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# A possibilistic semantic for quantum phenomena

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## Abstract

We develop a possibilistic semantic for quantum phenomena in an operational perspective. This semantic is based on a Chu duality between preparation processes and yes/no tests, the target space being a three-valued set equipped with an informational interpretation.

After having defined the notion of states, we develop a precise axiomatic for the space of states. The 'information principle', proposed as a building block for some quantum axiomatic program, is translated into our framework to emphasize the quantum character of our description. This principle suffices to constrain the space of states to be a locally-boolean qualitative domain. The subset of pure states is then characterized within this domain structure.

After having carefully precised the notions of properties and measurements, we explore the notion of compatibility between measurements. The existence of minimally-disturbing measurements is then emphasized as a key axiom to describe the space of yes/no tests. The conditions of existence of such measurements is translated into a simple property on the space of states, and an explicit formula for these measurements is given. Having reduced the space of yes/no tests to this set of minimally-disturbing operations, our Chu space becomes bi-extensional. The space of 'descriptions', associated to families of compatible measurements, inherits a structure of coherence domain. A subset of properties corresponding to 'classical properties' is identified and explored.

Endly, the symmetries of the system are characterized as a general sub-class of Chu morphisms. We prove that these symmetries preserve the class of minimally-disturbing measurements and the orthogonality relation between states. Explicit expressions of these symmetries are identified for a large class of them, called 'linear symmetries'.

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# 1 Introduction

The basic description of an 'experimental act' relies generally on (i) a description of a given *preparation setting* allowing to produce *samples* of a given physical object, along some well-established procedures, and (ii) a particular set of *operations/tests* that can be realized by the *observer* on these prepared samples. Each prepared sample is associated to a set of *information*, checked along the preparation process and recorded on devices, and to a set of *instructions*, processed by a computer or followed by the human operator. These information/instructions are naturally attached to macroscopically observable events occurring during the preparation process and to macroscopically distinguishable modes of the preparation apparatuses used along this preparation process. Analogously, the operations/tests, realized on the prepared samples, must be attached unambiguously to some *knowledge/belief* of the observer about the outcomes of this experimental step and to a new *state of facts* concerning the outcomes of the whole experiment, in order for these operations/tests to give a sense to any 'sorting' of these outcomes. As soon as these operations/tests do not 'destroy' or 'alter irremediably' the samples under study, we can then consider the whole experimental protocol<sup>[1]</sup> as a completed preparation procedure for forthcoming experiments. The information characterizing the chosen operation, and the information retrieved by the observer during these tests, should then be recorded as new information/instructions entering this so-completed preparation procedure.<sup>[2]</sup> This *syntactic* description of preparation procedures and operations/tests is a fundamental ingredient of the physical description. It is a *sine qua non condition* for the experimental protocol to satisfy a basic requirement for it : *the reproducibility*. As noted by Kraus [42], there exist macroscopic devices undergoing macroscopic changes when interacting with micro-systems, and the observation of micro-objects always requires this inter-mediation. This fundamental empirical fact justifies the attempt to establish such an *operational description* for quantum experiments as well. Explicitly, the different procedures designed to prepare collection of similar quantum micro-systems may combine measurements and filtering operations (associated to the unambiguously measured properties) in order to produce collections of samples, that may be submitted to forthcoming measurements.

Obviously, different preparation procedures may be used to produce distinct samples, to which the observer would nevertheless attach the same *informational content*. It is the case, precisely, if this observer do not know any experiments, that could be realized conjointly on these differently prepared samples, and which would produce 'unambiguously-incompatible' logical conclusions. A physical description (of the objects submitted to experiment) is an attempt to establish a *semantic perspective* adapted to previous *syntactic* description of the process of preparation/measurement. The notion of *physical state* occupies a central position in this semantic construction. The physical states are abstract names for the different possible realizations of the object under study. Adopting this ontological perspective, the observer may associate an element of the *space of states* to any preparation process, which will *a priori* characterize any particular sample 'prepared' along this process. The logical truth of a proposition about the 'similarity' of two given samples should then be directly linked to the logical truth of the proposition regarding the identity of the associated states. However, the ontological notion of state has to be faced with its epistemological counterpart. From an ontological perspective, we consider that a given physical system is necessarily in a particular realization, but from an epistemological perspective, the observer should test the hypothesis that this system may indeed be described by this state.<sup>[3]</sup> Adopting this perspective, it seems that a given physical state could also be meant as a *denotation* for the set of preparations, that the observer is led to identify empirically. From a strictly operational point of view, the observer will always establish the equivalence between different preparation procedures, by testing conjointly the corresponding prepared samples through a 'well-chosen' collection of control-tests. In other words, the state could be defined by the set of common facts, that could be established by realizing these control-tests on the corresponding samples. Adopting another perspective, we could also consider

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<sup>1</sup>i.e. the initial preparation followed by the operation/test step as a global preparation process for subsequent tests

<sup>2</sup>We note that the description of preparation/measurement process should then exploit some tools of recursion theory.

<sup>3</sup>Note that, concretely, the observer is rather led to associate a 'mixture' according to his limited knowledge about this sample.

that the state should encode the determined aspects relative to the possible results of forthcoming experiments. There should be no problem with such a 'versatile' perspective in the operational description of classical experiments, as soon as the properties, established as 'actual' during the preparation process, characterize the sample in a way that will be questioned neither by any control-test realized on it, nor by any future experiments, as soon as these measurements do not 'alter' the sample. The operational description of quantum experiments is in fact significantly more intricate, due to some annoying features of the measurement operation. Indeed, it is a fundamental fact of quantum experiments that, whichever set of properties has been checked as 'actual' by the preparation process, the outcomes of an irreducible part of the measurements, that can be made on these prepared samples, remain completely indeterminate. More than that, if some of these measurements are realized in order to establish some new properties, it generically occurs that the measured samples do not exhibit anymore, afterwards, some of the properties that had been established, beforehand, on the prepared samples.

Despite this indeterministic character of quantum theory, it is an empirical fact that the distinct outcomes of these measurements, operated on a large collection of samples, prepared according to the same experimental procedure, have reproducible relative frequencies. This fundamental fact has led physicists to consider, as basic objects of the physical description, large collections of statistically independent experimental sequences rather than a single experiment on a singular realization of the object under study (see [55] for a reference book). According to *Generalized Probabilistic Theory*<sup>[4]</sup>, a physical state (corresponding to a class of operationally equivalent preparation procedures) is defined by a vector of probabilities associated to the outcomes of a maximal and irredundant set of fiducial tests that can be effectuated on collections of samples produced by any of these preparation procedures.<sup>[5]</sup> In other words, two distinct collections of prepared samples will be considered as operationally equivalent if they lead to the same probabilities for the outcomes of any test on them. The physical description consists then in a set of prescriptions allowing to define sophisticated constructs from elementary ones. In particular, combination rules are defined for the concrete mixtures of states and for the allowed operations/tests. The different attempts to reconstruct quantum mechanics along this path ([33, 34, 35][18][53][13]) proceed by the determination of a minimal set of plausible constraints, imposed on the space of states, sufficient to 'derive' the usual Von Neumann's axiomatic of quantum theory. Although this probabilistic approach is now accepted as a standard conceptual framework for the reconstruction of quantum theory, the adopted perspective appears puzzling for different reasons. First of all, the observer contributes fundamentally to give an intuitive meaning to the notions of preparation, operation and measurement on physical systems. However, the concrete process of 'acquisition of information' (by the observer / on the system) has no real place in this description. Secondly, the definition of the state has definitively lost its meaning for a singular prepared sample, the physical state is now intrinsically attached to large collections of similarly prepared samples. This point has concentrated many critics since the original article of Einstein, Podolsky and Rosen [24], although the empirical testing of quantum theory in EPR experiments has led physicists to definitively accept the traditional probabilistic interpretation. The GPT's approach adopts without discussion the probabilistic description of quantum phenomena without trying to explain why we have to do so. Thirdly, in order to clarify the requirements about the basic set of fiducial tests, necessary and sufficient to define the space of states, this approach must proceed along a technical analysis which limits fundamentally this description to 'finite dimensional' systems (finite dimensional Hilbert space of states). Endly, the axioms chosen to characterize quantum theory, among other theories encompassed by GPT's formalism, must exhibit some 'naturalness' that is still missing - to our point of view - in the existing proposals.

Alternative research programs have tried to overcome some of these conceptual problems. In particular, they try to put the emphasis on the 'informational' relation emerging between the observer and the

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<sup>4</sup>this formalism has its origins in the pioneering works of Mackey [48], Ludwig [45, 46, 47] and Kraus [42]. See [39] for a recent review

<sup>5</sup>The description of quantum theory in this framework has then to deal with the problem to define the notions of consistency, completeness and irredundancy of the set of control-tests defining an element of the quantum space of states

system, through the concrete set of 'yes/no tests' that can be addressed by this observer to this system, and to characterize quantum theory in these terms through a small number of basic semantic requirements. It must be noted that, although these programs try to clarify the central notion of "information" in the quantum description an observer can elaborate about the system under study, they basically adopt the 'probabilistic' interpretation about measurements. The fundamental limitation of the information, that an observer can retrieve from a given quantum system through yes-no experiments, has been taken by different physicists as a central principle for the reconstruction of quantum theory (see the reference paper [64, Chap III], and see [71] for another perspective on this basic principle) :

**Information Principle** *'Information is a discrete quantity: there is a minimum amount of information exchangeable (a single bit, or the information that distinguishes between just two alternatives).[...] Since information is discrete, any process of acquisition of information can be decomposed into acquisitions of elementary bits of information.'* [64, p.1655].

We intent to adopt our own version of this fundamental pre-requisite to build the quantum space of states. To be more explicit, according to C.Rovelli, quantum mechanics appears to be governed by two seemingly incompatible principles (Postulate 1 and 2 of [64, Chap III]). On a first part, the amount of independent informations that can be retrieved from a 'bounded' quantum system is fundamentally 'finite'. On another part, the test of any observable property, that has not been stated as 'actual' beforehand on a given state<sup>[6]</sup>, remains fundamentally undeterminate (of course, this test will establish an actual value for this observable property, valid after the measurement). Nevertheless, this 'new information' established through this measurement operation will have been paid by the restoration of an indeterminacy on some of the properties that could have been established as actual beforehand. Another interesting analysis regarding this 'balance' principle concerning the knowledge of the observer about the quantum system is given in [66]. It must be noted that the combinatorics of the 'incompatibilities' between different measurements can be exploited to explore the algebraic structures behind quantum theory and to proceed to the 'reconstruction' of this theory [36, 37]. Nevertheless, these reconstruction programs stay technically imprisoned by a finite dimensional analysis. A third postulate in Rovelli's axiomatic proposal prescribes the nature of automorphisms acting on the space of states (the continuous unitary transformations corresponding, in particular, to Schrodinger's dynamics). However, the form of this last postulate is not plainly satisfactory as soon as it imposes some intricate relations on the probabilities associated to transitions between states.

Adopting another perspective, the *operational quantum logic* program tries to overcome the introduction of probabilities and explores the relevant categorical structures underlying the space of states and the set of properties of a quantum system. In this description, probabilities appear only as a derived concept.<sup>[7]</sup> Following G. Birkhoff and J. Von Neumann [15] and G. W. Mackey [48], this approach focuses on the structured space of 'testable properties' of a physical system.<sup>[8]</sup> Mackey identifies axioms on the set of yes/no questions sufficient to relate this set to the set of closed subspaces of a complex Hilbert space. Later, C.Piron proposed a set of axioms that (almost) lead back to the general framework

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<sup>6</sup>In Von Neumann's formalism, this point means : the quantum state of the system is not an eigenstate of the associated operator

<sup>7</sup>For the sake of usual quantum formalism, quantum probabilities can be recovered from this formalism, under the assumption of the existence of some well-behaved measurements using Gleason's theorem [31] (see [56] for an historical and pedagogical presentation of these elements).

<sup>8</sup>In Von Neumann's quantum mechanics, to each entity is associated a complex Hilbert space  $H$ . A state  $\psi$  of this entity is defined by a ray  $v(\psi)$  in  $H$ , and an observable is defined by a self-adjoint operator on  $H$ . In particular, a yes/no test  $t$  is represented by an orthogonal projector  $\Pi_t$  or equivalently by the closed subspace  $\mathfrak{A}_t$  defined as the range of  $\Pi_t$ . The answer "yes" (resp. "no") is obtained with certainty for the yes/no test  $t$ , if and only if the state  $\psi$  is such that  $v(\psi)$  is included in  $\mathfrak{A}_t$  (resp. in the orthogonal of  $\mathfrak{A}_t$ ). Birkhoff and von Neumann proposed to focus not on the structure of the Hilbert space itself, but on the structure of the set of closed subspaces of  $H$ . The mathematical structure associated to the set of quantum propositions defined by the closed subspaces of  $H$  is not a Boolean algebra (contrary to the case encountered in classical mechanics). By shifting the attention to the set of closed subspaces of  $H$  instead of  $H$ , the possibility is open to build an operational approach to quantum mechanics, because the basic elements of this description are yes/no experiments.

of quantum mechanics (see [20] for an historical perspective of the abundant literature inherited from the original Piron's works [56, 57]). Piron's framework has been developed into a full operational approach and the categories underlying this approach were analyzed (see [52, 51] for a detailed account of this categorical perspective). It must be noticed that these constructions are established in reference to some general results of projective geometry [25] and are not imprisoned into a finite dimensional perspective. Despite some beautiful results (in particular the restriction of the division ring associated to Piron's reconstruction of the Hilbert space from Piron's propositional lattices [38][8]) and the attractiveness of a completely categorical approach (see [67] for an analysis of the main results on propositional systems), many problems have been encountered by this approach. Among these problems, we may cite the difficulties to build a consistent description of compound systems due to no-go results relative to the existence of a tensor product of Piron's propositional systems [63, 26][6, 7]. These works have cast doubts on the adequacy of Piron's choice of an "orthomodular complete lattice" structure for the set of properties of the system. D.Foulis, C.Piron and C.Randall [27] have produced a commonly refined version of their respective approach in order to rule out these problems (see also [69]). This description emphasizes the centrality of the treatment of the incompatibilities between measurements.<sup>[9]</sup> The algebraic structures introduced in this work are necessarily weaker than those playing a central role in Piron's main program. Our work will emphasize the necessity to replace the lattice structures, introduced to describe the set of propositions about the quantum system, by domains. Another central problem with the logico-algebraic approach was its inability to describe the dynamic aspect of the measurement operation. The operational quantum logic program has then been developed later on in different categorical perspectives in order to clarify the links between quantum logic, on one part, and modal/dynamic logic [9, 10, 11][19, 21, 23][73, 14], or linear logic [22][29, 30], on another part.

Other categorical formalisms, adapted to the axiomatic study of quantum theory, have been developed more recently [3] and their relation with the 'operational approach' have been partly explored [1, 2, 4]. In [1, Theorem 3.15], S. Abramsky makes explicit the fact that the *Projective quantum symmetry groupoid*  $PSymmH$ <sup>[10]</sup> is fully and faithfully represented by the category  $bmChu_{[0,1]}$ , i.e. by the sub-category of the category of bi-extensional Chu spaces associated to the evaluation set  $[0, 1]$  obtained by restricting to Chu morphisms  $(f_*, f^*)$  for which  $f_*$  is injective. This result suggests the central role of Chu categories in the construction of axiomatic quantum mechanics, as soon as they provide a natural characterization of the automorphisms of the theory. More surprisingly, and interestingly for us, in [1] S. Abramsky shows that the previously mentioned representation of  $PSymmH$  is full and faithful 'already' if we replace the evaluation space of the Chu category by a three element set, where the three values represent "definitely yes", "definitely no" and "maybe" [1, Theorem 4.4] :

*'The results on reduction to finite value sets are also intriguing. Not only is the bare Chu condition on morphisms sufficient to whittle them down to the semi-unitaries, this is even the case when the discriminations on which the condition is based are reduced to three values. The general case for two values remains open, but we have shown that the two standard possibilistic reductions both fail to preserve fullness. A negative answer for two-valued semantics in general would suggest an unexpected rôle for three-valued logic in the foundations of Quantum Mechanics.'*

Although such three-valued-logic approach (in the rest of this paper we refer to this approach as a 'possibilistic' approach to distinguish it from the 'probabilistic' one) should present similarities with what

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<sup>9</sup>'It is our contention that the realistic view implicit in classical physics need not be abandoned to accommodate the contemporary conceptions of quantum physics. All that must be abandoned is the presumption that each set of experiments possesses a common refinement (that is, the experiments are compatible). As we shall argue, this in no way excludes the notion of physical systems existing exterior to an observer, nor does it imply that the properties of such systems depend on the knowledge of the observer.' [27, p.813]

<sup>10</sup>the objects of this category are the natural space of states in quantum mechanics, i.e. the Hilbert spaces of dimension greater than two, and the morphisms are the orbits under the  $U(1)$  group action on semi-unitary maps (i.e. unitary or anti-unitary), which are the relevant symmetries of Hilbert spaces from the point of view of Quantum Mechanics.

has been formalized in the *operational quantum logic* program (this point is emphasized by S. Abramsky and its study is postponed to a future paper that remains unpublished till now). We pretend that, *stricto sensu*, such a construction is still missing in the literature and could be established as a proper axiomatic for quantum mechanics.

In the present paper, we intent to develop a possibilistic semantic formalism for quantum phenomena. This new approach tries to exploit the advantages of the different programs presented above. In particular, our formalism is based on a Chu duality between preparation processes and quantum tests. This Chu duality refers to a three-valued target space. This three valued target space is equipped with a 'possibilist' semantic which leads to an 'informational' interpretation on the set of preparations.

In a first part of our study, we will formulate a precise semantic description of the space of states. The 'Information Principle' introduced by C. Rovelli plays a central role in this formalism. After having introduced a basic set of axioms about the space of states, we will show that the space of states is naturally equipped with a 'domain' structure. The technical details concerning the properties of this domain structure are given in an appendix to this paper. The space of pure states will then be characterized.

In a second part of the study, we clarify successively the notion of property and the notion of measurement associated to a given property of the system. We explore the consequences of the incompatibilities existing between measurements. We clarify in particular the conditions of existence of minimally-disturbing measurements in terms of the nature of the space of states and the space of properties. These results lead to a construction of a domain structure on the space of descriptions.

In the third part of this paper, we build the the set of symmetries of the system as a particular sub- algebra of Chu morphisms. These symmetries appear to let stable this subset of minimally-disturbing measurement operations.

Concerning the basic notations, notions and results relative to order theory, used in the present paper, we refer the reader to the section 7.

## 2 Preparations and states

### 2.1 Operational formalism

Adopting the operational perspective on quantum experiments, we will introduce the following definitions :

**Notion 1.** A *preparation process* is an objectively defined, and then 'repeatable', experimental sequence allowing to produce singular samples of a certain physical system, in such a way that we are able to submit them to tests. We will denote by  $\mathfrak{P}$  the set of preparation processes (each element of  $\mathfrak{P}$  can be equivalently considered as the collection of samples produced along this preparation procedure). The informations corresponding to macroscopic events/operations detailing the procedure depend on an observer  $O$ . If this dependence has to be made explicit, we will adopt the notation  $\mathfrak{P}^{(O)}$  to denote the set of preparation processes defined by the observer  $O$ .

**Notion 2.** For each *property*, that the observer pretends to test macroscopically on *any particular sample* of the considered micro-system, it will be assumed that the observer is able to define (i) some detailed 'procedure', in reference to the modes of use of some experimental apparatuses chosen to proceed to the operation/test, and (ii) a 'rule' allowing to extract the answer 'yes' if the macroscopic outcome of the experiment is conform with the expectation of the observer, when the test is performed on any input sample (as soon as this experimental procedure can be opportunely applied to this particular sample). These operations/tests, dedicated to determine the occurrence of a given property for a given sample,

will be called *yes/no tests associated to this property* (it is also called *definite experimental project* in [58]). If a yes/no test, associated to a given property, is effectuated according to the established procedure, and if the positive result is actually obtained for a given sample, we will say that *this property has been measured* for this sample. The set of 'yes/no tests' at the disposal of the observer will be denoted by  $\mathfrak{T}$ . If the dependence with respect to the observer  $O$  has to be made explicit, we will adopt the notation  $\mathfrak{T}^{(O)}$  to denote the set of tests defined by the observer  $O$ .

We will be basically interested by the informations gathered by the observer through the implementation of some yes/no tests, designated by elements in  $\mathfrak{T}$ , on finite collections of samples prepared similarly along any of the preparation procedures, given as elements of  $\mathfrak{P}$ . Under this perspective, we have to abandon any reference to the probabilistic interpretation<sup>[11]</sup>. Nevertheless, the observer is still able to distinguish the situations where he can pronounce a statement with 'certainty', from the situations where he can judge the result as 'undeterminate', on the basis of the knowledge he has gathered beforehand.

**Notion 3.** A yes/no test  $t \in \mathfrak{T}$  will be said to be *positive with certainty* (resp. *negative with certainty*) relatively to a preparation process  $p \in \mathfrak{P}$  iff the observer is led to affirm that the result of this test, realized on any of the particular samples that could be prepared according to this preparation process, would be 'positive with certainty' (resp. would be 'negative with certainty'), 'should' this test be effectuated. If the yes/no test can not be stated as 'certain', this yes/no test will be said to be *undeterminate*. Concretely, the observer can establish his 'certainty', about the result of a given yes/no test on any given sample issued from a given preparation procedure, by running the same test on a sufficiently large (but finite) collection of samples issued from this same preparation process : if the outcome is always the same, he will be led to claim that similarly prepared 'new' samples would also produce the same result, if the experiment was effectuated.

To summarize, for any preparation process  $p$  and any yes/no test  $t$ , the element  $e(p, t) \in \mathfrak{B} := \{\perp, \mathbf{Y}, \mathbf{N}\}$  will be defined to be  $\perp$  (resp.  $\mathbf{Y}$  resp.  $\mathbf{N}$ ) if the outcome of the yes/no test  $t$  on any sample prepared according to the preparation procedure  $p$  is judged as 'undeterminate' (resp. 'positive with certainty' resp. 'negative with certainty') by the observer.

$$e : \mathfrak{P} \times \mathfrak{T} \longrightarrow \mathfrak{B} := \{\perp, \mathbf{Y}, \mathbf{N}\} \quad (1)$$

$$(p, t) \longmapsto e(p, t).$$

Several remarks have to be done after previous definitions.

*Remark 1.* It is essential to note the *counter-factual* aspect of these definitions : in the 'determinate' case, the observer is led to predict the result of the test *before this test and no matter if this test will be effectuated*. Of course, any 'determinate statement' (positive or negative) produced by the observer, about the result of any forthcoming yes/no test relatively to a given preparation process, is a strictly falsifiable statement : it may be proved to be false after some test realized on a *finite collection* of new similarly prepared sample<sup>[12]</sup>.

*Remark 2.* The 'certainty' of the observer about the occurrence of the considered 'property' is intrinsically attached to *any singular sample* prepared along this preparation process and can be falsified as a property of this sample. In other words, it is not necessary to consider a *statistical ensemble* of similarly prepared samples to give a meaning to these notions and to the logical perspective adopted to confront these statements to the measurable state of facts.

<sup>11</sup>the finite character of the tested collection of prepared samples render any notion of relative frequency of the outcomes 'meaningless'

<sup>12</sup>if the observer is certain of the positive result after having operated a given yes/no test on a finite number of similarly prepared samples, a negative result obtained for any newly tested sample will lead the observer to revise his prediction and to consider this yes/no test as being 'undeterminate' for this preparation



*Remark 3.* When the determinacy of a yes/no test is established for an observer, we can consider that this observer owns some elementary 'information' about the state of the system, whence, in the 'undeterminate case', he has none (relatively to the occurrence of the considered property).

**Notion 4.** The set  $\mathfrak{B}$  will be equipped with the following poset structure, characterizing the 'information' gathered by the observer:

$$\forall u, v \in \mathfrak{B}, \quad (u \leq v) :\Leftrightarrow (u = \perp \text{ or } u = v). \quad (2)$$

$(\mathfrak{B}, \leq)$  will be called *flat boolean domain* in the rest of this paper.

**Notion 5.**  $(\mathfrak{B}, \leq)$  is equipped with the following involution map :

$$\overline{\perp} := \perp \quad \overline{\mathbf{Y}} := \mathbf{N} \quad \overline{\mathbf{N}} := \mathbf{Y}. \quad (3)$$

The conjugate of a yes/no test  $t \in \mathfrak{T}$  is the yes/no test denoted  $\bar{t}$  and defined from  $t$  by exchanging the roles of  $\mathbf{Y}$  and  $\mathbf{N}$  in every results obtained by applying  $t$  to any given input sample. In other words,

$$\forall t \in \mathfrak{T}, \forall p \in \mathfrak{P}, \quad \epsilon(p, \bar{t}) := \overline{\epsilon(p, t)}. \quad (4)$$

**Notion 6.** For any yes/no test  $t$ , the set of preparation processes  $p$  for which this test is established as *actual*, i.e. 'positive with certainty', will be denoted  $\mathfrak{A}_t$ .

$$\forall t \in \mathfrak{T}, \quad \mathfrak{A}_t := \{p \in \mathfrak{P} \mid \epsilon(p, t) = \mathbf{Y}\}. \quad (5)$$

**Notion 7.** For a given yes/no test  $t$ , we define the subset  $\mathfrak{Q}_t$  of preparation processes that are known by the observer to produce collections of samples leading some positive results to the yes/no test  $t$ . Concerning these prepared samples, the observer is then led to pronounce a statement about any future result of this test on similarly prepared new samples : 'positive' or 'undeterminate'. The collections of samples, outcoming from these preparation processes, may then be filtered in order to select collections of samples that are known by the observer to have 'passed positively the yes/no test  $t$ '. If a preparation process  $p$  is in  $\mathfrak{Q}_t$ , we will say that the property associated to the yes/no test  $t$  is *potential* for the samples produced along  $p$  (or  $p$  is *questionable* by  $t$ ). The subset  $\mathfrak{Q}_t$  is given by

$$\mathfrak{Q}_t := \{p \in \mathfrak{P} \mid \epsilon(p, t) \leq \mathbf{Y}\} \subseteq \mathfrak{P}. \quad (6)$$

The evaluation map  $\epsilon$  defines a particular 'duality' between the spaces  $\mathfrak{P}$  and  $\mathfrak{T}$ . Formally,  $(\mathfrak{P}, \mathfrak{T}, \epsilon)$  defines a Chu Space<sup>[13][14]</sup>. The set of preparations  $\mathfrak{P}$  (resp. the set of yes/no tests  $\mathfrak{T}$ ) will be *a priori* interpreted as the set of *points* (resp. the set of *opens*) of this Chu space.<sup>[15]</sup> Indeed, the preparation processes are naturally considered as 'coexisting entities' distinguished by the properties they own, whereas the yes/no tests are naturally interpreted as 'alternative predicates' relative to the properties attached to these prepared samples.

According to the perspective adopted by [55], we will define the *states* of the physical system as follows :

<sup>13</sup>see [12] for a reference pape and [61][62] for a basic presentation

<sup>14</sup>In this section, we are concerned with the duality aspect and the situation of Chu morphisms will be treated later

<sup>15</sup>These designations are reminiscent of the basic fact that Chu spaces are generalizations of topological spaces. However this distinction is largely obsolete, as soon as the Chu space construction establishes a duality between these two sets.

**Notion 8.** An equivalence relation, denoted  $\sim_{\mathfrak{P}}$ , is defined on the set of preparations  $\mathfrak{P}$  :

Two preparation processes are identified if the statements established by the observer about the corresponding prepared samples are identical.

A *state* of the physical system is an equivalence class of preparation processes corresponding to the same informational content, i.e. a class of preparation processes that are not distinguished by the statements established by the observer in reference to the tests he has realized on finite collections of samples produced along these preparation processes.

The set of equivalence classes, modulo the relation  $\sim_{\mathfrak{P}}$ , will be denoted  $\mathfrak{S}$ . In other words,

$$\forall p_1, p_2 \in \mathfrak{P}, (p_1 \sim_{\mathfrak{P}} p_2) \Leftrightarrow (\forall t \in \mathfrak{T}, \epsilon(p_1, t) = \epsilon(p_2, t)), \quad (7)$$

$$\sim_{\mathfrak{P}} \text{ is an equivalence relation,} \quad (8)$$

$$[p] := \{p' \in \mathfrak{P} \mid p' \sim_{\mathfrak{P}} p\}, \quad (9)$$

$$\mathfrak{S} := \{[p] \mid p \in \mathfrak{P}\}. \quad (10)$$

*Remark 4.* It must be noticed that a given yes/no test  $t$  can be applied separately on the two distinct collections of samples prepared respectively along the two distinct preparation procedures  $p_1$  and  $p_2$ . The corresponding counter-factual statements  $\epsilon(p_1, t)$  and  $\epsilon(p_2, t)$ , established by the observer about  $p_1$  and  $p_2$ , are then formulated 'consistently' after these two independent experimental sequences.

We will derive a map  $\tilde{\epsilon}$  from the evaluation map  $\epsilon$  according to the following definition :

$$\begin{aligned} \tilde{\epsilon} : \mathfrak{T} &\rightarrow \mathfrak{B}^{\mathfrak{S}} \\ t &\mapsto \tilde{\epsilon}_t \mid \tilde{\epsilon}_t([p]) := \epsilon(p, t), \forall p \in \mathfrak{P}. \end{aligned} \quad (11)$$

As a result of this quotienting operation on the space of preparation processes, it appears that we have the following natural property of our Chu space.

**Lemma 1.** The Chu space  $(\mathfrak{S}, \mathfrak{T}, \tilde{\epsilon})$  is *separated*. Different preparation procedures are indeed 'identified' by the observer as soon as this observer attributes the same statements to the differently prepared samples. In other words,

$$\forall \sigma_1, \sigma_2 \in \mathfrak{S}, (\forall t \in \mathfrak{T}, \tilde{\epsilon}_t(\sigma_1) = \tilde{\epsilon}_t(\sigma_2)) \Rightarrow (\sigma_1 = \sigma_2). \quad (12)$$

■

## 2.2 Mixtures

We will define a binary operation on the set of preparations.

**Requirement 1.** If we consider two preparation processes  $p_1$  and  $p_2$  in  $\mathfrak{P}$ , we can define a new preparation procedure, called *mixture* and denoted  $p_1 \sqcap_{\mathfrak{P}} p_2$ , as follows :

the samples produced from the preparation procedure  $p_1 \sqcap_{\mathfrak{P}} p_2$  are obtained by a random mixing of the samples issued from the preparation processes  $p_1$  and  $p_2$  indifferently.

As a consequence, the statements that the observer can establish after a sequence of tests  $t \in \mathfrak{T}$  on these samples produced along the procedure  $p_1 \sqcap_{\mathfrak{P}} p_2$  is given as the infimum of the statements that the observer can establish for  $p_1$  and  $p_2$  separately. In other words,

$$\forall p_1, p_2 \in \mathfrak{P}, \exists! (p_1 \sqcap_{\mathfrak{P}} p_2) \in \mathfrak{P} \mid (\forall t \in \mathfrak{T}, \epsilon((p_1 \sqcap_{\mathfrak{P}} p_2), t) = \epsilon(p_1, t) \wedge \epsilon(p_2, t)), \quad (13)$$

where  $\wedge$  denotes the infimum of a couple of elements in the poset  $(\mathfrak{B}, \leq)$ .

We note the following obvious properties deduced from the literal definitions of the random mixing operation  $\sqcap_{\mathfrak{P}}$  and the equivalence relation  $\sim_{\mathfrak{P}}$  defining the space of states.

**Lemma 2.** For any  $p_1, p_2, p_3 \in \mathfrak{P}$ , we have

$$p_1 \sqcap_{\mathfrak{P}} (p_2 \sqcap_{\mathfrak{P}} p_3) \sim_{\mathfrak{P}} (p_1 \sqcap_{\mathfrak{P}} p_2) \sqcap_{\mathfrak{P}} p_3 \quad (14)$$

$$(p_1 \sqcap_{\mathfrak{P}} p_1) \sim_{\mathfrak{P}} p_1, \quad (15)$$

$$(p_2 \sqcap_{\mathfrak{P}} p_1) \sim_{\mathfrak{P}} (p_1 \sqcap_{\mathfrak{P}} p_2) \quad (16)$$

■

More generally, a pre-order relation can be defined on the set  $\mathfrak{P}$  of preparation processes.

**Notion 9.** A preparation process  $p_2 \in \mathfrak{P}$  is said to be *sharper* than another preparation process  $p_1 \in \mathfrak{P}$  (this fact will be denoted  $p_1 \sqsubseteq_{\mathfrak{P}} p_2$ ) iff any yes/no test  $t \in \mathfrak{T}$  that is 'determinate' for the samples prepared along  $p_1$  is also necessarily 'determinate' with the same value for the samples prepared along  $p_2$ , i.e.

$$\forall p_1, p_2 \in \mathfrak{P}, (p_1 \sqsubseteq_{\mathfrak{P}} p_2) :\Leftrightarrow (\forall t \in \mathfrak{T}, \epsilon(p_1, t) \leq \epsilon(p_2, t)), \quad (17)$$

If  $p_1 \sqsubseteq_{\mathfrak{P}} p_2$  (i.e.  $p_2$  is 'sharper' than  $p_1$ ),  $p_1$  is said to be 'coarser' than  $p_2$ .

**Lemma 3.**  $(\mathfrak{P}, \sqsubseteq_{\mathfrak{P}})$  is a pre-ordered set. ■

The equivalence relation defined in **Notion 8** derives from this pre-order :

$$\forall p_1, p_2 \in \mathfrak{P}, (p_1 \sqsubseteq_{\mathfrak{P}} p_2 \text{ and } p_1 \supseteq_{\mathfrak{P}} p_2) \Rightarrow (p_1 \sim_{\mathfrak{P}} p_2). \quad (18)$$

The properties of the equivalence relation  $\sim_{\mathfrak{P}}$  with respect to the pre-order  $\sqsubseteq_{\mathfrak{P}}$  and the random-mixing binary operation  $\sqcap_{\mathfrak{P}}$  leads to the following properties :

**Lemma 4.** The binary operation  $\sqcap_{\mathfrak{P}}$  being literally designed to satisfy the properties (14)(15)(16), the binary relation  $\sqsubseteq_{\mathfrak{P}}$  is then equivalently defined by :

$$\forall p_1, p_2 \in \mathfrak{P}, (p_1 \sqsubseteq_{\mathfrak{P}} p_2) \Leftrightarrow (p_1 \sim_{\mathfrak{P}} (p_1 \sqcap_{\mathfrak{P}} p_2)). \quad (19)$$

The following properties of the pre-order  $\sqsubseteq_{\mathfrak{P}}$  are direct consequences of this fact :

$$\forall p, p_1, p_2 \in \mathfrak{P}, (p_1 \sqcap_{\mathfrak{P}} p_2) \sqsubseteq_{\mathfrak{P}} p_1, \quad (20)$$

$$(p \sqsubseteq_{\mathfrak{P}} p_1 \text{ and } p \sqsubseteq_{\mathfrak{P}} p_2) \Rightarrow (p \sqsubseteq_{\mathfrak{P}} (p_1 \sqcap_{\mathfrak{P}} p_2)). \quad (21)$$

■

Quite naturally, we will assume the existence and uniqueness of a bottom element in  $\mathfrak{P}$ :

**Requirement 2.** it exists a unique preparation process, that can be interpreted as a 'randomly-selected' collection of 'un-prepared samples'. This element leads to a complete undeterminacy for any yes/no test realized on it. In other words, the following axiom will be imposed

$$\exists! p_{\perp} \in \mathfrak{P} \mid (\forall t \in \mathfrak{T}, \epsilon(p_{\perp}, t) = \perp). \quad (22)$$

**Lemma 5.** The space of states  $\mathfrak{S}$  is partially ordered

$$\forall \sigma_1, \sigma_2 \in \mathfrak{S}, (\sigma_1 \sqsubseteq_{\mathfrak{S}} \sigma_2) :\Leftrightarrow (\forall p_1, p_2 \in \mathfrak{P}, (\sigma_1 = [p_1], \sigma_2 = [p_2]) \Rightarrow (p_1 \sqsubseteq_{\mathfrak{P}} p_2)), \quad (23)$$

$$(\mathfrak{S}, \sqsubseteq_{\mathfrak{S}}) \text{ is a partial order.} \quad (24)$$

Moreover, the existence of the preparation procedure  $p_1 \sqcap_{\mathfrak{P}} p_2$  satisfying the property (13) leads to the following definition. ■

**Notion 10.**

$$\forall p_1, p_2 \in \mathfrak{P}, \quad [p_1] \sqcap_{\mathfrak{S}} [p_2] := [p_1 \sqcap_{\mathfrak{P}} p_2]. \quad (25)$$

**Lemma 6.**  $(\mathfrak{S}, \sqcap_{\mathfrak{S}})$  is an Inf semi-lattice. ■

**Lemma 7.**  $(\mathfrak{S}, \sqsubseteq_{\mathfrak{S}})$  admits a bottom element denoted  $\perp_{\mathfrak{S}}$ :

$$\perp_{\mathfrak{S}} := [p_{\perp}] \quad \text{is the bottom element of } (\mathfrak{S}, \sqsubseteq_{\mathfrak{S}}). \quad (26)$$

**Notion 11.** Two preparation processes  $p_1, p_2 \in \mathfrak{P}$  will be said to be *consistent* (this fact will be denoted  $\widehat{p_1 p_2}^{\mathfrak{P}}$ ) iff they can be considered as two different incomplete preparations of the same targeted collection of prepared samples, i.e. iff it exists a preparation process  $p \in \mathfrak{P}$  which is simultaneously sharper than  $p_1$  and  $p_2$  (i.e. is a common upper-bound in  $\mathfrak{P}$ ). In other words,

$$\forall p_1, p_2 \in \mathfrak{P}, \quad \widehat{p_1 p_2}^{\mathfrak{P}} \quad :\Leftrightarrow \quad (\exists p \in \mathfrak{P} \mid p_1 \sqsubseteq_{\mathfrak{P}} p, p_2 \sqsubseteq_{\mathfrak{P}} p). \quad (27)$$

The consistency relation is obviously reflexive and symmetric.

Due to the following relation  $\forall p_1, p_2, p_3 \in \mathfrak{P}, (\widehat{p_1 p_2}^{\mathfrak{P}} \text{ and } p_2 \sim_{\mathfrak{P}} p_3) \Rightarrow \widehat{p_1 p_3}^{\mathfrak{P}}$ , the consistency relation can be defined between states as follows.

$$\forall \sigma_1, \sigma_2 \in \mathfrak{S}, \quad \widehat{\sigma_1 \sigma_2}^{\mathfrak{S}} \quad :\Leftrightarrow \quad (\forall p_1, p_2 \in \mathfrak{P}, \sigma_1 = [p_1], \sigma_2 = [p_2] \Rightarrow \widehat{p_1 p_2}^{\mathfrak{P}}). \quad (28)$$

For any  $\sigma_1, \sigma_2 \in \mathfrak{S}$ , we will denote  $\widetilde{\sigma_1 \sigma_2}^{\mathfrak{S}}$  the property  $\neg \widehat{\sigma_1 \sigma_2}^{\mathfrak{S}}$ .

### 2.3 Physical requirements

The previous construction of the space of states, albeit usual, appears a bit strange from an operational perspective. Indeed, concretely, the observer is never confronted to a given 'state' (i.e. to a generically-infinite class of preparation processes, indistinguishable by the generically-infinite set of tests that can be realized on them) in order to decide if it is consistent (or not) to affirm with 'certainty' the occurrence of a given 'property' for a given sample corresponding to this state. The observer is rather confronted to a restricted set of preparation processes, allowing to produce mixtures, which generically lead to undetermined results when it is confronted to a family of tests.

In order to produce 'determinacy', relative to the occurrence of a given property for a given state of the system, the observer extracts (within the distinguished family of preparation processes at his disposal and which are sensed to produce samples corresponding more-or-less to the chosen state) some sub-families realizing concretely *a sharpening* of the parameters defining the preparation setting/procedure, according to a given set of pre-requisites concerning the samples that will be submitted to the test. Through each 'sharpening' of its preparation procedures, the observer is pretending to fix 'unambiguously', but 'inductively', a 'state' of the system. This limit process is understood in terms of the limit taken for every statements that can be pronounced about the selected samples (i.e. the samples prepared according to any of the preparation processes, which are elements of the chosen sharpening family).

**Notion 12.** A family  $\Omega \subseteq \mathfrak{P}$  is a *sharpening family of preparation processes* (this fact will be denoted  $\Omega \subseteq_{Chain} \mathfrak{P}$ ) iff every couple of elements of  $\Omega$  are ordered by  $\sqsubseteq_{\mathfrak{P}}$ , i.e. for any  $p_1$  and  $p_2$  in  $\Omega$ , we have necessarily  $p_1 \sqsubseteq_{\mathfrak{P}} p_2$  or  $p_2 \sqsubseteq_{\mathfrak{P}} p_1$ .

**Requirement 3.** For any family  $\Omega$  in  $\mathfrak{P}$  defining a 'sharpening', it exists a state  $\sigma$  in  $\mathfrak{S}$  which is the supremum of the chain of states corresponding to the elements of  $\Omega$ .

$$\forall \Omega \subseteq_{Chain} \mathfrak{P}, \quad \text{the supremum } (\bigsqcup_{\mathfrak{S}} [\Omega]) \text{ exists in the partially ordered set } \mathfrak{S} \quad (29)$$

In other words,  $\mathfrak{S}$  will be required to be a *chain-complete partial order*.

Let us then fix a yes/no test  $t$ . If the observer intents to designate the corresponding property as an 'element of reality' attached to the system itself, and not as a datum depending on the explicit operational requirements used to define the state, the following condition must be satisfied.

**Requirement 4.** The observer is authorized to formulate a 'determinate' statement, about the occurrence of a given property, for the 'limit state' induced from a given sharpening family of preparation processes, iff he was already led to formulate this same statement for some preparation process, element of the chosen sharpening family (and then also for any preparation process sharper than this one). In other words,

$$\forall t \in \mathfrak{T}, \forall \mathfrak{C} \subseteq_{Chain} \mathfrak{S}, \quad ((\bigsqcup_{\mathfrak{S}} \mathfrak{C}) \in [\mathfrak{A}_t]) \Rightarrow (\exists \sigma \in \mathfrak{C} \mid \sigma \in [\mathfrak{A}_t]). \quad (30)$$

The set of states  $[\mathfrak{A}_t]$ , for which the property tested by  $t$  is actual, must be a 'Scott-open filter' in the dcpo  $(\mathfrak{S}, \sqsubseteq_{\mathfrak{S}})$ .

We can reformulate previous requirement in terms of a continuity property of the evaluation map with respect to the sharpening process.

**Lemma 8.** For any  $t \in \mathfrak{T}$ , the map  $\tilde{e}_t$  is chain-continuous, i.e. continuous with respect to the Scott-topology on  $\mathfrak{P}$  and  $\mathfrak{B}$ . In other words,

$$\forall t \in \mathfrak{T}, \forall \mathfrak{C} \subseteq_{Chain} \mathfrak{S}, \quad \bigvee_{\sigma \in \mathfrak{C}} \tilde{e}_t(\sigma) = \tilde{e}_t(\bigsqcup_{\mathfrak{S}} \mathfrak{C}). \quad (31)$$

■

Previous topological requirements have to be completed by another one, intimately attached to the specific character of quantum systems. C.Rovelli pretends to reconstruct quantum mechanics from the following conceptual proposal [64, Chap.III]:

*'Information is a discrete quantity: there is a minimum amount of information exchangeable (a single bit, or the information that distinguishes between just two alternatives). [...] Since information is discrete, any process of acquisition of information can be decomposed into acquisitions of elementary bits of information.'* [64, p.1655].

We will translate Rovelli's conceptual proposal as follows.

**Notion 13.** A preparation process  $p_2$  is said to *contain one more bit of information than* another preparation process  $p_1$  (this fact will be denoted  $p_1 \bar{\sqsubseteq}_{\mathfrak{P}} p_2$ ), iff (i)  $p_2$  is strictly sharper than  $p_1$ , and (ii) there is no preparation process strictly separating  $p_1$  and  $p_2$ . In other words,

$$\forall p_1, p_2 \in \mathfrak{P}, (p_1 \bar{\sqsubseteq}_{\mathfrak{P}} p_2) :\Leftrightarrow \begin{cases} (p_1 \sqsubset_{\mathfrak{P}} p_2) \\ (\forall p \in \mathfrak{P}, (p_1 \sqsubseteq_{\mathfrak{P}} p \sqsubseteq_{\mathfrak{P}} p_2) \Rightarrow (p_1 \sim_{\mathfrak{P}} p \text{ or } p_2 \sim_{\mathfrak{P}} p)). \end{cases} \quad (32)$$

The discreteness of the informational content, encoded in the pre-order  $\sqsubseteq_{\mathfrak{P}}$  defined on the set of preparation processes  $\mathfrak{P}$ , is translated into the following topological condition.

**Requirement 5.** For any preparation process admitting some sharper preparation processes, it exists another preparation process which contains one more 'bit of information' than this preparation process. In other words, the pre-order  $(\mathfrak{P}, \sqsubseteq_{\mathfrak{P}})$  satisfies

$$\forall p_1 \in \mathfrak{P} \setminus \text{Max}(\mathfrak{P}), \quad (\exists p_2 \in \mathfrak{P} \mid p_1 \sqsubseteq_{\mathfrak{P}} p_2). \quad (33)$$

However, this requirement is not sufficient to capture all the aspects of Rovelli's proposal. In Rovelli's terms, the exchangeable information is made of a collection of distinguishable 'bits' and the acquisition of an informational content is reduced to the acquisition of these bits of information.

**Requirement 6.** For any preparation process  $p_2 \in \mathfrak{P}$  and any other  $p'_2 \in \mathfrak{P}$  containing one more bit of information (i.e.  $p_2 \sqsubseteq_{\mathfrak{P}} p'_2$ ), the couple  $(p_2, p'_2)$  'defines' a bit of information entering into the 'decomposition' of  $p'_2$ . More explicitly, for any  $p_1 \in \mathfrak{P}$  'coarser' than  $p_2$ , it exists a preparation process  $p'_1$ , unique from the point of view of the statements that can be produced about it, and which (i) contains one more bit of information than  $p_1$ , (ii) is not coarser than  $p_2$ , (iii) is coarser than  $p'_2$ . In other words,

$$\forall p_1, p_2, p'_2 \in \mathfrak{P}, \quad (p_1 \sqsubseteq_{\mathfrak{P}} p_2 \sqsubseteq_{\mathfrak{P}} p'_2) \Rightarrow \begin{cases} \exists p'_1 \mid (p'_1 \not\sqsubseteq_{\mathfrak{P}} p_2 \text{ and } p_1 \sqsubseteq_{\mathfrak{P}} p'_1 \sqsubseteq_{\mathfrak{P}} p'_2) \\ \text{and } p'_1 \text{ is unique up to } \sim_{\mathfrak{P}}. \end{cases} \quad (34)$$

Due to **Requirement 5** and **Requirement 6**, the space of states  $(\mathfrak{S}, \sqsubseteq_{\mathfrak{S}})$  satisfies the following properties

$$\forall \sigma \in \mathfrak{S}, \quad (\sigma \notin \text{Max}(\mathfrak{S})) \Rightarrow (\exists \sigma' \in \mathfrak{S}, \sigma \sqsubseteq \sigma'). \quad (35)$$

$$\forall \sigma_1, \sigma_2, \sigma'_2 \in \mathfrak{S}, \quad (\sigma_1 \sqsubseteq \sigma_2 \sqsubseteq \sigma'_2) \Rightarrow (\exists! \sigma'_1 \in \mathfrak{S}, \mid (\sigma_1 \sqsubseteq \sigma'_1 \sqsubseteq \sigma'_2 \text{ and } \sigma'_1 \not\sqsubseteq \sigma_2)). \quad (36)$$

## 2.4 The space of states as a domain

Let us now exploit the previous requirements to characterize the structure of the space of states. Let us firstly summarize the collected information.

$(\mathfrak{S}, \sqsubseteq_{\mathfrak{S}})$  is a partial order (property (24)). Due to **Requirement 2**, this partial order is pointed (property (26)). Due to **Requirement 1**,  $(\mathfrak{S}, \sqsubseteq_{\mathfrak{S}})$  is also an Inf semi-lattice (Lemma 6). Moreover, due to **Requirement 3**, this partial order is chain-complete (property (29)).

An interesting consequence of the continuity property (31) formalizing the **Requirement 8** is the following property of  $(\mathfrak{S}, \sqsubseteq_{\mathfrak{S}})$ .

**Lemma 9.** The chain-complete Inf semi-lattice  $(\mathfrak{S}, \sqsubseteq_{\mathfrak{S}})$  is meet-continuous, i.e.

$$\forall \mathcal{C} \subseteq_{\text{Chain}} \mathfrak{S}, \forall \sigma \in \mathfrak{S}, \quad \sigma \sqcap_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} \mathcal{C}) = \bigsqcup_{\sigma' \in \mathcal{C}} (\sigma \sqcap_{\mathfrak{S}} \sigma'). \quad (37)$$

■

*Proof.* This fact is easily established using the continuity of the map  $\tilde{\mathfrak{e}}_t$  and the meet-continuity of the dcpo  $\mathfrak{B}$ . For any  $\mathcal{C} \subseteq_{\text{Chain}} \mathfrak{S}$  and any  $\sigma \in \mathfrak{S}$ ,  $(\sigma \sqcap_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} \mathcal{C}))$  and  $\bigsqcup_{\sigma' \in \mathcal{C}} (\sigma \sqcap_{\mathfrak{S}} \sigma')$  exist as elements of  $\mathfrak{S}$ . Moreover,

$$\begin{aligned} \forall t \in \mathfrak{T}, \quad \tilde{\mathfrak{e}}_t(\sigma \sqcap_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} \mathcal{C})) &= \tilde{\mathfrak{e}}_t(\sigma) \wedge \tilde{\mathfrak{e}}_t(\bigsqcup_{\sigma' \in \mathcal{C}} \sigma') \\ &= \tilde{\mathfrak{e}}_t(\sigma) \wedge \bigvee_{\sigma' \in \mathcal{C}} \tilde{\mathfrak{e}}_t(\sigma') \\ &= \bigvee_{\sigma' \in \mathcal{C}} (\tilde{\mathfrak{e}}_t(\sigma) \wedge \tilde{\mathfrak{e}}_t(\sigma')) \\ &= \bigvee_{\sigma' \in \mathcal{C}} \tilde{\mathfrak{e}}_t(\sigma \sqcap_{\mathfrak{S}} \sigma') \\ &= \tilde{\mathfrak{e}}_t(\bigsqcup_{\sigma' \in \mathcal{C}} (\sigma \sqcap_{\mathfrak{S}} \sigma')). \end{aligned}$$

We finally use the separation of the Chu space  $(\mathfrak{B}, \mathfrak{T}, \epsilon)$  (i.e. Lemma 1) to conclude that  $(\sigma \sqcap_{\mathfrak{S}} (\bigsqcup_{\mathfrak{C}})) = \bigsqcup_{\sigma' \in \mathfrak{C}} (\sigma \sqcap_{\mathfrak{S}} \sigma')$ .  $\square$

Due to the **Requirement 5**, the space of states  $(\mathfrak{S}, \sqsubseteq_{\mathfrak{S}})$  satisfies the semi-artinian property (35). Endly, due to the **Requirement 6**, the space of states  $(\mathfrak{S}, \sqsubseteq_{\mathfrak{S}})$  satisfies the polarization property (36).

As a consequence of previous properties, the space of states can be characterized as follows

**Definition 1. [Main axiomatic for selection structures]** The poset  $(\mathfrak{S}, \sqsubseteq_{\mathfrak{S}})$  will be said to be a *selection structure* iff it satisfies the following axioms

$$\exists \perp_{\mathfrak{S}} \in \mathfrak{S}, \forall x \in \mathfrak{S}, \perp_{\mathfrak{S}} \sqsubseteq_{\mathfrak{S}} x, \quad (38)$$

$$\forall x, y \in \mathfrak{S}, (x \sqsubset_{\mathfrak{S}} y) \Rightarrow (\exists t \in \mathfrak{S}, x \bar{\sqsubseteq}_{\mathfrak{S}} t \sqsubseteq_{\mathfrak{S}} y), \quad (39)$$

$$\forall x, y, z \in \mathfrak{S}, (z \bar{\sqsubseteq}_{\mathfrak{S}} x \text{ and } z \sqsubset_{\mathfrak{S}} y \text{ and } x \parallel_{\mathfrak{S}} y) \Rightarrow (z = x \sqcap_{\mathfrak{S}} y), \quad (40)$$

$$\forall x, u, v \in \mathfrak{S}, (x \sqsubset_{\mathfrak{S}} u \bar{\sqsubseteq}_{\mathfrak{S}} v) \Rightarrow (\exists! y \in \mathfrak{S}, \text{ such that } (x \bar{\sqsubseteq}_{\mathfrak{S}} y \sqsubset_{\mathfrak{S}} v \text{ and } y \parallel_{\mathfrak{S}} u)), \quad (41)$$

$$\forall C \subseteq_{\text{chain}} \mathfrak{S}, (\bigsqcup_{\mathfrak{S}} C) \text{ exists in } \mathfrak{S} \quad (42)$$

$$\forall C \subseteq_{\text{chain}} \mathfrak{S}, \forall x, y \in \mathfrak{S} \mid (x \bar{\sqsubseteq}_{\mathfrak{S}} y, x \sqsubset_{\mathfrak{S}} C), (y \parallel_{\mathfrak{S}} C) \Rightarrow (y \parallel_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} C)). \quad (43)$$

The set of selection structures is denoted  $\mathcal{S}$ .

