On the homogeneous ergodic bilinear averages with 1-bounded multiplicative weights

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ON THE HOMOGENEOUS ERGODIC BILINEAR AVERAGES WITH 1-BOUNDED MULTIPLICATIVE WEIGHTS

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Abstract. We establish a generalization of Bourgain double recurrence theorem and ergodic Bourgain-Sarnak's theorem by proving that for any aperiodic 1-bounded multiplicative function $\boldsymbol{\nu}$, for any map T acting on a probability space $(X, \mathcal{A}, \mu)$, for any integers a, b , for any $f, g \in L^2(X)$, and for almost all $x \in X$, we have

$$\frac{1}{N}\sum_{n=1}^{N} \boldsymbol{\nu}(n) f(T^{an}x) g(T^{bn}x) \xrightarrow[N\to+\infty]{} 0.$$

We further present with proof the key ingredients of Bourgain's proof of his double recurrence theorem.

1. Introduction

The purpose of this paper is to consider the homogeneous ergodic bilinear averages with arithmetical weight. But, we focus only on the case when the weight is given by 1-bounded multiplicative function. This will give an another generalization of Bourgain's double recurrence theorem (BDRT) [9] and Bourgain-Sarnak's ergodic theorem [27]. For a simple proof of this later theorem, we refer to [3]. But, we stress that our proof follows that of Bourgain. As a consequence, we present with proof its key ingredients.

A careful understanding of Bourgain's proof gives that for any invertible measure preserving transformation T, acting on a probability space $(X, \mathcal{B}, \mu)$, for any $f \in L^r(X, \mu)$, $g \in L^{r'}(X, \mu)$ such that $\frac{1}{r} + \frac{1}{r'} = 1$, for any non-constant polynomials $P(n), n \in \mathbb{Z}$, taking integer values, and for almost all $x \in X$, we have,

$$\frac{1}{N}\sum_{n=1}^{N} f(T^{P(n)}x) g(T^{n}x).$$

(See Lemma 3.4 (equation (8)) and Lemma 5.4). The study of the almost everywhere convergence of the homogeneous ergodic bilinear averages with weight was initiated by I. Assani, D. Duncan, and R. Moore in [5]. Therein, the authors proved a Wiener-Wintner version of BDRT, that is, the exponential sequences $(e^{2\pi i n t})_{n\in\mathbb{Z}}$ are good weight for the homogeneous ergodic bilinear averages. Subsequently, I. Assani and R. Moore showed that the polynomials exponential sequences $\left(e^{2\pi i P(n)}\right)_{n\in\mathbb{Z}}$ are also uniformly good weights for the homogeneous ergodic bilinear averages [6]. One year later, I. Assani [7] and P. Zorin-Kranich [34] proved independently that the nilsequences are uniformly good weights for the homogeneous ergodic bilinear averages. Their proofs depend also on Bourgain's theorem. Very

2010 *Mathematics Subject Classification.* Primary: 37A30; Secondary: 28D05, 5D10, 11B30, 11N37, 37A45.

Key words and phrases. Fürstenberg's multilinear ergodic averages, Multiplicative function, , Birkhoff ergodic theorem, Bourgain's double recurrence theorem, Zhan's estimation, Davenport-Hua's estimation, Bourgain-Sarnak's ergodic theorem.
December 9, 2020.

recently, the author extended Bourgain-Sarnak theorem by proving that the Möbius and Liouville functions are a good weight for the homogeneous ergodic bilinear averages [2]. But there is a gap in the proof. Here, we will generalize that theorem to the bounded multiplicative function and we will thus fill the gap. Let us point out that therein, the author extended Assani's theorem [4] by proving , independently of the gap, that the Möbius and Liouville functions are a good weight for the homogeneous Fürstenberg's multilinear ergodic averages provided that the map is weakly mixing and its restriction to its Pinsker algebra has singular spectrum.

Here, our proof follows closely Bourgain's proof of his double recurrence theorem which in turn is based on his method and the Calderón transference principal. For a nice account on this method, we refer to [26], [31]. Despite the fact that the classical spectral analysis can not be applied to study of Fürstenberg's multilinear ergodic average, some kind of spectral analysis tools based on the Fourier transform can be adapted for its studies. This is accomplished by applying Caldéron principal and the discrete Fourier transform which can be seen as a spectral isomorphism. We notice that in this setting, the dynamics on the diagonal in Fürstenberg's ergodic average is interpreted as an operation on the kernels. The kernel is an average mass on the particles $X = \left\{x_1, \cdots, x_N\right\}$ and the operation act on the diagonal of $X \times X$ and it is given as a kernel the Dirac mass (x_i, x_i), $i = 1, \cdots, N$. We emphasize that the product on the observable functions in ℓ^2 is interpreted as a convolution. We thus believe that, as in Bourgain's proof, the harmonic analyis methods can be useful to address the problem of the convergence almost everywhere of Fürstenberg's multilinear ergodic average. This approach is raised in [1].

