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THE MONOTONICITY OF THE SYSTOLE OF CONVEX RIEMANNIAN TWO-SPHERES

ALBERTO ABBONDANDOLO AND MARCO MAZZUCHELLI

ABSTRACT. We prove that the systole of the smooth boundary of a strictly convex ball in \mathbb{R}^3 is monotone with respect to the inclusion.

Throughout this note, the notion of convexity must be understood in the differentiable sense: A compact three-ball $B \subset \mathbb{R}^3$ with smooth boundary is strictly convex when there exists a smooth function $F : \mathbb{R}^3 \rightarrow [0, \infty)$ with positive definite Hessian at every point and such that $\partial B = F^{-1}(1)$. Equivalently, the boundary sphere $M = \partial B$, which will always be equipped with the Riemannian metric g that is the restriction of the ambient Euclidean metric, has strictly positive Gaussian curvature. The systole $\text{sys}(M) > 0$ is the length of the shortest closed geodesic of (M, g) . Our main result answers in dimension 3 a question that was posed to us by Yaron Ostrover:

Theorem 0.1. *Let $B_1 \subseteq B_2$ be two compact strictly convex three-balls in \mathbb{R}^3 with smooth boundary. Then $\text{sys}(\partial B_1) \leq \text{sys}(\partial B_2)$.*

The main ingredient of the proof is the observation that the systole of positively curved Riemannian two-spheres coincides with the classical Birkhoff min-max, as we will now prove. Let (M, g) be a Riemannian two-sphere. We denote the energy functional on the $W^{1,2}$ free loop space by

$$E : \Lambda M = W^{1,2}(S^1, M) \rightarrow [0, \infty), \quad E(\zeta) = \int_{S^1} \|\dot{\zeta}(t)\|_g^2 dt.$$

Here and in the following, we denote by $S^1 = \mathbb{R}/\mathbb{Z}$ the 1-periodic circle. We consider the unit sphere $S^2 \subset \mathbb{R}^3$. For each $z \in [-1, 1]$, we denote by $\gamma_z : S^1 \rightarrow S^2$ the parallel at latitude z , parametrized as

$$\gamma_z(t) = \left(\sqrt{1-z^2} \cos(2\pi t), \sqrt{1-z^2} \sin(2\pi t), z \right).$$

For each continuous map $u : [-1, 1] \rightarrow \Lambda M$ such that $E(u(0)) = E(u(1)) = 0$ there exists a unique continuous map $\tilde{u} : S^2 \rightarrow M$ such that $u(z) = \tilde{u} \circ \gamma_z$ for each $z \in [-1, 1]$. We denote by \mathcal{U} the space of such maps u whose associated \tilde{u} has degree 1. The Birkhoff min-max value

$$\text{bir}(M, g) = \inf_{u \in \mathcal{U}} \max_{z \in [-1, 1]} E(u(z))^{1/2}$$

is the length of some closed geodesic of (M, g) .

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Lemma 0.2. *On every positively curved closed Riemannian two-sphere (M, g) , we have $\text{bir}(M, g) = \text{sys}(M, g)$.*

Proof. Let $\gamma : S^1 \rightarrow M$ be a shortest closed geodesic of (M, g) parametrized with constant speed, so that $E(\gamma) = L(\gamma)^2 = \text{sys}(M, g)^2$. A theorem of Calabi–Cao [CC92] implies that γ is simple, that is, an embedding $\gamma : S^1 \hookrightarrow M$. We fix an orientation on M , and consider the corresponding complex structure of (M, g) . Namely, for every non-zero $v \in T_x M$, the tangent vector $Jv \in T_x M$ is obtained by rotating v in the positive direction of an angle $\pi/2$. We consider the vector field $\nu(t) = J\dot{\gamma}(t)$ orthogonal to $\dot{\gamma}(t)$. Notice that ν is a parallel vector field, since the complex structure J is parallel. If K_g denotes the Gaussian curvature of (M, g) , we have

$$d^2 E(\gamma)[\nu, \nu] = \int_{S^1} (\|\nabla_t \nu\|_g^2 - K_g \|\dot{\gamma}\|_g^2 \|\nu\|_g^2) dt = - \int_{S^1} K_g \|\dot{\gamma}\|_g^4 dt < 0. \quad (0.1)$$

We now consider Morse’s finite dimensional approximation of the free loop space (see, e.g., [Mil63]). We fix a positive integer k that is large enough so that $d(\zeta(t_0), \zeta(t_1)) < \text{inrad}(M, g)$ for all $\zeta \in \Lambda M$ with $E(\zeta) \leq E(\gamma) = \text{sys}(M, g)^2$ and for all $t_0, t_1 \in \mathbb{R}$ with $|t_1 - t_0| < 1/k$. Here, d denotes the Riemannian distance on (M, g) . We consider the open finite dimensional manifold

$$\Lambda_k M = \{ \mathbf{x} = (x_0, \dots, x_{k-1}) \in M \times \dots \times M \mid d(x_i, x_{i+1}) < \text{inrad}(M, g) \ \forall i \in \mathbb{Z}_k \}.$$

