32)}(\mathfrak{t}_3)). \tag{181}$$

Secondly, we have

$$\forall \mathfrak{p}_1 \in \mathfrak{P}^{(O_1)}, \forall \mathfrak{t}_3 \in \mathfrak{T}^{(O_3)},\ \ (g_{(23)} \circ f_{(12)})(\mathfrak{p}_1)\centerdot\mathfrak{t}_3 = g_{(23)}(f_{(12)}(\mathfrak{p}_1)\centerdot g^{(32)}(\mathfrak{t}_3)) = (g_{(23)} \circ f_{(12)})((\mathfrak{p}_1)\centerdot(f^{(21)} \circ g^{(32)})(\mathfrak{t}_3)). \tag{182}$$

The properties (176), (177), and (178) are trivially preserved by composition. □

**Lemma 44.** The dictionary $(h_{(12)},h^{(21)})$ defined by

$$\begin{array}{rccc} h_{(12)}: & \mathfrak{P} & \to & \mathfrak{S} \\ & \mathfrak{p} & \mapsto & \lceil \mathfrak{p} \rceil \end{array} \qquad \begin{array}{rccc} h^{(21)}: & \mathfrak{T} & \to & \mathfrak{T} \\ & \mathfrak{t} & \mapsto & \mathfrak{t} \end{array} \tag{183}$$

satisfies properties (173), (175), (176), (177) and (178). In other words, this dictionary relates by a symmetry $(\mathfrak{P},\mathfrak{T},\mathfrak{e})$ to $(\mathfrak{S},\mathfrak{T},\widetilde{\mathfrak{e}})$.

$$(h_{(12)},h^{(21)}) \in \mathbf{Sym}\left[(\mathfrak{P},\mathfrak{T},\mathfrak{e}) \to (\mathfrak{S},\mathfrak{T},\widetilde{\mathfrak{e}})\right] \tag{184}$$

■

*Proof.* Property (173) is a direct consequence of the definition (11) of $\widetilde{\mathfrak{e}}$.
Property (175) is a direct consequence of property (67).
Property (176) is tautologically verified.
Property (177) relies on the definition of the map $\lceil\cdot\rceil$.
Property (178) is trivial. □

As a consequence of Lemma 44 and Theorem 17, we can define the following notion.

**Notion 54.** For any dictionary $(f_{(12)}, f^{(21)}) \in \mathbf{Sym}\left[(\mathfrak{P}^{(O_1)}, \mathfrak{T}^{(O_1)}, \mathfrak{e}^{(O_1)}) \to (\mathfrak{P}^{(O_2)}, \mathfrak{T}^{(O_2)}, \mathfrak{e}^{(O_2)})\right]$, we will associate the dictionary $(\widetilde{f}_{(12)}, f^{(21)})$ defined by

$$\forall \mathfrak{p} \in \mathfrak{P}^{(O_1)}, \qquad \widetilde{f}_{(12)}(\lceil \mathfrak{p} \rceil_1) := \lceil f_{(12)}(\mathfrak{p}) \rceil_2. \tag{185}$$

We have

$$(\widetilde{f}_{(12)}, f^{(21)}) \in \mathbf{Sym}\left[(\mathfrak{S}^{(O_1)}, \mathfrak{T}^{(O_1)}, \widetilde{\mathfrak{e}}^{(O_1)}) \to (\mathfrak{S}^{(O_2)}, \mathfrak{T}^{(O_2)}, \widetilde{\mathfrak{e}}^{(O_2)})\right]. \tag{186}$$

Explicitly, $(\widetilde{f}_{(12)}, f^{(21)})$ has to satisfy the following requirements :

$$\forall \sigma_1 \in \mathfrak{S}^{(O_1)}, \forall \mathfrak{t}_2 \in \mathfrak{T}^{(O_2)},\ \widetilde{\mathfrak{e}}^{(O_2)}_{\mathfrak{t}_2}(\widetilde{f}_{(12)}(\sigma_1)) = \widetilde{\mathfrak{e}}^{(O_1)}_{f^{(21)}(\mathfrak{t}_2)}(\sigma_1), \tag{187}$$

$$\forall \sigma_1 \in \mathfrak{S}^{(O_1)}, \forall \mathfrak{t}_2 \in \mathfrak{T}^{(O_2)},\ \widetilde{f}_{(12)}(\sigma_1).\mathfrak{t}_2 \ = \ \widetilde{f}_{(12)}(\sigma_1.f^{(21)}(\mathfrak{t}_2)), \tag{188}$$

$$\forall \mathfrak{t}, \mathfrak{t}' \in \mathfrak{T}^{(O_2)},\ f^{(21)}(\mathfrak{t}.\mathfrak{t}') = f^{(21)}(\mathfrak{t}).f^{(21)}(\mathfrak{t}'), \tag{189}$$

$$\widetilde{f}_{(12)} \text{ injective} \tag{190}$$

$$f^{(21)} \text{ surjective} \tag{191}$$

**Lemma 45.** $\widetilde{f}_{(12)}$ is as map satisfying

$$\widetilde{f}_{(12)} \in \left[\mathfrak{S}^{(O_1)} \to \mathfrak{S}^{(O_2)}\right]^{\overline{\wedge}}_{\sqcap} \quad \text{and} \quad \widetilde{f}_{(12)}(\bot_{\mathfrak{S}^{(O_1)}}) = \bot_{\mathfrak{S}^{(O_2)}}. \tag{192}$$

In particular, $\widetilde{f}_{(12)}$ is order-preserving. ■

*Proof.*

$$\forall \mathfrak{t}_2 \in \mathfrak{T}^{(O_2)},\ \widetilde{\mathfrak{e}}^{(O_2)}_{\mathfrak{t}_2}(\widetilde{f}_{(12)}(\bot_{\mathfrak{S}^{(O_1)}})) = \widetilde{\mathfrak{e}}^{(O_1)}_{f^{(21)}(\mathfrak{t}_2)}(\bot_{\mathfrak{S}^{(O_1)}}) = \bot \quad \Rightarrow \quad \widetilde{f}_{(12)}(\bot_{\mathfrak{S}^{(O_1)}}) = \bot_{\mathfrak{S}^{(O_2)}}. \tag{193}$$

Secondly, from Theorem 4, we know that $\forall \mathfrak{t} \in \mathfrak{T}^{(O_2)}$, $\widetilde{\mathfrak{e}}^{(O_2)}_{\mathfrak{t}}$ is an order-preserving and Lawson-continuous map (i.e., $\widetilde{\mathfrak{e}}^{(O_1)}_{\mathfrak{t}} \in \left[\mathfrak{S}^{(O_1)} \to \mathfrak{B}\right]_{\sqcap}$). We can then deduce that $\widetilde{f}_{(12)}$ is an order-preserving and Lawson-continuous map. Let us first check the Scott continuity of $\widetilde{f}_{(12)}$ :

$$\begin{aligned}
\forall \mathfrak{t} \in \mathfrak{T}^{(O_2)}, \forall \mathfrak{C} \subseteq_{Chain} \mathfrak{S}^{(O_1)},\quad \widetilde{\mathfrak{e}}^{(O_2)}_{\mathfrak{t}}(\widetilde{f}_{(12)}(\textstyle\bigsqcup_{\mathfrak{S}^{(O_1)}} \mathfrak{C})) &= \widetilde{\mathfrak{e}}^{(O_1)}_{f^{(21)}(\mathfrak{t})}(\textstyle\bigsqcup_{\mathfrak{S}^{(O_1)}} \mathfrak{C}) && \text{from eq. (187)} \\
&= \textstyle\bigsqcup_{\sigma \in \mathfrak{C}} \widetilde{\mathfrak{e}}^{(O_1)}_{f^{(21)}(\mathfrak{t})}(\sigma) && \text{from eq. (31)} \\
&= \textstyle\bigsqcup_{\sigma \in \mathfrak{C}} \widetilde{\mathfrak{e}}^{(O_2)}_{\mathfrak{t}}(f_{(12)}(\sigma)) && \text{from eq. (187)} \\
&= \widetilde{\mathfrak{e}}^{(O_2)}_{\mathfrak{t}}(\textstyle\bigsqcup_{\sigma \in \mathfrak{C}} f_{(12)}(\sigma)) && \text{from eq. (31).}
\end{aligned} \tag{194}$$

Then, using Lemma 1, we conclude that $\widetilde{f}_{(12)}$ is Scott continuous (and in particular order preserving)

$$\forall \mathfrak{C} \subseteq_{Chain} \mathfrak{S}^{(O_1)},\ \widetilde{f}_{(12)}(\textstyle\bigsqcup_{\mathfrak{S}^{(O_1)}} \mathfrak{C}) \ = \ \textstyle\bigsqcup_{\sigma \in \mathfrak{C}} \widetilde{f}_{(12)}(\sigma). \tag{195}$$

We can prove property (196) below, using properties (51) and (187), along the same line of proof :

$$\forall \mathfrak{Q} \subseteq \mathfrak{S}^{(O_1)},\ \widetilde{f}_{(12)}(\textstyle\bigsqcap_{\mathfrak{S}^{(O_1)}} \mathfrak{Q}) \ = \ \textstyle\bigsqcap_{\sigma \in \mathfrak{Q}} \widetilde{f}_{(12)}(\sigma). \tag{196}$$

□

**Theorem 18. [Preservation of the class of minimally disturbing measurements by symmetry]** Let $(\widetilde{f}_{(12)}, f^{(21)})$ be a dictionary relating by symmetry $(\mathfrak{S}^{(O_1)}, \mathfrak{T}^{(O_1)}, \widetilde{\mathfrak{e}}^{(O_1)})$ to $(\mathfrak{S}^{(O_2)}, \mathfrak{T}^{(O_2)}, \widetilde{\mathfrak{e}}^{(O_2)})$, and let $\mathfrak{t} \in \mathfrak{T}^{(O_2)}_{IFKM}$ be a yes/no test leading to an ideal first-kind measurement, therefore $f^{(21)}(\mathfrak{t})$ is a yes/no test that leads to an ideal first-kind measurement, i.e., $f^{(21)}(\mathfrak{t}) \in \mathfrak{T}^{(O_1)}_{IFKM}$. In other words,

$$\forall (\widetilde{f}_{(12)}, f^{(21)}) \in \mathbf{Sym}\left[(\mathfrak{S}^{(O_1)}, \mathfrak{T}^{(O_1)}, \widetilde{\mathfrak{e}}^{(O_1)}) \to (\mathfrak{S}^{(O_2)}, \mathfrak{T}^{(O_2)}, \widetilde{\mathfrak{e}}^{(O_2)})\right], \quad \mathfrak{t} \in \mathfrak{T}^{(O_2)}_{IFKM} \quad \Rightarrow \quad f^{(21)}(\mathfrak{t}) \in \mathfrak{T}^{(O_1)}_{IFKM}. \tag{197}$$

$$\text{i.e.} \qquad (\widetilde{f}_{(12)}, f^{(21)}) \in \mathbf{Sym}\left[(\mathfrak{S}^{(O_1)}, \mathfrak{T}^{(O_1)}_{min}, \widetilde{\mathfrak{e}}^{(O_1)}) \to (\mathfrak{S}^{(O_2)}, \mathfrak{T}^{(O_2)}_{min}, \widetilde{\mathfrak{e}}^{(O_2)})\right]. \tag{198}$$

■

*Proof.* Let us first prove that the symmetry preserves the Lawson-continuity property of measurement maps. Let us prove it is true for the Scott continuity:

$$\begin{aligned}
\forall \mathfrak{C} \subseteq_{Chain} \mathfrak{Q}^{(O_1)}_{f^{(21)}(\mathfrak{t})},\ \widetilde{f}_{(12)}((\bigsqcup\nolimits_{\mathfrak{S}^{(O_1)}} \mathfrak{C}).f^{(21)}(\mathfrak{t})) &= \widetilde{f}_{(12)}(\bigsqcup\nolimits_{\mathfrak{S}^{(O_1)}} \mathfrak{C}).\mathfrak{t} && \text{from eq. (188)} \\
&= (\bigsqcup\nolimits_{\sigma\in\mathfrak{C}} \widetilde{f}_{(12)}(\sigma)).\mathfrak{t} && \text{from eq. (195)} \\
&= \bigsqcup\nolimits_{\sigma\in\mathfrak{C}} (\widetilde{f}_{(12)}(\sigma).\mathfrak{t}) && \text{from eq. (72)} \\
&= \bigsqcup\nolimits_{\sigma\in\mathfrak{C}} \widetilde{f}_{(12)}(\sigma.f^{(21)}(\mathfrak{t})) && \text{from eq. (188)} \\
&= \widetilde{f}_{(12)}(\bigsqcup\nolimits_{\sigma\in\mathfrak{C}} (\sigma.f^{(21)}(\mathfrak{t}))) && \text{from eq. (195)}
\end{aligned} \tag{199}$$

We now use the injectivity property (190) to confirm the preservation of Scott continuity of measurement maps by symmetries:

$$\forall \mathfrak{C} \subseteq_{Chain} \mathfrak{Q}^{(O_1)}_{f^{(21)}(\mathfrak{t})},\ (\bigsqcup\nolimits_{\mathfrak{S}^{(O_1)}} \mathfrak{C}).f^{(21)}(\mathfrak{t}) = \bigsqcup\nolimits_{\sigma\in\mathfrak{C}} (\sigma.f^{(21)}(\mathfrak{t})) \tag{200}$$

The second continuity property is proved along the same line of proof, using properties (188), (196), (71) and (190)

$$\forall \mathfrak{Q} \subseteq \mathfrak{Q}^{(O_1)}_{f^{(21)}(\mathfrak{t})},\ (\bigsqcap\nolimits_{\mathfrak{S}^{(O_1)}} \mathfrak{Q}).f^{(21)}(\mathfrak{t}) = \bigsqcap\nolimits_{\sigma\in\mathfrak{Q}} (\sigma.f^{(21)}(\mathfrak{t})). \tag{201}$$

Secondly, let us prove that $f^{(21)}(\mathfrak{t}) \in \mathfrak{T}^{(O_1)}_{FKM}$. Let us consider any $\sigma \in \mathfrak{S}^{(O_1)}$, and $\mathfrak{t} \in \mathfrak{T}^{(O_2)}_{FKM}$.
The preservation of equation (75 (i)) is proved as follows

$$\begin{aligned}
\widetilde{\mathfrak{e}}^{(O_1)}_{f^{(21)}(\mathfrak{t})}(\sigma) \leq \mathrm{Y} \quad &\Rightarrow \quad \widetilde{\mathfrak{e}}^{(O_1)}_{\mathfrak{t}}(\widetilde{f}_{(12)}(\sigma)) \leq \mathrm{Y} && \text{from eq. (187)} \\
&\Rightarrow \quad \widetilde{\mathfrak{e}}^{(O_1)}_{\mathfrak{t}}(\widetilde{f}_{(12)}(\sigma).\mathfrak{t}) = \mathrm{Y} && \text{from } \mathfrak{t} \in \mathfrak{T}^{(O_2)}_{FKM} \\
&\Rightarrow \quad \widetilde{\mathfrak{e}}^{(O_1)}_{\mathfrak{t}}(\widetilde{f}_{(12)}(\sigma.f^{(21)}(\mathfrak{t}))) = \mathrm{Y} && \text{from eq. (188)} \\
&\Rightarrow \quad \widetilde{\mathfrak{e}}^{(O_1)}_{f^{(21)}(\mathfrak{t})}(\sigma.f^{(21)}(\mathfrak{t})) = \mathrm{Y} && \text{from eq. (187)}
\end{aligned} \tag{202}$$

The preservation of equation (75 (ii)) is proved as follows

$$\begin{aligned}
\widetilde{\mathfrak{e}}^{(O_1)}_{f^{(21)}(\mathfrak{t})}(\sigma) = \mathrm{Y} \quad &\Rightarrow \quad \widetilde{\mathfrak{e}}^{(O_1)}_{\mathfrak{t}}(\widetilde{f}_{(12)}(\sigma)) = \mathrm{Y} && \text{from eq. (187)} \\
&\Rightarrow \quad \widetilde{f}_{(12)}(\sigma).\mathfrak{t} = \widetilde{f}_{(12)}(\sigma) && \text{from } \mathfrak{t} \in \mathfrak{T}^{(O_2)}_{FKM} \\
&\Rightarrow \quad \widetilde{f}_{(12)}(\sigma.f^{(21)}(\mathfrak{t})) = \widetilde{f}_{(12)}(\sigma) && \text{from eq. (188)} \\
&\Rightarrow \quad \sigma.f^{(21)}(\mathfrak{t}) = \sigma && \text{from eq. (190)}
\end{aligned} \tag{203}$$

Thirdly, it remains to be shown that $f^{(21)}(\mathfrak{t}) \in \mathfrak{T}^{(O_1)}_{Ideal}$. Let us consider $\mathfrak{u} \in \mathfrak{T}^{(O_1)}$ such that $\overbrace{\lfloor f^{(21)}(\mathfrak{t})\rfloor \lfloor \mathfrak{u} \rfloor}$. The surjectivity of $f^{(21)}$ (equation (191)) implies that there exists $\mathfrak{t}' \in \mathfrak{T}^{(O_2)}$ such that $\mathfrak{u} = f^{(21)}(\mathfrak{t}')$.

The compatibility relation $\overbrace{\lfloor f^{(21)}(\mathfrak{t})\rfloor\lfloor f^{(21)}(\mathfrak{t}')\rfloor}$ implies the compatibility relation $\overbrace{\lfloor \mathfrak{t}\rfloor\lfloor \mathfrak{t}'\rfloor}$. Indeed

$$\begin{aligned}
\overbrace{\lfloor f^{(21)}(\mathfrak{t})\rfloor\lfloor f^{(21)}(\mathfrak{t}')\rfloor} \quad &\Leftrightarrow \quad \exists\sigma\in\mathfrak{S} \mid \widetilde{\mathfrak{e}}^{(O_1)}_{f^{(21)}(\mathfrak{t})}(\sigma)=\widetilde{\mathfrak{e}}^{(O_1)}_{f^{(21)}(\mathfrak{t}')}(\sigma)=\mathrm{Y} \\
&\Rightarrow \quad \exists\sigma'=\widetilde{f}_{(12)}(\sigma)\in\mathfrak{S} \mid \widetilde{\mathfrak{e}}^{(O_1)}_{\mathfrak{t}}(\sigma')=\widetilde{\mathfrak{e}}^{(O_1)}_{\mathfrak{t}'}(\sigma')=\mathrm{Y} \qquad \text{from eq. (187)} \\
&\Rightarrow \quad \overbrace{\lfloor \mathfrak{t}\rfloor\lfloor \mathfrak{t}'\rfloor}. \qquad (204)
\end{aligned}$$

Let us now consider any $\sigma\in\mathfrak{S}$ and $\mathfrak{t}'\in\mathfrak{T}^{(O_2)}$ such that $\overbrace{\lfloor f^{(21)}(\mathfrak{t})\rfloor\lfloor f^{(21)}(\mathfrak{t}')\rfloor}$, we then have

$$\begin{aligned}
(\widetilde{\mathfrak{e}}^{(O_1)}_{f^{(21)}(\mathfrak{t})}(\sigma)\leq\mathrm{Y} \ \text{ and } \ \widetilde{\mathfrak{e}}^{(O_1)}_{f^{(21)}(\mathfrak{t}')}(\sigma)=\mathrm{Y}) \quad &\Rightarrow \quad (\widetilde{\mathfrak{e}}^{(O_1)}_{\mathfrak{t}}(\widetilde{f}_{(12)}(\sigma))\leq\mathrm{Y} \ \text{ and } \ \widetilde{\mathfrak{e}}^{(O_1)}_{\mathfrak{t}'}(\widetilde{f}_{(12)}(\sigma))=\mathrm{Y}) \qquad \text{from eq. (187)} \\
&\Rightarrow \quad (\widetilde{\mathfrak{e}}^{(O_1)}_{\mathfrak{t}'}(\widetilde{f}_{(12)}(\sigma)\boldsymbol{.}\mathfrak{t})=\mathrm{Y}) \qquad \text{from } \mathfrak{t}\in\mathfrak{T}^{(O_2)}_{Ideal} \text{ and } \widehat{\mathfrak{t}\mathfrak{t}'} \\
&\Rightarrow \quad (\widetilde{\mathfrak{e}}^{(O_1)}_{\mathfrak{t}'}(\widetilde{f}_{(12)}(\sigma\boldsymbol{.}f^{(21)}(\mathfrak{t})))=\mathrm{Y}) \qquad \text{from eq. (188)} \\
&\Rightarrow \quad (\widetilde{\mathfrak{e}}^{(O_1)}_{f^{(21)}(\mathfrak{t}')}(\sigma\boldsymbol{.}f^{(21)}(\mathfrak{t}))=\mathrm{Y}) \qquad \text{from eq. (187)} \qquad (205)
\end{aligned}$$

□

**Lemma 46.** $\widetilde{f}_{(12)}$ is surjective. ■

*Proof.* Let us introduce the following map on $\mathfrak{S}$:

$$\begin{aligned}
f^{\downarrow}_{(21)} \quad : \quad \mathfrak{S}^{(O_2)} \quad &\longrightarrow \quad \mathfrak{S}^{(O_1)} \\
\Sigma \quad &\mapsto \quad f^{\downarrow}_{(21)}(\Sigma):=\bigsqcap\nolimits_{\mathfrak{S}^{(O_1)}}\{\sigma \mid \widetilde{\mathfrak{e}}^{(O_1)}_{f^{(21)}(\mathfrak{t}_{(\Sigma,\Sigma')})}(\sigma)=\mathbf{Y}\}, \qquad (206)
\end{aligned}$$

where $\mathfrak{t}_{(\Sigma,\Sigma')}$ designates the unique discriminating yes/no test with $\Sigma_{\mathfrak{t}_{(\Sigma,\Sigma')}}=\Sigma$.
We have

$$\begin{aligned}
\forall\Sigma\in\mathfrak{S}^{(O_2)},\Sigma'\in\mathfrak{S}^{(O_1)}, \quad f^{\downarrow}_{(21)}(\Sigma)\sqsubseteq_{\mathfrak{S}^{(O_1)}}\Sigma' \quad &\Leftrightarrow \quad \widetilde{\mathfrak{e}}^{(O_1)}_{f^{(21)}(\mathfrak{t}_{(\Sigma,\Sigma')})}(\Sigma')=\mathbf{Y} \\
&\Leftrightarrow \quad \widetilde{\mathfrak{e}}^{(O_1)}_{\mathfrak{t}_{(\Sigma,\Sigma')}}(\widetilde{f}_{(12)}(\Sigma'))=\mathbf{Y} \\
&\Leftrightarrow \quad \Sigma\sqsubseteq_{\mathfrak{S}^{(O_2)}}\widetilde{f}_{(12)}(\Sigma') \qquad (207)
\end{aligned}$$

$\widetilde{f}_{(12)}$ is then the right Galois adjunct of $f^{\downarrow}_{(21)}$. Then, $\widetilde{f}_{(12)}$ is surjective and preserves infima (this last property was already proved as a part of Lemma 45). □

**Theorem 19.**

$$\forall\mathfrak{t}_{(\Sigma,\Sigma')}\in\widetilde{\mathfrak{T}}^{(O_2)}_{disc}, \qquad f^{(21)}(\mathfrak{t}_{(\Sigma,\Sigma')})\in\widetilde{\mathfrak{T}}^{(O_1)}_{disc} \qquad (208)$$

Moreover, we have

$$f^{(21)}(\mathfrak{t}_{(\Sigma,\Sigma')})=\mathfrak{t}_{\left(\widetilde{f}^{-1}_{(12)}(\Sigma),\widetilde{f}^{-1}_{(12)}(\Sigma')\right)} \qquad (209)$$

■

*Proof.* Let us introduce $\kappa, \kappa' \in \mathfrak{S}^{(O_1)}$ as follows

$$\kappa := \bigsqcap_{\mathfrak{S}} \{ \sigma \mid \widetilde{\mathfrak{e}}^{(O_1)}_{f^{(21)}(\mathfrak{t}_{(\Sigma,\Sigma')})}(\sigma) = \mathbf{Y} \} = \bigsqcap_{\mathfrak{S}} \{ \sigma \mid \widetilde{\mathfrak{e}}^{(O_2)}_{\mathfrak{t}_{(\Sigma,\Sigma')}}(\widetilde{f}_{(12)}(\sigma)) = \mathbf{Y} \} \tag{210}$$

$$\kappa' := \bigsqcap_{\mathfrak{S}} \{ \sigma \mid \widetilde{\mathfrak{e}}^{(O_1)}_{f^{(21)}(\mathfrak{t}_{(\Sigma,\Sigma')})}(\sigma) = \mathbf{N} \} = \bigsqcap_{\mathfrak{S}} \{ \sigma \mid \widetilde{\mathfrak{e}}^{(O_2)}_{\mathfrak{t}_{(\Sigma,\Sigma')}}(\widetilde{f}_{(12)}(\sigma)) = \mathbf{N} \} \tag{211}$$

Using the surjectivity of $\widetilde{f}_{(12)}$ (Lemma 46) and the fact that $\widetilde{f}_{(12)}$ preserves infima, we obtain

$$\widetilde{f}_{(12)}(\kappa) = \Sigma \quad \text{and} \quad \widetilde{f}_{(12)}(\kappa') = \Sigma'. \tag{212}$$

$\mathfrak{t}_{(\Sigma,\Sigma')} \in \widetilde{\mathfrak{T}}^{(O_2)}_{disc}$ means that $(\Sigma \bowtie_{\mathfrak{S}} \Sigma')$ and $\widetilde{\Sigma\Sigma'}^{\mathfrak{S}}$. Using the bijectivity of $\widetilde{f}_{(12)}$ (Lemma 46 and property (190)) and its order-preserving property (Lemma 45), with the explicit definition of the relations $\cdot \bowtie_{\mathfrak{S}} \cdot$ and $\widetilde{\cdot\cdot}^{\mathfrak{S}}$, we deduce that $\kappa \bowtie_{\mathfrak{S}} \kappa'$ and $\widetilde{\kappa\kappa'}^{\mathfrak{S}}$. As a result, $f^{(21)}(\mathfrak{t}_{(\Sigma,\Sigma')}) \in \widetilde{\mathfrak{T}}^{(O_1)}_{disc}$. Moreover,

$$\kappa = \widetilde{f}^{-1}_{(12)}(\Sigma) \quad \text{and} \quad \kappa' = \widetilde{f}^{-1}_{(12)}(\Sigma'). \tag{213}$$

□

**Lemma 47.** For any $(\widetilde{f}_{(12)}, f^{(21)}) \in \mathbf{Sym}\left[(\mathfrak{S}^{(O_1)}, \mathfrak{T}^{(O_1)}, \widetilde{\mathfrak{e}}^{(O_1)}) \to (\mathfrak{S}^{(O_2)}, \mathfrak{T}^{(O_2)}, \widetilde{\mathfrak{e}}^{(O_2)})\right]$ and $\mathfrak{t} \in \mathfrak{T}^{(O_2)}$

$$f^{(21)}(\overline{\mathfrak{t}}) = \overline{f^{(21)}(\mathfrak{t})}. \tag{214}$$

■

*Proof.*

$$\begin{aligned}
\forall \sigma \in \mathfrak{S}^{(O_1)},\ \widetilde{\mathfrak{e}}^{(O_1)}_{f^{(21)}(\overline{\mathfrak{t}})}(\sigma) &= \widetilde{\mathfrak{e}}^{(O_2)}_{\overline{\mathfrak{t}}}(\widetilde{f}_{(12)}(\sigma)) && \text{from eq. (187)} \\
&= \overline{\widetilde{\mathfrak{e}}^{(O_2)}_{\mathfrak{t}}(\widetilde{f}_{(12)}(\sigma))} && \text{from eq. (4)} \\
&= \overline{\widetilde{\mathfrak{e}}^{(O_1)}_{f^{(21)}(\mathfrak{t})}(\sigma)} && \text{from eq. (187)} \\
&= \widetilde{\mathfrak{e}}^{(O_1)}_{\overline{f^{(21)}(\mathfrak{t})}}(\sigma) && \text{from eq. (4)}
\end{aligned} \tag{215}$$

□

**Theorem 20. [Preservation of the orthogonality of states]**

$$\forall (\widetilde{f}_{(12)}, f^{(21)}) \in \mathbf{Sym}\left[(\mathfrak{S}^{(O_1)}, \mathfrak{T}^{(O_1)}, \widetilde{\mathfrak{e}}^{(O_1)}) \to (\mathfrak{S}^{(O_2)}, \mathfrak{T}^{(O_2)}, \widetilde{\mathfrak{e}}^{(O_2)})\right], \forall \sigma_1, \sigma_2 \in \mathfrak{S}^{(O_1)}, \quad \sigma_1 \perp \sigma_2 \Rightarrow \widetilde{f}_{(12)}(\sigma_1) \perp \widetilde{f}_{(12)}(\sigma_2). \tag{216}$$

As an immediate consequence, we have $\forall \sigma \in \mathfrak{S}^{(O_1)}, \quad \widetilde{f}_{(12)}(\sigma^\star) = \widetilde{f}_{(12)}(\sigma)^\star$. ■