*Remark 5.* Note that, due to property (40), the property (43) can be rewritten equivalently as:

$$\forall x, y \in \mathfrak{S} \mid x \bar{\sqsubseteq}_{\mathfrak{S}} y, \forall C \subseteq_{\text{chain}} \mathfrak{S}, (\forall c \in C, x = (y \sqcap_{\mathfrak{S}} c)) \Rightarrow (x = (y \sqcap_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} C))). \quad (44)$$

*Remark 6.* Note that the property (41) will be equivalently rewritten as:

$$\forall x, u, v \in \mathfrak{S}, (x \sqsubset_{\mathfrak{S}} u \bar{\sqsubseteq}_{\mathfrak{S}} v) \Rightarrow (\exists! y \in \mathfrak{S}, \text{ such that } (x \bar{\sqsubseteq}_{\mathfrak{S}} y \sqsubseteq_{\mathfrak{S}} v \text{ and } y \not\sqsubseteq_{\mathfrak{S}} u)). \quad (45)$$

**Theorem 1.**  $(\mathfrak{S}, \sqsubseteq_{\mathfrak{S}})$  is a selection structure.  $\blacksquare$

*Proof.* Note that, for technical reasons, the presentation of selection structures, given in Definition 1, adopts a more elementary form. However, we can easily show the strict equivalence of the two presentations and then the announced result. More precisely, we can adopt an equivalent axiomatic for selection structures, obtained by replacing the couple of axioms (strong-atomicity property (39) and weak-continuity property (44)) by the couple of axioms (meet-continuity property (37) and the semi-artinian property (35)), and by replacing the property (40) by the Inf semi-lattice axiom.

We firstly note that the semi-artinian property (35), is clearly a weakening of the strong-atomicity property (property 39).

Secondly, the weak-continuity property (44) is a weakening of meet-continuity property (37)). Indeed, if the meet-continuity property (37) is satisfied by the selection structure  $\mathfrak{S}$ , we have for any  $s \in \mathfrak{S}$ ,  $(s \sqcap_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} C)) = (\bigsqcup_{\mathfrak{S}} \{s \sqcap_{\mathfrak{S}} c \mid c \in C\}) = (\bigsqcup_{\mathfrak{S}} \{(s \sqcap_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} C)) \sqcap_{\mathfrak{S}} c \mid c \in C\})$ . Hence, for any  $u \sqsubseteq_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} C)$  (i.e. any  $u$  that can be rewritten  $u = u \sqcap_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} C)$ ), we have  $u = (\bigsqcup_{\mathfrak{S}} \{u \sqcap_{\mathfrak{S}} c \mid c \in C\})$ . Now, we obtain the property (44) as a special case of this reformulation of meet-continuity property, where  $u$  is only authorized to satisfy  $u \bar{\sqsubseteq}_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} C)$  and not  $u \sqsubseteq_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} C)$  in general.

Thirdly, a meet-continuous semi-artinian poset is necessarily strongly atomic (i.e. combined properties (35) and (37)) imply property (39)). More precisely, we intent to check the strong atomicity of  $\mathfrak{S}$  (i.e. property (39)) using exclusively the axioms (38)(40)(41)(42) and (37)(35).

To begin, note that if  $]x, y[ = \emptyset$ , then  $x \bar{\sqsubseteq}_{\mathfrak{S}} y$  and the result is trivially established. We will then only be interested in the non trivial case,  $]x, y[ \neq \emptyset$ . Let us begin with some considerations about  $\text{Max}(\mathfrak{S})$ .

We will denote for any  $s \in \mathfrak{S}$ ,  $\hat{s} := \text{Max}(\mathfrak{S}) \cap ]s[$ .

First of all, for any  $v \in ]x, y[$ , we have necessarily  $\hat{x} \supseteq \hat{v} \supseteq \hat{y} \neq \emptyset$ . Now, we can distinguish two cases:

either  $(\exists v \in ]x, y], \hat{x} = \hat{v})$  or  $(\forall v \in ]x, y], \hat{v} \subset \hat{x})$ .

In the first case, we have necessarily  $]x) = ]x, v] \cup ]v)$ , and then,  $\mathfrak{S}$  being semi-artinian (equation 35),  $\exists t \in \mathfrak{S}, x \bar{\sqsubseteq}_{\mathfrak{S}} t \sqsubseteq_{\mathfrak{S}} v \sqsubseteq_{\mathfrak{S}} y$  which is the announced result.

In the second case, let us choose any  $v \in ]x, y]$ , such that  $\hat{v} \subset \hat{x}$ , and choose  $u \in \hat{v}$ . We have obviously  $x \sqsubseteq_{\mathfrak{S}} v \sqsubseteq_{\mathfrak{S}} u$  and  $x \sqsubseteq_{\mathfrak{S}} v \sqsubseteq_{\mathfrak{S}} (u \sqcap_{\mathfrak{S}} y)$ .

Let us consider the subset  $Q$  of  $P$  given by  $Q := \{s \in \mathfrak{S} \mid s \sqsubseteq_{\mathfrak{S}} u \text{ and } y \sqcap_{\mathfrak{S}} s = x\}$ .  $Q$  is not empty because  $x$  is an element of  $Q$ . Let us consider  $C \sqsubseteq_{\text{chain}} Q$ . We remark that, due to the definition of supremum, we have  $(\bigsqcup_{\mathfrak{S}} C) \sqsubseteq_{\mathfrak{S}} u$  and, due to meet-continuity of  $\mathfrak{S}$  (equation 37), we have  $y \sqcap_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} C) = \bigsqcup_{\mathfrak{S}} \{(y \sqcap_{\mathfrak{S}} c) \mid c \in C\} = x$ . As a result,  $(\bigsqcup_{\mathfrak{S}} C) \in Q$ . From Zorn's lemma, we deduce that  $Q$  has some maximal elements. Let us choose  $z \in \text{Max}(Q)$ . We first note that  $z \sqsubseteq_{\mathfrak{S}} u$  by definition of  $Q$ , but the equality  $z = u$  is excluded because  $(u \sqcap_{\mathfrak{S}} y) \neq x$ . Hence, our element  $u \in \mathfrak{S}$  satisfies  $z \sqsubseteq_{\mathfrak{S}} u$ . In other words,  $z \notin \text{Max}(\mathfrak{S})$ .  $\mathfrak{S}$  being a semi-artinian (equation 35), it then exists  $w \in \mathfrak{S}$ , such that  $z \bar{\sqsubseteq}_{\mathfrak{S}} w$ . For any choice of  $w$  and  $y$ , we can define the element  $t := w \sqcap_{\mathfrak{S}} y$  which satisfies  $t \sqsubseteq_{\mathfrak{S}} w$  and  $t \sqsubseteq_{\mathfrak{S}} y$ . We note that  $z \sqcap_{\mathfrak{S}} t = z \sqcap_{\mathfrak{S}} (w \sqcap_{\mathfrak{S}} y) = (z \sqcap_{\mathfrak{S}} y) \sqcap_{\mathfrak{S}} w = x \sqcap_{\mathfrak{S}} w$ . The lower-covering property (Lemma 40) has been shown without using strong-atomicity of  $\mathfrak{S}$ , i.e. equation 39), the property  $z \bar{\sqsubseteq}_{\mathfrak{S}} w$  implies that  $(z \sqcap_{\mathfrak{S}} y) \bar{\sqsubseteq}_{\mathfrak{S}} (w \sqcap_{\mathfrak{S}} y)$  or  $(z \sqcap_{\mathfrak{S}} y) = (w \sqcap_{\mathfrak{S}} y)$ , i.e.  $x \bar{\sqsubseteq}_{\mathfrak{S}} (w \sqcap_{\mathfrak{S}} y)$  or  $x = (w \sqcap_{\mathfrak{S}} y)$ . However, previous equality is forbidden by the maximality condition imposed on  $z$  (i.e.  $z \in \text{Max}(Q)$  implies  $z \bar{\sqsubseteq}_{\mathfrak{S}} w \notin Q$ ). As a result, we deduce that  $x \bar{\sqsubseteq}_{\mathfrak{S}} t \sqsubseteq_{\mathfrak{S}} y$ . This concludes the proof of strong-atomicity of  $\mathfrak{S}$ .

Endly, the replacement of property (40) by the Inf semi-lattice axiom is allowed by Theorem 19 in Section 6.  $\square$

In the following, we will extensively use the definitions and results relative to selection structures and detailed in the appendix of this paper. In particular, we recall the following fundamental result about the space of states :

**Corollary 1. [The space of states is a locally-boolean qualitative domain]**

The space of states  $(\mathfrak{S}, \sqsubseteq_{\mathfrak{S}})$  is a 'qualitative-domain' (i.e. a conditionally-distributive and finitary Scott-domain in which the prime elements are the atoms of  $\mathfrak{S}$ ) such that, for any finite and then compact element  $\sigma$ , the subset  $(\downarrow_{\mathfrak{S}} \sigma)$  is a finite boolean lattice constituted of finite elements.  $\blacksquare$

*Proof.* this conclusion has been obtained in Theorem 25 completed by Remark 22.  $\square$

**Definition 2. Sub- selection structures**

Let  $(\mathfrak{S}, \sqsubseteq_{\mathfrak{S}})$  be a selection structure and  $(\mathfrak{S}', \sqsubseteq_{\mathfrak{S}'})$  be a sub-poset of  $(\mathfrak{S}, \sqsubseteq_{\mathfrak{S}})$ .

$(\mathfrak{S}', \sqsubseteq_{\mathfrak{S}'})$  will be said to be a *sub- selection structure of  $(\mathfrak{S}, \sqsubseteq_{\mathfrak{S}})$*  iff : (i)  $\mathfrak{S}'$  is a downset of  $(\mathfrak{S}, \sqsubseteq_{\mathfrak{S}})$ , and (ii)  $(\mathfrak{S}', \sqsubseteq_{\mathfrak{S}'})$  is a selection structure in itself. The set of sub- selection structures of a given selection structure  $(\mathfrak{S}, \sqsubseteq_{\mathfrak{S}})$  will be denoted  $\mathcal{O}_{\mathfrak{S}}$ .

**Lemma 10.**  $[\Omega_I]$  is a sub- selection structure of  $\mathfrak{S}$ .  $\blacksquare$

*Proof.*  $[\Omega_I]$  is a Scott-closed subset in  $\mathfrak{S}$ , as the reverse image of the Scott-closed subset  $\{\perp, \mathbf{Y}\}$  by the Scott-continuous map  $\tilde{v}_I$ . Using Lemma 56, we conclude that  $[\Omega_I]$  is a sub- selection structure of  $\mathfrak{S}$ .  $\square$

## 2.5 Pure states

**Notion 14.** A state is said to be a *pure state* if and only if it can not be build as a mixture of other states (the set of pure states will be denoted  $\mathfrak{S}_{\text{pure}}$ ). More explicitly,

$$\sigma \in \mathfrak{S}_{\text{pure}} \quad :\Leftrightarrow \quad (\forall S \subseteq^{\neq \emptyset} \mathfrak{S}, (\sigma = \bigsqcap_{\mathfrak{S}} S) \Rightarrow (\sigma \in S)). \quad (46)$$



In other words, pure states are associated to complete meet-irreducible elements in the selection structure  $\mathfrak{S}$ .

*Remark 7.* Complete meet-irreducibility implies meet-irreducibility. In other words,

$$\sigma \in \mathfrak{S}_{\text{pure}} \Rightarrow \forall \sigma_1, \sigma_2 \in \mathfrak{S}, (\sigma = \sigma_1 \sqcap_{\mathfrak{S}} \sigma_2) \Rightarrow (\sigma = \sigma_1 \text{ or } \sigma = \sigma_2). \quad (47)$$

A simple characterization of completely meet-irreducible elements within posets is adopted in [28, Definition I-4.21]. This characterization is equivalent to previous one for bounded complete inf semi-lattice ( $\mathfrak{S}$  is bounded-complete due to Remark 20). We have explicitly

**Theorem 2. [First characterization of pure states]**

$$\sigma \in \mathfrak{S}_{\text{pure}} \Leftrightarrow (\sigma \in \text{Max}(\mathfrak{S}) \text{ or } (\uparrow^{\mathfrak{S}} \sigma) \setminus \{\sigma\} \text{ admits a minimum element}). \quad (48)$$

■

**Theorem 3. [Every state is a mixture of pure states and the set of pure states is unambiguously defined]**

Every state can be written as a mixture of pure states. In other words,

$$\forall \sigma \in \mathfrak{S} \quad \sigma = \bigsqcap_{\mathfrak{S}} (\mathfrak{S}_{\text{pure}} \cap (\uparrow^{\mathfrak{S}} \sigma)). \quad (49)$$

Moreover,  $\mathfrak{S}_{\text{pure}}$  is the unique smallest subset of states generating any state by mixture. In other words,  $\mathfrak{S}_{\text{pure}}$  is the unique smallest order-generating subset in  $\mathfrak{S}$  (i.e. the unique smallest subset of  $\mathfrak{S}$  satisfying property (49)). ■

*Proof.*  $\mathfrak{S}$  being, in particular, a bounded complete algebraic domain, this result is a direct consequence of [28, Theorem I-4.26]. □

A second characterization of completely meet-irreducible elements, valid for bounded complete algebraic domains ( $\mathfrak{S}$  is bounded-complete due to Remark 20 and algebraic due to Theorem 21), is adopted in [28, Proposition I-4.27]:

**Theorem 4. [Second characterization of pure states]**

$$\sigma \in \mathfrak{S}_{\text{pure}} \Leftrightarrow (\exists c \in \mathfrak{S}_c \mid \sigma \in \text{Max}(\mathfrak{S} \setminus (\uparrow^{\mathfrak{S}} c))). \quad (50)$$

■

## 3 Properties and measurements

### 3.1 Properties and States

Let us now focus on the set of yes/no tests. Adopting our perspective on the Chu duality between  $\mathfrak{P}$  and  $\mathfrak{T}$ , it is natural to introduce the following equivalence relation on  $\mathfrak{T}$ .

**Notion 15.** An equivalence relation, denoted  $\sim_{\mathfrak{T}}$ , is defined on the set of yes/no tests  $\mathfrak{T}$  :

Two yes/no tests are identified iff the corresponding statements established by the observer about any given preparation process are the same.

A *property* of the physical system is an equivalence class of yes/no tests, i.e. a class of yes/no tests that are not distinguished from the point of view of the statements that the observer can produce by using these yes/no tests on finite collections of samples.

The set of equivalence classes of yes/no tests, modulo the relation  $\sim_{\mathfrak{T}}$ , will be denoted  $\mathcal{L}$ .

In other words,

$$\forall t_1, t_2 \in \mathfrak{T}, (t_1 \sim_{\bar{x}} t_2) \Leftrightarrow (\forall p \in \mathfrak{P}, e(p, t_1) = e(p, t_2)), \quad (51)$$

$$\sim_{\bar{x}} \text{ is an equivalence relation,} \quad (52)$$

$$[t] := \{t' \in \mathfrak{T} \mid t' \sim_{\bar{x}} t\}, \quad (53)$$

$$\mathcal{L} := \{[t] \mid t \in \mathfrak{T}\}. \quad (54)$$

The following equivalence justifies the use of the notion of 'property' in the literal definitions of 'potentiality' and 'actuality' :

$$\forall t_1, t_2 \in \mathfrak{T}, (t_1 \sim_{\bar{x}} t_2) \Leftrightarrow (\Omega_{t_1} = \Omega_{t_2} \text{ and } \mathfrak{A}_{t_1} = \mathfrak{A}_{t_2}). \quad (55)$$

Hence,

**Notion 16.** for any  $l \in \mathcal{L}$ , we will currently denote by  $\Omega_l$  (resp.  $\mathfrak{A}_l$ ) the set  $\Omega_t$  (resp.  $\mathfrak{A}_t$ ) taken for any  $t$  such that  $l = [t]$ .

Moreover,

**Notion 17.** for any  $l \in \mathcal{L}$ , we will currently denote by  $\tilde{e}_l$  the evaluation map defined on  $\mathfrak{G}$  and defined by  $\tilde{e}_l := \tilde{e}_t$  for any  $t$  such that  $l = [t]$ .

**Notion 18.** A property  $l \in \mathcal{L}$  will be said to be *testable* iff it can be revealed as 'actual' at least for some collections of prepared samples. In other words,

$$l \text{ is 'testable' } \Leftrightarrow \mathfrak{A}_l \neq \emptyset. \quad (56)$$

We check immediatly that  $\forall t_1, t_2 \in \mathfrak{T}, (t_1 \sim_{\bar{x}} t_2) \Leftrightarrow (\bar{t}_2 \sim_{\bar{x}} \bar{t}_1)$ . As a consequence, the bar involution will be defined on the space of properties simply by requiring

**Notion 19.**

$$\forall l \in \mathcal{L}, \quad \bar{l} := \{\bar{t} \mid l = [t]\}. \quad (57)$$

$(\mathfrak{G}, \sqsubseteq_{\mathfrak{G}})$  being a selection structure, we know that  $(\mathfrak{G}, \sqsubseteq_{\mathfrak{G}})$  is a bounded-complete Inf semi-lattice, and in particular is closed under arbitrary Infima. In other words,

**Lemma 11.**

$$\forall \Omega \subseteq^{\neq \emptyset} \mathfrak{P}, \quad \text{the infimum } (\bigsqcap_{\mathfrak{G}} [\Omega]) \text{ exists in } \mathfrak{G}, \quad (58)$$

■

Moreover, we inherit from Lemma 8 the following continuity property :

**Lemma 12.**

$$\forall \mathfrak{R} \subseteq^{\neq \emptyset} \mathfrak{G}, \forall t \in \mathfrak{T}, \quad \tilde{e}_t(\bigsqcap_{\mathfrak{G}} \mathfrak{R}) = \bigwedge_{\sigma \in \mathfrak{M}_{\mathfrak{R}}} \tilde{e}_t(\sigma). \quad (59)$$

■

*Proof.*  $\forall \mathfrak{R} \subseteq^{\neq \emptyset} \mathfrak{G}$ , we define  $\mathfrak{M}_{\mathfrak{R}} := \{\sigma \in \mathfrak{G} \mid \sigma \sqsubseteq_{\mathfrak{G}} \mathfrak{R}\}$ .  $\mathfrak{M}_{\mathfrak{R}}$  is obviously directed and  $\bigsqcap_{\mathfrak{G}} \mathfrak{R} = \bigsqcap_{\mathfrak{G}} \mathfrak{M}_{\mathfrak{R}}$ . Now, using the Scott-continuity of  $\tilde{e}_t$ , we have  $\tilde{e}_t(\bigsqcap_{\mathfrak{G}} \mathfrak{R}) = \bigvee_{\sigma \in \mathfrak{M}_{\mathfrak{R}}}^{\bar{\wedge}} \tilde{e}_t(\sigma)$ . The monotonicity of  $\tilde{e}_t$  and the fact that the target space of  $\tilde{e}_t$  is the boolean domain  $\mathfrak{B}$  implies moreover that  $\bigvee_{\sigma \in \mathfrak{M}_{\mathfrak{R}}}^{\bar{\wedge}} \tilde{e}_t(\sigma) = \bigwedge_{\sigma \in \mathfrak{M}_{\mathfrak{R}}} \tilde{e}_t(\sigma)$ . □

**Theorem 5.** For any property  $l \in \mathcal{L}$ , the evaluation map  $\tilde{e}_l$  is order-preserving and continuous with respect to the Lawson topologies on  $\mathfrak{S}$  and  $\mathfrak{B}$ . ■

*Proof.* From previous lemma we have that, for any  $t \in \mathfrak{T}$ , the map  $\tilde{e}_t$  is continuous with respect to the lower-topologies on  $\mathfrak{P}$  and  $\mathfrak{B}$ .

Due to Lemma 8 and property (59), and using [28, Theorem III-1.8 p.213], we then prove the announced continuity property. □

We can then deduce the form of the subsets  $[\mathfrak{A}_t]$  determining the sub-space of states for which the property tested by  $t$  is actual.

**Theorem 6. [Property-state]**

For any  $t$  in  $\mathfrak{T}$ , corresponding to the testable property  $[t]$ , it exists an element  $\Sigma_t \in \mathfrak{S}_c$  (i.e. a compact element in the algebraic domain  $\mathfrak{S}$ ), such that the Scott-open filter  $[\mathfrak{A}_t]$  is the principal filter associated to  $\Sigma_t$  :

$$\forall t \in \mathfrak{T} \mid [t] \text{ is testable, } \exists \Sigma_t \in \mathfrak{S}_c \mid [\mathfrak{A}_t] = (\uparrow^{\mathfrak{S}} \Sigma_t). \quad (60)$$

In particular, we have

$$\forall t \in \mathfrak{T}, \forall \sigma \in \mathfrak{S}, \quad \tilde{e}_t(\sigma) = \mathbf{Y} \Leftrightarrow \Sigma_t \sqsubseteq_{\mathfrak{S}} \sigma. \quad (61)$$

■

*Proof.* We have already remarked that  $[\mathfrak{A}_t]$  has to be a Scott-open filter. Using the fact that  $\mathfrak{S}$  is a complete Inf semi-lattice and the property (59), we also note that the element  $\Sigma_t := \bigcap_{\mathfrak{S}} [\mathfrak{A}_t]$  obeys  $\tilde{e}_t(\Sigma_t) = \bigwedge_{\sigma \in [\mathfrak{A}_t]} \tilde{e}_t(\sigma) = \mathbf{Y}$ . Then,  $\Sigma_t$  is the 'minimum' of  $[\mathfrak{A}_t]$ . As a consequence, the filter  $[\mathfrak{A}_t]$  reveals to be the principal filter  $(\uparrow^{\mathfrak{S}} \Sigma_t)$ . From [28, Remark I-4.24], we deduce that  $\Sigma_t$  is in  $\mathfrak{S}_c$ . □

**Notion 20.** For any yes/no test  $t \in \mathfrak{T}$ , corresponding to the testable property  $[t]$ , the state  $\Sigma_t$  is defined to be the minimal element of the principal filter  $[\mathfrak{A}_t]$  in  $(\mathfrak{S}, \sqsubseteq_{\mathfrak{S}})$ .

$$\forall t \in \mathfrak{T} \mid [t] \text{ is testable, } \Sigma_t := \bigcap_{\mathfrak{S}} [\mathfrak{A}_t] = \bigcap_{\mathfrak{S}} \{ \sigma \mid \tilde{e}_t(\sigma) = \mathbf{Y} \}. \quad (62)$$

The state  $\Sigma_t$  depends only on the testable property  $[t]$  associated to  $t$ . This state will then be called the *property-state* associated to  $t$  and we will currently adopt the following abuse of notation  $\Sigma_{[t]} := \Sigma_t$ .

### 3.2 Tests and measurements

The measurement process offers a new perspective on the relation between the spaces  $\mathfrak{P}$  and  $\mathfrak{T}$  (the first one was a duality relation) : this perspective emphasizes the 'recursive' aspect of preparation process. Indeed, a given yes/no test  $t \in \mathfrak{T}$  can be used to complete a given preparation procedure  $p \in \mathfrak{P}$  in order to produce a new preparation procedure, as soon as the 'filtering operation' associated to  $t$  actually operates on a collection of produced samples that can exhibit the desired property associated to  $t$ .

**Notion 21.** For any yes/no test  $t \in \mathfrak{T}$  and any preparation procedure  $p \in \mathfrak{P}$ , we can define a preparation procedure denoted  $p.t$  and defined as follows :

the samples, previously prepared along the procedure  $p$ , are actually submitted to the measurement operation defined according to the yes/no test  $t$ , the out-coming samples of the 'whole preparation process' (i.e. the initial preparation  $p$  followed by the filtering operation defined by  $t$ ), denoted  $(p.t)$ , are the samples 'actually measured as positive' through the yes/no test  $t$ .

To any yes/no test  $t \in \mathfrak{T}$ , we will then associate the partial map denoted  $\cdot t$  and defined by (its domain will be denoted  $Dom_{\cdot t}^{\mathfrak{P}} \subseteq \mathfrak{P}$ , and its range  $Im_{\cdot t}^{\mathfrak{P}}$ ):

$$\begin{aligned} \cdot t : \mathfrak{P} &\dashrightarrow \mathfrak{P} & Dom_{\cdot t}^{\mathfrak{P}} &:= \Omega_t \\ p &\mapsto p \cdot t \end{aligned} \quad (63)$$

These maps define the measurement operation associated to a given property. The map  $\cdot t$  associated to a yes/no test  $t$  is called *the measurement map* associated to  $t$ .

To begin, let us clarify the notion of 'succession' of measurements.

**Notion 22.** For any yes/no tests  $t_1, t_2 \in \mathfrak{T}$ , we can build a new yes/no test denoted  $(t_1 \cdot t_2) \in \mathfrak{T}$  and called *the succession of  $t_1$  by  $t_2$* . It is defined as follows :

to begin, the incoming sample is tested along the yes/no test  $t_1$ ; if the result of this test is negative, then the whole yes/no test  $(t_1 \cdot t_2)$  is declared 'negative', otherwise the outgoing sample having been positively measured by  $t_1$  is submitted to the yes/no test  $t_2$ ; the result of this test is then attributed to the whole yes/no test  $(t_1 \cdot t_2)$  for the given prepared sample. In other words,

$$\forall t_1, t_2 \in \mathfrak{T}, \exists (t_1 \cdot t_2) \in \mathfrak{T} \quad | \quad \mathfrak{A}_{t_1 \cdot t_2} = \{ p \in \mathfrak{A}_{t_1} \mid (p \cdot t_1) \in \mathfrak{A}_{t_2} \} \quad (64)$$

$$\Omega_{t_1 \cdot t_2} = \{ p \in \Omega_{t_1} \mid (p \cdot t_1) \in \Omega_{t_2} \} \quad (65)$$

$$\forall p \in Dom_{\cdot (t_1 \cdot t_2)}^{\mathfrak{P}} = \Omega_{t_1 \cdot t_2}, \quad p \cdot (t_1 \cdot t_2) = (p \cdot t_1) \cdot t_2. \quad (66)$$

We note the following natural relations.

**Lemma 13.**

$$\forall t_1, t_2 \in \mathfrak{T}, \quad \forall p \in \Omega_{t_1}, \quad e(p \cdot t_1, t_2) \geq e(p, t_1 \cdot t_2), \quad (67)$$

$$\overline{t_1 \cdot t_2} = \overline{t_1} \cdot \overline{t_2} \quad (68)$$

■

**Requirement 7.** The 'succession' of yes/no tests satisfies the following associativity property:

$$\forall t_1, t_2, t_3 \in \mathfrak{T}, \quad (t_1 \cdot t_2) \cdot t_3 = t_1 \cdot (t_2 \cdot t_3). \quad (69)$$

As a basic requirement about measurement maps, we will impose that they are monotonic maps on their domain. The following simple analysis justifies this requirement.

If the preparation processes  $p_1$  and  $p_2$  are ordered by  $\sqsubseteq_{\mathfrak{P}}$  ( $p_2$  being sharper than  $p_1$ ), every statements pronounced by the observer about  $p_1$  are also necessarily pronounced about  $p_2$ , i.e.  $\forall u \in \mathfrak{T}, e(p_1, u) \leq e(p_2, u)$ . This is true in particular for the statements that can be pronounced by the observer about the corresponding collections of samples after having separately measured the property associated to a given yes/no test  $t$  on each collection of samples beforehand. More precisely, we must then have  $\forall v \in \mathfrak{T}, e(p_1 \cdot t, v) \leq e(p_2 \cdot t, v)$ . In other words,

**Requirement 8.** a measurement operation associated to any yes/no test  $t$  respects the ordering of information established by the observer about the collection of samples on which it is realized.

$$\forall t \in \mathfrak{T}, \quad \forall p_1, p_2 \in \Omega_t, \quad (p_1 \sqsubseteq_{\mathfrak{P}} p_2) \Rightarrow ((p_1 \cdot t) \sqsubseteq_{\mathfrak{P}} (p_2 \cdot t)). \quad (70)$$

The measurement map  $(\cdot t)$  is an order-preserving map on  $(\mathfrak{P}, \sqsubseteq_{\mathfrak{P}})$ .

**Lemma 14.** As a consequence, the measurement operation  $(\cdot t)$  associated to a given yes/no test  $t$  can not distinguish different collection of samples on which it acts, as soon as these collections of samples correspond to the same state of the system, i.e.

$$\forall t \in \mathfrak{T}, \forall p_1, p_2 \in \Omega_t, \quad (p_1 \sim_{\mathfrak{P}} p_2) \Rightarrow ((p_1 \cdot t) \sim_{\mathfrak{P}} (p_2 \cdot t)). \quad (71)$$

■

**Notion 23.** The measurement operation will then be defined to act on states as follows:

$$\forall t \in \mathfrak{T}, \forall p \in \Omega_t, \quad [p] \cdot t := [p \cdot t] \quad (72)$$

We will adopt the following notations  $Dom_t^{\mathfrak{S}} := [Dom_t^{\mathfrak{P}}]$  and  $Im_t^{\mathfrak{S}} := [Im_t^{\mathfrak{P}}]$ .

**Lemma 15.** The measurement map  $(\cdot t)$  is an order-preserving map on  $(\mathfrak{S}, \sqsubseteq_{\mathfrak{S}})$

$$\forall t \in \mathfrak{T}, \forall \sigma_1, \sigma_2 \in \mathfrak{S}_t, \quad (\sigma_1 \sqsubseteq_{\mathfrak{S}} \sigma_2) \Rightarrow ((\sigma_1 \cdot t) \sqsubseteq_{\mathfrak{S}} (\sigma_2 \cdot t)). \quad (73)$$

■

We will complete the previous requirement of monotonicity by a requirement concerning the random mixing.

**Requirement 9.** The operation of measurement 'commutes' with the operation of random mixing on the measured samples. More precisely, the collection of samples issued from the random mixing of two collections of samples obtained by separately measuring by  $(\cdot t)$  the collections of samples prepared respectively along the preparation processes  $p_1$  and  $p_2$ , which can potentially exhibit the property  $[t]$ , cannot be distinguished by the statements of the observer from the collection of samples issued from the measuring operation by  $(\cdot t)$  of the collection of samples prepared beforehand by the random mixing of the collections prepared along  $p_1$  and  $p_2$ . In other words,

$$\forall t \in \mathfrak{T}, \forall p_1, p_2 \in \Omega_t, \quad (p_1 \sqcap_{\mathfrak{P}} p_2) \cdot t \sim_{\mathfrak{P}} (p_1 \cdot t) \sqcap_{\mathfrak{P}} (p_2 \cdot t). \quad (74)$$

We will more generally require

$$\forall t \in \mathfrak{T}, \forall Q \subseteq \Omega_t, \quad (\bigsqcap_{\mathfrak{P}} Q) \cdot t \sim_{\mathfrak{P}} \bigsqcap_{p \in Q} (p \cdot t). \quad (75)$$

**Lemma 16.** As an immediate consequence of previous requirement, for any yes/no test  $t$ , the measurement map  $(\cdot t)$  is a partial map preserving arbitrary infima on the space of states. In other words,

$$\forall t \in \mathfrak{T}, \forall S \subseteq [\Omega_t], \quad (\bigsqcap_{\mathfrak{S}} S) \cdot t = \bigsqcap_{\sigma \in S} (\sigma \cdot t). \quad (76)$$

■

Endly, let us consider a sharpening family of preparation processes  $\Omega$ . The existence of the limit state  $(\bigsqcap_{\mathfrak{S}} [\Omega])$  is ensured by the collection of properties  $\bigvee_{\sigma \in [\Omega]} \tilde{\mathfrak{e}}_u(\sigma) = \tilde{\mathfrak{e}}_u(\bigsqcap_{\mathfrak{S}} [\Omega])$  considered for every yes/no test  $u \in \mathfrak{T}$ . If we consider, in particular, the subset of statements associated to any yes/no test  $t_2$  pronounced by the observer concerning the states outcoming from the measurement associated to a yes/no test  $t_1$ , we deduce  $\bigvee_{\sigma \in [\Omega]} \tilde{\mathfrak{e}}_{t_2}(\sigma \cdot t_1) = \tilde{\mathfrak{e}}_{t_2}((\bigsqcap_{\mathfrak{S}} [\Omega]) \cdot t_1)$ . And then, for any  $t_1, t_2 \in \mathfrak{T}$ , we obtain  $\tilde{\mathfrak{e}}_{t_2}(\bigsqcap_{\sigma \in [\Omega]} (\sigma \cdot t_1)) = \tilde{\mathfrak{e}}_{t_2}((\bigsqcap_{\mathfrak{S}} [\Omega]) \cdot t_1)$ . As a conclusion, we will require the following property.

**Requirement 10.** The operation of measurement respects the induction process of a limit state from any sharpening family. In other words, for any yes/no test  $t$ , the measurement map  $(\cdot)t$  is a Scott-continuous partial map.

$$\forall t \in \mathcal{T}, \forall \mathcal{C} \subseteq_{Chain} [\Omega_t], \quad \bigsqcup_{\sigma \in \mathcal{C}} (\sigma.t) = \left( \bigsqcup_{\sigma \in \mathcal{C}} \sigma \right).t. \quad (77)$$

**Definition 3.** The set of partial maps defined from the selection structure  $\mathfrak{S}_1$  (the domain of the partial map has to be a Scott-closed subset of  $\mathfrak{S}_1$ ) to the selection structure  $\mathfrak{S}_2$ , which are order-preserving and Lawson-continuous (i.e. Scott-continuous and preserve arbitrary-meets) will be denoted  $[\mathfrak{S}_1 \dashrightarrow \mathfrak{S}_2]_{\sqcap}$ .

To summarize **Requirements 8, 9, 10**, we will impose :

$$\forall t \in \mathcal{T}, \quad \cdot t \in [\mathfrak{S} \dashrightarrow \mathfrak{S}]_{\sqcap} \mid Dom_{\cdot t}^{\mathfrak{S}} = [\Omega_t]. \quad (78)$$

### 3.3 Minimally-disturbing measurements

As analyzed before, the 'certainty' of the observer, about the occurrence (or not) of a given 'property' for a given state, is formulated as a counter-factual statement ('actuality' or 'impossibility') about the tests that 'could' be realized on any sample corresponding to this state (and this certainty has been produced after having tested this property on similarly prepared samples). *Stricto sensu* this statement is then formulated without disturbing in any way the considered new sample, according to the definition of the 'elements of reality' for the system given in the celebrated paper of A. Einstein, B. Podolsky, and N. Rosen :

*'If, without in any way disturbing a system, we can predict with certainty (i.e., with probability equal to unity) the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity.'* [24].

Nevertheless, in order to establish an interpretation of 'properties' as elements of reality, the observer must be able to establish and confirm 'conjointly' his statements about the different properties that he is considering as 'simultaneously' actual for a given collection of similarly prepared samples. It is then necessary to restrict the measurement operations, that will be used by the observer, regarding the disturbance they provoke on the measured sample. These measurement operations should guaranty that the state of the system after measurement be characterized by the properties established as actual beforehand, through 'successive' measurement operations. The possibility (and necessity) to characterize minimally-disturbing measurements exists in the classical and in the quantum situation as well. However, although the existence of such measurements does not pose any conceptual problem in the classical situation, things are more complex in the quantum situation as soon as the measurement process irreducibly 'alters' the state of the measured system.

Despite the indeterministic character of quantum measurements, we note that the realization of a 'careful' yes/no test do allow the observer to pronounce some statements about the state 'after' the measurement, although it appears hazardous to extend these conclusions to the state of the system 'before' the measurement (due to the irreducible alteration of the state during the measurement operation). At least, a 'careful' measurement of a given property on a given sample<sup>[16]</sup> may guaranty the actuality of this property just after the measurement. As a consequence, the immediate repetition of this test should produce with certainty the same 'answer'. This sort of careful measurements do exist in the classical or in the quantum situation as well, they have been called *first kind measurements* by W. Pauli :

<sup>16</sup>Here we mean that the test is effectuated on the considered sample, according to the procedure defined by a yes/no test associated to this property, and the 'answer' received by the observer is 'positive'.

'The method of measurement [...] has the property that a repetition of measurement gives the same value for the quantity measured as in the first measurement. In other words, if the result of using the measuring apparatus is not known, but only the fact of its use is known [...], the probability that the quantity measured has a certain value is the same, both before and after the measurement. We shall call such measurements the measurements of the first kind.' [54, p.75]<sup>17</sup>

We will adopt the following formal definition for this type of measurements.

**Notion 24.** A yes/no test  $t \in \mathfrak{T}$  is said to lead to a *first kind measurement* associated to the corresponding testable property  $[t]$  iff (i) a positive result to the test  $t$  realized on any input sample is necessarily confirmed by an immediate repetition of this test realized on the samples out-coming from the first test, and (ii) the observer cannot establish if this new 'check' has been operated or not on the basis of the new tests he could operate on the out-coming samples of the experiment. In other words, the testable property  $[t]$  can be considered as 'actual' after the measurement by  $t$ , and the actuality of the property  $[t]$  can be 'confirmed' through any repetition of the measurement by  $t$ , because this repeated measurement let unchanged the state of the system.

The subset of yes/no tests leading to first-kind measurements will be denoted  $\mathfrak{T}_{FKM}$ . In other words,

$$\forall t \in \mathfrak{T}, \quad t \in \mathfrak{T}_{FKM} \quad :\Leftrightarrow \quad \begin{cases} \text{(i)} & \forall p \in \Omega_t, (p.t) \in \mathfrak{A}_t \\ \text{(ii)} & \forall p \in \mathfrak{A}_t, (p.t) \sim_{\mathfrak{P}} p \end{cases} \quad (79)$$

**Lemma 17.** In terms of the action on the space of states, we then have (using **Notion 23**) :

$$\forall t \in \mathfrak{T}_{FKM}, \quad \begin{cases} \text{(i)} & \forall \sigma \in [\Omega_t], (\sigma.t) \in [\mathfrak{A}_t] \\ \text{(ii)} & \forall \sigma \in [\mathfrak{A}_t], (\sigma.t) = \sigma \end{cases} \quad (80)$$

■

As a remark, we also note the following trivial lemmas.

**Lemma 18.**

$$\forall t \in \mathfrak{T}_{FKM}, \quad (p_{\perp}.t) \in \mathfrak{A}_t \neq \emptyset \quad (81)$$

$$\forall t \in \mathfrak{T}_{FKM}, \forall p \in \Omega_t, \quad ((p.t).t) \sim_{\mathfrak{P}} (p.t) \quad (82)$$

$$\forall t \in \mathfrak{T}_{FKM}, \forall p \in \Omega_t, \quad ((p.t) \sim_{\mathfrak{P}} p) \Rightarrow p \in \mathfrak{A}_t \quad (83)$$

$$\forall t \in \mathfrak{T}_{FKM}, \quad [\mathfrak{A}_t] = Im_t^{\mathfrak{S}} = \{ \sigma \in \mathfrak{S} \mid \sigma = (\sigma.t) \}. \quad (84)$$

■

<sup>17</sup>Note the distinction made by W.Pauli with the measurements of the second kind : 'On the other hand it can also happen that the system is changed but in a controllable fashion by the measurement — even when, in the state before the measurement, the quantity measured had with certainty a definite value. In this method, the result of a repeated measurement is not the same as that of the first measurement. But still it may be that from the result of this measurement, an un-ambiguous conclusion can be drawn regarding the quantity being measured for the concerned system before the measurement. Such measurements, we call the measurements of the second kind.' [54, p.75] To be concrete: (i) the determination of the position of a particle by a test of the presence of the particle in a given box appears to be a measurement of the first kind, (ii) the determination of the momentum of a particle by the evaluation of the 'impact' of this particle on a given detector appears to be a measurement of the second kind.

**Notion 25.** The subset of testable properties that can be tested through first-kind measurements will be denoted  $\mathcal{L}_{FKM}$ . The definition of this subset of  $\mathcal{L}$  is then summarized as follows

$$l \in \mathcal{L}_{FKM} \quad :\Leftrightarrow \quad \exists (\cdot, t) \in [\mathfrak{S} \dashrightarrow \mathfrak{S}]_{\cap} \quad | \quad \begin{cases} Dom_{\cdot, t}^{\mathfrak{S}} = [\Omega_l] \\ Im_{\cdot, t}^{\mathfrak{S}} = [\mathfrak{A}_l] = (\uparrow^{\mathfrak{S}} \Sigma_l) \\ \forall \sigma \in Dom_{\cdot, t}^{\mathfrak{S}}, (\sigma \cdot t) \cdot t = \sigma \cdot t \end{cases} \quad (85)$$

In order to pronounce synthetic statements concerning the actuality of a 'collection' of properties for a given state, it appears necessary to clarify how 'successive' measurements of different properties can be used to pronounce the actuality of these properties 'conjointly' for a given sample. Indeed, an annoying aspect of quantum measurements emerges when different properties are tested successively on a given sample : it generically happens that the effective measurements associated to some different properties 'interfere', forbidding the actuality for two such 'incompatible' properties to be affirmed simultaneously for a given preparation process (in this context, the definition of first-kind measurements appears to be insufficient for our purpose, as soon as this subset of measurement maps is not closed under the succession of distinct measurement operations).

This limitation of effective measurement leads to no-go theorems in the pathway to the construction of a classical logic for the description of the properties of the system, as remarked originally by E.Specker :

*'Is it possible to extend the description of a quantum mechanical system through the introduction of supplementary — fictitious — propositions in such a way that in the extended domain the classical propositional logic holds (whereby, of course, for simultaneously decidable proposition negation, conjunction and disjunction must retain their meaning)? The answer to this question is negative, except in the case of Hilbert spaces of dimension 1 and 2.'*[65]

The origins of this puzzling fact can be presented in terms of concrete measurements operated on the system. Let us consider any preparation procedure  $p \in \mathfrak{P}$  guarantying the actuality of a given property  $l \in \mathcal{L}$  for its outcoming samples (i.e.  $p \in \mathfrak{A}_l$ ). The actuality of this property may then be checked by applying any yes/no test  $t \in \mathfrak{T}$  associated to this property (i.e.  $l = [t]$ ) on these prepared samples before any other experiment as soon as this yes/no test leads to a first-kind measurement, i.e.  $(p \cdot t) \in \mathfrak{A}_l$ . However, it generically exists another property  $l' \in \mathcal{L}$  (and a yes/no test  $t' \in \mathfrak{T}$  associated to it, i.e.  $l' = [t']$ ) such that: (i) the result of any yes/no test associated to this second property on these prepared samples is fundamentally indeterminate (i.e.  $\epsilon(p, t') = \perp$ ), (ii) if we select among the out-comes of these yes/no tests the samples exhibiting this second property (i.e. the samples produced along the preparation procedure  $(p \cdot t') \in \mathfrak{A}_{l'}$ ), then, any forthcoming measurement relative to the first property on these selected samples will exhibit an indeterminacy (i.e.  $\epsilon((p \cdot t'), t) = \perp$ ). In other words, the actuality of the first property can not be affirmed anymore (i.e.  $(p \cdot t') \notin \mathfrak{A}_l$ ) after the actuality of the second property has been effectively established by a measurement, even if the actuality of the first property had been established beforehand on these same prepared samples! As a summary, contrary to the context of classical measurements, for  $p \in \mathfrak{P}$  and  $t \in \mathfrak{T}$ , we can not affirm that  $p \sqsubseteq_{\mathfrak{P}} (p \cdot t)$ .

Despite this severe limitation on the determination process of the actual properties of a quantum system, it is however possible to singularize some measurements, chosen for their ability to 'perturb minimally the system'. C.Piron summarizes the proposal for these ideal measurements as follows :

*'In general if we test a property  $a$  by performing  $\alpha$ , one of the corresponding questions, we disturb completely the given physical system even if  $a$  is actually true. We will say that a question  $\alpha$  is an ideal measurement if, when we perform it, we can assert that (i) If the answer is "yes", then the corresponding proposition  $a$  is true afterwards, and (ii) If the answer is "yes" and if a property  $b$  is true and compatible with  $a$ , then  $b$  is still true afterwards.'*[59]



Nevertheless, the definitions of 'compatible properties' and of 'ideal first-kind measurements' seem to be trapped in a vicious circle : two properties are *compatible* as soon as they can be 'simultaneously' stated as actual using successive ideal first-kind measurements, and measurements are defined to be ideal as soon as they respect the actuality of the properties that are compatible with the measured property!<sup>[18]</sup> In order to establish a consistent description, it appears necessary to clarify these notions in our vocabulary.

**Notion 26.** A family of testable properties  $\mathcal{L} = (l_i)_{i \in I} \subseteq \mathcal{L}$  will be said to be a *compatible family of properties* (this fact will be denoted  $\widehat{\mathcal{L}}$ ), iff it exists at least one preparation process  $p \in \mathfrak{P}$  producing collections of samples exhibiting all of these properties as 'actual' (the statements about the occurrence of the properties  $(l_i)_{i \in I}$  for the samples prepared along  $p$  will all be simultaneously 'positive with certainty'). In other words,

$$\widehat{\mathcal{L}} \quad :\Leftrightarrow \quad \bigcap_{l \in \mathcal{L}} \mathfrak{A}_l \neq \emptyset \quad (86)$$

In particular, a property  $l_1$  is said to be compatible with a property  $l_2$  iff  $\widehat{l_1 l_2}$ . This defines a binary relation called *compatibility relation* on  $\mathcal{L}$ .

The compatibility relation is a reflexive and symmetric relation. Moreover,  $t_1 \sqsubseteq_{\mathfrak{P}} t_2$  implies  $\widehat{[t_1][t_2]}$ .

The *ideal measurements* will be characterized as follows.

**Notion 27.** A yes/no test  $t_1 \in \mathfrak{T}$  is said to *lead to an ideal measurement* of the corresponding testable property  $[t_1]$  (this fact is denoted  $t_1 \in \mathfrak{T}_{ideal}$ ) iff, for any property  $[t_2]$  compatible with  $[t_1]$ , the statement pronounced by the observer beforehand concerning the 'actuality' of the property  $[t_2]$  is conserved after the measurement operation associated to  $t_1$  has been realized, i.e.

$$\forall t_1 \in \mathfrak{T}, \quad t_1 \in \mathfrak{T}_{ideal} \quad :\Leftrightarrow \quad (\forall t_2 \in \mathfrak{T} \mid \widehat{[t_1][t_2]}, \forall p \in \mathfrak{Q}_{t_1}, \mathfrak{e}(p, t_2) = \mathbf{Y} \Rightarrow \mathfrak{e}(p \cdot t_1, t_2) = \mathbf{Y}) \quad (87)$$

$$\Leftrightarrow \quad (\forall t_2 \in \mathfrak{T}, \widehat{[t_1][t_2]} \Rightarrow (\mathfrak{Q}_{t_1} \cap \mathfrak{A}_{t_2}) \cdot t_1 \subseteq \mathfrak{A}_{t_2}) \quad (88)$$

We will adopt the following notation

$$\mathfrak{T}_{IFKM} \quad := \quad \mathfrak{T}_{ideal} \cap \mathfrak{T}_{FKM}. \quad (89)$$

*Remark 8.* When an ideal measurement operation associated to a yes/no test  $t_1$  is operated on a given collection of samples similarly prepared in such a way that the property  $[t_2]$  was actual before this measurement operation, the outcoming samples are such that the observer can not distinguish if the property  $[t_2]$  has been checked after this measurement (or not). Indeed,

$$\forall t_1, t_2 \in \mathfrak{T}_{IFKM} \mid \widehat{[t_1][t_2]} \quad \forall p \in \mathfrak{A}_{t_2} \cap \mathfrak{Q}_{t_1}, \quad p \cdot t_1 \cdot t_2 \sim_{\mathfrak{P}} p \cdot t_1. \quad (90)$$

<sup>18</sup>A basic solution to this problem has been formalized by C. Piron [59]. This construction relies on an orthomodular lattice structure introduced on the space of properties (Note: we do not expect any such a construction in our perspective). In Piron's vocabulary,

two properties are compatible as soon as they form a boolean sub-algebra in the orthomodular lattice of properties (this requirement about the sub-boolean structure is a remnant of the particular structure on the space of properties in the classical situation).[56, p.295]

**Notion 28.** The subset of testable properties that can be tested through ideal first-kind measurements will be denoted  $\mathcal{L}_{IFKM}$ . The definition of this subset of  $\mathcal{L}$  is then summarized as follows

$$l \in \mathcal{L}_{IFKM} \quad :\Leftrightarrow \quad \exists (\cdot, t) \in [\mathfrak{G} \dashrightarrow \mathfrak{G}]_n \mid \left\{ \begin{array}{l} \text{Dom}_{\cdot, t}^{\mathfrak{G}} = [\Omega_l] \\ \text{Im}_{\cdot, t}^{\mathfrak{G}} = [\mathfrak{A}_l] = (\uparrow^{\mathfrak{G}} \Sigma_l) \\ \forall \sigma \in \text{Dom}_{\cdot, t}^{\mathfrak{G}}, (\sigma \cdot t) \cdot t = \sigma \cdot t \\ \forall \sigma \in \mathfrak{G} \mid \widehat{\sigma \Sigma_l}^{\mathfrak{G}}, (\text{Dom}_{\cdot, t}^{\mathfrak{G}} \cap (\uparrow^{\mathfrak{G}} \sigma)) \cdot t \subseteq ((\uparrow^{\mathfrak{G}} \sigma) \cap [\mathfrak{A}_l]) \end{array} \right. \quad (91)$$

The motivation to introduce 'ideal first-kind measurements' is very clear from an operational point of view.<sup>[19]</sup> If such measurement operations exist for a basic set of compatible properties, they can be used to build preparation processes devoted to produce collection of samples for which these properties will reveal to be 'conjointly actual'. It is also completely clear that ideal first-kind measurements exist in concrete quantum mechanical experiments [60] and it is then natural to impose their existence at the center of a quantum axiomatic. Nevertheless, nothing guaranties that such a measurement operation exists for any given property  $l$  (or, equivalently, that a measurement map satisfying property (91) exists for the property  $l$ ). Our aim, until the end the present subsection, will be to prove that an ideal first-kind measurement map exists as soon as the corresponding 'property' satisfies a quasi-classicality criterion.

**Notion 29.** A preparation process  $p \in \mathfrak{P}$  is said to be *consistent with the actuality of a given testable property*  $l \in \mathcal{L}$  iff it exists a preparation process  $p' \in \mathfrak{P}$ , sharper than  $p$ , and for which the property  $l$  is actual. We denote by  $\mathfrak{K}_l$  the set of preparation processes consistent with the actuality of the testable property  $l$ , i.e.

$$\forall l \in \mathcal{L}, \quad \mathfrak{K}_l := \{p \in \mathfrak{P} \mid \exists p' \in \mathfrak{A}_l, p \sqsubseteq_{\mathfrak{P}} p'\}. \quad (92)$$

$[\mathfrak{K}_l]$  will be called *the consistency-domain of the property*  $l$ . We have

$$\forall l \in \mathcal{L}, \quad [\mathfrak{K}_l] = \downarrow_{\mathfrak{G}} [\mathfrak{A}_l] = \{\sigma' \in \mathfrak{G} \mid \widehat{\Sigma_l \sigma'}^{\mathfrak{G}}\}. \quad (93)$$

**Lemma 19.** For any testable property  $l \in \mathcal{L}$ , the consistency domain  $[\mathfrak{K}_l]$  is a Scott-closed subset of  $[\Omega_l]$ , and then equivalently a sub-selection structure of  $[\Omega_l]$ .  $\blacksquare$

*Proof.* First of all, for any  $\sigma' \in [\mathfrak{K}_l]$  the property  $\widehat{\Sigma_l \sigma'}^{\mathfrak{G}}$  implies  $\exists \sigma'' \in [\mathfrak{A}_l]$  with  $\sigma' \sqsubseteq_{\mathfrak{G}} \sigma''$ , and then  $\tilde{\epsilon}_t(\sigma') \leq \tilde{\epsilon}_t(\sigma'') = \mathbf{Y}$  for any  $t$  such that  $l = [t]$ . As a consequence,  $[\mathfrak{K}_l] \subseteq [\Omega_l]$ . Moreover,  $[\mathfrak{K}_l]$  is obviously a downset. Endly, let us consider  $\mathcal{C} \subseteq_{\text{chain}} \mathfrak{G}$  such that  $(\forall c \in \mathcal{C}, \widehat{\Sigma_l c}^{\mathfrak{G}})$ . For any  $c \in \mathcal{C}$ , we can define the element  $(\Sigma_l \sqcup_{\mathfrak{G}} c)$ . The chain  $\mathcal{C}' := \{\Sigma_l \sqcup_{\mathfrak{G}} c \mid c \in \mathcal{C}\}$  admits a supremum in  $\mathfrak{G}$  satisfying (i)  $\forall c' \in \mathcal{C}', \Sigma_l \sqsubseteq_{\mathfrak{G}} c'$  and then  $\Sigma_l \sqsubseteq_{\mathfrak{G}} \sqcup_{\mathfrak{G}} \mathcal{C}'$ , and (ii)  $\forall c \in \mathcal{C}, \exists c' \in \mathcal{C}' \mid c \sqsubseteq_{\mathfrak{G}} c'$  and then  $c \sqsubseteq_{\mathfrak{G}} \sqcup_{\mathfrak{G}} \mathcal{C}'$  and then  $\sqcup_{\mathfrak{G}} \mathcal{C} \sqsubseteq_{\mathfrak{G}} \sqcup_{\mathfrak{G}} \mathcal{C}'$ . As a result, we have obtained  $(\widehat{\Sigma_l \sqcup_{\mathfrak{G}} \mathcal{C}}^{\mathfrak{G}})$  and then  $(\sqcup_{\mathfrak{G}} \mathcal{C}) \in [\mathfrak{K}_l]$ . This chain-completeness property implies the directed-completeness of  $[\mathfrak{K}_l]$ .