Such a manifold admits an embedding

$$\iota : \Lambda_k M \hookrightarrow \Lambda M, \quad \iota(\mathbf{x}) = \gamma_{\mathbf{x}},$$

where each restriction $\gamma_{\mathbf{x}}|_{[i/k, (i+1)/k]}$ is the shortest geodesic parametrized with constant speed joining x_i and x_{i+1} . We denote the restricted energy functional by

$$E_k = E \circ \iota : \Lambda_k M \rightarrow [0, \infty), \quad E_k(\mathbf{x}) = k \sum_{i \in \mathbb{Z}_k} d(x_i, x_{i+1})^2.$$

Let $\mathbf{x} := \iota^{-1}(\gamma)$. We consider the tangent vector $\mathbf{v} := (v_0, \dots, v_{k-1}) \in T_{\mathbf{x}}(\Lambda_k M)$ such that $v_i = \nu(i/k)$ for all $i \in \mathbb{Z}_k$. Inequality (0.1) readily implies that $d\iota(\mathbf{x})\mathbf{v}$ lies in the negative cone of the Hessian $d^2 E(\gamma)$, since

$$\begin{aligned} d^2 E_k(\mathbf{x})[\mathbf{v}, \mathbf{v}] &= \frac{d^2}{dz^2} \Big|_{z=0} E(\iota(\exp_{\mathbf{x}}(z\mathbf{v}))) \\ &\leq \frac{d^2}{dz^2} \Big|_{z=0} E(\exp_{\gamma(\cdot)}(z\nu(\cdot))) \\ &= d^2 E(\gamma)[\nu, \nu] \\ &< 0. \end{aligned} \quad (0.2)$$

Here, the exponential map in $\Lambda_k M$ is the one associated with the natural Riemannian metric $g \oplus \dots \oplus g$.

The complement $M \setminus \gamma$ has two connected components B_+ and B_- , each one diffeomorphic to a two-ball. The vector field ν points into one of them, say B_+ . We define the continuous map

$$w : [-1/3, 1/3] \rightarrow \Lambda_k M, \quad w(z) = \exp_{\mathbf{x}}(z\epsilon\mathbf{v}).$$

Notice that $w(0) = \mathbf{x}$. We fix $\epsilon > 0$ small enough so that, for all $z \in (0, 1/3]$, the loop $\iota(w(\pm z))$ is entirely contained in the open ball B_{\pm} , and by Equation (0.2) we have

$$E_k(w(z)) < E_k(w(0)) = \text{sys}(M, g)^2, \quad \forall z \in [-1/3, 1/3] \setminus \{0\}.$$

We now consider the open subspaces $U_+, U_- \subset \Lambda_k M$ given by

$$U_{\pm} = \Lambda_k M \cap (B_{\pm} \times \dots \times B_{\pm}).$$

We have $w(\pm 1/3) \in U_{\pm}$. The flow ϕ_s of the anti-gradient $-\nabla E_k$ is complete in positive time s in the sublevel set $E_k^{-1}([0, \text{sys}(M, g)^2])$. We claim that

$$\phi_s(w(\pm 1/3)) \in U_{\pm}, \quad \forall s \geq 0.$$

Indeed, assume by contradiction that there exists $s_0 > 0$ such that $\phi_{s_0}(w(\pm 1/3)) \in \partial U_{\pm}$, and take s_0 to be the minimal such time. If $\mathbf{y} := \phi_{s_0}(w(\pm 1/3))$, the components of the anti-gradient vector $\mathbf{z} := -\nabla E_k(\mathbf{y})$ are given by

$$z_i = 2(\dot{\gamma}_{\mathbf{y}}(\frac{i}{k}^+) - \dot{\gamma}_{\mathbf{y}}(\frac{i}{k}^-)), \quad \forall i \in \mathbb{Z}_k.$$

Since $\mathbf{y} \in \partial U_{\pm}$, at least one of its components y_i must belong to ∂B_{\pm} . Assume that all the y_i 's belong to ∂B_{\pm} , and therefore they are of the form $y_i = \gamma(t_i)$ for some $t_i \in S^1$. In this case, we have $z_i = \lambda_i \dot{\gamma}(t_i)$ for some $\lambda_i \in \mathbb{R}$; but this is impossible, since it would imply that all the components of $\phi_s(w(\pm 1/3))$ belong to ∂B_{\pm} for all $s \in \mathbb{R}$, and thus that $\phi_s(w(\pm 1/3))$ belong to ∂U_{\pm} for all $s \in \mathbb{R}$. Therefore at least one component $y_i \in \partial B_{\pm}$ is adjacent to a component in the interior $y_{i-1} \in B_{\pm}$. However, this implies that the vector z_i points inside B_{\pm} , and therefore $\phi_{s_0-\delta}(w(\pm 1/3)) \notin U_{\pm}$ for all $\delta > 0$ small enough, contradicting the minimality of s_0 .