*Proof.*

$$\begin{aligned}
\sigma_1 \perp \sigma_2 \quad &\Leftrightarrow \quad \exists \sigma \in \mathfrak{S}^{(O_1)}, \exists \mathfrak{t} \in \mathfrak{T}^{(O_1)}_{disc} \mid (\sigma_1 \sqsupseteq_{\mathfrak{S}} \sigma . \mathfrak{t} \quad \text{and} \quad \sigma_2 \sqsupseteq_{\mathfrak{S}} \sigma . \overline{\mathfrak{t}}) \\
&\Leftrightarrow \quad \exists \sigma \in \mathfrak{S}^{(O_1)}, \exists \mathfrak{u} \in \mathfrak{T}^{(O_2)}_{disc} \mid (\sigma_1 \sqsupseteq_{\mathfrak{S}} \sigma . f^{(21)}(\mathfrak{u}) \quad \text{and} \quad \sigma_2 \sqsupseteq_{\mathfrak{S}} \sigma . \overline{f^{(21)}(\mathfrak{u})}) && \text{from eq. (191) and theorem 19} \\
&\Rightarrow \quad \exists \sigma \in \mathfrak{S}^{(O_1)}, \exists \mathfrak{u} \in \mathfrak{T}^{(O_2)}_{disc} \mid (\widetilde{f}_{(12)}(\sigma_1) \sqsupseteq_{\mathfrak{S}} \widetilde{f}_{(12)}(\sigma . f^{(21)}(\mathfrak{u})) \text{ and } \widetilde{f}_{(12)}(\sigma_2) \sqsupseteq_{\mathfrak{S}} \widetilde{f}_{(12)}(\sigma . \overline{f^{(21)}(\mathfrak{u})})) && \text{from Lemma 45} \\
&\Rightarrow \quad \exists \sigma \in \mathfrak{S}^{(O_1)}, \exists \mathfrak{u} \in \mathfrak{T}^{(O_2)}_{disc} \mid (\widetilde{f}_{(12)}(\sigma_1) \sqsupseteq_{\mathfrak{S}} \widetilde{f}_{(12)}(\sigma . f^{(21)}(\mathfrak{u})) \text{ and } \widetilde{f}_{(12)}(\sigma_2) \sqsupseteq_{\mathfrak{S}} \widetilde{f}_{(12)}(\sigma . f^{(21)}(\overline{\mathfrak{u}}))) && \text{from Lemma 47} \\
&\Rightarrow \quad \exists \sigma' = \widetilde{f}_{(12)}(\sigma) \in \mathfrak{S}^{(O_1)}, \exists \mathfrak{u} \in \mathfrak{T}^{(O_2)}_{disc} \mid (\widetilde{f}_{(12)}(\sigma_1) \sqsupseteq_{\mathfrak{S}} \sigma' . \mathfrak{u} \quad \text{and} \quad \widetilde{f}_{(12)}(\sigma_2) \sqsupseteq_{\mathfrak{S}} \sigma' . \overline{\mathfrak{u}}) && \text{from eq. (188)} \\
&\Leftrightarrow \quad \widetilde{f}_{(12)}(\sigma_1) \perp \widetilde{f}_{(12)}(\sigma_2).
\end{aligned} \tag{217}$$

□

**Lemma 48.**

$$\begin{array}{rcccl} \varphi_L & : & \mathscr{C}(\mathfrak{S}_{pure}) & \longrightarrow & \mathfrak{S} \\ & & \mathfrak{c} & \mapsto & \bigsqcap_{\mathfrak{S}} \mathfrak{c} \end{array} \qquad \begin{array}{rcccl} \varphi_D & : & \mathfrak{S} & \longrightarrow & \mathscr{C}(\mathfrak{S}_{pure}) \\ & & \sigma & \mapsto & \underline{\{\sigma\}} \end{array} \tag{218}$$

satisfy the following Galois adjunction relation

$$\forall \mathfrak{c} \in \mathscr{C}(\mathfrak{S}_{pure}), \forall \sigma \in \mathfrak{S}, \qquad (\varphi_L(\mathfrak{c}) \sqsubseteq_{\mathfrak{S}} \sigma) \Leftrightarrow (\mathfrak{c} \subseteq \varphi_D(\sigma)), \tag{219}$$

and

$$\varphi^L \circ \varphi^D = id_{\mathfrak{S}}. \tag{220}$$

■

**Notion 55.** Using the elements of a symmetry $(\widetilde{f}_{(12)}, f^{(21)}) \in \mathbf{Sym}\left[(\mathfrak{S}^{(O_1)}, \mathfrak{T}^{(O_1)}, \widetilde{\mathfrak{e}}^{(O_1)}) \to (\mathfrak{S}^{(O_2)}, \mathfrak{T}^{(O_2)}, \widetilde{\mathfrak{e}}^{(O_2)})\right]$, we define the following maps

$$\begin{array}{rcccl} \underline{\widetilde{f}_{(12)}} & : & \mathscr{C}(\mathfrak{S}^{(O_1)}_{pure}) & \longrightarrow & \mathscr{C}(\mathfrak{S}^{(O_2)}_{pure}) \\ & & \mathfrak{c} & \mapsto & \bigvee\{\widetilde{f}_{(12)}(\sigma) \mid \sigma \in \mathfrak{c}\} \end{array} \qquad \begin{array}{rcccl} \underline{f^{(21)}} & : & \mathscr{C}(\mathfrak{S}^{(O_2)}_{pure}) & \longrightarrow & \mathscr{C}(\mathfrak{S}^{(O_1)}_{pure}) \\ & & \mathfrak{c} & \mapsto & (\varphi_D^{(O_1)} \circ f^{\downarrow}_{(21)} \circ \varphi_L^{(O_2)})(\mathfrak{c}) \end{array} \tag{221}$$

Note that $f^{\downarrow}_{(21)}$ has been defined in equation (206).

**Lemma 49.** $(\underline{\widetilde{f}_{(12)}}, \underline{f^{(21)}})$ forms a Galois connection. ■

*Proof.* For any $\mathfrak{c}_1 \in \mathscr{C}(\mathfrak{S}^{(O_1)}_{pure})$ and $\mathfrak{c}_2 \in \mathscr{C}(\mathfrak{S}^{(O_2)}_{pure})$, we have

$$\begin{aligned}
(\underline{\widetilde{f}_{(12)}}(\mathfrak{c}_1) \subseteq \mathfrak{c}_2) \quad &\Leftrightarrow \quad (\forall \sigma \in \mathfrak{c}_1,\ \{\widetilde{f}_{(12)}(\sigma)\} \subseteq \mathfrak{c}_2) \\
&\Leftrightarrow \quad (\forall \sigma \in \mathfrak{c}_1,\ \varphi_L^{(O_2)}(\mathfrak{c}_2) \sqsubseteq_{\mathfrak{S}^{(O_2)}} \widetilde{f}_{(12)}(\sigma)) \\
&\Leftrightarrow \quad (\forall \sigma \in \mathfrak{c}_1,\ \widetilde{\mathfrak{e}}^{(O_2)}(\widetilde{f}_{(12)}(\sigma), \mathfrak{t}_{(\varphi_L^{(O_2)}(\mathfrak{c}_2), (\varphi_L^{(O_2)}(\mathfrak{c}_2))^{\star})}) = \mathbf{Y}) \\
&\Leftrightarrow \quad (\forall \sigma \in \mathfrak{c}_1,\ \widetilde{\mathfrak{e}}^{(O_1)}(\sigma, f^{(21)}(\mathfrak{t}_{(\varphi_L^{(O_2)}(\mathfrak{c}_2), (\varphi_L^{(O_2)}(\mathfrak{c}_2))^{\star})})) = \mathbf{Y}) \\
&\Leftrightarrow \quad (\forall \sigma \in \mathfrak{c}_1,\ \{\sigma\} \subseteq \underline{\left\{\bigsqcap_{\mathfrak{S}^{(O_1)}} \{\sigma' \in \mathfrak{S}^{(O_1)}_{pure} \mid \widetilde{\mathfrak{e}}^{(O_1)}(\sigma, f^{(21)}(\mathfrak{t}_{(\varphi_L^{(O_2)}(\mathfrak{c}_2), (\varphi_L^{(O_2)}(\mathfrak{c}_2))^{\star})})) = \mathbf{Y}\}\right\}}) \\
&\Leftrightarrow \quad (\forall \sigma \in \mathfrak{c}_1,\ \{\sigma\} \subseteq (\varphi_D^{(O_1)} \circ f^{\downarrow}_{(21)} \circ \varphi_L^{(O_2)})(\mathfrak{c}_2)) \\
&\Leftrightarrow \quad (\mathfrak{c}_1 \subseteq \underline{f^{(21)}}(\mathfrak{c}_2))
\end{aligned} \tag{222}$$

□

**Theorem 21.** $\underline{\widetilde{f}_{(12)}}$ is an injective map from $\mathscr{C}(\mathfrak{S}^{(O_1)}_{pure})$ to $\mathscr{C}(\mathfrak{S}^{(O_2)}_{pure})$ preserving suprema, mapping atoms to atoms and preserving orthogonality (i.e. $\forall \mathfrak{c} \in \mathscr{C}(\mathfrak{S}^{(O_1)}_{pure})$, $\underline{\widetilde{f}_{(12)}}(\mathfrak{c}^{\perp}) \subseteq (\underline{\widetilde{f}_{(12)}}(\mathfrak{c}))^{\perp}$).
It is then an ortho-morphism of Hilbert lattice (see [73, Definition 5.12 and Definition 5.17]). ■

*Proof.* $\underline{\widetilde{f}_{(12)}}$ preserves atoms as an extension of $\widetilde{f}_{(12)}$ from $\mathfrak{S}^{(O_1)}_{pure}$ to $\mathscr{C}(\mathfrak{S}^{(O_1)}_{pure})$. It is injective and preserves suprema as a left Galois adjunct. It preserves the orthogonality relation according to Theorem 20. □

# 5 Conclusion

We aim to develop a new axiomatic approach to quantum theory and this article is designed as a first decisive step for this axiomatic program. A precise semantic description of the space of preparations and of the associated 'mixed states' of the system was formulated. This semantic formalism is based on a Chu space construction involving the set of preparations, the set of yes/no tests and an evaluation map with a three-valued target space. The values taken by this map are associated with counterfactual statements of the observer for a given yes/no test and a given prepared sample. The three values are interpreted in a possibilistic perspective, i.e., as 'certainly yes', 'certainly no' and 'maybe'. This domain structure on the target space led to an 'informational' interpretation of the set of preparations. The space of preparations was equipped with a notion of 'mixtures', expressed in terms of the meet operation on this poset. From natural requirements about the inductive definition of states, it appeared that this Inf semi-lattice was also a pointed directed-complete partial order. Then, an 'Information Principle' was introduced in the form of two topological requirements on the space of states. Although new in its form, this principle is very standard in different quantum axiomatic programs. This basic set of axioms was shown to be sufficient to constrain the structure of the space of states to be a 'locally-boolean qualitative domain'. The space of pure states was then basically identified in terms of maximal elements of this domain.Then, the relation between yes/no tests and states was studied from two perspectives: using the notion of Chu duality and using the notion of 'measurement'. Adopting the first perspective, the notion of 'properties' of the system was defined and a 'property-state' was identified for any property, as in Piron's construction. The second perspective emphasizes the recursive aspect of preparation processes. In order to identify a subclass of measurement operations corresponding to minimally disturbing measurements, it appeared necessary to clarify the notion of 'compatibility between measurements'. The compatibility between two measurements was defined in terms of the existence of preparations that simultaneously exhibited the two corresponding properties as actual. This notion was used to define 'ideal first-kind measurements' and to characterize them as 'minimally disturbing measurements'. We finally proved the existence of a 'bi-extensional $Chu_3$ duality' between the space of minimally disturbing yes/no tests and the space of states. The conditions of existence of a general class of minimally disturbing yes/no tests, called 'discriminating yes/no tests', were clarified in terms of a condition on the space of states (the 'quasi-coherence condition') that achieves the axiomatic presentation of its domain structure. This domain structure generalizes the notion of 'concrete domains'. This domain structure on the space of states is the first main achievement of this paper. The simultaneous ideal first-kind measurements of compatible properties was then studied and 'Specker's principle' was proved. Using this result, we obtained a 'coherent domain' structure on the space of 'descriptions' formalizing the families of compatible properties used to define a state of the system. An orthogonality relation was then defined on the space of states in terms of the class of discriminating yes/no tests. Using the properties of the domain structure established on the space of states, we deduced that the set of ortho-closed subsets of pure states, equipped with the induced orthogonality relation, inherits a structure of Hilbert lattice. This is the first part of our reconstruction theorem. In Section 4, we explored the properties of Chu morphisms with respect to previous notions. As a central result, a sub-algebra of the algebra of Chu morphisms, corresponding to 'symmetries' of the system, was defined. These symmetries appear to preserve the class of minimally disturbing yes/no tests and the orthogonality of states. The link between these Chu symmetries and the morphisms of Hilbert lattice, defined on the space of ortho-closed subsets of pure states, is finally emphasized. This is the second part of our reconstruction theorem.

Despite its self-contained character and the fact that it achieves to produce a reconstruction theorem from very basic premises for quantum theory, this paper requires more developments to be plainly satisfactory. First of all, it appears necessary to study the ways to recover a probabilistic formalism from a possibilistic one. Secondly, it appears necessary to pursue the construction to the case of compound systems by studying the tensor products structures on the domains appearing in the present paper. Endly, it seems unavoidable to study extensively the category of 'quantum domains' extracted from our set of axioms. These problems will be attacked in our forthcoming papers.

# 6 Appendix : Basic properties of selection structures

**Definition 4. [Main set of axioms for selection structures]** The poset $(\mathfrak{S}, \sqsubseteq_{\mathfrak{S}})$ will be said to be *a selection structure* iff it satisfies the following axioms

$$\exists \perp_{\mathfrak{S}} \in \mathfrak{S},\ \forall x \in \mathfrak{S},\ \perp_{\mathfrak{S}} \sqsubseteq_{\mathfrak{S}} x, \tag{223}$$

$$\forall x, y \in \mathfrak{S},\ (x \sqsubset_{\mathfrak{S}} y) \Rightarrow (\exists t \in \mathfrak{S},\ x \overline{\sqsubset}_{\mathfrak{S}} t \sqsubseteq_{\mathfrak{S}} y), \tag{224}$$

$$\forall x, y, z \in \mathfrak{S},\ (z \overline{\sqsubset}_{\mathfrak{S}} x \text{ and } z \sqsubset_{\mathfrak{S}} y \text{ and } x \parallel_{\mathfrak{S}} y) \Rightarrow (z = x \sqcap_{\mathfrak{S}} y), \tag{225}$$

$$\forall x, u, v \in \mathfrak{S}, (x \sqsubset_{\mathfrak{S}} u \overline{\sqsubset}_{\mathfrak{S}} v) \Rightarrow (\exists!\, y \in \mathfrak{S}, \text{ such that } (x \overline{\sqsubset}_{\mathfrak{S}} y \sqsubset_{\mathfrak{S}} v \text{ and } y \parallel_{\mathfrak{S}} u)), \tag{226}$$

$$\forall C \subseteq_{chain} \mathfrak{S},\ (\bigsqcup\nolimits_{\mathfrak{S}} C) \text{ exists in } \mathfrak{S} \tag{227}$$

$$\forall C \subseteq_{chain} \mathfrak{S}, \forall x, y \in \mathfrak{S} \,|\, (x \overline{\sqsubset}_{\mathfrak{S}} y,\ x \sqsubset_{\mathfrak{S}} C),\ (y \parallel_{\mathfrak{S}} C) \Rightarrow (y \parallel_{\mathfrak{S}} (\bigsqcup\nolimits_{\mathfrak{S}} C)). \tag{228}$$

The *set of selection structures* is denoted $\mathscr{S}$.

*Remark* 20. Note that, due to property (225), property (228) can be equivalently rewritten as:

$$\forall x, y \in \mathfrak{S} \,|\, x \overline{\sqsubset}_{\mathfrak{S}} y, \forall C \subseteq_{chain} \mathfrak{S},\ (\forall c \in C, x = (y \sqcap_{\mathfrak{S}} c)) \Rightarrow (x = (y \sqcap_{\mathfrak{S}} (\bigsqcup\nolimits_{\mathfrak{S}} C))). \tag{229}$$

*Remark* 21. Note that the property (226) may be equivalently rewritten as:

$$\forall x, u, v \in \mathfrak{S}, (x \sqsubseteq_{\mathfrak{S}} u \overline{\sqsubset}_{\mathfrak{S}} v) \Rightarrow (\exists!\, y \in \mathfrak{S}, \text{ such that } (x \overline{\sqsubset}_{\mathfrak{S}} y \sqsubseteq_{\mathfrak{S}} v \text{ and } y \not\sqsubseteq_{\mathfrak{S}} u)). \tag{230}$$

**Lemma 50. [Selection structures are directed-complete] [53]** As soon as the selection structure $\mathfrak{S}$ is chain-complete, it is necessarily directed-complete. ■

The way-below relation for the dcpo $(\mathfrak{S}, \sqsubseteq_{\mathfrak{S}})$ will be denoted $\ll_{\mathfrak{S}}$.

**Lemma 51. [Basic joins]** Let $(\mathfrak{S}, \sqsubseteq_{\mathfrak{S}})$ be a selection structure. For any $x, y \in \mathfrak{S}$, we have

$$(\exists z \in \mathfrak{S} \,|\, (y \overline{\sqsubset}_{\mathfrak{S}} z \text{ and } x \sqsubset_{\mathfrak{S}} z \text{ and } x \parallel_{\mathfrak{S}} y)) \quad \Rightarrow \quad ((x \sqcup_{\mathfrak{S}} y) \text{ exists in } \mathfrak{S} \text{ and } x \sqcup_{\mathfrak{S}} y = z). \tag{231}$$

■

*Proof.* First of all, the hypothesis $y \overline{\sqsubset}_{\mathfrak{S}} z$, $x \sqsubseteq_{\mathfrak{S}} z$ means that $z$ is a common upper bound of $x$ and $y$. Let us then consider another common upper-bound of $x$ and $y$ denoted $z'$ and let us suppose $z \not\sqsubseteq_{\mathfrak{S}} z'$.
The case $z' \sqsubset_{\mathfrak{S}} z$ is immediatly eliminated because the property $y \sqsubseteq_{\mathfrak{S}} z' \sqsubset_{\mathfrak{S}} z$, combined with $y \overline{\sqsubset}_{\mathfrak{S}} z$, implies $z' = y$, i.e. $x \sqsubseteq_{\mathfrak{S}} y$ which is contradictory with the hypothesis $x \parallel_{\mathfrak{S}} y$.
If we suppose now $z' \parallel_{\mathfrak{S}} z$, we deduce from property (225) and the basic facts $y \overline{\sqsubset}_{\mathfrak{S}} z$ and $y \sqsubseteq_{\mathfrak{S}} z'$ that $y = z \sqcap_{\mathfrak{S}} z'$. However, we have also $x \sqsubseteq_{\mathfrak{S}} z$ and $x \sqsubseteq_{\mathfrak{S}} z'$, which would then imply $x \sqsubseteq_{\mathfrak{S}} y$ and then would contradict $x \parallel_{\mathfrak{S}} y$. Hence, necessarily $z \sqsubset_{\mathfrak{S}} z'$ and $z$ is the supremum $x \sqcup_{\mathfrak{S}} y$. □

**Lemma 52. [Natural expression of polarization]** The polarization axiom (i.e. property (226)) is equivalent to the following axiom:

$$\forall b, u, v \in \mathfrak{S}, \text{ with } b \sqsubset_{\mathfrak{S}} u \overline{\sqsubset}_{\mathfrak{S}} v,\ \exists!\, r \in \mathfrak{S}, \text{ such that } (u \sqcap_{\mathfrak{S}} r = b \text{ and } u \sqcup_{\mathfrak{S}} r = v). \tag{232}$$

■

*Proof.* Let us first suppose that $\mathfrak{S}$ verifies equation (226). Note that, if $b \sqsubset_{\mathfrak{S}} u \overline{\sqsubset}_{\mathfrak{S}} v$, $b \overline{\sqsubset}_{\mathfrak{S}} r \sqsubset_{\mathfrak{S}} v$ and $r \parallel_{\mathfrak{S}} u$, then, using Lemma 51 and property (225), we deduce $u \sqcap_{\mathfrak{S}} r = b$ and $r \sqcup_{\mathfrak{S}} u = v$. As a result, we have then proved property (232).

Conversely, if we suppose that property (232) is verified, let us denote $r$ the (unique) element such that $(u \sqcap_{\mathfrak{S}} r = b \text{ and } r \sqcup_{\mathfrak{S}} u = v)$. We can remark that $u \sqcap_{\mathfrak{S}} r = b$ and $b \sqsubset_{\mathfrak{S}} u$ implies necessarily $r \parallel_{\mathfrak{S}} u$.
It remains to show that $b \overline{\sqsubset}_{\mathfrak{S}} r \sqsubset_{\mathfrak{S}} v$. First of all, $u \sqcap_{\mathfrak{S}} r = b$ implies $b \sqsubseteq_{\mathfrak{S}} r$. However, we cannot have $b = r$ because it would imply $r \sqsubseteq_{\mathfrak{S}} u$, and then $r \sqcup_{\mathfrak{S}} u = u$ which is contradicted by the property $u \overline{\sqsubset}_{\mathfrak{S}} v$. Hence, we have $b \sqsubset_{\mathfrak{S}} r$. Let us now suppose that $b \not\overline{\sqsubset}_{\mathfrak{S}} r$. Then, $\exists w \in \mathfrak{S}$ such that $b \overline{\sqsubset}_{\mathfrak{S}} w \sqsubset_{\mathfrak{S}} r$. Necessarily, using property (225), we deduce $b = (u \sqcap_{\mathfrak{S}} w)$. From Lemma 51, we observe that, as soon as $w \sqsubset_{\mathfrak{S}} r \sqsubset_{\mathfrak{S}} v$ and $u \overline{\sqsubset}_{\mathfrak{S}} v$, then $w \sqcup_{\mathfrak{S}} u$ exists and is equal to $v$. This contradicts the uniqueness of $r$ in the property (232). As a conclusion, we have necessarily $b \overline{\sqsubset}_{\mathfrak{S}} r$. Moreover, it exists a unique $r$ satisfying $b \overline{\sqsubset}_{\mathfrak{S}} r \sqsubset_{\mathfrak{S}} v$ and $u \parallel_{\mathfrak{S}} r$, which concludes the proof of property (226). □

**Lemma 53. [Basic meets]** Let $(\mathfrak{S}, \sqsubseteq_{\mathfrak{S}})$ be a selection structure. For any $y, u, v \in \mathfrak{S}$, we have

$$(y \sqsubset_{\mathfrak{S}} v,\ u \overline{\sqsubset}_{\mathfrak{S}} v,\ y \parallel_{\mathfrak{S}} u) \quad \Rightarrow \quad ((y \sqcap_{\mathfrak{S}} u) \text{ exists in } \mathfrak{S} \text{ and } (y \sqcap_{\mathfrak{S}} u) \overline{\sqsubset}_{\mathfrak{S}} y). \tag{233}$$

■

*Proof.* Let us consider $y, u, v \in \mathfrak{S}$, such that $(y \sqsubset_{\mathfrak{S}} v,\ u \overline{\sqsubset}_{\mathfrak{S}} v,\ y \parallel_{\mathfrak{S}} u)$. We do not know yet if $(y \sqcap_{\mathfrak{S}} u)$ exists. However, we can consider the set $m_{y,u} := \{s \in \mathfrak{S} \mid s \sqsubseteq_{\mathfrak{S}} y \text{ and } s \sqsubseteq_{\mathfrak{S}} u\}$. $\bot_{\mathfrak{S}} \in m_{y,u}$ and then $m_{y,u}$ is non-empty. Moreover, every chain of elements of $m_{y,u}$ admits a supremum in $m_{y,u}$, and then, from Zorn's Lemma, $m_{y,u}$ admits maximal elements. Let us consider $m \in Max(m_{y,u})$. We have $m \sqsubset_{\mathfrak{S}} u \overline{\sqsubset}_{\mathfrak{S}} v$, and then, using property (226), we deduce that it exists a unique element $r$ such that $m \overline{\sqsubset}_{\mathfrak{S}} r \sqsubset_{\mathfrak{S}} v$ and $r \parallel_{\mathfrak{S}} u$. As soon as it exists an element $y'$ such that $m \overline{\sqsubset}_{\mathfrak{S}} y' \sqsubseteq_{\mathfrak{S}} y$, due to property (224), the unicity of $r$ implies that $r = y'$. In other words, we have necessarily $r \sqsubseteq_{\mathfrak{S}} y$. Let us suppose that $r \sqsubset_{\mathfrak{S}} y$. From property (224), it then exist $r'$ such that $r \overline{\sqsubset}_{\mathfrak{S}} r' \sqsubseteq_{\mathfrak{S}} y$, however, using property (226), we would deduce the existence of an element $r''$ satisfying $m \overline{\sqsubset}_{\mathfrak{S}} r'' \sqsubset_{\mathfrak{S}} r' \sqsubseteq_{\mathfrak{S}} y$ and $r'' \parallel_{\mathfrak{S}} r$. Moreover, the unicity of $r$ as solution of the relations $m \overline{\sqsubset}_{\mathfrak{S}} r \sqsubset_{\mathfrak{S}} v$ and $r \parallel_{\mathfrak{S}} u$, imposes necessarily $r'' \sqsubseteq_{\mathfrak{S}} u$ (the option $u \sqsubset_{\mathfrak{S}} r''$ would contradict the property $r'' \sqsubset_{\mathfrak{S}} v$). But then $r''$ would be a common lower-bound of $u$ and $y$ and this point contradicts the maximality of $m$. As a result, we have necessarily $r = y$, and then $m \overline{\sqsubset}_{\mathfrak{S}} y$. Now, we use the property (225) to conclude that $m$ is in fact the infimum $(y \sqcap_{\mathfrak{S}} u)$. □

**Lemma 54. [Lower-covering property]** The following property satisfied for any $x, u, v \in \mathfrak{S}$ will be called *lower-covering property*:

$$\begin{aligned}(u \overline{\sqsubset}_{\mathfrak{S}} v \text{ and } (x \sqcap_{\mathfrak{S}} v) \text{ exists in } \mathfrak{S}) \quad \Rightarrow \quad & ((x \sqcap_{\mathfrak{S}} u) \text{ exists in } \mathfrak{S} \\ & \text{and } ((x \sqcap_{\mathfrak{S}} u) \overline{\sqsubset}_{\mathfrak{S}} (x \sqcap_{\mathfrak{S}} v) \text{ or } (x \sqcap_{\mathfrak{S}} u) = (x \sqcap_{\mathfrak{S}} v))).\end{aligned} \tag{234}$$

■

*Proof.* Let us consider $x, u, v \in \mathfrak{S}$ with $u \overline{\sqsubset}_{\mathfrak{S}} v$ and such that $(x \sqcap_{\mathfrak{S}} v)$ exists in $\mathfrak{S}$.
We will distinguish three different cases : $u \sqsubseteq_{\mathfrak{S}} x$, $x \sqsubset_{\mathfrak{S}} u$ and $x \parallel_{\mathfrak{S}} u$.

- If $u \sqsubseteq_{\mathfrak{S}} x$, then $(x \sqcap_{\mathfrak{S}} u)$ obviously exists in $\mathfrak{S}$ and $(x \sqcap_{\mathfrak{S}} u) = u$. We distinguish two sub-cases: $v \sqsubseteq_{\mathfrak{S}} x$ or $v \parallel_{\mathfrak{S}} x$.
  - If $v \sqsubseteq_{\mathfrak{S}} x$, then $(v \sqcap_{\mathfrak{S}} x) = v$. We have $(x \sqcap_{\mathfrak{S}} u) = u \overline{\sqsubset}_{\mathfrak{S}} v = (x \sqcap_{\mathfrak{S}} v)$.
  - If $v \parallel_{\mathfrak{S}} x$, using $u \sqsubseteq_{\mathfrak{S}} x$, $u \overline{\sqsubset}_{\mathfrak{S}} v$, and property (225), we deduce that $(x \sqcap_{\mathfrak{S}} v) = u$, i.e. $(x \sqcap_{\mathfrak{S}} v) = (x \sqcap_{\mathfrak{S}} u)$.
- If $x \sqsubset_{\mathfrak{S}} u \overline{\sqsubset}_{\mathfrak{S}} v$, $(x \sqcap_{\mathfrak{S}} u)$ obviously exists in $\mathfrak{S}$ and $(x \sqcap_{\mathfrak{S}} u) = x$. We have then $x = (x \sqcap_{\mathfrak{S}} u) = (x \sqcap_{\mathfrak{S}} v)$.
- If $x \parallel_{\mathfrak{S}} u$, we will distinguish two sub-cases $x \sqsubset_{\mathfrak{S}} v$ or $x \parallel_{\mathfrak{S}} v$ (indeed, $u \overline{\sqsubset}_{\mathfrak{S}} v \sqsubseteq_{\mathfrak{S}} x$ is incompatible with $u \parallel_{\mathfrak{S}} x$).