$[\mathfrak{K}_l]$  is then equivalently a sub-selection structure of  $[\Omega_l]$ , using Lemma 56.  $\square$

**Lemma 20.** For any testable property  $l \in \mathcal{L}$ , the map

$$\begin{aligned} \theta_l &: [\mathfrak{K}_l] \rightarrow [\mathfrak{A}_l] \\ \sigma &\mapsto \theta_l(\sigma) := \Sigma_l \sqcup_{\mathfrak{G}} \sigma. \end{aligned} \quad (94)$$

<sup>19</sup>These measurements were playing a fundamental role in the traditional Mackey's axiomatic approach to quantum theory [17]

is an idempotent, order-preserving, Scott-continuous map. It preserves arbitrary-infima and existing suprema.  $\blacksquare$

*Proof.* Firstly, from the basic properties of  $\sqcup_{\mathfrak{S}}$ , we know that  $\theta_l$  is idempotent and order-preserving. Secondly, if we denote  $\iota$  the inclusion of  $[\mathfrak{A}_l]$  in  $[\mathfrak{R}_l]$ , we have  $id \sqsubseteq \iota \circ \theta_l$  and  $\theta_l \circ \iota = id$ . As a result, the right-adjunct  $\theta_l$  of this Galois connection preserves existing suprema. In particular,  $\theta_l$  is Scott-continuous.

Thirdly, from the conditional-distributivity property satisfied in  $\mathfrak{S}$  (property (232)), we deduce that  $\theta_l$  preserves finite-meets. Furthermore,  $\theta_l$  preserves filtered-infima. Indeed, let us consider any filtered-set  $F \subseteq_{Fil} [\mathfrak{R}_l]$  and let us denote  $\sigma \in [\mathfrak{R}_l] \mid F, \Sigma_l \sqsubseteq_{\mathfrak{S}} \sigma$ , we have, using the properties of complementation in  $\mathfrak{S}$  (property (227)) and meet-continuity,  $\theta_l(\prod_{\mathfrak{S}}^{\vee} F) = \Sigma_l \sqcap_{\mathfrak{S}} \prod_{\mathfrak{S}}^{\vee} F = \sigma \setminus ((\sigma \setminus \Sigma_l) \sqcap_{\mathfrak{S}} (\sigma \setminus (\prod_{\mathfrak{S}}^{\vee} F))) = \sigma \setminus ((\sigma \setminus \Sigma_l) \sqcap_{\mathfrak{S}} \bigsqcup_{\sigma' \in F}^{\wedge} (\sigma \setminus \sigma')) = \sigma \setminus \bigsqcup_{\sigma' \in F}^{\wedge} ((\sigma \setminus \Sigma_l) \sqcap_{\mathfrak{S}} (\sigma \setminus \sigma')) = \prod_{\sigma' \in F}^{\vee} (\Sigma_l \sqcup_{\mathfrak{S}} \sigma')$ . As a conclusion of these two basic properties, we obtain that  $\theta_l$  preserves arbitrary-infima.  $\square$

**Notion 30.** A testable property  $l \in \mathcal{L}$  is said to be *quasi-classical* (this fact is denoted  $l \in \mathcal{L}_{q-cl}$ ) iff the consistency-domain  $[\mathfrak{R}_l]$  is a continuous retract of the domain  $[\mathfrak{Q}_l]$ , i.e.

$$\forall l \in \mathcal{L}, l \in \mathcal{L}_{q-cl} : \Leftrightarrow \exists \pi_l : [\mathfrak{Q}_l] \longrightarrow [\mathfrak{R}_l] \mid \begin{array}{l} \pi_l \text{ is Scott-continuous} \\ \forall \sigma \in [\mathfrak{R}_l], \pi_l(\sigma) = \sigma \\ \forall \sigma' \in [\mathfrak{Q}_l], \pi_l(\sigma') \sqsubseteq_{\mathfrak{S}} \sigma'. \end{array} \quad (95)$$

$$\forall \sigma \in [\mathfrak{R}_l], \pi_l(\sigma) = \sigma \quad (96)$$

$$\forall \sigma' \in [\mathfrak{Q}_l], \pi_l(\sigma') \sqsubseteq_{\mathfrak{S}} \sigma'. \quad (97)$$

**Definition 4. [Idealized sub- selection structures]** Let  $(\mathfrak{S}, \sqsubseteq_{\mathfrak{S}})$  be a selection structure and  $(\mathfrak{S}', \sqsubseteq_{\mathfrak{S}'})$  be a sub- selection structure of  $(\mathfrak{S}, \sqsubseteq_{\mathfrak{S}})$ .  $\mathfrak{S}' \in \mathcal{O}_{\mathfrak{S}}$  is said to be *idealized* iff

$$\forall S \subseteq_{fin} \mathfrak{S}', \quad \widehat{S}^{\mathfrak{S}} \Rightarrow \widehat{S}^{\mathfrak{S}'}. \quad (98)$$

The set of idealized sub- selection structures of a given selection structure  $(\mathfrak{S}, \sqsubseteq_{\mathfrak{S}})$  is denoted  $\widehat{\mathcal{O}}_{\mathfrak{S}}$ .

**Lemma 21.** Let  $l$  be a quasi-classical property. Then,

$$[\mathfrak{R}_l] \text{ is an 'idealized sub- selection structure' in the 'sub- selection structure' } [\mathfrak{Q}_l]. \quad (99)$$

Conversely, if  $[\mathfrak{R}_l]$  is an 'idealized sub- selection structure' in the 'sub- selection structure'  $[\mathfrak{Q}_l]$ , then  $l$  is a quasi-classical property. Moreover, the retraction  $\pi_l$  is given by

$$\forall \sigma \in [\mathfrak{Q}_l], \pi_l(\sigma) = \bigsqcup_{\mathfrak{S}} ([\mathfrak{R}_l] \cap (\downarrow_{\mathfrak{S}} \sigma)). \quad (100)$$

The retraction  $\pi_l$  is a surjective, order-preserving map which preserves arbitrary infima and existing suprema.  $\blacksquare$

*Remark 9.*  $\forall \sigma \in [\mathfrak{Q}_l], \perp_{\mathfrak{S}} \in ([\mathfrak{R}_l] \cap (\downarrow_{\mathfrak{S}} \sigma)) \neq \emptyset$

*Remark 10.*  $[\mathfrak{Q}_l]$  is a downset containing  $[\mathfrak{A}_l]$ . Hence,  $[\mathfrak{Q}_l]$  contains  $[\mathfrak{R}_l] = \downarrow_{\mathfrak{S}} [\mathfrak{A}_l]$ .

*Proof.* Let us consider  $\sigma_1, \sigma_2 \in [\mathfrak{R}_l]$  and let us suppose that  $\exists \sigma \in [\mathfrak{Q}_l] \mid \sigma_1, \sigma_2 \sqsubseteq_{\mathfrak{S}} \sigma$ . The monotonicity of  $\pi_l$  implies  $\pi_l(\sigma_1) \sqsubseteq_{\mathfrak{S}} \pi_l(\sigma)$  and  $\pi_l(\sigma_2) \sqsubseteq_{\mathfrak{S}} \pi_l(\sigma)$ . Secondly, property (96) implies  $\sigma_1 = \pi_l(\sigma_1)$  and  $\sigma_2 = \pi_l(\sigma_2)$ . Endly, property (97) implies  $\pi_l(\sigma) \sqsubseteq_{\mathfrak{S}} \sigma$ . As a conclusion,  $\exists \sigma' = \pi_l(\sigma) \in [\mathfrak{R}_l] \mid \sigma_1, \sigma_2 \sqsubseteq_{\mathfrak{S}} \sigma'$ . Furthermore, from Lemma 19, we know already that  $[\mathfrak{R}_l]$  is Scott-closed. As a result, we then conclude that  $[\mathfrak{R}_l]$  is a Scott-ideal in  $[\mathfrak{Q}_l]$ , i.e. an idealized sub- selection structure of  $[\mathfrak{Q}_l]$ . Conversely, if  $[\mathfrak{R}_l]$  is a Scott-ideal in  $[\mathfrak{Q}_l]$ , we can use the Lemma 57 to conclude that

$$\forall \sigma \in [\mathfrak{Q}_l], \pi_l(\sigma) := \bigsqcup_{\mathfrak{S}} ([\mathfrak{R}_l] \cap (\downarrow_{\mathfrak{S}} \sigma)) \in [\mathfrak{R}_l]. \quad (101)$$

$\pi$  is a map defined from  $[\Omega_t]$  to  $[\mathfrak{K}_t]$  which satisfies trivially the properties (95), (96) and (97).  $\iota$  is then a quasi-classical property and the expression of the retraction  $\pi_t$  is given by (100).

$\pi_t$  is also the Galois right-adjunct of the inclusion map from  $[\mathfrak{K}_t]$  to  $[\Omega_t]$ . As an immediate consequence,  $\pi_t$  is a surjective, order-preserving map which preserves arbitrary infima.

Endly, the conditional-distributivity property satisfied in the sub-selection structure  $[\Omega_t]$  leads to the preservation of existing finite joins by  $\pi_t$ . Indeed, let us consider  $\sigma_1, \sigma_2 \in [\Omega_t]$  such that  $(\sigma_1 \sqcup_{\mathfrak{E}} \sigma_2)$  exists. We remark that  $\pi_t(\sigma_1) = \sigma_1 \sqcap_{\mathfrak{E}} \pi_t(\sigma_1 \sqcup_{\mathfrak{E}} \sigma_2)$ . Indeed, (i)  $\pi_t(\sigma_1) \sqsubseteq_{\mathfrak{E}} \sigma_1$  and  $\pi_t(\sigma_1) = \pi_t(\sigma_1 \sqcup_{\mathfrak{E}} \sigma_2)$  implies  $\pi_t(\sigma_1) \sqsubseteq_{\mathfrak{E}} \sigma_1 \sqcap_{\mathfrak{E}} \pi_t(\sigma_1 \sqcup_{\mathfrak{E}} \sigma_2)$ , and (ii)  $\sigma_1 \sqcap_{\mathfrak{E}} \pi_t(\sigma_1 \sqcup_{\mathfrak{E}} \sigma_2)$  is an element of  $[\mathfrak{K}_t]$  below  $\sigma_1$  and  $\pi_t(\sigma_1)$  is the supremum of all these elements and then  $\pi_t(\sigma_1) \sqsupseteq_{\mathfrak{E}} \sigma_1 \sqcap_{\mathfrak{E}} \pi_t(\sigma_1 \sqcup_{\mathfrak{E}} \sigma_2)$ . With the same type of arguments, we show  $\pi_t(\sigma_2) = \sigma_2 \sqcap_{\mathfrak{E}} \pi_t(\sigma_1 \sqcup_{\mathfrak{E}} \sigma_2)$ . We then obtain :

$$\begin{aligned} \pi_t(\sigma_1) \sqcup_{\mathfrak{E}} \pi_t(\sigma_2) &= (\sigma_1 \sqcap_{\mathfrak{E}} \pi_t(\sigma_1 \sqcup_{\mathfrak{E}} \sigma_2)) \sqcup_{\mathfrak{E}} (\sigma_2 \sqcap_{\mathfrak{E}} \pi_t(\sigma_1 \sqcup_{\mathfrak{E}} \sigma_2)) \\ &= (\sigma_1 \sqcup_{\mathfrak{E}} \sigma_2) \sqcap_{\mathfrak{E}} \pi_t(\sigma_1 \sqcup_{\mathfrak{E}} \sigma_2) \quad (\text{conditional-distributivity}) \\ &= \pi_t(\sigma_1 \sqcup_{\mathfrak{E}} \sigma_2) \end{aligned} \quad (102)$$

Any existing supremum being the directed supremum of the suprema of finite subsets of the considered family, and  $\pi_t$  preserving directed suprema and finite joins, we obtain the preservation of any existing suprema.  $\square$

**Notion 31.** A yes/no test  $t \in \mathfrak{T}$  is said to *lead to a minimally-disturbing measurement* of the corresponding testable property  $[t]$  (this fact is denoted  $t \in \mathfrak{T}_{min}$ ) iff (i) this measurement is first-kind, and (ii) for any preparation  $p$  'consistent with the actuality of  $[t]$ ', the observer is allowed to pronounce statements about the measured state  $[p, t]$  which are the minimal statements simultaneously finer than the statements pronounced separately about  $[p]$  and about  $\Sigma_{[t]}$  before the measurement. In other words,

$$\forall t \in \mathfrak{T}, \quad t \in \mathfrak{T}_{min} \Leftrightarrow \begin{cases} Dom_{.t}^{\mathfrak{P}} = \Omega_{[t]} \\ t \in \mathfrak{T}_{FKM} \\ \forall p \in \mathfrak{K}_{[t]}, [p, t] = [p] \sqcup_{\mathfrak{E}} \Sigma_{[t]}. \end{cases} \quad (103)$$

**Notion 32.** The subset of properties that can be tested through minimally-disturbing measurements will be denoted  $\mathcal{L}_{min}$ . In other words,

$$\forall \iota \in \mathcal{L}, \quad \iota \in \mathcal{L}_{min} \Leftrightarrow \exists (\cdot, t) \in [\mathfrak{E} \dashrightarrow \mathfrak{E}]_{\cap} \mid \begin{cases} Dom_{.t}^{\mathfrak{E}} = [\Omega_t] \\ Im_{.t}^{\mathfrak{E}} = [\mathfrak{A}_t] = (\uparrow^{\mathfrak{E}} \Sigma_t) \\ \forall \sigma \in Dom_{.t}^{\mathfrak{E}}, (\sigma, t), t = \sigma \cdot t \\ \forall \sigma \in [\mathfrak{K}_t], \sigma \cdot t = \sigma \sqcup_{\mathfrak{E}} \Sigma_t. \end{cases} \quad (104)$$

*Remark 11.* We note that  $\sigma \in [\mathfrak{K}_t]$  means  $\widehat{\sigma \Sigma_t}^{\mathfrak{E}}$  and then the supremum  $(\sigma \sqcup_{\mathfrak{E}} \Sigma_t)$  exists, due to the consistent-completeness of  $\mathfrak{E}$ .

*Remark 12.* Note the following implicit property of minimally-disturbing measurement maps :

$$t \in \mathfrak{T}_{min} \Rightarrow Im_{.t}^{\mathfrak{P}} = \mathfrak{A}_{[t]} \quad (105)$$

Let us indeed distinguish two sub-cases.

As a first sub-case, let us suppose that  $p \in (\downarrow_{\mathfrak{P}} \mathfrak{A}_t)$ . Applying property (103), we deduce immediatly that  $[p, t] \sqsupseteq_{\mathfrak{E}} \Sigma_t$ , i.e.  $[p, t] \in (\uparrow^{\mathfrak{E}} \Sigma_t) = [\mathfrak{A}_t]$ , and then  $(p, t) \in \mathfrak{A}_t$ .

As a second sub-case, we now suppose  $p \in \Omega_t \setminus (\downarrow_{\mathfrak{P}} \mathfrak{A}_t)$ . Let us denote by  $p''$  an element of  $(\downarrow_{\mathfrak{P}} p) \cap \mathfrak{K}_{[t]}$  which is non-empty (Remark 9). Firstly, we have  $p'' \sqsubseteq_{\mathfrak{P}} p$ . We note in particular that  $p''$  is then an element of  $\Omega_t$ , because  $\Omega_t$  is a downset. Moreover, the monotonicity of the measurement map associated to  $t$  implies  $(p'', t) \sqsubseteq_{\mathfrak{P}} (p, t)$ . Secondly, we have  $p'' \in (\downarrow_{\mathfrak{P}} \mathfrak{A}_t) = \mathfrak{K}_t$ , and then  $(p'', t) \in \mathfrak{A}_t$  as we have proved it in the first sub-case. We now use the fact that  $\mathfrak{A}_t$  is an upper-set to conclude that  $(p, t) \in \mathfrak{A}_t$ .

**Lemma 22.**

$$\mathcal{L}_{q-cl} \subseteq \mathcal{L}_{min} \quad (106)$$

More precisely, if  $l \in \mathcal{L}_{q-cl}$ , the map  $\bullet\Theta_l$  defined by

$$\begin{aligned} \bullet\Theta_l & : \mathfrak{S} \dashrightarrow \mathfrak{S} \\ [\mathfrak{Q}_l] \ni \sigma & \mapsto \sigma \bullet\Theta_l := \Sigma_l \sqcup_{\mathfrak{S}} \pi_l(\sigma). \end{aligned} \quad (107)$$

is an idempotent, order-preserving partial map from  $[\mathfrak{Q}_l]$  to  $[\mathfrak{R}_l]$ , which preserves arbitrary infima and existing suprema, and satisfies  $\forall \sigma \in [\mathfrak{R}_l], \sigma \bullet\Theta_l = \sigma \sqcup_{\mathfrak{S}} \Sigma_l$ . ■

*Proof.* Let us consider  $l \in \mathcal{L}_{q-cl}$ .  $\pi_l$  is an idempotent, order-preserving partial map from  $[\mathfrak{Q}_l]$  to  $[\mathfrak{R}_l]$ , which preserves arbitrary infima and existing suprema as shown in Lemma 21.

As proved in Lemma 20, the map

$$\begin{aligned} [\mathfrak{R}_l] & \longrightarrow [\mathfrak{Q}_l] \\ \sigma & \mapsto \Sigma_l \sqcup_{\mathfrak{S}} \sigma \end{aligned} \quad (108)$$

is an idempotent, order-preserving partial map, which preserves arbitrary infima and existing suprema. As a result,  $\bullet\Theta_l$  is an idempotent, order-preserving partial map, which preserves arbitrary-infima and existing suprema.

Moreover, for any  $\sigma \in [\mathfrak{R}_l] = (\downarrow_{\mathfrak{S}} [\mathfrak{Q}_l])$ , we have  $\pi_l(\sigma) = \sigma$ . Hence, we obtain  $\forall \sigma \in (\downarrow_{\mathfrak{S}} [\mathfrak{Q}_l]), \sigma \bullet\Theta_l = \sigma \sqcup_{\mathfrak{S}} \Sigma_l$ .

As a result,  $\bullet\Theta_l$  satisfies all properties mentioned in (104), i.e. the properties required to conclude that  $l \in \mathcal{L}_{min}$ . As a conclusion,  $\mathcal{L}_{q-cl} \subseteq \mathcal{L}_{min}$ . □

**Lemma 23.**

$$\mathcal{L}_{q-cl} \supseteq \mathcal{L}_{min} \quad (109)$$

More precisely, if  $l \in \mathcal{L}_{min}$  and  $\bullet t$  is a measurement map defined to satisfy the minimality requirement (104), then the partial map  $\rho_t$  defined on  $[\mathfrak{Q}_l]$  by

$$\begin{aligned} \rho_t & : \mathfrak{S} \dashrightarrow \mathfrak{S} \\ [\mathfrak{Q}_l] \ni \sigma & \mapsto \rho_t(\sigma) := \sigma \sqcap_{\mathfrak{S}} (\sigma \bullet t). \end{aligned} \quad (110)$$

is a Scott-continuous retraction from  $[\mathfrak{Q}_l]$  to  $[\mathfrak{R}_l]$ . ■

*Proof.* We firstly note the obvious property  $\forall \sigma' \in [\mathfrak{Q}_l], \rho_t(\sigma') \sqsubseteq_{\mathfrak{S}} \sigma'$ .

Secondly, we note that, due to monotonicity requirement on the measurement map  $(\bullet t)$ ,  $(\sigma \bullet t)$  is in  $[\mathfrak{Q}_l]$ .

We then deduce that the range of  $\rho_t$  is included in  $(\downarrow_{\mathfrak{S}} [\mathfrak{Q}_l])$ , i.e. included in  $[\mathfrak{R}_l]$ .

Thirdly, due to the property (104) satisfied by  $\bullet t$ , we know that  $\forall \sigma \in [\mathfrak{R}_l], \rho_t(\sigma) = \sigma \sqcap_{\mathfrak{S}} (\sigma \sqcup_{\mathfrak{S}} \Sigma_l) = \sigma$ .

The Scott-continuity of  $\rho_t$  is derived from the Scott-continuity of the measurement map  $(\bullet t)$ , the meet-continuity property in  $\mathfrak{S}$  and [5, Proposition 2.1.12] :

$$\begin{aligned} \forall \mathfrak{C} \subseteq_{Chain} [\mathfrak{Q}_l], \quad \rho_t(\bigsqcup_{\mathfrak{S}} \mathfrak{C}) & = (\bigsqcup_{\mathfrak{S}} \mathfrak{C}) \sqcap_{\mathfrak{S}} ((\bigsqcup_{\mathfrak{S}} \mathfrak{C}) \bullet t) \\ & = (\bigsqcup_{c \in \mathfrak{C}} c) \sqcap_{\mathfrak{S}} (\bigsqcup_{c' \in \mathfrak{C}} (c' \bullet t)) \\ & = \bigsqcup_{c \in \mathfrak{C}} \bigsqcup_{c' \in \mathfrak{C}} (c \sqcap_{\mathfrak{S}} (c' \bullet t)) \\ & = \bigsqcup_{c \in \mathfrak{C}} (c \sqcap_{\mathfrak{S}} (c \bullet t)) = \bigsqcup_{c \in \mathfrak{C}} \rho_t(c). \end{aligned}$$

Finally, Chain continuity is equivalent to Scott-continuity.

We have then checked properties (95), (96), (97) for  $\rho_t$ . As a result,  $l \in \mathcal{L}_{q-cl}$  and then  $\mathcal{L}_{q-cl} \supseteq \mathcal{L}_{min}$ . □

**Lemma 24.** Let  $l$  be a testable quasi-classical property and let  $t$  be a yes/no test leading to a minimally-disturbing measurement of the property  $l = \lfloor t \rfloor$  (i.e.  $l = \lfloor t \rfloor$  and  $t \in \mathfrak{T}_{min}$ ), then necessarily  $t$  leads to ideal first-kind measurements of the property  $l$ . As a conclusion,

$$t \in \mathfrak{T}_{min} \Rightarrow t \in \mathfrak{T}_{IFKM}. \quad (111)$$

■

*Proof.* First of all,  $t$  being in  $\mathfrak{T}_{min}$ , we know that  $\lfloor t \rfloor$  is in  $\mathcal{L}_{min}$  and then in  $\mathcal{L}_{q-cl}$  using Lemma 23. We now intent to prove that the yes/no tests  $t$  satisfying the properties given in (103) satisfy also the properties (79 (i)), (79 (ii)) and (88), as soon as the property  $\lfloor t \rfloor$  is quasi-classical.

We have proved the property (79 (i)) in Remark 12.

Using property (103) for the particular case  $p \in \mathfrak{A}_t$ , we obtain  $\lceil p.t \rceil = \lceil p \rceil$ , because by definition  $\Sigma_t = \bigsqcup_{\mathfrak{E}} \mathfrak{A}_t$ . We have then proved the property (79 (ii)).

Let us now consider a second yes/no test  $t' \in \mathfrak{T}$  such that  $\widehat{\lfloor t \rfloor \lfloor t' \rfloor}$ , i.e.  $\mathfrak{A}_t \cap \mathfrak{A}_{t'} \neq \emptyset$ . We note in particular that this compatibility property implies  $\tilde{e}_{t'}(\Sigma_t) \leq \mathbf{Y}$ . Let us also consider  $p \in \mathfrak{Q}_t$  such that  $e(p, t') = \mathbf{Y}$ , i.e.  $p \in \mathfrak{A}_{t'}$ . We will distinguish two sub-cases as before.

As a first sub-case, let us suppose that  $p \in (\downarrow_{\mathfrak{P}} \mathfrak{A}_t)$ . Applying property (103), we have  $e(p.t, t') \geq e(p, t') = \mathbf{Y}$  using the hypotheses.

As a second sub-case, we now suppose  $p \in \mathfrak{Q}_t \setminus (\downarrow_{\mathfrak{P}} \mathfrak{A}_t)$ . Let us consider once again any preparation process  $p'' \in \mathfrak{P}$  which state corresponds to the supremum  $\bigsqcup_{\mathfrak{E}} ((\downarrow_{\mathfrak{E}} \lceil p \rceil) \cap (\downarrow_{\mathfrak{E}} \lceil \mathfrak{A}_t \rceil))$ . We conclude as before that  $p'' \in \mathfrak{Q}_t$  and  $(p''.t) \sqsubseteq_{\mathfrak{P}} (p.t)$ . In particular  $e(p''.t, t') \leq e(p.t, t')$ .

We know also that  $p'' \in (\downarrow_{\mathfrak{P}} \mathfrak{A}_t)$  and then, using property (103),  $e(p''.t, t') \geq e(p'', t')$ .

Endly, we know that  $\Sigma_{t'} \sqsubseteq_{\mathfrak{E}} \lceil p'' \rceil = \bigsqcup_{\mathfrak{E}} ((\downarrow_{\mathfrak{E}} \lceil p \rceil) \cap (\downarrow_{\mathfrak{E}} \lceil \mathfrak{A}_t \rceil))$  because (i)  $p \in \mathfrak{A}_{t'}$  implies  $\Sigma_{t'} \sqsubseteq_{\mathfrak{E}} \lceil p \rceil$  implies  $\Sigma_{t'} \in (\downarrow_{\mathfrak{E}} \lceil p \rceil)$ , and (ii)  $\mathfrak{A}_t \cap \mathfrak{A}_{t'} \neq \emptyset$  implies  $\Sigma_{t'} \in (\downarrow_{\mathfrak{E}} \lceil \mathfrak{A}_t \rceil)$ . But  $\Sigma_{t'} \sqsubseteq_{\mathfrak{E}} \lceil p'' \rceil$  is equivalently rewritten as  $e(p'', t') = \mathbf{Y}$ . We then deduce  $e(p''.t, t') = \mathbf{Y}$ .

Using the two intermediary results, we conclude this analysis of the second sub-case by  $e(p.t, t') = \mathbf{Y}$ .

As a conclusion, we have for any  $p \in \mathfrak{Q}_t$ ,  $e(p.t, t') = \mathbf{Y}$ . We have then shown the property (88).  $\square$

**Lemma 25.**

$$\forall t \in \mathfrak{T}_{IFKM}, \forall \sigma \in \lceil \mathfrak{R}_{\lfloor t \rfloor} \rceil, \quad (\sigma.t) = (\sigma \sqcup_{\mathfrak{E}} \Sigma_{\lfloor t \rfloor}). \quad (112)$$

In other words,

$$t \in \mathfrak{T}_{IFKM} \Rightarrow t \in \mathfrak{T}_{min}. \quad (113)$$

■

*Proof.* Let us consider  $t \in \mathfrak{T}_{IFKM}$  and  $\sigma \in \{ \sigma' \mid \widehat{\sigma \Sigma_{\lfloor t \rfloor}}^{\mathfrak{E}} \} = \lceil \mathfrak{R}_{\lfloor t \rfloor} \rceil$ . Using the fact that the measurement map  $(\cdot.t)$  preserves arbitrary infima, we deduce

$$\begin{aligned} (\sigma.t) &= \left( \prod_{\mathfrak{E}} (\uparrow^{\mathfrak{E}} \sigma) \right).t \\ &= \prod_{\mathfrak{E}} ((\text{Dom}_{\cdot t}^{\mathfrak{E}} \cap (\uparrow^{\mathfrak{E}} \sigma)).t) \\ &\sqsupseteq_{\mathfrak{E}} \prod_{\mathfrak{E}} ((\uparrow^{\mathfrak{E}} \sigma) \cap \lceil \mathfrak{A}_{\lfloor t \rfloor} \rceil) = (\sigma \sqcup_{\mathfrak{E}} \Sigma_{\lfloor t \rfloor}) \end{aligned}$$

Using  $\sigma \sqsubseteq_{\mathfrak{E}} (\sigma \sqcup_{\mathfrak{E}} \Sigma_{\lfloor t \rfloor})$  and the monotonicity of the measurement map, and the property (79 (ii)), we also obtain, for any  $\sigma \in \lceil \mathfrak{R}_{\lfloor t \rfloor} \rceil$ , the property  $\sigma.t \sqsubseteq_{\mathfrak{E}} (\sigma \sqcup_{\mathfrak{E}} \Sigma_{\lfloor t \rfloor}).t = \sigma \sqcup_{\mathfrak{E}} \Sigma_{\lfloor t \rfloor}$ .  $\square$

**Theorem 7. [Characterization of minimally-disturbing measurements]**

A property  $l$  can be measured through ideal first-kind measurements iff  $l$  is a testable quasi-classical property (i.e. iff  $[\mathfrak{K}_l]$  is an idealized sub-selection structure in the selection structure  $[\mathfrak{Q}_l]$ ). In other words,

$$\mathcal{L}_{q-cl} = \mathcal{L}_{min} = \mathcal{L}_{IFKM}. \quad (114)$$

For any quasi-classical property  $l \in \mathcal{L}_{q-cl}$ , the map given by

$$\begin{aligned} \cdot\Theta_l & : \mathfrak{S} \dashrightarrow \mathfrak{S} \\ [\mathfrak{Q}_l] \ni \sigma & \mapsto \sigma \cdot \Theta_l := \Sigma_l \sqcup_{\mathfrak{S}} \sqcup_{\mathfrak{S}} ([\mathfrak{K}_l] \cap (\downarrow_{\mathfrak{S}} \sigma)). \end{aligned} \quad (115)$$

defines an idempotent, order-preserving, Scott-continuous partial map from  $[\mathfrak{Q}_l]$  to  $[\mathfrak{K}_l]$ , preserving arbitrary infima and satisfying  $\forall \sigma \in [\mathfrak{K}_l], \sigma \cdot \Theta_l = \sigma \sqcup_{\mathfrak{S}} \Sigma_l$ . This is the explicit form of the minimally-disturbing measurement map (it is also an ideal first-kind measurement map) associated to the property  $l$ .

The measurement map  $\cdot\Theta_l$  satisfies moreover the following 'linearity' property :

$$\forall \sigma_1, \sigma_2 \in \mathfrak{S} \mid (\sigma_1 \sqcup_{\mathfrak{S}} \sigma_2) \in [\mathfrak{Q}_l], \quad (\sigma_1 \sqcup_{\mathfrak{S}} \sigma_2) \cdot \Theta_l = (\sigma_1 \cdot \Theta_l) \sqcup_{\mathfrak{S}} (\sigma_2 \cdot \Theta_l). \quad (116)$$

■

*Proof.* Direct consequence of Lemma 24, Lemma 25, Lemma 23, Lemma 22 and Lemma 21. □

*Remark 13.* We note also the following 'regularity' property

$$\forall \sigma_1, \sigma_2 \in [\mathfrak{Q}_l], \quad (\sigma_1 \bar{\sqcup}_{\mathfrak{S}} \sigma_2) \Rightarrow ((\sigma_1 \cdot \Theta_l) = (\sigma_2 \cdot \Theta_l) \text{ or } (\sigma_1 \cdot \Theta_l) \bar{\sqcup}_{\mathfrak{S}} (\sigma_2 \cdot \Theta_l)). \quad (117)$$

Indeed, we firstly note that  $\sigma_2 = \sigma_1 \sqcup_{\mathfrak{S}} (\sigma_2 \setminus \sigma_1)$ . Secondly,  $\sigma_2 \setminus \sigma_1$  is an atom in  $[\mathfrak{Q}_l]$ , and then  $\pi_l(\sigma_2 \setminus \sigma_1)$  is equal either to  $\perp_{\mathfrak{S}}$  or to an atom of  $[\mathfrak{K}_l]$ . Endly, we use the property (116) to obtain  $(\sigma_2 \cdot \Theta_l) = (\sigma_1 \cdot \Theta_l) \sqcup_{\mathfrak{S}} \pi_l(\sigma_2 \setminus \sigma_1)$  and conclude.

**Theorem 8. [The Chu space  $(\mathfrak{S}, \mathfrak{T}_{min}, \tilde{\mathfrak{e}})$  is bi-extensional]** If we denote by  $[\mathfrak{S} \rightarrow \mathfrak{B}]_{\sqcap}^{testable}$  the set of order-preserving Lawson-continuous maps  $f$  from  $\mathfrak{S}$  to  $\mathfrak{B}$ , satisfying  $f^{-1}(\mathbf{Y}) \neq \emptyset$ . The map

$$\begin{aligned} \mu & : \mathfrak{T}_{min} \longrightarrow [\mathfrak{S} \rightarrow \mathfrak{B}]_{\sqcap}^{testable} \\ t & \mapsto \tilde{\mathfrak{e}}_t \end{aligned} \quad (118)$$

is injective. As a result, the Chu space  $(\mathfrak{S}, \mathfrak{T}_{min}, \tilde{\mathfrak{e}})$  is *extensional*, i.e.

$$\forall t_1, t_2 \in \mathfrak{T}_{min}, \quad (\forall \sigma \in \mathfrak{S}, \tilde{\mathfrak{e}}_{t_1}(\sigma) = \tilde{\mathfrak{e}}_{t_2}(\sigma)) \Rightarrow (t_1 = t_2) \quad (119)$$

As a consequence of Lemma 1, this Chu space is *bi-extensional*. ■

*Proof.* Once the evaluation map is given, we obtain unambiguously (1) the Scott-closed subset  $[\mathfrak{Q}_t]$  of  $\mathfrak{S}$  as the reverse image of the subset  $\{\perp, \mathbf{Y}\}$  by the Scott-continuous map  $\tilde{\mathfrak{e}}_t$ , and (2) the element  $\Sigma_t$  as the Infimum of the Scott-open filter determined as the non-empty reverse image of the subset  $\{\mathbf{Y}\}$  by the order-preserving and infima-preserving map  $\tilde{\mathfrak{e}}_t$ .

Let us now consider two minimally-disturbing tests  $t_1, t_2 \in \mathfrak{T}_{min}$  such that  $\Sigma_{t_1} = \Sigma_{t_2}$  and  $[\mathfrak{Q}_{t_1}] = [\mathfrak{Q}_{t_2}]$ . We have then firstly :

$$Dom_{\cdot t_1}^{\mathfrak{S}} = [\mathfrak{Q}_{t_1}] = [\mathfrak{Q}_{t_2}] = Dom_{\cdot t_2}^{\mathfrak{S}}. \quad (120)$$

Secondly, we have for any  $\sigma$  in  $Dom_{\cdot t_1}^{\mathfrak{S}} = Dom_{\cdot t_2}^{\mathfrak{S}}$

$$\sigma \cdot t_1 = \Sigma_{t_1} \sqcup_{\mathfrak{S}} \sqcup_{\mathfrak{S}} ((\downarrow_{\mathfrak{S}} \uparrow^{\mathfrak{S}} \Sigma_{t_1}) \cap (\downarrow_{\mathfrak{S}} \sigma)) = \Sigma_{t_2} \sqcup_{\mathfrak{S}} \sqcup_{\mathfrak{S}} ((\downarrow_{\mathfrak{S}} \uparrow^{\mathfrak{S}} \Sigma_{t_2}) \cap (\downarrow_{\mathfrak{S}} \sigma)) = \sigma \cdot t_2 \quad (121)$$

As a consequence,  $t_1 = t_2$ . As a result, the map  $\mu$  is injective, and the Chu space  $(\mathfrak{G}, \mathfrak{T}_{min}, \tilde{\epsilon})$  is then extensional. Then, using Lemma 1, we deduce that the Chu space  $(\mathfrak{G}, \mathfrak{T}_{min}, \tilde{\epsilon})$  is bi-extensional.  $\square$

Theorem 8 suggests to transfer some structures, that can be put on  $[\mathfrak{G} \rightarrow \mathfrak{B}]_{\square}^{testable}$ , to the set of minimally-disturbing yes/no tests  $\mathfrak{T}_{min}$ .

**Lemma 26.** The binary relation  $\sqsubseteq_{\mathfrak{T}_{min}}$  defined as follows

$$\forall t_1, t_2 \in \mathfrak{T}_{min}, \quad t_1 \sqsubseteq_{\mathfrak{T}_{min}} t_2 \quad :\Leftrightarrow \quad (\forall \sigma \in \mathfrak{G}, \tilde{\epsilon}_{t_1}(\sigma) \leq \tilde{\epsilon}_{t_2}(\sigma)) \quad (122)$$

$$\Leftrightarrow \quad (\mathfrak{A}_{t_1} \subseteq \mathfrak{A}_{t_2} \text{ and } \mathfrak{A}_{\bar{t}_1} \subseteq \mathfrak{A}_{\bar{t}_2}) \quad (123)$$

is reflexive, transitive and anti-symmetric.  $(\mathfrak{T}_{min}, \sqsubseteq_{\mathfrak{T}_{min}})$  is then a partially ordered set.  $\blacksquare$

The structure of this poset can be studied in itself. However, from an operational perspective, it appears more natural to study the space of 'descriptions' of the system, i.e. the set of families of compatible properties, and its domain structure. This is the subject of the subsection 3.4.

### 3.4 The space of descriptions as a domain

**Lemma 27.** For any  $t_1, t_2 \in \mathfrak{T}_{IFKM}$  two yes/no tests leading to ideal first-kind measurements such that  $t_1$  is compatible with  $t_2$ , the yes/no test  $(t_1 \cdot t_2)$  leads to ideal first-kind measurements associated to the conjunction of the properties  $[t_1]$  and  $[t_2]$ . In other words,

$$\forall t_1, t_2 \in \mathfrak{T}_{IFKM} \mid \overbrace{[t_1][t_2]}, \quad (t_1 \cdot t_2) \in \mathfrak{T}_{IFKM} \quad (124)$$

$$\mathfrak{A}_{t_1 \cdot t_2} = (\mathfrak{A}_{t_1} \cap \mathfrak{A}_{t_2}). \quad (125)$$

$\blacksquare$

*Proof.* Using equations (64) and (79 (ii)), we deduce (125). Let us now prove that  $(t_1 \cdot t_2) \in \mathfrak{T}_{IFKM}$ . For any  $p \in \mathfrak{Q}_{(t_1 \cdot t_2)}$ , we know from equation (65) that  $(p \cdot t_1) \in \mathfrak{A}_{t_1} \cap \mathfrak{A}_{t_2}$ . Using the equations (79 (i)) and (88), we then deduce that  $((p \cdot t_1) \cdot t_2) \in \mathfrak{A}_{t_1} \cap \mathfrak{A}_{t_2} = \mathfrak{A}_{(t_1 \cdot t_2)}$  (using (125)), i.e. equation (79 (i)) applied to the yes/no test  $(t_1 \cdot t_2)$ .

For any  $p \in \mathfrak{A}_{(t_1 \cdot t_2)} = (\mathfrak{A}_{t_1} \cap \mathfrak{A}_{t_2})$  (using (125)), we deduce from equation (79 (ii)) and (66) that  $(p \cdot (t_1 \cdot t_2)) = ((p \cdot t_1) \cdot t_2) \sim_{\mathfrak{P}} (p \cdot t_2) \sim_{\mathfrak{P}} p$ , i.e. equation (79 (ii)) applied to the yes/no test  $(t_1 \cdot t_2)$ .

Let us now consider a yes/no test  $t \in \mathfrak{T}$  compatible with the yes/no test  $(t_1 \cdot t_2)$ . In other words, we have  $\mathfrak{A}_t \cap \mathfrak{A}_{(t_1 \cdot t_2)} \neq \emptyset$ . As a result, we have then  $\mathfrak{A}_t \cap \mathfrak{A}_{t_1} \neq \emptyset$  and  $\mathfrak{A}_t \cap \mathfrak{A}_{t_2} \neq \emptyset$ , i.e.  $\overbrace{[t][t_1]}$  and  $\overbrace{[t][t_2]}$ .

Using successively (66), (65), the compatibility relations  $\overbrace{[t][t_1]}$  and  $\overbrace{[t][t_2]}$  coupled with the property (88), we obtain  $(\mathfrak{Q}_{(t_1 \cdot t_2)} \cap \mathfrak{A}_t) \cdot (t_1 \cdot t_2) \subseteq ((\mathfrak{Q}_{t_2} \cap \mathfrak{A}_t) \cdot t_2) \subseteq \mathfrak{A}_t$ , i.e. the property (88) for the compatibility property  $\overbrace{[t][t_1][t_2]}$ .  $\square$

**Corollary 2.** Let us consider two yes/no tests  $t_1, t_2 \in \mathfrak{T}_{min}$  leading to minimally-disturbing measurements of the corresponding quasi-classical properties  $[t_1], [t_2]$ . Let us moreover suppose that these properties are compatible (i.e.  $\overbrace{[t_1][t_2]}$ ). The property  $[t_1 \cdot t_2]$  is also quasi-classical. In other words,

$$\begin{aligned} \forall t_1, t_2 \in \mathfrak{T}_{q-cl} \mid \overbrace{[t_1][t_2]}, \quad & (t_1 \cdot t_2) \in \mathfrak{T}_{q-cl} \text{ and } \pi_{[t_1 \cdot t_2]} = \pi_{[t_2]} \circ \pi_{[t_1]} \text{ is a continuous retraction} \\ & \text{from } [\mathfrak{Q}_{[t_1 \cdot t_2]}] = \text{Dom}_{\pi_{[t_2]} \circ \pi_{[t_1]}}^{\mathfrak{S}} \\ & \text{to } [\mathfrak{R}_{[t_1 \cdot t_2]}] = [\mathfrak{R}_{[t_1]}] \cap [\mathfrak{R}_{[t_2]}]. \end{aligned} \quad (126)$$



*Proof.*  $t_1, t_2 \in \mathfrak{T}_{\min}$  lead to minimally-disturbing measurements of the corresponding quasi-classical properties  $[t_1], [t_2]$ , and then also to ideal first-kind measurements of this property (using Theorem 7). If these properties are compatible (i.e.  $\widehat{[t_1][t_2]}$ ), from Theorem 27, we know that  $(t_1 \cdot t_2)$  leads to ideal first-kind measurements of the property  $[t_1 \cdot t_2]$ , and then also leads to a minimally-disturbing measurement of this property. As a result the property  $[t_1 \cdot t_2]$  is also quasi-classical.

From equation (64), we know that  $\mathfrak{A}_{t_1 \cdot t_2} = \{p \in \mathfrak{A}_{t_1} \mid p \cdot t_1 \in \mathfrak{A}_{t_2}\}$ . We know from Lemma 24 that the measurement map  $(\cdot t_1)$  verifies in particular (79 (ii)), and then  $\mathfrak{A}_{t_1 \cdot t_2} = \mathfrak{A}_{t_1} \cap \mathfrak{A}_{t_2}$ . Then, we obtain  $[\mathfrak{R}_{[t_1 \cdot t_2]}] = [(\downarrow_{\mathfrak{P}} \mathfrak{A}_{t_1 \cdot t_2})] = [\downarrow_{\mathfrak{P}} (\mathfrak{A}_{t_1} \cap \mathfrak{A}_{t_2})] = [(\downarrow_{\mathfrak{P}} \mathfrak{A}_{t_1}) \cap (\downarrow_{\mathfrak{P}} \mathfrak{A}_{t_2})] = [\mathfrak{R}_{[t_1]}] \cap [\mathfrak{R}_{[t_2]}]$ . Secondly, we note that  $\{\sigma' \in [\mathfrak{Q}_{[t_1]}] \mid (\sigma' \cdot t_1) \in [\mathfrak{Q}_{[t_2]}]\} = \{\sigma' \in [\mathfrak{Q}_{[t_1]}] \mid \pi_{[t_1]}(\sigma') \in [\mathfrak{Q}_{[t_2]}]\}$ , because  $[\mathfrak{Q}_{[t_2]}]$  is a downset and  $(\sigma' \cdot t_1) = \Sigma_{[t_1]} \sqcup_{\mathfrak{S}} \pi_{[t_1]}(\sigma') \sqsupseteq_{\mathfrak{S}} \pi_{[t_1]}(\sigma')$ . Then,  $\pi_{[t_2]} \circ \pi_{[t_1]}$  is a well-defined retraction with domain  $[\mathfrak{Q}_{[t_1 \cdot t_2]}]$  and with range  $\pi_{[t_2]}([\mathfrak{R}_{[t_1]}]) = [\mathfrak{R}_{[t_1]}] \cap [\mathfrak{R}_{[t_2]}]$ .  $\square$

**Lemma 28. [”Specker’s principle”]** Let us consider a finite family of testable properties  $\mathfrak{L} = (l_k)_{k \in \{1, \dots, N\}}$  with  $l_k \in \mathcal{L}_{IFKM}$  for any  $k \in \{1, \dots, N\}$ , such that the elements of this family are pairwise compatible, i.e.  $\forall k, k' \in \{1, \dots, N\}, \widehat{l_k l_{k'}}$ . The family  $\mathfrak{L}$  is then a compatible family. In other words,

$$\forall \mathfrak{L} = (l_k)_{k \in \{1, \dots, N\}} \mid l_k \in \mathcal{L}_{IFKM}, \quad \widehat{\mathfrak{L}} \Leftrightarrow (\forall k, k' \in \{1, \dots, N\}, \widehat{l_k l_{k'}}). \quad (127)$$

*Proof.* Considering previous requirement, we consider  $T = (t_k)_{k \in \{1, \dots, N\}}$  with  $t_k \in \mathfrak{T}_{IFKM}$  and  $l_k = [t_k]$  for any  $k = 1, \dots, N$ .

First of all, using the compatibility relations  $\widehat{l_i l_k}$  for  $k = 1, \dots, i-1$  and Lemma 27, we prove (i)  $[t_1 \cdot \dots \cdot t_{i-1}] \in \mathcal{L}_{IFKM}$ , (ii) the compatibility relation  $\widehat{l_i [t_1 \cdot \dots \cdot t_{i-1}]}$  and (iii)  $\mathfrak{A}_{t_1 \cdot \dots \cdot t_{i-1}} = \mathfrak{A}_{t_1} \cap \dots \cap \mathfrak{A}_{t_{i-1}}$  for any  $i = 1, \dots, N$ .

Let us now consider  $p \in \mathfrak{Q}_{t_1 \cdot (t_2 \cdot (\dots (t_{N-1} \cdot (t_N)) \dots))}$  (we note that  $\perp_{\mathfrak{S}} \in \mathfrak{Q}_{t_1 \cdot (t_2 \cdot (\dots (t_{N-1} \cdot (t_N)) \dots))} \neq \emptyset$ ). We have then from equation (65),  $(p \in \mathfrak{Q}_{t_1}$  and  $(p \cdot t_1) \in \mathfrak{Q}_{t_2 \cdot (\dots (t_{N-1} \cdot (t_N)) \dots)}$ ). Let us denote by  $(P_i)$  the statement:  $((p \cdot t_1 \cdot \dots \cdot t_{i-1}) \in \mathfrak{A}_{t_1} \cap \dots \cap \mathfrak{A}_{t_{i-1}} \cap \mathfrak{Q}_{t_i}$  and  $(p \cdot t_1 \cdot \dots \cdot t_i) \in \mathfrak{Q}_{t_{i+1} \cdot (\dots (t_{N-1} \cdot (t_N)) \dots)}$ ). Let us suppose that  $(P_i)$  is satisfied.

Using the compatibility relation  $\widehat{l_i [t_1 \cdot \dots \cdot t_{i-1}]}$  and the fact that  $t_i$  leads to an ideal measurement and then satisfies property (88), we deduce that  $(p \cdot t_1 \cdot \dots \cdot t_i) \in \mathfrak{A}_{t_1} \cap \dots \cap \mathfrak{A}_{t_{i-1}}$ . Using the fact that  $t_i$  leads to a first-kind measurement and then satisfies (79 (i)), we have also  $(p \cdot t_1 \cdot \dots \cdot t_i) \in \mathfrak{A}_{t_i}$ . Using now the expression (65), we conclude that the property  $(P_{i+1})$  is satisfied.

As a result, we conclude by recursion that  $(p \cdot t_1 \cdot \dots \cdot t_N) \in \mathfrak{A}_{t_1} \cap \dots \cap \mathfrak{A}_{t_N}$ . The set  $\mathfrak{A}_{t_1} \cap \dots \cap \mathfrak{A}_{t_N}$  is then non-empty, and the family  $\mathfrak{L}$  is then a compatible family of properties.  $\square$

*Remark 14.* The history of this ”principle” is recalled in [16] and some results are given in this paper that ”suggest that the principle that pairwise decidable propositions are jointly decidable, together with Boole’s condition that the sum of probabilities of jointly exclusive propositions cannot be higher than one  $[\dots]$ , which I will collectively call Specker’s principle, may explain quantum contextuality.” [20]

**Notion 33.** We denote by  $\mathcal{D}$  the following subset of the powerset  $\mathcal{P}(\mathcal{L}_{IFKM})$  :

$$\mathcal{D} := \{ \mathfrak{L} \in \mathcal{P}(\mathcal{L}_{IFKM}) \mid \widehat{\mathfrak{L}} \}. \quad (128)$$

An element of  $\mathcal{D}$  defines a family of properties that can be checked by ’conjoint measure-

<sup>20</sup>see [41] for the original results on non-contextuality in quantum mechanics

ments' as being 'simultaneously actual' on a given prepared sample. Hence,  $\mathcal{D}$  will be called *the space of descriptions*.

**Theorem 9. [The space of descriptions is a coherence domain]**

The space of descriptions  $\mathcal{D}$  is a 'coherence domain' [30] associated to the 'web'  $\mathcal{L}_{IFKM}$  and to the coherence relation  $\widehat{\cdot \cdot}$ . In other words,  $\mathcal{D} \subseteq \mathcal{P}(\mathcal{L}_{IFKM})$  satisfies

$$\forall \mathcal{L}_1, \mathcal{L}_2 \in \mathcal{P}(\mathcal{L}_{IFKM}), (\mathcal{L}_1 \subseteq \mathcal{L}_2 \text{ and } \mathcal{L}_2 \in \mathcal{D}) \Rightarrow (\mathcal{L}_1 \in \mathcal{D}) \quad (129)$$

$$\forall l \in \mathcal{L}_{IFKM}, \{l\} \in \mathcal{D} \quad (130)$$

$$\forall L \in \mathcal{P}(\mathcal{D}) \mid (\forall \mathcal{L}_1, \mathcal{L}_2 \in L, \widehat{\mathcal{L}_1 \cup \mathcal{L}_2}), \bigcup L \in \mathcal{D} \quad (131)$$

■

*Proof.* The first property is a direct consequence of the definition of  $\mathcal{D}$ . Indeed,  $\forall \mathcal{L}_1, \mathcal{L}_2 \in \mathcal{P}(\mathcal{L}_{IFKM})$ , the properties  $\mathcal{L}_1 \subseteq \mathcal{L}_2$  and  $\mathcal{L}_2 \in \mathcal{D}$  imply  $\emptyset \neq \bigcap_{l \in \mathcal{L}_2} \mathcal{A}_l \subseteq \bigcap_{l \in \mathcal{L}_1} \mathcal{A}_l$ .

The second property is a trivial consequence of the definition of consistency and the fact that we consider testable properties.

The property  $(\forall M \in \mathcal{P}_{fin}(\mathcal{D}) \mid (\forall \mathcal{L}_1, \mathcal{L}_2 \in M, \widehat{\mathcal{L}_1 \cup \mathcal{L}_2}), \bigcup M \in \mathcal{D})$  is a direct consequence of Lemma 28. Hence, for any  $L \in \mathcal{P}(\mathcal{D}) \mid (\forall \mathcal{L}_1, \mathcal{L}_2 \in L, \widehat{\mathcal{L}_1 \cup \mathcal{L}_2})$ , we deduce that  $N \in \mathcal{D}$  for any  $N \subseteq_{fin} \bigcup L$ , which allows to deduce the third property. □

### 3.5 The subset of 'classical' properties

**Notion 34.** A testable property  $l \in \mathcal{L}$  is said to be *classical* (this fact is denoted  $l \in \mathcal{L}_{cl}$ ) iff any preparation, which exhibits potentially the property  $l$ , is 'consistent with the actuality' of this same property, i.e. iff the consistency-domain  $[\mathfrak{K}_l]$  is identical to the domain  $[\mathfrak{Q}_l]$ . In other words,

$$l \in \mathcal{L}_{cl} \quad :\Leftrightarrow \quad [\mathfrak{K}_l] = [\mathfrak{Q}_l]. \quad (132)$$

$\mathfrak{T}_{cl}$  denotes the corresponding set of yes/no tests.