We recall that the problem of the convergence almost everywhere (a.e.) of the ergodic multilinear averages was initiated by Fürstenberg in [18, Question 1 p.96]. Bourgain answered that question by proving the following:

Let T be a map acting on a probability space $(X, \mathcal{A}, \nu)$, and $a, b \in \mathbb{Z}$, then for any $f, g \in L^\infty(X)$, the averages

$$\frac{1}{N}\sum_{n=1}^{N} f(T^{an}x)g(T^{bn}x)$$

converge for almost every x. Here, we will state and formulate the fundamental ingredients in Bourgain's proof of his theorem. We will also present the proof of the main ingredients. In fact, our proof is essentially based on those ingredients.

2. Set up and Tools

An arithmetical function is a complex-valued function defined on the positive integers. It is also called a number-theoretic function. The function $\boldsymbol{\nu}$ is called multiplicative if $\boldsymbol{\nu}$ is not identically zero and if

$$\boldsymbol{\nu}(nm) = \boldsymbol{\nu}(n)\boldsymbol{\nu}(m) \text{ whenever } (n, m) = 1.$$

(n, m) stand for the greatest common divisor of n and m.

$\boldsymbol{\nu}$ is called completely multiplicative if we also have

$$\boldsymbol{\nu}(nm) = \boldsymbol{\nu}(n)\boldsymbol{\nu}(m), \ \forall n, m \in \mathbb{N}.$$

An easy example of function which is multiplicative but not completely multiplicative can be given by the function $f(n) = \lfloor\sqrt{n}\rfloor - \lfloor\sqrt{n-1}\rfloor$, where, as customary, $\lfloor \, . \, \rfloor$ is the integer part.

For the bounded completely multiplicative, the Liouville function is the famous example. This function is defined for the positive integers n by

$$\boldsymbol{\lambda}(n) = (-1)^{\Omega(n)},$$

where $\Omega(n)$ is the length of the word n is the alphabet of prime, that is, $\Omega(n)$ is the number of prime factors of n counted with multiplicities.
Its allies is the Möbius function which is only multiplicative. It is given by

$$\boldsymbol{\mu}(n) = \begin{cases} 1 \text{ if } n = 1; \\ \boldsymbol{\lambda}(n) \text{ if } n \text{ is the product of } r \text{ distinct primes;} \\ 0 \text{ if not} \end{cases} \tag{1}$$

These two functions are of great importance in number theory since the Prime Number Theorem is equivalent to

$$\sum_{n\leq N} \boldsymbol{\lambda}(n) = \mathrm{o}(N) = \sum_{n\leq N} \boldsymbol{\mu}(n). \tag{2}$$

Furthermore, there is a connection between these two functions and Riemann zeta function, namely

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\boldsymbol{\mu}(n)}{n^s} \text{ for any } s \in \mathbb{C} \text{ with } \mathrm{Re}(s) > 1.$$

Moreover, Littlewood proved that the estimate

$$\left| \sum_{n=1}^{x} \boldsymbol{\mu}(n) \right| = O\left(x^{\frac{1}{2}+\varepsilon}\right) \qquad \text{as} \quad x \longrightarrow +\infty, \quad \forall \varepsilon > 0$$

is equivalent to the Riemann Hypothesis (RH) ([30, pp.315]).

We recall that the proof of Sarnak-Bourgain theorem [27] is based on the following Davenport-Hua's estimation [14], [19, Theorem 10.]: for each $A > 0$, for any $k \geq 1$, we have

$$\max_{z\in\mathbb{T}} \left| \sum_{n\leq N} z^{n^k} \boldsymbol{\mu}(n) \right| \leq C_A \frac{N}{\log^A N} \text{ for some } C_A > 0. \tag{3}$$

We refer to [3] and [13] for this proof.

The inequality (3) can be established also for the Liouville function by applying carefully the following identity:

$$\boldsymbol{\lambda}(n) = \sum_{d:d^2|n} \boldsymbol{\mu}\Big(\frac{n}{d^2}\Big).$$

We further recall that the multiplicative function $\boldsymbol{\nu}$ is said to be aperiodic if

$$\frac{1}{N} \sum_{n=1}^{N} \boldsymbol{\nu}(an+b) \xrightarrow[N\to+\infty]{} 0,$$

for any $(a, b) \in \mathbb{N}^* \times \mathbb{N}$. By Davenport's theorem [14] and Bateman-Chowla's theorem [8], the Möbius and Liouville functions are aperiodic.