We set $\delta := \min\{\text{injrad}(M, g), \text{sys}(M)/(4k)\}$. Since $E_k(\phi_s(w(\pm 1/3))) < \text{sys}(M, g)^2$ for all $s \geq 0$, and since $\text{sys}(M, g)^2$ is the smallest positive critical value of E_k , we can fix a large enough $s > 0$ such that $E_k(\phi_s(w(\pm 1/3))) < \delta^2$. We extend w to a map $w : [-2/3, 2/3] \rightarrow \Lambda_k M$ by setting

$$w(\pm z) = \phi_{(3z-1)s}(w(\pm 1/3)), \quad \forall z \in [1/3, 2/3].$$

Notice that $w(\pm z) \in U_{\pm}$ for all $z \in (0, 2/3]$, and $E_k(w(\pm 2/3)) < \delta^2$. We set

$$\mathbf{y}^{\pm} = (y_0^{\pm}, \dots, y_{k-1}^{\pm}) := w(\pm 2/3).$$

For each $r \in [0, 1]$, we define $\mathbf{y}^{\pm}(r) = (y_0^{\pm}(r), \dots, y_{k-1}^{\pm}(r))$ by

$$y_i^{\pm}(r) := \exp_{y_0^{\pm}}((1-r) \exp_{y_0^{\pm}}^{-1}(y_i^{\pm})).$$

Notice that $\mathbf{y}^{\pm}(0) = \mathbf{y}^{\pm}$, $\mathbf{y}^{\pm}(r) \in U_{\pm}$, and

$$E_k(\mathbf{y}^{\pm}(r)) = k \sum_{i \in \mathbb{Z}_k} d(y_i^{\pm}(r), y_{i+1}^{\pm}(r))^2 < 4k^2 \delta^2 \leq \text{sys}(M, g)^2, \quad \forall r \in [0, 1],$$

$$E_k(\mathbf{y}^{\pm}(1)) = 0.$$

We extend w to a continuous map $w : [-1, 1] \rightarrow \Lambda_k M$ by setting

$$w(\pm z) = \mathbf{y}^{\pm}(3z-2), \quad \forall z \in [2/3, 1].$$

Finally, we define $u := \iota \circ w : [-1, 1] \rightarrow \Lambda M$. Notice that the associated continuous map $\tilde{u} : S^2 \rightarrow M$ has degree 1; indeed, the preimage $u^{-1}(\gamma(t))$ is a singleton for every $t \in S^1$, and the restriction of u to a neighborhood of $u^{-1}(\gamma)$ is a homeomorphism onto its image. Therefore $u \in \mathcal{U}$, and

$$\text{bir}(M, g) \leq \max_{z \in [-1, 1]} E(u(z))^{1/2} = E(u(0))^{1/2} = \text{sys}(M, g).$$

On the other hand, $\text{bir}(M, g)^2$ is a positive critical value of E , and therefore

$$\text{bir}(M, g) \geq \text{sys}(M, g). \quad \square$$

Proof of Theorem 0.1. We set $M_i := \partial B_i$, $i = 1, 2$. Since the regions $B_1 \subset B_2$ are strictly convex, for each $x \in M_2$ there exists a unique $\pi(x) \in M_1$ such that

$$\|x - \pi(x)\| = \min_{y \in M_1} \|x - y\|.$$

The map $\pi : M_2 \rightarrow M_1$ is a 1-Lipschitz homeomorphism with respect to the Riemannian metrics g_i on M_i that are restriction of the ambient Euclidean metric. In particular, for every $W^{1,2}$ curve $\gamma_2 : S^1 \rightarrow M_2$, if we denote by $\gamma_1 := \pi \circ \gamma_2$ its image in M_1 , we have

$$\int_{S^1} \|\dot{\gamma}_2(t)\|^2 dt \geq \int_{S^1} \|\dot{\gamma}_1(t)\|^2 dt$$

We denote by \mathcal{U}_1 and \mathcal{U}_2 the family of maps involved in the definition of the Birkhoff min-max values of M_1 and M_2 respectively. Notice that $\pi \circ u \in \mathcal{U}_1$ for all $u \in \mathcal{U}_2$. Therefore, if we denote the energy of $W^{1,2}$ loops $\gamma : S^1 \rightarrow \mathbb{R}^3$ by

$$E(\gamma) = \int_{S^1} \|\dot{\gamma}(t)\|^2 dt,$$

we have

$$\text{bir}(M_2) = \inf_{u \in \mathcal{U}_2} \max_{z \in [-1,1]} E(u(z))^{1/2} \geq \inf_{u \in \mathcal{U}_2} \max_{z \in [-1,1]} E(\pi \circ u(z))^{1/2} \geq \text{bir}(M_1).$$

This, together with Lemma 0.2, implies that $\text{sys}(M_2) \geq \text{sys}(M_1)$. \square

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