- If $x \sqsubset_{\mathfrak{S}} v$, we have $x \sqcap_{\mathfrak{S}} v = x$. To summarize, we have $(x \sqcap_{\mathfrak{S}} u) \sqsubset_{\mathfrak{S}} x \sqsubset_{\mathfrak{S}} v$, $x \parallel_{\mathfrak{S}} u$, $(x \sqcap_{\mathfrak{S}} u) \sqsubset_{\mathfrak{S}} u \,\overline{\sqsubset}_{\mathfrak{S}} v$. We are then in the configuration of property (233). As a result, $(x \sqcap_{\mathfrak{S}} u)$ exists in $\mathfrak{S}$ and $(x \sqcap_{\mathfrak{S}} u) \,\overline{\sqsubset}_{\mathfrak{S}} x = (x \sqcap_{\mathfrak{S}} v)$.
- If $x \parallel_{\mathfrak{S}} v$, we will distinguish two sub-sub-cases: $(x \sqcap_{\mathfrak{S}} v) \sqsubset_{\mathfrak{S}} u$ and $(x \sqcap_{\mathfrak{S}} v) \parallel_{\mathfrak{S}} u$. Indeed, the option $u \sqsubseteq_{\mathfrak{S}} (x \sqcap_{\mathfrak{S}} v)$ is excluded, because this hypothesis would imply $u \sqsubseteq_{\mathfrak{S}} x$ which is false.
  - if $(x \sqcap_{\mathfrak{S}} v) \sqsubset_{\mathfrak{S}} u$, we note that $\forall z \in \mathfrak{S}$, $(z \sqsubseteq_{\mathfrak{S}} u) \Rightarrow (z \sqsubseteq_{\mathfrak{S}} v)$ and then $(z \sqsubseteq_{\mathfrak{S}} \{x,u\}) \Rightarrow (z \sqsubseteq_{\mathfrak{S}} (x \sqcap_{\mathfrak{S}} v))$. But precisely $(x \sqcap_{\mathfrak{S}} v) \sqsubseteq_{\mathfrak{S}} \{x,u\})$. As a result, $(x \sqcap_{\mathfrak{S}} u)$ exists in $\mathfrak{S}$ and $(x \sqcap_{\mathfrak{S}} u) = (x \sqcap_{\mathfrak{S}} v)$.
  - if $(x \sqcap_{\mathfrak{S}} v) \parallel_{\mathfrak{S}} u$. We have then $u \,\overline{\sqsubset}_{\mathfrak{S}} v$, $(x \sqcap_{\mathfrak{S}} v) \sqsubset_{\mathfrak{S}} v$ and $(x \sqcap_{\mathfrak{S}} v) \parallel_{\mathfrak{S}} u$, which is the exact configuration of property (233), if we denote $y := (x \sqcap_{\mathfrak{S}} v)$. Hence, $(y \sqcap_{\mathfrak{S}} u)$ exists in $\mathfrak{S}$ and $(y \sqcap_{\mathfrak{S}} u) \,\overline{\sqsubset}_{\mathfrak{S}} y$, i.e. $(x \sqcap_{\mathfrak{S}} u)$ exists in $\mathfrak{S}$ and $(x \sqcap_{\mathfrak{S}} u) \,\overline{\sqsubset}_{\mathfrak{S}} (x \sqcap_{\mathfrak{S}} v)$.

This concludes the complete analysis of the different cases for the relation (234) and then the proof of the announced result. □

**Lemma 55. [Dual formulation of polarization axiom]** A selection structure $\mathfrak{S}$ satisfies necessarily the following dual formulation of polarization axiom:

$$\begin{aligned}
&\forall r,b,t \in \mathfrak{S}, \text{ with } b \,\overline{\sqsubset}_{\mathfrak{S}} r \sqsubset_{\mathfrak{S}} t,\\
&u := \bigsqcup\nolimits_{\mathfrak{S}} \{ s \in \mathfrak{S} \mid b \sqsubseteq_{\mathfrak{S}} s \sqsubseteq_{\mathfrak{S}} t \text{ and } s \parallel_{\mathfrak{S}} r \} \text{ exists in } \mathfrak{S}\\
&\text{Moreover, we have } (r \sqcap_{\mathfrak{S}} u = b \text{ and } b \sqsubset_{\mathfrak{S}} u \,\overline{\sqsubset}_{\mathfrak{S}} (u \sqcup_{\mathfrak{S}} r) = t),
\end{aligned} \tag{235}$$

■

*Proof.* Let us then consider $r,b,t \in \mathfrak{S}$, with $b \,\overline{\sqsubset}_{\mathfrak{S}} r \sqsubset_{\mathfrak{S}} t$. We have to prove that the subset

$$Q^r_{(b,t)} := \{ s \in \mathfrak{S} \mid b \sqsubseteq_{\mathfrak{S}} s \sqsubseteq_{\mathfrak{S}} t \text{ and } s \parallel_{\mathfrak{S}} r \}.$$

admits a supremum in $\mathfrak{S}$ (obviously, this supremum is unique). To begin, we will study the existence and explore the properties of the maximum elements of this set.
We note, first of all, that $Q^r_{(b,t)}$ is non-empty. Indeed, we know from property (224) that it exists an element $r'$ such that $r \,\overline{\sqsubset}_{\mathfrak{S}} r' \sqsubseteq_{\mathfrak{S}} t$, and then from property (226) that it exists an element $r''$ such that $r \,\overline{\sqsubset}_{\mathfrak{S}} r'' \sqsubset_{\mathfrak{S}} r' \sqsubseteq_{\mathfrak{S}} t$ and $r' \parallel_{\mathfrak{S}} r''$, i.e. also $r'' \sqcap_{\mathfrak{S}} r = b$ using (225).
For any chain $C \sqsubseteq_{chain} Q^r_{(b,t)}$, we can consider its supremum $(\bigsqcup_{\mathfrak{S}} C)$ which exists in $\mathfrak{S}$ due to chain-completeness property of $\mathfrak{S}$ (equation 227). As soon as $\forall c \in C$, $b \sqsubseteq_{\mathfrak{S}} c \sqsubseteq_{\mathfrak{S}} t$, we obtain immediatly $b \sqsubseteq_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} C) \sqsubseteq_{\mathfrak{S}} t$, by definition of the supremum. Moreover, using the weak-continuity of $\mathfrak{S}$ (equation (229)), we know that $\forall c \in C$, $c \parallel_{\mathfrak{S}} r = b$ implies $r \parallel_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} C)$. As a result, $(\bigsqcup_{\mathfrak{S}} C) \in Q^r_{(b,t)}$. Then, from Zorn's lemma, $Q^r_{(b,t)}$ admits some maximal elements.

$$Max(Q^r_{(b,t)}) \neq \varnothing.$$

We note that, for any $s \in Max(Q^r_{(b,t)})$, we have $s \sqsubseteq_{\mathfrak{S}} t$ but the equality $s = t$ is excluded because $t \sqcap_{\mathfrak{S}} r = r \neq b$, i.e. $t \notin Q^r_{(b,t)}$. As a result, we obtain:

$$Max(Q^r_{(b,t)}) \cap Max(\mathfrak{S}) = \varnothing.$$

Then, due to strong-atomicity of $\mathfrak{S}$ (equation (224)), we have $\forall u \in Max(Q^r_{(b,t)}), \exists v \in \mathfrak{S}, u \,\overline{\sqsubset}_{\mathfrak{S}} v \sqsubseteq_{\mathfrak{S}} t$. We cannot have $r \parallel_{\mathfrak{S}} v$, because it would contradict the maximality property of $u$. We cannot have $v \sqsubseteq_{\mathfrak{S}} r$ either, because it would contradict $u \parallel_{\mathfrak{S}} r$ and $u \,\overline{\sqsubset}_{\mathfrak{S}} v$. We have then established:

$$\forall u \in Max(Q^r_{(b,t)}), \qquad \forall v \in \mathfrak{S}, \ (u \,\overline{\sqsubset}_{\mathfrak{S}} v \sqsubseteq_{\mathfrak{S}} t) \Rightarrow (r \sqsubset_{\mathfrak{S}} v).$$

More precisely, using Lemma 51, we have in fact

$$\forall u \in Max(Q^r_{(b,t)}), \qquad \forall v \in \mathfrak{S},\ (u \overline{\sqsubset}_{\mathfrak{S}} v \sqsubseteq_{\mathfrak{S}} t) \Rightarrow (v = u \sqcup_{\mathfrak{S}} r).$$

Let us now show that $v = t$. Let us then suppose that $v$ is different from $t$ and let us exhibit a contradiction. If $u \overline{\sqsubset}_{\mathfrak{S}} v \sqsubset_{\mathfrak{S}} t$, then it exists $v'$ such that $u \overline{\sqsubset}_{\mathfrak{S}} v \sqsubset_{\mathfrak{S}} t$ and $v \parallel_{\mathfrak{S}} v'$ as a direct consequence of property (226). But then we should have $u = v \sqcap_{\mathfrak{S}} v'$ (using property (225)), which would lead to the following contradiction: $b = (u \sqcap_{\mathfrak{S}} r) = (v \sqcap_{\mathfrak{S}} v') \sqcap_{\mathfrak{S}} r = (v \sqcap_{\mathfrak{S}} (v' \sqcap_{\mathfrak{S}} r)) = (v \sqcap_{\mathfrak{S}} r) = r$. We have then obtained $u \overline{\sqsubset}_{\mathfrak{S}} v = t$, $u \parallel_{\mathfrak{S}} r$ and $r \sqsubset_{\mathfrak{S}} t$, and then $v = t = u \sqcup_{\mathfrak{S}} r$ using Lemma 51. As a summary, we have

$$\forall u \in Max(Q^r_{(b,t)}), \qquad \forall v \in \mathfrak{S},\ (u \overline{\sqsubset}_{\mathfrak{S}} v \sqsubseteq_{\mathfrak{S}} t) \Rightarrow (v = t = u \sqcup_{\mathfrak{S}} r).$$

Let us now consider $u_1$ and $u_2$, two elements of $Max(Q^r_{(b,t)})$. We intent to prove that necessarily $u_1 = u_2$. We will suppose $u_1 \neq u_2$ and try to exhibit a contradiction. From $u_1, u_2 \overline{\sqsubset}_{\mathfrak{S}} t$, we deduce $u_1 \parallel_{\mathfrak{S}} u_2$. Moreover, from previous results, we have $b \sqsubset_{\mathfrak{S}} u_k \overline{\sqsubset}_{\mathfrak{S}} t$, $u_k \parallel_{\mathfrak{S}} r$ for $k = 1,2$. From property (53), we deduce that $(u_1 \sqcap_{\mathfrak{S}} u_2)$ exists in $\mathfrak{S}$ and $b \sqsubset_{\mathfrak{S}} (u_1 \sqcap_{\mathfrak{S}} u_2) \overline{\sqsubset}_{\mathfrak{S}} u_1, u_2$. We now use property (226) to deduce that it exists a unique $w$ such that $b \overline{\sqsubset}_{\mathfrak{S}} w \sqsubset_{\mathfrak{S}} u_1 \sqsubset_{\mathfrak{S}} t$ and $(u_1 \sqcap_{\mathfrak{S}} u_2) \parallel_{\mathfrak{S}} w$. We have then $b \overline{\sqsubset}_{\mathfrak{S}} w \sqsubset_{\mathfrak{S}} u_1 \sqsubset_{\mathfrak{S}} t$ and $w \parallel_{\mathfrak{S}} u_2$. The unicity requirement should impose $w = r$ which is contradictory because $w \sqsubset_{\mathfrak{S}} u_1$ is incompatible with $r \parallel_{\mathfrak{S}} u_1$. As a result, $u_1 = u_2$. This establishes the unicity of $u$ in equation (235). □

**Lemma 56. [$D - D^*$ properties of Selection structures]** A selection structure $\mathfrak{S}$ satisfies the following properties :

$$(D^*) \quad \forall s_1, s_2, t \in \mathfrak{S}, \quad ((s_1 \sqcap_{\mathfrak{S}} s_2) \overline{\sqsubset}_{\mathfrak{S}} s_1 \sqsubset_{\mathfrak{S}} t \text{ and } (s_1 \sqcap_{\mathfrak{S}} s_2) \overline{\sqsubset}_{\mathfrak{S}} s_2 \sqsubset_{\mathfrak{S}} t) \Rightarrow$$
$$((s_1 \sqcup_{\mathfrak{S}} s_2) \text{ exists in } \mathfrak{S} \quad \text{and} \quad [s_1 \sqcap_{\mathfrak{S}} s_2, s_1 \sqcup_{\mathfrak{S}} s_2] \cong B_2), \tag{236}$$

$$(D) \quad \forall s_1, s_2, t \in \mathfrak{S}, \quad ((s_1 \sqcup_{\mathfrak{S}} s_2) \text{ exists}, \quad s_1 \overline{\sqsubset}_{\mathfrak{S}} (s_1 \sqcup_{\mathfrak{S}} s_2) \text{ and } s_2 \overline{\sqsubset}_{\mathfrak{S}} (s_1 \sqcup_{\mathfrak{S}} s_2)) \Rightarrow$$
$$([s_1 \sqcap_{\mathfrak{S}} s_2, s_1 \sqcup_{\mathfrak{S}} s_2] \cong B_2) \tag{237}$$

In the equations (236) and (237), we have denoted by $B_2$ the lattice of only four elements

$$B_2 := (\{a \sqcap_{\mathfrak{S}} b, a, b, a \sqcup_{\mathfrak{S}} b\}, \sqcap, \sqcup) \text{ with } a \parallel_{\mathfrak{S}} b.$$

■

*Remark* 22. The proof of the equivalence of these $D - D^*$ properties and the distributivity property, in the special case where $\mathfrak{S}$ is a lattice, is established in the two papers [47][48].

*Remark* 23. We note that the domain $\mathfrak{S}$ satisfies the 'property $R_T$' [44, p.52] and equivalently satisfies the 'property $R$' defining Concrete domains (we use the equivalence [44, Theorem 5.1 p.52]).

*Proof.* Let us consider $s_1, s_2, t \in \mathfrak{S}$ satisfying $s_1 \parallel_{\mathfrak{S}} s_2$, $(s_1 \sqcap_{\mathfrak{S}} s_2) \overline{\sqsubset}_{\mathfrak{S}} s_1 \sqsubset_{\mathfrak{S}} t$ and $(s_1 \sqcap_{\mathfrak{S}} s_2) \overline{\sqsubset}_{\mathfrak{S}} s_2 \sqsubset_{\mathfrak{S}} t$. Using Lemma 55 and the ordering relation $(s_1 \sqcap_{\mathfrak{S}} s_2) \overline{\sqsubset}_{\mathfrak{S}} s_2 \sqsubset_{\mathfrak{S}} t$, we deduce that :

$$\exists! u \in \mathfrak{S}, u \overline{\sqsubset}_{\mathfrak{S}} t \quad | \quad \forall s \in \mathfrak{S}, ((s_1 \sqcap_{\mathfrak{S}} s_2) \sqsubset_{\mathfrak{S}} s \sqsubset_{\mathfrak{S}} t) \text{ and } (s \parallel_{\mathfrak{S}} s_2)) \Leftrightarrow (s \sqsubseteq_{\mathfrak{S}} u). \tag{238}$$

Moreover, the unique element $u$, defined according to the property (238), satisfies $t = (u \sqcup_{\mathfrak{S}} s_2)$. Endly, using the defining property of $u$ (i.e. property (238)) and the properties satisfied by $s_1$ (i.e. $(s_1 \sqcap_{\mathfrak{S}} s_2) \sqsubseteq_{\mathfrak{S}} s_1 \sqsubseteq_{\mathfrak{S}} t$) and $(s_1 \parallel_{\mathfrak{S}} s_2)$), we conclude that $s_1 \sqsubseteq_{\mathfrak{S}} u$.
We can distinguish different cases, relative to $u$.
If $s_1 = u$, we have then $u = s_1 \overline{\sqsubset}_{\mathfrak{S}} t$, $s_2 \sqsubset_{\mathfrak{S}} t$, $s_1 \parallel_{\mathfrak{S}} s_2$ and then, using Lemma 53, $t = (s_1 \sqcup_{\mathfrak{S}} s_2)$. As a result, we have then established that $(s_1 \sqcup_{\mathfrak{S}} s_2)$ exists in $\mathfrak{S}$.
If $s_1 \sqsubset_{\mathfrak{S}} u \overline{\sqsubset}_{\mathfrak{S}} t$, we can apply property (226) to deduce that it exists a unique $r \in \mathfrak{S}$, such that $(s_1 \overline{\sqsubset}_{\mathfrak{S}} r \sqsubset_{\mathfrak{S}} t$ and $r \parallel_{\mathfrak{S}} u)$. Let us show that necessarily $s_2 \sqsubset_{\mathfrak{S}} r$. We firstly note that the option $r \sqsubseteq_{\mathfrak{S}} s_2$ is excluded

since $(s_1 \sqcap_{\mathfrak{S}} s_2) \overline{\sqsubset}_{\mathfrak{S}} s_2$ and $(s_1 \sqcap_{\mathfrak{S}} s_2) \overline{\sqsubset}_{\mathfrak{S}} s_1 \overline{\sqsubset}_{\mathfrak{S}} r$. Secondly, the option $r \parallel_{\mathfrak{S}} s_2$ is also excluded, because $(s_1 \sqcap_{\mathfrak{S}} s_2) \overline{\sqsubset}_{\mathfrak{S}} s_1 \overline{\sqsubset}_{\mathfrak{S}} r \sqsubset_{\mathfrak{S}} t$ and $r \parallel_{\mathfrak{S}} s_2$ would imply $(r \sqsubseteq_{\mathfrak{S}} u)$, because of property (238), and this fact would then contradict the hypothesis $r \parallel_{\mathfrak{S}} u$. As a result, we have then $s_2 \sqsubset_{\mathfrak{S}} r$. To summarize, we have $s_1 \overline{\sqsubset}_{\mathfrak{S}} r$, $s_2 \sqsubset_{\mathfrak{S}} r$ and $s_1 \parallel_{\mathfrak{S}} s_2$, and then, using Lemma 53, we deduce that $(s_1 \sqcup_{\mathfrak{S}} s_2)$ exists in $\mathfrak{S}$ and $r = (s_1 \sqcup_{\mathfrak{S}} s_2)$.
As a conclusion, in any configurations, the hypotheses $s_1 \parallel_{\mathfrak{S}} s_2$, $(s_1 \sqcap_{\mathfrak{S}} s_2) \overline{\sqsubset}_{\mathfrak{S}} s_1 \sqsubset_{\mathfrak{S}} t$ and $(s_1 \sqcap_{\mathfrak{S}} s_2) \overline{\sqsubset}_{\mathfrak{S}} s_2 \sqsubset_{\mathfrak{S}} t$ imply that $(s_1 \sqcup_{\mathfrak{S}} s_2)$ exists in $\mathfrak{S}$. Obviously, using Lemma 55, we have also explicitely $s_1 \overline{\sqsubset}_{\mathfrak{S}} (s_1 \sqcup_{\mathfrak{S}} s_2)$ and $s_2 \overline{\sqsubset}_{\mathfrak{S}} (s_1 \sqcup_{\mathfrak{S}} s_2)$.
If we now suppose that it exists $z \in \mathfrak{S}$ with $(s_1 \sqcap_{\mathfrak{S}} s_2) \sqsubset_{\mathfrak{S}} z \sqsubset_{\mathfrak{S}} (s_1 \sqcup_{\mathfrak{S}} s_2)$, and $z \neq s_1, z \neq s_2$. Then necessarily, $z \parallel_{\mathfrak{S}} s_1$ because of the covering property $(s_1 \sqcap_{\mathfrak{S}} s_2) \overline{\sqsubset}_{\mathfrak{S}} s_1 \overline{\sqsubset}_{\mathfrak{S}} (s_1 \sqcup_{\mathfrak{S}} s_2)$. However, from property (226), we know that it exists a unique $r \in \mathfrak{S}$, such that $((s_1 \sqcap_{\mathfrak{S}} s_2) \overline{\sqsubset}_{\mathfrak{S}} r \sqsubset_{\mathfrak{S}} (s_1 \sqcup_{\mathfrak{S}} s_2)$ and $r \parallel_{\mathfrak{S}} s_1)$. Hence, the property $(s_1 \sqcap_{\mathfrak{S}} s_2) \overline{\sqsubset}_{\mathfrak{S}} z$ is impossible because it would impose $z = s_2$. And if $(s_1 \sqcap_{\mathfrak{S}} s_2) \not\overline{\sqsubset}_{\mathfrak{S}} z$ then strong atomicity of $\mathfrak{S}$ (i.e. property (224)) imposes the existence of some $z' \in \mathfrak{S}$ with $(s_1 \sqcap_{\mathfrak{S}} s_2) \overline{\sqsubset}_{\mathfrak{S}} z' \sqsubset_{\mathfrak{S}} z \sqsubset_{\mathfrak{S}} (s_1 \sqcup_{\mathfrak{S}} s_2)$. The same unicity argument can be applied to $z'$ which must be equal to $s_2$. Then the covering property $s_2 \sqsubset_{\mathfrak{S}} z \sqsubset_{\mathfrak{S}} (s_1 \sqcup_{\mathfrak{S}} s_2)$ appears to be contradictory with $s_2 \overline{\sqsubset}_{\mathfrak{S}} (s_1 \sqcup_{\mathfrak{S}} s_2)$. As a result, $[s_1 \sqcap_{\mathfrak{S}} s_2, s_1 \sqcup_{\mathfrak{S}} s_2] \cong B_2$. This concludes the proof of property (236).

Equation (237) can also be shown to be satisfied by selection structures. Indeed, let us consider $s_1, s_2 \in \mathfrak{S}$ with $s_1 \overline{\sqsubset}_{\mathfrak{S}} (s_1 \sqcup_{\mathfrak{S}} s_2)$ and $s_2 \overline{\sqsubset}_{\mathfrak{S}} (s_1 \sqcup_{\mathfrak{S}} s_2)$. From Lemma 53, we deduce that $(s_1 \sqcap_{\mathfrak{S}} s_2)$ exists. Moreover, $(s_1 \sqcap_{\mathfrak{S}} s_2) \overline{\sqsubset}_{\mathfrak{S}} s_1$ and $(s_1 \sqcap_{\mathfrak{S}} s_2) \overline{\sqsubset}_{\mathfrak{S}} s_2$. The property $[s_1 \sqcap_{\mathfrak{S}} s_2, s_1 \sqcup_{\mathfrak{S}} s_2] \cong B_2$ is then deduced from the previous analysis. □

**Definition 5. [Atoms and finite elements of a selection structure]** If $\mathfrak{S}$ is a selection structure, $\mathfrak{S}$ is *atomic*, i.e.

$$\forall s \in \mathfrak{S},\ \exists a \in \mathscr{A}^*_{\mathfrak{S}},\ a \sqsubseteq_{\mathfrak{S}} s, \qquad \mathscr{A}^*_{\mathfrak{S}} := \{a \in \mathfrak{S} \mid \bot_{\mathfrak{S}} \overline{\sqsubset}_{\mathfrak{S}} a\}. \tag{239}$$

The set of atoms will be denoted $\mathscr{A}^*_{\mathfrak{S}}$ and we will denote naturally $\mathscr{A}_{\mathfrak{S}} := \mathscr{A}^*_{\mathfrak{S}} \cup \{\bot_{\mathfrak{S}}\}$.
The sub- poset $(\mathscr{A}_{\mathfrak{S}}, \sqsubseteq_{\mathfrak{S}})$ equipped with the induced partial order is a flat selection structure, i.e.$\forall x, y \in \mathscr{A}_{\mathfrak{S}},\ (x \sqsubseteq_{\mathfrak{S}} y) \Leftrightarrow (x = \bot_{\mathfrak{S}}$ or $x = y)$.

For any $s \in \mathfrak{S}$, we will denote

$$\mathbb{A}^*(s) := \mathscr{A}^*_{\mathfrak{S}} \cap \downarrow s \qquad \mathbb{A}(s) := \mathbb{A}^*(s) \cup \{\bot_{\mathfrak{S}}\}. \tag{240}$$

Elements $s$ of $\mathfrak{S}$ such that $\mathbb{A}^*(s) \subseteq_{fin} \mathscr{A}_{\mathfrak{S}}$ will be called *finite elements of* $\mathfrak{S}$.

**Lemma 57. [Pre-complementation]** A selection structure $\mathfrak{S}$ satisfies necessarily the following *pre-complementation property* :

$$\forall s_1, s_2 \in \mathfrak{S}, \quad (s_1 \sqsubset_{\mathfrak{S}} s_2) \ \Rightarrow\ (\exists a \in \mathfrak{S},\ \bot_{\mathfrak{S}} \overline{\sqsubset}_{\mathfrak{S}} a \sqsubset_{\mathfrak{S}} s_2 \text{ and } a \sqcap_{\mathfrak{S}} s_1 = \bot_{\mathfrak{S}}), \tag{241}$$

$$\Rightarrow\ (\exists a \in \mathfrak{S},\ \bot_{\mathfrak{S}} \overline{\sqsubset}_{\mathfrak{S}} a \sqsubset_{\mathfrak{S}} s_2 \text{ and } a \not\sqsubseteq_{\mathfrak{S}} s_1). \tag{242}$$

■

*Proof.* $\mathfrak{S}$ being strongly atomic (property (224)), $\exists s \in \mathfrak{S}, s_1 \overline{\sqsubset}_{\mathfrak{S}} s \sqsubseteq_{\mathfrak{S}} s_2$. Then, from property (226), it exists a unique element $a$ such that $\bot_{\mathfrak{S}} \overline{\sqsubset}_{\mathfrak{S}} a \sqsubset_{\mathfrak{S}} s \sqsubseteq_{\mathfrak{S}} s_2$ and $s_1 \parallel_{\mathfrak{S}} a$. From Lemma 53, we have also $s_1 \sqcap_{\mathfrak{S}} a = \bot_{\mathfrak{S}}$. □

**Lemma 58. [Atoms are compact elements]** Atoms of a selection structure $\mathfrak{S}$ are compact element of this dcpo. ■

*Proof.* Let us consider an atom $a \in \mathscr{A}^*_{\mathfrak{S}}$. For any chain $C \subseteq_{chain} \mathfrak{S}$ such that $a \sqsubseteq_{\mathfrak{S}} \bigsqcup_{\mathfrak{S}} C$, the property $\forall c \in C, a \not\sqsubseteq_{\mathfrak{S}} c$ (i.e. $a \sqcap_{\mathfrak{S}} c = \bot_{\mathfrak{S}}$) is then contradictory, because of property (229). Then, it exists necessarily $c \in C$ such that $a \sqsubseteq_{\mathfrak{S}} c$. Hence, $a$ is compact. □

**Lemma 59. [Consistent joins of finite elements]** For any finite upper-bounded family of atoms $F \subseteq_{Con} \mathscr{A}^*_{\mathfrak{S}}$, $(\bigsqcup_{\mathfrak{S}} F)$ exists in $\mathfrak{S}$ and $\mathbb{A}(\bigsqcup_{\mathfrak{S}} F) = F$. In other words,

$$\forall F \subseteq_{Con} \mathscr{A}^*_{\mathfrak{S}}, \qquad (\bigsqcup\nolimits_{\mathfrak{S}} F) \text{ exists in } \mathfrak{S} \quad \text{and} \quad \mathbb{A}(\bigsqcup\nolimits_{\mathfrak{S}} F) = F. \tag{243}$$

Let $f$ be a finite element of $\mathfrak{S}$ (i.e. $\mathbb{A}(f) \subseteq_{Con} \mathscr{A}^*_{\mathfrak{S}}$). Then, $(\bigsqcup_{\mathfrak{S}} \mathbb{A}(f))$ exists in $\mathfrak{S}$ and $f = (\bigsqcup_{\mathfrak{S}} \mathbb{A}(f))$.
Let $w_1, w_2$ be finite elements of $\mathfrak{S}$ (i.e. $\mathbb{A}(w_1), \mathbb{A}(w_2) \subseteq_{Con} \mathscr{A}^*_{\mathfrak{S}}$). If $\widehat{w_1 w_2}^{\mathfrak{S}}$, then $(w_1 \sqcup_{\mathfrak{S}} w_2)$ exists in $\mathfrak{S}$, $\mathbb{A}(w_1 \sqcup_{\mathfrak{S}} w_2) = \mathbb{A}(w_1) \cup \mathbb{A}(w_2)$ and $(w_1 \sqcup_{\mathfrak{S}} w_2) = (\bigsqcup_{\mathfrak{S}} (\mathbb{A}(w_1) \cup \mathbb{A}(w_2)))$. ■

*Proof.* The property (243) will be proved by induction on $Card(F)$.
Let $F$ be any consistent family of atoms of cardinal $n$, explicitely given by $F = \{ a_i \mid i \in \{1, \cdots, n\} \}$ and let us consider that we have established the property:

$$(R_{\sqsubset_{\mathfrak{S}} n}) \qquad \forall F \subseteq_{Con} \mathscr{A}^*_{\mathfrak{S}} \text{ with } Card(F) \sqsubset_{\mathfrak{S}} n, \ (\bigsqcup\nolimits_{\mathfrak{S}} F) \text{ exists in } \mathfrak{S} \quad \text{and} \quad \mathbb{A}(\bigsqcup\nolimits_{\mathfrak{S}} F) = F. \tag{244}$$