3. Wiener-Wintner's version of Daboussi-Katai-Bourgain-Sarnak-Ziegler's (DKBSZ) criterion (WWDKBSZ)

For the proof of our main result, we need a straightforward generalization à la Wiener-Wintner of the so-called DKBSZ criterion. This criterion is based on the results of Bourgain-Sarnak-Ziegler [10, Theorem 2], and Katai [21] , which in turn develop some ideas of Daboussi (presented in [15], [16]). we state now the WWDKBSZ criterion in the following form.

Proposition 3.1 (WWKBSZ criterion)**.** Let $(X, \mathcal{A}, \mu)$ be a Lebesgue probability space and T be an invertible measure preserving transformation. Let $\boldsymbol{\nu}$ be a multiplicative function, f be in L^∞ with $\|f\|_\infty \leq 1$ and $\varepsilon > 0$. Suppose that for almost all point $x \in X$ and for all different prime numbers p and q less than $\exp(1/\varepsilon)$, we have

$$\limsup_{N\to\infty} \sup_t \left| \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i n(p-q)t} f(T^{pn}x) f(T^{qn}x) \right| < \varepsilon, \tag{4}$$

then, for almost all $x \in X$, we have

$$\limsup_{N\to\infty} \sup_t \left| \frac{1}{N} \sum_{n=1}^{N} \boldsymbol{\nu}(n) e^{2\pi i n t} f(T^n x) \right| < 2\sqrt{\varepsilon \log 1/\varepsilon}. \tag{5}$$

Proof. The proof is verbatim the same as [10, Theorem 2], except that at the equation (2.7) one need to apply the following elementary inequality: for any two bounded positive functions F and G, we have

$$\sup(F(x) + G(x)) \leq \sup(F(x)) + \sup(G(x)).$$

□

3.1. **Gowers Norms.** Gowers norms are a great tools in additive number theory, combinatorics and ergodic theory. We present its definition for the shift on the set of integers.

Let $d \geq 1$ and $C_d = \{0,1\}^d$. If $\boldsymbol{h} \in \mathbb{Z}^d$ and $\boldsymbol{c} \in C_d$, then $\boldsymbol{c}.\boldsymbol{h} = \sum_{i=1}^d c_i h_i$. Let $\big(f_{\boldsymbol{c}}\big)_{\boldsymbol{c}\in C_d}$ be a family of bounded functions that are finitely supported, that is, for each $\boldsymbol{c} \in C_d$, $f_{\boldsymbol{c}}$ is in $L_c^\infty(\mathbb{Z})$ the subspace of functions that are finitely supported. The Gowers inner product is given by

$$\Big\langle \big(f_{\boldsymbol{c}}\big) \Big\rangle_{U^d(\mathbb{Z})} = \sum_{\boldsymbol{x},\boldsymbol{h}\in G^{d+1}} \prod_{\boldsymbol{c}\in C_d} \mathcal{C}^{|\boldsymbol{c}|} f_{\boldsymbol{c}}(g + \boldsymbol{c}.\boldsymbol{h}),$$

where $|\boldsymbol{c}| = \boldsymbol{c}.\mathbf{1}$, $\mathbf{1} = (1, 1, \cdots, 1) \in C_d$ and $\mathcal{C}$ is the conjugacy anti-linear operator. If all $f_{\boldsymbol{c}}$ are the same function f then the Gowers norms of f is defined by

$$||f||_{U^d(\mathbb{Z})}^{2^d} = \Big\langle \big(f\big) \Big\rangle_{U^d(\mathbb{Z})}.$$

The fact that $||.||_{U^d(\mathbb{Z})}$ is a norm for $d \geq 2$ follows from the following generalization of Cauchy-Bunyakovski-Schwarz inequality for the Gowers inner product.