We note the following trivial theorem.

**Corollary 3.** A classical property is necessarily quasi-classical. As a consequence, any classical property can be measured using an ideal first-kind measurement operation. The corresponding measurement map is defined by

$$\begin{aligned} \bullet_{\Theta_l} : \mathfrak{S} &\dashrightarrow \mathfrak{S} \\ [\mathfrak{K}_l] = [\mathfrak{Q}_l] \ni \sigma &\mapsto \sigma \bullet_{\Theta_l} := \Sigma_l \sqcup_{\mathfrak{S}} \sigma. \end{aligned} \quad (133)$$

■

*Proof.* As soon as  $[\mathfrak{Q}_l] = [\mathfrak{K}_l]$ , we have necessarily that  $[\mathfrak{K}_l]$  is a continuous retract of  $[\mathfrak{Q}_l]$ .

We know from the results of Subsection 3.3, that, for any classical property  $l \in \mathcal{L}_{cl}$ , the associated measurement map is given by equation (133).

The range of this map is  $[\mathfrak{A}_l]$ . The measurement map  $\bullet_{\Theta_l}$  is an idempotent, order-preserving, Scott-continuous partial map, which preserves arbitrary infima and existing suprema. It moreover satisfies  $\forall \sigma \in [\mathfrak{Q}_l] = [\mathfrak{K}_l], ([\mathfrak{K}_l] \cap (\uparrow^{\mathfrak{S}} \sigma)) \bullet_{\Theta_l} \subseteq ((\uparrow^{\mathfrak{S}} \sigma) \cap [\mathfrak{A}_l])$  as soon as  $\forall \sigma \in [\mathfrak{Q}_l] = [\mathfrak{K}_l], \sigma \bullet_{\Theta_l} \sqsupseteq_{\mathfrak{S}} \sigma$ . It then defines an ideal first-kind measurement map. □

**Lemma 29.**

$$l \in \mathcal{L}_{cl} \Leftrightarrow \bar{l} \in \mathcal{L}_{cl} \quad (134)$$

$$\text{Max}(\mathfrak{S}) = \text{Max}(\lceil \mathfrak{A}_1 \rceil) \cup \text{Max}(\lceil \mathfrak{A}_2 \rceil) \quad (135)$$

$$\emptyset = \text{Max}(\lceil \mathfrak{A}_1 \rceil) \cap \text{Max}(\lceil \mathfrak{A}_2 \rceil) \quad (136)$$

■

*Proof.* By definition  $l \in \mathcal{L}_{cl}$  means  $\lceil \mathfrak{K}_l \rceil = \lceil \mathfrak{Q}_l \rceil = \mathfrak{S} \setminus \lceil \mathfrak{A}_l \rceil$ . In other words,  $(\lceil \mathfrak{K}_l \rceil \cup \lceil \mathfrak{A}_l \rceil = \mathfrak{S}$  and  $\lceil \mathfrak{K}_l \rceil \cap \lceil \mathfrak{A}_l \rceil = \emptyset$ ). Using  $\text{Max}(\lceil \mathfrak{K}_l \rceil) = \text{Max}(\lceil \mathfrak{A}_l \rceil)$ , we deduce  $\text{Max}(\mathfrak{S}) = \text{Max}(\lceil \mathfrak{A}_1 \rceil) \cup \text{Max}(\lceil \mathfrak{A}_2 \rceil)$  and  $\text{Max}(\lceil \mathfrak{A}_1 \rceil) \cap \text{Max}(\lceil \mathfrak{A}_2 \rceil) = \emptyset$ . Let us consider  $\sigma \in \mathfrak{S} \setminus \lceil \mathfrak{A}_1 \rceil$ . We cannot have  $\text{Max}(\uparrow^\mathfrak{S} \sigma) \subseteq \text{Max}(\lceil \mathfrak{A}_1 \rceil)$  because  $\sigma = \prod_{\mathfrak{S}} \text{Max}(\uparrow^\mathfrak{S} \sigma)$  and  $\lceil \mathfrak{A}_1 \rceil$  is an ideal. Hence, using  $\text{Max}(\mathfrak{S}) = \text{Max}(\lceil \mathfrak{A}_1 \rceil) \cup \text{Max}(\lceil \mathfrak{A}_2 \rceil)$ , we deduce that it exists  $\sigma' \in \text{Max}(\lceil \mathfrak{A}_2 \rceil) \cap (\uparrow^\mathfrak{S} \sigma)$ . In other words, necessarily,  $\sigma \in \lceil \mathfrak{K}_2 \rceil$ . As a result, we have proved  $\mathfrak{S} \setminus \lceil \mathfrak{A}_1 \rceil \subseteq \lceil \mathfrak{K}_2 \rceil$ .

Conversely, for any  $\sigma'' \in \lceil \mathfrak{A}_1 \rceil$ , we cannot have  $\text{Max}(\uparrow^\mathfrak{S} \sigma'') \cap \text{Max}(\lceil \mathfrak{A}_2 \rceil) \neq \emptyset$  because  $\text{Max}(\lceil \mathfrak{A}_2 \rceil) \cap \text{Max}(\lceil \mathfrak{A}_1 \rceil) = \emptyset$ . Hence,  $\sigma'' \notin \lceil \mathfrak{K}_2 \rceil$ . As a result,  $\lceil \mathfrak{A}_1 \rceil \subseteq \mathfrak{S} \setminus \lceil \mathfrak{K}_2 \rceil$ , i.e.  $\mathfrak{S} \setminus \lceil \mathfrak{A}_1 \rceil \supseteq \lceil \mathfrak{K}_2 \rceil$ .

As a final result, we have proved  $\lceil \mathfrak{K}_2 \rceil = \mathfrak{S} \setminus \lceil \mathfrak{A}_1 \rceil = \lceil \mathfrak{Q}_2 \rceil$ . □

The following theorem justifies the name "classical" for these properties.

**Theorem 10.** Let us consider a property  $l_2 \in \mathcal{L}$  and a classical property  $l_1 \in \mathcal{L}_{cl}$ . The property  $l_2$  is necessarily compatible with  $l_1$  or compatible with  $\bar{l}_1$ . In other words,

$$\forall l_1 \in \mathcal{L}_{cl}, \forall l_2 \in \mathcal{L}, \quad (\widehat{l_1 l_2} \text{ or } \widehat{\bar{l}_1 l_2}). \quad (137)$$

More precisely, a classical property  $l_1 \in \mathcal{L}_{cl}$  is compatible with any property  $l_2 \in \mathcal{L}_{min}$  that is not finer than  $\bar{l}_1$ . The actuality of  $l_1$  can be established independently of any minimally-disturbing measurement operations associated to  $l_2$ . In other words,

$$\forall l_1 \in \mathcal{L}_{cl}, \forall l_2 \in \mathcal{L} \mid \bar{l}_1 \not\subseteq_{\mathfrak{S}} l_2, \quad \widehat{l_1 l_2}. \quad (138)$$

If  $l_2$  is finer than  $\bar{l}_1$ ,  $l_1 \in \mathcal{L}$  is simply stated as 'impossible' when  $l_2$  is stated as actual. ■

*Proof.* Using property (135), we deduce  $\text{Max}(\lceil \mathfrak{A}_1 \rceil) \cap \text{Max}(\lceil \mathfrak{A}_2 \rceil) \neq \emptyset$  or  $\text{Max}(\lceil \mathfrak{A}_1 \rceil) \cap \text{Max}(\lceil \mathfrak{A}_2 \rceil) = \emptyset$ , which proves the property (137).

If  $l_2$  is not finer than  $\bar{l}_1$ , we have  $\lceil \mathfrak{A}_2 \rceil \not\subseteq \lceil \mathfrak{A}_1 \rceil$  and in particular  $\text{Max}(\lceil \mathfrak{A}_2 \rceil) \not\subseteq \text{Max}(\lceil \mathfrak{A}_1 \rceil)$ . Hence, using property (135), we deduce  $\text{Max}(\lceil \mathfrak{A}_2 \rceil) \cap \text{Max}(\lceil \mathfrak{A}_1 \rceil) \neq \emptyset$ , which proves the second result. □

**Theorem 11.** Let  $l_1 \in \mathcal{L}_{cl}$  be a classical property and  $l_2 \in \mathcal{L}_{min}$  be a minimal property such that  $\bar{l}_1 \not\subseteq_{\mathfrak{S}} l_2$ , i.e.  $\widehat{l_1 l_2}$ . Let  $\bullet_{\Theta_{l_1}}$  and  $\bullet_{\Theta_{l_2}}$  be the associated measurement maps. We have

$$\text{Dom}_{\bullet_{\Theta_{l_1} \bullet_{\Theta_{l_2}}}}^{\mathfrak{S}} \subseteq \text{Dom}_{\bullet_{\Theta_{l_2} \bullet_{\Theta_{l_1}}}}^{\mathfrak{S}}, \quad (139)$$

$$\forall \sigma \in \text{Dom}_{\bullet_{\Theta_{l_1} \bullet_{\Theta_{l_2}}}}^{\mathfrak{S}}, \quad \sigma \bullet_{\Theta_{l_1} \bullet_{\Theta_{l_2}}} = \sigma \bullet_{\Theta_{l_2} \bullet_{\Theta_{l_1}}} = \pi_{l_2}(\sigma) \sqcup_{\mathfrak{S}} \Sigma_{l_2} \sqcup_{\mathfrak{S}} \Sigma_{l_1}. \quad (140)$$

■

*Proof.*  $\text{Dom}_{\bullet_{\Theta_{l_1} \bullet_{\Theta_{l_2}}}}^{\mathfrak{S}} = \{\sigma \in \text{Dom}_{\bullet_{\Theta_{l_1}}}^{\mathfrak{S}} \mid (\sigma \bullet_{\Theta_{l_1}}) \in \text{Dom}_{\bullet_{\Theta_{l_2}}}^{\mathfrak{S}}\} = \{\sigma \mid \widehat{\sigma \Sigma_{l_1}}^{\mathfrak{S}} \text{ and } (\sigma \sqcup_{\mathfrak{S}} \Sigma_{l_1}) \in \text{Dom}_{\bullet_{\Theta_{l_2}}}^{\mathfrak{S}}\}$ . Using  $\sigma \sqsubseteq (\sigma \sqcup_{\mathfrak{S}} \Sigma_{l_1})$  and the fact that  $\text{Dom}_{\bullet_{\Theta_{l_2}}}^{\mathfrak{S}}$  is a downset, we deduce :  $\sigma \in \text{Dom}_{\bullet_{\Theta_{l_1} \bullet_{\Theta_{l_2}}}}^{\mathfrak{S}}$  implies  $\sigma \in \text{Dom}_{\bullet_{\Theta_{l_2}}}^{\mathfrak{S}}$ . From  $(\sigma \sqcup_{\mathfrak{S}} \Sigma_{l_1}) \in \text{Dom}_{\bullet_{\Theta_{l_2}}}^{\mathfrak{S}}$  we deduce that  $\pi_{l_2}(\sigma \sqcup_{\mathfrak{S}} \Sigma_{l_1})$  exists as an element of  $\lceil \mathfrak{K}_2 \rceil$  and then  $\pi_{l_2}(\sigma \sqcup_{\mathfrak{S}} \Sigma_{l_1}) \sqcup_{\mathfrak{S}} \Sigma_{l_2}$  exists. Using the linearity property of the retraction  $\pi_{l_2}$ , we deduce  $\pi_{l_2}(\sigma \sqcup_{\mathfrak{S}}$

$\Sigma_{l_1} = \pi_{l_2}(\sigma) \sqcup_{\mathfrak{E}} \pi_{l_2}(\Sigma_{l_1})$ . Using the compatibility  $\widehat{l_1 l_2}$ , we know that  $\Sigma_{l_1} \in [\mathfrak{K}_{l_2}]$  and then  $\pi_{l_2}(\Sigma_{l_1}) = \Sigma_{l_1}$ . As a consequence,  $(\pi_{l_2}(\sigma) \sqcup_{\mathfrak{E}} \Sigma_{l_2}) \sqcup_{\mathfrak{E}} \Sigma_{l_1}$  exists and then  $(\sigma \cdot \Theta_{l_2}) \in \text{Dom}_{\cdot \Theta_{l_1}}^{\mathfrak{E}}$ . As a final result,  $\{\sigma \in \text{Dom}_{\cdot \Theta_{l_1}}^{\mathfrak{E}} \mid (\sigma \cdot \Theta_{l_1}) \in \text{Dom}_{\cdot \Theta_{l_2}}^{\mathfrak{E}}\} \subseteq \{\sigma \in \text{Dom}_{\cdot \Theta_{l_2}}^{\mathfrak{E}} \mid (\sigma \cdot \Theta_{l_2}) \in \text{Dom}_{\cdot \Theta_{l_1}}^{\mathfrak{E}}\}$  and for any  $\sigma \in \text{Dom}_{\cdot (\Theta_{l_1}, \Theta_{l_2})}^{\mathfrak{E}}$ ,  $\sigma \cdot (\Theta_{l_1}, \Theta_{l_2}) = \pi_{l_2}(\sigma \sqcup_{\mathfrak{E}} \Sigma_{l_1}) \sqcup_{\mathfrak{E}} \Sigma_{l_2} = (\pi_{l_2}(\sigma) \sqcup_{\mathfrak{E}} \Sigma_{l_2}) \sqcup_{\mathfrak{E}} \Sigma_{l_1} = \sigma \cdot (\Theta_{l_2}, \Theta_{l_1})$   $\square$

**Theorem 12.** Let  $l_1, l_2 \in \mathcal{L}_{cl}$  be two classical properties, which are supposed to be compatible, i.e.  $\widehat{l_1 l_2}$ . Let  $\cdot \Theta_{l_1}$  and  $\cdot \Theta_{l_2}$  be the associated measurement maps. We have

$$\text{Im}_{\cdot (\Theta_{l_1}, \Theta_{l_2})}^{\mathfrak{E}} = [\mathfrak{A}_{l_1}] \cap [\mathfrak{A}_{l_2}] = \uparrow^{\mathfrak{E}}(\Sigma_{l_1} \sqcup_{\mathfrak{E}} \Sigma_{l_2}), \quad (141)$$

$$\text{Dom}_{\cdot (\Theta_{l_1}, \Theta_{l_2})}^{\mathfrak{E}} = \text{Dom}_{\cdot (\Theta_{l_2}, \Theta_{l_1})}^{\mathfrak{E}} = [\mathfrak{K}_{l_1}] \cap [\mathfrak{K}_{l_2}] = \downarrow_{\mathfrak{E}} \uparrow^{\mathfrak{E}}(\Sigma_{l_1} \sqcup_{\mathfrak{E}} \Sigma_{l_2}), \quad (142)$$

$$\forall \sigma \in \text{Dom}_{\cdot (\Theta_{l_1}, \Theta_{l_2})}^{\mathfrak{E}}, \quad \sigma \cdot \Theta_{l_1} \cdot \Theta_{l_2} = \sigma \cdot \Theta_{l_2} \cdot \Theta_{l_1} = \sigma \sqcup_{\mathfrak{E}} (\Sigma_{l_1} \sqcup_{\mathfrak{E}} \Sigma_{l_2}) \quad (143)$$

■

*Proof.* The equation (141) has already been proved in Lemma 27.

Concerning the second equality, we note that previous theorem implies immediately  $\text{Dom}_{\cdot (\Theta_{l_1}, \Theta_{l_2})}^{\mathfrak{E}} = \text{Dom}_{\cdot (\Theta_{l_2}, \Theta_{l_1})}^{\mathfrak{E}}$ . Let us explicitly compute the expression of  $\text{Dom}_{\cdot (\Theta_{l_1}, \Theta_{l_2})}^{\mathfrak{E}} = \{\sigma \in \text{Dom}_{\cdot \Theta_{l_1}}^{\mathfrak{E}} \mid (\sigma \cdot \Theta_{l_1}) \in \text{Dom}_{\cdot \Theta_{l_2}}^{\mathfrak{E}}\} = \{\sigma \mid \widehat{\sigma \Sigma_{l_1}}^{\mathfrak{E}} \text{ and } \widehat{(\sigma \sqcup_{\mathfrak{E}} \Sigma_{l_1}) \Sigma_{l_2}}^{\mathfrak{E}}\}$ . The compatibility relation  $\widehat{(\sigma \sqcup_{\mathfrak{E}} \Sigma_{l_1}) \Sigma_{l_2}}^{\mathfrak{E}}$  implies immediately  $\widehat{\sigma \Sigma_{l_2}}^{\mathfrak{E}}$  because  $\sigma \sqsubseteq_{\mathfrak{E}} (\sigma \sqcup_{\mathfrak{E}} \Sigma_{l_1})$ . As a consequence,  $\text{Dom}_{\cdot (\Theta_{l_1}, \Theta_{l_2})}^{\mathfrak{E}} \subseteq [\mathfrak{K}_{l_1}] \cap [\mathfrak{K}_{l_2}] = \downarrow_{\mathfrak{E}} \uparrow^{\mathfrak{E}}(\Sigma_{l_1} \sqcup_{\mathfrak{E}} \Sigma_{l_2})$ . Conversely,  $\forall \sigma \in \downarrow_{\mathfrak{E}} \uparrow^{\mathfrak{E}}(\Sigma_{l_1} \sqcup_{\mathfrak{E}} \Sigma_{l_2}) = [\mathfrak{K}_{l_1}] \cap [\mathfrak{K}_{l_2}]$ , we have by definition  $\widehat{\sigma(\Sigma_{l_1} \sqcup_{\mathfrak{E}} \Sigma_{l_2})}^{\mathfrak{E}}$ , and then  $\sigma \sqcup_{\mathfrak{E}} (\Sigma_{l_1} \sqcup_{\mathfrak{E}} \Sigma_{l_2})$  exists. As a consequence, we obtain  $\widehat{(\sigma \sqcup_{\mathfrak{E}} \Sigma_{l_1}) \Sigma_{l_2}}^{\mathfrak{E}}$ . Moreover,  $[\mathfrak{K}_{l_1}] \cap [\mathfrak{K}_{l_2}] \subseteq [\mathfrak{K}_{l_1}]$ . As a result, we obtain  $[\mathfrak{K}_{l_1}] \cap [\mathfrak{K}_{l_2}] \subseteq \text{Dom}_{\cdot (\Theta_{l_1}, \Theta_{l_2})}^{\mathfrak{E}}$ .

As a final result, we obtain the equation (142).

The property (143) is a direct consequence of property (140).  $\square$

### 3.6 Perfect measurements and the orthogonality relation on the space of states

In a classical perspective, distinct prepared samples can be considered as corresponding to two 'orthogonal states' as soon as the observer can produce a statement that unambiguously distinguishes them. In other words, two states  $\sigma_1, \sigma_2 \in \mathfrak{S}$  are said to be *orthogonal* (this fact will be denoted  $\sigma_1 \perp \sigma_2$ ) iff it exists a property  $l \in \mathcal{L}$  such that  $l$  is simultaneously 'actual' on  $\sigma_1$  and 'impossible' on  $\sigma_2$ .

Note however that, from an operational perspective, this definition do not guaranty that the observer can prepare systematically pairs of samples corresponding to pairs of orthogonal states associated to a given property. In this context, it appears natural to introduce a more restrictive class of properties.

**Notion 35.** A property  $l \in \mathcal{L}$  will be said to be a *perfect property* (this fact will be denoted  $l \in \mathcal{L}_{perfect}$ ) iff it exists two conjugate testable yes/no tests  $t$  and  $\bar{t}$  leading to ideal first-kind measurements of the respective properties  $l = [t]$  and  $\bar{l} = [\bar{t}]$ . The corresponding measurement  $\cdot t$  is said to be a *perfect measurement*.

$$\forall t \in \mathfrak{T}, \quad t \in \mathfrak{T}_{perfect} \Leftrightarrow (t \in \mathfrak{T}_{IFKM} \text{ and } \bar{t} \in \mathfrak{T}_{IFKM}) \quad (144)$$

$$\forall l \in \mathcal{L}, \quad l \in \mathcal{L}_{perfect} \Leftrightarrow \exists t \in \mathfrak{T}_{perfect} \mid l = [t]. \quad (145)$$

**Lemma 30.**

$$(l \in \mathcal{L}_{q-cl} \text{ and } \bar{l} \in \mathcal{L}_{q-cl}) \Leftrightarrow l \in \mathcal{L}_{perfect}. \quad (146)$$

*Proof.* If  $l \in \mathcal{L}_{perfect}$ , then, by definition,  $l \in \mathcal{L}_{IFKM}$  and  $\bar{l} \in \mathcal{L}_{IFKM}$ , and then, using Theorem 7,  $l \in \mathcal{L}_{q-cl}$  and  $\bar{l} \in \mathcal{L}_{q-cl}$ . ■

Conversely, let us suppose that  $l \in \mathcal{L}_{q-cl}$  and  $\bar{l} \in \mathcal{L}_{q-cl}$ . Then, it exists a Scott-continuous retraction  $\pi_l$  (resp.  $\pi_{\bar{l}}$ ) from  $[\Omega_l] = \mathfrak{S} \setminus [\mathfrak{A}_{\bar{l}}]$  to  $[\mathfrak{K}_l] = \downarrow_{\mathfrak{S}} [\mathfrak{A}_l]$  (resp. from  $[\Omega_{\bar{l}}] = \mathfrak{S} \setminus [\mathfrak{A}_l]$  to  $[\mathfrak{K}_{\bar{l}}] = \downarrow_{\mathfrak{S}} [\mathfrak{A}_{\bar{l}}]$ ). We can then define the minimally-disturbing measurement maps  $\cdot\Theta_l$  (resp.  $\cdot\Theta_{\bar{l}}$ ) according to equation (107). These measurements are ideal first-kind measurements, as proved in Lemma 24. By construction,  $[\mathfrak{A}_{\Theta_l}] = [\mathfrak{A}_l]$  and  $[\mathfrak{A}_{\Theta_{\bar{l}}}] = [\mathfrak{A}_{\bar{l}}]$ , and then  $\Theta_l = \Theta_{\bar{l}}$ . □

The space of states can be equipped with an orthogonality relation.

**Notion 36.** Two states  $\sigma_1, \sigma_2 \in \mathfrak{S}$  are said to be *orthogonal* (this fact will be denoted  $\sigma_1 \perp \sigma_2$ ) iff they are the output states  $\sigma_1 = \sigma \cdot t$  and  $\sigma_2 = \sigma \cdot \bar{t}$  of two conjugate perfect measurements (i.e.  $t \in \mathfrak{T}_{perfect}$ ), associated respectively to a perfect property  $l = [t]$  and to the conjugate property  $\bar{l} = [\bar{t}]$ , and operated on the same input state  $\sigma$ . In other words,

$$\forall \sigma_1, \sigma_2 \in \mathfrak{S},$$

$$(\sigma_1 \perp \sigma_2) \quad :\Leftrightarrow \quad (\exists t \in \mathfrak{T}_{perfect}, \exists \sigma \in [\Omega_{[t]}] \cap [\Omega_{[\bar{t}}] \mid (\sigma_1 = \sigma \cdot t \text{ and } \sigma_2 = \sigma \cdot \bar{t})). \quad (147)$$

This relation is symmetric and anti-reflexive (i.e.  $\sigma_1 \perp \sigma_2$  implies  $\sigma_1 \neq \sigma_2$ ).

**Lemma 31.** If two states  $\sigma_1, \sigma_2 \in \mathfrak{S}$  are orthogonal, then they can be distinguished unambiguously by a statement associated to a perfect property

$$\forall \sigma_1, \sigma_2 \in \mathfrak{S}, (\sigma_1 \perp \sigma_2) \quad \Rightarrow \quad (\exists l \in \mathcal{L}_{perfect} \mid \sigma_1 \in [\mathfrak{A}_l] \text{ and } \sigma_2 \in [\mathfrak{A}_{\bar{l}}]). \quad (148)$$

Moreover,

$$\forall \sigma_1, \sigma_2 \in \mathfrak{S}, \quad \sigma_1 \perp \sigma_2 \quad \Rightarrow \quad \widehat{\sigma_1 \sigma_2}^{\mathfrak{S}}. \quad (149)$$

*Proof.* The first statement is a direct consequence of property (79 (i)) and the definition of the orthogonality relation. ■

Following directly the definition of consistency relation, we then note that, for any  $\sigma_1, \sigma_2 \in \mathfrak{S}$ , the property  $\widehat{\sigma_1 \sigma_2}^{\mathfrak{S}}$  implies immediatly  $\forall t \in \mathfrak{T}, \epsilon(\sigma_1, t)\epsilon(\sigma_2, t)$ . By negation, we obtain that  $(\exists l \in \mathcal{L} \mid \sigma_1 \in [\mathfrak{A}_l] \text{ and } \sigma_2 \in [\mathfrak{A}_{\bar{l}}])$  implies  $\widehat{\sigma_1 \sigma_2}^{\mathfrak{S}}$ . □

## 4 Symmetries

Let us consider two observers  $O_1$  and  $O_2$  that intent to formalize 'transactions' concerning their experimental results about the system.

**Notion 37.** The observer  $O_1$  has chosen a preparation process  $p_1 \in \mathfrak{P}^{(O_1)}$  and intents to describe it to the observer  $O_2$ . The observer  $O_2$  is able to interpret the macroscopic data defining  $p_1$  in terms of the elements of  $\mathfrak{P}^{(O_2)}$ , using a map  $f_{(12)} : \mathfrak{P}^{(O_1)} \rightarrow \mathfrak{P}^{(O_2)}$  (i.e.  $O_2$  knows how to identify a preparation procedure  $f_{(12)}(p_1)$  corresponding to any  $p_1$ ).

The observer  $O_2$  has chosen a yes/no test  $t_2 \in \mathfrak{T}^{(O_2)}$  and intents to address the corresponding question to  $O_1$ . The observer  $O_1$  is able to interpret the macroscopic data defining  $t_2$  in terms of the elements of  $\mathfrak{T}^{(O_1)}$ , using a map  $f^{(21)} : \mathfrak{T}^{(O_2)} \rightarrow \mathfrak{T}^{(O_1)}$  (i.e.  $O_1$  identifies knows

how to fix a test  $f^{(21)}(t_2)$  corresponding to any  $t_2$ ).

The couple of maps  $(f_{(12)}, f^{(21)})$  where  $f_{(12)} \in \mathfrak{P}^{(O_1)} \rightarrow \mathfrak{P}^{(O_2)}$  and  $f^{(21)} : \mathfrak{T}^{(O_2)} \rightarrow \mathfrak{T}^{(O_1)}$  defines a *dictionary* formalizing the transaction from  $O_1$  to  $O_2$ .

The first task these observers want to accomplish is to confront their knowledge, i.e. to compare their 'statements' about the system.

**Notion 38.** As soon as the transaction is formalized using a dictionary, each observer can formulate his statements and confront them to the statements of the other.

Firstly, the observer  $O_1$  can interpret the macroscopic data defining  $t_2$  using the map  $f^{(21)}$ . Then, he produces the statement  $\epsilon^{(O_1)}(p_1, f^{(21)}(t_2))$  concerning the results of this test on the chosen samples.

Secondly, the observer  $O_2$  can interpret the macroscopic data defining  $p_1$  using the map  $f_{(12)}$ .

Then, the observer  $O_2$  pronounces his statement  $\epsilon^{(O_2)}(f_{(12)}(p_1), t_2)$  concerning the results of the test  $t_2$  on the correspondingly prepared samples.

The two observers  $O_1$  and  $O_2$  are said to *agree about their statements* iff

$$\forall p_1 \in \mathfrak{P}^{(O_1)}, \forall t_2 \in \mathfrak{T}^{(O_2)}, \quad \epsilon^{(O_2)}(f_{(12)}(p_1), t_2) = \epsilon^{(O_1)}(p_1, f^{(21)}(t_2)), \quad (150)$$

i.e. iff the adjoint pair  $(f_{(12)}, f^{(21)})$  defines a morphism of Chu spaces from  $(\mathfrak{P}^{(O_1)}, \mathfrak{T}^{(O_1)}, \epsilon^{(O_1)})$  to  $(\mathfrak{P}^{(O_2)}, \mathfrak{T}^{(O_2)}, \epsilon^{(O_2)})$  [62].

**Lemma 32.** As soon as the dictionary  $(f_{(12)}, f^{(21)})$  satisfies the property (150) (i.e. the adjoint pair defines a Chu morphism from  $(\mathfrak{P}^{(O_1)}, \mathfrak{T}^{(O_1)}, \epsilon^{(O_1)})$  to  $(\mathfrak{P}^{(O_2)}, \mathfrak{T}^{(O_2)}, \epsilon^{(O_2)})$ ) we have immediately

$$\forall p, p' \in \mathfrak{P}^{(O_1)}, \quad (p \sim_{\mathfrak{P}^{(O_1)}} p') \Rightarrow (f_{(12)}(p) \sim_{\mathfrak{P}^{(O_2)}} f_{(12)}(p')) \quad (151)$$

■

*Proof.*  $p \sim_{\mathfrak{P}^{(O_1)}} p'$  implies, in particular,  $\epsilon^{(O_1)}(p, f^{(21)}(t_2)) = \epsilon^{(O_1)}(p', f^{(21)}(t_2))$  for any  $t_2 \in \mathfrak{T}^{(O_2)}$ . Using property (150), we obtain  $\epsilon^{(O_2)}(f_{(12)}(p), t_2) = \epsilon^{(O_2)}(f_{(12)}(p'), t_2)$  for any  $t_2 \in \mathfrak{T}^{(O_2)}$ , i.e.  $f_{(12)}(p) \sim_{\mathfrak{P}^{(O_2)}} f_{(12)}(p')$ . □

**Notion 39.** In order for the observers  $O_1$  and  $O_2$  to be in *complete agreement* about the system (once they do agree about their statements), it is necessary for them to be unable to distinguish the outcomes of the control tests, realized to confirm (or not) their statements.

Firstly, the measurement realized by the observer  $O_2$  is given by  $(f_{(12)}(p_1) \cdot t_2)$ .

Secondly, the observer  $O_2$  interpret, using  $f_{(12)}$ , the measurement  $(p_1 \cdot f^{(21)}(t_2))$  realized by the observer  $O_1$ .

In other words, it is necessary for the dictionary  $(f_{(12)}, f^{(21)})$  to satisfy also the following property

$$\forall p_1 \in \mathfrak{P}^{(O_1)}, \forall t_2 \in \mathfrak{T}^{(O_2)}, \quad f_{(12)}(p_1) \cdot t_2 \sim_{\mathfrak{S}^{(O_2)}} f_{(12)}(p_1 \cdot f^{(21)}(t_2)), \quad (152)$$

**Notion 40.** If the dictionary  $(f_{(12)}, f^{(21)})$  satisfies the properties (150), (152), the property (153) below

$$\forall t, t' \in \mathfrak{T}^{(O_2)}, \quad f^{(21)}(t \cdot t') = f^{(21)}(t) \cdot f^{(21)}(t'), \quad (153)$$

and the properties (154), (155) below

$$\forall \mathfrak{p}, \mathfrak{p}' \in \mathfrak{P}^{(o_1)}, (f_{(12)}(\mathfrak{p}) \sim_{\mathfrak{P}^{(o_2)}} f_{(12)}(\mathfrak{p}')) \Rightarrow (\mathfrak{p} \sim_{\mathfrak{P}^{(o_1)}} \mathfrak{p}'), \quad (154)$$

$$f^{(21)} \text{ surjective,} \quad (155)$$

then this dictionary is said to *relate by a symmetry*  $(\mathfrak{P}^{(o_1)}, \mathfrak{T}^{(o_1)}, \mathfrak{e}^{(o_1)})$  to  $(\mathfrak{P}^{(o_2)}, \mathfrak{T}^{(o_2)}, \mathfrak{e}^{(o_2)})$ .

This fact will be denoted

$$(f_{(12)}, f^{(21)}) \in \mathbf{Sym} [(\mathfrak{P}^{(o_1)}, \mathfrak{T}^{(o_1)}, \mathfrak{e}^{(o_1)}) \rightarrow (\mathfrak{P}^{(o_2)}, \mathfrak{T}^{(o_2)}, \mathfrak{e}^{(o_2)})]. \quad (156)$$

*Remark 15.* We note that the axiom (153) has been designed to preserve the associativity property of the succession rule. Indeed, for any  $\mathfrak{p} \in \mathfrak{P}^{(o_1)}$ , and any  $\mathfrak{t}, \mathfrak{t}' \in \mathfrak{T}^{(o_2)}$ , we have  $f_{(12)}(\mathfrak{p}) \cdot (\mathfrak{t} \cdot \mathfrak{t}') = (f_{(12)}(\mathfrak{p}) \cdot \mathfrak{t}) \cdot \mathfrak{t}' = f_{(12)}(\mathfrak{p} \cdot f^{(21)}(\mathfrak{t})) \cdot \mathfrak{t}' = f_{(12)}((\mathfrak{p} \cdot f^{(21)}(\mathfrak{t})) \cdot f^{(21)}(\mathfrak{t}')) = f_{(12)}(\mathfrak{p} \cdot (f^{(21)}(\mathfrak{t}) \cdot f^{(21)}(\mathfrak{t}')) = f_{(12)}(\mathfrak{p} \cdot f^{(21)}(\mathfrak{t} \cdot \mathfrak{t}'))$ .

*Remark 16.* The properties (154) and (155) have been introduced in order to be able to derive the Theorem 14.

**Theorem 13. [Composition of symmetries]**

$$\left. \begin{aligned} (f_{(12)}, f^{(21)}) &\in \mathbf{Sym} [(\mathfrak{P}^{(o_1)}, \mathfrak{T}^{(o_1)}, \mathfrak{e}^{(o_1)}) \rightarrow (\mathfrak{P}^{(o_2)}, \mathfrak{T}^{(o_2)}, \mathfrak{e}^{(o_2)})] \\ (g_{(23)}, g^{(32)}) &\in \mathbf{Sym} [(\mathfrak{P}^{(o_2)}, \mathfrak{T}^{(o_2)}, \mathfrak{e}^{(o_2)}) \rightarrow (\mathfrak{P}^{(o_3)}, \mathfrak{T}^{(o_3)}, \mathfrak{e}^{(o_3)})] \end{aligned} \right\} \\ \Rightarrow (g_{(23)} \circ f_{(12)}, f^{(21)} \circ g^{(32)}) \in \mathbf{Sym} [(\mathfrak{P}^{(o_1)}, \mathfrak{T}^{(o_1)}, \mathfrak{e}^{(o_1)}) \rightarrow (\mathfrak{P}^{(o_3)}, \mathfrak{T}^{(o_3)}, \mathfrak{e}^{(o_3)})]. \quad (157)$$

■

*Proof.* We firstly note that

$$\forall \mathfrak{p}_1 \in \mathfrak{P}^{(o_1)}, \forall \mathfrak{t}_3 \in \mathfrak{T}^{(o_3)}, \mathfrak{e}^{(o_3)}(g_{(23)} \circ f_{(12)}(\mathfrak{p}_1), \mathfrak{t}_3) = \mathfrak{e}^{(o_2)}(f_{(12)}(\mathfrak{p}_1), g^{(32)}(\mathfrak{t}_3)) = \mathfrak{e}^{(o_1)}(\mathfrak{p}_1, f^{(21)} \circ g^{(32)}(\mathfrak{t}_3)). \quad (158)$$

Secondly, we have

$$\forall \mathfrak{p}_1 \in \mathfrak{P}^{(o_1)}, \forall \mathfrak{t}_3 \in \mathfrak{T}^{(o_3)}, (g_{(23)} \circ f_{(12)})(\mathfrak{p}_1) \cdot \mathfrak{t}_3 = g_{(23)}(f_{(12)}(\mathfrak{p}_1) \cdot g^{(32)}(\mathfrak{t}_3)) = (g_{(23)} \circ f_{(12)})((\mathfrak{p}_1) \cdot (f^{(21)} \circ g^{(32)})(\mathfrak{t}_3)). \quad (159)$$

The properties (153), (154), and (155) are trivially preserved by composition. □

**Lemma 33.** The dictionary  $(h_{(12)}, h^{(21)})$  defined by

$$\begin{array}{ccc} h_{(12)} : \mathfrak{P} & \rightarrow & \mathfrak{G} & & h^{(21)} : \mathfrak{T} & \rightarrow & \mathfrak{T} \\ & & \mathfrak{p} & \mapsto & [\mathfrak{p}] & & \mathfrak{t} & \mapsto & \mathfrak{t} \end{array} \quad (160)$$

satisfies the properties (150), (152), (153), (154), (155). In other words, this dictionary relates by a symmetry  $(\mathfrak{P}, \mathfrak{T}, \mathfrak{e})$  to  $(\mathfrak{G}, \mathfrak{T}, \tilde{\mathfrak{e}})$ .

$$(h_{(12)}, h^{(21)}) \in \mathbf{Sym} [(\mathfrak{P}, \mathfrak{T}, \mathfrak{e}) \rightarrow (\mathfrak{G}, \mathfrak{T}, \tilde{\mathfrak{e}})] \quad (161)$$

■

*Proof.* The property (150) is a direct consequence of the definition (11) of  $\tilde{\mathfrak{e}}$ .

The property (152) is a direct consequence of property (72).

The property (153) is tautologically verified.

The property (154) relies on the definition of the map  $[\cdot]$ .

The property (155) is trivial. □

As a consequence of Lemma 33 and Theorem 13, we can define the following notion.

**Notion 41.** To any dictionary  $(f_{(12)}, f^{(21)}) \in \mathbf{Sym} [(\mathfrak{P}^{(o_1)}, \mathfrak{T}^{(o_1)}, \mathfrak{e}^{(o_1)}) \rightarrow (\mathfrak{P}^{(o_2)}, \mathfrak{T}^{(o_2)}, \mathfrak{e}^{(o_2)})]$ , we will associate the dictionary  $(\tilde{f}_{(12)}, f^{(21)})$  defined by

$$\forall \mathfrak{p} \in \mathfrak{P}^{(o_1)}, \quad \tilde{f}_{(12)}([\mathfrak{p}]_1) := [f_{(12)}(\mathfrak{p})]_2. \quad (162)$$

We have

$$(\tilde{f}_{(12)}, f^{(21)}) \in \mathbf{Sym} [(\mathfrak{S}^{(o_1)}, \mathfrak{T}^{(o_1)}, \tilde{\mathfrak{e}}^{(o_1)}) \rightarrow (\mathfrak{S}^{(o_2)}, \mathfrak{T}^{(o_2)}, \tilde{\mathfrak{e}}^{(o_2)})] \quad (163)$$

Explicitely,

$$\forall \sigma_1 \in \mathfrak{S}^{(o_1)}, \forall \mathfrak{t}_2 \in \mathfrak{T}^{(o_2)}, \quad \tilde{\mathfrak{e}}_{\mathfrak{t}_2}^{(o_2)}(\tilde{f}_{(12)}(\sigma_1)) = \tilde{\mathfrak{e}}_{f_{(12)}(\mathfrak{t}_2)}^{(o_1)}(\sigma_1), \quad (164)$$

$$\forall \sigma_1 \in \mathfrak{S}^{(o_1)}, \forall \mathfrak{t}_2 \in \mathfrak{T}^{(o_2)}, \quad \tilde{f}_{(12)}(\sigma_1) \cdot \mathfrak{t}_2 = \tilde{f}_{(12)}(\sigma_1 \cdot f^{(21)}(\mathfrak{t}_2)), \quad (165)$$

$$\forall \mathfrak{t}, \mathfrak{t}' \in \mathfrak{T}^{(o_2)}, \quad f^{(21)}(\mathfrak{t} \cdot \mathfrak{t}') = f^{(21)}(\mathfrak{t}) \cdot f^{(21)}(\mathfrak{t}'), \quad (166)$$

$$\tilde{f}_{(12)} \text{ injective} \quad (167)$$

$$f^{(21)} \text{ surjective} \quad (168)$$

**Lemma 34.**  $\tilde{f}_{(12)}$  is an order-preserving Lawson-continuous map with  $\tilde{f}_{(12)}(\perp_{\mathfrak{S}^{(o_1)}}) = \perp_{\mathfrak{S}^{(o_2)}}$ , i.e.

$$\tilde{f}_{(12)} \in [\mathfrak{S}^{(o_1)} \rightarrow \mathfrak{S}^{(o_2)}]_{\square}^{\perp}. \quad (169)$$

■

*Proof.*

$$\forall \mathfrak{t}_2 \in \mathfrak{T}^{(o_2)}, \quad \tilde{\mathfrak{e}}_{\mathfrak{t}_2}^{(o_2)}(\tilde{f}_{(12)}(\perp_{\mathfrak{S}^{(o_1)}})) = \tilde{\mathfrak{e}}_{f_{(12)}(\mathfrak{t}_2)}^{(o_1)}(\perp_{\mathfrak{S}^{(o_1)}}) = \perp \quad \Rightarrow \quad \tilde{f}_{(12)}(\perp_{\mathfrak{S}^{(o_1)}}) = \perp_{\mathfrak{S}^{(o_2)}}. \quad (170)$$

Secondly, from Theorem 5, we know that  $\forall \mathfrak{t} \in \mathfrak{T}^{(o_2)}, \tilde{\mathfrak{e}}_{\mathfrak{t}}^{(o_2)}$  is an order-preserving and Lawson-continuous map (i.e.  $\tilde{\mathfrak{e}}_{\mathfrak{t}}^{(o_1)} \in [\mathfrak{S}^{(o_1)} \rightarrow \mathfrak{B}]_{\square}$ ). Then, using properties (164) and Lemma 1, we can deduce that  $\tilde{f}_{(12)}$  is an order-preserving and Lawson-continuous map. For example, let us check the Scott-continuity of  $\tilde{f}_{(12)}$  :

$$\begin{aligned} \forall \mathfrak{t} \in \mathfrak{T}^{(o_2)}, \forall \mathfrak{C} \subseteq_{Chain} \mathfrak{S}^{(o_1)}, \quad \tilde{\mathfrak{e}}_{\mathfrak{t}}^{(o_2)}(\tilde{f}_{(12)}(\bigsqcup_{\mathfrak{S}^{(o_1)}} \mathfrak{C})) &= \tilde{\mathfrak{e}}_{f_{(12)}(\mathfrak{t})}^{(o_1)}(\bigsqcup_{\mathfrak{S}^{(o_1)}} \mathfrak{C}) && \text{from eq. (164)} \\ &= \bigsqcup_{\sigma \in \mathfrak{C}} \tilde{\mathfrak{e}}_{f_{(12)}(\mathfrak{t})}^{(o_1)}(\sigma) && \text{from Theorem 5} \\ &= \bigsqcup_{\sigma \in \mathfrak{C}} \tilde{\mathfrak{e}}_{\mathfrak{t}}^{(o_2)}(f_{(12)}(\sigma)) && \text{from eq. (164)} \\ &= \tilde{\mathfrak{e}}_{\mathfrak{t}}^{(o_2)}(\bigsqcup_{\sigma \in \mathfrak{C}} f_{(12)}(\sigma)) && \text{from Theorem 5.} \end{aligned} \quad (171)$$

Then, using Lemma 1, we conclude

$$\forall \mathfrak{C} \subseteq_{Chain} \mathfrak{S}^{(o_1)}, \quad \tilde{f}_{(12)}(\bigsqcup_{\mathfrak{S}^{(o_1)}} \mathfrak{C}) = \bigsqcup_{\sigma \in \mathfrak{C}} \tilde{f}_{(12)}(\sigma). \quad (172)$$

We can prove with the same tricks :

$$\forall \mathfrak{Q} \subseteq \mathfrak{S}^{(o_1)}, \quad \tilde{f}_{(12)}(\prod_{\mathfrak{S}^{(o_1)}} \mathfrak{Q}) = \prod_{\sigma \in \mathfrak{Q}} \tilde{f}_{(12)}(\sigma). \quad (173)$$

□



**Theorem 14. [Preservation of the class of minimally-disturbing measurements by symmetry]**

Let  $(\tilde{f}_{(12)}, f^{(21)})$  be a dictionary relating by symmetry  $(\mathfrak{S}^{(O_1)}, \mathfrak{T}^{(O_1)}, \tilde{\mathfrak{e}}^{(O_1)})$  to  $(\mathfrak{S}^{(O_2)}, \mathfrak{T}^{(O_2)}, \tilde{\mathfrak{e}}^{(O_2)})$ , and let  $\mathfrak{t} \in \mathfrak{T}_{IFKM}^{(O_2)}$  be a yes/no test leading to an ideal first-kind measurement, then  $f^{(21)}(\mathfrak{t})$  is a yes/no test leading to an ideal first-kind measurements, i.e.  $f^{(21)}(\mathfrak{t}) \in \mathfrak{T}_{IFKM}^{(O_1)}$ . In other words,

$$\forall (\tilde{f}_{(12)}, f^{(21)}) \in \mathbf{Sym} [(\mathfrak{S}^{(O_1)}, \mathfrak{T}^{(O_1)}, \tilde{\mathfrak{e}}^{(O_1)}) \rightarrow (\mathfrak{S}^{(O_2)}, \mathfrak{T}^{(O_2)}, \tilde{\mathfrak{e}}^{(O_2)})], \quad \mathfrak{t} \in \mathfrak{T}_{IFKM}^{(O_2)} \Rightarrow f^{(21)}(\mathfrak{t}) \in \mathfrak{T}_{IFKM}^{(O_1)} \quad (174)$$

$$\text{i.e.} \quad (\tilde{f}_{(12)}, f^{(21)}) \in \mathbf{Sym} [(\mathfrak{S}^{(O_1)}, \mathfrak{T}_{min}^{(O_1)}, \tilde{\mathfrak{e}}^{(O_1)}) \rightarrow (\mathfrak{S}^{(O_2)}, \mathfrak{T}_{min}^{(O_2)}, \tilde{\mathfrak{e}}^{(O_2)})]. \quad (175)$$

■

*Proof.* Let us firstly prove that the symmetry preserves the Lawson-continuity property of measurement maps. Let us prove it is true for the Scott-continuity (the other proof is exactly analog) :

$$\begin{aligned} \forall \mathfrak{C} \subseteq_{Chain} \mathfrak{Q}_{f^{(21)}(\mathfrak{t})}^{(O_1)}, \quad \tilde{f}_{(12)}(\bigsqcup_{\mathfrak{S}^{(O_1)}} \mathfrak{C}) \cdot f^{(21)}(\mathfrak{t}) &= \tilde{f}_{(12)}(\bigsqcup_{\mathfrak{S}^{(O_1)}} \mathfrak{C}) \cdot \mathfrak{t} && \text{from eq. (165)} \\ &= \bigsqcup_{\sigma \in \mathfrak{C}} \tilde{f}_{(12)}(\sigma) \cdot \mathfrak{t} && \text{from eq. (172)} \\ &= \bigsqcup_{\sigma \in \mathfrak{C}} (\tilde{f}_{(12)}(\sigma) \cdot \mathfrak{t}) && \text{from eq. (77)} \\ &= \bigsqcup_{\sigma \in \mathfrak{C}} \tilde{f}_{(12)}(\sigma \cdot f^{(21)}(\mathfrak{t})) && \text{from eq. (165)} \\ &= \tilde{f}_{(12)}(\bigsqcup_{\sigma \in \mathfrak{C}} (\sigma \cdot f^{(21)}(\mathfrak{t}))) && \text{from eq. (172)} \end{aligned} \quad (176)$$

We now use the injectivity property (167) to conclude on the preservation of Scott-continuity of measurement-maps by symmetries :

$$\forall \mathfrak{C} \subseteq_{Chain} \mathfrak{Q}_{f^{(21)}(\mathfrak{t})}^{(O_1)}, \quad (\bigsqcup_{\mathfrak{S}^{(O_1)}} \mathfrak{C}) \cdot f^{(21)}(\mathfrak{t}) = \bigsqcup_{\sigma \in \mathfrak{C}} (\sigma \cdot f^{(21)}(\mathfrak{t})) \quad (177)$$

The second continuity-property is proved in an analog way

$$\forall \mathfrak{Q} \subseteq \mathfrak{Q}_{f^{(21)}(\mathfrak{t})}^{(O_1)}, \quad (\prod_{\mathfrak{S}^{(O_1)}} \mathfrak{Q}) \cdot f^{(21)}(\mathfrak{t}) = \prod_{\sigma \in \mathfrak{Q}} (\sigma \cdot f^{(21)}(\mathfrak{t})). \quad (178)$$

Secondly, let us prove that  $f^{(21)}(\mathfrak{t}) \in \mathfrak{T}_{FKM}^{(O_1)}$ . Let us consider any  $\sigma \in \mathfrak{S}^{(O_1)}$ , and  $\mathfrak{t} \in \mathfrak{T}_{FKM}^{(O_2)}$ . The preservation of the equation (80 (i)) is proved as follows

$$\begin{aligned} \tilde{\mathfrak{e}}_{f^{(21)}(\mathfrak{t})}^{(O_1)}(\sigma) \leq Y &\Rightarrow \tilde{\mathfrak{e}}_{\mathfrak{t}}^{(O_1)}(\tilde{f}_{(12)}(\sigma)) \leq Y && \text{from eq. (164)} \\ &\Rightarrow \tilde{\mathfrak{e}}_{\mathfrak{t}}^{(O_1)}(\tilde{f}_{(12)}(\sigma) \cdot \mathfrak{t}) = Y && \text{from } \mathfrak{t} \in \mathfrak{T}_{FKM}^{(O_2)} \\ &\Rightarrow \tilde{\mathfrak{e}}_{\mathfrak{t}}^{(O_1)}(\tilde{f}_{(12)}(\sigma \cdot f^{(21)}(\mathfrak{t}))) = Y && \text{from eq. (165)} \\ &\Rightarrow \tilde{\mathfrak{e}}_{f^{(21)}(\mathfrak{t})}^{(O_1)}(\sigma \cdot f^{(21)}(\mathfrak{t})) = Y && \text{from eq. (164)} \end{aligned} \quad (179)$$

The preservation of the equation (80 (ii)) is proved as follows

$$\begin{aligned} \tilde{\mathfrak{e}}_{f^{(21)}(\mathfrak{t})}^{(O_1)}(\sigma) = Y &\Rightarrow \tilde{\mathfrak{e}}_{\mathfrak{t}}^{(O_1)}(\tilde{f}_{(12)}(\sigma)) = Y && \text{from eq. (164)} \\ &\Rightarrow \tilde{f}_{(12)}(\sigma) \cdot \mathfrak{t} = \tilde{f}_{(12)}(\sigma) && \text{from } \mathfrak{t} \in \mathfrak{T}_{FKM}^{(O_2)} \\ &\Rightarrow \tilde{f}_{(12)}(\sigma \cdot f^{(21)}(\mathfrak{t})) = \tilde{f}_{(12)}(\sigma) && \text{from eq. (165)} \\ &\Rightarrow \sigma \cdot f^{(21)}(\mathfrak{t}) = \sigma && \text{from eq. (167)} \end{aligned} \quad (180)$$

Thirdly, it remains to show that  $f^{(21)}(\mathfrak{t}) \in \mathfrak{T}_{Ideal}^{(O_1)}$ . Let us consider  $u \in \mathfrak{T}^{(O_1)}$  such that  $\overbrace{f^{(21)}(\mathfrak{t})u}$ . The surjectivity of  $f^{(21)}$  (equation (155)) implies that it exists  $\mathfrak{t}' \in \mathfrak{T}^{(O_2)}$  such that  $u = f^{(21)}(\mathfrak{t}')$ .