We want to prove the property $(R_{\sqsubset_{\mathfrak{S}} (n+1)})$.
Let us consider a consistent family of atoms $G := \{ a_i \mid i \in \{1, \cdots, n\} \}$, and let us introduce $G^{\triangleleft} := \{ a_i \mid i \in \{1, \cdots, n-1\} \}$ and $G^{\triangleright} := \{ a_i \mid i \in \{2, \cdots, n\} \}$ which are both consistent family of atoms of cardinal $n-1$, and $G^{\triangledown} := \{ a_i \mid i \in \{2, \cdots, n-1\} \}$ which is a consistent family of atoms of cardinal $n-2$. From the recursion hypothesis (property (244)), we obtain that $(\bigsqcup_{\mathfrak{S}} G^{\triangleleft})$, $(\bigsqcup_{\mathfrak{S}} G^{\triangleright})$, and $(\bigsqcup_{\mathfrak{S}} G^{\triangledown})$ all exist in $\mathfrak{S}$. Moreover, if $M$ is an upper-bound of $G$, it is also an upper-bound of $G^{\triangleleft}$ and $G^{\triangleright}$, and then $M$ is above $(\bigsqcup_{\mathfrak{S}} G^{\triangleleft})$ and $(\bigsqcup_{\mathfrak{S}} G^{\triangleright})$. Hence, the set $\{ (\bigsqcup_{\mathfrak{S}} G^{\triangleleft}), (\bigsqcup_{\mathfrak{S}} G^{\triangleright}) \}$ is upper-bounded.
The properties $(\bigsqcup_{\mathfrak{S}} G^{\triangledown}) \sqsubseteq_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} G^{\triangleleft})$ and $(\bigsqcup_{\mathfrak{S}} G^{\triangledown}) \sqsubseteq_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} G^{\triangleright})$ are obvious, but we want to check $(\bigsqcup_{\mathfrak{S}} G^{\triangledown}) \overline{\sqsubset}_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} G^{\triangleleft})$ and $(\bigsqcup_{\mathfrak{S}} G^{\triangledown}) \overline{\sqsubset}_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} G^{\triangleright})$. Let us consider any $x \in \mathfrak{S}$ such that $(\bigsqcup_{\mathfrak{S}} G^{\triangledown}) \sqsubseteq_{\mathfrak{S}} x \sqsubseteq_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} G^{\triangleleft})$, and then in particular $G^{\triangledown} \subseteq \mathbb{A}(x) \subseteq G^{\triangleleft}$. If $\mathbb{A}(x) = G^{\triangledown}$, then $(\bigsqcup_{\mathfrak{S}} \mathbb{A}(x))$ exists and $\mathbb{A}(\bigsqcup_{\mathfrak{S}} \mathbb{A}(x)) = \mathbb{A}(x)$ (from recursion hypothesis). Moreover, $(\bigsqcup_{\mathfrak{S}} G^{\triangledown}) = (\bigsqcup_{\mathfrak{S}} \mathbb{A}(x)) \sqsubseteq_{\mathfrak{S}} x$, but the option $(\bigsqcup_{\mathfrak{S}} \mathbb{A}(x)) \sqsubset_{\mathfrak{S}} x$ is excluded, because the pre-complementation property (Lemma 57) would impose the existence of an atom $b$ such that $\bot_{\mathfrak{S}} \overline{\sqsubset}_{\mathfrak{S}} b \sqsubset_{\mathfrak{S}} x$ (i.e. $b \in \mathbb{A}(x)$) and $b \not\sqsubseteq_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} \mathbb{A}(x))$ (i.e. $b \notin \mathbb{A}(x)$). Hence, $\mathbb{A}(x) = G^{\triangledown}$ implies $(\bigsqcup_{\mathfrak{S}} \mathbb{A}(x)) = x$ and then $x = (\bigsqcup_{\mathfrak{S}} G^{\triangledown})$. If $\mathbb{A}(x) \subsetneq G^{\triangledown}$, then necessarily $\mathbb{A}(x) = G^{\triangleleft}$. $(\bigsqcup_{\mathfrak{S}} \mathbb{A}(x))$ exists and $\mathbb{A}(\bigsqcup_{\mathfrak{S}} \mathbb{A}(x)) = \mathbb{A}(x)$ (from recursion hypothesis). Moreover, $(\bigsqcup_{\mathfrak{S}} \mathbb{A}(x)) \sqsubseteq_{\mathfrak{S}} x$ and $x \sqsubseteq_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} G^{\triangleleft})$ implies $(\bigsqcup_{\mathfrak{S}} G^{\triangleleft}) = x$. As a conclusion, $(\bigsqcup_{\mathfrak{S}} G^{\triangledown}) \overline{\sqsubset}_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} G^{\triangleleft})$, and in the same way $(\bigsqcup_{\mathfrak{S}} G^{\triangledown}) \overline{\sqsubset}_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} G^{\triangleright})$.
Using property (225), we have more precisely : $(\bigsqcup_{\mathfrak{S}} G^{\triangleleft}) \sqcap_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} G^{\triangleright}) = (\bigsqcup_{\mathfrak{S}} G^{\triangledown})$.
Now, using the property $(D^*)$, i.e. equation (236), we deduce that $(\bigsqcup_{\mathfrak{S}} G^{\triangleleft}) \sqcup_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} G^{\triangleright})$ exists and that $(\bigsqcup_{\mathfrak{S}} G^{\triangleleft}) \overline{\sqsubset}_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} G^{\triangleleft}) \sqcup_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} G^{\triangleright})$, $(\bigsqcup_{\mathfrak{S}} G^{\triangleright}) \overline{\sqsubset}_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} G^{\triangleleft}) \sqcup_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} G^{\triangleright})$.
The second part of property $(D^*)$ is also essential. Let $b$ be an element of $\mathbb{A}^*_{\mathfrak{S}}$ such that $b \sqsubseteq_{\mathfrak{S}} ((\bigsqcup_{\mathfrak{S}} G^{\triangleleft}) \sqcup_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} G^{\triangleright}))$, and let us suppose that $(b \not\sqsubseteq_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} G^{\triangleleft})$ and $b \not\sqsubseteq_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} G^{\triangleright}))$. The family $(G^{\triangledown} \cup \{b\})$ is a finite family of cardinal $(n-1)$, upper-bounded by $((\bigsqcup_{\mathfrak{S}} G^{\triangleleft}) \sqcup_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} G^{\triangleright}))$, and then, due to the recursion hypothesis, the join $u := b \sqcup_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} G^{\triangledown})$ exists. Moreover, using previous analysis and the fact that $b \not\sqsubseteq_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} G^{\triangledown})$, we have also $(\bigsqcup_{\mathfrak{S}} G^{\triangledown}) \overline{\sqsubset}_{\mathfrak{S}} u$. Endly, $u$ satisfies $u \sqsubseteq_{\mathfrak{S}} ((\bigsqcup_{\mathfrak{S}} G^{\triangleleft}) \sqcup_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} G^{\triangleright}))$. We now exploit the second part of property $(D^*)$. We have necessarily $(u = ((\bigsqcup_{\mathfrak{S}} G^{\triangleleft}) \sqcup_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} G^{\triangleright}))$ or $u = (\bigsqcup_{\mathfrak{S}} G^{\triangleleft})$ or $u = (\bigsqcup_{\mathfrak{S}} G^{\triangleright}))$. The second and the third options are excluded, because it has been supposed that $(b \not\sqsubseteq_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} G^{\triangleleft})$ and $b \not\sqsubseteq_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} G^{\triangleright}))$, and the first option is excluded because$(\bigsqcup_{\mathfrak{S}} G^{\triangledown}) \overline{\sqsubset}_{\mathfrak{S}} u$. We have then obtained a contradiction. As a result, if an atom $b \in \mathscr{A}^*_{\mathfrak{S}}$ is such that $b \sqsubseteq_{\mathfrak{S}} ((\bigsqcup_{\mathfrak{S}} G^{\triangleleft}) \sqcup_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} G^{\triangleright}))$,

then $b$ is necessarily an element of $\mathbb{A}(\bigsqcup_{\mathfrak{S}} G^{\triangleleft}) \cup \mathbb{A}(\bigsqcup_{\mathfrak{S}} G^{\triangleright})$, i.e. $b \in \{a_i \mid i \in \{1, \cdots, n\}\} = \mathbb{A}(G)$. In other words, $(\bigsqcup_{\mathfrak{S}} G^{\triangleleft}) \sqcup_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} G^{\triangleright})$ is an upper-bound of $G$.
Conversely, if an element of $\mathfrak{S}$ is an upper-bound of $G$, it is a common upper-bound of $(\bigsqcup_{\mathfrak{S}} G^{\triangleleft})$ and $(\bigsqcup_{\mathfrak{S}} G^{\triangleright})$ because $G^{\triangleleft}$ and $G^{\triangleright}$ are subsets of $G$.
As a conclusion, $(\bigsqcup_{\mathfrak{S}} G^{\triangleleft}) \sqcup_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} G^{\triangleright})$ is the least upper-bound of $G$ and we have then proved the property $(R_{\sqsubset_{\mathfrak{S}}(n+1)})$.
The property (243) is then proved by recursion.
If we now consider a finite element $f \in \mathfrak{S}$ (i.e. $\mathbb{A}(s) \subseteq_{Con} \mathscr{A}^*_{\mathfrak{S}}$). $(\bigsqcup_{\mathfrak{S}} \mathbb{A}(f))$ exists in $\mathfrak{S}$, due to previous result. Moreover, $(\bigsqcup_{\mathfrak{S}} \mathbb{A}(f)) \sqsubseteq_{\mathfrak{S}} f$. The option $(\bigsqcup_{\mathfrak{S}} \mathbb{A}(f)) \sqsubset_{\mathfrak{S}} f$ is nevertheless excluded, because Lemma 57 would impose the existence of an atom $b$ such that $\bot_{\mathfrak{S}} \overline{\sqsubset}_{\mathfrak{S}} b \sqsubset_{\mathfrak{S}} f$ (i.e. $b \in \mathbb{A}(f)$) and $b \not\sqsubseteq_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} \mathbb{A}(f))$ (i.e. $b \notin \mathbb{A}(f)$), which is contradictory. This concludes the proof that a finite element is a finite join of its atoms.
Let $w_1, w_2$ be finite elements of $\mathfrak{S}$ (i.e. $\mathbb{A}(w_1), \mathbb{A}(w_2) \subseteq_{Con} \mathscr{A}^*_{\mathfrak{S}}$). If $\widehat{w_1 w_2}^{\mathfrak{S}}$, then we consider the consistent family $B := \mathbb{A}(w_1) \cup \mathbb{A}(w_2)$. $(\bigsqcup_{\mathfrak{S}} B)$ exists, due to previous results. We have $w_1 = (\bigsqcup_{\mathfrak{S}} \mathbb{A}(w_1)) \sqsubseteq_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} B)$ and $w_2 = (\bigsqcup_{\mathfrak{S}} \mathbb{A}(w_2)) \sqsubseteq_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} B)$. Hence, if $(w_1 \sqcup_{\mathfrak{S}} w_2)$ exists, we have $(w_1 \sqcup_{\mathfrak{S}} w_2) \sqsubseteq_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} B)$ and then $\mathbb{A}(w_1 \sqcup_{\mathfrak{S}} w_2) \subseteq \mathbb{A}(\bigsqcup_{\mathfrak{S}} B) = B$. We have also $\mathbb{A}(w_1), \mathbb{A}(w_1) \subseteq \mathbb{A}(w_1 \sqcup_{\mathfrak{S}} w_2)$ and then $B \subseteq \mathbb{A}(w_1 \sqcup_{\mathfrak{S}} w_2)$. As a result, $\mathbb{A}(w_1 \sqcup_{\mathfrak{S}} w_2) = \mathbb{A}(w_1) \cup \mathbb{A}(w_2)$ and $(w_1 \sqcup_{\mathfrak{S}} w_2) = (\bigsqcup_{\mathfrak{S}} (\mathbb{A}(w_1) \cup \mathbb{A}(w_2)))$. □

**Theorem 22. [Selection structures are atomistic]** Any element of a selection structure $\mathfrak{S}$ is a join of finite elements in $\mathfrak{S}$.

$$\forall s \in \mathfrak{S}, \qquad s = \bigsqcup_{\mathfrak{S}} \mathbb{A}(s) = \bigsqcup_{\mathfrak{S}}^{\bar{\wedge}} \{\bigsqcup_{\mathfrak{S}} F \mid F \subseteq_{fin} \mathbb{A}(s)\}. \tag{245}$$

Note that $\forall u, v \in \mathfrak{S}$, $(\mathbb{A}(u) \subseteq \mathbb{A}(v)) \Rightarrow (u \sqsubseteq_{\mathfrak{S}} v)$. ■

*Proof.* Let us consider any $u$ in $\mathfrak{S}^*$. For any $F \subseteq_{fin} \mathbb{A}(u)$, the elements of $F$ are upper-bounded by $u$, and then the supremum $(\bigsqcup_{\mathfrak{S}} F)$ exists in $\mathfrak{S}$, due to Lemma 59). Moreover, we have $(\bigsqcup_{\mathfrak{S}} F) \sqsubseteq_{\mathfrak{S}} u$. Due to the formula $\forall F_1, F_2 \subseteq_{fin} \mathbb{A}(u)$, $(\bigsqcup_{\mathfrak{S}} F_1) \sqcup_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} F_2) = (\bigsqcup_{\mathfrak{S}} (F_1 \cup F_2))$ also proved in Lemma 59), we deduce that the set $\mathbb{B}(u) := \{\bigsqcup_{\mathfrak{S}} F \mid F \subseteq_{fin} \mathbb{A}(u)\}$ is a directed subset of $\mathfrak{S}$. $\mathfrak{S}$ being directed complete, from Lemma 50, the directed supremum $M(u) := \bigsqcup_{\mathfrak{S}}^{\bar{\wedge}_{\mathfrak{S}}} \mathbb{B}(u)$ exists in $\mathfrak{S}$. As a supremum of elements which are less or equal than $u$, we have necessarily $M(u) \sqsubseteq_{\mathfrak{S}} u$. By construction, we have for any atom $a \in \mathbb{A}(u)$, necessarily $a \sqsubseteq_{\mathfrak{S}} M(u)$. Then, we can not have $M(u) \sqsubset_{\mathfrak{S}} u$. Indeed, due to the pre-complementation property (Lemma 57), we would have: $\exists b \in \mathscr{A}^*_{\mathfrak{S}}$ such that $b \sqsubseteq_{\mathfrak{S}} u$ (i.e. $b \in \mathbb{A}(u)$) and $b \not\sqsubseteq_{\mathfrak{S}} M(u)$, which is contradictory. Hence, $M(u) = u$. □

**Lemma 60. [Compacity is identical to finiteness]** Compact elements are finite elements and reciprocally. ■

*Proof.* Let $w$ be a compact element of $\mathfrak{S}$. By definition, we have

$$\forall D \subseteq_{Dir} \mathfrak{S},\ (w \sqsubseteq_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}}^{\bar{\wedge}} D)) \Rightarrow (\exists d \in D,\ w \sqsubseteq_{\mathfrak{S}} d).$$

However, from Theorem 22 , we can write

$$w = \bigsqcup_{\mathfrak{S}}^{\bar{\wedge}} \{\bigsqcup_{\mathfrak{S}} F \mid F \subseteq_{fin} \mathbb{A}(w)\}. \tag{246}$$

Hence, it exists $F \subseteq_{fin} \mathbb{A}(\bigsqcup_{\mathfrak{S}} C)$ such that $w \sqsubseteq_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} F)$ and $f := (\bigsqcup_{\mathfrak{S}} F)$ is a finite element.
Conversely, any finite element is compact. Indeed, let $g := (\bigsqcup_{\mathfrak{S}} G)$ for $G \subseteq_{fin} \mathscr{A}^*_{\mathfrak{S}}$ be a finite element of $\mathfrak{S}$. And let us consider a certain directed subset $D \subseteq_{Dir} \mathfrak{S}$, such that $g \sqsubseteq_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}}^{\bar{\wedge}} D)$. For any $a \in G$, it exists $d_a \in D$ such that $a \sqsubseteq_{\mathfrak{S}} d_a$. Hence, $D$ being directed, it exists an element $d_G \in D$ common upper-bound of $\{d_a \mid a \in G\}$. As a result, $g \sqsubseteq_{\mathfrak{S}} d_G$. This concludes the proof. □

*Remark* 24. We note that previous property is equivalent to 'property I' in [44, p.13].

**Lemma 61. [Selection structures are C-complete]** Selection structures are C-complete, i.e.

$$\forall w_1, w_2 \in \mathfrak{S}_c, \qquad \widehat{w_1 w_2}^{\mathfrak{S}} \Rightarrow (w_1 \sqcup_{\mathfrak{S}} w_2) \text{ exists in } \mathfrak{S}, \tag{247}$$

■

*Proof.* Direct consequence of Lemma 59 and Lemma 60. □

**Lemma 62. [Selection structures are algebraic]** Selection structures are algebraic, i.e. compactly generated. In other words,

$$\forall s \in \mathfrak{S}, \qquad s = \bigsqcup_{\mathfrak{S}} \{ w \in \mathfrak{S}_c \mid w \sqsubseteq_{\mathfrak{S}} s \} = \bigsqcup_{\mathfrak{S}}^{\bar{\wedge}} \{ \bigsqcup_{\mathfrak{S}} F \mid F \subseteq_{fin} \mathbb{A}(s) \}. \tag{248}$$

■

*Proof.* Direct consequence of Lemma 60 and Theorem 22. □

*Remark* 25. The property (248) implies trivially that $\mathfrak{S}$ is *compactly separated*, i.e.

$$\forall s_1, s_2 \in \mathfrak{S}, \qquad (s_1 \not\sqsubseteq_{\mathfrak{S}} s_2) \Rightarrow (\exists w \in \mathfrak{S}_c, w \sqsubseteq_{\mathfrak{S}} s_1 \text{ and } w \not\sqsubseteq_{\mathfrak{S}} s_2).$$

**Theorem 23. [Selection structures are Inf semi-lattice]** For any $s_1$ and $s_2$ elements of $\mathfrak{S}$, the infimum $s_1 \sqcap_{\mathfrak{S}} s_2$ exists in $\mathfrak{S}$. ■

*Proof.* Let us consider two elements of $\mathfrak{S}$ denoted $s_1$ and $s_2$. Let us introduce $B := \mathbb{A}(s_1) \cap \mathbb{A}(s_2)$. For any $F \subseteq_{fin} B$, we have necessarily $\forall a \in F$, $a \in \mathbb{A}(s_1)$, i.e. $a \sqsubseteq_{\mathfrak{S}} s_1$, and then $F$ is a consistent subset of $\mathfrak{S}$. As a consequence, $(\bigsqcup_{\mathfrak{S}} F)$ exists in $\mathfrak{S}$ from Lemma 59. For any $F_1, F_2 \subseteq_{fin} B$, we know also from Lemma 59 that $(F_1 \cup F_2)$ is consistent $\mathfrak{S}$ and $(\bigsqcup_{\mathfrak{S}} F_1) \sqcup_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} F_2) = (\bigsqcup_{\mathfrak{S}} (F_1 \cup F_2))$. The set $\{ \bigsqcup_{\mathfrak{S}} F \mid F \subseteq_{fin} \mathbb{A}(s) \}$ is then directed. We denote by $s$ its directed supremum in $\mathfrak{S}$. Necessarily, $s \sqsubseteq_{\mathfrak{S}} s_1$ and $s \sqsubseteq_{\mathfrak{S}} s_2$, because of formula (245) and the fact that $B \subseteq \mathbb{A}(s_1)$ and $B \subseteq \mathbb{A}(s_2)$. Moreover, for any $s'$ common lower-bound of $s_1$ and $s_2$, we know that necessarily $\mathbb{A}(s') \subseteq \mathbb{A}(s_1)$ and $\mathbb{A}(s') \subseteq \mathbb{A}(s_2)$, i.e. $\mathbb{A}(s') \subseteq B$, which implies $s' \sqsubseteq_{\mathfrak{S}} s$. The element $s$ is then the infimum $(s_1 \sqcap_{\mathfrak{S}} s_2)$. □

**Theorem 24. [Selection structures are meet-continuous]** A selection structure is necessarily meet-continuous, i.e.

$$\forall C \subseteq_{chain} \mathfrak{S}, \forall s \in \mathfrak{S}, \qquad s \sqcap_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} C) = \bigsqcup_{\mathfrak{S}} \{ s \sqcap_{\mathfrak{S}} c \mid c \in C \} \tag{249}$$

■

*Proof.* Let us consider a chain $C \subseteq_{chain} \mathfrak{S}$. The following inequality between suprema $(\bigsqcup_{\mathfrak{S}} \{ x \sqcap_{\mathfrak{S}} c \mid c \in C \}) \sqsubseteq_{\mathfrak{S}} (x \sqcap_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} C))$ is always true. Let us suppose that this inequality is strict. By assumption, it exists a compact element $u \in \mathfrak{S}_c$ with $u \not\sqsubseteq_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} \{ x \sqcap_{\mathfrak{S}} c \mid c \in C \})$ and $u \sqsubseteq_{\mathfrak{S}} (x \sqcap_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} C))$. But $u \sqsubseteq_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} C)$ implies $\exists c \in c, u \sqsubseteq_{\mathfrak{S}} c$ because $u$ is compact. Now, using $u \sqsubseteq_{\mathfrak{S}} x$, we have $u \sqsubseteq_{\mathfrak{S}} (x \sqcap_{\mathfrak{S}} c) \sqsubseteq_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} \{ x \sqcap_{\mathfrak{S}} c \mid c \in C \})$ which contradicts our hypothesis. As a result, $\mathfrak{S}$ is then meet-continuous. □

**Theorem 25. [Selection structures are algebraic Scott-domains]** The set of compact elements of $\mathfrak{S}$ (denoted $\mathfrak{S}_c$) is a basis of the dcpo $\mathfrak{S}$. Selection structure are then algebraic dcpo. Moreover, selections structures are $C-$complete. Hence, selection structures are algebraic Scott-domains. ■

*Proof.* Consequence of Lemma 61 and Lemma 62. □

**Lemma 63. [Selection structures are finitary domains]** A selection structure is necessarily a finitary Scott-domain. ■

*Proof.* The basis of compact elements is finitary, simply because compact elements are finite elements. Indeed, $\forall w \in \mathfrak{S}_c, \exists W \subseteq_{fin} \mathscr{A}_{\mathfrak{S}} \mid w = (\bigsqcup_{\mathfrak{S}} W)$ and $(\downarrow_{(D_c, \sqsubseteq_{\mathfrak{S}})} w) = \{ (\bigsqcup_{\mathfrak{S}} F) \mid F \subseteq_{fin} W \} \subseteq_{fin} \mathfrak{S}$. □

**Lemma 64.**

$$\{ w \in \mathfrak{S}_c \mid w \sqsubseteq_{\mathfrak{S}} s_1 \text{ and } w \sqcap_{\mathfrak{S}} s_2 = \bot_{\mathfrak{S}} \} \quad \subseteq_{Dir} \quad \mathfrak{S} \tag{250}$$

■

*Proof.* Let us denote $U_{(s_1,s_2)} := \{ w \in \mathfrak{S}_c \mid w \sqsubseteq_{\mathfrak{S}} s_1 \text{ and } w \sqcap_{\mathfrak{S}} s_2 = \bot_{\mathfrak{S}} \}$. Let us consider $w, w' \in U_{(s_1,s_2)}$ and let us denote $W, W' \subseteq_{fin} \mathscr{A}^*_{\mathfrak{S}}$ such that $W := \mathbb{A}(w)$ and $W' := \mathbb{A}(w')$, or in other words $w = \bigsqcup_{\mathfrak{S}} W$ and $w' = \bigsqcup_{\mathfrak{S}} W'$. The finite elements $w$ and $w'$ have a common upper-bound, $w \sqsubseteq_{\mathfrak{S}} s_1$ and $w' \sqsubseteq_{\mathfrak{S}} s_1$. Hence, from Lemma 59, the join $w \sqcup_{\mathfrak{S}} w'$ exists and is less or equal than $s_1$. Moreover, $w \sqcap_{\mathfrak{S}} s_2 = \bot_{\mathfrak{S}}$ and $w' \sqcap_{\mathfrak{S}} s_2 = \bot_{\mathfrak{S}}$ means that $\forall a \in W \cup W', a \not\sqsubseteq_{\mathfrak{S}} s_2$, i.e. $(w \sqcup_{\mathfrak{S}} w') \sqcap_{\mathfrak{S}} s_2 = \bot_{\mathfrak{S}}$. As a result, $(w \sqcup_{\mathfrak{S}} w')$, which is a common upper-bound of $w$ and $w'$, is an element of $U_{(s_1,s_2)}$. In other words, $U_{(s_1,s_2)}$ is directed. □

**Definition 6. [Relative complementation]** For any elements $s_1$ and $s_2$ in $\mathfrak{S}$ such that $s_2 \sqsubseteq_{\mathfrak{S}} s_1$, we define *the complement of $s_2$ relative to $s_1$* to be the element of $\mathfrak{S}$ denoted $(s_1 \smallsetminus s_2)$ and defined by

$$(s_1 \smallsetminus s_2) \quad := \quad \bigsqcup_{\mathfrak{S}}^{\bar{\wedge}} \{ w \in \mathfrak{S}_c \mid w \sqsubseteq_{\mathfrak{S}} s_1 \text{ and } w \sqcap_{\mathfrak{S}} s_2 = \bot_{\mathfrak{S}} \} \tag{251}$$

*Remark* 26. $\mathfrak{S}$ being directed complete (indeed, $\mathfrak{S}$ is chain-complete), the supremum of the set $\{ w \in \mathfrak{S}_c \mid w \sqsubseteq_{\mathfrak{S}} s_1 \text{ and } w \sqcap_{\mathfrak{S}} s_2 = \bot_{\mathfrak{S}} \}$ then exists.