Proposition 3.2. (Cauchy-Bunyakovskii-Gowers-Schwarz inequality)

$$\Big\langle \big(f_c\big) \Big\rangle_{U^d(\mathbb{Z})} \leq \prod_{c\in C_q} ||f_c||_{U^d(\mathbb{Z})}.$$

The proof of Cauchy-Bunyakovskii-Gowers-Schwarz inequality can be obtain easily by applying inductively Cauchy-Bunyakovskii-Schwarz inequality. Indeed, it easy to check that we have

$$\Big\langle (f_c) \Big\rangle_{U^d(\mathbb{Z})} \leq \prod_{j=0,1} \Big| \langle (f_{\pi_{i,j}(\boldsymbol{c})}) \Big\rangle_{U^d(G)} \Big|^{\frac{1}{2}},$$

For all $i = 1, \cdots, d-1$, where $\pi_{i,j}(\boldsymbol{c}) \in C_d$ is formed from $\boldsymbol{c}$ by replacing the i^{th} coordinates with j. Iterated this, we obtain the complete proof of Proposition 3.2.

Combining Cauchy-Bunyakovskii-Gowers-Schwarz inequality with the binomial formula and the multilinearity of the Gowers inner product one can easily check that the triangle inequality for $\|.\|_{U^d(G)}$ holds. We further have

$$\|f\|_{U^d(\mathbb{Z})} \leq \|f\|_{U^{d+1}(\mathbb{Z})},$$

by applying Cauchy-Bunyakovskii-Gowers-Schwarz inequality with $f_{\boldsymbol{c}} = 1$ if $\boldsymbol{c}_d = 0$ and $f_{\boldsymbol{c}} = 1$ if $\boldsymbol{c}_d = 1$. The Gowers norms are also invariant under the shift and conjugacy.

We define also the discrete derivative of function $f : \mathbb{Z} \to \mathbb{C}$ by putting

$$\partial_h(f) = f(x+h).\overline{f}(x),$$

for all $h, x \in \mathbb{Z}$. We can thus write the Gowers norm of f as follows

$$\|f\|^{2^d}_{U^d(\mathbb{Z})} = \int_{G^{d+1}} \partial_{h_1}\partial_{h_2}\cdots\partial_{h_d}(f)(x)d\boldsymbol{h}dx.$$

If further f take values on $\mathbb{R}/\mathbb{Z}$ and

$$\partial_{h_1}\partial_{h_2}\cdots\partial_{h_{d+1}}(f)(x) = 0,$$

for all $h_1, \cdots, h_{d+1}, x \in G$, then f is said to be a polynomial function of degree at most d. The degree of f is denoted by $d^\circ(f)$.

According to this it is easy to see that for any function f and any polynomial function ϕ of degree at most d, we have

$$\big|\big|e^{2\pi i\phi(x)}f(x)\big|\big|_{U^d(\mathbb{Z})} = \big|\big|f\big|\big|_{U^d(\mathbb{Z})}.$$

Therefore

$$\sup_{\phi,\ d^\circ(\phi)\leq d} \Big| \int_G e^{2\pi i\phi(x)}f(x)dx \Big| \leq \big|\big|f\big|\big|_{U^d(\mathbb{Z})}. \tag{6}$$

In application and here we need to define the Gowers norms for a bounded function defined on $\big\{1, \cdots, N\big\}$. For that, if f is a bounded function defined on $\big\{1, \cdots, N-1\big\}$, we put

$$\big|\big|f\big|\big|_{U^d[N]} = \frac{\big|\big|\widetilde{f}\big|\big|_{U^d(\mathbb{Z}/2^d.N\mathbb{Z})}}{\big|\big|\,\mathbb{1}_{[N]}\big|\big|_{U^d(\mathbb{Z}/2^d.N\mathbb{Z})}},$$

where $\widetilde{f} = f(x).\,\mathbb{1}_{[N]}$, $x \in \mathbb{Z}/2^d.N\mathbb{Z}$, $\mathbb{1}_{[N]}$ is the indicator function of $\big\{1, \cdots, N\big\}$. For more details on Gowers norms, we refer to [28], [29].

The sequence f is said to have a small Gowers norms if for any $d \geq 1$,

$$\big|\big|f\big|\big|_{U^d[N]} \xrightarrow[N\to+\infty]{} 0.$$

An example of sequences of small Gowers norms that we shall need here is given by Thue-Morse and Rudin-Shapiro sequences. This result is due J. Konieczny [24].

The notion of Gowers norms can be extended to the dynamical systems as follows.

Let T is be an ergodic measure preserving transformation on X. Then, for any $k \geq 1$, the Gowers seminorms on $L^\infty(X)$ are defined inductively as follows

$$|||f|||_1 = \Big| \int f d\mu \Big|;$$

$$|||f|||_{k+1}^{2^{k+1}} = \lim \frac{1}{H} \sum_{l=1}^{H} |||\overline{f}.f \circ T^l|||_k^{2^k}.$$

For each $k \geq 1$, the seminorm $|||.|||_k$ is well defined. Those norms are also called Gowers-Host-Kra's seminorms. For details, we refer the reader to [20]. Notice that the definitions of Gowers seminorms can be also easily extended to non-ergodic maps.