The compatibility relation  $\overbrace{f^{(21)}(t)f^{(21)}(t')}$  implies the compatibility relation  $\widehat{tt'}$ . Indeed

$$\begin{aligned} \overbrace{f^{(21)}(t)f^{(21)}(t')} &\Leftrightarrow \exists \sigma \in \mathfrak{S} \mid \tilde{\mathfrak{e}}_{f^{(21)}(t)}^{(O_1)}(\sigma) = \tilde{\mathfrak{e}}_{f^{(21)}(t')}^{(O_1)}(\sigma) = Y \\ &\Rightarrow \exists \sigma' = \tilde{f}_{(12)}(\sigma) \in \mathfrak{S} \mid \tilde{\mathfrak{e}}_t^{(O_1)}(\sigma') = \tilde{\mathfrak{e}}_{t'}^{(O_1)}(\sigma') = Y \quad \text{from eq. (164)} \\ &\Rightarrow \widehat{tt'}. \end{aligned} \quad (181)$$

Let us now consider any  $\sigma \in \mathfrak{S}$  and  $t' \in \mathfrak{T}^{(O_2)}$  such that  $\overbrace{f^{(21)}(t)f^{(21)}(t')}$ , we then have

$$\begin{aligned} (\tilde{\mathfrak{e}}_{f^{(21)}(t)}^{(O_1)}(\sigma) \leq Y \text{ and } \tilde{\mathfrak{e}}_{f^{(21)}(t')}^{(O_1)}(\sigma) = Y) &\Rightarrow (\tilde{\mathfrak{e}}_t^{(O_1)}(\tilde{f}_{(12)}(\sigma)) \leq Y \text{ and } \tilde{\mathfrak{e}}_{t'}^{(O_1)}(\tilde{f}_{(12)}(\sigma)) = Y) \quad \text{from eq. (164)} \\ &\Rightarrow (\tilde{\mathfrak{e}}_{t'}^{(O_1)}(\tilde{f}_{(12)}(\sigma) \cdot t) = Y) \quad \text{from } t \in \mathfrak{T}_{Ideal}^{(O_2)} \text{ and } \widehat{tt'} \\ &\Rightarrow (\tilde{\mathfrak{e}}_{t'}^{(O_1)}(\tilde{f}_{(12)}(\sigma \cdot f^{(21)}(t))) = Y) \quad \text{from eq. (165)} \\ &\Rightarrow (\tilde{\mathfrak{e}}_{f^{(21)}(t')}^{(O_1)}(\sigma \cdot f^{(21)}(t)) = Y) \quad \text{from eq. (164)} \end{aligned} \quad (182)$$

□

**Lemma 35.** For any  $(\tilde{f}_{(12)}, f^{(21)}) \in \mathbf{Sym} [(\mathfrak{S}^{(O_1)}, \mathfrak{T}^{(O_1)}, \tilde{\mathfrak{e}}^{(O_1)}) \rightarrow (\mathfrak{S}^{(O_2)}, \mathfrak{T}^{(O_2)}, \tilde{\mathfrak{e}}^{(O_2)})]$  and  $t \in \mathfrak{T}^{(O_2)}$

$$f^{(21)}(\bar{t}) = \overline{f^{(21)}(t)}. \quad (183)$$

■

*Proof.*

$$\begin{aligned} \forall \sigma \in \mathfrak{S}^{(O_1)}, \tilde{\mathfrak{e}}_{f^{(21)}(\bar{t})}^{(O_1)}(\sigma) &= \tilde{\mathfrak{e}}_{\bar{t}}^{(O_2)}(f_{(21)}(\sigma)) \quad \text{from eq. (164)} \\ &= \overline{\tilde{\mathfrak{e}}_t^{(O_2)}(f_{(12)}(\sigma))} \quad \text{from eq. (4)} \\ &= \overline{\tilde{\mathfrak{e}}_{f^{(21)}(t)}^{(O_1)}(\sigma)} \quad \text{from eq. (164)} \\ &= \tilde{\mathfrak{e}}_{f^{(21)}(t)}^{(O_1)}(\sigma) \quad \text{from eq. (4)} \end{aligned} \quad (184)$$

□

**Theorem 15. [Preservation of the orthogonality of states]**

$$\forall (\tilde{f}_{(12)}, f^{(21)}) \in \mathbf{Sym} [(\mathfrak{S}^{(O_1)}, \mathfrak{T}^{(O_1)}, \tilde{\mathfrak{e}}^{(O_1)}) \rightarrow (\mathfrak{S}^{(O_2)}, \mathfrak{T}^{(O_2)}, \tilde{\mathfrak{e}}^{(O_2)})], \forall \sigma_1, \sigma_2 \in \mathfrak{S}^{(O_1)}, \sigma_1 \perp \sigma_2 \Rightarrow \tilde{f}_{(12)}(\sigma_1) \perp \tilde{f}_{(12)}(\sigma_2). \quad (185)$$

■

*Proof.*

$$\begin{aligned} \sigma_1 \perp \sigma_2 &\Leftrightarrow \exists \sigma \in \mathfrak{S}^{(O_1)}, \exists t \in \mathfrak{T}^{(O_1)} \mid (\sigma_1 = \sigma \cdot t \text{ and } \sigma_2 = \sigma \cdot \bar{t}) \\ &\Leftrightarrow \exists \sigma \in \mathfrak{S}^{(O_1)}, \exists u \in \mathfrak{T}^{(O_2)} \mid (\sigma_1 = \sigma \cdot f^{(21)}(u) \text{ and } \sigma_2 = \sigma \cdot \overline{f^{(21)}(u)}) \quad \text{from eq. (155)} \\ &\Rightarrow \exists \sigma \in \mathfrak{S}^{(O_1)}, \exists u \in \mathfrak{T}^{(O_2)} \mid (\tilde{f}_{(12)}(\sigma_1) = \tilde{f}_{(12)}(\sigma \cdot f^{(21)}(u)) \text{ and } \tilde{f}_{(12)}(\sigma_2) = \tilde{f}_{(12)}(\sigma \cdot f^{(21)}(\bar{u}))) \quad \text{from Lemma 35} \\ &\Rightarrow \exists \sigma' = \tilde{f}_{(12)}(\sigma) \in \mathfrak{S}^{(O_2)}, \exists u \in \mathfrak{T}^{(O_2)} \mid (\tilde{f}_{(12)}(\sigma_1) = \sigma' \cdot u \text{ and } \tilde{f}_{(12)}(\sigma_2) = \sigma' \cdot \bar{u}) \quad \text{from eq. (165)} \\ &\Leftrightarrow \tilde{f}_{(12)}(\sigma_1) \perp \tilde{f}_{(12)}(\sigma_2). \end{aligned} \quad (186)$$

□

From now, we want to restrict the class of symmetries in order to take into consideration the extra-properties (116)(117) satisfied by minimally-disturbing measurement maps.

**Notion 42.** The symmetry  $(\tilde{f}_{(12)}, f^{(21)}) \in \mathbf{Sym} [(\mathfrak{S}^{(o_1)}, \mathfrak{T}_{min}^{(o_1)}, \tilde{\mathfrak{e}}^{(o_1)}) \rightarrow (\mathfrak{S}^{(o_2)}, \mathfrak{T}_{min}^{(o_2)}, \tilde{\mathfrak{e}}^{(o_2)})]$  is said to be *linear* iff  $f_{(12)}$  satisfies

$$\forall \sigma_1, \sigma_2 \in \mathfrak{S}^{(o_1)}, \widehat{\sigma_1 \sigma_2}^{\mathfrak{S}^{(o_1)}} \Rightarrow \tilde{f}_{(12)}(\sigma_1 \sqcup_{\mathfrak{S}^{(o_1)}} \sigma_2) = \tilde{f}_{(12)}(\sigma_1) \sqcup_{\mathfrak{S}^{(o_1)}} \tilde{f}_{(12)}(\sigma_2) \quad (187)$$

$$\tilde{f}_{(12)} \text{ bijective.} \quad (188)$$

**Theorem 16.** Let the dictionary  $(\tilde{f}_{(12)}, f^{(21)})$  define a 'linear symmetry' from  $(\mathfrak{S}^{(o_1)}, \mathfrak{T}_{min}^{(o_1)}, \tilde{\mathfrak{e}}^{(o_1)})$  to  $(\mathfrak{S}^{(o_2)}, \mathfrak{T}_{min}^{(o_2)}, \tilde{\mathfrak{e}}^{(o_2)})$ . Then, an explicit formula for  $f^{(21)}$  can be found in terms of  $\tilde{f}_{(12)}$  :

$$\forall \sigma_1 \in \mathfrak{S}^{(o_1)}, \forall t_2 \in \mathfrak{T}_{min}^{(o_2)},$$

$$\sigma_1 \cdot f^{(21)}(t_2) = \tilde{f}_{(12)}^{-1}(\tilde{f}_{(12)}(\sigma_1) \cdot t_2) = \tilde{f}_{(12)}^{-1}(\Sigma_{t_2}) \sqcup_{\mathfrak{S}} \bigsqcup_{\mathfrak{S}} ((\downarrow_{\mathfrak{S}} \uparrow^{\mathfrak{S}} \tilde{f}_{(12)}^{-1}(\Sigma_{t_2})) \cap (\downarrow_{\mathfrak{S}} \sigma)), \quad (189)$$

$$[\mathfrak{A}_{f^{(21)}}(t_2)] = \uparrow^{\mathfrak{S}} \tilde{f}_{(12)}^{-1}(\Sigma_{t_2}), \quad [\mathfrak{Q}_{f^{(21)}}(t_2)] = \mathfrak{S} \setminus (\uparrow^{\mathfrak{S}} \tilde{f}_{(12)}^{-1}(\Sigma_{t_2})). \quad (190)$$

■

*Proof.* For any  $t_2 \in \mathfrak{T}_{min}^{(o_2)}$ , we have

$$\begin{aligned} [\mathfrak{A}_{f^{(21)}}(t_2)] &= \{\sigma \mid \tilde{\mathfrak{e}}_{f^{(21)}(t_2)}^{(o_1)}(\sigma) = \mathbf{Y}\} = \{\sigma \mid \tilde{\mathfrak{e}}_{t_2}^{(o_1)}(f_{(12)}(\sigma)) = \mathbf{Y}\} \quad \text{from eq. (164)} \\ &= \{\sigma \mid f_{(12)}(\sigma) \sqsubseteq_{\mathfrak{S}^{(o_1)}} \Sigma_{t_2}\} \quad \text{from eq. (62)} \\ &= \{\sigma \mid \sigma \sqsubseteq_{\mathfrak{S}^{(o_1)}} f_{(12)}^{-1}(\Sigma_{t_2})\} = \uparrow^{\mathfrak{S}} \tilde{f}_{(12)}^{-1}(\Sigma_{t_2}) \quad \text{from } f_{(12)} \text{ order-preserving} \\ [\mathfrak{Q}_{f^{(21)}}(t_2)] &= \mathfrak{S} \setminus \{\sigma \mid \tilde{\mathfrak{e}}_{f^{(21)}(t_2)}^{(o_1)}(\sigma) = \mathbf{N}\} = \mathfrak{S} \setminus \{\sigma \mid \tilde{\mathfrak{e}}_{t_2}^{(o_1)}(f_{(12)}(\sigma)) = \mathbf{N}\} \quad \text{from eq. (164)} \\ &= \mathfrak{S} \setminus \{\sigma \mid \overline{\tilde{\mathfrak{e}}_{t_2}^{(o_1)}(f_{(12)}(\sigma))} = \mathbf{Y}\} = \mathfrak{S} \setminus \{\sigma \mid \tilde{\mathfrak{e}}_{t_2}^{(o_1)}(f_{(12)}(\sigma)) = \mathbf{Y}\} \quad \text{from eq. (4)} \\ &= \mathfrak{S} \setminus \{\sigma \mid f_{(12)}(\sigma) \sqsubseteq_{\mathfrak{S}^{(o_1)}} \Sigma_{t_2}\} \quad \text{from eq. (62)} \\ &= \mathfrak{S} \setminus \{\sigma \mid \sigma \sqsubseteq_{\mathfrak{S}^{(o_1)}} f_{(12)}^{-1}(\Sigma_{t_2})\} = \mathfrak{S} \setminus \uparrow^{\mathfrak{S}} \tilde{f}_{(12)}^{-1}(\Sigma_{t_2}) \quad \text{from } f_{(12)} \text{ order-preserving} \end{aligned} \quad (191)$$

For any  $\sigma_1 \in \mathfrak{S}^{(o_1)}$  and any  $t_2 \in \mathfrak{T}_{min}^{(o_2)}$ , we have (during this proof, we will denote shortly  $g := \tilde{f}_{(12)}$ )

$$\begin{aligned} \sigma_1 \cdot f^{(21)}(t_2) &= g^{-1}(g(\sigma_1) \cdot t_2) \quad \text{from eq. (165) and eq. (188)} \\ &= g^{-1} \left( \Sigma_{t_2} \sqcup_{\mathfrak{S}^{(o_1)}} \bigsqcup_{\mathfrak{S}^{(o_1)}} \{ \sigma' \mid \widehat{\Sigma_{t_2} \sigma'}^{\mathfrak{S}^{(o_1)}} \text{ and } \sigma' \sqsubseteq_{\mathfrak{S}^{(o_1)}} g(\sigma_1) \} \right) \quad \text{from eq. (115)} \\ &= g^{-1}(\Sigma_{t_2}) \sqcup_{\mathfrak{S}^{(o_1)}} \bigsqcup_{\mathfrak{S}^{(o_1)}} \{ g^{-1}(\sigma') \mid \widehat{\Sigma_{t_2} \sigma'}^{\mathfrak{S}^{(o_1)}} \text{ and } \sigma' \sqsubseteq_{\mathfrak{S}^{(o_1)}} g(\sigma_1) \} \quad \text{from eq. (187) and (172)} \\ &= g^{-1}(\Sigma_{t_2}) \sqcup_{\mathfrak{S}^{(o_1)}} \bigsqcup_{\mathfrak{S}^{(o_1)}} \{ \sigma'' \mid \widehat{\Sigma_{t_2} g(\sigma'')}^{\mathfrak{S}^{(o_1)}} \text{ and } g(\sigma'') \sqsubseteq_{\mathfrak{S}^{(o_1)}} g(\sigma_1) \} \\ &= g^{-1}(\Sigma_{t_2}) \sqcup_{\mathfrak{S}^{(o_1)}} \bigsqcup_{\mathfrak{S}^{(o_1)}} \{ \sigma'' \mid \widehat{g^{-1}(\Sigma_{t_2}) \sigma''}^{\mathfrak{S}^{(o_1)}} \text{ and } \sigma'' \sqsubseteq_{\mathfrak{S}^{(o_1)}} \sigma_1 \} \quad \text{from } g \text{ order-preserving} \end{aligned} \quad (192)$$

By construction,  $(g, f^{(21)})$  satisfies the property (165). We have moreover

$$\begin{aligned}
\sigma_1 \cdot f^{(21)}(t_2 \cdot t'_2) &= g^{-1}(g(\sigma_1) \cdot t_2 \cdot t'_2) \\
&= g^{-1}(g(g^{-1}(g(\sigma_1) \cdot t_2)) \cdot t'_2) \\
&= g^{-1}(g(\sigma_1 \cdot f^{(21)}(t_2)) \cdot t'_2) \\
&= (\sigma_1 \cdot f^{(21)}(t_2)) \cdot f^{(21)}(t'_2) \\
&= \sigma_1 \cdot (f^{(21)}(t_2) \cdot f^{(21)}(t'_2))
\end{aligned} \tag{193}$$

Endly,  $f^{(21)}$  is obviously bijective.  $\square$

**Notion 43.** The symmetry  $(\tilde{f}_{(12)}, f^{(21)}) \in \mathbf{Sym}[(\mathfrak{S}^{(o_1)}, \mathfrak{I}_{min}^{(o_1)}, \tilde{\mathfrak{e}}^{(o_1)}) \rightarrow (\mathfrak{S}^{(o_2)}, \mathfrak{I}_{min}^{(o_2)}, \tilde{\mathfrak{e}}^{(o_2)})]$  is said to be *regular* iff (i) it is linear and (ii)  $\tilde{f}_{(12)}$  satisfies moreover

$$\forall \sigma_1, \sigma_2 \in \mathfrak{S}^{(o_1)}, \quad (\sigma_1 \sqsubseteq_{\mathfrak{S}^{(o_1)}} \sigma_2) \Rightarrow (\tilde{f}_{(12)}(\sigma_1) \sqsubseteq_{\mathfrak{S}^{(o_2)}} \tilde{f}_{(12)}(\sigma_2) \text{ or } \tilde{f}_{(12)}(\sigma_1) = \tilde{f}_{(12)}(\sigma_2)) \tag{194}$$

**Theorem 17. [Reconstruction of regular symmetries]** For any map  $\tilde{f}_{(12)}$  defining a regular symmetry  $(\tilde{f}_{(12)}, f^{(21)}) \in \mathbf{Sym}[(\mathfrak{S}^{(o_1)}, \mathfrak{I}_{min}^{(o_1)}, \tilde{\mathfrak{e}}^{(o_1)}) \rightarrow (\mathfrak{S}^{(o_2)}, \mathfrak{I}_{min}^{(o_2)}, \tilde{\mathfrak{e}}^{(o_2)})]$ , we consider the restriction of  $\tilde{f}_{(12)}$  to  $\mathcal{A}_{\mathfrak{S}^{(o_1)}}$  denoted  $\underline{\tilde{f}}_{(12)}$ . We have

$$\forall \sigma_1 \in \mathfrak{S}^{(o_1)} \quad \underline{\tilde{f}}_{(12)}(\sigma_1) = \bigsqcup_{\mathfrak{S}^{(o_1)}}^{\bar{\lambda}} \{ \bigsqcup_{a \in W} \underline{\tilde{f}}_{(12)}(a) \mid W \subseteq_{fin} \mathcal{A}_{\mathfrak{S}^{(o_1)}} \cap (\downarrow_{\mathfrak{S}^{(o_1)}} \sigma_1) \}. \tag{195}$$

*Proof.* From the property  $\tilde{f}_{(12)}(\perp_{\mathfrak{S}^{(o_1)}}) = \perp_{\mathfrak{S}^{(o_2)}}$  and the property (194) we deduce that the range of  $\underline{\tilde{f}}_{(12)}$  is included in  $\mathcal{A}_{\mathfrak{S}^{(o_1)}}$ .

For any  $W \subseteq_{fin} \mathcal{A}_{\mathfrak{S}^{(o_1)}} \cap (\downarrow_{\mathfrak{S}^{(o_1)}} \sigma_1)$ , we know obviously that the elements of  $W$  have a common upper-bound in  $\mathfrak{S}^{(o_1)}$ , which is  $\sigma_1$ . Hence, using property (187), we know that  $\underline{\tilde{f}}_{(12)}(\bigsqcup_{\mathfrak{S}^{(o_1)}} W) = \bigsqcup_{a \in W} \underline{\tilde{f}}_{(12)}(a)$ .

We now use the following formula inherited from the property (216) of the selection structure  $\mathfrak{S}^{(o_1)}$

$$\forall \sigma_1 \in \mathfrak{S}^{(o_1)} \quad \sigma_1 = \bigsqcup_{\mathfrak{S}^{(o_1)}}^{\bar{\lambda}} \{ \bigsqcup_{\mathfrak{S}^{(o_1)}} W \mid W \subseteq_{fin} \mathcal{A}_{\mathfrak{S}^{(o_1)}} \cap (\downarrow_{\mathfrak{S}^{(o_1)}} \sigma_1) \} \tag{196}$$

and the Scott-continuity of  $\underline{\tilde{f}}_{(12)}$  to prove the announced result.  $\square$

## 5 Conclusion

We intent to develop a new axiomatic approach of quantum theory and this article has been designated as an introduction, a semantic clarification for this axiomatic program. We then propose a semantic formalism adapted to the possibilistic description of quantum phenomena in the operational perspective.

In Section 2, a precise semantic description of the space of preparations and of the associated 'mixed states' of the system has been formulated. This semantic is based on a Chu space construction involving the set of preparations, the set of yes/no tests and an evaluation map with a three-valued target space. The values taken by this map are associated to counterfactual statements of the observer for a given yes/no test and a given prepared sample. The three values are interpreted in a possibilistic perspective, i.e. as 'certainly yes', 'certainly not' and 'may be'. This domain structure on the target space led to an 'informational' interpretation on the set of preparations. The space of preparations has then been equipped with a

notion of 'mixtures', expressed in terms of the meet operation on this poset. From natural requirements about the inductive definition of states, it appeared that this Inf semi-lattice was also a directed-complete partial order. Then, an 'Information Principle' has been introduced under the form of two topological requirements on the space of states. Although new in its form, this principle is very standard in different quantum axiomatic program. This basic set of axioms revealed to be sufficient to constrain the structure of the space of states to be a 'locally-boolean qualitative domain'. The space of pure states was then basically identified in terms of maximal elements of this domain.

In Section 3, the relation between yes/no tests and states has been studied in a double perspective : using the notion of Chu duality and using the notion of 'measurement'. Adopting the first perspective, the notion of 'properties' of the system has been defined and a 'property-state' has been identified for any property, as in Piron's construction. The second perspective emphasizes the recursive aspect of preparation process. In order to identify a sub-class of measurement operations corresponding to minimally-disturbing measurements it appeared necessary to clarify the notion of 'compatibility between measurements'. The compatibility between two measurements was defined in terms of the existence of preparations exhibiting simultaneously the two corresponding properties as actual. This notion was used to define 'ideal first-kind measurements' and to characterize them as 'minimally-disturbing measurements'. The conditions of existence of such minimally-disturbing measurements were clarified in terms of a quasi-classicality condition on the space of states. We finally proved the existence of a 'bi-extensional Chu duality' between the space of minimally-disturbing measurements and the space of states. The succession rule for ideal first-kind measurements was then studied and the 'Specker's principle' was proved. Using this result, we finally obtained a 'coherent domain' structure on the space of 'descriptions' formalizing the families of compatible properties used to define a state of the system. Endly, a sub-class of measurement operations corresponding to the measurement of 'classical' properties was identified.

In Section 4, we have explored the properties of Chu morphisms with respect to previous notions. As a central result, a sub-algebra of the algebra of Chu morphisms, corresponding to 'symmetries' of the system, was defined. It appeared that these symmetries preserve the class of minimally-disturbing measurements and preserve the orthogonality of states. A restricted class of symmetries has then been considered to take into account the linearity and regularity properties of measurement-maps. Using the atomisticity properties of the space of states, these symmetries can then be completely 'computed'.

## 6 Appendix : Basic properties of selection structures

Selection structures are defined in Definition 1.

**Lemma 36. [Selection structures are directed-complete] [49]** As soon as the selection structure  $\mathfrak{S}$  is chain-complete, it is necessarily directed-complete. ■

The way-below relation for the dcpo  $(\mathfrak{S}, \sqsubseteq_{\mathfrak{S}})$  will be denoted  $\sqsubset_{\mathfrak{S}}$ .

We will denote  $\underline{\mathcal{L}}$  the subset of  $\mathcal{S}$  defined as follows (elements of  $\underline{\mathcal{L}}$  will be called *flat selection structures*) :

$$\begin{aligned} \underline{\mathcal{L}} &\subseteq \mathcal{S}, \\ \forall (\mathfrak{S}, \sqsubseteq_{\mathfrak{S}}) \in \underline{\mathcal{L}}, \quad \forall x, y \in \mathfrak{S}, (x \sqsubset_{\mathfrak{S}} y) &\Leftrightarrow (x = \perp_{\mathfrak{S}} \text{ or } x = y). \end{aligned} \quad (197)$$

**Lemma 37. [Basic joins]** Let  $(\mathfrak{S}, \sqsubseteq_{\mathfrak{S}})$  be a selection structure. For any  $x, y \in \mathfrak{S}$ , we have

$$(\exists z \in \mathfrak{S} \mid (y \bar{\sqsubseteq}_{\mathfrak{S}} z \text{ and } x \sqsubset_{\mathfrak{S}} z \text{ and } x \parallel_{\mathfrak{S}} y)) \Rightarrow ((x \sqcup_{\mathfrak{S}} y) \text{ exists in } \mathfrak{S} \text{ and } x \sqcup_{\mathfrak{S}} y = z). \quad (198)$$

■

*Proof.* First of all, the hypothesis  $y \bar{\sqsubseteq}_{\mathfrak{S}} z, x \sqsubseteq_{\mathfrak{S}} z$  means that  $z$  is a common upper bound of  $x$  and  $y$ . Let us then consider another common upper-bound of  $x$  and  $y$  denoted  $z'$  and let us suppose  $z \not\sqsubseteq_{\mathfrak{S}} z'$ . The case  $z' \sqsubseteq_{\mathfrak{S}} z$  is immediatly eliminated because the property  $y \sqsubseteq_{\mathfrak{S}} z' \sqsubseteq_{\mathfrak{S}} z$ , combined with  $y \bar{\sqsubseteq}_{\mathfrak{S}} z$ , implies  $z' = y$ , i.e.  $x \sqsubseteq_{\mathfrak{S}} y$  which is contradictory with the hypothesis  $x \parallel_{\mathfrak{S}} y$ . If we suppose now  $z' \parallel_{\mathfrak{S}} z$ , we deduce from property (40) and the basic facts  $y \bar{\sqsubseteq}_{\mathfrak{S}} z$  and  $y \sqsubseteq_{\mathfrak{S}} z'$  that  $y = z \sqcap_{\mathfrak{S}} z'$ . However, we have also  $x \sqsubseteq_{\mathfrak{S}} z$  and  $x \sqsubseteq_{\mathfrak{S}} z'$ , which would then imply  $x \sqsubseteq_{\mathfrak{S}} y$  and then would contradict  $x \parallel_{\mathfrak{S}} y$ . Hence, necessarily  $z \sqsubseteq_{\mathfrak{S}} z'$  and  $z$  is the supremum  $x \sqcup_{\mathfrak{S}} y$ .  $\square$

**Lemma 38. [Natural expression of polarization]** The polarization axiom (i.e. property (41)) is equivalent to the following axiom:

$$\forall b, u, v \in \mathfrak{S}, \text{ with } b \sqsubseteq_{\mathfrak{S}} u \bar{\sqsubseteq}_{\mathfrak{S}} v, \exists! r \in \mathfrak{S}, \text{ such that } (u \sqcap_{\mathfrak{S}} r = b \text{ and } u \sqcup_{\mathfrak{S}} r = v). \quad (199)$$

■

*Proof.* Let us first suppose that  $\mathfrak{S}$  verifies equation (41). Note that, if  $b \sqsubseteq_{\mathfrak{S}} u \bar{\sqsubseteq}_{\mathfrak{S}} v, b \bar{\sqsubseteq}_{\mathfrak{S}} r \sqsubseteq_{\mathfrak{S}} v$  and  $r \parallel_{\mathfrak{S}} u$ , then, using Lemma 37 and property (40), we deduce  $u \sqcap_{\mathfrak{S}} r = b$  and  $r \sqcup_{\mathfrak{S}} u = v$ . As a result, we have then proved property (199).

Conversely, if we suppose that property (199) is verified, let us denote  $r$  the (unique) element such that  $(u \sqcap_{\mathfrak{S}} r = b \text{ and } r \sqcup_{\mathfrak{S}} u = v)$ . We can remark that  $u \sqcap_{\mathfrak{S}} r = b$  and  $b \sqsubseteq_{\mathfrak{S}} u$  implies necessarily  $r \parallel_{\mathfrak{S}} u$ .

It remains to show that  $b \bar{\sqsubseteq}_{\mathfrak{S}} r \sqsubseteq_{\mathfrak{S}} v$ . First of all,  $u \sqcap_{\mathfrak{S}} r = b$  implies  $b \sqsubseteq_{\mathfrak{S}} r$ . However, we cannot have  $b = r$  because it would imply  $r \sqsubseteq_{\mathfrak{S}} u$ , and then  $r \sqcup_{\mathfrak{S}} u = u$  which is contradicted by the property  $u \bar{\sqsubseteq}_{\mathfrak{S}} v$ . Hence, we have  $b \sqsubseteq_{\mathfrak{S}} r$ . Let us now suppose that  $b \not\bar{\sqsubseteq}_{\mathfrak{S}} r$ . Then,  $\exists w \in \mathfrak{S}$  such that  $b \bar{\sqsubseteq}_{\mathfrak{S}} w \sqsubseteq_{\mathfrak{S}} r$ . Necessarily, using property (40), we deduce  $b = (u \sqcap_{\mathfrak{S}} w)$ . From Lemma 37, we observe that, as soon as  $w \sqsubseteq_{\mathfrak{S}} r \sqsubseteq_{\mathfrak{S}} v$  and  $u \bar{\sqsubseteq}_{\mathfrak{S}} v$ , then  $w \sqcup_{\mathfrak{S}} u$  exists and is equal to  $v$ . This contradicts the uniqueness of  $r$  in the property (199). As a conclusion, we have necessarily  $b \bar{\sqsubseteq}_{\mathfrak{S}} r$ . Moreover, it exists a unique  $r$  satisfying  $b \bar{\sqsubseteq}_{\mathfrak{S}} r \sqsubseteq_{\mathfrak{S}} v$  and  $u \parallel_{\mathfrak{S}} r$ , which concludes the proof of property (41).  $\square$

**Lemma 39. [Basic meets]** Let  $(\mathfrak{S}, \sqsubseteq_{\mathfrak{S}})$  be a selection structure. For any  $y, u, v \in \mathfrak{S}$ , we have

$$(y \sqsubseteq_{\mathfrak{S}} v, u \bar{\sqsubseteq}_{\mathfrak{S}} v, y \parallel_{\mathfrak{S}} u) \Rightarrow ((y \sqcap_{\mathfrak{S}} u) \text{ exists in } \mathfrak{S} \text{ and } (y \sqcap_{\mathfrak{S}} u) \bar{\sqsubseteq}_{\mathfrak{S}} y). \quad (200)$$

■

*Proof.* Let us consider  $y, u, v \in \mathfrak{S}$ , such that  $(y \sqsubseteq_{\mathfrak{S}} v, u \bar{\sqsubseteq}_{\mathfrak{S}} v, y \parallel_{\mathfrak{S}} u)$ . We do not know yet if  $(y \sqcap_{\mathfrak{S}} u)$  exists. However, we can consider the set  $m_{y,u} := \{s \in \mathfrak{S} \mid s \sqsubseteq_{\mathfrak{S}} y \text{ and } s \sqsubseteq_{\mathfrak{S}} u\}$ .  $\perp_{\mathfrak{S}} \in m_{y,u}$  and then  $m_{y,u}$  is non-empty. Moreover, every chain of elements of  $m_{y,u}$  admits a supremum in  $m_{y,u}$ , and then, from Zorn's Lemma,  $m_{y,u}$  admits maximal elements. Let us consider  $m \in \text{Max}(m_{y,u})$ . We have  $m \sqsubseteq_{\mathfrak{S}} u \bar{\sqsubseteq}_{\mathfrak{S}} v$ , and then, using property (41), we deduce that it exists a unique element  $r$  such that  $m \bar{\sqsubseteq}_{\mathfrak{S}} r \sqsubseteq_{\mathfrak{S}} v$  and  $r \parallel_{\mathfrak{S}} u$ . As soon as it exists an element  $y'$  such that  $m \bar{\sqsubseteq}_{\mathfrak{S}} y' \sqsubseteq_{\mathfrak{S}} y$ , due to property (39), the unicity of  $r$  implies that  $r = y'$ . In other words, we have necessarily  $r \sqsubseteq_{\mathfrak{S}} y$ . Let us suppose that  $r \not\sqsubseteq_{\mathfrak{S}} y$ . From property (39), it then exist  $r'$  such that  $r \bar{\sqsubseteq}_{\mathfrak{S}} r' \sqsubseteq_{\mathfrak{S}} y$ , however, using property (41), we would deduce the existence of an element  $r''$  satisfying  $m \bar{\sqsubseteq}_{\mathfrak{S}} r'' \sqsubseteq_{\mathfrak{S}} r' \sqsubseteq_{\mathfrak{S}} y$  and  $r'' \parallel_{\mathfrak{S}} r$ . Moreover, the unicity of  $r$  as solution of the relations  $m \bar{\sqsubseteq}_{\mathfrak{S}} r \sqsubseteq_{\mathfrak{S}} v$  and  $r \parallel_{\mathfrak{S}} u$ , imposes necessarily  $r'' \sqsubseteq_{\mathfrak{S}} u$  (the option  $u \sqsubseteq_{\mathfrak{S}} r''$  would contradict the property  $r'' \sqsubseteq_{\mathfrak{S}} v$ ). But then  $r''$  would be a common lower-bound of  $u$  and  $y$  and this point contradicts the maximality of  $m$ . As a result, we have necessarily  $r = y$ , and then  $m \bar{\sqsubseteq}_{\mathfrak{S}} y$ . Now, we use the property (40) to conclude that  $m$  is in fact the infimum  $(y \sqcap_{\mathfrak{S}} u)$ .  $\square$

**Lemma 40. [Lower-covering property]** The following property satisfied for any  $x, u, v \in \mathfrak{S}$  will

be called *lower-covering property*:

$$(u \bar{\sqsubseteq}_{\mathfrak{S}} v \text{ and } (x \sqcap_{\mathfrak{S}} v) \text{ exists in } \mathfrak{S}) \Rightarrow ((x \sqcap_{\mathfrak{S}} u) \text{ exists in } \mathfrak{S} \\ \text{and } ((x \sqcap_{\mathfrak{S}} u) \bar{\sqsubseteq}_{\mathfrak{S}} (x \sqcap_{\mathfrak{S}} v) \text{ or } (x \sqcap_{\mathfrak{S}} u) = (x \sqcap_{\mathfrak{S}} v))). \quad (201)$$

■

*Proof.* Let us consider  $x, u, v \in \mathfrak{S}$  with  $u \bar{\sqsubseteq}_{\mathfrak{S}} v$  and such that  $(x \sqcap_{\mathfrak{S}} v)$  exists in  $\mathfrak{S}$ . We will distinguish three different cases :  $u \sqsubseteq_{\mathfrak{S}} x$ ,  $x \sqsubseteq_{\mathfrak{S}} u$  and  $x \parallel_{\mathfrak{S}} u$ .

- If  $u \sqsubseteq_{\mathfrak{S}} x$ , then  $(x \sqcap_{\mathfrak{S}} u)$  obviously exists in  $\mathfrak{S}$  and  $(x \sqcap_{\mathfrak{S}} u) = u$ . We distinguish two sub-cases:  $v \sqsubseteq_{\mathfrak{S}} x$  or  $v \parallel_{\mathfrak{S}} x$ .
  - If  $v \sqsubseteq_{\mathfrak{S}} x$ , then  $(v \sqcap_{\mathfrak{S}} x) = v$ . We have  $(x \sqcap_{\mathfrak{S}} u) = u \bar{\sqsubseteq}_{\mathfrak{S}} v = (x \sqcap_{\mathfrak{S}} v)$ .
  - If  $v \parallel_{\mathfrak{S}} x$ , using  $u \sqsubseteq_{\mathfrak{S}} x$ ,  $u \bar{\sqsubseteq}_{\mathfrak{S}} v$ , and property (40), we deduce that  $(x \sqcap_{\mathfrak{S}} v) = u$ , i.e.  $(x \sqcap_{\mathfrak{S}} v) = (x \sqcap_{\mathfrak{S}} u)$ .
- If  $x \sqsubseteq_{\mathfrak{S}} u \bar{\sqsubseteq}_{\mathfrak{S}} v$ ,  $(x \sqcap_{\mathfrak{S}} u)$  obviously exists in  $\mathfrak{S}$  and  $(x \sqcap_{\mathfrak{S}} u) = x$ . We have then  $x = (x \sqcap_{\mathfrak{S}} u) = (x \sqcap_{\mathfrak{S}} v)$ .
- If  $x \parallel_{\mathfrak{S}} u$ , we will distinguish two sub-cases  $x \sqsubseteq_{\mathfrak{S}} v$  or  $x \parallel_{\mathfrak{S}} v$  (indeed,  $u \bar{\sqsubseteq}_{\mathfrak{S}} v \sqsubseteq_{\mathfrak{S}} x$  is incompatible with  $u \parallel_{\mathfrak{S}} x$ ).
  - If  $x \sqsubseteq_{\mathfrak{S}} v$ , we have  $x \sqcap_{\mathfrak{S}} v = x$ . To summarize, we have  $(x \sqcap_{\mathfrak{S}} u) \sqsubseteq_{\mathfrak{S}} x \sqsubseteq_{\mathfrak{S}} v$ ,  $x \parallel_{\mathfrak{S}} u$ ,  $(x \sqcap_{\mathfrak{S}} u) \sqsubseteq_{\mathfrak{S}} u \bar{\sqsubseteq}_{\mathfrak{S}} v$ . We are then in the configuration of property (200). As a result,  $(x \sqcap_{\mathfrak{S}} u)$  exists in  $\mathfrak{S}$  and  $(x \sqcap_{\mathfrak{S}} u) \bar{\sqsubseteq}_{\mathfrak{S}} x = (x \sqcap_{\mathfrak{S}} v)$ .
  - If  $x \parallel_{\mathfrak{S}} v$ , we will distinguish two sub-sub-cases:  $(x \sqcap_{\mathfrak{S}} v) \sqsubseteq_{\mathfrak{S}} u$  and  $(x \sqcap_{\mathfrak{S}} v) \parallel_{\mathfrak{S}} u$ . Indeed, the option  $u \sqsubseteq_{\mathfrak{S}} (x \sqcap_{\mathfrak{S}} v)$  is excluded, because this hypothesis would imply  $u \sqsubseteq_{\mathfrak{S}} x$  which is false.
    - \* if  $(x \sqcap_{\mathfrak{S}} v) \sqsubseteq_{\mathfrak{S}} u$ , we note that  $\forall z \in \mathfrak{S}$ ,  $(z \sqsubseteq_{\mathfrak{S}} u) \Rightarrow (z \sqsubseteq_{\mathfrak{S}} v)$  and then  $(z \sqsubseteq_{\mathfrak{S}} \{x, u\}) \Rightarrow (z \sqsubseteq_{\mathfrak{S}} (x \sqcap_{\mathfrak{S}} v))$ . But precisely  $(x \sqcap_{\mathfrak{S}} v) \sqsubseteq_{\mathfrak{S}} \{x, u\}$ . As a result,  $(x \sqcap_{\mathfrak{S}} u)$  exists in  $\mathfrak{S}$  and  $(x \sqcap_{\mathfrak{S}} u) = (x \sqcap_{\mathfrak{S}} v)$ .
    - \* if  $(x \sqcap_{\mathfrak{S}} v) \parallel_{\mathfrak{S}} u$ . We have then  $u \bar{\sqsubseteq}_{\mathfrak{S}} v$ ,  $(x \sqcap_{\mathfrak{S}} v) \sqsubseteq_{\mathfrak{S}} v$  and  $(x \sqcap_{\mathfrak{S}} v) \parallel_{\mathfrak{S}} u$ , which is the exact configuration of property (200), if we denote  $y := (x \sqcap_{\mathfrak{S}} v)$ . Hence,  $(y \sqcap_{\mathfrak{S}} u)$  exists in  $\mathfrak{S}$  and  $(y \sqcap_{\mathfrak{S}} u) \bar{\sqsubseteq}_{\mathfrak{S}} y$ , i.e.  $(x \sqcap_{\mathfrak{S}} u)$  exists in  $\mathfrak{S}$  and  $(x \sqcap_{\mathfrak{S}} u) \bar{\sqsubseteq}_{\mathfrak{S}} (x \sqcap_{\mathfrak{S}} v)$ .

This concludes the complete analysis of the different cases for the relation (201) and then the proof of the announced result. □

**Lemma 41. [Dual formulation of polarization axiom]** A selection structure  $\mathfrak{S}$  satisfies necessarily the following dual formulation of polarization axiom:

$$\forall r, b, t \in \mathfrak{S}, \text{ with } b \bar{\sqsubseteq}_{\mathfrak{S}} r \sqsubseteq_{\mathfrak{S}} t, \\ u := \bigsqcup_{\mathfrak{S}} \{s \in \mathfrak{S} \mid b \sqsubseteq_{\mathfrak{S}} s \sqsubseteq_{\mathfrak{S}} t \text{ and } s \parallel_{\mathfrak{S}} r\} \text{ exists in } \mathfrak{S} \\ \text{Moreover, we have } (r \sqcap_{\mathfrak{S}} u = b \text{ and } b \sqsubseteq_{\mathfrak{S}} u \bar{\sqsubseteq}_{\mathfrak{S}} (u \sqcup_{\mathfrak{S}} r) = t), \quad (202)$$

■

*Proof.* Let us then consider  $r, b, t \in \mathfrak{S}$ , with  $b \bar{\sqsubseteq}_{\mathfrak{S}} r \sqsubseteq_{\mathfrak{S}} t$ . We have to prove that the subset

$$Q_{(b,t)}^r := \{s \in \mathfrak{S} \mid b \sqsubseteq_{\mathfrak{S}} s \sqsubseteq_{\mathfrak{S}} t \text{ and } s \parallel_{\mathfrak{S}} r\}.$$

admits a supremum in  $\mathfrak{S}$  (obviously, this supremum is unique). To begin, we will study the existence and explore the properties of the maximum elements of this set.

We note, first of all, that  $Q_{(b,t)}^r$  is non-empty. Indeed, we know from property (39) that it exists an element  $r'$  such that  $r \bar{\sqsubseteq}_{\mathfrak{S}} r' \sqsubseteq_{\mathfrak{S}} t$ , and then from property (41) that it exists an element  $r''$  such that  $r \bar{\sqsubseteq}_{\mathfrak{S}} r'' \sqsubseteq_{\mathfrak{S}} r' \sqsubseteq_{\mathfrak{S}} t$  and  $r' \parallel_{\mathfrak{S}} r''$ , i.e. also  $r'' \sqcap_{\mathfrak{S}} r = b$  using (40).

For any chain  $C \sqsubseteq_{\text{chain}} Q_{(b,t)}^r$ , we can consider its supremum  $(\sqcup_{\mathfrak{S}} C)$  which exists in  $\mathfrak{S}$  due to chain-completeness property of  $\mathfrak{S}$  (equation 42). As soon as  $\forall c \in C, b \sqsubseteq_{\mathfrak{S}} c \sqsubseteq_{\mathfrak{S}} t$ , we obtain immediately  $b \sqsubseteq_{\mathfrak{S}} (\sqcup_{\mathfrak{S}} C) \sqsubseteq_{\mathfrak{S}} t$ , by definition of the supremum. Moreover, using the weak-continuity of  $\mathfrak{S}$  (equation (44)), we know that  $\forall c \in C, c \parallel_{\mathfrak{S}} r = b$  implies  $r \parallel_{\mathfrak{S}} (\sqcup_{\mathfrak{S}} C)$ . As a result,  $(\sqcup_{\mathfrak{S}} C) \in Q_{(b,t)}^r$ . Then, from Zorn's lemma,  $Q_{(b,t)}^r$  admits some maximal elements.

$$\text{Max}(Q_{(b,t)}^r) \neq \emptyset.$$

We note that, for any  $s \in \text{Max}(Q_{(b,t)}^r)$ , we have  $s \sqsubseteq_{\mathfrak{S}} t$  but the equality  $s = t$  is excluded because  $t \sqcap_{\mathfrak{S}} r = r \neq b$ , i.e.  $t \notin Q_{(b,t)}^r$ . As a result, we obtain:

$$\text{Max}(Q_{(b,t)}^r) \cap \text{Max}(\mathfrak{S}) = \emptyset.$$

Then, due to strong-atomicity of  $\mathfrak{S}$  (equation (39)), we have  $\forall u \in \text{Max}(Q_{(b,t)}^r), \exists v \in \mathfrak{S}, u \bar{\sqsubseteq}_{\mathfrak{S}} v \sqsubseteq_{\mathfrak{S}} t$ . We cannot have  $r \parallel_{\mathfrak{S}} v$ , because it would contradict the maximality property of  $u$ . We cannot have  $v \sqsubseteq_{\mathfrak{S}} r$  either, because it would contradict  $u \parallel_{\mathfrak{S}} r$  and  $u \bar{\sqsubseteq}_{\mathfrak{S}} v$ . We have then established:

$$\forall u \in \text{Max}(Q_{(b,t)}^r), \quad \forall v \in \mathfrak{S}, (u \bar{\sqsubseteq}_{\mathfrak{S}} v \sqsubseteq_{\mathfrak{S}} t) \Rightarrow (r \sqsubseteq_{\mathfrak{S}} v).$$

More precisely, using Lemma 37, we have in fact

$$\forall u \in \text{Max}(Q_{(b,t)}^r), \quad \forall v \in \mathfrak{S}, (u \bar{\sqsubseteq}_{\mathfrak{S}} v \sqsubseteq_{\mathfrak{S}} t) \Rightarrow (v = u \sqcup_{\mathfrak{S}} r).$$

Let us now show that  $v = t$ . Let us then suppose that  $v$  is different from  $t$  and let us exhibit a contradiction. If  $u \bar{\sqsubseteq}_{\mathfrak{S}} v \sqsubseteq_{\mathfrak{S}} t$ , then it exists  $v'$  such that  $u \bar{\sqsubseteq}_{\mathfrak{S}} v \sqsubseteq_{\mathfrak{S}} t$  and  $v \parallel_{\mathfrak{S}} v'$  as a direct consequence of property (41). But then we should have  $u = v \sqcap_{\mathfrak{S}} v'$  (using property (40)), which would lead to the following contradiction:  $b = (u \sqcap_{\mathfrak{S}} r) = (v \sqcap_{\mathfrak{S}} v') \sqcap_{\mathfrak{S}} r = (v \sqcap_{\mathfrak{S}} (v' \sqcap_{\mathfrak{S}} r)) = (v \sqcap_{\mathfrak{S}} r) = r$ . We have then obtained  $u \bar{\sqsubseteq}_{\mathfrak{S}} v = t, u \parallel_{\mathfrak{S}} r$  and  $r \sqsubseteq_{\mathfrak{S}} t$ , and then  $v = t = u \sqcup_{\mathfrak{S}} r$  using Lemma 37. As a summary, we have

$$\forall u \in \text{Max}(Q_{(b,t)}^r), \quad \forall v \in \mathfrak{S}, (u \bar{\sqsubseteq}_{\mathfrak{S}} v \sqsubseteq_{\mathfrak{S}} t) \Rightarrow (v = t = u \sqcup_{\mathfrak{S}} r).$$

Let us now consider  $u_1$  and  $u_2$ , two elements of  $\text{Max}(Q_{(b,t)}^r)$ . We intent to prove that necessarily  $u_1 = u_2$ . We will suppose  $u_1 \neq u_2$  and try to exhibit a contradiction. From  $u_1, u_2 \bar{\sqsubseteq}_{\mathfrak{S}} t$ , we deduce  $u_1 \parallel_{\mathfrak{S}} u_2$ . Moreover, from previous results, we have  $b \sqsubseteq_{\mathfrak{S}} u_k \bar{\sqsubseteq}_{\mathfrak{S}} t, u_k \parallel_{\mathfrak{S}} r$  for  $k = 1, 2$ . From property (39), we deduce that  $(u_1 \sqcap_{\mathfrak{S}} u_2)$  exists in  $\mathfrak{S}$  and  $b \sqsubseteq_{\mathfrak{S}} (u_1 \sqcap_{\mathfrak{S}} u_2) \bar{\sqsubseteq}_{\mathfrak{S}} u_1, u_2$ . We now use property (41) to deduce that it exists a unique  $w$  such that  $b \bar{\sqsubseteq}_{\mathfrak{S}} w \sqsubseteq_{\mathfrak{S}} u_1 \sqsubseteq_{\mathfrak{S}} t$  and  $(u_1 \sqcap_{\mathfrak{S}} u_2) \parallel_{\mathfrak{S}} w$ . We have then  $b \bar{\sqsubseteq}_{\mathfrak{S}} w \sqsubseteq_{\mathfrak{S}} u_1 \sqsubseteq_{\mathfrak{S}} t$  and  $w \parallel_{\mathfrak{S}} u_2$ . The unicity requirement should impose  $w = r$  which is contradictory because  $w \sqsubseteq_{\mathfrak{S}} u_1$  is incompatible with  $r \parallel_{\mathfrak{S}} u_1$ . As a result,  $u_1 = u_2$ . This establishes the unicity of  $u$  in equation (202).  $\square$

**Lemma 42. [D – D\* properties of Selection structures]** A selection structure  $\mathfrak{S}$  satisfies the following properties :

$$(D^*) \quad \forall s_1, s_2, t \in \mathfrak{S}, ((s_1 \sqcap_{\mathfrak{S}} s_2) \bar{\sqsubseteq}_{\mathfrak{S}} s_1 \sqsubseteq_{\mathfrak{S}} t \text{ and } (s_1 \sqcap_{\mathfrak{S}} s_2) \bar{\sqsubseteq}_{\mathfrak{S}} s_2 \sqsubseteq_{\mathfrak{S}} t) \Rightarrow \\ ((s_1 \sqcup_{\mathfrak{S}} s_2) \text{ exists in } \mathfrak{S} \text{ and } [s_1 \sqcap_{\mathfrak{S}} s_2, s_1 \sqcup_{\mathfrak{S}} s_2] \cong B_2), \quad (203)$$

$$(D) \quad \forall s_1, s_2, t \in \mathfrak{S}, ((s_1 \sqcup_{\mathfrak{S}} s_2) \text{ exists, } s_1 \bar{\sqsubseteq}_{\mathfrak{S}} (s_1 \sqcup_{\mathfrak{S}} s_2) \text{ and } s_2 \bar{\sqsubseteq}_{\mathfrak{S}} (s_1 \sqcup_{\mathfrak{S}} s_2)) \Rightarrow \\ ([s_1 \sqcap_{\mathfrak{S}} s_2, s_1 \sqcup_{\mathfrak{S}} s_2] \cong B_2) \quad (204)$$

In the equations (203) and (204), we have denoted by  $B_2$  the lattice of only four elements

$$B_2 := (\{a \sqcap_{\mathfrak{S}} b, a, b, a \sqcup_{\mathfrak{S}} b\}, \sqcap, \sqcup) \text{ with } a \parallel_{\mathfrak{S}} b.$$



■

*Remark 17.* The proof of the equivalence of these  $D - D^*$  properties and the distributivity property, in the special case where  $\mathfrak{S}$  is a lattice, is established in the couple of papers [43][44].

*Proof.* Let us consider  $s_1, s_2, t \in \mathfrak{S}$  satisfying  $s_1 \parallel_{\mathfrak{S}} s_2$ ,  $(s_1 \sqcap_{\mathfrak{S}} s_2) \bar{\sqsubseteq}_{\mathfrak{S}} s_1 \sqsubseteq_{\mathfrak{S}} t$  and  $(s_1 \sqcap_{\mathfrak{S}} s_2) \bar{\sqsubseteq}_{\mathfrak{S}} s_2 \sqsubseteq_{\mathfrak{S}} t$ . Using Lemma 41 and the ordering relation  $(s_1 \sqcap_{\mathfrak{S}} s_2) \bar{\sqsubseteq}_{\mathfrak{S}} s_2 \sqsubseteq_{\mathfrak{S}} t$ , we deduce that :

$$\exists! u \in \mathfrak{S}, u \bar{\sqsubseteq}_{\mathfrak{S}} t \mid \forall s \in \mathfrak{S}, ((s_1 \sqcap_{\mathfrak{S}} s_2) \sqsubseteq_{\mathfrak{S}} s \sqsubseteq_{\mathfrak{S}} t) \text{ and } (s \parallel_{\mathfrak{S}} s_2) \Leftrightarrow (s \sqsubseteq_{\mathfrak{S}} u). \quad (205)$$

Moreover, the unique element  $u$ , defined according to the property (205), satisfies  $t = (u \sqcup_{\mathfrak{S}} s_2)$ . Endly, using the defining property of  $u$  (i.e. property (205)) and the properties satisfied by  $s_1$  (i.e.  $(s_1 \sqcap_{\mathfrak{S}} s_2) \bar{\sqsubseteq}_{\mathfrak{S}} s_1 \sqsubseteq_{\mathfrak{S}} t$  and  $(s_1 \parallel_{\mathfrak{S}} s_2)$ ), we conclude that  $s_1 \sqsubseteq_{\mathfrak{S}} u$ .

We can distinguish different cases, relative to  $u$ .

If  $s_1 = u$ , we have then  $u = s_1 \bar{\sqsubseteq}_{\mathfrak{S}} t$ ,  $s_2 \sqsubseteq_{\mathfrak{S}} t$ ,  $s_1 \parallel_{\mathfrak{S}} s_2$  and then, using Lemma 39,  $t = (s_1 \sqcup_{\mathfrak{S}} s_2)$ . As a result, we have then established that  $(s_1 \sqcup_{\mathfrak{S}} s_2)$  exists in  $\mathfrak{S}$ .

If  $s_1 \sqsubseteq_{\mathfrak{S}} u \bar{\sqsubseteq}_{\mathfrak{S}} t$ , we can apply property (41) to deduce that it exists a unique  $r \in \mathfrak{S}$ , such that  $(s_1 \bar{\sqsubseteq}_{\mathfrak{S}} r \sqsubseteq_{\mathfrak{S}} t$  and  $r \parallel_{\mathfrak{S}} u$ ). Let us show that necessarily  $s_2 \sqsubseteq_{\mathfrak{S}} r$ . We firstly note that the option  $r \bar{\sqsubseteq}_{\mathfrak{S}} s_2$  is excluded since  $(s_1 \sqcap_{\mathfrak{S}} s_2) \bar{\sqsubseteq}_{\mathfrak{S}} s_2$  and  $(s_1 \sqcap_{\mathfrak{S}} s_2) \bar{\sqsubseteq}_{\mathfrak{S}} s_1 \bar{\sqsubseteq}_{\mathfrak{S}} r$ . Secondly, the option  $r \parallel_{\mathfrak{S}} s_2$  is also excluded, because  $(s_1 \sqcap_{\mathfrak{S}} s_2) \bar{\sqsubseteq}_{\mathfrak{S}} s_1 \bar{\sqsubseteq}_{\mathfrak{S}} r \sqsubseteq_{\mathfrak{S}} t$  and  $r \parallel_{\mathfrak{S}} s_2$  would imply  $(r \sqsubseteq_{\mathfrak{S}} u)$ , because of property (205), and this fact would then contradict the hypothesis  $r \parallel_{\mathfrak{S}} u$ . As a result, we have then  $s_2 \sqsubseteq_{\mathfrak{S}} r$ . To summarize, we have  $s_1 \bar{\sqsubseteq}_{\mathfrak{S}} r$ ,  $s_2 \sqsubseteq_{\mathfrak{S}} r$  and  $s_1 \parallel_{\mathfrak{S}} s_2$ , and then, using Lemma 39, we deduce that  $(s_1 \sqcup_{\mathfrak{S}} s_2)$  exists in  $\mathfrak{S}$  and  $r = (s_1 \sqcup_{\mathfrak{S}} s_2)$ .