**Lemma 65. [Properties of relative complementation]** $\forall s_1, s_2, s_3 \in \mathfrak{S}$ with $s_3 \sqsubseteq_{\mathfrak{S}} s_1$ and $s_2 \sqsubseteq_{\mathfrak{S}} s_1$, we have:

$$(s_1 \smallsetminus \bot_{\mathfrak{S}}) = s_1, \qquad (s_1 \smallsetminus s_1) = \bot_{\mathfrak{S}} \tag{252}$$

$$s_3 \sqsubseteq_{\mathfrak{S}} s_2 \quad \Rightarrow \quad (s_1 \smallsetminus s_2) \sqsubseteq_{\mathfrak{S}} (s_1 \smallsetminus s_3), \tag{253}$$

$$(s_1 \smallsetminus s_2) \sqsubseteq_{\mathfrak{S}} s_1 \tag{254}$$

$$s_2 \sqcap_{\mathfrak{S}} (s_1 \smallsetminus s_2) = \bot_{\mathfrak{S}}, \tag{255}$$

$$(s_1 \smallsetminus (s_1 \smallsetminus s_2)) = s_2. \tag{256}$$

For any $s_1, s_2, s_3 \in \mathfrak{S}$ with $s_3 \sqsubseteq_{\mathfrak{S}} s_2$, we have

$$s_1 \sqcap_{\mathfrak{S}} (s_2 \smallsetminus s_3) = (s_1 \sqcap_{\mathfrak{S}} s_2) \smallsetminus (s_1 \sqcap_{\mathfrak{S}} s_3). \tag{257}$$

■

*Proof.* The first property of (252) is obvious. Indeed, $(s_1 \smallsetminus \bot_{\mathfrak{S}}) = \bigsqcup_{\mathfrak{S}}^{\bar{\wedge}} \{ w \in \mathfrak{S}_c \mid w \sqsubseteq_{\mathfrak{S}} s_1 \} = s_1$, because $s_1$ is the supremum of its compact lower-bounds (i.e. algebraicity property 248).
The second property of (252) is obtained as follows. For any $s \in \mathfrak{S}$, $s \sqsubseteq_{\mathfrak{S}} s_1$ implies $s \sqcap_{\mathfrak{S}} s_1 = s$, and then $\bot_{\mathfrak{S}} = s \sqcap_{\mathfrak{S}} s_1$ implies $s = \bot_{\mathfrak{S}}$. Hence, $(s_1 \smallsetminus s_1) = \bot_{\mathfrak{S}}$.
The proof of property (253) is a simple ordering property. For any $s$ in $\mathfrak{S}_c$ (it is true for any $s$ in $\mathfrak{S}$), $s \sqcap_{\mathfrak{S}} s_2 = \bot_{\mathfrak{S}}$ and $s_3 \sqsubseteq_{\mathfrak{S}} s_2$ imply $s \sqcap_{\mathfrak{S}} s_3 = \bot_{\mathfrak{S}}$. As a result,

$$\{ s \in \mathfrak{S}_c \mid s \sqsubseteq_{\mathfrak{S}} s_1 \text{ and } s \sqcap_{\mathfrak{S}} s_2 = \bot_{\mathfrak{S}} \} \subseteq \{ s \in \mathfrak{S}_c \mid s \sqsubseteq_{\mathfrak{S}} s_1 \text{ and } s \sqcap_{\mathfrak{S}} s_3 = \bot_{\mathfrak{S}} \}.$$

Passing to the suprema, we obtain the relation $(s_1 \smallsetminus s_2) \sqsubseteq_{\mathfrak{S}} (s_1 \smallsetminus s_3)$.
To obtain property (254), we take $s_3 := \bot_{\mathfrak{S}}$ in property 253. Using $(s_1 \smallsetminus \bot_{\mathfrak{S}}) = s_1$, we note that $\forall s_1, s_2 \in \mathfrak{S}$, such that $s_2 \sqsubseteq_{\mathfrak{S}} s_1$, we have $(s_1 \smallsetminus s_2) \sqsubseteq_{\mathfrak{S}} s_1$.
The property (255) can be deduced as follows. Let us suppose $\bot_{\mathfrak{S}} \sqsubset_{\mathfrak{S}} s_2 \sqcap_{\mathfrak{S}} (s_1 \smallsetminus s_2)$. From atomicity of $\mathfrak{S}$, it exists $a \in \mathbb{A}^*_{\mathfrak{S}}$ such that $a \sqsubseteq_{\mathfrak{S}} s_2 \sqcap_{\mathfrak{S}} (s_1 \smallsetminus s_2)$. We then have simultaneously $a \sqsubseteq_{\mathfrak{S}} s_2$ and $a \sqsubseteq_{\mathfrak{S}} (s_1 \smallsetminus s_2)$. The atoms of $\mathfrak{S}$ being compact elements, the property

$$a \sqsubseteq_{\mathfrak{S}} (s_1 \smallsetminus s_2) = \bigsqcup\nolimits^{\bar{\wedge}}_{\mathfrak{S}} \{ w \in \mathfrak{S}_c \mid w \sqsubseteq_{\mathfrak{S}} s_1 \text{ and } w \sqcap_{\mathfrak{S}} s_2 = \bot_{\mathfrak{S}} \}$$

implies $\exists w \in \mathfrak{S}_c$ such that $w \sqsubseteq_{\mathfrak{S}} s_1$, $w \sqcap_{\mathfrak{S}} s_2 = \bot_{\mathfrak{S}}$ and $a \sqsubseteq_{\mathfrak{S}} w$. As a summary, we obtain in particular $a \sqsubseteq_{\mathfrak{S}} s_2$ and $a \sqcap_{\mathfrak{S}} s_2 = \bot_{\mathfrak{S}}$ which is impossible for an atom. Then, necessarily $s_2 \sqcap_{\mathfrak{S}} (s_1 \smallsetminus s_2) = \bot_{\mathfrak{S}}$.
Endly, let us prove the non-trivial property (256). Due to the property $(s_1 \smallsetminus s_2) \sqsubseteq_{\mathfrak{S}} s_1$, we can form the following object: $(s_1 \smallsetminus (s_1 \smallsetminus s_2))$. By definition,

$$(s_1 \smallsetminus (s_1 \smallsetminus s_2)) = \bigsqcup\nolimits^{\bar{\wedge}}_{\mathfrak{S}} \{ s \in \mathfrak{S}_c \mid s \sqsubseteq_{\mathfrak{S}} s_1 \text{ and } s \sqcap_{\mathfrak{S}} (s_1 \smallsetminus s_2) = \bot_{\mathfrak{S}} \}.$$

We have assumed $s_2 \sqsubseteq_{\mathfrak{S}} s_1$ and we have already established $s_2 \sqcap_{\mathfrak{S}} (s_1 \smallsetminus s_2) = \bot_{\mathfrak{S}}$. We deduce from these properties that, for any $w \in \mathfrak{S}_c$, $(w \sqsubseteq_{\mathfrak{S}} s_2)$ implies $(w \sqsubseteq_{\mathfrak{S}} s_1 \text{ and } w \sqcap_{\mathfrak{S}} (s_1 \smallsetminus s_2) = \bot_{\mathfrak{S}})$. Hence, passing to supremum, we obtain

$$s_2 = \bigsqcup\nolimits_{\mathfrak{S}} \{ w \in \mathfrak{S}_c \mid w \sqsubseteq_{\mathfrak{S}} s_2 \} \sqsubseteq_{\mathfrak{S}} \bigsqcup\nolimits^{\bar{\wedge}}_{\mathfrak{S}} \{ w \in \mathfrak{S}_c \mid w \sqsubseteq_{\mathfrak{S}} s_1 \text{ and } w \sqcap_{\mathfrak{S}} (s_1 \smallsetminus s_2) = \bot_{\mathfrak{S}} \} = (s_1 \smallsetminus (s_1 \smallsetminus s_2)).$$

Let us consider any $w \in \mathfrak{S}_c$ such that $w \sqsubseteq_{\mathfrak{S}} s_1$ and $w \sqcap_{\mathfrak{S}} (s_1 \smallsetminus s_2) = \bot_{\mathfrak{S}}$. Then, for any atom $a \sqsubseteq_{\mathfrak{S}} w$, we have $a \sqsubseteq_{\mathfrak{S}} s_1$ and $a \sqcap_{\mathfrak{S}} (s_1 \smallsetminus s_2) = \bot_{\mathfrak{S}}$. Using meet-continuity property of $\mathfrak{S}$ (equation 249), we have

$$a \sqcap_{\mathfrak{S}} (s_1 \smallsetminus s_2) = \bigsqcup\nolimits^{\bar{\wedge}}_{\mathfrak{S}} \{ (a \sqcap_{\mathfrak{S}} u) \mid u \in \mathfrak{S}_c,\ u \sqsubseteq_{\mathfrak{S}} s_1 \text{ and } u \sqcap_{\mathfrak{S}} s_2 = \bot_{\mathfrak{S}} \}.$$

Hence, for any atom $a \sqsubseteq_{\mathfrak{S}} w$, we must have $\forall u \in \mathfrak{S}_c$, $(u \sqsubseteq_{\mathfrak{S}} s_1 \text{ and } u \sqcap_{\mathfrak{S}} s_2 = \bot_{\mathfrak{S}}) \Rightarrow (a \sqcap_{\mathfrak{S}} u = \bot_{\mathfrak{S}})$, or in other words $\forall b \in \mathbb{A}^*_{\mathfrak{S}}$, $(b \sqsubseteq_{\mathfrak{S}} s_1 \text{ and } b \not\sqsubseteq_{\mathfrak{S}} s_2) \Rightarrow (a \neq b)$. As a conclusion, for any atom $a \sqsubseteq_{\mathfrak{S}} w$ we have necessarily $a \sqsubseteq_{\mathfrak{S}} s_2$, and then $w \sqsubseteq_{\mathfrak{S}} s_2$. As a result, we obtain

$$\bigsqcup\nolimits^{\bar{\wedge}}_{\mathfrak{S}} \{ w \in \mathfrak{S}_c \mid w \sqsubseteq_{\mathfrak{S}} s_1 \text{ and } w \sqcap_{\mathfrak{S}} (s_1 \smallsetminus s_2) = \bot_{\mathfrak{S}} \} \sqsubseteq_{\mathfrak{S}} s_2.$$

We have then established the formula $(s_1 \smallsetminus (s_1 \smallsetminus s_2)) = s_2$.

$\forall a \in \mathscr{A}^*_{\mathfrak{S}}$, $(a \sqsubseteq_{\mathfrak{S}} (s_1 \sqcap_{\mathfrak{S}} (s_2 \smallsetminus s_3))) \Rightarrow (a \sqsubseteq_{\mathfrak{S}} s_1 \text{ and } a \sqsubseteq (s_2 \smallsetminus s_3)) \Rightarrow (a \sqsubseteq_{\mathfrak{S}} s_1 \text{ and } (\exists w \in \mathfrak{S}_c, a \sqsubseteq_{\mathfrak{S}} w \text{ and } w \sqsubseteq_{\mathfrak{S}} s_2 \text{ and } w \sqcap_{\mathfrak{S}} s_3 = \bot_{\mathfrak{S}})) \Rightarrow (a \sqsubseteq_{\mathfrak{S}} (s_1 \sqcap_{\mathfrak{S}} s_2) \text{ and } a \sqcap_{\mathfrak{S}} (s_1 \sqcap_{\mathfrak{S}} s_3) = \bot_{\mathfrak{S}}) \Rightarrow (a \sqsubseteq ((s_1 \sqcap_{\mathfrak{S}} s_2) \smallsetminus (s_1 \sqcap_{\mathfrak{S}} s_3)))$. Hence, $(s_1 \sqcap_{\mathfrak{S}} (s_2 \smallsetminus s_3)) \sqsubseteq_{\mathfrak{S}} (s_1 \sqcap_{\mathfrak{S}} s_2) \smallsetminus (s_1 \sqcap_{\mathfrak{S}} s_3)$. Conversely, let us consider $a \in \mathscr{A}^*_{\mathfrak{S}}$ such that $a \sqsubseteq_{\mathfrak{S}} (s_1 \sqcap_{\mathfrak{S}} s_2) \smallsetminus (s_1 \sqcap_{\mathfrak{S}} s_3)$. We have in particular, $a \sqsubseteq_{\mathfrak{S}} (s_1 \sqcap_{\mathfrak{S}} s_2) \sqsubseteq_{\mathfrak{S}} s_1$. Let us suppose that $a \not\sqsubseteq_{\mathfrak{S}} (s_1 \sqcap_{\mathfrak{S}} (s_2 \smallsetminus s_3))$, then, necessarily, $a \sqsubseteq_{\mathfrak{S}} s_3$ and then $a \sqsubseteq_{\mathfrak{S}} (s_1 \sqcap_{\mathfrak{S}} s_3)$, which contradicts $a \sqsubseteq_{\mathfrak{S}} (s_1 \sqcap_{\mathfrak{S}} s_2) \smallsetminus (s_1 \sqcap_{\mathfrak{S}} s_3)$. Hence, $a \sqsubseteq_{\mathfrak{S}} (s_1 \sqcap_{\mathfrak{S}} (s_2 \smallsetminus s_3))$. This concludes the proof of property (257). □

**Lemma 66. [Relative co-atoms]** For any $s \in \mathfrak{S}$ and any atom $a \in \mathbb{A}^*(s)$ we have

$$(s \smallsetminus a) \,\overline{\sqsubset}_{\mathfrak{S}}\, s. \tag{258}$$

For any $s_1, s_2 \in \mathfrak{S}$ verifying $s_2 \sqsubset_{\mathfrak{S}} s_1$, it exists $s_3 \in \mathfrak{S}$ with $s_2 \sqsubseteq_{\mathfrak{S}} s_3 \,\overline{\sqsubset}_{\mathfrak{S}}\, s_1$. ■

*Proof.* From Lemma 55, we know that the supremum $u := \{ t \in \mathfrak{S} \mid \bot_{\mathfrak{S}} \sqsubseteq_{\mathfrak{S}} t \sqsubseteq_{\mathfrak{S}} s \text{ and } t \sqcap_{\mathfrak{S}} a = \bot_{\mathfrak{S}} \}$ exists in $\mathfrak{S}$ (it is unique, by definition of the supremum) and this supremum verifies $u \sqcap_{\mathfrak{S}} a = \bot_{\mathfrak{S}}$ and $u \,\overline{\sqsubset}_{\mathfrak{S}} (u \sqcup_{\mathfrak{S}} a) \sqsubseteq_{\mathfrak{S}} s$. By definition of $(s \smallsetminus a)$, we have immediatly $(s \smallsetminus a) \sqsubseteq_{\mathfrak{S}} u$. Hence, $s = (a \sqcup_{\mathfrak{S}} (s \smallsetminus a)) \sqsubseteq_{\mathfrak{S}} (u \sqcup_{\mathfrak{S}} a) \sqsubseteq_{\mathfrak{S}} s$ implies immediatly $(u \sqcup_{\mathfrak{S}} a) = s$ and then $u \,\overline{\sqsubset}_{\mathfrak{S}} s$. From Lemma 52 and $\bot_{\mathfrak{S}} \sqsubset_{\mathfrak{S}} u \,\overline{\sqsubset}_{\mathfrak{S}} s$, we know that $a$ is the unique element of $\mathfrak{S}$ such that $a \sqcap_{\mathfrak{S}} a = \bot_{\mathfrak{S}}$ and $u \sqcup_{\mathfrak{S}} a = s$. $a$ being an atom, and then also a compact element, we obtain immediatly $(s \smallsetminus u) = a$. Now, using property (256),

we deduce that $u = (s \smallsetminus a)$. As a final conclusion, we obtain $(s \smallsetminus a) \overline{\sqsubset}_{\mathfrak{S}} s$.
From lemma 57, if $s_2 \sqsubset_{\mathfrak{S}} s_1$, it then exists $a \in \mathbb{A}^*(s_1)$ satisfying $a \sqcap_{\mathfrak{S}} s_2 = \bot_{\mathfrak{S}}$. Let us suppose that $s_2 \not\sqsubseteq_{\mathfrak{S}} (s_1 \smallsetminus a)$. It would then exists an atom $b$ such that $b \sqsubseteq_{\mathfrak{S}} s_2$ but $b \not\sqsubseteq_{\mathfrak{S}} (s_1 \smallsetminus a)$. $b$ has to be different from $a$ because $a \sqcap_{\mathfrak{S}} s_2 = \bot_{\mathfrak{S}}$. Hence, $b \not\sqsubseteq_{\mathfrak{S}} s_1$. We have then obtained a contradiction of the hypothesis $s_2 \sqsubset_{\mathfrak{S}} s_1$. □

**Lemma 67. [Consistent-completeness of selection structures and relative complementation]** For any $s_1, s_2, s_3 \in \mathfrak{S}$ such that $s_3 \sqsubseteq_{\mathfrak{S}} s_1$ and $s_2 \sqsubseteq_{\mathfrak{S}} s_1$, the supremum $s_2 \sqcup_{\mathfrak{S}} s_3$ exists and we have

$$s_2 \sqcup_{\mathfrak{S}} s_3 = (s_1 \smallsetminus ((s_1 \smallsetminus s_2) \sqcap_{\mathfrak{S}} (s_1 \smallsetminus s_3))). \tag{259}$$

We note that this supremum does not depend on the choice of the upper-bound $s_1$. We note the following basic relative complementation property:

$$s_2 \sqcup_{\mathfrak{S}} (s_1 \smallsetminus s_2) = s_1. \tag{260}$$

A selection structure is then always consistently complete. ■

*Proof.* Using $((s_1 \smallsetminus s_2) \sqcap_{\mathfrak{S}} (s_1 \smallsetminus s_3))) \sqsubseteq_{\mathfrak{S}} (s_1 \smallsetminus s_i)$ for $i = 2,3$ and equation (253), we obtain $s_i = (s_1 \smallsetminus (s_1 \smallsetminus s_i)) \sqsubseteq_{\mathfrak{S}} (s_1 \smallsetminus ((s_1 \smallsetminus s_2) \sqcap_{\mathfrak{S}} (s_1 \smallsetminus s_3)))$ for $i = 2,3$, i.e. $(s_1 \smallsetminus ((s_1 \smallsetminus s_2) \sqcap_{\mathfrak{S}} (s_1 \smallsetminus s_3)))$ is a common upper-bound of $s_2$ and $s_3$. Now, if we consider any $M$ such that $s_i \sqsubseteq_{\mathfrak{S}} M \sqsubseteq_{\mathfrak{S}} s_1$ for $i = 2,3$, then $(s_1 \smallsetminus M) \sqsubseteq_{\mathfrak{S}} (s_1 \smallsetminus s_i)$ for $i = 2,3$, and then $(s_1 \smallsetminus M) \sqsubseteq_{\mathfrak{S}} ((s_1 \smallsetminus s_2) \sqcap_{\mathfrak{S}} (s_1 \smallsetminus s_3))$, and finally $(s_1 \smallsetminus ((s_1 \smallsetminus s_2) \sqcap_{\mathfrak{S}} (s_1 \smallsetminus s_3))) \sqsubseteq_{\mathfrak{S}} M$, which shows that $(s_1 \smallsetminus ((s_1 \smallsetminus s_2) \sqcap_{\mathfrak{S}} (s_1 \smallsetminus s_3)))$ is the least upper-bound of $\{s_2, s_3\}$. The uniqueness of the supremum implies that the result does not depend on the choice of $s_1$ as an upper-bound of $s_2$ and $s_3$. This concludes the proof of property (259).
Taking $s_3 = (s_1 \smallsetminus s_2)$ in the equation (259) and using successively the equations (256), (255), (252) we obtain $s_2 \sqcup_{\mathfrak{S}} (s_1 \smallsetminus s_2) = s_1$. This concludes the proof of property (260). □

*Remark* 27. Let us now consider $x, y \in \mathfrak{S}$ such that $\widehat{xy}^{\mathfrak{S}}$. We deduce, using successively the property $y = (y \sqcup_{\mathfrak{S}} (x \sqcap_{\mathfrak{S}} y))$[25] and the property (260): $y \sqcup_{\mathfrak{S}} (x \smallsetminus (x \sqcap_{\mathfrak{S}} y)) = (y \sqcup_{\mathfrak{S}} (x \sqcap_{\mathfrak{S}} y)) \sqcup_{\mathfrak{S}} (x \smallsetminus (x \sqcap_{\mathfrak{S}} y)) = y \sqcup_{\mathfrak{S}} ((x \sqcap_{\mathfrak{S}} y) \sqcup_{\mathfrak{S}} (x \smallsetminus (x \sqcap_{\mathfrak{S}} y))) = y \sqcup_{\mathfrak{S}} x$. As a result, we deduce the covering property (called 'property $\widehat{C}$' in [44, p.23 section 3]):

$$\forall x, y \in \mathfrak{S} \mid \widehat{xy}^{\mathfrak{S}}, \quad (x \sqcap_{\mathfrak{S}} y) \overline{\sqsubset}_{\mathfrak{S}} x \quad \Rightarrow \quad y \overline{\sqsubset}_{\mathfrak{S}} (x \sqcup_{\mathfrak{S}} y). \tag{261}$$

**Theorem 26. [Coherent completeness and down-completeness of selection structures]**

$$\forall S \subseteq_{Coh} \mathfrak{S}, \qquad \bigsqcup\nolimits_{\mathfrak{S}} S := \bigsqcup\nolimits_{\mathfrak{S}}^{\overline{\wedge}} \{ \bigsqcup\nolimits_{\mathfrak{S}} R \mid R \subseteq_{fin} S \} \text{ exists in } \mathfrak{S} \tag{262}$$

$$\forall S \subseteq \mathfrak{S}, \qquad (\bigsqcap\nolimits_{\mathfrak{S}} S) \text{ exists in } \mathfrak{S} \tag{263}$$

■

*Proof.* The directed-completeness coupled with the consistent-completeness of the selection structure $\mathfrak{S}$ implies directly the coherent-completeness of $\mathfrak{S}$.
The consistent-completeness of $\mathfrak{S}$ implies that the set of lower-bounds of a given subset $S$ of $\mathfrak{S}$ is a directed-subset of $\mathfrak{S}$. As a result, $\mathfrak{S}$ is down-complete. □

*Remark* 28. The down-completeness implies immediately the bounded-completeness of $\mathfrak{S}$.

[25] which is a direct consequence of the property $(x \sqcap_{\mathfrak{S}} y) \sqsubseteq_{\mathfrak{S}} y$ using $(y \sqcup_{\mathfrak{S}} (x \sqcap_{\mathfrak{S}} y)) = y \smallsetminus ((y \smallsetminus y) \sqcap_{\mathfrak{S}} (y \smallsetminus (x \sqcap_{\mathfrak{S}} y))) = y \smallsetminus \bot_{\mathfrak{S}} = y$

**Theorem 27. [Conditional-distributivity in selection structures]** Let $\mathfrak{S}$ be a selection structure. For any $x,y,z \in \mathfrak{S}$, we have the following conditional-distributivity properties:

$$\widehat{xy}^{\mathfrak{S}} \quad \Rightarrow \quad z \sqcap_{\mathfrak{S}} (y \sqcup_{\mathfrak{S}} x) = (z \sqcap_{\mathfrak{S}} y) \sqcup_{\mathfrak{S}} (z \sqcap_{\mathfrak{S}} x), \tag{264}$$

$$\widehat{xy}^{\mathfrak{S}} \text{ and } \widehat{xz}^{\mathfrak{S}} \quad \Rightarrow \quad x \sqcup_{\mathfrak{S}} (y \sqcap_{\mathfrak{S}} z) = (x \sqcup_{\mathfrak{S}} y) \sqcap_{\mathfrak{S}} (x \sqcup_{\mathfrak{S}} z). \tag{265}$$

■

*Proof.* Let us choose $w$ upper-bound of $y$ and $z$. Using mainly the properties (259) and (257), we obtain $z \sqcap_{\mathfrak{S}} (x \sqcup_{\mathfrak{S}} y) = z \sqcap_{\mathfrak{S}} (w \smallsetminus ((w \smallsetminus x) \sqcap_{\mathfrak{S}} (w \smallsetminus y))) = ((z \sqcap_{\mathfrak{S}} w) \smallsetminus (z \sqcap_{\mathfrak{S}} (w \smallsetminus x) \sqcap_{\mathfrak{S}} (w \smallsetminus y))) = ((z \sqcap_{\mathfrak{S}} w) \smallsetminus (((z \sqcap_{\mathfrak{S}} w) \smallsetminus (z \sqcap_{\mathfrak{S}} x))) \sqcap_{\mathfrak{S}} ((z \sqcap_{\mathfrak{S}} w) \smallsetminus (z \sqcap_{\mathfrak{S}} y)))))$. Choosing now $(z \sqcap_{\mathfrak{S}} w)$ as upper-bound of $(z \sqcap_{\mathfrak{S}} x)$ and $(z \sqcap_{\mathfrak{S}} y)$ to develop $(z \sqcap_{\mathfrak{S}} x) \sqcup_{\mathfrak{S}} (z \sqcap_{\mathfrak{S}} y)$, we obtain the same result. This concludes the proof of the first property.
If we consider $w$ upper-bound of $x$, $y$ and $z$. We have $z \sqcup_{\mathfrak{S}} (x \sqcap_{\mathfrak{S}} y) = (w \smallsetminus ((w \smallsetminus z) \sqcap_{\mathfrak{S}} (w \smallsetminus (x \sqcap_{\mathfrak{S}} y)))) = (w \smallsetminus ((w \smallsetminus z) \sqcap_{\mathfrak{S}} (w \smallsetminus x) \sqcup_{\mathfrak{S}} (w \smallsetminus y))))$, and the second distributivity property is deduced from the first one: $x \sqcup_{\mathfrak{S}} (y \sqcap_{\mathfrak{S}} z) = (x \sqcap_{\mathfrak{S}} y) \sqcup_{\mathfrak{S}} (x \sqcap_{\mathfrak{S}} z)$. □

**Lemma 68. [Join-prime elements are the atoms]** For any selection structure $\mathfrak{S}$, the join-prime elements are the atoms of $\mathfrak{S}$. ■

*Proof.* Atoms are obviously join-irreducible. Atoms are even join-prime, but only because $\mathfrak{S}$ is distributive. Indeed, for any atom $a$, if $a \sqsubseteq_{\mathfrak{S}} (x \sqcup_{\mathfrak{S}} y)$, then $a = a \sqcap_{\mathfrak{S}} (x \sqcup_{\mathfrak{S}} y) = (a \sqcap_{\mathfrak{S}} x) \sqcup_{\mathfrak{S}} (a \sqcap_{\mathfrak{S}} y)$ and the join-irreducibility of $a$ implies $a \sqsubseteq_{\mathfrak{S}} x$ or $a \sqsubseteq_{\mathfrak{S}} y$.
Conversely, let $y$ be a non-atomic element, i.e. it exists an atom $x \in D$ such that $\bot_D \sqsubseteq_{\mathfrak{S}} x \sqsubset_{\mathfrak{S}} y$. We have $y = x \sqcup_{\mathfrak{S}} (y \smallsetminus x)$ but $y \neq x$, $y \neq (y \smallsetminus x)$. Hence, $y$ is not join-irreducible, and then *a fortiori* not join-prime. □

**Theorem 28. [Selection structures are qualitative domains]** A selection structure $(\mathfrak{S}, \sqsubseteq_{\mathfrak{S}})$ is a qualitative domain, i.e. a dI-domain in which the join-prime elements are atomic. ■

*Proof.* As a direct consequence of theorem 25, Lemma 63, Lemma 27 and Lemma 68, we obtain that $(\mathfrak{S}, \sqsubseteq_{\mathfrak{S}})$ is a qualitative domain. □

**Lemma 69. [Finitary Scott-domain are algebraic]** If a Scott-domain $(\mathfrak{S}, \sqsubseteq_{\mathfrak{S}})$ admits a finitary basis $B$, then, necessarily, $\mathfrak{S}$ is an algebraic Scott-domain. The finitary property implies in fact the following stronger property:

$$\forall s \in \mathfrak{S}_c, \qquad (\downarrow_{\mathfrak{S}} s) \text{ is a finite lattice included in } \mathfrak{S}_c. \tag{266}$$

■

*Proof.* Let $x$ be an element of $B$ and let us suppose that $x$ is not compact. Due to the finitary character of the basis $B$, i.e.

$$\forall x \in B, \qquad \{y \in B \,|\, y \sqsubset\!\!\sqsubset_{\mathfrak{S}} x\} \subseteq_{fin} B, \tag{267}$$

the continuity property applied in $x$ (i.e. $x = \bigsqcup_{\mathfrak{S}}^{\bar{\wedge}} (\downarrow_{(B, \sqsubset\!\!\sqsubset_{\mathfrak{S}})} x)$) expresses $x$ as a finite join of basis elements in $(\downarrow_{(B, \sqsubset\!\!\sqsubset_{\mathfrak{S}})} x)$. It is important to note that, $x$ being not-compact, $x$ is not element of the subset $(\downarrow_{(B, \sqsubset\!\!\sqsubset_{\mathfrak{S}})} x)$. Because finite joins of compact elements are also compact, it must then exist $y \in (\downarrow_{(B, \sqsubset\!\!\sqsubset_{\mathfrak{S}})} x)$ (and then necessarily $y \sqsubset_{\mathfrak{S}} x$) with $y \notin \mathfrak{S}_c$. Let us consider the subset $\{z \in (\downarrow_{(B, \sqsubset\!\!\sqsubset_{\mathfrak{S}})} x) \mid z \notin \mathfrak{S}_c\}$. As a finite and non-empty subset, it admits at least a minimal element denoted $x'$. This element $x'$ satisfies: $x' \notin \mathfrak{S}_c$,

$x' \sqsubset_{\mathfrak{S}} x$ and $(\downarrow_{(B,\mathbb{C}_{\mathfrak{S}})} x) \cap (\downarrow x') = \{x'\}$. However, we know that $x' \sqsubset_{\mathfrak{S}} x$ implies $(\downarrow_{(B,\mathbb{C}_{\mathfrak{S}})} x') \subseteq (\downarrow_{(B,\mathbb{C}_{\mathfrak{S}})} x)$. We have also obviously $(\downarrow_{(B,\mathbb{C}_{\mathfrak{S}})} x') \subseteq (\downarrow x')$. Hence, $(\downarrow_{(B,\mathbb{C}_{\mathfrak{S}})} x') \subseteq (\downarrow_{(B,\mathbb{C}_{\mathfrak{S}})} x) \cap (\downarrow x') = \{x'\}$, and then $(\downarrow_{(B,\mathbb{C}_{\mathfrak{S}})} x') = \varnothing$ (since $x' \notin \mathfrak{S}_c$) which is contradictory. As a conclusion, $\mathfrak{S}$ is an algebraic Scott-domain.

$$((\forall x \in \mathfrak{S}, (\downarrow_{(B,\mathbb{C}_{\mathfrak{S}})} x) \subseteq_{Dir} \mathfrak{S} \text{ and } x = \bigsqcup\nolimits_{\mathfrak{S}}^{\bar{\wedge}} (\downarrow_{(B,\mathbb{C}_{\mathfrak{S}})} x)) \text{ and } (\forall y \in B, (\downarrow_{(B,\mathbb{C}_{\mathfrak{S}})} y) \subseteq_{fin} B)) \Rightarrow B = \mathfrak{S}_c. \quad (268)$$

Using $B = \mathfrak{S}_c$, the finitary property (267) is simplified: $\forall s \in \mathfrak{S}_c$, $(\downarrow_{(\mathfrak{S}_c,\sqsubseteq_{\mathfrak{S}})} s) \subseteq_{fin} \mathfrak{S}_c$.
It is in fact possible to go further and observe that: $\forall s \in \mathfrak{S}_c, \forall x \in \mathfrak{S}$, $(x \sqsubseteq_{\mathfrak{S}} s) \Rightarrow (x \in \mathfrak{S}_c)$. Indeed, if we had $(s \in \mathfrak{S}_c, \forall x \in \mathfrak{S}, x \sqsubseteq_{\mathfrak{S}} s)$ and $x \notin \mathfrak{S}_c$, the set $(\downarrow_{(\mathfrak{S}_c,\sqsubseteq_{\mathfrak{S}})} x)$ would be an infinite subset of the finite set $(\downarrow_{(\mathfrak{S}_c,\sqsubseteq_{\mathfrak{S}})} s)$, which is obviously contradictory. As a conclusion,

$$\forall s \in \mathfrak{S}_c, \qquad (\downarrow_{\mathfrak{S}} s) \subseteq_{fin} \mathfrak{S}_c. \quad (269)$$