As a conclusion, in any configurations, the hypotheses  $s_1 \parallel_{\mathfrak{S}} s_2$ ,  $(s_1 \sqcap_{\mathfrak{S}} s_2) \bar{\sqsubseteq}_{\mathfrak{S}} s_1 \sqsubseteq_{\mathfrak{S}} t$  and  $(s_1 \sqcap_{\mathfrak{S}} s_2) \bar{\sqsubseteq}_{\mathfrak{S}} s_2 \sqsubseteq_{\mathfrak{S}} t$  imply that  $(s_1 \sqcup_{\mathfrak{S}} s_2)$  exists in  $\mathfrak{S}$ . Obviously, using Lemma 41, we have also explicitly  $s_1 \bar{\sqsubseteq}_{\mathfrak{S}} (s_1 \sqcup_{\mathfrak{S}} s_2)$  and  $s_2 \bar{\sqsubseteq}_{\mathfrak{S}} (s_1 \sqcup_{\mathfrak{S}} s_2)$ .

If we now suppose that it exists  $z \in \mathfrak{S}$  with  $(s_1 \sqcap_{\mathfrak{S}} s_2) \sqsubseteq_{\mathfrak{S}} z \sqsubseteq_{\mathfrak{S}} (s_1 \sqcup_{\mathfrak{S}} s_2)$ , and  $z \neq s_1$ ,  $z \neq s_2$ . Then necessarily,  $z \parallel_{\mathfrak{S}} s_1$  because of the covering property  $(s_1 \sqcap_{\mathfrak{S}} s_2) \bar{\sqsubseteq}_{\mathfrak{S}} s_1 \bar{\sqsubseteq}_{\mathfrak{S}} (s_1 \sqcup_{\mathfrak{S}} s_2)$ . However, from property (41), we know that it exists a unique  $r \in \mathfrak{S}$ , such that  $((s_1 \sqcap_{\mathfrak{S}} s_2) \bar{\sqsubseteq}_{\mathfrak{S}} r \sqsubseteq_{\mathfrak{S}} (s_1 \sqcup_{\mathfrak{S}} s_2)$  and  $r \parallel_{\mathfrak{S}} s_1$ ). Hence, the property  $(s_1 \sqcap_{\mathfrak{S}} s_2) \bar{\sqsubseteq}_{\mathfrak{S}} z$  is impossible because it would impose  $z = s_2$ . And if  $(s_1 \sqcap_{\mathfrak{S}} s_2) \bar{\sqsubseteq}_{\mathfrak{S}} z$  then strong atomicity of  $\mathfrak{S}$  (i.e. property (39)) imposes the existence of some  $z' \in \mathfrak{S}$  with  $(s_1 \sqcap_{\mathfrak{S}} s_2) \bar{\sqsubseteq}_{\mathfrak{S}} z' \sqsubseteq_{\mathfrak{S}} z \sqsubseteq_{\mathfrak{S}} (s_1 \sqcup_{\mathfrak{S}} s_2)$ . The same unicity argument can be applied to  $z'$  which must be equal to  $s_2$ . Then the covering property  $s_2 \sqsubseteq_{\mathfrak{S}} z \sqsubseteq_{\mathfrak{S}} (s_1 \sqcup_{\mathfrak{S}} s_2)$  appears to be contradictory with  $s_2 \bar{\sqsubseteq}_{\mathfrak{S}} (s_1 \sqcup_{\mathfrak{S}} s_2)$ . As a result,  $[s_1 \sqcap_{\mathfrak{S}} s_2, s_1 \sqcup_{\mathfrak{S}} s_2] \cong B_2$ . This concludes the proof of property (203).

Equation (204) can also be shown to be satisfied by selection structures. Indeed, let us consider  $s_1, s_2 \in \mathfrak{S}$  with  $s_1 \bar{\sqsubseteq}_{\mathfrak{S}} (s_1 \sqcup_{\mathfrak{S}} s_2)$  and  $s_2 \bar{\sqsubseteq}_{\mathfrak{S}} (s_1 \sqcup_{\mathfrak{S}} s_2)$ . From Lemma 39, we deduce that  $(s_1 \sqcap_{\mathfrak{S}} s_2)$  exists. Moreover,  $(s_1 \sqcap_{\mathfrak{S}} s_2) \bar{\sqsubseteq}_{\mathfrak{S}} s_1$  and  $(s_1 \sqcap_{\mathfrak{S}} s_2) \bar{\sqsubseteq}_{\mathfrak{S}} s_2$ . The property  $[s_1 \sqcap_{\mathfrak{S}} s_2, s_1 \sqcup_{\mathfrak{S}} s_2] \cong B_2$  is then deduced from the previous analysis.  $\square$

**Definition 5. [Atoms and finite elements of a selection structure]** If  $\mathfrak{S}$  is a selection structure,  $\mathfrak{S}$  is *atomic*, i.e.

$$\forall s \in \mathfrak{S}, \exists a \in \mathcal{A}_{\mathfrak{S}}^*, a \sqsubseteq_{\mathfrak{S}} s, \quad \mathcal{A}_{\mathfrak{S}}^* := \{a \in \mathfrak{S} \mid \perp_{\mathfrak{S}} \bar{\sqsubseteq}_{\mathfrak{S}} a\}. \quad (206)$$

The set of atoms will be denoted  $\mathcal{A}_{\mathfrak{S}}^*$  and we will denote naturally  $\mathcal{A}_{\mathfrak{S}} := \mathcal{A}_{\mathfrak{S}}^* \cup \{\perp_{\mathfrak{S}}\}$ .

The sub- poset  $(\mathcal{A}_{\mathfrak{S}}, \sqsubseteq_{\mathfrak{S}})$  equipped with the induced partial order is a flat selection structure

$$(\mathcal{A}_{\mathfrak{S}}, \sqsubseteq_{\mathfrak{S}}) \in \underline{\mathcal{L}} \quad (207)$$

Accordingly to definition property (197), we will eventually denote  $(\underline{\mathfrak{S}}, \sqsubseteq_{\mathfrak{S}})$  the selection structure  $(\mathcal{A}_{\mathfrak{S}}, \sqsubseteq_{\mathfrak{S}})$  associated to the atoms of  $\mathfrak{S}$ .

For any  $s \in \mathfrak{S}$ , we will denote

$$\mathbb{A}^*(s) := \mathcal{A}_{\mathfrak{S}}^* \cap \downarrow s \quad \mathbb{A}(s) := \mathbb{A}^*(s) \cup \{\perp_{\mathfrak{S}}\}. \quad (208)$$

Elements  $s$  of  $\mathfrak{S}$  such that  $\mathbb{A}^*(s) \subseteq_{fin} \mathcal{A}_{\mathfrak{S}}$  will be called *finite elements of  $\mathfrak{S}$* .

**Lemma 43. [Pre-complementation]** A selection structure  $\mathfrak{S}$  satisfies necessarily the following *pre-complementation property* :

$$\forall s_1, s_2 \in \mathfrak{S}, \quad (s_1 \sqsubset_{\mathfrak{S}} s_2) \Rightarrow (\exists a \in \mathfrak{S}, \perp_{\mathfrak{S}} \bar{\sqsubset}_{\mathfrak{S}} a \sqsubset_{\mathfrak{S}} s_2 \text{ and } a \sqcap_{\mathfrak{S}} s_1 = \perp_{\mathfrak{S}}), \quad (209)$$

$$\Rightarrow (\exists a \in \mathfrak{S}, \perp_{\mathfrak{S}} \bar{\sqsubset}_{\mathfrak{S}} a \sqsubset_{\mathfrak{S}} s_2 \text{ and } a \not\sqsubset_{\mathfrak{S}} s_1). \quad (210)$$

■

*Proof.*  $\mathfrak{S}$  being strongly atomic (property (39)),  $\exists s \in \mathfrak{S}, s_1 \bar{\sqsubset}_{\mathfrak{S}} s \sqsubset_{\mathfrak{S}} s_2$ . Then, from property (41), it exists a unique element  $a$  such that  $\perp_{\mathfrak{S}} \bar{\sqsubset}_{\mathfrak{S}} a \sqsubset_{\mathfrak{S}} s \sqsubset_{\mathfrak{S}} s_2$  and  $s_1 \parallel_{\mathfrak{S}} a$ . From Lemma 39, we have also  $s_1 \sqcap_{\mathfrak{S}} a = \perp_{\mathfrak{S}}$ . □

**Lemma 44. [Atoms are compact elements]** Atoms of a selection structure  $\mathfrak{S}$  are compact element of this dcpo. ■

*Proof.* Let us consider an atom  $a \in \mathcal{A}_{\mathfrak{S}}^*$ . For any chain  $C \subseteq_{chain} \mathfrak{S}$  such that  $a \sqsubseteq_{\mathfrak{S}} \bigsqcup_{\mathfrak{S}} C$ , the property  $\forall c \in C, a \not\sqsubset_{\mathfrak{S}} c$  (i.e.  $a \sqcap_{\mathfrak{S}} c = \perp_{\mathfrak{S}}$ ) is then contradictory, because of property (44). Then, it exists necessarily  $c \in C$  such that  $a \sqsubseteq_{\mathfrak{S}} c$ . Hence,  $a$  is compact. □

**Lemma 45. [Consistent joins of finite elements]** For any finite upper-bounded family of atoms  $F \subseteq_{Con} \mathcal{A}_{\mathfrak{S}}^*$ ,  $(\bigsqcup_{\mathfrak{S}} F)$  exists in  $\mathfrak{S}$  and  $\mathbb{A}(\bigsqcup_{\mathfrak{S}} F) = F$ . In other words,

$$\forall F \subseteq_{Con} \mathcal{A}_{\mathfrak{S}}^*, \quad (\bigsqcup_{\mathfrak{S}} F) \text{ exists in } \mathfrak{S} \quad \text{and} \quad \mathbb{A}(\bigsqcup_{\mathfrak{S}} F) = F. \quad (211)$$

Let  $f$  be a finite element of  $\mathfrak{S}$  (i.e.  $\mathbb{A}(f) \subseteq_{Con} \mathcal{A}_{\mathfrak{S}}^*$ ). Then,  $(\bigsqcup_{\mathfrak{S}} \mathbb{A}(f))$  exists in  $\mathfrak{S}$  and  $f = (\bigsqcup_{\mathfrak{S}} \mathbb{A}(f))$ .

Let  $w_1, w_2$  be finite elements of  $\mathfrak{S}$  (i.e.  $\mathbb{A}(w_1), \mathbb{A}(w_2) \subseteq_{Con} \mathcal{A}_{\mathfrak{S}}^*$ ). If  $\widehat{w_1 w_2}_{\mathfrak{S}}$ , then  $(w_1 \sqcup_{\mathfrak{S}} w_2)$  exists in  $\mathfrak{S}$ ,  $\mathbb{A}(w_1 \sqcup_{\mathfrak{S}} w_2) = \mathbb{A}(w_1) \cup \mathbb{A}(w_2)$  and  $(w_1 \sqcup_{\mathfrak{S}} w_2) = (\bigsqcup_{\mathfrak{S}} (\mathbb{A}(w_1) \cup \mathbb{A}(w_2)))$ . ■

*Proof.* The property (211) will be proved by recursion on  $Card(F)$ .

Let  $F$  be any consistent family of atoms of cardinal  $n$ , explicitly given by  $F = \{a_i \mid i \in \{1, \dots, n\}\}$  and let us consider that we have established the property:

$$(R_{\sqsubset_{\mathfrak{S}} n}) \quad \forall F \subseteq_{Con} \mathcal{A}_{\mathfrak{S}}^* \text{ with } Card(F) \sqsubset_{\mathfrak{S}} n, \quad (\bigsqcup_{\mathfrak{S}} F) \text{ exists in } \mathfrak{S} \quad \text{and} \quad \mathbb{A}(\bigsqcup_{\mathfrak{S}} F) = F. \quad (212)$$

We want to prove the property  $(R_{\sqsubset_{\mathfrak{S}} (n+1)})$ .

Let us consider a consistent family of atoms  $G := \{a_i \mid i \in \{1, \dots, n\}\}$ , and let us introduce  $G^{\triangleleft} := \{a_i \mid i \in \{1, \dots, n-1\}\}$  and  $G^{\triangleright} := \{a_i \mid i \in \{2, \dots, n\}\}$  which are both consistent family of atoms of cardinal  $n-1$ , and  $G^{\nabla} := \{a_i \mid i \in \{2, \dots, n-1\}\}$  which is a consistent family of atoms of cardinal  $n-2$ . From the recursion hypothesis (property (212)), we obtain that  $(\bigsqcup_{\mathfrak{S}} G^{\triangleleft})$ ,  $(\bigsqcup_{\mathfrak{S}} G^{\triangleright})$ , and  $(\bigsqcup_{\mathfrak{S}} G^{\nabla})$  all exist in  $\mathfrak{S}$ . Moreover, if  $M$  is an upper-bound of  $G$ , it is also an upper-bound of  $G^{\triangleleft}$  and  $G^{\triangleright}$ , and then  $M$  is above  $(\bigsqcup_{\mathfrak{S}} G^{\triangleleft})$  and  $(\bigsqcup_{\mathfrak{S}} G^{\triangleright})$ . Hence, the set  $\{(\bigsqcup_{\mathfrak{S}} G^{\triangleleft}), (\bigsqcup_{\mathfrak{S}} G^{\triangleright})\}$  is upper-bounded.

The properties  $(\bigsqcup_{\mathfrak{S}} G^{\nabla}) \sqsubseteq_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} G^{\triangleleft})$  and  $(\bigsqcup_{\mathfrak{S}} G^{\nabla}) \sqsubseteq_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} G^{\triangleright})$  are obvious, but we want to check  $(\bigsqcup_{\mathfrak{S}} G^{\nabla}) \bar{\sqsubset}_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} G^{\triangleleft})$  and  $(\bigsqcup_{\mathfrak{S}} G^{\nabla}) \bar{\sqsubset}_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} G^{\triangleright})$ . Let us consider any  $x \in \mathfrak{S}$  such that  $(\bigsqcup_{\mathfrak{S}} G^{\nabla}) \sqsubseteq_{\mathfrak{S}} x \sqsubseteq_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} G^{\triangleleft})$ , and then in particular  $G^{\nabla} \subseteq \mathbb{A}(x) \subseteq G^{\triangleleft}$ . If  $\mathbb{A}(x) = G^{\nabla}$ , then  $(\bigsqcup_{\mathfrak{S}} \mathbb{A}(x))$  exists and

$\mathbb{A}(\bigsqcup_{\mathfrak{S}} \mathbb{A}(x)) = \mathbb{A}(x)$  (from recursion hypothesis). Moreover,  $(\bigsqcup_{\mathfrak{S}} G^{\nabla}) = (\bigsqcup_{\mathfrak{S}} \mathbb{A}(x)) \sqsubseteq_{\mathfrak{S}} x$ , but the option  $(\bigsqcup_{\mathfrak{S}} \mathbb{A}(x)) \sqsubset_{\mathfrak{S}} x$  is excluded, because the pre-complementation property (Lemma 43) would impose the existence of an atom  $b$  such that  $\perp_{\mathfrak{S}} \bar{\sqsubseteq}_{\mathfrak{S}} b \sqsubset_{\mathfrak{S}} x$  (i.e.  $b \in \mathbb{A}(x)$ ) and  $b \not\sqsubseteq_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} \mathbb{A}(x))$  (i.e.  $b \notin \mathbb{A}(x)$ ). Hence,  $\mathbb{A}(x) = G^{\nabla}$  implies  $(\bigsqcup_{\mathfrak{S}} \mathbb{A}(x)) = x$  and then  $x = (\bigsqcup_{\mathfrak{S}} G^{\nabla})$ . If  $\mathbb{A}(x) \not\sqsubseteq G^{\nabla}$ , then necessarily  $\mathbb{A}(x) = G^{\triangleleft}$ .  $(\bigsqcup_{\mathfrak{S}} \mathbb{A}(x))$  exists and  $\mathbb{A}(\bigsqcup_{\mathfrak{S}} \mathbb{A}(x)) = \mathbb{A}(x)$  (from recursion hypothesis). Moreover,  $(\bigsqcup_{\mathfrak{S}} \mathbb{A}(x)) \sqsubseteq_{\mathfrak{S}} x$  and  $x \sqsubseteq_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} G^{\triangleleft})$  implies  $(\bigsqcup_{\mathfrak{S}} G^{\triangleleft}) = x$ . As a conclusion,  $(\bigsqcup_{\mathfrak{S}} G^{\nabla}) \bar{\sqsubseteq}_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} G^{\triangleleft})$ , and in the same way  $(\bigsqcup_{\mathfrak{S}} G^{\nabla}) \bar{\sqsubseteq}_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} G^{\triangleright})$ .

Using property (40), we have more precisely :  $(\bigsqcup_{\mathfrak{S}} G^{\triangleleft}) \sqcap_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} G^{\triangleright}) = (\bigsqcup_{\mathfrak{S}} G^{\nabla})$ .

Now, using the property  $(D^*)$ , i.e. equation (203), we deduce that  $(\bigsqcup_{\mathfrak{S}} G^{\triangleleft}) \sqcup_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} G^{\triangleright})$  exists and that  $(\bigsqcup_{\mathfrak{S}} G^{\triangleleft}) \bar{\sqsubseteq}_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} G^{\triangleleft}) \sqcup_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} G^{\triangleright})$ ,  $(\bigsqcup_{\mathfrak{S}} G^{\triangleright}) \bar{\sqsubseteq}_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} G^{\triangleleft}) \sqcup_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} G^{\triangleright})$ .

The second part of property  $(D^*)$  is also essential. Let  $b$  be an element of  $\mathbb{A}_{\mathfrak{S}}^*$  such that  $b \sqsubseteq_{\mathfrak{S}} ((\bigsqcup_{\mathfrak{S}} G^{\triangleleft}) \sqcup_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} G^{\triangleright}))$ , and let us suppose that  $(b \not\sqsubseteq_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} G^{\triangleleft})$  and  $b \not\sqsubseteq_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} G^{\triangleright}))$ . The family  $(G^{\nabla} \cup \{b\})$  is a finite family of cardinal  $(n-1)$ , upper-bounded by  $((\bigsqcup_{\mathfrak{S}} G^{\triangleleft}) \sqcup_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} G^{\triangleright}))$ , and then, due to the recursion hypothesis, the join  $u := b \sqcup_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} G^{\nabla})$  exists. Moreover, using previous analysis and the fact that  $b \not\sqsubseteq_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} G^{\nabla})$ , we have also  $(\bigsqcup_{\mathfrak{S}} G^{\nabla}) \bar{\sqsubseteq}_{\mathfrak{S}} u$ . Endly,  $u$  satisfies  $u \sqsubseteq_{\mathfrak{S}} ((\bigsqcup_{\mathfrak{S}} G^{\triangleleft}) \sqcup_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} G^{\triangleright}))$ . We now exploit the second part of property  $(D^*)$ . We have necessarily  $(u = ((\bigsqcup_{\mathfrak{S}} G^{\triangleleft}) \sqcup_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} G^{\triangleright}))$  or  $u = (\bigsqcup_{\mathfrak{S}} G^{\triangleleft})$  or  $u = (\bigsqcup_{\mathfrak{S}} G^{\triangleright})$ . The second and the third options are excluded, because it has been supposed that  $(b \not\sqsubseteq_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} G^{\triangleleft})$  and  $b \not\sqsubseteq_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} G^{\triangleright})$ , and the first option is excluded because  $(\bigsqcup_{\mathfrak{S}} G^{\nabla}) \bar{\sqsubseteq}_{\mathfrak{S}} u$ . We have then obtained a contradiction. As a result, if an atom  $b \in \mathcal{A}_{\mathfrak{S}}^*$  is such that  $b \sqsubseteq_{\mathfrak{S}} ((\bigsqcup_{\mathfrak{S}} G^{\triangleleft}) \sqcup_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} G^{\triangleright}))$ , then  $b$  is necessarily an element of  $\mathbb{A}(\bigsqcup_{\mathfrak{S}} G^{\triangleleft}) \cup \mathbb{A}(\bigsqcup_{\mathfrak{S}} G^{\triangleright})$ , i.e.  $b \in \{a_i \mid i \in \{1, \dots, n\}\} = \mathbb{A}(G)$ . In other words,  $(\bigsqcup_{\mathfrak{S}} G^{\triangleleft}) \sqcup_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} G^{\triangleright})$  is an upper-bound of  $G$ .

Conversely, if an element of  $\mathfrak{S}$  is an upper-bound of  $G$ , it is a common upper-bound of  $(\bigsqcup_{\mathfrak{S}} G^{\triangleleft})$  and  $(\bigsqcup_{\mathfrak{S}} G^{\triangleright})$  because  $G^{\triangleleft}$  and  $G^{\triangleright}$  are subsets of  $G$ .

As a conclusion,  $(\bigsqcup_{\mathfrak{S}} G^{\triangleleft}) \sqcup_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} G^{\triangleright})$  is the least upper-bound of  $G$  and we have then proved the property  $(R_{\sqsubseteq_{\mathfrak{S}}(n+1)})$ .

The property (211) is then proved by recursion.

If we now consider a finite element  $f \in \mathfrak{S}$  (i.e.  $\mathbb{A}(f) \subseteq_{\text{Con}} \mathcal{A}_{\mathfrak{S}}^*$ ).  $(\bigsqcup_{\mathfrak{S}} \mathbb{A}(f))$  exists in  $\mathfrak{S}$ , due to previous result. Moreover,  $(\bigsqcup_{\mathfrak{S}} \mathbb{A}(f)) \sqsubseteq_{\mathfrak{S}} f$ . The option  $(\bigsqcup_{\mathfrak{S}} \mathbb{A}(f)) \sqsubset_{\mathfrak{S}} f$  is nevertheless excluded, because Lemma 43 would impose the existence of an atom  $b$  such that  $\perp_{\mathfrak{S}} \bar{\sqsubseteq}_{\mathfrak{S}} b \sqsubset_{\mathfrak{S}} f$  (i.e.  $b \in \mathbb{A}(f)$ ) and  $b \not\sqsubseteq_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} \mathbb{A}(f))$  (i.e.  $b \notin \mathbb{A}(f)$ ), which is contradictory. This concludes the proof that a finite element is a finite join of its atoms.

Let  $w_1, w_2$  be finite elements of  $\mathfrak{S}$  (i.e.  $\mathbb{A}(w_1), \mathbb{A}(w_2) \subseteq_{\text{Con}} \mathcal{A}_{\mathfrak{S}}^*$ ). If  $\widehat{w_1 w_2}^{\mathfrak{S}}$ , then we consider the consistent family  $B := \mathbb{A}(w_1) \cup \mathbb{A}(w_2)$ .  $(\bigsqcup_{\mathfrak{S}} B)$  exists, due to previous results. We have  $w_1 = (\bigsqcup_{\mathfrak{S}} \mathbb{A}(w_1)) \sqsubseteq_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} B)$  and  $w_2 = (\bigsqcup_{\mathfrak{S}} \mathbb{A}(w_2)) \sqsubseteq_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} B)$ . Hence, if  $(w_1 \sqcup_{\mathfrak{S}} w_2)$  exists, we have  $(w_1 \sqcup_{\mathfrak{S}} w_2) \sqsubseteq_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} B)$  and then  $\mathbb{A}(w_1 \sqcup_{\mathfrak{S}} w_2) \subseteq \mathbb{A}(\bigsqcup_{\mathfrak{S}} B) = B$ . We have also  $\mathbb{A}(w_1), \mathbb{A}(w_2) \subseteq \mathbb{A}(w_1 \sqcup_{\mathfrak{S}} w_2)$  and then  $B \subseteq \mathbb{A}(w_1 \sqcup_{\mathfrak{S}} w_2)$ . As a result,  $\mathbb{A}(w_1 \sqcup_{\mathfrak{S}} w_2) = \mathbb{A}(w_1) \cup \mathbb{A}(w_2)$  and  $(w_1 \sqcup_{\mathfrak{S}} w_2) = (\bigsqcup_{\mathfrak{S}} (\mathbb{A}(w_1) \cup \mathbb{A}(w_2)))$ .  $\square$

**Theorem 18. [Selection structures are atomistic]** Any element of a selection structure  $\mathfrak{S}$  is a join of finite elements in  $\mathfrak{S}$ .

$$\forall s \in \mathfrak{S}, \quad s = \bigsqcup_{\mathfrak{S}} \mathbb{A}(s) = \bigsqcup_{\mathfrak{S}} \bar{\sqsubseteq}_{\mathfrak{S}} \{ \bigsqcup_{\mathfrak{S}} F \mid F \subseteq_{\text{fin}} \mathbb{A}(s) \}. \quad (213)$$

Note that  $\forall u, v \in \mathfrak{S}$ ,  $(\mathbb{A}(u) \subseteq \mathbb{A}(v)) \Rightarrow (u \sqsubseteq_{\mathfrak{S}} v)$ .  $\blacksquare$

*Proof.* Let us consider any  $u$  in  $\mathfrak{S}^*$ . For any  $F \subseteq_{\text{fin}} \mathbb{A}(u)$ , the elements of  $F$  are upper-bounded by  $u$ , and then the supremum  $(\bigsqcup_{\mathfrak{S}} F)$  exists in  $\mathfrak{S}$ , due to Lemma 45). Moreover, we have  $(\bigsqcup_{\mathfrak{S}} F) \sqsubseteq_{\mathfrak{S}} u$ . Due to the formula  $\forall F_1, F_2 \subseteq_{\text{fin}} \mathbb{A}(u)$ ,  $(\bigsqcup_{\mathfrak{S}} F_1) \sqcup_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} F_2) = (\bigsqcup_{\mathfrak{S}} (F_1 \cup F_2))$  also proved in Lemma 45), we deduce that the set  $\mathbb{B}(u) := \{ \bigsqcup_{\mathfrak{S}} F \mid F \subseteq_{\text{fin}} \mathbb{A}(u) \}$  is a directed subset of  $\mathfrak{S}$ .  $\mathfrak{S}$  being directed complete, from Lemma 36, the directed supremum  $M(u) := \bigsqcup_{\mathfrak{S}} \bar{\sqsubseteq}_{\mathfrak{S}} \mathbb{B}(u)$  exists in  $\mathfrak{S}$ . As a supremum of elements which are less or equal than  $u$ , we have necessarily  $M(u) \sqsubseteq_{\mathfrak{S}} u$ . By construction, we have for any atom  $a \in \mathbb{A}(u)$ , necessarily  $a \sqsubseteq_{\mathfrak{S}} M(u)$ . Then, we can not have  $M(u) \sqsubset_{\mathfrak{S}} u$ . Indeed, due to the

pre-complementation property (Lemma 43), we would have:  $\exists b \in \mathcal{A}_{\mathfrak{G}}^*$  such that  $b \sqsubseteq_{\mathfrak{G}} u$  (i.e.  $b \in \mathbb{A}(u)$ ) and  $b \not\sqsubseteq_{\mathfrak{G}} M(u)$ , which is contradictory. Hence,  $M(u) = u$ .  $\square$

**Lemma 46. [Compactness is identical to finiteness]** Compact elements are finite elements and reciprocally.  $\blacksquare$

*Proof.* Let  $w$  be a compact element of  $\mathfrak{G}$ . By definition, we have

$$\forall D \subseteq_{Dir} \mathfrak{G}, (w \sqsubseteq_{\mathfrak{G}} (\bigsqcup_{\mathfrak{G}}^{\bar{\cdot}} D)) \Rightarrow (\exists d \in D, w \sqsubseteq_{\mathfrak{G}} d).$$

However, from Theorem 18, we can write

$$w = \bigsqcup_{\mathfrak{G}}^{\bar{\cdot}} \{ \bigsqcup_{\mathfrak{G}} F \mid F \subseteq_{fin} \mathbb{A}(w) \}. \quad (214)$$

Hence, it exists  $F \subseteq_{fin} \mathbb{A}(\bigsqcup_{\mathfrak{G}} C)$  such that  $w \sqsubseteq_{\mathfrak{G}} (\bigsqcup_{\mathfrak{G}} F)$  and  $f := (\bigsqcup_{\mathfrak{G}} F)$  is a finite element. Conversely, any finite element is compact. Indeed, let  $g := (\bigsqcup_{\mathfrak{G}} G)$  for  $G \subseteq_{fin} \mathcal{A}_{\mathfrak{G}}^*$  be a finite element of  $\mathfrak{G}$ . And let us consider a certain directed subset  $D \subseteq_{Dir} \mathfrak{G}$ , such that  $g \sqsubseteq_{\mathfrak{G}} (\bigsqcup_{\mathfrak{G}}^{\bar{\cdot}} D)$ . For any  $a \in G$ , it exists  $d_a \in D$  such that  $a \sqsubseteq_{\mathfrak{G}} d_a$ . Hence,  $D$  being directed, it exists an element  $d_G \in D$  common upper-bound of  $\{d_a \mid a \in G\}$ . As a result,  $g \sqsubseteq_{\mathfrak{G}} d_G$ . This concludes the proof.  $\square$

**Lemma 47. [Selection structures are C-complete]** Selection structures are C-complete, i.e.

$$\forall w_1, w_2 \in \mathfrak{G}_c, \quad \widehat{w_1 w_2}^{\mathfrak{G}} \Rightarrow (w_1 \sqcup_{\mathfrak{G}} w_2) \text{ exists in } \mathfrak{G}, \quad (215)$$

*Proof.* Direct consequence of Lemma 45 and Lemma 46.  $\square$

**Lemma 48. [Selection structures are algebraic]** Selection structures are algebraic, i.e. compactly generated. In other words,

$$\forall s \in \mathfrak{G}, \quad s = \bigsqcup_{\mathfrak{G}} \{ w \in \mathfrak{G}_c \mid w \sqsubseteq_{\mathfrak{G}} s \} = \bigsqcup_{\mathfrak{G}}^{\bar{\cdot}} \{ \bigsqcup_{\mathfrak{G}} F \mid F \subseteq_{fin} \mathbb{A}(s) \}. \quad (216)$$

*Proof.* Direct consequence of Lemma 46 and Theorem 18.  $\square$

*Remark 18.* The property (216) implies trivially that  $\mathfrak{G}$  is *compactly separated*, i.e.

$$\forall s_1, s_2 \in \mathfrak{G}, \quad (s_1 \not\sqsubseteq_{\mathfrak{G}} s_2) \Rightarrow (\exists w \in \mathfrak{G}_c, w \sqsubseteq_{\mathfrak{G}} s_1 \text{ and } w \not\sqsubseteq_{\mathfrak{G}} s_2).$$

**Theorem 19. [Selection structures are Inf semi-lattice]** For any  $s_1$  and  $s_2$  elements of  $\mathfrak{G}$ , the infimum  $s_1 \sqcap_{\mathfrak{G}} s_2$  exists in  $\mathfrak{G}$ .  $\blacksquare$

*Proof.* Let us consider two elements of  $\mathfrak{G}$  denoted  $s_1$  and  $s_2$ . Let us introduce  $B := \mathbb{A}(s_1) \cap \mathbb{A}(s_2)$ . For any  $F \subseteq_{fin} B$ , we have necessarily  $\forall a \in F, a \in \mathbb{A}(s_1)$ , i.e.  $a \sqsubseteq_{\mathfrak{G}} s_1$ , and then  $F$  is a consistent subset of  $\mathfrak{G}$ . As a consequence,  $(\bigsqcup_{\mathfrak{G}} F)$  exists in  $\mathfrak{G}$  from Lemma 45. For any  $F_1, F_2 \subseteq_{fin} B$ , we know also from Lemma 45 that  $(F_1 \cup F_2)$  is consistent  $\mathfrak{G}$  and  $(\bigsqcup_{\mathfrak{G}} F_1) \sqcup_{\mathfrak{G}} (\bigsqcup_{\mathfrak{G}} F_2) = (\bigsqcup_{\mathfrak{G}} (F_1 \cup F_2))$ . The set  $\{ \bigsqcup_{\mathfrak{G}} F \mid F \subseteq_{fin} \mathbb{A}(s) \}$  is then directed. We denote by  $s$  its directed supremum in  $\mathfrak{G}$ . Necessarily,  $s \sqsubseteq_{\mathfrak{G}} s_1$  and  $s \sqsubseteq_{\mathfrak{G}} s_2$ , because of formula (213) and the fact that  $B \subseteq \mathbb{A}(s_1)$  and  $B \subseteq \mathbb{A}(s_2)$ . Moreover, for any  $s'$  common lower-bound of  $s_1$  and  $s_2$ , we know that necessarily  $\mathbb{A}(s') \subseteq \mathbb{A}(s_1)$  and  $\mathbb{A}(s') \subseteq \mathbb{A}(s_2)$ , i.e.  $\mathbb{A}(s') \subseteq B$ , which implies  $s' \sqsubseteq_{\mathfrak{G}} s$ . The element  $s$  is then the infimum  $(s_1 \sqcap_{\mathfrak{G}} s_2)$ .  $\square$

**Theorem 20. [Selection structures are meet-continuous]** A selection structure is necessarily meet-continuous, i.e.

$$\forall C \subseteq_{chain} \mathfrak{S}, \forall s \in \mathfrak{S}, \quad s \sqcap_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} C) = \bigsqcup_{\mathfrak{S}} \{s \sqcap_{\mathfrak{S}} c \mid c \in C\} \quad (217)$$

■

*Proof.* Let us consider a chain  $C \subseteq_{chain} \mathfrak{S}$ . The following inequality between suprema  $(\bigsqcup_{\mathfrak{S}} \{x \sqcap_{\mathfrak{S}} c \mid c \in C\}) \sqsubseteq_{\mathfrak{S}} (x \sqcap_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} C))$  is always true. Let us suppose that this inequality is strict. By hypothesis, it exists a compact element  $u \in \mathfrak{S}_c$  with  $u \not\sqsubseteq_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} \{x \sqcap_{\mathfrak{S}} c \mid c \in C\})$  and  $u \sqsubseteq_{\mathfrak{S}} (x \sqcap_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} C))$ . But  $u \sqsubseteq_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} C)$  implies  $\exists c \in C, u \sqsubseteq_{\mathfrak{S}} c$  because  $u$  is compact. Now, using  $u \sqsubseteq_{\mathfrak{S}} x$ , we have  $u \sqsubseteq_{\mathfrak{S}} (x \sqcap_{\mathfrak{S}} c) \sqsubseteq_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} \{x \sqcap_{\mathfrak{S}} c \mid c \in C\})$  which contradicts our hypothesis. As a result,  $\mathfrak{S}$  is then meet-continuous.  $\square$

**Theorem 21. [Selection structures are algebraic Scott-domains]** The set of compact elements of  $\mathfrak{S}$  (denoted  $\mathfrak{S}_c$ ) is a basis of the dcpo  $\mathfrak{S}$ . Selection structure are then algebraic dcpo. Moreover, selections structures are  $C$ -complete. Hence, selection structures are algebraic Scott-domains. ■

*Proof.* Consequence of Lemma 47 and Lemma 48.  $\square$

**Lemma 49. [Selection structures are finitary domains]** A selection structure is necessarily a finitary Scott-domain. ■

*Proof.* The basis of compact elements is finitary, simply because compact elements are finite elements. Indeed,  $\forall w \in \mathfrak{S}_c, \exists W \subseteq_{fin} \mathcal{A}_{\mathfrak{S}} \mid w = (\bigsqcup_{\mathfrak{S}} W)$  and  $(\downarrow_{(D_c, \sqsubseteq_{\mathfrak{S}})} w) = \{(\bigsqcup_{\mathfrak{S}} F) \mid F \subseteq_{fin} W\} \subseteq_{fin} \mathfrak{S}$ .  $\square$

**Lemma 50.**

$$\{w \in \mathfrak{S}_c \mid w \sqsubseteq_{\mathfrak{S}} s_1 \text{ and } w \sqcap_{\mathfrak{S}} s_2 = \perp_{\mathfrak{S}}\} \subseteq_{Dir} \mathfrak{S} \quad (218)$$

■

*Proof.* Let us denote  $U_{(s_1, s_2)} := \{w \in \mathfrak{S}_c \mid w \sqsubseteq_{\mathfrak{S}} s_1 \text{ and } w \sqcap_{\mathfrak{S}} s_2 = \perp_{\mathfrak{S}}\}$ . Let us consider  $w, w' \in U_{(s_1, s_2)}$  and let us denote  $W, W' \subseteq_{fin} \mathcal{A}_{\mathfrak{S}}^*$  such that  $W := \mathbb{A}(w)$  and  $W' := \mathbb{A}(w')$ , or in other words  $w = \bigsqcup_{\mathfrak{S}} W$  and  $w' = \bigsqcup_{\mathfrak{S}} W'$ . The finite elements  $w$  and  $w'$  have a common upper-bound,  $w \sqsubseteq_{\mathfrak{S}} s_1$  and  $w' \sqsubseteq_{\mathfrak{S}} s_1$ . Hence, from Lemma 45, the join  $w \sqcup_{\mathfrak{S}} w'$  exists and is less or equal than  $s_1$ . Moreover,  $w \sqcap_{\mathfrak{S}} s_2 = \perp_{\mathfrak{S}}$  and  $w' \sqcap_{\mathfrak{S}} s_2 = \perp_{\mathfrak{S}}$  means that  $\forall a \in W \cup W', a \not\sqsubseteq_{\mathfrak{S}} s_2$ , i.e.  $(w \sqcup_{\mathfrak{S}} w') \sqcap_{\mathfrak{S}} s_2 = \perp_{\mathfrak{S}}$ . As a result,  $(w \sqcup_{\mathfrak{S}} w')$ , which is a common upper-bound of  $w$  and  $w'$ , is an element of  $U_{(s_1, s_2)}$ . In other words,  $U_{(s_1, s_2)}$  is directed.  $\square$

**Definition 6. [Relative complementation]** For any elements  $s_1$  and  $s_2$  in  $\mathfrak{S}$  such that  $s_2 \sqsubseteq_{\mathfrak{S}} s_1$ , we define the complement of  $s_2$  relative to  $s_1$  to be the element of  $\mathfrak{S}$  denoted  $(s_1 \setminus s_2)$  and defined by

$$(s_1 \setminus s_2) := \bigsqcup_{\mathfrak{S}}^{\bar{\Lambda}} \{w \in \mathfrak{S}_c \mid w \sqsubseteq_{\mathfrak{S}} s_1 \text{ and } w \sqcap_{\mathfrak{S}} s_2 = \perp_{\mathfrak{S}}\} \quad (219)$$

*Remark 19.*  $\mathfrak{S}$  being directed complete (indeed,  $\mathfrak{S}$  is chain-complete), the supremum of the set  $\{w \in \mathfrak{S}_c \mid w \sqsubseteq_{\mathfrak{S}} s_1 \text{ and } w \sqcap_{\mathfrak{S}} s_2 = \perp_{\mathfrak{S}}\}$  then exists.

**Lemma 51. [Properties of relative complementation]**  $\forall s_1, s_2, s_3 \in \mathfrak{S}$  with  $s_3 \sqsubseteq_{\mathfrak{S}} s_1$  and  $s_2 \sqsubseteq_{\mathfrak{S}} s_1$ ,

we have:

$$(s_1 \searrow \perp_{\mathfrak{G}}) = s_1, \quad (s_1 \searrow s_1) = \perp_{\mathfrak{G}} \quad (220)$$

$$s_3 \sqsubseteq_{\mathfrak{G}} s_2 \Rightarrow (s_1 \searrow s_2) \sqsubseteq_{\mathfrak{G}} (s_1 \searrow s_3), \quad (221)$$

$$(s_1 \searrow s_2) \sqsubseteq_{\mathfrak{G}} s_1 \quad (222)$$

$$s_2 \sqcap_{\mathfrak{G}} (s_1 \searrow s_2) = \perp_{\mathfrak{G}}, \quad (223)$$

$$(s_1 \searrow (s_1 \searrow s_2)) = s_2. \quad (224)$$

For any  $s_1, s_2, s_3 \in \mathfrak{G}$  with  $s_3 \sqsubseteq_{\mathfrak{G}} s_2$ , we have

$$s_1 \sqcap_{\mathfrak{G}} (s_2 \searrow s_3) = (s_1 \sqcap_{\mathfrak{G}} s_2) \searrow (s_1 \sqcap_{\mathfrak{G}} s_3). \quad (225)$$

■

*Proof.* The first property of (220) is obvious. Indeed,  $(s_1 \searrow \perp_{\mathfrak{G}}) = \bigsqcup_{\mathfrak{G}}^{\wedge} \{w \in \mathfrak{G}_c \mid w \sqsubseteq_{\mathfrak{G}} s_1\} = s_1$ , because  $s_1$  is the supremum of its compact lower-bounds (i.e. algebraicity property 216).

The second property of (220) is obtained as follows. For any  $s \in \mathfrak{G}$ ,  $s \sqsubseteq_{\mathfrak{G}} s_1$  implies  $s \sqcap_{\mathfrak{G}} s_1 = s$ , and then  $\perp_{\mathfrak{G}} = s \sqcap_{\mathfrak{G}} s_1$  implies  $s = \perp_{\mathfrak{G}}$ . Hence,  $(s_1 \searrow s_1) = \perp_{\mathfrak{G}}$ .

The proof of property (221) is a simple ordering property. For any  $s$  in  $\mathfrak{G}_c$  (it is true for any  $s$  in  $\mathfrak{G}$ ),  $s \sqcap_{\mathfrak{G}} s_2 = \perp_{\mathfrak{G}}$  and  $s_3 \sqsubseteq_{\mathfrak{G}} s_2$  imply  $s \sqcap_{\mathfrak{G}} s_3 = \perp_{\mathfrak{G}}$ . As a result,

$$\{s \in \mathfrak{G}_c \mid s \sqsubseteq_{\mathfrak{G}} s_1 \text{ and } s \sqcap_{\mathfrak{G}} s_2 = \perp_{\mathfrak{G}}\} \subseteq \{s \in \mathfrak{G}_c \mid s \sqsubseteq_{\mathfrak{G}} s_1 \text{ and } s \sqcap_{\mathfrak{G}} s_3 = \perp_{\mathfrak{G}}\}.$$

Passing to the suprema, we obtain the relation  $(s_1 \searrow s_2) \sqsubseteq_{\mathfrak{G}} (s_1 \searrow s_3)$ .

To obtain property (222), we take  $s_3 := \perp_{\mathfrak{G}}$  in property 221. Using  $(s_1 \searrow \perp_{\mathfrak{G}}) = s_1$ , we note that  $\forall s_1, s_2 \in \mathfrak{G}$ , such that  $s_2 \sqsubseteq_{\mathfrak{G}} s_1$ , we have  $(s_1 \searrow s_2) \sqsubseteq_{\mathfrak{G}} s_1$ .

The property (223) can be deduced as follows. Let us suppose  $\perp_{\mathfrak{G}} \sqsubseteq_{\mathfrak{G}} s_2 \sqcap_{\mathfrak{G}} (s_1 \searrow s_2)$ . From atomicity of  $\mathfrak{G}$ , it exists  $a \in \mathbb{A}_{\mathfrak{G}}^*$  such that  $a \sqsubseteq_{\mathfrak{G}} s_2 \sqcap_{\mathfrak{G}} (s_1 \searrow s_2)$ . We then have simultaneously  $a \sqsubseteq_{\mathfrak{G}} s_2$  and  $a \sqsubseteq_{\mathfrak{G}} (s_1 \searrow s_2)$ . The atoms of  $\mathfrak{G}$  being compact elements, the property

$$a \sqsubseteq_{\mathfrak{G}} (s_1 \searrow s_2) = \bigsqcup_{\mathfrak{G}}^{\wedge} \{w \in \mathfrak{G}_c \mid w \sqsubseteq_{\mathfrak{G}} s_1 \text{ and } w \sqcap_{\mathfrak{G}} s_2 = \perp_{\mathfrak{G}}\}$$

implies  $\exists w \in \mathfrak{G}_c$  such that  $w \sqsubseteq_{\mathfrak{G}} s_1$ ,  $w \sqcap_{\mathfrak{G}} s_2 = \perp_{\mathfrak{G}}$  and  $a \sqsubseteq_{\mathfrak{G}} w$ . As a summary, we obtain in particular  $a \sqsubseteq_{\mathfrak{G}} s_2$  and  $a \sqcap_{\mathfrak{G}} s_2 = \perp_{\mathfrak{G}}$  which is impossible for an atom. Then, necessarily  $s_2 \sqcap_{\mathfrak{G}} (s_1 \searrow s_2) = \perp_{\mathfrak{G}}$ .

Endly, let us prove the non-trivial property (224). Due to the property  $(s_1 \searrow s_2) \sqsubseteq_{\mathfrak{G}} s_1$ , we can form the following object:  $(s_1 \searrow (s_1 \searrow s_2))$ . By definition,

$$(s_1 \searrow (s_1 \searrow s_2)) = \bigsqcup_{\mathfrak{G}}^{\wedge} \{s \in \mathfrak{G}_c \mid s \sqsubseteq_{\mathfrak{G}} s_1 \text{ and } s \sqcap_{\mathfrak{G}} (s_1 \searrow s_2) = \perp_{\mathfrak{G}}\}.$$

We have assumed  $s_2 \sqsubseteq_{\mathfrak{G}} s_1$  and we have already established  $s_2 \sqcap_{\mathfrak{G}} (s_1 \searrow s_2) = \perp_{\mathfrak{G}}$ . We deduce from these properties that, for any  $w \in \mathfrak{G}_c$ ,  $(w \sqsubseteq_{\mathfrak{G}} s_2)$  implies  $(w \sqsubseteq_{\mathfrak{G}} s_1 \text{ and } w \sqcap_{\mathfrak{G}} (s_1 \searrow s_2) = \perp_{\mathfrak{G}})$ . Hence, passing to supremum, we obtain

$$s_2 = \bigsqcup_{\mathfrak{G}} \{w \in \mathfrak{G}_c \mid w \sqsubseteq_{\mathfrak{G}} s_2\} \sqsubseteq_{\mathfrak{G}} \bigsqcup_{\mathfrak{G}}^{\wedge} \{w \in \mathfrak{G}_c \mid w \sqsubseteq_{\mathfrak{G}} s_1 \text{ and } w \sqcap_{\mathfrak{G}} (s_1 \searrow s_2) = \perp_{\mathfrak{G}}\} = (s_1 \searrow (s_1 \searrow s_2)).$$

Let us consider any  $w \in \mathfrak{G}_c$  such that  $w \sqsubseteq_{\mathfrak{G}} s_1$  and  $w \sqcap_{\mathfrak{G}} (s_1 \searrow s_2) = \perp_{\mathfrak{G}}$ . Then, for any atom  $a \sqsubseteq_{\mathfrak{G}} w$ , we have  $a \sqsubseteq_{\mathfrak{G}} s_1$  and  $a \sqcap_{\mathfrak{G}} (s_1 \searrow s_2) = \perp_{\mathfrak{G}}$ . Using meet-continuity property of  $\mathfrak{G}$  (equation 217), we have

$$a \sqcap_{\mathfrak{G}} (s_1 \searrow s_2) = \bigsqcup_{\mathfrak{G}}^{\wedge} \{(a \sqcap_{\mathfrak{G}} u) \mid u \in \mathfrak{G}_c, u \sqsubseteq_{\mathfrak{G}} s_1 \text{ and } u \sqcap_{\mathfrak{G}} s_2 = \perp_{\mathfrak{G}}\}.$$

Hence, for any atom  $a \sqsubseteq_{\mathfrak{G}} w$ , we must have  $\forall u \in \mathfrak{G}_c, (u \sqsubseteq_{\mathfrak{G}} s_1 \text{ and } u \sqcap_{\mathfrak{G}} s_2 = \perp_{\mathfrak{G}}) \Rightarrow (a \sqcap_{\mathfrak{G}} u = \perp_{\mathfrak{G}})$ , or in other words  $\forall b \in \mathbb{A}_{\mathfrak{G}}^*, (b \sqsubseteq_{\mathfrak{G}} s_1 \text{ and } b \not\sqsubseteq_{\mathfrak{G}} s_2) \Rightarrow (a \neq b)$ . As a conclusion, for any atom  $a \sqsubseteq_{\mathfrak{G}} w$  we have necessarily  $a \sqsubseteq_{\mathfrak{G}} s_2$ , and then  $w \sqsubseteq_{\mathfrak{G}} s_2$ . As a result, we obtain

$$\bigsqcup_{\mathfrak{G}}^{\wedge} \{w \in \mathfrak{G}_c \mid w \sqsubseteq_{\mathfrak{G}} s_1 \text{ and } w \sqcap_{\mathfrak{G}} (s_1 \searrow s_2) = \perp_{\mathfrak{G}}\} \sqsubseteq_{\mathfrak{G}} s_2.$$

We have then established the formula  $(s_1 \setminus (s_1 \setminus s_2)) = s_2$ .

$\forall a \in \mathcal{A}_{\mathfrak{G}}^*$ ,  $(a \sqsubseteq_{\mathfrak{G}} (s_1 \sqcap_{\mathfrak{G}} (s_2 \setminus s_3))) \Rightarrow (a \sqsubseteq_{\mathfrak{G}} s_1 \text{ and } a \sqsubseteq (s_2 \setminus s_3)) \Rightarrow (a \sqsubseteq_{\mathfrak{G}} s_1 \text{ and } (\exists w \in \mathfrak{G}_c, a \sqsubseteq_{\mathfrak{G}} w \text{ and } w \sqsubseteq_{\mathfrak{G}} s_2 \text{ and } w \sqcap_{\mathfrak{G}} s_3 = \perp_{\mathfrak{G}})) \Rightarrow (a \sqsubseteq_{\mathfrak{G}} (s_1 \sqcap_{\mathfrak{G}} s_2) \text{ and } a \sqcap_{\mathfrak{G}} (s_1 \sqcap_{\mathfrak{G}} s_3) = \perp_{\mathfrak{G}}) \Rightarrow (a \sqsubseteq ((s_1 \sqcap_{\mathfrak{G}} s_2) \setminus (s_1 \sqcap_{\mathfrak{G}} s_3)))$ . Hence,  $(s_1 \sqcap_{\mathfrak{G}} (s_2 \setminus s_3)) \sqsubseteq_{\mathfrak{G}} (s_1 \sqcap_{\mathfrak{G}} s_2) \setminus (s_1 \sqcap_{\mathfrak{G}} s_3)$ . Conversely, let us consider  $a \in \mathcal{A}_{\mathfrak{G}}^*$  such that  $a \sqsubseteq_{\mathfrak{G}} (s_1 \sqcap_{\mathfrak{G}} s_2) \setminus (s_1 \sqcap_{\mathfrak{G}} s_3)$ . We have in particular,  $a \sqsubseteq_{\mathfrak{G}} (s_1 \sqcap_{\mathfrak{G}} s_2) \sqsubseteq_{\mathfrak{G}} s_1$ . Let us suppose that  $a \not\sqsubseteq_{\mathfrak{G}} (s_1 \sqcap_{\mathfrak{G}} (s_2 \setminus s_3))$ , then, necessarily,  $a \sqsubseteq_{\mathfrak{G}} s_3$  and then  $a \sqsubseteq_{\mathfrak{G}} (s_1 \sqcap_{\mathfrak{G}} s_3)$ , which contradicts  $a \sqsubseteq_{\mathfrak{G}} (s_1 \sqcap_{\mathfrak{G}} s_2) \setminus (s_1 \sqcap_{\mathfrak{G}} s_3)$ . Hence,  $a \sqsubseteq_{\mathfrak{G}} (s_1 \sqcap_{\mathfrak{G}} (s_2 \setminus s_3))$ . This concludes the proof of property (225).  $\square$

**Lemma 52. [Relative co-atoms]** For any  $s \in \mathfrak{G}$  and any atom  $a \in \mathbb{A}^*(s)$  we have

$$(s \setminus a) \bar{\sqsubseteq}_{\mathfrak{G}} s. \quad (226)$$

For any  $s_1, s_2 \in \mathfrak{G}$  verifying  $s_2 \sqsubset_{\mathfrak{G}} s_1$ , it exists  $s_3 \in \mathfrak{G}$  with  $s_2 \sqsubseteq_{\mathfrak{G}} s_3 \bar{\sqsubseteq}_{\mathfrak{G}} s_1$ .  $\blacksquare$

*Proof.* From Lemma 41, we know that the supremum  $u := \{t \in \mathfrak{G} \mid \perp_{\mathfrak{G}} \sqsubseteq_{\mathfrak{G}} t \sqsubseteq_{\mathfrak{G}} s \text{ and } t \sqcap_{\mathfrak{G}} a = \perp_{\mathfrak{G}}\}$  exists in  $\mathfrak{G}$  (it is unique, by definition of the supremum) and this supremum verifies  $u \sqcap_{\mathfrak{G}} a = \perp_{\mathfrak{G}}$  and  $u \bar{\sqsubseteq}_{\mathfrak{G}} (u \sqcup_{\mathfrak{G}} a) \sqsubseteq_{\mathfrak{G}} s$ . By definition of  $(s \setminus a)$ , we have immediatly  $(s \setminus a) \sqsubseteq_{\mathfrak{G}} u$ . Hence,  $s = (a \sqcup_{\mathfrak{G}} (s \setminus a)) \sqsubseteq_{\mathfrak{G}} (u \sqcup_{\mathfrak{G}} a) \sqsubseteq_{\mathfrak{G}} s$  implies immediatly  $(u \sqcup_{\mathfrak{G}} a) = s$  and then  $u \bar{\sqsubseteq}_{\mathfrak{G}} s$ . From Lemma 38 and  $\perp_{\mathfrak{G}} \sqsubseteq_{\mathfrak{G}} u \bar{\sqsubseteq}_{\mathfrak{G}} s$ , we know that  $a$  is the unique element of  $\mathfrak{G}$  such that  $a \sqcap_{\mathfrak{G}} a = \perp_{\mathfrak{G}}$  and  $u \sqcup_{\mathfrak{G}} a = s$ .  $a$  being an atom, and then also a compact element, we obtain immediatly  $(s \setminus u) = a$ . Now, using property (224), we deduce that  $u = (s \setminus a)$ . As a final conclusion, we obtain  $(s \setminus a) \bar{\sqsubseteq}_{\mathfrak{G}} s$ .

From lemma 43, if  $s_2 \sqsubset_{\mathfrak{G}} s_1$ , it then exists  $a \in \mathbb{A}^*(s_1)$  satisfying  $a \sqcap_{\mathfrak{G}} s_2 = \perp_{\mathfrak{G}}$ . Let us suppose that  $s_2 \not\sqsubseteq_{\mathfrak{G}} (s_1 \setminus a)$ . It would then exists an atom  $b$  such that  $b \sqsubseteq_{\mathfrak{G}} s_2$  but  $b \not\sqsubseteq_{\mathfrak{G}} (s_1 \setminus a)$ .  $b$  has to be different from  $a$  because  $a \sqcap_{\mathfrak{G}} s_2 = \perp_{\mathfrak{G}}$ . Hence,  $b \not\sqsubseteq_{\mathfrak{G}} s_1$ . We have then obtained a contradiction of the hypothesis  $s_2 \sqsubset_{\mathfrak{G}} s_1$ .  $\square$

**Lemma 53. [Consistent-completeness of selection structures and relative complementation]**

For any  $s_1, s_2, s_3 \in \mathfrak{G}$  such that  $s_3 \sqsubseteq_{\mathfrak{G}} s_1$  and  $s_2 \sqsubseteq_{\mathfrak{G}} s_1$ , the supremum  $s_2 \sqcup_{\mathfrak{G}} s_3$  exists and we have

$$s_2 \sqcup_{\mathfrak{G}} s_3 = (s_1 \setminus ((s_1 \setminus s_2) \sqcap_{\mathfrak{G}} (s_1 \setminus s_3))). \quad (227)$$

We note that this supremum does not depend on the choice of the upper-bound  $s_1$ . We note the following basic relative complementation property:

$$s_2 \sqcup_{\mathfrak{G}} (s_1 \setminus s_2) = s_1. \quad (228)$$

A selection structure is then always consistently complete.  $\blacksquare$

*Proof.* Using  $((s_1 \setminus s_2) \sqcap_{\mathfrak{G}} (s_1 \setminus s_3)) \sqsubseteq_{\mathfrak{G}} (s_1 \setminus s_i)$  for  $i = 2, 3$  and equation (221), we obtain  $s_i = (s_1 \setminus (s_1 \setminus s_i)) \sqsubseteq_{\mathfrak{G}} (s_1 \setminus ((s_1 \setminus s_2) \sqcap_{\mathfrak{G}} (s_1 \setminus s_3)))$  for  $i = 2, 3$ , i.e.  $(s_1 \setminus ((s_1 \setminus s_2) \sqcap_{\mathfrak{G}} (s_1 \setminus s_3)))$  is a common upper-bound of  $s_2$  and  $s_3$ . Now, if we consider any  $M$  such that  $s_i \sqsubseteq_{\mathfrak{G}} M \sqsubseteq_{\mathfrak{G}} s_1$  for  $i = 2, 3$ , then  $(s_1 \setminus M) \sqsubseteq_{\mathfrak{G}} (s_1 \setminus s_i)$  for  $i = 2, 3$ , and then  $(s_1 \setminus M) \sqsubseteq_{\mathfrak{G}} ((s_1 \setminus s_2) \sqcap_{\mathfrak{G}} (s_1 \setminus s_3))$ , and finally  $(s_1 \setminus ((s_1 \setminus s_2) \sqcap_{\mathfrak{G}} (s_1 \setminus s_3))) \sqsubseteq_{\mathfrak{G}} M$ , which shows that  $(s_1 \setminus ((s_1 \setminus s_2) \sqcap_{\mathfrak{G}} (s_1 \setminus s_3)))$  is the least upper-bound of  $\{s_2, s_3\}$ . The uniqueness of the supremum implies that the result does not depend on the choice of  $s_1$  as an upper-bound of  $s_2$  and  $s_3$ . This concludes the proof of property (227).

Taking  $s_3 = (s_1 \setminus s_2)$  in the equation (227) and using successively the equations (224), (223), (220) we obtain  $s_2 \sqcup_{\mathfrak{G}} (s_1 \setminus s_2) = s_1$ . This concludes the proof of property (228).  $\square$

**Theorem 22. [Coherent completeness and down-completeness of selection structures]**

$$\forall S \subseteq_{Coh} \mathfrak{S}, \quad \bigsqcup_{\mathfrak{S}} S := \bigsqcup_{\mathfrak{S}}^{\bar{\lambda}} \{ \bigsqcup_{\mathfrak{S}} R \mid R \subseteq_{fin} S \} \text{ exists in } \mathfrak{S} \quad (229)$$

$$\forall S \subseteq \mathfrak{S}, \quad (\prod_{\mathfrak{S}} S) \text{ exists in } \mathfrak{S} \quad (230)$$

■

*Proof.* The directed-completeness coupled with the consistent-completeness of the selection structure  $\mathfrak{S}$  implies directly the coherent-completeness of  $\mathfrak{S}$ .

The consistent-completeness of  $\mathfrak{S}$  implies that the set of lower-bounds of a given subset  $S$  of  $\mathfrak{S}$  is a directed-subset of  $\mathfrak{S}$ . As a result,  $\mathfrak{S}$  is down-complete.  $\square$

*Remark 20.* The down-completeness implies immediatly the bounded-completeness of  $\mathfrak{S}$ .

**Theorem 23. [Conditional-distributivity in selection structures]** Let  $\mathfrak{S}$  be a selection structure. For any  $x, y, z \in \mathfrak{S}$ , we have the following conditional-distributivity properties:

$$\widehat{xy}^{\mathfrak{S}} \Rightarrow z \sqcap_{\mathfrak{S}} (y \sqcup_{\mathfrak{S}} x) = (z \sqcap_{\mathfrak{S}} y) \sqcup_{\mathfrak{S}} (z \sqcap_{\mathfrak{S}} x), \quad (231)$$

$$\widehat{xy}^{\mathfrak{S}} \text{ and } \widehat{xz}^{\mathfrak{S}} \Rightarrow x \sqcup_{\mathfrak{S}} (y \sqcap_{\mathfrak{S}} z) = (x \sqcup_{\mathfrak{S}} y) \sqcap_{\mathfrak{S}} (x \sqcup_{\mathfrak{S}} z). \quad (232)$$

■

*Proof.* Let us choose  $w$  upper-bound of  $y$  and  $z$ . Using mainly the properties (227) and (225), we obtain  $z \sqcap_{\mathfrak{S}} (x \sqcup_{\mathfrak{S}} y) = z \sqcap_{\mathfrak{S}} (w \setminus ((w \setminus x) \sqcap_{\mathfrak{S}} (w \setminus y))) = ((z \sqcap_{\mathfrak{S}} w) \setminus (z \sqcap_{\mathfrak{S}} (w \setminus x) \sqcap_{\mathfrak{S}} (w \setminus y))) = ((z \sqcap_{\mathfrak{S}} w) \setminus (((z \sqcap_{\mathfrak{S}} w) \setminus (z \sqcap_{\mathfrak{S}} x)) \sqcap_{\mathfrak{S}} ((z \sqcap_{\mathfrak{S}} w) \setminus (z \sqcap_{\mathfrak{S}} y))))$ . Choosing now  $(z \sqcap_{\mathfrak{S}} w)$  as upper-bound of  $(z \sqcap_{\mathfrak{S}} x)$  and  $(z \sqcap_{\mathfrak{S}} y)$  to develop  $(z \sqcap_{\mathfrak{S}} x) \sqcup_{\mathfrak{S}} (z \sqcap_{\mathfrak{S}} y)$ , we obtain the same result. This concludes the proof of the first property.

If we consider  $w$  upper-bound of  $x$ ,  $y$  and  $z$ . We have  $z \sqcup_{\mathfrak{S}} (x \sqcap_{\mathfrak{S}} y) = (w \setminus ((w \setminus z) \sqcap_{\mathfrak{S}} (w \setminus (x \sqcap_{\mathfrak{S}} y)))) = (w \setminus ((w \setminus z) \sqcap_{\mathfrak{S}} (w \setminus x) \sqcup_{\mathfrak{S}} (w \setminus y))))$ , and the second distributivity property is deduced from the first one:  $x \sqcup_{\mathfrak{S}} (y \sqcap_{\mathfrak{S}} z) = (x \sqcap_{\mathfrak{S}} y) \sqcup_{\mathfrak{S}} (x \sqcap_{\mathfrak{S}} z)$ .  $\square$

**Lemma 54. [Join-prime elements are the atoms]** For any selection structure  $\mathfrak{S}$ , the join-prime elements are the atoms of  $\mathfrak{S}$ .  $\blacksquare$

*Proof.* Atoms are obviously join-irreducible. Atoms are even join-prime, but only because  $\mathfrak{S}$  is distributive. Indeed, for any atom  $a$ , if  $a \sqsubseteq_{\mathfrak{S}} (x \sqcup_{\mathfrak{S}} y)$ , then  $a = a \sqcap_{\mathfrak{S}} (x \sqcup_{\mathfrak{S}} y) = (a \sqcap_{\mathfrak{S}} x) \sqcup_{\mathfrak{S}} (a \sqcap_{\mathfrak{S}} y)$  and the join-irreducibility of  $a$  implies  $a \sqsubseteq_{\mathfrak{S}} x$  or  $a \sqsubseteq_{\mathfrak{S}} y$ .

Conversely, let  $y$  be a non-atomic element, i.e. it exists an atom  $x \in D$  such that  $\perp_D \bar{\sqsubseteq}_{\mathfrak{S}} x \sqsubseteq_{\mathfrak{S}} y$ . We have  $y = x \sqcup_{\mathfrak{S}} (y \setminus x)$  but  $y \neq x$ ,  $y \neq (y \setminus x)$ . Hence,  $y$  is not join-irreducible, and then *a fortiori* not join-prime.  $\square$

**Theorem 24. [Selection structures are qualitative domains]** A selection structure  $(\mathfrak{S}, \sqsubseteq_{\mathfrak{S}})$  is a qualitative domain, i.e. a dI-domain in which the join-prime elements are atomic.  $\blacksquare$

*Proof.* As a direct consequence of theorem 21, Lemma 49, Lemma 23 and Lemma 54, we obtain that  $(\mathfrak{S}, \sqsubseteq_{\mathfrak{S}})$  is a qualitative domain.  $\square$

**Lemma 55. [Finitary Scott-domain are algebraic]** If a Scott-domain  $(\mathfrak{S}, \sqsubseteq_{\mathfrak{S}})$  admits a finitary basis  $B$ , then, necessarily,  $\mathfrak{S}$  is an algebraic Scott-domain. The finitary property implies in fact the following stronger property:

$$\forall s \in \mathfrak{S}_c, \quad (\downarrow_{\mathfrak{S}} s) \text{ is a finite lattice included in } \mathfrak{S}_c. \quad (233)$$



■

*Proof.* Let  $x$  be an element of  $B$  and let us suppose that  $x$  is not compact. Due to the finitary character of the basis  $B$ , i.e.

$$\forall x \in B, \quad \{y \in B \mid y \sqsubseteq_{\mathfrak{S}} x\} \subseteq_{fin} B, \quad (234)$$

the continuity property applied in  $x$  (i.e.  $x = \bigsqcup_{\mathfrak{S}}^{\wedge} (\downarrow_{(B, \sqsubseteq_{\mathfrak{S}})} x)$ ) expresses  $x$  as a finite join of basis elements in  $(\downarrow_{(B, \sqsubseteq_{\mathfrak{S}})} x)$ . It is important to note that,  $x$  being not-compact,  $x$  is not element of the subset  $(\downarrow_{(B, \sqsubseteq_{\mathfrak{S}})} x)$ . Because finite joins of compact elements are also compact, it must then exist  $y \in (\downarrow_{(B, \sqsubseteq_{\mathfrak{S}})} x)$  (and then necessarily  $y \sqsubseteq_{\mathfrak{S}} x$ ) with  $y \notin \mathfrak{S}_c$ . Let us consider the subset  $\{z \in (\downarrow_{(B, \sqsubseteq_{\mathfrak{S}})} x) \mid z \notin \mathfrak{S}_c\}$ . As a finite and non-empty subset, it admits at least a minimal element denoted  $x'$ . This element  $x'$  satisfies:  $x' \notin \mathfrak{S}_c$ ,  $x' \sqsubseteq_{\mathfrak{S}} x$  and  $(\downarrow_{(B, \sqsubseteq_{\mathfrak{S}})} x) \cap (\downarrow x') = \{x'\}$ . However, we know that  $x' \sqsubseteq_{\mathfrak{S}} x$  implies  $(\downarrow_{(B, \sqsubseteq_{\mathfrak{S}})} x') \subseteq (\downarrow_{(B, \sqsubseteq_{\mathfrak{S}})} x)$ . We have also obviously  $(\downarrow_{(B, \sqsubseteq_{\mathfrak{S}})} x') \subseteq (\downarrow x')$ . Hence,  $(\downarrow_{(B, \sqsubseteq_{\mathfrak{S}})} x') \subseteq (\downarrow_{(B, \sqsubseteq_{\mathfrak{S}})} x) \cap (\downarrow x') = \{x'\}$ , and then  $(\downarrow_{(B, \sqsubseteq_{\mathfrak{S}})} x') = \emptyset$  (since  $x' \notin \mathfrak{S}_c$ ) which is contradictory. As a conclusion,  $\mathfrak{S}$  is an algebraic Scott-domain.

$$((\forall x \in \mathfrak{S}, (\downarrow_{(B, \sqsubseteq_{\mathfrak{S}})} x) \subseteq_{Dir} \mathfrak{S} \text{ and } x = \bigsqcup_{\mathfrak{S}}^{\wedge} (\downarrow_{(B, \sqsubseteq_{\mathfrak{S}})} x)) \text{ and } (\forall y \in B, (\downarrow_{(B, \sqsubseteq_{\mathfrak{S}})} y) \subseteq_{fin} B)) \Rightarrow B = \mathfrak{S}_c. \quad (235)$$

Using  $B = \mathfrak{S}_c$ , the finitary property (234) is simplified:  $\forall s \in \mathfrak{S}_c, (\downarrow_{(\mathfrak{S}_c, \sqsubseteq_{\mathfrak{S}})} s) \subseteq_{fin} \mathfrak{S}_c$ .

It is in fact possible to go further and observe that:  $\forall s \in \mathfrak{S}_c, \forall x \in \mathfrak{S}, (x \sqsubseteq_{\mathfrak{S}} s) \Rightarrow (x \in \mathfrak{S}_c)$ . Indeed, if we had  $(s \in \mathfrak{S}_c, \forall x \in \mathfrak{S}, x \sqsubseteq_{\mathfrak{S}} s)$  and  $x \notin \mathfrak{S}_c$ , the set  $(\downarrow_{(\mathfrak{S}_c, \sqsubseteq_{\mathfrak{S}})} x)$  would be an infinite subset of the finite set  $(\downarrow_{(\mathfrak{S}_c, \sqsubseteq_{\mathfrak{S}})} s)$ , which is obviously contradictory. As a conclusion,

$$\forall s \in \mathfrak{S}_c, \quad (\downarrow_{\mathfrak{S}} s) \subseteq_{fin} \mathfrak{S}_c. \quad (236)$$

For any  $x$  and  $y$  in  $\mathfrak{S}_c$ ,  $(x \sqcap_{\mathfrak{S}} y)$  exists in  $\mathfrak{S}$ , because  $\mathfrak{S}$  is an inf semi-lattice ( $\mathfrak{S}$  is pointed and bounded-complete). Using property (236) and  $(x \sqcap_{\mathfrak{S}} y) \sqsubseteq_{\mathfrak{S}} x$  implies  $(x \sqcap_{\mathfrak{S}} y) \in \mathfrak{S}_c$ .  $\mathfrak{S}$  is then an arithmetic domain.

$$\forall x, y \in \mathfrak{S}_c, \quad (x \sqcap_{\mathfrak{S}} y) \in \mathfrak{S}_c. \quad (237)$$

Consider, once again,  $x$  and  $y$  two compact elements of  $\mathfrak{S}$  such that  $\{x, y\}$  is upper-bounded in  $\mathfrak{S}$ . The join  $(x \sqcup_{\mathfrak{S}} y)$  exists in  $\mathfrak{S}$ , since  $\mathfrak{S}$  is consistently complete (as a Scott-domain). Let us then consider  $E \subseteq_{Dir} \mathfrak{S}$  such that  $(x \sqcup_{\mathfrak{S}} y) \sqsubseteq_{\mathfrak{S}} \bigsqcup_{\mathfrak{S}}^{\wedge} E$ . The compactness of  $x$  and  $y$ , and the ordering relations  $x \sqsubseteq_{\mathfrak{S}} \bigsqcup_{\mathfrak{S}}^{\wedge} E$  and  $y \sqsubseteq_{\mathfrak{S}} \bigsqcup_{\mathfrak{S}}^{\wedge} E$ , imply that it exists  $x'$  and  $y'$  in  $E$  such that  $x \sqsubseteq_{\mathfrak{S}} x'$  and  $y \sqsubseteq_{\mathfrak{S}} y'$ . The subset  $E$  being directed, it exists  $z \in E$  such that  $x' \sqsubseteq_{\mathfrak{S}} z$  and  $y' \sqsubseteq_{\mathfrak{S}} z$ . Hence, it exists  $z \in E$  such that  $(x \sqcup_{\mathfrak{S}} y) \sqsubseteq_{\mathfrak{S}} z$ . This proves that  $(x \sqcup_{\mathfrak{S}} y)$  is compact. As a conclusion,  $\forall s \in \mathfrak{S}_c, (\downarrow_{\mathfrak{S}} s)$  is a finite lattice included in  $\mathfrak{S}_c$ .  $\square$

**Theorem 25. [Selection structures as locally boolean qualitative domains]** If  $(\mathfrak{S}, \sqsubseteq_{\mathfrak{S}})$  is a qualitative-domain satisfying the following additional property

$$\forall x, y \in \mathfrak{S}_c, (\perp_{\mathfrak{S}} \sqsubseteq_{\mathfrak{S}} x \sqsubseteq_{\mathfrak{S}} y) \Rightarrow (\exists z \in \mathfrak{S}_c, \perp_{\mathfrak{S}} \sqsubseteq_{\mathfrak{S}} z \sqsubseteq_{\mathfrak{S}} y \text{ and } z \parallel_{\mathfrak{S}} x), \quad (238)$$

then  $(\mathfrak{S}, \sqsubseteq_{\mathfrak{S}})$  is necessarily a selection structure.

Conversely, if  $(\mathfrak{S}, \sqsubseteq_{\mathfrak{S}})$  is a selection structure, then  $(\mathfrak{S}, \sqsubseteq_{\mathfrak{S}})$  is necessarily a qualitative-domain satisfying the additional property (238).  $\blacksquare$

*Remark 21.* The property (238) is equivalent to:

$$\forall x, y \in \mathfrak{S}_c, (\perp_{\mathfrak{S}} \sqsubseteq_{\mathfrak{S}} x \sqsubseteq_{\mathfrak{S}} y) \Rightarrow (\exists z \in \mathfrak{S}_c, \perp_{\mathfrak{S}} \sqsubseteq_{\mathfrak{S}} z \sqsubseteq_{\mathfrak{S}} y \text{ and } z \sqcap_{\mathfrak{S}} x = \perp_{\mathfrak{S}}). \quad (239)$$

Let us consider,  $x_1, y_0 \in \mathfrak{S}_c$  with  $\perp_{\mathfrak{S}} \sqsubseteq_{\mathfrak{S}} x_1 \sqsubseteq_{\mathfrak{S}} y_0$ . From property (238), it exists  $y_1 \in \mathfrak{S}_c$  such that  $\perp_{\mathfrak{S}} \sqsubseteq_{\mathfrak{S}} y_1 \sqsubseteq_{\mathfrak{S}} y_0$  and  $y_1 \parallel_{\mathfrak{S}} x_1$ . Let us denote  $x_2 := (y_1 \sqcap_{\mathfrak{S}} x_1)$ . Obviously  $\perp_{\mathfrak{S}} \sqsubseteq_{\mathfrak{S}} x_2, x_2 \sqsubseteq_{\mathfrak{S}} x_1$  and

$x_2 \sqsubseteq_{\mathfrak{S}} y_1$ . If  $x_2$  is different from  $\perp_{\mathfrak{S}}$ , we can apply once again the property (238) to deduce that it exists  $y_2 \in \mathfrak{S}_c$  such that  $\perp_{\mathfrak{S}} \sqsubseteq_{\mathfrak{S}} y_2 \sqsubseteq_{\mathfrak{S}} y_1 \sqsubseteq_{\mathfrak{S}} y_0$  and  $y_2 \parallel_{\mathfrak{S}} x_2$ . We note that  $(y_2 \sqcap_{\mathfrak{S}} x_2) = (y_2 \sqcap_{\mathfrak{S}} x_1)$ , since  $y_2 \sqsubseteq_{\mathfrak{S}} y_1$  and  $x_2 = (y_1 \sqcap_{\mathfrak{S}} x_1)$ . The subset  $(\downarrow_{(\mathfrak{S}_c, \sqsubseteq_{\mathfrak{S}})} y_0)$  being finite, the strictly descending chains  $(x_n)_{n \geq 1}$  and  $(y_n)_{n \geq 1}$  are necessarily finite, and then  $(x_n)_{n \geq 1}$  must terminate on  $\perp_{\mathfrak{S}}$ . This establishes the property (239).

*Remark 22.* Let us suppose that  $(\mathfrak{S}, \sqsubseteq_{\mathfrak{S}})$  is a qualitative-domain satisfying the property (239), then  $(\mathfrak{S}, \sqsubseteq_{\mathfrak{S}})$  satisfies the property

$$\forall s \in \mathfrak{S}_c, \quad (\downarrow_{\mathfrak{S}} s) \text{ is a finite boolean lattice included in } \mathfrak{S}_c \quad (240)$$

and conversely.

Indeed, let us consider any  $s \in \mathfrak{S}_c$  and any  $t \in (\downarrow_{\mathfrak{S}} s)$ . Using property (239), we know that  $\{w \in \mathfrak{S}_c \mid w \sqsubseteq_{\mathfrak{S}} s \text{ and } w \sqcap_{\mathfrak{S}} t = \perp_{\mathfrak{S}}\} \neq \emptyset$ . Then, using the definition property (219), we deduce that  $(s \setminus t) = \bigsqcup_{\mathfrak{S}}^{\bar{\cdot}} \{w \in \mathfrak{S}_c \mid w \sqsubseteq_{\mathfrak{S}} s \text{ and } w \sqcap_{\mathfrak{S}} t = \perp_{\mathfrak{S}}\}$  exists. Using property (233), we conclude that  $(s \setminus t)$  is an element of  $(\downarrow_{\mathfrak{S}} s)$ . Properties of relative complementation proven in allow to conclude that  $(s \setminus t) \sqcap_{\mathfrak{S}} t = \perp_{\mathfrak{S}}$  and  $(s \setminus t) \sqcup_{\mathfrak{S}} t = s$ . The lattice  $(\downarrow_{\mathfrak{S}} s)$  is then boolean.

Conversely, if  $(\mathfrak{S}, \sqsubseteq_{\mathfrak{S}})$  is a dI-domain satisfying the property (240), then  $(\mathfrak{S}, \sqsubseteq_{\mathfrak{S}})$  trivially satisfies the property (239) (take for example  $z := (y \setminus x)$ ).

*Proof.* Let us firstly consider the converse result. If  $(\mathfrak{S}, \sqsubseteq_{\mathfrak{S}})$  is a selection structure, then  $(\mathfrak{S}, \sqsubseteq_{\mathfrak{S}})$  is necessarily a dI-domain from Theorem 24. Moreover, the property (239) is trivially satisfied for a selection-structure (take for  $z$  any atom in  $\mathbb{A}(y) \setminus \mathbb{A}(x)$ ).

Let us now suppose that  $(\mathfrak{S}, \sqsubseteq_{\mathfrak{S}})$  is a dI-domain satisfying the additional property (238). Let us then check every axioms of selection structures for  $(\mathfrak{S}, \sqsubseteq_{\mathfrak{S}})$ .

The chain-completeness property (42) is also a trivial consequence of directed-completeness property satisfied by the Scott-domain  $\mathfrak{S}$ .

This directed-completeness of  $\mathfrak{S}$  supposes implicitly that  $\mathfrak{S}$  is pointed ( $\perp_{\mathfrak{S}} = \bigsqcup_{\mathfrak{S}}^{\bar{\cdot}} \emptyset$ ). The property (38) is then satisfied. We will denote  $\perp_{\mathfrak{S}}$  the bottom element of  $\mathfrak{S}$ .

A Scott-domain is necessarily bounded-complete. In particular, it is then an Inf semi-lattice. Hence, property (40) is satisfied by  $\mathfrak{S}$ .

From the algebraicity property of the Scott-domain  $\mathfrak{S}$ , we deduce that the meet-continuity property (217) is satisfied, and then *a fortiori* the weak-continuity property (44) is satisfied.

Let us now establish the property (41), i.e.

$$\forall x, y \in \mathfrak{S}, \perp_{\mathfrak{S}} \sqsubseteq_{\mathfrak{S}} x \bar{\sqsubseteq}_{\mathfrak{S}} y \quad \exists! z \in \mathfrak{S}_c, \perp_{\mathfrak{S}} \bar{\sqsubseteq}_{\mathfrak{S}} z \sqsubseteq_{\mathfrak{S}} y \text{ and } z \sqcap_{\mathfrak{S}} x = \perp_{\mathfrak{S}}. \quad (241)$$

In the following, we will adopt the following notation:

$$\forall x, y \in \mathfrak{S} \mid \perp_{\mathfrak{S}} \sqsubseteq_{\mathfrak{S}} x \sqsubseteq_{\mathfrak{S}} y, \quad B_{x,y} := \{z \in \mathfrak{S}_c \mid z \sqsubseteq_{\mathfrak{S}} y \text{ and } z \sqcap_{\mathfrak{S}} x = \perp_{\mathfrak{S}}\} \quad B_{x,y}^* := B_{x,y} \setminus \{\perp_{\mathfrak{S}}\}. \quad (242)$$

We note that, for any  $x, y \in \mathfrak{S}$ , we have

$$\forall x, y \in \mathfrak{S}, \perp_{\mathfrak{S}} \sqsubseteq_{\mathfrak{S}} x \sqsubseteq_{\mathfrak{S}} y, \quad B_{x,y} \text{ is a downset in } (\mathfrak{S}, \sqsubseteq_{\mathfrak{S}}). \quad (243)$$

Indeed, for any  $v \in B_{x,y}$  and any  $t \in \mathfrak{S}$ ,  $t \sqsubseteq_{\mathfrak{S}} v$  implies  $t \sqsubseteq_{\mathfrak{S}} v \sqsubseteq_{\mathfrak{S}} y$  and  $t \sqcap_{\mathfrak{S}} x \sqsubseteq_{\mathfrak{S}} v \sqcap_{\mathfrak{S}} x = \perp_{\mathfrak{S}}$ , i.e.  $t \in B_{x,y}$ .

We begin to prove the property (240) for  $x, y \in \mathfrak{S}_c$ .

First of all, from property (239), we know that  $B_{x,y}^*$  is non-empty.

$$\forall x, y \in \mathfrak{S}_c, \perp_{\mathfrak{S}} \sqsubseteq_{\mathfrak{S}} x \sqsubseteq_{\mathfrak{S}} y, \quad B_{x,y}^* \neq \emptyset. \quad (244)$$

If  $y \in \mathfrak{S}_c$ , the set  $B_{x,y}$  is finite, due to property (233). Hence,  $B_{x,y}^*$  admits minimal elements. For any  $z \in \min_{\mathfrak{S}}(B_{x,y}^*)$ ,  $z$  is an atom of  $\mathfrak{S}$ , since  $B_{x,y}$  is a finite downset in  $\mathfrak{S}$ .<sup>[21]</sup> In other words,

$$\forall x, y \in \mathfrak{S}_c, (\perp_{\mathfrak{S}} \sqsubseteq_{\mathfrak{S}} x \sqsubseteq_{\mathfrak{S}} y) \Rightarrow (\forall z \in \min_{\mathfrak{S}}(B_{x,y}^*), \perp_{\mathfrak{S}} \bar{\sqsubseteq}_{\mathfrak{S}} z). \quad (245)$$

Let us now focus on the case  $x \bar{\sqsubseteq}_{\mathfrak{S}} y$ . For any  $z \in B_{x,y}^*$ , the supremum  $(z \sqcup_{\mathfrak{S}} x)$  exists because  $\mathfrak{S}_c$  is consistently complete and  $y$  is a common upper-bound of  $x$  and  $z$ . Moreover,

$$\forall x, y \in \mathfrak{S}_c \mid x \bar{\sqsubseteq}_{\mathfrak{S}} y, \quad \forall z \in B_{x,y}^*, (z \sqcup_{\mathfrak{S}} x) = y. \quad (246)$$

Indeed, by definition, we know that  $z \sqcup_{\mathfrak{S}} x \sqsubseteq_{\mathfrak{S}} y$ , but  $z \sqcup_{\mathfrak{S}} x = x$  is excluded because  $z \sqcap_{\mathfrak{S}} x = \perp_{\mathfrak{S}}$  would imply  $z = \perp_{\mathfrak{S}}$ , and then,  $x \bar{\sqsubseteq}_{\mathfrak{S}} y$  imposes  $z \sqcup_{\mathfrak{S}} x = y$ .

This property suffices to establish the uniqueness of  $z \in \min_{\mathfrak{S}}(B_{x,y}^*)$ , in the case  $x \bar{\sqsubseteq}_{\mathfrak{S}} y$ . Indeed, if we consider  $a_1, a_2 \in \min_{\mathfrak{S}}(B_{x,y}^*)$  and suppose  $a_1 \neq a_2$ , we would then have  $\perp_{\mathfrak{S}} \sqsubseteq_{\mathfrak{S}} (a_1 \sqcap_{\mathfrak{S}} a_2) \sqsubseteq_{\mathfrak{S}} a_1$  and then  $(a_1 \sqcap_{\mathfrak{S}} a_2) = \perp_{\mathfrak{S}}$ , since  $\perp_{\mathfrak{S}} \bar{\sqsubseteq}_{\mathfrak{S}} a_1$ . Moreover, using distributivity property and  $a_1 \sqcap_{\mathfrak{S}} x = a_2 \sqcap_{\mathfrak{S}} x = y$ , we would then deduce  $x = \perp_{\mathfrak{S}} \sqcup_{\mathfrak{S}} x = (a_1 \sqcap_{\mathfrak{S}} a_2) \sqcup_{\mathfrak{S}} x = a_1 \sqcup_{\mathfrak{S}} x = y$ , which is false. As a result

$$\forall x, y \in \mathfrak{S}_c, \perp_{\mathfrak{S}} \sqsubseteq_{\mathfrak{S}} x \bar{\sqsubseteq}_{\mathfrak{S}} y, \quad \exists! z \in \mathfrak{S}_c, \perp_{\mathfrak{S}} \bar{\sqsubseteq}_{\mathfrak{S}} z \sqsubseteq_{\mathfrak{S}} y \text{ and } z \sqcap_{\mathfrak{S}} x = \perp_{\mathfrak{S}}. \quad (247)$$

Let us now consider  $u, v \in \mathfrak{S}$  with  $\perp_{\mathfrak{S}} \sqsubseteq_{\mathfrak{S}} u \bar{\sqsubseteq}_{\mathfrak{S}} v$ , but now we will assume that  $v \notin \mathfrak{S}_c$ . The algebraicity of  $\mathfrak{S}$  implies  $\exists y \in \mathfrak{S}_c$  such that  $y \sqsubseteq_{\mathfrak{S}} v$  and  $y \not\sqsubseteq_{\mathfrak{S}} u$ . The element  $v \in \mathfrak{S}$  being non-compact, we have then  $y \sqsubseteq_{\mathfrak{S}} v$ . Moreover, the properties  $\perp_{\mathfrak{S}} \sqsubseteq_{\mathfrak{S}} u \bar{\sqsubseteq}_{\mathfrak{S}} v$  and  $y \not\sqsubseteq_{\mathfrak{S}} u$  impose  $y \parallel_{\mathfrak{S}} u$ , i.e.  $\perp_{\mathfrak{S}} \sqsubseteq_{\mathfrak{S}} y$  and  $(y \sqcap_{\mathfrak{S}} u) \sqsubseteq_{\mathfrak{S}} y$ . Using property (236) and  $(y \sqcap_{\mathfrak{S}} u) \sqsubseteq_{\mathfrak{S}} y \in \mathfrak{S}_c$ , we deduce that  $(y \sqcap_{\mathfrak{S}} u)$  is in  $\mathfrak{S}_c$ . We distinguish two subcases:

- If  $(y \sqcap_{\mathfrak{S}} u) = \perp_{\mathfrak{S}}$ , we have then found  $z := y \in \mathfrak{S}_c$  such that  $\perp_{\mathfrak{S}} \sqsubseteq_{\mathfrak{S}} z \sqsubseteq_{\mathfrak{S}} v$  and  $z \sqcap_{\mathfrak{S}} u = \perp_{\mathfrak{S}}$ , i.e.  $B_{u,v}^*$  is not empty.
- If  $(y \sqcap_{\mathfrak{S}} u) \neq \perp_{\mathfrak{S}}$ , we use property (239) to deduce that it exists  $z \in \mathfrak{S}_c$  such that  $\perp_{\mathfrak{S}} \sqsubseteq_{\mathfrak{S}} z \sqsubseteq_{\mathfrak{S}} y$  and  $z \sqcap_{\mathfrak{S}} (y \sqcap_{\mathfrak{S}} u) = \perp_{\mathfrak{S}}$ , and then *a fortiori*  $\perp_{\mathfrak{S}} \sqsubseteq_{\mathfrak{S}} z \sqsubseteq_{\mathfrak{S}} v$  and  $z \sqcap_{\mathfrak{S}} u = \perp_{\mathfrak{S}}$ , i.e.  $B_{u,v}^*$  is not empty.

From properties (236) and (243), we know that  $\forall s \in B_{u,v}^*, (B_{u,v}^* \cap (\downarrow s))$  is a finite and non-empty subset in  $(\mathfrak{S}, \sqsubseteq_{\mathfrak{S}})$ . Hence, for any  $s \in B_{u,v}^*, (B_{u,v}^* \cap (\downarrow s))$  admits minimal elements. Note that any minimal element of  $(B_{u,v}^* \cap (\downarrow s))$  is also a minimal element of  $B_{u,v}^*$ . As a result,

$$\forall u, v \in \mathfrak{S}, \perp_{\mathfrak{S}} \sqsubseteq_{\mathfrak{S}} u \bar{\sqsubseteq}_{\mathfrak{S}} v, \quad \forall s \in B_{u,v}^*, \exists t \in \min_{\mathfrak{S}}(B_{u,v}^*) \mid t \sqsubseteq_{\mathfrak{S}} s. \quad (248)$$

As shown before, for any  $z \in \min_{\mathfrak{S}}(B_{u,v}^* \cap (\downarrow s))$ ,  $z$  is an atom of  $\mathfrak{S}$ . Indeed, let us consider  $z \in \min_{\mathfrak{S}}(B_{u,v}^*)$  and  $w$  in  $\mathfrak{S}$  such that  $\perp_{\mathfrak{S}} \sqsubseteq_{\mathfrak{S}} w \sqsubseteq_{\mathfrak{S}} z$ .  $w$  is necessarily compact (property (233)), and satisfies  $\perp_{\mathfrak{S}} \sqsubseteq_{\mathfrak{S}} w \sqsubseteq_{\mathfrak{S}} z \sqsubseteq_{\mathfrak{S}} v$  and  $w \sqcap_{\mathfrak{S}} u \sqsubseteq_{\mathfrak{S}} z \sqcap_{\mathfrak{S}} u = \perp_{\mathfrak{S}}$ , hence  $w \in B_{x,y}^*$  which is contradictory with the minimality requirement on  $z$ . To summarize:

$$\forall u, v \in \mathfrak{S}, \perp_{\mathfrak{S}} \sqsubseteq_{\mathfrak{S}} u \bar{\sqsubseteq}_{\mathfrak{S}} v, \quad B_{u,v}^* \neq \emptyset \text{ and } \forall z \in \min_{\mathfrak{S}}(B_{u,v}^*), \perp_{\mathfrak{S}} \bar{\sqsubseteq}_{\mathfrak{S}} z. \quad (249)$$

For any  $y \in B_{u,v}^*$ , the supremum  $(u \sqcup_{\mathfrak{S}} y)$  exists in  $\mathfrak{S}$ , since  $\mathfrak{S}$  is consistently-complete and  $v$  is a common upper-bound for  $u$  and  $y$ . For any  $y \in B_{u,v}^*$ , the property  $y \sqcap_{\mathfrak{S}} u = \perp_{\mathfrak{S}}$  means  $y \parallel_{\mathfrak{S}} u$  and then  $u \sqsubseteq_{\mathfrak{S}} (u \sqcup_{\mathfrak{S}} y)$ . We have also,  $(u \sqcup_{\mathfrak{S}} y) \sqsubseteq_{\mathfrak{S}} v$  because  $v$  is a common upper-bound of  $u$  and  $y$ . The property  $u \bar{\sqsubseteq}_{\mathfrak{S}} v$  implies then  $u \sqcup_{\mathfrak{S}} y = v$ . As a result,

$$\forall u, v \in \mathfrak{S}, \perp_{\mathfrak{S}} \sqsubseteq_{\mathfrak{S}} u \bar{\sqsubseteq}_{\mathfrak{S}} v, \quad \forall y \in B_{u,v}^*, u \sqcup_{\mathfrak{S}} y = v. \quad (250)$$

Let us consider  $y_1$  and  $y_2$  elements of  $\min_{\mathfrak{S}}(B_{u,v}^* \cap (\downarrow s))$ . From property (249), we recall that  $(y_1 \neq y_2)$  implies  $(y_1 \sqcap_{\mathfrak{S}} y_2) = \perp_{\mathfrak{S}}$ . However, this fact contradicts the distributivity property in  $\mathfrak{S}_c$ . Indeed, if we define the following map

$$\alpha : (\downarrow_{(\mathfrak{S}_c, \sqsubseteq_{\mathfrak{S}})} u)^{\times 2} \rightarrow \mathfrak{S}_c, (c_1, c_2) \mapsto (y_1 \sqcup_{\mathfrak{S}} c_1) \sqcap_{\mathfrak{S}} (y_2 \sqcup_{\mathfrak{S}} c_2).$$

<sup>21</sup>Indeed, let us consider  $z \in \min_{\mathfrak{S}}(B_{x,y}^*)$  and  $w$  in  $\mathfrak{S}$  such that  $\perp_{\mathfrak{S}} \sqsubseteq_{\mathfrak{S}} w \sqsubseteq_{\mathfrak{S}} z$ .  $w$  is necessarily compact (property (233)), we have also  $\perp_{\mathfrak{S}} \sqsubseteq_{\mathfrak{S}} w \sqsubseteq_{\mathfrak{S}} y$  and  $z \sqcap_{\mathfrak{S}} x = \perp_{\mathfrak{S}}$ , hence  $w \in B_{x,y}^*$  which is contradictory to the minimality requirement on  $z$ .

$\alpha$  is obviously a monotone net and then, using successively the property (250), two times the meet-continuity of  $\mathfrak{S}$ , [5, Proposition 2.1.12], distributivity property, once again meet-continuity of  $\mathfrak{S}$ , and the property  $(y_1 \sqcap_{\mathfrak{S}} y_2) = \perp_{\mathfrak{S}}$ , we obtain the following contradiction:

$$\begin{aligned}
v &= (y_1 \sqcup_{\mathfrak{S}} u) \sqcap_{\mathfrak{S}} (y_2 \sqcup_{\mathfrak{S}} u) = \\
&= \bigsqcup_{c_1 \in \mathfrak{S}_c \cap \downarrow u}^{\bar{\wedge}} ((y_1 \sqcup_{\mathfrak{S}} c_1) \sqcap_{\mathfrak{S}} (y_2 \sqcup_{\mathfrak{S}} u)) = \\
&= \bigsqcup_{c_1 \in \mathfrak{S}_c \cap \downarrow u}^{\bar{\wedge}} \bigsqcup_{c_2 \in \mathfrak{S}_c \cap \downarrow u}^{\bar{\wedge}} ((y_1 \sqcup_{\mathfrak{S}} c_1) \sqcap_{\mathfrak{S}} (y_2 \sqcup_{\mathfrak{S}} c_2)) = \\
&= \bigsqcup_{c \in \mathfrak{S}_c \cap \downarrow u}^{\bar{\wedge}} ((y_1 \sqcup_{\mathfrak{S}} c) \sqcap_{\mathfrak{S}} (y_2 \sqcup_{\mathfrak{S}} c)) = \\
&= \bigsqcup_{c \in \mathfrak{S}_c \cap \downarrow u}^{\bar{\wedge}} ((y_1 \sqcap_{\mathfrak{S}} y_2) \sqcup_{\mathfrak{S}} c) = \\
&= ((y_1 \sqcap_{\mathfrak{S}} y_2) \sqcup_{\mathfrak{S}} u) = \\
&= \perp_{\mathfrak{S}} \sqcup_{\mathfrak{S}} u = u.
\end{aligned}$$

We then conclude  $\text{Card}(\min_{\mathfrak{S}}(B_{u,v}^*)) = 1$ . Conversely, if  $y$  is an element of  $\mathfrak{S}$  such that  $\perp_{\mathfrak{S}} \bar{\sqsubseteq}_{\mathfrak{S}} y \sqsubseteq_{\mathfrak{S}} v$  and  $u \parallel_{\mathfrak{S}} y$ , we have necessarily  $u \sqcap_{\mathfrak{S}} y = \perp_{\mathfrak{S}}$  and  $y \in \mathfrak{S}_c$  [22], i.e.  $y \in \min(B_{u,v}^*)$ . As a conclusion,

$$\forall u, v \in \mathfrak{S}, v \notin \mathfrak{S}_c \mid u \bar{\sqsubseteq}_{\mathfrak{S}} v, \quad \exists! y \in \mathfrak{S} \mid (\perp_{\mathfrak{S}} \bar{\sqsubseteq}_{\mathfrak{S}} y \sqsubseteq_{\mathfrak{S}} v \text{ and } u \parallel_{\mathfrak{S}} y). \quad (251)$$

This concludes the proof of property (240).

We now intent to prove the final result

$$\forall b, u, v \in \mathfrak{S} \mid b \sqsubseteq_{\mathfrak{S}} u \bar{\sqsubseteq}_{\mathfrak{S}} v, \quad \exists! r \mid (b \bar{\sqsubseteq}_{\mathfrak{S}} r \sqsubseteq_{\mathfrak{S}} v \text{ and } r \sqcap_{\mathfrak{S}} u = \perp_{\mathfrak{S}}). \quad (252)$$

Let us begin to show the existence of  $r$  in previous property.

From property (240), it exists an element  $y \in \mathfrak{S}$  such that  $\perp_{\mathfrak{S}} \bar{\sqsubseteq}_{\mathfrak{S}} y \sqsubseteq_{\mathfrak{S}} v$  ( $y$  is atomic and then compact) and  $u \sqcap_{\mathfrak{S}} y = \perp_{\mathfrak{S}}$ . We denote  $r := b \sqcup_{\mathfrak{S}} y$ , which exists since  $\mathfrak{S}$  is consistently complete and  $v$  is a common upper-bound of  $b$  and  $y$ .

The following simple property

$$\forall b, y, v \in \mathfrak{S} \quad (\perp_{\mathfrak{S}} \bar{\sqsubseteq}_{\mathfrak{S}} y \sqsubseteq_{\mathfrak{S}} v \text{ and } \perp_{\mathfrak{S}} \sqsubseteq_{\mathfrak{S}} b \sqsubseteq_{\mathfrak{S}} v \text{ and } b \sqcap_{\mathfrak{S}} y = \perp_{\mathfrak{S}}) \Rightarrow (b \bar{\sqsubseteq}_{\mathfrak{S}} (b \sqcup_{\mathfrak{S}} y) \sqsubseteq_{\mathfrak{S}} v) \quad (253)$$

is a simple consequence of the distributivity property satisfied by  $\mathfrak{S}$ . Indeed, let us assume that it exists  $x \in \mathfrak{S}$  with  $b \sqsubseteq_{\mathfrak{S}} x \sqsubseteq_{\mathfrak{S}} (b \sqcup_{\mathfrak{S}} y)$  and let us exhibit a contradiction.

$b \sqsubseteq_{\mathfrak{S}} x$  implies  $(b \sqcup_{\mathfrak{S}} y) \sqsubseteq_{\mathfrak{S}} (x \sqcup_{\mathfrak{S}} y)$ , and  $x \sqsubseteq_{\mathfrak{S}} (b \sqcup_{\mathfrak{S}} y)$  implies  $(x \sqcup_{\mathfrak{S}} y) \sqsubseteq_{\mathfrak{S}} (b \sqcup_{\mathfrak{S}} y) \sqcup_{\mathfrak{S}} y = (b \sqcup_{\mathfrak{S}} y)$ . As a result, we have necessarily  $(x \sqcup_{\mathfrak{S}} y) = (b \sqcup_{\mathfrak{S}} y)$ .

We have also  $(x \sqcap_{\mathfrak{S}} y) = \perp_{\mathfrak{S}}$ . Indeed, we have always  $\perp_{\mathfrak{S}} \sqsubseteq_{\mathfrak{S}} (x \sqcap_{\mathfrak{S}} y) \sqsubseteq_{\mathfrak{S}} y$ , but  $(x \sqcap_{\mathfrak{S}} y) = y$  would mean  $y \sqsubseteq_{\mathfrak{S}} x$  and then  $(x \sqcup_{\mathfrak{S}} y) = x$  which is contradictory with  $x \sqsubseteq_{\mathfrak{S}} (x \sqcup_{\mathfrak{S}} y) = (b \sqcup_{\mathfrak{S}} y)$ . Now we use  $\perp_{\mathfrak{S}} \bar{\sqsubseteq}_{\mathfrak{S}} y$  to conclude that  $(x \sqcap_{\mathfrak{S}} y) = \perp_{\mathfrak{S}}$ .

Endly, using distributivity of  $\mathfrak{S}$ , we have  $b = b \sqcup_{\mathfrak{S}} \perp_{\mathfrak{S}} = b \sqcup_{\mathfrak{S}} (x \sqcap_{\mathfrak{S}} y) = (b \sqcup_{\mathfrak{S}} x) \sqcap_{\mathfrak{S}} (b \sqcup_{\mathfrak{S}} y) = x \sqcap_{\mathfrak{S}} (b \sqcup_{\mathfrak{S}} y) = (b \sqcup_{\mathfrak{S}} y)$ , i.e.  $y \sqsubseteq_{\mathfrak{S}} b$  which is false since  $y \sqcap_{\mathfrak{S}} b = \perp_{\mathfrak{S}}$  and  $y$  is an atom.

This concludes the proof of property (253).

Now, we can use the property  $u \sqcap_{\mathfrak{S}} y = \perp_{\mathfrak{S}}$  to prove that  $u \parallel_{\mathfrak{S}} r$ , and even that  $u \sqcap_{\mathfrak{S}} r = b$ . Using distributivity of  $\mathfrak{S}$ , we have indeed  $u \sqcap_{\mathfrak{S}} r = (u \sqcap_{\mathfrak{S}} b) \sqcup_{\mathfrak{S}} (u \sqcap_{\mathfrak{S}} y) = b \sqcup_{\mathfrak{S}} \perp_{\mathfrak{S}} = b$  (here, we have used  $b \sqsubseteq_{\mathfrak{S}} u$  and  $(u \sqcap_{\mathfrak{S}} y) = \perp_{\mathfrak{S}}$ ).

We have then established the existence of  $r$  in property (252). Note that  $r$  satisfies  $(r \sqcup_{\mathfrak{S}} u) = v$  (the properties  $r \sqsubseteq_{\mathfrak{S}} v$  and  $u \sqsubseteq_{\mathfrak{S}} v$  implies  $u \sqsubseteq_{\mathfrak{S}} (r \sqcup_{\mathfrak{S}} u) \sqsubseteq_{\mathfrak{S}} v$ , however  $(r \sqcup_{\mathfrak{S}} u) = u$  is excluded by the property  $u \parallel_{\mathfrak{S}} r$ , and  $u \bar{\sqsubseteq}_{\mathfrak{S}} v$  leads to the conclusion).

It is now easy to show the uniqueness of this element  $r$ . Let us suppose that we have  $r_1$  and  $r_2$  distinct elements of  $\mathfrak{S}$  satisfying property (252). We would then have  $(r_1 \sqcap_{\mathfrak{S}} r_2) = b$  and  $u \sqcup_{\mathfrak{S}} r_1 = u \sqcup_{\mathfrak{S}} r_2 = v$ . However, the distributivity of  $\mathfrak{S}$  would imply  $b = u \sqcup_{\mathfrak{S}} b = u \sqcup_{\mathfrak{S}} (r_1 \sqcap_{\mathfrak{S}} r_2) = (u \sqcup_{\mathfrak{S}} r_1) \sqcap_{\mathfrak{S}} (u \sqcup_{\mathfrak{S}} r_2) = v$ ,

<sup>22</sup>Indeed,  $\forall E \subseteq_{Dir} \mathfrak{S} \mid y \sqsubseteq_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}}^{\bar{\wedge}} E)$ , we have  $y = y \sqcup_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}}^{\bar{\wedge}} E) = \bigsqcup_{\mathfrak{S}}^{\bar{\wedge}} \{(y \sqcap_{\mathfrak{S}} e) \mid e \in E\}$ . Hence, if we had  $\forall e \in E, y \not\sqsubseteq_{\mathfrak{S}} e$ , we would have  $\forall e \in E, y \sqcap_{\mathfrak{S}} e = \perp_{\mathfrak{S}}$ , and then  $y = \perp_{\mathfrak{S}}$ , which is contradictory.

which is false. The unicity of  $r$  in property (252) is then established, which concludes the proof of property (41). The last property to verify is the property (39).

Let  $x$  and  $y$  be elements of  $\mathfrak{S}_c$  such that  $x \sqsubset_{\mathfrak{S}} y$ . From properties (244) and (245), it exists an atom  $z$  such that  $z \sqsubset_{\mathfrak{S}} y$  and  $(x \sqcap_{\mathfrak{S}} z) = \perp_{\mathfrak{S}}$ . Using now the property (253), we deduce that  $x \bar{\sqsubset}_{\mathfrak{S}} (x \sqcup_{\mathfrak{S}} z) \sqsubseteq_{\mathfrak{S}} y$ .