For any $x$ and $y$ in $\mathfrak{S}_c$, $(x \sqcap_{\mathfrak{S}} y)$ exists in $\mathfrak{S}$, because $\mathfrak{S}$ is an inf semi-lattice ($\mathfrak{S}$ is pointed and bounded complete). Using property (269) and $(x \sqcap_{\mathfrak{S}} y) \sqsubseteq_{\mathfrak{S}} x$ implies $(x \sqcap_{\mathfrak{S}} y) \in \mathfrak{S}_c$. $\mathfrak{S}$ is then an arithmetic domain.

$$\forall x, y \in \mathfrak{S}_c, \qquad (x \sqcap_{\mathfrak{S}} y) \in \mathfrak{S}_c. \quad (270)$$

Consider, once again, $x$ and $y$ two compact elements of $\mathfrak{S}$ such that $\{x, y\}$ is upper-bounded in $\mathfrak{S}$. The join $(x \sqcup_{\mathfrak{S}} y)$ exists in $\mathfrak{S}$, since $\mathfrak{S}$ is consistently complete (as a Scott-domain). Let us then consider $E \subseteq_{Dir} \mathfrak{S}$ such that $(x \sqcup_{\mathfrak{S}} y) \sqsubseteq_{\mathfrak{S}} \bigsqcup_{\mathfrak{S}}^{\bar{\wedge}} E$. The compacity of $x$ and $y$, and the ordering relations $x \sqsubseteq_{\mathfrak{S}} \bigsqcup_{\mathfrak{S}}^{\bar{\wedge}} E$ and $y \sqsubseteq_{\mathfrak{S}} \bigsqcup_{\mathfrak{S}}^{\bar{\wedge}} E$, imply that it exists $x'$ and $y'$ in $E$ such that $x \sqsubseteq_{\mathfrak{S}} x'$ and $y \sqsubseteq_{\mathfrak{S}} y'$. The subset $E$ being directed, it exists $z \in E$ such that $x' \sqsubseteq_{\mathfrak{S}} z$ and $y' \sqsubseteq_{\mathfrak{S}} z$. Hence, it exists $z \in E$ such that $(x \sqcup_{\mathfrak{S}} y) \sqsubseteq_{\mathfrak{S}} z$. This proves that $(x \sqcup_{\mathfrak{S}} y)$ is compact. As a conclusion, $\forall s \in \mathfrak{S}_c$, $(\downarrow_{\mathfrak{S}} s)$ is a finite lattice included in $\mathfrak{S}_c$. □

**Theorem 29. [Selection structures as locally boolean qualitative domains]** If $(\mathfrak{S}, \sqsubseteq_{\mathfrak{S}})$ is a qualitative-domain satisfying the following additional property

$$\forall x, y \in \mathfrak{S}_c, \ (\bot_{\mathfrak{S}} \sqsubset_{\mathfrak{S}} x \sqsubset_{\mathfrak{S}} y) \Rightarrow (\exists z \in \mathfrak{S}_c, \bot_{\mathfrak{S}} \sqsubset_{\mathfrak{S}} z \sqsubset_{\mathfrak{S}} y \text{ and } z \parallel_{\mathfrak{S}} x), \quad (271)$$

then $(\mathfrak{S}, \sqsubseteq_{\mathfrak{S}})$ is necessarily a selection structure.
Conversely, if $(\mathfrak{S}, \sqsubseteq_{\mathfrak{S}})$ is a selection structure, then $(\mathfrak{S}, \sqsubseteq_{\mathfrak{S}})$ is necessarily a qualitative-domain satisfying the additional property (271). ■

*Remark* 29. The property (271) is equivalent to:

$$\forall x, y \in \mathfrak{S}_c, \qquad (\bot_{\mathfrak{S}} \sqsubset_{\mathfrak{S}} x \sqsubset_{\mathfrak{S}} y) \Rightarrow (\exists z \in \mathfrak{S}_c, \bot_{\mathfrak{S}} \sqsubset_{\mathfrak{S}} z \sqsubset_{\mathfrak{S}} y \text{ and } z \sqcap_{\mathfrak{S}} x = \bot_{\mathfrak{S}}). \quad (272)$$

Let us consider, $x_1, y_0 \in \mathfrak{S}_c$ with $\bot_{\mathfrak{S}} \sqsubset_{\mathfrak{S}} x_1 \sqsubset_{\mathfrak{S}} y_0$. From property (271), it exists $y_1 \in \mathfrak{S}_c$ such that $\bot_{\mathfrak{S}} \sqsubset_{\mathfrak{S}} y_1 \sqsubset_{\mathfrak{S}} y_0$ and $y_1 \parallel_{\mathfrak{S}} x_1$. Let us denote $x_2 := (y_1 \sqcap_{\mathfrak{S}} x_1)$. Obviously $\bot_{\mathfrak{S}} \sqsubseteq_{\mathfrak{S}} x_2$, $x_2 \sqsubset_{\mathfrak{S}} x_1$ and $x_2 \sqsubset_{\mathfrak{S}} y_1$. If $x_2$ is different from $\bot_{\mathfrak{S}}$, we can apply once again the property (271) to deduce that it exists $y_2 \in \mathfrak{S}_c$ such that $\bot_{\mathfrak{S}} \sqsubset_{\mathfrak{S}} y_2 \sqsubset_{\mathfrak{S}} y_1 \sqsubset_{\mathfrak{S}} y_0$ and $y_2 \parallel_{\mathfrak{S}} x_2$. We note that $(y_2 \sqcap_{\mathfrak{S}} x_2) = (y_2 \sqcap_{\mathfrak{S}} x_1)$, since $y_2 \sqsubset_{\mathfrak{S}} y_1$ and $x_2 = (y_1 \sqcap_{\mathfrak{S}} x_1)$. The subset $(\downarrow_{(\mathfrak{S}_c,\sqsubseteq_{\mathfrak{S}})} y_0)$ being finite, the strictly descending chains $(x_n)_{n\geq 1}$ and $(y_n)_{n\geq 1}$ are necessarily finite, and then $(x_n)_{n\geq 1}$ must terminate on $\bot_{\mathfrak{S}}$. This establishes the property (272).

*Remark* 30. Let us suppose that $(\mathfrak{S}, \sqsubseteq_{\mathfrak{S}})$ is a qualitative-domain satisfying the property (272), then $(\mathfrak{S}, \sqsubseteq_{\mathfrak{S}})$ satisfies the property

$$\forall s \in \mathfrak{S}_c, \qquad (\downarrow_{\mathfrak{S}} s) \text{ is a finite boolean lattice included in } \mathfrak{S}_c \quad (273)$$

and conversely.
Indeed, let us consider any $s \in \mathfrak{S}_c$ and any $t \in (\downarrow_{\mathfrak{S}} s)$. Using property (272), we know that $\{ w \in \mathfrak{S}_c \mid w \sqsubseteq_{\mathfrak{S}}$

$s$ and $w \sqcap_{\mathfrak{S}} t = \bot_{\mathfrak{S}} \} \neq \varnothing$. Then, using the definition property (251), we deduce that $(s \setminus t) = \bigsqcap_{\mathfrak{S}}^{\bar{\wedge}} \{ w \in \mathfrak{S}_c \mid w \sqsubseteq_{\mathfrak{S}} s$ and $w \sqcap_{\mathfrak{S}} t = \bot_{\mathfrak{S}} \}$ exists. Using property (266), we conclude that $(s \setminus t)$ is an element of $(\downarrow_{\mathfrak{S}} s)$. Properties of relative complementation proven in allow to conclude that $(s \setminus t) \sqcap_{\mathfrak{S}} t = \bot_{\mathfrak{S}}$ and $(s \setminus t) \sqcup_{\mathfrak{S}} t = s$. The lattice $(\downarrow_{\mathfrak{S}} s)$ is then boolean.
Conversely, if $(\mathfrak{S}, \sqsubseteq_{\mathfrak{S}})$ is a dI-domain satisfying the property (273), then $(\mathfrak{S}, \sqsubseteq_{\mathfrak{S}})$ trivially satisfies the property (272) (take for example $z := (y \setminus x)$).

*Proof.* Let us firstly consider the converse result. If $(\mathfrak{S}, \sqsubseteq_{\mathfrak{S}})$ is a selection structure, then $(\mathfrak{S}, \sqsubseteq_{\mathfrak{S}})$ is necessarily a dI-domain from Theorem 28. Moreover, the property (272) is trivially satisfied for a selection-structure (take for $z$ any atom in $\mathbb{A}(y) \setminus \mathbb{A}(x)$).

Let us now suppose that $(\mathfrak{S}, \sqsubseteq_{\mathfrak{S}})$ is a dI-domain satisfying the additional property (271). Let us then check every axioms of selection structures for $(\mathfrak{S}, \sqsubseteq_{\mathfrak{S}})$.
The chain-completeness property (227) is also a trivial consequence of directed-completeness property satisfied by the Scott-domain $\mathfrak{S}$.
This directed-completeness of $\mathfrak{S}$ supposes implicitly that $\mathfrak{S}$ is pointed ($\bot_{\mathfrak{S}} = \bigsqcap_{\mathfrak{S}}^{\bar{\wedge}} \varnothing$). The property (223) is then satisfied. We will denote $\bot_{\mathfrak{S}}$ the bottom element of $\mathfrak{S}$.
A Scott-domain is necessarily bounded complete. In particular, it is then an Inf semi-lattice. Hence, property (225) is satisfied by $\mathfrak{S}$.
From the algebraicity property of the Scott-domain $\mathfrak{S}$, we deduce that the meet-continuity property (249) is satisfied, and then *a fortiori* the weak-continuity property (229) is satisfied.
Let us now establish the property :

$$\forall x, y \in \mathfrak{S},\ \bot_{\mathfrak{S}} \sqsubset_{\mathfrak{S}} x \,\overline{\sqsubset}_{\mathfrak{S}}\, y \qquad \exists ! z \in \mathfrak{S}_c,\ \bot_{\mathfrak{S}} \,\overline{\sqsubset}_{\mathfrak{S}}\, z \sqsubset_{\mathfrak{S}} y \text{ and } z \sqcap_{\mathfrak{S}} x = \bot_{\mathfrak{S}}. \tag{274}$$

In the following, we will adopt the notation:

$$\forall x, y \in \mathfrak{S} \,|\, \bot_{\mathfrak{S}} \sqsubset_{\mathfrak{S}} x \sqsubset_{\mathfrak{S}} y, \qquad B_{x,y} := \{ z \in \mathfrak{S}_c \mid z \sqsubset_{\mathfrak{S}} y \text{ and } z \sqcap_{\mathfrak{S}} x = \bot_{\mathfrak{S}} \} \qquad B^*_{x,y} := B_{x,y} \setminus \{ \bot_{\mathfrak{S}} \}. \tag{275}$$

We note that, for any $x, y \in \mathfrak{S}$, we have

$$\forall x, y \in \mathfrak{S},\ \bot_{\mathfrak{S}} \sqsubset_{\mathfrak{S}} x \sqsubset_{\mathfrak{S}} y, \qquad B_{x,y} \text{ is a downset in } (\mathfrak{S}, \sqsubseteq_{\mathfrak{S}}). \tag{276}$$

Indeed, for any $v \in B_{x,y}$ and any $t \in \mathfrak{S}$, $t \sqsubseteq_{\mathfrak{S}} v$ implies $t \sqsubseteq_{\mathfrak{S}} v \sqsubset_{\mathfrak{S}} y$ and $t \sqcap_{\mathfrak{S}} x \sqsubseteq_{\mathfrak{S}} v \sqcap_{\mathfrak{S}} x = \bot_{\mathfrak{S}}$, i.e. $t \in B_{x,y}$.
We begin to prove the property (273) for $x, y \in \mathfrak{S}_c$.
First of all, from property (272), we know that $B^*_{x,y}$ is non-empty.

$$\forall x, y \in \mathfrak{S}_c, \,|\, \bot_{\mathfrak{S}} \sqsubset_{\mathfrak{S}} x \sqsubset_{\mathfrak{S}} y, \qquad B^*_{x,y} \neq \varnothing. \tag{277}$$

If $y \in \mathfrak{S}_c$, the set $B_{x,y}$ is finite, due to property (266). Hence, $B^*_{x,y}$ admits minimal elements. For any $z \in min_{\mathfrak{S}}(B^*_{x,y})$, $z$ is an atom of $\mathfrak{S}$, since $B_{x,y}$ is a finite downset in $\mathfrak{S}$.[26] In other words,

$$\forall x, y \in \mathfrak{S}_c, (\bot_{\mathfrak{S}} \sqsubset_{\mathfrak{S}} x \sqsubset_{\mathfrak{S}} y) \quad \Rightarrow \quad (\forall z \in min_{\mathfrak{S}}(B^*_{x,y}),\ \bot_{\mathfrak{S}} \,\overline{\sqsubset}_{\mathfrak{S}}\, z). \tag{278}$$

Let us now focus on the case $x \,\overline{\sqsubset}_{\mathfrak{S}}\, y$. For any $z \in B^*_{x,y}$, the supremum $(z \sqcup_{\mathfrak{S}} x)$ exists because $\mathfrak{S}_c$ is consistently complete and $y$ is a common upper-bound of $x$ and $z$. Moreover,

$$\forall x, y \in \mathfrak{S}_c \,|\, x \,\overline{\sqsubset}_{\mathfrak{S}}\, y, \qquad \forall z \in B^*_{x,y}, (z \sqcup_{\mathfrak{S}} x) = y. \tag{279}$$

Indeed, by definition, we know that $z \sqcup_{\mathfrak{S}} x \sqsubseteq_{\mathfrak{S}} y$, but $z \sqcup_{\mathfrak{S}} x = x$ is excluded because $z \sqcap_{\mathfrak{S}} x = \bot_{\mathfrak{S}}$ would imply $z = \bot_{\mathfrak{S}}$, and then, $x \,\overline{\sqsubset}_{\mathfrak{S}}\, y$ imposes $z \sqcup_{\mathfrak{S}} x = y$.

[26] Indeed, let us consider $z \in min_{\mathfrak{S}}(B^*_{x,y})$ and $w$ in $\mathfrak{S}$ such that $\bot_{\mathfrak{S}} \sqsubset_{\mathfrak{S}} w \sqsubset_{\mathfrak{S}} z$. $w$ is necessarily compact (property (266)), we have also $\bot_{\mathfrak{S}} \sqsubset_{\mathfrak{S}} w \sqsubset_{\mathfrak{S}} y$ and $z \sqcap_{\mathfrak{S}} x = \bot_{\mathfrak{S}}$, hence $w \in B^*_{x,y}$ which is contradictory to the minimality requirement on $z$.

This property suffices to establish the uniqueness of $z \in min_{\mathfrak{S}}(B^*_{x,y})$, in the case $x \,\overline{\sqsubset}_{\mathfrak{S}} y$. Indeed, if we consider $a_1, a_2 \in min_{\mathfrak{S}}(B^*_{x,y})$ and suppose $a_1 \neq a_2$, we would then have $\bot_{\mathfrak{S}} \sqsubseteq_{\mathfrak{S}} (a_1 \sqcap_{\mathfrak{S}} a_2) \sqsubset_{\mathfrak{S}} a_1$ and then $(a_1 \sqcap_{\mathfrak{S}} a_2) = \bot_{\mathfrak{S}}$, since $\bot_{\mathfrak{S}} \,\overline{\sqsubset}_{\mathfrak{S}} a_1$. Moreover, using distributivity property and $a_1 \sqcap_{\mathfrak{S}} x = a_2 \sqcap_{\mathfrak{S}} x = y$, we would then deduce $x = \bot_{\mathfrak{S}} \sqcup_{\mathfrak{S}} x = (a_1 \sqcap_{\mathfrak{S}} a_2) \sqcup_{\mathfrak{S}} x = a_1 \sqcup_{\mathfrak{S}} x = y$, which is false. As a result

$$\forall x, y \in \mathfrak{S}_c,\ \bot_{\mathfrak{S}} \sqsubset_{\mathfrak{S}} x \,\overline{\sqsubset}_{\mathfrak{S}} y, \qquad \exists! z \in \mathfrak{S}_c,\ \bot_{\mathfrak{S}} \,\overline{\sqsubset}_{\mathfrak{S}} z \sqsubset_{\mathfrak{S}} y \text{ and } z \sqcap_{\mathfrak{S}} x = \bot_{\mathfrak{S}}. \tag{280}$$

Let us now consider $u, v \in \mathfrak{S}$ with $\bot_{\mathfrak{S}} \sqsubset_{\mathfrak{S}} u \,\overline{\sqsubset}_{\mathfrak{S}} v$, but now we will assume that $v \notin \mathfrak{S}_c$. The algebraicity of $\mathfrak{S}$ implies $\exists y \in \mathfrak{S}_c$ such that $y \sqsubseteq_{\mathfrak{S}} v$ and $y \not\sqsubseteq_{\mathfrak{S}} u$. The element $v \in \mathfrak{S}$ being non-compact, we have then $y \sqsubset_{\mathfrak{S}} v$. Moreover, the properties $\bot_{\mathfrak{S}} \sqsubset_{\mathfrak{S}} u \,\overline{\sqsubset}_{\mathfrak{S}} v$ and $y \not\sqsubseteq_{\mathfrak{S}} u$ impose $y \parallel_{\mathfrak{S}} u$, i.e. $\bot_{\mathfrak{S}} \sqsubset_{\mathfrak{S}} y$ and $(y \sqcap_{\mathfrak{S}} u) \sqsubset_{\mathfrak{S}} y$. Using property (269) and $(y \sqcap_{\mathfrak{S}} u) \sqsubset_{\mathfrak{S}} y \in \mathfrak{S}_c$, we deduce that $(y \sqcap_{\mathfrak{S}} u)$ is in $\mathfrak{S}_c$. We distinguish two subcases:

- If $(y \sqcap_{\mathfrak{S}} u) = \bot_{\mathfrak{S}}$, we have then found $z := y \in \mathfrak{S}_c$ such that $\bot_{\mathfrak{S}} \sqsubset_{\mathfrak{S}} z \sqsubset_{\mathfrak{S}} v$ and $z \sqcap_{\mathfrak{S}} u = \bot_{\mathfrak{S}}$, i.e. $B^*_{u,v}$ is not empty.
- If $(y \sqcap_{\mathfrak{S}} u) \neq \bot_{\mathfrak{S}}$, we use property (272) to deduce that it exists $z \in \mathfrak{S}_c$ such that $\bot_{\mathfrak{S}} \sqsubset_{\mathfrak{S}} z \sqsubset_{\mathfrak{S}} y$ and $z \sqcap_{\mathfrak{S}} (y \sqcap_{\mathfrak{S}} u) = \bot_{\mathfrak{S}}$, and then *a fortiori* $\bot_{\mathfrak{S}} \sqsubset_{\mathfrak{S}} z \sqsubset_{\mathfrak{S}} v$ and $z \sqcap_{\mathfrak{S}} u = \bot_{\mathfrak{S}}$, i.e. $B^*_{u,v}$ is not empty.

From properties (269) and (276), we know that $\forall s \in B^*_{u,v}$, $(B^*_{u,v} \cap (\downarrow s))$ is a finite and non-empty subset in $(\mathfrak{S}, \sqsubseteq_{\mathfrak{S}})$. Hence, for any $s \in B^*_{u,v}$, $(B^*_{u,v} \cap (\downarrow s))$ admits minimal elements. Note that any minimal element of $(B^*_{u,v} \cap (\downarrow s))$ is also a minimal element of $B^*_{u,v}$. As a result,

$$\forall u, v \in \mathfrak{S},\ \bot_{\mathfrak{S}} \sqsubset_{\mathfrak{S}} u \,\overline{\sqsubset}_{\mathfrak{S}} v, \qquad \forall s \in B^*_{u,v},\ \exists t \in min_{\mathfrak{S}}(B^*_{u,v}) \ \mid\ t \sqsubseteq_{\mathfrak{S}} s. \tag{281}$$

As shown before, for any $z \in min_{\mathfrak{S}}(B^*_{u,v} \cap (\downarrow s))$, $z$ is an atom of $\mathfrak{S}$. Indeed, let us consider $z \in min_{\mathfrak{S}}(B^*_{x,y})$ and $w$ in $\mathfrak{S}$ such that $\bot_{\mathfrak{S}} \sqsubset_{\mathfrak{S}} w \sqsubset_{\mathfrak{S}} z$. $w$ is necessarily compact (property (266)), and satisfies $\bot_{\mathfrak{S}} \sqsubset_{\mathfrak{S}} w \sqsubset_{\mathfrak{S}} z \sqsubset_{\mathfrak{S}} v$ and $w \sqcap_{\mathfrak{S}} u \sqsubseteq_{\mathfrak{S}} z \sqcap_{\mathfrak{S}} u = \bot_{\mathfrak{S}}$, hence $w \in B^*_{x,y}$ which is contradictory with the minimality requirement on $z$. To summarize:

$$\forall u, v \in \mathfrak{S},\ \bot_{\mathfrak{S}} \sqsubset_{\mathfrak{S}} u \,\overline{\sqsubset}_{\mathfrak{S}} v, \qquad B^*_{u,v} \neq \varnothing \text{ and } \forall z \in min_{\mathfrak{S}}(B^*_{u,v}),\ \bot_{\mathfrak{S}} \,\overline{\sqsubset}_{\mathfrak{S}} z. \tag{282}$$

For any $y \in B^*_{u,v}$, the supremum $(u \sqcup_{\mathfrak{S}} y)$ exists in $\mathfrak{S}$, since $\mathfrak{S}$ is consistently-complete and $v$ is a common upper-bound for $u$ and $y$. For any $y \in B^*_{u,v}$, the property $y \sqcap_{\mathfrak{S}} u = \bot_{\mathfrak{S}}$ means $y \parallel_{\mathfrak{S}} u$ and then $u \sqsubset_{\mathfrak{S}} (u \sqcup_{\mathfrak{S}} y)$. We have also, $(u \sqcup_{\mathfrak{S}} y) \sqsubseteq_{\mathfrak{S}} v$ because $v$ is a common upper-bound of $u$ and $y$. The property $u \,\overline{\sqsubset}_{\mathfrak{S}} v$ implies then $u \sqcup_{\mathfrak{S}} y = v$. As a result,

$$\forall u, v \in \mathfrak{S},\ \bot_{\mathfrak{S}} \sqsubset_{\mathfrak{S}} u \,\overline{\sqsubset}_{\mathfrak{S}} v, \qquad \forall y \in B^*_{u,v},\ u \sqcup_{\mathfrak{S}} y = v. \tag{283}$$

Let us consider $y_1$ and $y_2$ elements of $min_{\mathfrak{S}}(B^*_{u,v} \cap (\downarrow s))$. From property (282), we recall that $(y_1 \neq y_2)$ implies $(y_1 \sqcap_{\mathfrak{S}} y_2) = \bot_{\mathfrak{S}}$. However, this fact contradicts the distributivity property in $\mathfrak{S}_c$. Indeed, if we define the following map

$$\alpha : (\downarrow_{(\mathfrak{S}_c, \sqsubseteq_{\mathfrak{S}})} u)^{\times 2} \to \mathfrak{S}_c,\ (c_1, c_2) \mapsto (y_1 \sqcup_{\mathfrak{S}} c_1) \sqcap_{\mathfrak{S}} (y_2 \sqcup_{\mathfrak{S}} c_2).$$

$\alpha$ is obviously a monotone net and then, using successively the property (283), two times the meet-continuity of $\mathfrak{S}$, [5, Proposition 2.1.12], distributivity property, once again meet-continuity of $\mathfrak{S}$, and the property $(y_1 \sqcap_{\mathfrak{S}} y_2) = \bot_{\mathfrak{S}}$, we obtain the following contradiction:

$$\begin{aligned}
v &= (y_1 \sqcup_{\mathfrak{S}} u) \sqcap_{\mathfrak{S}} (y_2 \sqcup_{\mathfrak{S}} u) = \\
&= \bigsqcap^{\overline{\wedge}}_{c_1 \in \mathfrak{S}_c \cap \downarrow u} ((y_1 \sqcup_{\mathfrak{S}} c_1) \sqcap_{\mathfrak{S}} (y_2 \sqcup_{\mathfrak{S}} u)) = \\
&= \bigsqcap^{\overline{\wedge}}_{c_1 \in \mathfrak{S}_c \cap \downarrow u} \bigsqcap^{\overline{\wedge}}_{c_2 \in \mathfrak{S}_c \cap \downarrow u} ((y_1 \sqcup_{\mathfrak{S}} c_1) \sqcap_{\mathfrak{S}} (y_2 \sqcup_{\mathfrak{S}} c_2)) = \\
&= \bigsqcap^{\overline{\wedge}}_{c \in \mathfrak{S}_c \cap \downarrow u} ((y_1 \sqcup_{\mathfrak{S}} c) \sqcap_{\mathfrak{S}} (y_2 \sqcup_{\mathfrak{S}} c)) = \\
&= \bigsqcap^{\overline{\wedge}}_{c \in \mathfrak{S}_c \cap \downarrow u} ((y_1 \sqcap_{\mathfrak{S}} y_2) \sqcup_{\mathfrak{S}} c) = \\
&= ((y_1 \sqcap_{\mathfrak{S}} y_2) \sqcup_{\mathfrak{S}} u) = \\
&= \bot_{\mathfrak{S}} \sqcup_{\mathfrak{S}} u = u.
\end{aligned}$$

We then conclude $Card(min_{\mathfrak{S}}(B^*_{u,v})) = 1$. Conversely, if $y$ is an element of $\mathfrak{S}$ such that $\bot_{\mathfrak{S}} \overline{\sqsubset}_{\mathfrak{S}} y \sqsubset_{\mathfrak{S}} v$ and $u \parallel_{\mathfrak{S}} y$, we have necessarily $u \sqcap_{\mathfrak{S}} y = \bot_{\mathfrak{S}}$ and $y \in \mathfrak{S}_c$[27], i.e. $y \in min(B^*_{u,v})$. As a conclusion,

$$\forall u, v \in \mathfrak{S}, v \notin \mathfrak{S}_c \mid u \overline{\sqsubset}_{\mathfrak{S}} v, \qquad \exists! y \in \mathfrak{S} \mid (\bot_{\mathfrak{S}} \overline{\sqsubset}_{\mathfrak{S}} y \sqsubset_{\mathfrak{S}} v \text{ and } u \parallel_{\mathfrak{S}} y). \tag{284}$$

This concludes the proof of the announced property.
We now intent to prove the final result

$$\forall b, u, v \in \mathfrak{S} \mid b \sqsubset_{\mathfrak{S}} u \overline{\sqsubset}_{\mathfrak{S}} v, \qquad \exists! r \mid (b \overline{\sqsubset}_{\mathfrak{S}} r \sqsubseteq_{\mathfrak{S}} v \text{ and } r \sqcap_{\mathfrak{S}} u = b). \tag{285}$$