Let  $x$  be an element of  $\mathfrak{S}_c$ ,  $y$  be an element of  $\mathfrak{S}$  but non-compact, and let us assume  $x \sqsubset_{\mathfrak{S}} y$ . From algebraicity of  $\mathfrak{S}$ , we know that it exists  $w \in \mathfrak{S}_c$  such that  $w \sqsubseteq_{\mathfrak{S}} y$  (in fact,  $w \sqsubset_{\mathfrak{S}} y$  because  $w$  is compact and  $y$  is non-compact) and  $w \not\sqsubset x$ . Then, necessarily  $x \sqsubset_{\mathfrak{S}} (x \sqcup_{\mathfrak{S}} w) \sqsubseteq_{\mathfrak{S}} y$  and  $y' := (x \sqcup_{\mathfrak{S}} w)$  is compact (in fact,  $(x \sqcup_{\mathfrak{S}} w) \sqsubset_{\mathfrak{S}} y$  because  $x$  and  $w$  are compact, but  $y$  is non-compact). We are now in the previous case, and it exists an atom  $z$  such that  $z \sqsubset_{\mathfrak{S}} y'$  and  $(x \sqcap_{\mathfrak{S}} z) = \perp_{\mathfrak{S}}$  and  $x \bar{\sqsubset}_{\mathfrak{S}} (x \sqcup_{\mathfrak{S}} z) \sqsubseteq_{\mathfrak{S}} y' \sqsubset_{\mathfrak{S}} y$ .

Endly, let us consider  $x$  and  $y$  be elements of  $\mathfrak{S}$  such that  $x \sqsubset_{\mathfrak{S}} y$  and let us assume that  $x$  is non-compact. From algebraicity of  $\mathfrak{S}$ , we know that it exists  $w \in \mathfrak{S}_c$  such that  $w \sqsubseteq_{\mathfrak{S}} y$  (in fact,  $w \sqsubset_{\mathfrak{S}} y$  because  $w$  is compact and  $y$  is non-compact) and  $w \not\sqsubset x$ .  $w$  being compact and  $x$  non-compact, we have necessarily  $w \parallel_{\mathfrak{S}} x$ . Then,  $(w \sqcap_{\mathfrak{S}} x)$  is compact (using  $(w \sqcap_{\mathfrak{S}} x) \sqsubset_{\mathfrak{S}} w$  and property (236)) and satisfies  $(w \sqcap_{\mathfrak{S}} x) \sqsubset_{\mathfrak{S}} w$ . We can then find an atom  $z$  such that  $z \sqsubset_{\mathfrak{S}} w$  and  $(x \sqcap_{\mathfrak{S}} z) = ((w \sqcap_{\mathfrak{S}} x) \sqcap_{\mathfrak{S}} z) = \perp_{\mathfrak{S}}$ . The property  $x \bar{\sqsubset}_{\mathfrak{S}} (x \sqcup_{\mathfrak{S}} z) \sqsubseteq_{\mathfrak{S}} y$  has been proved in property (253).

The case study is now finished, and we have obtained

$$\forall x, y \in \mathfrak{S}, \quad (x \sqsubset_{\mathfrak{S}} y) \Rightarrow (\exists u \in \mathfrak{S} \mid x \bar{\sqsubset}_{\mathfrak{S}} u \sqsubset_{\mathfrak{S}} y). \quad (254)$$

This concludes the proof.  $\square$

Sub- selection structures are defined in Definitions 2 and 4.

**Lemma 56. [Characterization of sub- selection structures]**

Let  $(\mathfrak{S}, \sqsubseteq)$  be a selection structure, and let us consider  $(\mathfrak{S}', \sqsubseteq)$  a sub-poset of  $(\mathfrak{S}, \sqsubseteq)$ .

We have the following equivalences:

$((\mathfrak{S}', \sqsubseteq)$  is a sub- selection structure of  $(\mathfrak{S}, \sqsubseteq)$ ) iff  $(\mathfrak{S}'$  is a Scott-closed subset of  $(\mathfrak{S}, \sqsubseteq)$ ), i.e. (1)  $\mathfrak{S}'$  is a lower-set of  $\mathfrak{S}$ , and (2) for any  $E \subseteq_{Dir} \mathfrak{S}'$ , the directed supremum  $\bigsqcup_{\mathfrak{S}}^{\wedge} E$  is an element of  $\mathfrak{S}'$ .

$((\mathfrak{S}', \sqsubseteq)$  is an idealized sub- selection structure of  $(\mathfrak{S}, \sqsubseteq)$ ) iff  $(\mathfrak{S}'$  is a Scott-ideal of  $(\mathfrak{S}, \sqsubseteq)$ ), i.e. (1)  $\mathfrak{S}'$  is a Scott-closed subset of  $\mathfrak{S}$ , (2) if a finite subset of  $\mathfrak{S}'$  is upper-bounded in  $\mathfrak{S}$ , it admits also an upper-bound in  $\mathfrak{S}'$  (Scott-ideals are called 'stable subdomains' in [40]).  $\blacksquare$

*Proof.* Let us suppose that  $(\mathfrak{S}, \sqsubseteq)$  is a selection structure, and then an algebraic Scott-domain. Let us consider a sub-poset of  $(\mathfrak{S}, \sqsubseteq)$  denoted  $(\mathfrak{S}', \sqsubseteq)$  which is Scott-closed in  $\mathfrak{S}$ . Let us then check the different axioms of selection structures (i.e. Definition 1) for  $(\mathfrak{S}', \sqsubseteq)$ .

$\mathfrak{S}'$  is closed under chain suprema, hence the chain supremum in  $\mathfrak{S}'$  of the empty set, i.e.  $b_{\mathfrak{S}'}$ , is in  $\mathfrak{S}'$ .

As a remark,  $b_{\mathfrak{S}'} = b_{\mathfrak{S}}$ , because  $\mathfrak{S}'$  contains  $b_{\mathfrak{S}}$  as a downset of  $\mathfrak{S}$ .

$\mathfrak{S}'$  being a downset of  $\mathfrak{S}$ , if  $s_1, s_2$  are in  $\mathfrak{S}'$ , then all common lower-bounds of  $s_1$  and  $s_2$  are in  $\mathfrak{S}'$  with the same inherited ordering as in  $\mathfrak{S}$ , then  $(s_1 \sqcap_{\mathfrak{S}'} s_2) = (s_1 \sqcap_{\mathfrak{S}} s_2)$  is also in  $\mathfrak{S}'$ .

For any  $x, y \in \mathfrak{S}'$  with  $x \sqsubset y$ , the strong-atomicity of  $\mathfrak{S}$  ensures that it exists  $t \in \mathfrak{S}$  with  $x \bar{\sqsubset} t \sqsubseteq_{\mathfrak{S}} y$ .  $\mathfrak{S}'$  being a downset of  $\mathfrak{S}$ ,  $t$  is in  $\mathfrak{S}'$ . Any element  $t' \in \mathfrak{S}' \subseteq \mathfrak{S}$  such that  $x \sqsubset t' \sqsubseteq_{\mathfrak{S}} t$  has to be also in  $\mathfrak{S}$  and then  $t'$  must be equal to  $x$  because of the property  $x \bar{\sqsubset} t$  in  $\mathfrak{S}$ . Hence,  $\mathfrak{S}'$  is also strongly atomic.

Let us consider  $b, u, v \in \mathfrak{S}' \subseteq \mathfrak{S}$  with  $b \sqsubset u \bar{\sqsubset} v$ . It exists a unique  $r \in \mathfrak{S}$  with  $b \bar{\sqsubset} r \sqsubset v$  and  $r \parallel u$ .  $\mathfrak{S}'$  being a downset of  $\mathfrak{S}$ ,  $r \sqsubset v \in \mathfrak{S}'$  implies  $r \in \mathfrak{S}'$ . Moreover,  $\mathfrak{S}'$  being a downset of  $\mathfrak{S}$ , if we have  $r \parallel u$  in  $\mathfrak{S}$  it is also necessarily true in  $\mathfrak{S}'$ .

For any chain  $C \subseteq_{chain} \mathfrak{S}'$ ,  $C$  is also trivially a directed subset of  $\mathfrak{S}'$ , and then  $(\bigsqcup_{\mathfrak{S}} C) \in \mathfrak{S}'$  exists and is in  $\mathfrak{S}'$ , because  $\mathfrak{S}'$  is closed under directed suprema in  $\mathfrak{S}$  as a Scott-closed subset of  $\mathfrak{S}$ .

For any  $s, s' \in \mathfrak{S}'$  with  $s \bar{\sqsubset} s'$ , and any chain  $C \subseteq_{chain} \mathfrak{S}'$ , we know that  $(\bigsqcup_{\mathfrak{S}} C)$  exists in  $\mathfrak{S}'$  and  $s' \sqcap_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} C)$  exists in  $\mathfrak{S}'$ . Hence,  $(\forall c \in C, s = (s' \sqcap_{\mathfrak{S}} c)) \Rightarrow (s = s' \sqcap_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} C))$ .

As a conclusion,  $(\mathfrak{S}', \sqsubseteq)$  satisfies the axioms of a selection-structure.  $\mathfrak{S}'$  being moreover a downset of  $\mathfrak{S}$ , we conclude that  $(\mathfrak{S}', \sqsubseteq)$  is a sub- selection structure of  $(\mathfrak{S}, \sqsubseteq)$ .

Conversely, let us consider a selection structure  $(\mathfrak{S}, \sqsubseteq)$ , and  $(\mathfrak{S}', \sqsubseteq)$  a sub-(selection-structure) of  $\mathfrak{S}$ .

By definition we already know that  $\mathfrak{S}'$  is a downset of  $(\mathfrak{S}, \sqsubseteq)$ . Hence, in order to show that  $\mathfrak{S}'$  is Scott-closed in  $(\mathfrak{S}, \sqsubseteq)$  it suffices to prove that  $\mathfrak{S}'$  is closed under directed-suprema in  $(\mathfrak{S}, \sqsubseteq)$ , i.e.

$$\forall D \subseteq_{Dir} \mathfrak{S}' \quad (\bigsqcup_{\mathfrak{S}}^{\bar{\cdot}} D) \in \mathfrak{S}'. \quad (255)$$

We firstly note that  $(\bigsqcup_{\mathfrak{S}}^{\bar{\cdot}} D)$  because the Scott-domain  $(\mathfrak{S}, \sqsubseteq)$  is directed complete.

Moreover, by definition,  $(\mathfrak{S}', \sqsubseteq)$  is itself a selection structure, and then a Scott-domain. Hence,  $(\bigsqcup_{\mathfrak{S}'}^{\bar{\cdot}} D)$  exists and

$$\forall D \subseteq_{Dir} \mathfrak{S}' \quad (\bigsqcup_{\mathfrak{S}'}^{\bar{\cdot}} D) \in \mathfrak{S}'.$$

Now, we obviously have  $(\bigsqcup_{\mathfrak{S}}^{\bar{\cdot}} D) \sqsubseteq_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}'}^{\bar{\cdot}} D)$ , because  $\mathfrak{S}' \subseteq \mathfrak{S}$ . Let us then define the following subset of  $\mathfrak{S}'$  for any  $D \subseteq_{Dir} \mathfrak{S}'$

$$U_D := \{t \in \mathfrak{S}' \mid \forall d \in D, (t \sqcup_{\mathfrak{S}'} d) \text{ exists in } \mathfrak{S}' \text{ and } (t \sqcup_{\mathfrak{S}'} d) \sqsubseteq_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} D)\}. \quad (256)$$

We observe that  $\mathfrak{b}_{\mathfrak{S}} = \mathfrak{b}_{\mathfrak{S}'} \in U_D$ , and then  $U$  is not empty.

Let us fix  $u \in U_D$  and  $d \in D$ .

For any  $d' \in D$ , it exists a common upper-bound  $d''$  of  $d$  and  $d'$ , with  $d'' \in D$  because  $D$  is directed. By definition of  $D$ , for any  $d, d' \in D$ , they have a common upper-bound in  $D$  denoted for example  $d''$ , and then, using consistent completeness of  $\mathfrak{S}'$ , we deduce that  $(d \sqcup_{\mathfrak{S}'} d')$  exists in  $\mathfrak{S}'$  and is less than  $d''$  and *a fortiori* less than  $(\bigsqcup_{\mathfrak{S}} D)$  because  $d'' \in D$ .

It is obvious that  $(u \sqcup_{\mathfrak{S}'} d'')$  is a common upper-bound of  $d' \in \mathfrak{S}'$  and  $(u \sqcup_{\mathfrak{S}'} d) \in \mathfrak{S}'$ . Then, consistent completeness of  $\mathfrak{S}'$  implies  $((u \sqcup_{\mathfrak{S}'} d) \sqcup_{\mathfrak{S}'} d')$  exists and is in  $\mathfrak{S}'$ . Moreover,  $((u \sqcup_{\mathfrak{S}'} d) \sqcup_{\mathfrak{S}'} d') = (u \sqcup_{\mathfrak{S}'} (d \sqcup_{\mathfrak{S}'} d')) \sqsubseteq_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} D)$  because  $u \in U_D$  implies  $u \sqsubseteq_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} D)$  and we have already shown that  $(d \sqcup_{\mathfrak{S}'} d') \sqsubseteq_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} D)$ . Hence,  $(u \sqcup_{\mathfrak{S}'} d) \in U_D$  for any  $u \in U_D$  and  $d \in D$ .

Let us now consider  $C \subseteq_{chain}^{\neq \emptyset} U_D$  and fix  $d \in D$ . Let us denote  $C' := \{(c \sqcup_{\mathfrak{S}'} d) \mid c \in C\}$ . Previous result, i.e.  $\forall u \in U_D, \forall d \in D, (u \sqcup_{\mathfrak{S}'} d) \in U$  implies  $C' \subseteq_{chain}^{\neq \emptyset} U_D$ .

As soon as  $C' \subseteq_{chain}^{\neq \emptyset} U_D$ , we have  $\forall c' \in C', (c' \sqcup d) \sqsubseteq_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} D)$  for any  $d \in D$ , and then  $c' \sqsubseteq_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} D)$ . Hence,  $(\bigsqcup_{\mathfrak{S}'} C') \sqsubseteq_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} D)$ .

Chain completeness property of  $\mathfrak{S}'$  (because  $\mathfrak{S}'$  is a selection structure) implies  $(\bigsqcup_{\mathfrak{S}'} C)$  and  $(\bigsqcup_{\mathfrak{S}'} C')$  exist in  $\mathfrak{S}'$ .

$(\bigsqcup_{\mathfrak{S}'} C')$  is a common upper-bound of  $d$  and  $(\bigsqcup_{\mathfrak{S}'} C)$ . Hence, consistent completeness of  $\mathfrak{S}'$  implies  $(d \sqcup_{\mathfrak{S}'} (\bigsqcup_{\mathfrak{S}'} C))$  exists in  $\mathfrak{S}'$  and  $(d \sqcup_{\mathfrak{S}'} (\bigsqcup_{\mathfrak{S}'} C)) \sqsubseteq_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}'} C') \sqsubseteq_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}} D)$ . Hence,  $(\bigsqcup_{\mathfrak{S}'} C)$  is in  $U_D$ .

Using Zorn's lemma, we conclude that  $U_D$  admits some maximal elements. Let us denote by  $M$  a maximal element of  $U_D$ . We have in particular,  $M \sqsubseteq_{\mathfrak{S}} (\bigsqcup_{\mathfrak{S}}^{\bar{\cdot}} D)$ . However,  $\forall d \in D, (M \sqcup d) \in U_D$  implies  $\forall d \in D, (M \sqcup d) \sqsubseteq_{\mathfrak{S}} M$  and then  $\forall d \in D, d \sqsubseteq_{\mathfrak{S}} M$ . As a conclusion, we obtain  $(\bigsqcup_{\mathfrak{S}}^{\bar{\cdot}} D) \sqsubseteq_{\mathfrak{S}} M$  and then  $(\bigsqcup_{\mathfrak{S}}^{\bar{\cdot}} D) = M \in U_D \subseteq \mathfrak{S}'$ . This concludes the proof of property (255).

We endly note that the property (98) is exactly the third axiom in the definition of Scott-ideals, which establishes the second equivalence announced in the Lemma.  $\square$

**Lemma 57. [Alternative description of idealized sub-selection structures]**

A sub-selection structure  $\mathfrak{S}'$  of a selection structure  $(\mathfrak{S}, \sqsubseteq)$  is idealized iff it satisfies the property:

$$\forall s \in \mathfrak{S}, \quad \bigsqcup_{\mathfrak{S}} ((\downarrow_{\mathfrak{S}} s) \sqcap_{\mathfrak{S}} \mathfrak{S}') \in \mathfrak{S}'. \quad (257)$$

■

*Proof.* Let us consider  $S \subseteq_{fin} \mathfrak{S}'$  such that it exists  $s \in \mathfrak{S}$  which is a common upper-bound of elements of  $S$ . We have then  $S \subseteq ((\downarrow_{\mathfrak{S}} s) \sqcap_{\mathfrak{S}} \mathfrak{S}')$ . Using property (257), the supremum  $\bigsqcup_{\mathfrak{S}} ((\downarrow_{\mathfrak{S}} s) \sqcap_{\mathfrak{S}} \mathfrak{S}')$  exists as an element of  $\mathfrak{S}'$ . It is then a common upper-bound of elements of  $S$  that belongs to  $\mathfrak{S}'$ . Hence, property

(257) implies that  $\mathfrak{S}'$  is idealized.

Conversely, let us suppose  $\mathfrak{S}'$  is idealized and let us fix any  $s \in \mathfrak{S}$ . The set  $((\downarrow_{\mathfrak{S}} s) \sqcap_{\mathfrak{S}} \mathfrak{S}')$  is upper-bounded by  $s$ . Using property (98), for any  $F \subseteq_{fin} ((\downarrow_{\mathfrak{S}} s) \sqcap_{\mathfrak{S}} \mathfrak{S}')$ , it exists  $M_F \in \mathfrak{S}'$  upper-bound of  $F$ . Hence,  $\bigsqcup_{\mathfrak{S}'} F$  exists and is in  $\mathfrak{S}'$ , and  $\bigsqcup_{\mathfrak{S}} F = \bigsqcup_{\mathfrak{S}'} F$ . As an example, for  $F \subseteq_{fin} \mathbb{A}(s) \sqcap_{\mathfrak{S}} \mathfrak{S}'$  We can build the following subset:  $\{\bigsqcup_{\mathfrak{S}} F \mid F \subseteq_{fin} \mathbb{A}(s) \sqcap_{\mathfrak{S}} \mathfrak{S}'\} \subseteq ((\downarrow_{\mathfrak{S}} s) \sqcap_{\mathfrak{S}} \mathfrak{S}')$ . It is a directed subset, by construction. It then admits a directed-supremum in  $\mathfrak{S}$  which is in  $\mathfrak{S}'$ , because  $\mathfrak{S}'$  is Scott-closed :  $\bigsqcup_{\mathfrak{S}} \{\bigsqcup_{\mathfrak{S}} F \mid F \subseteq_{fin} \mathbb{A}(s) \sqcap_{\mathfrak{S}} \mathfrak{S}'\} \in \mathfrak{S}'$ . Due to algebraicity of  $\mathfrak{S}$  (Theorem 21), we then obtain  $(\bigsqcup_{\mathfrak{S}} \{t \in \mathfrak{S}' \mid t \sqsubseteq_{\mathfrak{S}} s\}) \in \mathfrak{S}'$ . This achieves the proof of property (257).  $\square$

## 7 Appendix : Basic notions and results in order theory

We shall use “:=” (and sometimes “: $\Leftrightarrow$ ”) for a definitional equality. Thus,  $X := Y$  means “the expression denoted  $X$  is conventionnaly equal to  $Y$ ”.

We write  $X \subseteq_{fin} Y$  (resp.  $X \subseteq_{\neq \emptyset} Y$ ) for the assertion that  $X$  is a finite (resp. non-empty) subset of  $Y$ .

We will use  $\omega$  and  $\mathbb{N}$  to denote the natural numbers thought as an ordinal or as a set.

Given a set  $X$ , we write  $\mathcal{P}(X)$  (resp.  $\mathcal{P}_{fin}(X)$ , resp.  $\mathcal{P}_{fin}(X)^*$ ) for the set of subsets of  $X$  (resp. for the set of finite subsets of  $X$ , resp. for the set of finite non-empty subsets of  $X$ ).

The logical conjunction (resp. disjunction) of two propositions  $p_1, p_2$  will be written explicitly as “ $p_1$  and  $p_2$ ” (resp. “ $p_1$  or  $p_2$ ”) or denoted  $p_1 \cap p_2$  (resp.  $p_1 \cup p_2$ ). The logical negation of a proposition  $p$  will be explicitly denoted  $\neg p$ .

The symbols  $\forall$  (resp.  $\exists$ ) will be used with the usual meaning of “for any...” (resp. “it exists...”). The symbol  $\exists!$  will be used with the meaning “It exists a unique...”. The symbol “...|...” will be used with the meaning “... such that...”.

### 7.1 Order

A *proto-order* is a pair  $(B, \ll)$ , where  $B$  is a set and  $\ll$  is a *fully transitive binary relation*, i.e.

$$\left\{ \begin{array}{l} \text{(Full Transitivity)} \\ \forall x, y, z \in B, (x \ll y \text{ and } y \ll z) \Rightarrow x \ll z, \\ \forall x \in B, \forall A \subseteq_{fin} B, (A \ll x) \Rightarrow (\exists y \in B, A \ll y \ll x). \end{array} \right. \quad (258)$$

Here  $A \ll u$  means:  $a \ll u, \forall a \in A$ .

Let  $\mathbf{X} = (X, \ll)$  be a proto-order. A subset  $Q$  of  $X$  such that  $\forall x \in Q, \forall y \in X, (y \ll x) \Rightarrow (y \in Q)$  is said to be a *lower-set* of  $\mathbf{X}$ . An *upper-set* is a non-empty subset  $Q$  of  $X$  such that  $\forall x \in Q, \forall y \in X, (x \ll y) \Rightarrow (y \in Q)$ .

A nonempty subset  $Q$  is said to be a *directed subset* of  $\mathbf{X}$  (resp. a *filtered subset* of  $\mathbf{X}$ ), which is denoted  $Q \subseteq_{Dir} \mathbf{X}$  (resp.  $Q \subseteq_{Fil} \mathbf{X}$ ), iff  $Q \subseteq X$  and  $\forall x, y \in Q, \exists z \in Q, (x \ll z \text{ and } y \ll z)$  (resp.  $Q \subseteq X$  and  $\forall x, y \in Q, \exists z \in Q, (z \ll x \text{ and } z \ll y)$ ). Note that the empty set is directed and filtered.

A lower-set  $I$  (resp. an upper-set  $F$ ) of  $\mathbf{X}$  is an *ideal* (resp. a *filter*) of  $X$  iff it is a directed (resp. filtered) in  $\mathbf{X}$ .

The set formed by all ideals (resp. filters) of  $\mathbf{X}$  equipped with the set inclusion is denoted  $\mathcal{I}(\mathbf{X})$  (resp.  $\mathcal{F}(\mathbf{X})$ ).

An ideal  $I$  (resp. filter  $F$ ) is said to be *generated by a subset  $Q$  of  $X$*  iff  $I$  is the least<sup>[23]</sup> ideal (resp. filter) containing  $Q$ . In particular, we will denote  $(\downarrow_{(x, \ll)} x) := \{y \in X \mid y \ll x\}$  (resp.  $(\uparrow^{(x, \ll)} x) := \{y \in X \mid x \ll y\}$ ).

Let  $\mathbf{X} = (X, \ll)$  be a proto-order, and let  $Q$  be a subset of  $X$ .  $m \in X$  (resp.  $M \in X$ ) is a *lower bound* (resp. an *upper bound*) for  $Q$ , iff  $m \ll x$  (resp.  $x \ll M$ ),  $\forall x \in Q$ . If the subset  $Q$  of  $X$  admits an upper-bound (resp. a lower bound) in  $X$ , we will write  $Q \subseteq_{up-b} X$  (resp.  $Q \subseteq_{low-b} X$ ). The set of upper-bounds (esp. lower-bounds) of  $Q$  in  $X$  will be denoted  $ub_x(Q)$  (resp.  $lb_x(Q)$ ). A lower bound (resp. an upper bound)

<sup>23</sup>“least” with respect to set inclusion

for the whole set  $X$  will be called *the bottom element* (resp. *the top element*) of  $X$ . A finite upper-bounded subset  $Q$  will be said to be *consistent*, denoted  $Q \subseteq_{Con} P$  or shortly  $\widehat{Q}^P$ . The subset  $Q$  will be said to be *coherent* (denoted  $Q \subseteq_{Coh} P$ ) iff any finite subset of  $Q$  is upper-bounded in  $P$ .

A *pre-order* is a pair  $\mathbf{P} := (P, \preceq)^{[24]}$ , where  $P$  is a non-empty set (sometimes called *the universe of the pre-order*) and  $\preceq \subseteq P \times P$  is a binary relation satisfying:

$$\begin{cases} \text{(Transitivity)} & \forall x, y, z \in S, (x \preceq y \text{ and } y \preceq z) \Rightarrow x \preceq z, \\ \text{(Reflexivity)} & x \preceq x, \forall x \in P. \end{cases} \quad (259)$$

A pre-order is necessarily a proto-order. A pre-order  $(P, \preceq)$  is said to be *total* iff  $\forall x, y \in P, x \preceq y$  or  $y \preceq x$ . Let  $\mathbf{P} := (P, \preceq)$  be a pre-order. An element  $M$  (resp.  $m$ ) of  $Q$  will be said to be *a maximal element of  $Q$*  (resp. *a minimal element of  $Q$* ) iff  $\forall p \in Q, M \not\prec p$  (resp.  $\forall p \in Q, p \not\prec m$ ). The set of maximal elements (resp. minimal elements) of  $Q$  will be denoted  $Max(Q)$  (resp.  $min(Q)$ ). Any non-empty finite sub-poset admits obviously minimal and maximal elements.

A *partially ordered set* (or *poset*) is a pre-order  $\mathbf{P} := (P, \leq)$  satisfying moreover:

$$\text{(Anti-Symmetry)} \quad \forall x, y \in P \quad (x \leq y \text{ and } y \leq x) \Rightarrow (x = y). \quad (260)$$

Posets with a bottom element (resp. top element) are said to be *pointed* (resp. *unital*).

A poset being in particular a pre-order and then a proto-order, all definitions and results relative to proto-orders or pre-orders are transposed to posets literally.

Every subset  $Q$  of the universe of a poset  $\mathbf{P}$  has an associated partial order called the induced partial order of  $Q$ .

For any pre-order  $(P, \preceq)$ , we can define an equivalence relation

$$x \sim y :\Leftrightarrow (x \preceq y \text{ and } y \preceq x). \quad (261)$$

A pre-order  $(P, \preceq)$  leads to a partial-order  $(P/\sim, \leq)$  after quotienting by  $\sim$ :

$$[x]_{\sim} := \{y \in P \mid x \sim y\}, \quad P/\sim := \{[x]_{\sim} \mid x \in P\}, \quad ([x]_{\sim} \leq [x']_{\sim}) \Leftrightarrow (x \preceq x'). \quad (262)$$

A map  $f : P_1 \rightarrow P_2$  between two posets is said to be *an order preserving map*, or simply *a monotone map*, if for any  $x, y \in P_1, x \leq y \Rightarrow f(x) \leq f(y)$ .

The set  $[P \rightarrow Q]$  of all monotone functions between posets  $P$  and  $Q$ , when ordered pointwise (i.e.  $f \leq g$  if  $\forall x \in P, f(x) \leq g(x)$ ), gives rise to a poset: the monotone function space between  $P$  and  $Q$ . The set of posets equipped with monotone maps defines a category.

If a minimal (resp. a maximal) element  $m$  (resp.  $M$ ) of  $Q$  is also a lower-bound of  $Q$  (resp. an upper-bound of  $Q$ ), then  $m$  (resp.  $M$ ) will be called *the least element* (resp. *the greatest element*) of  $Q$  (the least/greatest element is necessarily unique).

For all  $x$  in a poset  $P$  we define the strict and closed initial segments, the strict and closed final segments, by  $\langle x \rangle := \downarrow_{(P, \leq)} x$ ,  $\langle x[ := \langle x \rangle \setminus \{x\}$ ,  $[x := \uparrow_{(P, \leq)} x$ ,  $]x := [x \setminus \{x\}$ , and, for all  $x$  and  $y$  in a poset  $P$  with  $x \leq y$ , we define the interval  $[x, y := \langle y \rangle \cap [x$ .

## 7.2 Induction

A *chain* of a poset  $\mathbf{P}$  is a subset of  $P$  which is totally ordered by the restriction of the order defined on  $P$ . An  $\omega$ -chain is a countable chain. If  $C$  is a chain (resp.  $\omega$ -chain) in the poset  $\mathbf{P}$ , we will write  $C \subseteq_{chain} \mathbf{P}$  (resp.  $C \subseteq_{\omega-chain} \mathbf{P}$ ). A descending (resp. ascending)  $\omega$ -chain is a chain  $(a_n)_{n \in \mathbb{N}}$  of elements of  $P$  such

<sup>24</sup>**Notations:** Different symbols ( $\leq, \subseteq, \preceq, \sqsubseteq, \dots$ ) will be used to designate the different pre-orders under study. When, for example, a symbol like  $\leq$  will be used, the mirrored symbol  $\geq$  will simply mean " $(y \geq x) \Leftrightarrow (x \leq y)$ ", the symbol  $<$  will mean " $(x < y) \Leftrightarrow ((x \leq y) \cap (x \neq y))$ " (the symbol  $\subsetneq$  will be sometimes used as an equivalent to  $\subset$ ). The symbol  $\not\prec$  will mean " $(x \not\prec y) \Leftrightarrow \neg(x \prec y)$ ".



that  $\forall i, j \in \mathbb{N}, (i \leq j) \Rightarrow (a_i \geq a_j)$  (resp.  $\forall i, j \in \mathbb{N}, (i \leq j) \Rightarrow (a_i \leq a_j)$ ). A descending/ascending chain is said to *stabilize* iff  $\exists k \in \mathbb{N}, \forall l \in \mathbb{N}, (k \leq l) \Rightarrow (a_k = a_l)$ .

A poset  $P$  is said to be *inductive* iff every chain admits an upper bound. Let  $P$  be an inductive poset, then  $P$  has a maximal element (possibly more than one).<sup>[25]</sup>

In a poset  $P$ , we will say that  $x \in P$  covers  $y \in P$  (denoted  $x \succ y$  or  $y \prec x$ ) iff

$$(Covering) \quad y \prec x \Leftrightarrow x \in \text{Min}(\ ]y ) \quad (263)$$

A poset  $P$  is said to be a *torsion* (or to be *semi-artinian*) iff

$$(P \text{ Semi-artinian}) \quad \forall y \in P, (y \in \text{Max}(P) \text{ or } \exists x \in P, y \prec x) \quad (264)$$

In a poset  $P$ , an element  $p$  is said to be a *torsion element* iff  $\langle x ]$  is a torsion. In a poset  $P$ , an element  $p$  is said to be *torsion free* iff  $\text{Min}(\ ]x ) = \emptyset$ .

A poset  $P$  is said to be *strongly atomic* iff

$$(Strong \text{ atomicity}) \quad \forall x, y \in P, (x < y) \Leftrightarrow (\exists z \in P, x \prec z \leq y). \quad (265)$$

A strongly atomic poset is necessarily semi-artinian.

A poset  $P$  is said to be *artinian* (resp. *noetherian*) iff every subset of  $P$  has a minimal element (resp. a maximal element).<sup>[26]</sup>

$$(P \text{ Artinian}) \quad \forall Q \subseteq P, \text{Min}(Q) \neq \emptyset. \quad (266)$$

If  $P$  is artinian (resp. noetherian), then any interval of  $P$  is artinian (resp. noetherian) as well (in particular the initial and final intervals). An artinian poset is necessarily also semi-artinian. Artinian (resp. noetherian) posets are defined equivalently as posets where every descending (resp. ascending)  $\omega$ -chain stabilizes.

### 7.3 Completeness

Let  $\mathbf{P} = (P, \leq)$  be a poset. The unique *infimum* (also called *meet*) of  $Q \subseteq P$  denoted  $\bigwedge_P Q$  is defined by

$$(\bigwedge_P Q \leq x, \forall x \in Q) \text{ and } (\forall y \in P, (y \leq x, \forall x \in Q) \Rightarrow y \leq \bigwedge_P Q). \quad (267)$$

We will denote  $x \wedge y := \bigwedge_P \{x, y\}, \forall x, y \in P$ .

The unique *supremum* (also called *join*) of  $Q \subseteq P$  denoted  $\bigvee_P Q$  is defined by

$$(\bigvee_P Q \geq x, \forall x \in Q) \text{ and } (\forall y \in P, (y \geq x, \forall x \in Q) \Rightarrow y \geq \bigvee_P Q). \quad (268)$$

We will denote  $x \vee y := \bigvee_P \{x, y\}, \forall x, y \in P$ .

If a directed set (resp. filtered set)  $A$  has a supremum (resp. an infimum), it will be eventually denoted  $\bigvee_P A$  (resp.  $\bigwedge_P A$ ) to recall the directedness (resp. the filteredness) of  $A$ .

A poset  $\mathbf{P} = (P, \leq)$  is said to be an *Inf semi-lattice* (resp. a *Sup semi-lattice*), if  $x \wedge y$  (resp.  $x \vee y$ ) exists for any  $x, y \in P$ , and then any finite subset of  $P$  has an infimum. An Inf semi-lattice can be

<sup>25</sup>This is *Zorn's lemma*, see [32] (page 62) for a pedagogical introduction and a proof. Zorn's lemma can be proved as a consequence of the "Axiom of Choice" defined as follows: If  $X$  is a set, and  $f$  is a function associating to every  $x \in X$  a non-empty set  $f(x)$ , then there exists a function  $g$  associating to every  $x \in X$  an element  $g(x) \in f(x)$ .

<sup>26</sup>Artinian posets are especially useful when we want to establish results by a suitable generalization of recursion methods [68, Theorem 7.1.5]. More precisely, let  $P$  be an artinian poset and  $Q$  be a subset of  $P$  such that  $\forall x \in Q, \langle x [ \in Q \Rightarrow x \in Q$ , then  $Q = P$ . More generally, let  $P$  be an artinian poset and  $M$  any set. Let us suppose that a particular construction has led to build a map  $\mathbf{c}$  which associates to any couple  $(u, f_{<u})$ , where  $u \in P$  and  $f_{<u}$  is a map from  $\langle x [$  to  $M$ , a uniquely defined element of  $M$  denoted  $\mathbf{c}(u, f_{<u})$ . Then there exists a unique function  $f$  from the whole  $P$  to  $M$  such that  $\forall u \in P, f(u) = \mathbf{c}(u, f_{<u})$ .

equivalently given as a commutative idempotent semi-group  $(P, \wedge)$ , the partial order being given as  $\forall p_1, p_2 \in P, (p_1 \wedge p_2 = p_1) \Leftrightarrow (p_1 \leq p_2)$ .

A poset  $\mathbf{P} = (P, \leq)$  is a *lattice*, if it is both an Inf semi-lattice and a Sup semi-lattice.

It is a central aspect of lattices that they can be jointly seen as an order and through an algebraic perspective.

Indeed, a lattice is also defined as an algebra  $(L, \wedge, \vee)$  such that  $\wedge, \vee$  satisfy conjointly the following properties:

$$(idempotency) \quad x \vee x = x, \quad x \wedge x = x \quad (269)$$

$$(commutativity) \quad x \vee y = y \vee x, \quad x \wedge y = y \wedge x \quad (270)$$

$$(associativity) \quad (x \vee y) \vee z = x \vee (y \vee z), \quad (x \wedge y) \wedge z = x \wedge (y \wedge z) \quad (271)$$

$$(absorption) \quad x \vee (x \wedge y) = x \quad \text{and} \quad x \wedge (x \vee y) = x \quad (272)$$

Starting from this algebraic definition, we can define the partial order  $\leq$  by

$$\forall x, y \in L, \quad x \leq y \quad \text{iff} \quad x \wedge y = x \quad \text{iff} \quad x \vee y = y$$

Equipped with this order,  $(L, \leq)$  is then a lattice with  $x \wedge y$  and  $x \vee y$  defined respectively as the supremum and the infimum of  $x, y$ .

A map preserving non-empty finite infima (resp. suprema) will be called a homomorphism of Inf semi-lattice (resp. Sup semi-lattice).

If  $(L, \vee, \wedge)$  and  $(L', \vee', \wedge')$  are lattices as algebras and  $f$  maps  $L$  into  $L'$ , then we call  $f$  a *homomorphism of lattices*<sup>[27]</sup> iff  $f$  is a meet-preserving and a join-preserving map. If  $(L, \leq)$  and  $(L', \leq')$  are lattices as orders and  $f$  maps  $L$  into  $L'$ , then we call  $f$  an isomorphism if it is one-to-one and onto and order-preserving. Note that the two concepts are equivalent. The lattices together with these morphisms form a category.

Let  $\mathbf{P} = (P, \leq)$  be a poset, and let  $Q$  be a subset of  $P$ .  $Q$  is said to be complete in  $P$  iff

$$(Q \subseteq_{Comp} P) \quad \forall R \subseteq Q, \forall u \in ub_p(R), \exists q \in Q \mid (\forall r \in R, r \leq q \leq u). \quad (273)$$

A poset  $\mathbf{P}$  is said to be *consistently complete* iff every consistent subset admits a supremum in  $\mathbf{P}$ . A poset  $\mathbf{P}$  is said to be *coherently complete* iff every coherent subset admits a supremum in  $\mathbf{P}$ .

A poset  $P$  is said to be *strictly inductive* (or *chain complete*) if it is non-empty and if every chain  $C \subseteq_{chain} P$  admits a supremum in  $P$ . A poset  $\mathbf{P}$  is  $\omega$ -chain complete, iff every countable directed subset of  $\mathbf{P}$  admits a supremum in  $\mathbf{P}$ .

A poset  $\mathbf{P} = (P, \leq)$  is said to be *directed complete*, and called a *directed complete partial order* (or *dcpo*), iff any of its directed subset  $Q \subseteq_{Dir} \mathbf{P}$  has a supremum in  $\mathbf{P}$ . A poset  $\mathbf{P} = (P, \leq)$  is said to be *filtered complete* iff any of its filtered subset  $Q \subseteq_{Fil} \mathbf{P}$  has an infimum in  $\mathbf{P}$ . A poset  $\mathbf{P} = (P, \leq)$  is said to be *bicomplete* iff it is simultaneously filtered-complete and directed-complete.

An Inf semi-lattice  $\mathbf{P}$  is said to be a *down-complete Inf semi-lattice*, iff any non-empty subset has an infimum in  $P$ . A poset  $\mathbf{P}$  is said to be *bounded complete* iff every non-empty upper-bounded subset  $Q$  of  $\mathbf{P}$ , i.e.  $Q \subseteq_{up-b} P$  has a supremum in  $\mathbf{P}$ , i.e.  $\bigvee_P Q \in P$ . It is equivalent to require that every non-empty lower-bounded subset  $Q$  of  $\mathbf{P}$ , i.e.  $Q \subseteq_{low-b} P$  has an infimum in  $\mathbf{P}$ , i.e.  $\bigwedge_P Q \in P$ . A poset  $\mathbf{P} = (P, \leq)$  is said to be *complete* iff every subset admits an infimum and a supremum in  $\mathbf{P}$ .

A poset  $\mathbf{P}$  is chain complete, iff it is directed complete. [49]

If the poset  $\mathbf{P} = (P, \leq)$  is directed-complete and consistently-complete,  $\mathbf{P}$  is then coherently-complete and

$$\forall Q \subseteq_{Coh} P, \quad \bigvee_P Q = \bigvee_P^{\wedge} \{ \bigvee_P R \mid R \subseteq_{fin} Q \}. \quad (274)$$

<sup>27</sup>If the map is one-to-one and onto, it is called an isomorphism. If  $f$  is one-to-one, it is called an embedding.

In particular,  $P$  is then also bounded-complete.

A monotone net in a poset  $P$  is a monotone function  $\alpha$  from a directed set  $I$  (the *index set* of  $\alpha$ ) into  $P$ . Let  $P$  be a poset and  $I$  be a directed set. Let  $\alpha : I \times I \rightarrow P$  be a monotone net. If the following suprema exist, then we have [5, Proposition 2.1.12] :

$$\bigsqcup_{i \in I}^{\wedge} \bigsqcup_{j \in I}^{\wedge} \alpha(i, j) = \bigsqcup_{j \in I}^{\wedge} \bigsqcup_{i \in I}^{\wedge} \alpha(i, j) = \bigsqcup_{i \in I}^{\wedge} \alpha(i, i). \quad (275)$$

Let  $C, C'$  be totally ordered sets ( $C \times C'$  is then naturally directed) and let us consider a monotone net  $\alpha : C \times C' \rightarrow P$ . If the following suprema exist, then we have :

$$\bigsqcup_{c \in C} \bigsqcup_{c' \in C'} \alpha(c, c') = \bigsqcup_{c' \in C'} \bigsqcup_{c \in C} \alpha(c, c'). \quad (276)$$

## 7.4 Continuity

A binary relation  $\ll$  on a pre-ordered set  $(P, \preceq)$  is called *an auxiliary relation* if it satisfies

$$\left\{ \begin{array}{l} \ll \text{ is a fully transitive binary relation,} \\ \forall x, y, \quad (x \ll y) \Rightarrow (x \preceq y), \\ \forall t, x, y, z, \quad (t \preceq x \ll y \preceq z) \Rightarrow (t \ll z), \\ \mathbf{b} \text{ bottom element of } P \Rightarrow \mathbf{b} \ll x. \end{array} \right. \quad (277)$$

Let  $\mathbf{P} := (P, \preceq)$  be a pre-ordered set,  $B$  a subset of  $P$ , and  $\ll$  an auxiliary relation on  $P$  such that  $\ll$  is fully transitive on  $B$  as well. Let us endly consider  $x \in P$ . We will denote

$$\downarrow_{(B, \ll)} x := \{y \in B \mid y \ll x\}. \quad (278)$$

Due to the full transitivity of  $\ll$  on  $B$ , the subset  $(\downarrow_{(B, \ll)} x)$  is a directed lower-set, i.e. an ideal of  $\mathbf{P}$ . More generally, for any  $X \subseteq P$  we denote  $(\downarrow_{(B, \ll)} X)$  the ideal  $\downarrow_{(B, \ll)} X := \{y \in B \mid \exists x \in X, y \ll x\}$ . The subset  $X$  is directed iff  $(\downarrow_{(B, \ll)} X)$  is an ideal.

If  $B = P$  and  $\ll = \preceq$ , the ideal  $(\downarrow_{(B, \ll)} x)$  will be denoted shortly  $\downarrow x$  and will be called *the principal ideal associated to  $x$* .

An auxiliary relation is said to be *approximating* iff

$$\forall x, y \in P, (x \not\ll y) \Rightarrow (\exists z \in P \mid z \ll x \text{ and } z \not\ll y). \quad (279)$$

Let  $(P, \preceq)$  be a pre-ordered set and let  $\ll$  be an approximating auxiliary relation on  $P$ . Then, we have

$$\forall x, y \in P, \quad (x \preceq y) \Leftrightarrow ((\downarrow_{(P, \ll)} x) \subseteq (\downarrow_{(P, \ll)} y)). \quad (280)$$

When a dcpo  $\mathbf{D} = (D, \leq)$  is given, a new fundamental binary relation  $\ll \subseteq D \times D$  can be defined, called *the way-below relation* (or *the order of approximation* of this dcpo). It is defined by<sup>[28]</sup>

$$\forall x, y \in D, \quad (x \ll y) :\Leftrightarrow (\forall E \subseteq_{\text{Dir}} D, y \leq \bigvee_D^{\wedge} E \Rightarrow \exists z \in E, x \leq z). \quad (281)$$

A subset  $B$  of a dcpo  $(D, \leq)$  is said to be *a basis of  $D$*  iff

$$(B \text{ basis of } D) \quad \forall x \in D, (\downarrow_{(B, \ll)} x) \subseteq_{\text{Dir}} B \text{ and } x = \bigvee_D^{\wedge} (\downarrow_{(B, \ll)} x). \quad (282)$$

<sup>28</sup>Directed subsets can be equivalently replaced by chains in this definition.

The elements of  $(\downarrow_{(B, \ll)} x)$  will be called *approximants of  $x$  relative to  $B$* .<sup>[29]</sup>

A dcpo  $(D, \leq)$  is said to be a *continuous dcpo* (or a *domain*) iff  $D$  admits a basis. In other words,  $(D, \leq)$  is continuous iff  $\ll$  is an approximating auxiliary relation.

A continuous Inf semi-lattice  $(D, \leq)$  is also necessarily meet-continuous, i.e. satisfies

$$(Meet\text{-}continuity) \quad \forall x \in L, \forall T \subseteq_{Dir} L, \quad x \wedge (\bigvee_L T) = \bigvee_L \{x \wedge t \mid t \in T\} \quad (283)$$

An element  $c$  in a directed complete partial order  $(D, \leq)$  is said to be *compact* (or *finite*) iff  $x \ll c$ . In other words,

$$(c\text{ compact}) \quad \forall E \subseteq_{Dir} D, (c \leq \bigvee_D E) \Rightarrow (\exists e \in E, c \leq e). \quad (284)$$

The subset of  $D$  formed by compact elements will be denoted  $D_c$ .

As soon as the join of a consistent set of compact elements exists, this join is necessarily compact.

Let  $(D, \leq)$  be a continuous dcpo and  $B$  a basis of  $D$ . The set  $D_c$  of compact elements of  $D$  is necessarily included in  $B$ .

$D$  is said to be an *algebraic dcpo* (or *algebraic domain*) iff the set  $D_c$  of compact elements of  $D$  forms a basis. In other words,  $D$  is algebraic iff any element of  $D$  is a join of compact elements, i.e.

$$(Algebraicity) \quad \forall x \in D, x = \bigvee_D \{u \in D_c \mid u \leq x\}. \quad (285)$$

An algebraic dcpo is also a continuous dcpo.

If  $(D, \leq)$  is an algebraic dcpo, the way-below relation can be reformulated as

$$\forall u, v \in D, u \ll v \Leftrightarrow (\exists c \in D_c, u \leq c \leq v). \quad (286)$$

A dcpo  $D$  is said to be a *continuous (resp. algebraic) Scott-domain* iff  $D$  is a consistently-complete continuous (resp. algebraic) dcpo.

A Scott-domain is necessarily bounded-complete.

Let  $(D, \leq)$  be a continuous dcpo. A basis  $B$  of  $D$  will be said to be *finitary* iff

$$\forall s \in B, (\downarrow_{(B, \ll)} s) \subseteq_{fin} D. \quad (287)$$

A continuous dcpo  $(D, \leq)$  will be said to be a *finitary domain* iff it admits a finitary basis.

## 7.5 Decomposition

An element  $x$  of an Inf semi-lattice  $L$  will be said to be a *meet-irreducible element* iff for  $y$  and  $z$  elements of  $L$  such that  $x = y \wedge z$ , we have necessarily  $x = y$  or  $x = z$ . An element  $x$  of an Inf semi-lattice  $L$  will be said to be a *meet-prime element* iff for  $y$  and  $z$  elements of  $L$  such that  $x \geq y \wedge z$ , we have necessarily  $x \geq y$  or  $x \geq z$ . We can define dually *join-irreducible elements* and *join-primes*. The set of meet-irreducible (resp. join-irreducible) elements in  $L$  will be denoted  $\wedge - IRR(L)$  (resp.  $\vee - IRR(L)$ ).

An element  $p$  in a poset is meet-irreducible iff  $p$  is maximal or  $\uparrow p \setminus \{p\}$  is a filter.

In a continuous Inf semi-lattice  $L$ , the set of non-identity meet-irreducibles  $\wedge - IRR(L) \setminus \{1\}$  is 'order-generating'. Note that a subset  $X$  is said to be order generating in  $L$  iff  $\forall x \in L, x = \bigwedge (\uparrow x \cap X)$ .

An element  $x$  of a down-complete Inf-semi lattice will be said to be a complete meet-irreducible element iff for any subset  $Q \subseteq P$ , we have  $(x = \bigwedge Q) \Rightarrow (\exists y \in Q \mid x = y)$ . The set of complete meet-irreducible elements will be denoted  $\wedge - Irr(L)$ .

Two other characterizations of completely meet-irreducible are given as follows. An element  $p$  of a poset

<sup>29</sup>We will adopt the following shortcut:  $\uparrow d := \uparrow_{(D, \ll)} d$  and  $\downarrow d := \downarrow_{(D, \ll)} d$ .

$L$  is completely irreducible iff either  $p$  is maximal in  $L$  but different from the top element or the set  $\uparrow p \setminus \{p\}$  has a least element [28, Definition I-4.21]. In a bounded complete algebraic domain  $L$ , an element  $p \in L$  is completely irreducible iff  $p$  is maximal in  $L \setminus \uparrow k$  for some compact element  $k$  [28, Proposition I-4.27].

In any bounded complete algebraic domain, the subset of non-identity completely meet-irreducible elements  $\wedge - Irr(L) \setminus \{1\}$  is the unique smallest order-generating subset [28, Theorem I-4.26].

The (complete) meet-prime elements in a bounded-complete Inf semi-lattice  $L$  are also (complete) meet-irreducible elements.

An element  $x$  of a coherently-complete partial order  $\mathbf{P} := (P, \leq)$  will be said to be a *complete join-prime element* of  $\mathbf{P}$  (resp. a complete join-irreducible element) iff for any subset  $Q \subseteq_{Coh} P$ , we have  $(x \sqsubseteq \bigvee Q) \Rightarrow (\exists y \in Q \mid x \leq y)$  (resp.  $(x = \bigvee Q) \Rightarrow (\exists y \in Q \mid x = y)$ ).

A domain  $D$  is said to be *prime algebraic* iff any element of  $D$  is a join of complete join-prime elements. In a finite lattice  $(L, \leq)$ , every element  $x \in L$  is a finite join of join-irreducible elements and a finite meet of meet-irreducible elements.

Let  $(L, \leq)$  be a lattice such that every element is the join of join-irreducible elements and the meet of meet-irreducible elements, then the property that every join-irreducible is join-prime<sup>[30]</sup> is equivalent to the distributivity of  $L$ , [50, Theorem 5], i.e.

$$\forall x, y, z \in L, \quad (x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z) \text{ and } x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z). \quad (288)$$

Let  $\mathbf{D} = (D, \leq)$  be a coherently-complete partial order. Then,  $D$  is prime algebraic iff it is a Scott domain and satisfies the conditional-distributivity laws [70]:

$$\forall X \subseteq_{Coh} D, \forall y \in D, \quad (\bigvee_D X) \wedge y = \bigvee_D \{(x \wedge y) \mid x \in X\}, \quad (289)$$

$$\forall X \subseteq D, \forall y \in D \mid \forall x \in X, \widehat{xy}^D, \quad (\bigwedge_D X) \vee y = \bigwedge_D \{(x \vee y) \mid x \in X\}. \quad (290)$$

Let  $\mathbf{D} = (D, \leq)$  be a finitary Scott domain. Then  $D$  is prime algebraic iff  $D$  satisfies the finite distributive law [70]:

$$\forall x, y, z \in D \mid \widehat{xy}^D, \quad (x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z). \quad (291)$$

A finitary domain  $(D, \leq)$  which satisfies the following conditional-distributivity properties

$$\widehat{xy}^D \Rightarrow z \wedge (y \vee x) = (z \wedge y) \vee (z \wedge x), \quad (292)$$

$$\widehat{xy}^D \text{ and } \widehat{xz}^D \Rightarrow x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z). \quad (293)$$

is called a *dI-domain* (see [72] for an extensive study of these domains).

Let  $\mathbf{P} = (P, \leq)$  be a pointed poset ( $\mathfrak{b}$  designates its bottom element). The set of *atoms* of  $\mathbf{P}$  is denoted  $\mathcal{A}_{\mathbf{P}} := \{a \in \mathbf{L} \mid a \succ \mathfrak{b}\}$ .  $\mathbf{P}$  is said to be *atomic*, iff

$$\forall x \in P, \quad \mathbb{A}(x) := (\mathcal{A}_{\mathbf{P}} \cap \downarrow_p x) \neq \emptyset. \quad (294)$$

If  $\mathbf{P} = (P, \leq)$  is a pointed coherently-complete poset,  $\mathbf{P}$  will be said to be *atomistic*, iff every element  $x$  of  $\mathbf{P}$  is the join of the atoms contained in  $x$ , i.e.

$$(\text{Atomisticity}) \quad \forall x \in P, \quad x = \bigvee_P \mathbb{A}(x) = \bigvee_P^{\bar{\wedge}} \{(\bigvee_P F) \mid F \subseteq_{fin} \mathbb{A}(s)\}. \quad (295)$$

A dI-domain in which the join-prime elements are atomic is called a *qualitative domain*. A qualitative domain is then necessarily atomistic.

Let  $\mathbf{L}$  be a coherently-complete partial order and an Inf semi-lattice, which is atomistic. Then, the following properties are equivalent: (1)  $L$  is meet-continuous, (2) the atoms of  $\mathbf{L}$  are compact, (3)  $\mathbf{L}$  is algebraic.

Let  $\mathbf{L}$  be a coherently-complete partial order and an Inf semi-lattice, which is atomistic and meet-continuous (or algebraic). Then,  $c$  is compact iff  $c$  is the supremum of finitely many atoms of  $\mathbf{L}$ .

<sup>30</sup>or every meet-irreducible is meet-prime

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