Let us begin to show the existence of $r$ in previous property.
From property (273), it exists an element $y \in \mathfrak{S}$ such that $\bot_{\mathfrak{S}} \overline{\sqsubset}_{\mathfrak{S}} y \sqsubset_{\mathfrak{S}} v$ ($y$ is atomic and then compact) and $u \sqcap_{\mathfrak{S}} y = \bot_{\mathfrak{S}}$. We denote $r := b \sqcup_{\mathfrak{S}} y$, which exists since $\mathfrak{S}$ is consistently complete and $v$ is a common upper-bound of $b$ and $y$.
The following simple property

$$\forall b, y, v \in \mathfrak{S} \qquad (\bot_{\mathfrak{S}} \overline{\sqsubset}_{\mathfrak{S}} y \sqsubset_{\mathfrak{S}} v \text{ and } \bot_{\mathfrak{S}} \sqsubseteq_{\mathfrak{S}} b \sqsubset_{\mathfrak{S}} v \text{ and } b \sqcap_{\mathfrak{S}} y = \bot_{\mathfrak{S}}) \Rightarrow (b \overline{\sqsubset}_{\mathfrak{S}} (b \sqcup_{\mathfrak{S}} y) \sqsubseteq_{\mathfrak{S}} v) \tag{286}$$

is a simple consequence of the distributivity property satisfied by $\mathfrak{S}$. Indeed, let us assume that it exists $x \in \mathfrak{S}$ with $b \sqsubset_{\mathfrak{S}} x \sqsubset_{\mathfrak{S}} (b \sqcup_{\mathfrak{S}} y)$ and let us exhibit a contradiction.
$b \sqsubset_{\mathfrak{S}} x$ implies $(b \sqcup_{\mathfrak{S}} y) \sqsubseteq_{\mathfrak{S}} (x \sqcup_{\mathfrak{S}} y)$, and $x \sqsubset_{\mathfrak{S}} (b \sqcup_{\mathfrak{S}} y)$ implies $(x \sqcup_{\mathfrak{S}} y) \sqsubseteq_{\mathfrak{S}} (b \sqcup_{\mathfrak{S}} y) \sqcup_{\mathfrak{S}} y = (b \sqcup_{\mathfrak{S}} y)$. As a result, we have necessarily $(x \sqcup_{\mathfrak{S}} y) = (b \sqcup_{\mathfrak{S}} y)$.
We have also $(x \sqcap_{\mathfrak{S}} y) = \bot_{\mathfrak{S}}$. Indeed, we have always $\bot_{\mathfrak{S}} \sqsubseteq_{\mathfrak{S}} (x \sqcap_{\mathfrak{S}} y) \sqsubseteq_{\mathfrak{S}} y$, but $(x \sqcap_{\mathfrak{S}} y) = y$ would mean $y \sqsubseteq_{\mathfrak{S}} x$ and then $(x \sqcup_{\mathfrak{S}} y) = x$ which is contradictory with $x \sqsubset_{\mathfrak{S}} (x \sqcup_{\mathfrak{S}} y) = (b \sqcup_{\mathfrak{S}} y)$. Now we use $\bot_{\mathfrak{S}} \overline{\sqsubset}_{\mathfrak{S}} y$ to conclude that $(x \sqcap_{\mathfrak{S}} y) = \bot_{\mathfrak{S}}$.
Endly, using distributivity of $\mathfrak{S}$, we have $b = b \sqcup_{\mathfrak{S}} \bot_{\mathfrak{S}} = b \sqcup_{\mathfrak{S}} (x \sqcap_{\mathfrak{S}} y) = (b \sqcup_{\mathfrak{S}} x) \sqcap_{\mathfrak{S}} (b \sqcup_{\mathfrak{S}} y) = x \sqcap_{\mathfrak{S}} (b \sqcup_{\mathfrak{S}} y) = (b \sqcup_{\mathfrak{S}} y)$, i.e. $y \sqsubseteq_{\mathfrak{S}} b$ which is false since $y \sqcap_{\mathfrak{S}} b = \bot_{\mathfrak{S}}$ and $y$ is an atom.
This concludes the proof of property (286).
Now, we can use the property $u \sqcap_{\mathfrak{S}} y = \bot_{\mathfrak{S}}$ to prove that $u \parallel_{\mathfrak{S}} r$, and even that $u \sqcap_{\mathfrak{S}} r = b$. Using distributivity of $\mathfrak{S}$, we have indeed $u \sqcap_{\mathfrak{S}} r = (u \sqcap_{\mathfrak{S}} b) \sqcup_{\mathfrak{S}} (u \sqcap_{\mathfrak{S}} y) = b \sqcup_{\mathfrak{S}} \bot_{\mathfrak{S}} = b$ (here, we have used $b \sqsubset_{\mathfrak{S}} u$ and $(u \sqcap_{\mathfrak{S}} y) = \bot_{\mathfrak{S}}$).
We have then established the existence of $r$ in property (285). Note that $r$ satisfies $(r \sqcup_{\mathfrak{S}} u) = v$ (the properties $r \sqsubset_{\mathfrak{S}} v$ and $u \sqsubset_{\mathfrak{S}} v$ implies $u \sqsubseteq_{\mathfrak{S}} (r \sqcup_{\mathfrak{S}} u) \sqsubseteq_{\mathfrak{S}} v$, however $(r \sqcup_{\mathfrak{S}} u) = u$ is excluded by the property $u \parallel_{\mathfrak{S}} r$, and $u \overline{\sqsubset}_{\mathfrak{S}} v$ leads to the conclusion).
It is now easy to show the uniqueness of this element $r$. Let us suppose that we have $r_1$ and $r_2$ distinct elements of $\mathfrak{S}$ satisfying property (285). We would then have $(r_1 \sqcap_{\mathfrak{S}} r_2) = b$ and $u \sqcup_{\mathfrak{S}} r_1 = u \sqcup_{\mathfrak{S}} r_2 = v$. However, the distributivity of $\mathfrak{S}$ would imply $b = u \sqcup_{\mathfrak{S}} b = u \sqcup_{\mathfrak{S}} (r_1 \sqcap_{\mathfrak{S}} r_2) = (u \sqcup_{\mathfrak{S}} r_1) \sqcap_{\mathfrak{S}} (u \sqcup_{\mathfrak{S}} r_2) = v$, which is false. The unicity of $r$ in property (285) is then established, which concludes the proof of property (226).
The last property to verify is the property (224). Let $x$ and $y$ be elements of $\mathfrak{S}_c$ such that $x \sqsubset_{\mathfrak{S}} y$. From properties (277) and (278), it exists an atom $z$ such that $z \sqsubset_{\mathfrak{S}} y$ and $(x \sqcap_{\mathfrak{S}} z) = \bot_{\mathfrak{S}}$. Using now the property (286), we deduce that $x \overline{\sqsubset}_{\mathfrak{S}} (x \sqcup_{\mathfrak{S}} z) \sqsubseteq_{\mathfrak{S}} y$.
Let $x$ be an element of $\mathfrak{S}_c$, $y$ be an element of $\mathfrak{S}$ but non-compact, and let us assume $x \sqsubset_{\mathfrak{S}} y$. From algebraicity of $\mathfrak{S}$, we know that it exists $w \in \mathfrak{S}_c$ such that $w \sqsubseteq_{\mathfrak{S}} y$ (in fact, $w \sqsubset_{\mathfrak{S}} y$ because $w$ is compact and $y$ is non-compact) and $w \nleq x$. Then, necessarily $x \sqsubset_{\mathfrak{S}} (x \sqcup_{\mathfrak{S}} w) \sqsubseteq_{\mathfrak{S}} y$ and $y' := (x \sqcup_{\mathfrak{S}} w)$ is compact (in fact, $(x \sqcup_{\mathfrak{S}} w) \sqsubset_{\mathfrak{S}} y$ because $x$ and $z$ are compact, but $y$ is non-compact). We are now in the previous case, and it exists an atom $z$ such that $z \sqsubset_{\mathfrak{S}} y'$ and $(x \sqcap_{\mathfrak{S}} z) = \bot_{\mathfrak{S}}$ and $x \overline{\sqsubset}_{\mathfrak{S}} (x \sqcup_{\mathfrak{S}} z) \sqsubseteq_{\mathfrak{S}} y' \sqsubset_{\mathfrak{S}} y$.
Endly, let us consider $x$ and $y$ be elements of $\mathfrak{S}$ such that $x \sqsubset_{\mathfrak{S}} y$ and let us assume that $x$ is non-compact. From algebraicity of $\mathfrak{S}$, we know that it exists $w \in \mathfrak{S}_c$ such that $w \sqsubseteq_{\mathfrak{S}} y$ (in fact, $w \sqsubset_{\mathfrak{S}} y$ because $w$ is compact and $y$ is non-compact) and $w \nleq x$. $w$ being compact and $x$ non-compact, we have necessarily $w \parallel_{\mathfrak{S}} x$. Then, $(w \sqcap_{\mathfrak{S}} x)$ is compact (using $(w \sqcap_{\mathfrak{S}} x) \sqsubset_{\mathfrak{S}} w$ and property (269)) and satisfies

[27] Indeed, $\forall E \subseteq_{Dir} \mathfrak{S} \mid y \sqsubseteq_{\mathfrak{S}} (\bigsqcup^{\uparrow}_{\mathfrak{S}} E)$, we have $y = y \sqcup_{\mathfrak{S}} (\bigsqcup^{\uparrow}_{\mathfrak{S}} E) = \bigsqcup^{\uparrow}_{\mathfrak{S}} \{ (y \sqcap_{\mathfrak{S}} e) \mid e \in E \}$. Hence, if we had $\forall e \in E, y \nleq e$, we would have $\forall e \in E, y \sqcap_{\mathfrak{S}} e = \bot_{\mathfrak{S}}$, and then $y = \bot_{\mathfrak{S}}$, which is contradictory.

$(w \sqcap_{\mathfrak{S}} x) \sqsubset_{\mathfrak{S}} w$. We can then find an atom $z$ such that $z \sqsubset_{\mathfrak{S}} w$ and $(x \sqcap_{\mathfrak{S}} z) = ((w \sqcap_{\mathfrak{S}} x) \sqcap_{\mathfrak{S}} z) = \bot_{\mathfrak{S}}$. The property $x \,\overline{\sqsubset}_{\mathfrak{S}} (x \sqcup_{\mathfrak{S}} z) \sqsubseteq_{\mathfrak{S}} y$ has been proved in property (286).
The case study is now finished, and we have obtained

$$\forall x, y \in \mathfrak{S}, \qquad (x \sqsubset_{\mathfrak{S}} y) \;\Rightarrow\; (\exists u \in \mathfrak{S} \,|\, x \,\overline{\sqsubset}_{\mathfrak{S}} u \sqsubset_{\mathfrak{S}} y). \tag{287}$$

This concludes the proof. □

Sub-selection structures are defined in Definitions 1 and 3.

**Lemma 70. [Characterization of sub-selection structures]**
Let $(\mathfrak{S}, \sqsubseteq)$ be a selection structure, and let us consider $(\mathfrak{S}', \sqsubseteq)$ a sub-poset of $(\mathfrak{S}, \sqsubseteq)$.
We have the following equivalences:
$((\mathfrak{S}', \sqsubseteq)$ is a sub-selection structure of $(\mathfrak{S}, \sqsubseteq))$ iff ($\mathfrak{S}'$ is a Scott-closed subset of $(\mathfrak{S}, \sqsubseteq)$), i.e. (1) $\mathfrak{S}'$ is a lower-set of $\mathfrak{S}$, and (2) for any $E \subseteq_{Dir} \mathfrak{S}'$, the directed supremum $\bigsqcup^{\overline{\wedge}}_{\mathfrak{S}} E$ is an element of $\mathfrak{S}'$.
$((\mathfrak{S}', \sqsubseteq)$ is an idealized sub-selection structure of $(\mathfrak{S}, \sqsubseteq))$ iff ($\mathfrak{S}'$ is a Scott-ideal of $(\mathfrak{S}, \sqsubseteq)$), i.e. (1) $\mathfrak{S}'$ is a Scott-closed subset of $\mathfrak{S}$, (2) if a finite subset of $\mathfrak{S}'$ is upper-bounded in $\mathfrak{S}$, it admits also an upper-bound in $\mathfrak{S}'$ (Scott-ideals are called 'stable subdomains' in [43]). ■

*Proof.* Let us suppose that $(\mathfrak{S}, \sqsubseteq)$ is a selection structure, and then an algebraic Scott-domain. Let us consider a sub-poset of $(\mathfrak{S}, \sqsubseteq)$ denoted $(\mathfrak{S}', \sqsubseteq)$ which is Scott-closed in $\mathfrak{S}$. Let us then check the different axioms of selection structures (i.e. Definition 4) for $(\mathfrak{S}', \sqsubseteq)$.
$\mathfrak{S}'$ is closed under chain suprema, hence the chain supremum in $\mathfrak{S}'$ of the empty set, i.e. $\bot_{\mathfrak{S}'}$, is in $\mathfrak{S}'$. As a remark, $\bot_{\mathfrak{S}'} = \bot_{\mathfrak{S}}$, because $\mathfrak{S}'$ contains $\bot_{\mathfrak{S}}$ as a downset of $\mathfrak{S}$.
$\mathfrak{S}'$ being a downset of $\mathfrak{S}$, if $s_1, s_2$ are in $\mathfrak{S}'$, then all common lower-bounds of $s_1$ and $s_2$ are in $\mathfrak{S}'$ with the same inherited ordering as in $\mathfrak{S}$, then $(s_1 \sqcap_{\mathfrak{S}'} s_2) = (s_1 \sqcap_{\mathfrak{S}} s_2)$ is also in $\mathfrak{S}'$.
For any $x, y \in \mathfrak{S}'$ with $x \sqsubset y$, the strong-atomicity of $\mathfrak{S}$ ensures that it exists $t \in \mathfrak{S}$ with $x \,\overline{\sqsubset}\, t \sqsubseteq_{\mathfrak{S}} y$. $\mathfrak{S}'$ being a downset of $\mathfrak{S}$, $t$ is in $\mathfrak{S}'$. Any element $t' \in \mathfrak{S}' \subseteq \mathfrak{S}$ such that $x \sqsubset t' \sqsubseteq_{\mathfrak{S}} t$ has to be also in $\mathfrak{S}$ and then $t'$ must be equal to $x$ because of the property $x \,\overline{\sqsubset}\, t$ in $\mathfrak{S}$. Hence, $\mathfrak{S}'$ is also strongly atomic.
Let us consider $b, u, v \in \mathfrak{S}' \subseteq \mathfrak{S}$ with $b \sqsubset u \,\overline{\sqsubset}\, v$. It exists a unique $r \in \mathfrak{S}$ with $b \,\overline{\sqsubset}\, r \sqsubset v$ and $r \parallel u$. $\mathfrak{S}'$ being a downset of $\mathfrak{S}$, $r \sqsubset v \in \mathfrak{S}'$ implies $r \in \mathfrak{S}'$. Moreover, $\mathfrak{S}'$ being a downset of $\mathfrak{S}$, if we have $r \parallel u$ in $\mathfrak{S}$ it is also necessarily true in $\mathfrak{S}'$.
For any chain $C \subseteq_{chain} \mathfrak{S}'$, $C$ is also trivially a directed subset of $\mathfrak{S}'$, and then $(\bigsqcup_{\mathfrak{S}} C) \in \mathfrak{S}'$ exists and is in $\mathfrak{S}'$, because $\mathfrak{S}'$ is closed under directed suprema in $\mathfrak{S}$ as a Scott-closed subset of $\mathfrak{S}$.
For any $s, s' \in \mathfrak{S}'$ with $s \,\overline{\sqsubset}\, s'$, and any chain $C \subseteq_{chain} \mathfrak{S}'$, we know that $(\bigsqcup_{\mathfrak{S}} C)$ exists in $\mathfrak{S}'$ and $s' \sqcap_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} C)$ exists in $\mathfrak{S}'$. Hence, $(\forall c \in C, s = (s' \sqcap_{\mathfrak{S}} c)) \Rightarrow (s = s' \sqcap_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} C)$.
As a conclusion, $(\mathfrak{S}', \sqsubseteq)$ satisfies the axioms of a selection-structure. $\mathfrak{S}'$ being moreover a downset of $\mathfrak{S}$, we conclude that $(\mathfrak{S}', \sqsubseteq)$ is a sub-selection structure of $(\mathfrak{S}, \sqsubseteq)$.
Conversely, let us consider a selection structure $(\mathfrak{S}, \sqsubseteq)$, and $(\mathfrak{S}', \sqsubseteq)$ a sub-(selection-structure) of $\mathfrak{S}$. By definition we already know that $\mathfrak{S}'$ is a downset of $(\mathfrak{S}, \sqsubseteq)$. Hence, in order to show that $\mathfrak{S}'$ is Scott-closed in $(\mathfrak{S}, \sqsubseteq)$ it suffices to prove that $\mathfrak{S}'$ is closed under directed-suprema in $(\mathfrak{S}, \sqsubseteq)$, i.e.

$$\forall D \subseteq_{Dir} \mathfrak{S}' \qquad (\bigsqcup^{\overline{\wedge}}_{\mathfrak{S}} D) \in \mathfrak{S}'. \tag{288}$$

We firstly note that $(\bigsqcup^{\overline{\wedge}}_{\mathfrak{S}} D)$ because the Scott-domain $(\mathfrak{S}, \sqsubseteq)$ is directed complete.
Moreover, by definition, $(\mathfrak{S}', \sqsubseteq)$ is itself a selection structure, and then a Scott-domain. Hence, $(\bigsqcup^{\overline{\wedge}}_{\mathfrak{S}'} D)$ exists and

$$\forall D \subseteq_{Dir} \mathfrak{S}' \qquad (\bigsqcup^{\overline{\wedge}}_{\mathfrak{S}'} D) \in \mathfrak{S}'.$$

Now, we obviously have $(\bigsqcup^{\overline{\wedge}}_{\mathfrak{S}} D) \sqsubseteq_{\mathfrak{S}} (\bigsqcup^{\overline{\wedge}}_{\mathfrak{S}'} D)$, because $\mathfrak{S}' \subseteq \mathfrak{S}$. Let us then define the following subset of $\mathfrak{S}'$ for any $D \subseteq_{Dir} \mathfrak{S}'$

$$U_D := \{ t \in \mathfrak{S}' \mid \forall d \in D,\ (t \sqcup_{\mathfrak{S}'} d) \text{ exists in } \mathfrak{S}' \text{ and } (t \sqcup_{\mathfrak{S}'} d) \sqsubseteq_{\mathfrak{S}} (\bigsqcup^{\overline{\wedge}}_{\mathfrak{S}} D) \}. \tag{289}$$

We observe that $\bot_{\mathfrak{S}} = \bot_{\mathfrak{S}'} \in U_D$, and then $U$ is not empty.
Let us fix $u \in U_D$ and $d \in D$.
For any $d' \in D$, it exists a common upper-bound $d''$ of $d$ and $d'$, with $d'' \in D$ because $D$ is directed. By definition of $D$, for any $d, d' \in D$, they have a common upper-bound in $D$ denoted for example $d''$, and then, using consistent completeness of $\mathfrak{S}'$, we deduce that $(d \sqcup_{\mathfrak{S}'} d')$ exists in $\mathfrak{S}'$ and is less than $d''$ and *a fortiori* less than $(\bigsqcup_{\mathfrak{S}} D)$ because $d'' \in D$.
It is obvious that $(u \sqcup_{\mathfrak{S}'} d'')$ is a common upper-bound of $d' \in \mathfrak{S}'$ and $(u \sqcup_{\mathfrak{S}'} d) \in \mathfrak{S}'$. Then, consistent completeness of $\mathfrak{S}'$ implies $((u \sqcup_{\mathfrak{S}'} d) \sqcup_{\mathfrak{S}'} d')$ exists and is in $\mathfrak{S}'$. Moreover, $((u \sqcup_{\mathfrak{S}'} d) \sqcup_{\mathfrak{S}'} d') = (u \sqcup_{\mathfrak{S}'} (d \sqcup_{\mathfrak{S}'} d')) \sqsubseteq_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} D)$ because $u \in U_D$ implies $u \sqsubseteq_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} D)$ and we have already shown that $(d \sqcup_{\mathfrak{S}'} d') \sqsubseteq_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} D)$. Hence, $(u \sqcup_{\mathfrak{S}'} d) \in U_D$ for any $u \in U_D$ and $d \in D$.
Let us now consider $C \subseteq^{\neq \varnothing}_{chain} U_D$ and fix $d \in D$. Let us denote $C' := \{ (c \sqcup_{\mathfrak{S}'} d) \mid c \in C \}$. Previous result, i.e. $\forall u \in U_D, \forall d \in D, (u \sqcup_{\mathfrak{S}'} d) \in U$ implies $C' \subseteq^{\neq \varnothing}_{chain} U_D$.
As soon as $C' \subseteq^{\neq \varnothing}_{chain} U_D$, we have $\forall c' \in C'$, $(c' \sqcup d) \sqsubseteq_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} D)$ for any $d \in D$, and then $c' \sqsubseteq_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} D)$. Hence, $(\bigsqcup_{\mathfrak{S}'} C') \sqsubseteq_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} D)$.
Chain completeness property of $\mathfrak{S}'$ (because $\mathfrak{S}'$ is a selection structure) implies $(\bigsqcup_{\mathfrak{S}'} C)$ and $(\bigsqcup_{\mathfrak{S}'} C')$ exist in $\mathfrak{S}'$.
$(\bigsqcup_{\mathfrak{S}'} C')$ is a common upper-bound of $d$ and $(\bigsqcup_{\mathfrak{S}'} C)$. Hence, consistent completeness of $\mathfrak{S}'$ implies $(d \sqcup_{\mathfrak{S}'} (\bigsqcup_{\mathfrak{S}'} C))$ exists in $\mathfrak{S}'$ and $(d \sqcup_{\mathfrak{S}'} (\bigsqcup_{\mathfrak{S}'} C)) \sqsubseteq_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}'} C') \sqsubseteq_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} D)$. Hence, $(\bigsqcup_{\mathfrak{S}'} C)$ is in $U_D$.
Using Zorn's lemma, we conclude that $U_D$ admits some maximal elements. Let us denote by $M$ a maximal element of $U_D$. We have in particular, $M \sqsubseteq_{\mathfrak{S}} (\bigsqcup^{\bar{\wedge}}_{\mathfrak{S}} D)$. However, $\forall d \in D$, $(M \sqcup d) \in U_D$ implies $\forall d \in D$, $(M \sqcup d) \sqsubseteq_{\mathfrak{S}} M$ and then $\forall d \in D$, $d \sqsubseteq_{\mathfrak{S}} M$. As a conclusion, we obtain $(\bigsqcup^{\bar{\wedge}}_{\mathfrak{S}} D) \sqsubseteq_{\mathfrak{S}} M$ and then $(\bigsqcup^{\bar{\wedge}}_{\mathfrak{S}} D) = M \in U_D \subseteq \mathfrak{S}'$. This concludes the proof of property (288).

We endly note that the property (95) is exactly the third axiom in the definition of Scott-ideals, which establishes the secund equivalence announced in the Lemma. □

**Lemma 71. [Alternative description of idealized sub-selection structures]**
A sub-selection structure $\mathfrak{S}'$ of a selection structure $(\mathfrak{S}, \sqsubseteq)$ is idealized iff it satisfies the property:

$$\forall s \in \mathfrak{S}, \qquad \bigsqcup_{\mathfrak{S}} ((\downarrow_{\mathfrak{S}} s) \sqcap_{\mathfrak{S}} \mathfrak{S}') \ \in \mathfrak{S}'. \tag{290}$$

■

*Proof.* Let us consider $S \subseteq_{fin} \mathfrak{S}'$ such that it exists $s \in \mathfrak{S}$ which is a common upper-bound of elements of $S$. We have then $S \subseteq ((\downarrow_{\mathfrak{S}} s) \sqcap_{\mathfrak{S}} \mathfrak{S}')$. Using property (290), the supremum $\bigsqcup_{\mathfrak{S}} ((\downarrow_{\mathfrak{S}} s) \sqcap_{\mathfrak{S}} \mathfrak{S}')$ exists as an element of $\mathfrak{S}'$. It is then a common upper-bound of elements of $S$ that belongs to $\mathfrak{S}'$. Hence, property (290) implies that $\mathfrak{S}'$ is idealized.
Conversely, let us suppose $\mathfrak{S}'$ is idealized and let us fix any $s \in \mathfrak{S}$. The set $((\downarrow_{\mathfrak{S}} s) \sqcap_{\mathfrak{S}} \mathfrak{S}')$ is upper-bounded by $s$. Using property (95), for any $F \subseteq_{fin} ((\downarrow_{\mathfrak{S}} s) \sqcap_{\mathfrak{S}} \mathfrak{S}')$, it exists $M_F \in \mathfrak{S}'$ upper-bound of $F$. Hence, $\bigsqcup_{\mathfrak{S}'} F$ exists and is in $\mathfrak{S}'$, and $\bigsqcup_{\mathfrak{S}} F = \bigsqcup_{\mathfrak{S}'} F$. As an example, for $F \subseteq_{fin} \mathbb{A}(s) \sqcap_{\mathfrak{S}} \mathfrak{S}'$ We can build the following subset: $\{ \bigsqcup_{\mathfrak{S}} F \mid F \subseteq_{fin} \mathbb{A}(s) \sqcap_{\mathfrak{S}} \mathfrak{S}' \} \subseteq ((\downarrow_{\mathfrak{S}} s) \sqcap_{\mathfrak{S}} \mathfrak{S}')$. It is a directed subset, by construction. It then admits a directed-supremum in $\mathfrak{S}$ which is in $\mathfrak{S}'$, because $\mathfrak{S}'$ is Scott-closed : $\bigsqcup^{\bar{\wedge}}_{\mathfrak{S}} \{ \bigsqcup_{\mathfrak{S}} F \mid F \subseteq_{fin} \mathbb{A}(s) \sqcap_{\mathfrak{S}} \mathfrak{S}' \} \in \mathfrak{S}'$. Due to algebraicity of $\mathfrak{S}$ (Theorem 25), we then obtain $(\bigsqcup_{\mathfrak{S}} \{ t \in \mathfrak{S}' \mid t \sqsubseteq_{\mathfrak{S}} s \}) \in \mathfrak{S}'$. This achieves the proof of property (290). □

# 7 Appendix : Basic notions and results in order theory

We shall use ":=" (and sometimes ":⇔") for a definitional equality. Thus, $X := Y$ means "the expression denoted $X$ is conventionnaly equal to $Y$".
We write $X \subseteq_{fin} Y$ (resp. $X \subseteq^{\neq \varnothing} Y$ ) for the assertion that $X$ is a finite (resp. non-empty) subset of $Y$ .
We will use $\omega$ and $\mathbb{N}$ to denote the natural numbers thought as an ordinal or as a set.

Given a set $X$, we write $\mathscr{P}(X)$ (resp. $\mathscr{P}_{fin}(X)$, resp. $\mathscr{P}_{fin}(X)^*$) for the set of subsets of $X$ (resp. for the set of finite subsets of $X$, resp. for the set of finite non-empty subsets of $X$).
The logical conjunction (resp. disjunction) of two propositions $p_1, p_2$ will be written explicitely as "$p_1$ and $p_2$" (resp. "$p_1$ or $p_2$") or denoted $p_1 \cap p_2$ (resp. $p_1 \cup p_2$). The logical negation of a proposition $p$ will be explicitely denoted $\neg\, p$.
The symbols $\forall$ (resp. $\exists$) will be used with the usual meaning of "for any$\cdots$" (resp. "it exists$\cdots$"). The symbol $\exists!$ will be used with the meaning "It exists a unique$\cdots$". The symbol "$\cdots | \cdots$" will be used with the meaning "$\cdots$such that$\cdots$".

### 7.1 Order

A *proto-order* is a pair $(B, \prec\!\!\!\prec)$, where $B$ is a set and $\prec\!\!\!\prec$ is *a fully transitive binary relation*, i.e.

$$\left\{ \begin{array}{l} \textit{(Full Transitivity)} \\ \forall x,y,z \in B,\;\; (\, x \prec\!\!\!\prec y \text{ and } y \prec\!\!\!\prec z\,) \Rightarrow x \prec\!\!\!\prec z, \\ \forall x \in B, \forall A \subseteq_{fin} B,\;\; (A \prec\!\!\!\prec x) \Rightarrow (\exists y \in B, A \prec\!\!\!\prec y \prec\!\!\!\prec x). \end{array} \right. \tag{291}$$

Here $A \prec\!\!\!\prec u$ means: $a \prec\!\!\!\prec u, \forall a \in A$.
Let $\mathbf{X} = (X, \prec\!\!\!\prec)$ be a proto-order. A subset $Q$ of $X$ such that $\forall x \in Q, \forall y \in X,\; (\, y \prec\!\!\!\prec x\,) \Rightarrow (y \in Q)$ is said to be a *lower-set* of $\mathbf{X}$. An *upper-set* is a non-empty subset $Q$ of $X$ such that $\forall x \in Q, \forall y \in X, (\, x \prec\!\!\!\prec y\,) \Rightarrow (y \in Q)$.
A nonempty subset $Q$ is said to be *a directed subset of* $\mathbf{X}$ (resp. *a filtered subset of* $\mathbf{X}$), which is denoted $Q \subseteq_{Dir} \mathbf{X}$) (resp. $Q \subseteq_{Fil} \mathbf{X}$), iff $Q \subseteq X$ and $\forall x,y \in Q, \exists z \in Q,\; (x \prec\!\!\!\prec z$ and $y \prec\!\!\!\prec z)$ (resp. $Q \subseteq X$ and $\forall x,y \in Q, \exists z \in Q,\; (z \prec\!\!\!\prec x$ and $z \prec\!\!\!\prec y)$). Note that the empty set is directed and filtered.
A lower-set $I$ (resp. an upper-set $F$) of $\mathbf{X}$ is *an ideal* (resp. *a filter*) of $X$ iff it is a directed (resp. filtered) in $\mathbf{X}$.
The set formed by all ideals (resp. filters) of $\mathbf{X}$ equipped with the set inclusion is denoted $\mathscr{I}(\mathbf{X})$ (resp. $\mathscr{F}(\mathbf{X})$).
An ideal $I$ (resp. filter $F$) is said to be *generated by a subset $Q$ of $X$* iff $I$ is the least[28] ideal (resp. filter) containing $Q$. In particular, we will denote $(\downarrow_{(X,\prec\!\!\!\prec)} x) := \{\, y \in X \mid y \prec\!\!\!\prec x\,\}$ (resp. $(\uparrow^{(X,\prec\!\!\!\prec)} x) := \{\, y \in X \mid x \prec\!\!\!\prec y\,\}$).
Let $\mathbf{X} = (X, \prec\!\!\!\prec)$ be a proto-order, and let $Q$ be a subset of $X$. $m \in X$ (resp. $M \in X$) is a *lower bound* (resp. an *upper bound*) for $Q$, iff $m \prec\!\!\!\prec x$ (resp. $x \prec\!\!\!\prec M$), $\forall x \in Q$. If the subset $Q$ of $X$ admits an upper-bound (resp. a lower bound) in $X$, we will write $Q \subseteq_{up-b} X$ (resp. $Q \subseteq_{low-b} X$). The set of upper-bounds (esp. lower-bounds) of $Q$ in $X$ will be denoted $ub_{X}(Q)$ (resp. $lb_{X}(Q)$). A lower bound (resp. an upper bound) for the whole set $X$ will be called *the bottom element* (resp. *the top element*) of $X$. A finite upper-bounded subset $Q$ will be said to be *consistent*, denoted $Q \subseteq_{Con} P$ or shortly $\widehat{Q}^{P}$. The subset $Q$ will be said to be *coherent* (denoted $Q \subseteq_{Coh} P$) iff any finite subset of $Q$ is upper-bounded in $P$.

A *pre-order* is a pair $\mathbf{P} := (P, \preccurlyeq)$[29], where $P$ is a non-empty set (sometimes called *the universe of the pre-order*) and $\preccurlyeq \subseteq P \times P$ is a binary relation satisfying:

$$\left\{ \begin{array}{ll} \textit{(Transitivity)} & \forall x,y,z \in S,\;\; (\, x \preccurlyeq y \text{ and } y \preccurlyeq z\,) \Rightarrow x \preccurlyeq z, \\ \textit{(Reflexivity)} & x \preccurlyeq x, \forall x \in P. \end{array} \right. \tag{292}$$

A pre-order is necessarily a proto-order. A pre-order $(P, \preccurlyeq)$ is said to be *total* iff $\forall x,y \in P, x \preccurlyeq y$ or $y \preccurlyeq x$.
Let $\mathbf{P} := (P, \preccurlyeq)$ be a pre-order. An element $M$ (resp. $m$) of $Q$ will be said to be *a maximal element of $Q$* (resp. *a minimal element of $Q$*) iff $\forall p \in Q, M \not\prec p$ (resp. $\forall p \in Q,\, p \not\prec m$). The set of maximal elements

[28]"least" with respect to set inclusion

[29]**Notations:** Different symbols $(\leqslant, \subseteq, \preccurlyeq, \sqsubseteq, \cdots)$ will be used to designate the different pre-orders under study. When, for example, a symbol like $\leq$ will be used, the mirrored symbol $\geq$ will simply mean "$(y \geq x) \Leftrightarrow (x \leq y)$", the symbol $<$ will mean "$(x < y) \Leftrightarrow (\,(x \leq y)\, \cap\, (x \neq y)\,)$" (the symbol $\subsetneq$ will be sometimes used as an equivalent to $\subset$). The symbol $\not<$ will mean "$(x \not< y) \Leftrightarrow \neg\,(x < y)$".

(resp. minimal elements) of $Q$ will be denoted $Max(Q)$ (resp. $min(Q)$). Any non-empty finite sub-poset admits obviously minimal and maximal elements.

A *partially ordered set* (or *poset*) is a pre-order $\mathbf{P} := (P, \leq)$ satisfying moreover:

$$(Anti-Symmetry) \quad \forall x, y \in P \qquad (\, x \leq y \text{ and } y \leq x \,) \Rightarrow (\, x = y \,). \tag{293}$$

Posets with a bottom element (resp. top element) are said to be *pointed* (resp. *unital*).
A poset being in particular a pre-order and then a proto-order, all definitions and results relative to proto-orders or pre-orders are transposed to posets litterally.
Every subset $Q$ of the universe of a poset $\mathbf{P}$ has an associated partial order called the induced partial order of $Q$.
For any pre-order $(P, \preccurlyeq)$, we can define an equivalence relation

$$x \sim y :\Leftrightarrow (x \preccurlyeq y \text{ and } y \preccurlyeq x). \tag{294}$$

A pre-order $(P, \preccurlyeq)$ leads to a partial-order $(P/\sim, \leq)$ after quotienting by $\sim$:

$$[x]_\sim := \{\, y \in P \mid x \sim y \,\}, \quad P/\sim := \{\, [x]_\sim \mid x \in P \,\}, \quad ([x]_\sim \leq [x']_\sim) \Leftrightarrow (x \preccurlyeq x'). \tag{295}$$

A map $f : P_1 \to P_2$ between two posets is said to be *an order preserving map*, or simply *a monotone map*, if for any $x, y \in P_1, x \leq y \Rightarrow f(x) \leq f(y)$.
The set $[P \to Q]$ of all monotone functions between posets $P$ and $Q$, when ordered pointwise (i.e. $f \leq g$ if $\forall x \in P, f(x) \leq g(x)$), gives rise to a poset: the monotone function space between $P$ and $Q$. The set of posets equipped with monotone maps defines a category.
If a minimal (resp. a maximal) element $m$ (resp. $M$) of $Q$ is also a lower-bound of $Q$ (resp. an upper-bound of $Q$), then $m$ (resp. $M$) will be called *the least element* (resp. *the greatest element*) of $Q$ (the least/greatest element is necessarily unique).
For all $x$ in a poset $P$ we define the strict and closed initial segments, the strict and closed final segments, by $\langle x\,] := \downarrow_{(P,\leq)} x$, $\langle x[ := \langle x\,] \setminus \{x\}$, $[\,x\rangle := \uparrow_{(P,\leq)} x$, $]x\rangle := [\,x\rangle \setminus \{x\}$, and, for all $x$ and $y$ in a poset $P$ with $x \leq y$, we define the interval $[x, y] := \langle y\,] \cap [\,x\rangle$.

## 7.2 Induction

A *chain* of a poset $\mathbf{P}$ is a subset of $P$ which is totally ordered by the restriction of the order defined on $P$. An $\omega-$chain is a countable chain. If $C$ is a chain (resp. $\omega-$chain) in the poset $\mathbf{P}$, we will write $C \subseteq_{chain} \mathbf{P}$ (resp. $C \subseteq_{\omega-chain} \mathbf{P}$. A descending (resp. ascending) $\omega-$chain is a chain $(a_n)_{n\in\mathbb{N}}$ of elements of $P$ such that $\forall i, j \in \mathbb{N},\ (i \leq j) \Rightarrow (a_i \geq a_j)$ (resp. $\forall i, j \in \mathbb{N},\ (i \leq j) \Rightarrow (a_i \leq a_j)$). A descending/ascending chain is said to *stabilize* iff $\exists k \in \mathbb{N}, \forall l \in \mathbb{N},\ (k \leq l) \Rightarrow (a_k = a_l)$.

A poset $P$ is said to be *inductive* iff every chain admits an upper bound. Let $P$ be an inductive poset, then $P$ has a maximal element (possibly more than one).[30]

In a poset $P$, we will say that $x \in P$ *covers* $y \in P$ (denoted $x \geqslant y$ or $y \leqslant x$) iff

$$(Covering) \qquad y \leqslant x \;\Leftrightarrow\; x \in Min(\,]y\rangle\,) \tag{296}$$

A poset $P$ is said to be *a torsion* (or to be *semi-artinian*) iff

$$(P \;\; Semi-artinian) \qquad \forall y \in P,\ (\, y \in Max(P) \text{ or } \exists x \in P, y \leqslant x \,) \tag{297}$$

In a poset $P$, an element $p$ is said to be *a torsion element* iff $\langle x\,]$ is a torsion. In a poset $P$, an element $p$ is said to be *torsion free* iff $Min(\,]x\rangle\,) = \varnothing$.
A poset $P$ is said to be *strongly atomic* iff

$$(Strong\ atomicity) \qquad \forall x, y \in P,\ (\, x < y \,) \Leftrightarrow (\, \exists z \in P,\ x \leqslant z \leq y \,). \tag{298}$$

[30] This is *Zorn's lemma*, see [34] (page 62) for a pedagogical introduction.

A strongly atomic poset is necessarily semi-artinian.
A poset $P$ is said to be *artinian* (resp. *noetherian*) iff every non-empty subset of $P$ has a minimal element (resp. a maximal element).[31]

$$(P\ Artinian) \qquad \forall Q \subseteq^{\neq \varnothing} P, \quad Min(Q) \neq \varnothing. \tag{299}$$

If $P$ is artinian (resp. noetherian), then any interval of $P$ is artinian (resp. noetherian) as well (in particular the initial and final intervals). An artinian poset is necessarily also semi-artinian. Artinian (resp. noetherian) posets are defined equivalently as posets where every descending (resp. ascending) $\omega-$chain stabilizes.

## 7.3 Completeness

Let $\mathbf{P} = (P, \leq)$ be a poset. The unique *infimum* (also called *meet*) of $Q \subseteq P$ denoted $\bigwedge_P Q$ is defined by

$$(\bigwedge_P Q \leq x, \forall x \in Q) \text{ and } (\forall y \in P, (y \leq x, \forall x \in Q) \Rightarrow y \leq \bigwedge_P Q). \tag{300}$$

We will denote $x \wedge y := \bigwedge_P \{x, y\}, \forall x, y \in P$.
The unique *supremum* (also called *join*) of $Q \subseteq P$ denoted $\bigvee_P Q$ is defined by

$$(\bigvee_P Q \geq x, \forall x \in Q) \text{ and } (\forall y \in P, (y \geq x, \forall x \in Q) \Rightarrow y \geq \bigvee_P Q). \tag{301}$$

We will denote $x \vee y := \bigvee_P \{x, y\}, \forall x, y \in P$.
If a directed set (resp. filtered set) $A$ has a supremum (resp. an infimum), it will be eventually denoted $\bigvee_P^{\overset{\wedge}{\vee}} A$ (resp. $\bigwedge_P^{\overset{\vee}{\wedge}} A$) to recall the directedness (resp. the filteredness) of $A$.

A poset $\mathbf{P} = (P, \leq)$ is said to be *an Inf semi-lattice* (resp. *a Sup semi-lattice*), if $x \wedge y$ (resp. $x \vee y$) exists for any $x, y \in P$, and then any finite subset of $P$ has an infimum. An Inf semi-lattice can be equivalently given as a commutative idempotent semi-group $(P, \wedge)$, the partial order being given as $\forall p_1, p_2 \in P,\ (p_1 \wedge p_2 = p_1) \Leftrightarrow (p_1 \leq p_2)$.
A poset $\mathbf{P} = (P, \leq)$ is a *lattice*, if it is both an Inf semi-lattice and a Sup semi-lattice.
It is a central aspect of lattices that they can be jointly seen as an order and through an algebraic perspective.
Indeed, a lattice is also defined as an algebra $(L, \wedge, \vee)$ such that $\wedge, \vee$ satisfy conjointly the following properties:

$$(idempotency) \qquad x \vee x = x, \quad x \wedge x = x \tag{302}$$

$$(commutativity) \qquad x \vee y = y \vee x, \quad x \wedge y = y \wedge x \tag{303}$$

$$(associativity) \qquad (x \vee y) \vee z = x \vee (y \vee z), \quad (x \wedge y) \wedge z = x \wedge (y \wedge z) \tag{304}$$

$$(absorption) \qquad x \vee (x \wedge y) = x \text{ and } x \wedge (x \vee y) = x \tag{305}$$

Starting from this algebraic definition, we can define the partial order $\leq$ by

$$\forall x, y \in L,\ x \leq y \text{ iff } x \wedge y = x \text{ iff } x \vee y = y$$

Equipped with this order, $(L, \leq)$ is then a lattice with $x \wedge y$ and $x \vee y$ defined respectively as the supremum and the infimum of $x, y$.

A map preserving non-empty finite infima (resp. suprema) will be called a homomorphism of Inf semi-lattice (resp. Sup semi-lattice).

[31] Artinian posets are especially useful when we want to establish results by a suitable generalization of recursion methods [74, Theorem 7.1.5]. More precisely, let $P$ be an artinian poset and $Q$ be a subset of $P$ such that $\forall x \in Q,\ \langle x\ [\ \in Q \Rightarrow x \in Q$, then $Q = P$. More generally, let $P$ be an artinian poset and $M$ any set. Let us suppose that a particular construction has led to build a map $\mathbf{c}$ which associates to any pair $(u, f_{<u})$, where $u \in P$ and $f_{<u}$ is a map from $\langle x\ [$ to $M$, a uniquely defined element of $M$ denoted $\mathbf{c}(u, f_{<u})$. Then there exists a unique function $f$ from the whole $P$ to $M$ such that $\forall u \in p, f(u) = \mathbf{c}(u, f_{<u})$.

If $(L,\vee,\wedge)$ and $(L',\vee',\wedge')$ are lattices as algebras and $f$ maps $L$ into $L'$, then we call $f$ a *homomorphism of lattices*[32] iff $f$ is a meet-preserving and a join-preserving map. If $(L,\leq)$ and $(L',\leq')$ are lattices as orders and $f$ maps $L$ into $L'$, then we call $f$ an isomorphism if it is one-to-one and onto and order preserving. Note that the two concepts are equivalent. The lattices together with these morphisms form a category.

Let $\mathbf{P}=(P,\leq)$ be a poset, and let $Q$ be a subset of $P$. $Q$ is said to be complete in $P$ iff

$$(Q \subseteq_{Comp} P) \qquad \forall R \subseteq Q,\ \forall u \in ub_p(R),\ \exists q \in Q \mid (\forall r \in R,\ r \leq q \leq u). \tag{306}$$

A poset $\mathbf{P}$ is said to be *consistently complete* iff every consistent subset admits a supremum in $\mathbf{P}$. A poset $\mathbf{P}$ is said to be *coherently complete* iff every coherent subset admits a supremum in $\mathbf{P}$.
A poset $P$ is said to be *strictly inductive* (or *chain complete*) if it is non-empty and if every chain $C \subseteq_{chain} \mathbf{P}$ admits a supremum in $P$. A poset $\mathbf{P}$ is $\omega-$chain complete, iff every countable directed subset of $\mathbf{P}$ admits a supremum in $\mathbf{P}$.
A poset $\mathbf{P}=(P,\leq)$ is said to be *directed complete*, and called a *directed complete partial order* (or *dcpo*), iff any of its directed subset $Q \subseteq_{Dir} \mathbf{P}$ has a supremum in $\mathbf{P}$. A poset $\mathbf{P}=(P,\leq)$ is said to be *filtered complete* iff any of its filtered subset $Q \subseteq_{Fil} \mathbf{P}$ has an infimum in $\mathbf{P}$. A poset $\mathbf{P}=(P,\leq)$ is said to be *bicomplete* iff it is simultaneously filtered-complete and directed-complete.
An Inf semi-lattice $\mathbf{P}$ is said to be *a down-complete Inf semi-lattice*, iff any non-empty subset has an infimum in $P$ A poset $\mathbf{P}$ is said to be *bounded complete* iff every non-empty upper-bounded subset $Q$ of $\mathbf{P}$, i.e $Q \subseteq^{\neq\varnothing}_{up-b} P$ has a supremum in $\mathbf{P}$, i.e. $\bigvee_P Q \in P$. It is equivalent to require that every non-empty lower-bounded subset $Q$ of $\mathbf{P}$, i.e $Q \subseteq^{\neq\varnothing}_{low-b} P$ has an infimum in $\mathbf{P}$, i.e. $\bigwedge_P Q \in P$. A poset $\mathbf{P}=(P,\leq)$ is said to be *complete* iff every subset admits an infimum and a supremum in $\mathbf{P}$.
A poset $\mathbf{P}$ is chain complete, iff it is directed complete. [53]
If the poset $\mathbf{P}=(P,\leq)$ is directed-complete and consistently-complete, $\mathbf{P}$ is then coherently-complete and

$$\forall Q \subseteq_{Coh} P, \qquad \bigvee_P Q = \bigvee_P^{\bar{\wedge}} \{ \bigvee_P R \mid R \subseteq_{fin} Q \}. \tag{307}$$

In particular, $P$ is then also bounded complete.

A monotone net in a poset $P$ is a monotone function $\alpha$ from a directed set $I$ (the *index set* of $\alpha$) into $P$. Let $P$ be a poset and $I$ be a directed set. Let $\alpha : I \times I \to P$ be a monotone net. If the following suprema exist, then we have [5, Proposition 2.1.12] :

$$\bigsqcup_{i\in I}^{\bar{\wedge}} \bigsqcup_{j\in I}^{\bar{\wedge}} \alpha(i,j) = \bigsqcup_{j\in I}^{\bar{\wedge}} \bigsqcup_{i\in I}^{\bar{\wedge}} \alpha(i,j) = \bigsqcup_{i\in I}^{\bar{\wedge}} \alpha(i,i). \tag{308}$$

Let $C, C'$ be totally ordered sets ($C \times C'$ is then naturally directed) and let us consider a monotone net $\alpha : C \times C' \to P$. If the following suprema exist, then we have :

$$\bigsqcup_{c\in C} \bigsqcup_{c'\in C'} \alpha(c,c') = \bigsqcup_{c'\in C'} \bigsqcup_{c\in C} \alpha(c,c'). \tag{309}$$

## 7.4 Continuity

A binary relation $\prec\!\!\!\prec$ on a pre-ordered set $(P,\preccurlyeq)$ is called *an auxiliary relation* if it satisfies

$$\begin{cases} \prec\!\!\!\prec \text{ is a fully transitive binary relation,} \\ \forall x,y, \quad (x \prec\!\!\!\prec y) \Rightarrow (x \preccurlyeq y), \\ \forall t,x,y,z, \quad (t \preccurlyeq x \prec\!\!\!\prec y \preccurlyeq z) \Rightarrow (t \prec\!\!\!\prec z), \\ \mathfrak{b} \text{ bottom element of } P \Rightarrow \mathfrak{b} \prec\!\!\!\prec x. \end{cases} \tag{310}$$

[32] If the map is one-to-one and onto, it is called an isomorphism. If $f$ is one-to-one, it is called an embedding.

Let $\mathbf{P} := (P, \preccurlyeq)$ be a pre-ordered set, $B$ a subset of $P$, and $\prec\!\!\!\!\prec$ an auxiliary relation on $P$ such that $\prec\!\!\!\!\prec$ is fully transitive on $B$ as well. Let us endly consider $x \in P$. We will denote

$$\downarrow_{(B,\prec\!\!\!\!\prec)} x \quad := \quad \{y \in B \mid y \prec\!\!\!\!\prec x\}. \tag{311}$$

Due to the full transitivity of $\prec\!\!\!\!\prec$ on $B$, the subset $(\downarrow_{(B,\prec\!\!\!\!\prec)} x)$ is a directed lower-set, i.e. an ideal of $\mathbf{P}$.
More generally, for any $X \subseteq P$ we denote $(\downarrow_{(B,\prec\!\!\!\!\prec)} X)$ the ideal $\downarrow_{(B,\prec\!\!\!\!\prec)} X := \{y \in B \mid \exists x \in X, y \prec\!\!\!\!\prec x\}$. The subset $X$ is directed iff $(\downarrow_{(B,\prec\!\!\!\!\prec)} X)$ is an ideal.
If $B = P$ and $\prec\!\!\!\!\prec = \preccurlyeq$, the ideal $(\downarrow_{(B,\prec\!\!\!\!\prec)} x)$ will be denoted shortly $\downarrow x$ and will be called *the principal ideal associated to x*.

An auxiliary relation is said to be *approximating* iff

$$\forall x, y \in P, (x \not\preccurlyeq y) \;\Rightarrow\; (\exists z \in P \mid z \prec\!\!\!\!\prec x \text{ and } z \not\preccurlyeq y). \tag{312}$$

Let $(P, \preccurlyeq)$ be a pre-ordered set and let $\prec\!\!\!\!\prec$ be an approximating auxiliary relation on $P$. Then, we have

$$\forall x, y \in P, \qquad (x \preccurlyeq y) \;\Leftrightarrow\; ((\downarrow_{(P,\prec\!\!\!\!\prec)} x) \subseteq (\downarrow_{(P,\prec\!\!\!\!\prec)} y)). \tag{313}$$

When a dcpo $\mathbf{D} = (D, \leq)$ is given, a new fundamental binary relation $\ll \subseteq D \times D$ can be defined, called the *way-below relation* (or *the order of approximation* of this dcpo). It is defined by[33]

$$\forall x, y \in D, \qquad (x \ll y) \;:\Leftrightarrow\; (\forall E \subseteq_{Dir} D,\; y \leq \bigvee\nolimits_D^{\bar{\wedge}} E \;\Rightarrow\; \exists z \in E,\; x \leq z). \tag{314}$$

A subset $B$ of a dcpo $(D, \leq)$ is said to be *a basis of D* iff

$$(B \textit{ basis of } D) \qquad \forall x \in D,\; (\downarrow_{(B,\ll)} x) \subseteq_{Dir} B \;\text{ and }\; x = \bigvee\nolimits_D^{\bar{\wedge}} (\downarrow_{(B,\ll)} x). \tag{315}$$

The elements of $(\downarrow_{(B,\ll)} x)$ will be called *approximants of x relative to B*.[34]

A dcpo $(D, \leq)$ is said to be *a continuous dcpo* (or *a domain*) iff $D$ admits a basis. In other words, $(D, \leq)$ is continuous iff $\ll$ is an approximating auxilliary relation.

A continuous Inf semi-lattice $(D, \leq)$ is also necessarily meet-continuous, i.e. satisfies

$$\textit{(Meet-continuity)} \quad \forall x \in L, \forall T \subseteq_{Dir} L, \qquad x \wedge (\bigvee\nolimits_L^{\bar{\wedge}} T) = \bigvee\nolimits_L^{\bar{\wedge}} \{x \wedge t \mid t \in T\} \tag{316}$$

An element $c$ in a directed complete partial order $(D, \leq)$ is said to be *compact* (or *finite*) iff $x \ll x$. In other words,

$$\textit{(c compact)} \qquad \forall E \subseteq_{Dir} D,\; (c \leq \bigvee\nolimits_D^{\bar{\wedge}} E) \;\Rightarrow\; (\exists e \in E, c \leq e). \tag{317}$$

The subset of $D$ formed by compact elements will be denoted $D_c$.
As soon as the join of a consistent set of compact elements exists, this join is necessarily compact.
Let $(D, \leq)$ be a continuous dcpo and $B$ a basis of $D$. The set $D_c$ of compact elements of $D$ is necessary included in $B$.
$D$ is said to be *an algebraic dcpo* (or *algebraic domain*) iff the set $D_c$ of compact elements of $D$ forms a basis. In other words, $D$ is algebraic iff any element of $D$ is a join of compact elements, i.e.

$$(\textit{Algebraicity}) \qquad \forall x \in D,\; x = \bigvee\nolimits_D^{\bar{\wedge}} \{u \in D_c \mid u \leq x\}. \tag{318}$$

[33] Directed subsets can be equivalently replaced by chains in this definition.
[34] We will adopt the following shortcut: $\forall d \in D,\; \Uparrow d := \uparrow_{(D,\ll)} d$ and $\Downarrow d := \downarrow_{(D,\ll)} d$.

An algebraic dcpo is also a continuous dcpo.
If $(D,\leq)$ is an algebraic dcpo, the way-below relation can be reformulated as

$$\forall u,v \in D,\ u \ll v \quad \Leftrightarrow \quad (\exists c \in D_c,\ u \leq c \leq v). \tag{319}$$

A dcpo $D$ is said to be *a continuous (resp. algebraic) Scott-domain* iff $D$ is a consistently-complete continuous (resp. algebraic) dcpo.
A Scott-domain is necessarily bounded complete.

Let $(D,\leq)$ be a continuous dcpo. A basis $B$ of $D$ will be said to be *finitary* iff

$$\forall s \in B, \qquad (\downarrow_{(B,\ll)} s) \subseteq_{fin} D. \tag{320}$$

A continuous dcpo $(D,\leq)$ will be said to be *a finitary domain* iff it admits a finitary basis.

## 7.5 Decomposition

An element $x$ of an Inf semi-lattice $L$ will be said to be *a meet-irreducible element* iff for $y$ and $z$ elements of $L$ such that $x = y \wedge z$, we have necessarily $x = y$ or $x = z$. An element $x$ of an Inf semi-lattice $L$ will be said to be *a meet-prime element* iff for $y$ and $z$ elements of $L$ such that $x \geq y \wedge z$, we have necessarily $x \geq y$ or $x \geq z$. We can define dually *join-irreducible elements* and *join-primes*. The set of meet-irreducible (resp. join-irreducible) elements in $L$ will be denoted $\wedge - IRR(L)$ (resp. $\vee - IRR(L)$).
An element $p$ in a poset is meet-irreducible iff $p$ is maximal or $\uparrow p \setminus \{p\}$ is a filter.
In a continuous Inf semi-lattice L, the set of non-identity meet- irreducibles $\wedge - IRR(L) \setminus \{1\}$ is 'order-generating'. Note that a subset $X$ is said to be order generating in $L$ iff $\forall x \in L,\ x = \bigwedge(\uparrow x \cap X)$.
An element $x$ of a down-complete Inf-semi lattice will be said to be a complete meet-irreducible element iff for any subset $Q \subseteq P$, we have $(x = \bigwedge Q) \Rightarrow (\exists y \in Q \mid x = y)$. The set of complete meet-irreducible elements will be denoted $\wedge - Irr(L)$.
Two other characterization of completely meet-irreducible are given as follows. An element $p$ of a poset $L$ is completely irreducible iff either $p$ is maximal in $L$ but different from the top element or the set $\uparrow p \setminus \{p\}$ has a least element [30, Definition I-4.21]. In a bounded-complete algebraic domain $L$, an element $p \in L$ is completely irreducible iff $p$ is maximal in $L \setminus \uparrow k$ for some compact element $k$ [30, Proposition I-4.27].
In any bounded-complete algebraic domain, the subset of non-identity completely meet-irreducible elements $\wedge - Irr(L) \setminus \{1\}$ is the unique smallest order-generating subset [30, Theorem I-4.26].
The (complete) meet-prime elements in a bounded-complete Inf semi-lattice $L$ are also (complete) meet-irreducible elements.

An element $x$ of a coherently-complete partial order $\mathbf{P} := (P,\leq)$ will be said to be *a complete join-prime element of* $\mathbf{P}$ (resp. a complete join-irreducible element) iff for any subset $Q \subseteq_{Coh} P$, we have $(x \sqsubseteq \bigvee Q) \Rightarrow (\exists y \in Q \mid x \leq y)$ (resp. $(x = \bigvee Q) \Rightarrow (\exists y \in Q \mid x = y)$).
A domain $D$ is said to be *prime algebraic* iff any element of $D$ is a join of complete join-prime elements.
In a finite lattice $(L,\leq)$, every element $x \in L$ is a finite join of join-irreducible elements and a finite meet of meet-irreducible elements.
Let $(L,\leq)$ be a lattice such that every element is the join of join-irreducible elements and the meet of meet-irreducible elements, then the property that every join-irreducible is join-prime[35] is equivalent to the distributivity of $L$,[54, Theorem 5], i.e.

$$\forall x,y,z \in L, \qquad (x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z) \text{ and } x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z). \tag{321}$$

[35] or every meet-irreducible is meet-prime

Let $\mathbf{D} = (D, \leq)$ be a coherently-complete partial order. Then, $D$ is prime algebraic iff it is a Scott domain and satisfies the conditional-distributivity laws [78]:

$$\forall X \subseteq_{Coh} D, \forall y \in D, \qquad (\bigvee_D X) \wedge y = \bigvee_D \{ (x \wedge y) \mid x \in X \}, \tag{322}$$

$$\forall X \subseteq D, \forall y \in D \,|\, \forall x \in X, \widehat{xy}^D, \qquad (\bigwedge_D X) \vee y = \bigwedge_D \{ (x \vee y) \mid x \in X \}. \tag{323}$$

Let $\mathbf{D} = (D, \leq)$ be a finitary Scott domain. Then $D$ is prime algebraic iff $D$ satisfies the finite distributive law [78]:

$$\forall x, y, z \in D \,|\, \widehat{xy}^D, \qquad (x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z). \tag{324}$$

A finitary domain $(D, \leq)$ which satisfies the following conditional-distributivity properties

$$\widehat{xy}^D \quad \Rightarrow \quad z \wedge (y \vee x) = (z \wedge y) \vee (z \wedge x), \tag{325}$$

$$\widehat{xy}^D \text{ and } \widehat{xz}^D \quad \Rightarrow \quad x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z). \tag{326}$$

is called *a dI-domain* (see [80] for an extensive study of these domains).

Let $\mathbf{P} = (P, \leq)$ be a pointed poset ($\mathfrak{b}$ designates its bottom element). The set of *atoms* of $\mathbf{P}$ is denoted $\mathscr{A}_{\mathbf{P}} := \{a \in \mathbf{L} \mid a \geqslant \mathfrak{b}\}$. $\mathbf{P}$ is said to be *atomic*, iff

$$\forall x \in P, \qquad \mathbb{A}(x) := (\mathscr{A}_{\mathbf{P}} \cap \downarrow_P x) \neq \varnothing. \tag{327}$$

If $\mathbf{P} = (P, \leq)$ is a pointed coherently-complete poset, $\mathbf{P}$ will be said to be *atomistic*, iff every element $x$ of $\mathbf{P}$ is the join of the atoms contained in $x$, i.e.

$$(Atomisticity) \quad \forall x \in P, \qquad x = \bigvee_P \mathbb{A}(x) = \bigvee_P^{\bar{\wedge}} \{ (\bigvee_P F) \mid F \subseteq_{fin} \mathbb{A}(s) \}. \tag{328}$$

A dI$-$domain in which the join-prime elements are atomic is called a *qualitative domain*. A qualitative domain is then necessarily atomistic.

Let $\mathbf{L}$ be a coherently-complete partial order and an Inf semi-lattice, which is atomistic. Then, the following properties are equivalent: (1) $L$ is meet-continuous, (2) the atoms of $\mathbf{L}$ are compact, (3) $\mathbf{L}$ is algebraic.
Let $\mathbf{L}$ be a coherently-complete partial order and an Inf semi-lattice, which is atomistic and meet-continuous (or algebraic). Then, $c$ is compact iff $c$ is the supremum of finitely many atoms of $\mathbf{L}$.

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