The importance of the Gowers-Host-Kra's seminorms in the study of the nonconventional multiple ergodic averages is due to the existence of T-invariant sub-σ-algebra $\mathcal{Z}_{k-1}$ of X that satisfies

$$\mathbb{E}(f|\mathcal{Z}_{k-1}) = 0 \Longleftrightarrow |||f|||_k = 0.$$

This was proved by Host and Kra in [20]. The existence of the factors $\mathcal{Z}_k$ was established by Host and Kra and independently by Ziegler in [33].

At this point, we are able to state the second main ingredient need it in our proof. It is due to Assani-Dacuna and Moore [5]. This result extend à la Wiener-Wintner Bourgain's double recurrence theorem (WWBDRT).

Proposition 3.3. Let (X, μ, T) be an ergodic dynamical system and a, b distinct non-zero integers. Then, for any $f, g \in L^\infty$ with $\min\{\|f_1\|_{U^2}, \|g\|_{U^2}\} = 0$, there exist a measurable set X' of full measure such ,for any $x \in X'$, we have

$$\sup_{|z|=1} \Big| \frac{1}{N} \sum_{n=0}^{N-1} z^n f(T^{an}x) g(T^{bn}x) \Big| \xrightarrow[N \to +\infty]{} 0.$$

Proposition 3.3 has been extended to the nilsequences independently by I. Assani [7] and A. Zorin-Kranich [34]. We refer to [7] or [34] for the definition of nilsequences.

We need also the following straightforward lemma.

Lemma 3.4. Let T is be an ergodic measure preserving transformation on $(X, \mathcal{B}, \mu)$ and $f \in L^\infty(X, \mu)$. Then

$$\limsup_{N \longrightarrow +\infty} \Big| \frac{1}{N} \sum_{n=0}^{N-1} z^n f(T^n x) \Big| \leq \|f\|_{U^2}. \tag{7}$$

The previous lemma can be extended to the nilsequences. We thus have, for any k-nilsequence (b_n).

$$\limsup_{N \longrightarrow +\infty} \Big| \frac{1}{N} \sum_{n=0}^{N-1} b_n f(T^n x) \Big| \leq \|f\|_{U^{k+1}}. \tag{8}$$

The proof can be obtained by applying by induction combined with the classical van der Corput inequality [23, p.25].

4. Some tools on the oscillation method and Calderón transference principle

Let $k \geq 2$ and $(X, \mathcal{A}, T_i, \mu)_{i=1}^k$ be a family of dynamical systems, that is, for each $i = 1, \cdots, k$, T_i is a measure-preserving transformation, $(\forall A \in \mathcal{A},\ \mu(T_i^{-1}A) = \mu(A).)$. The sequence of complex number (a_n) is said to be good weight in $L^{p_i}(X, \mu)$, $p_i \geq 1$, $i = 1, \cdots, k$, with $\sum_{i=1}^k \frac{1}{p_i} = 1$, if for any $f_i \in L^{p_i}(X, \mu)$, $i = 1, \cdots, k$, the ergodic k-multilinear averages

$$\frac{1}{N}\sum_{j=1}^{N} a_j \prod_{i=1}^{k} f_i(T_i^j x)$$

converges a.e.. The maximal multilinear ergodic inequality is said to hold in $L^{p_i}(X, \mu)$, $p_i \geq 1$, $i = 1, \cdots, k$, with $\sum_{i=1}^k \frac{1}{p_i} = 1$, if for any $f_i \in L^{p_i}(X, \mu)$, $i = 1, \cdots, k$, the maximal function given by

$$M(f_1, \cdots, f_k)(x) = \sup_{N\geq 1} \Big|\frac{1}{N}\sum_{j=1}^{N} a_j \prod_{i=1}^{k} f_i(T_i^j x)\Big|$$

satisfy the weak-type inequality

$$\sup_{\lambda>0}\left(\lambda\mu\Big\{x \ :\ M(f)(x) > \lambda\Big\}\right) \leq C\prod_{i=1}^{k}\|f_i\|_{p_i},$$

where C is an absolutely constant.

As far as the author know, it is seems that it is not known whether the classical maximal multilinear ergodic inequality ($a_n = 1$, for each n) holds for the general case $n \geq 3$. Nevertheless, we have the following Calderón transference principal in the homogeneous case.

Proposition 4.1. Let (a_n) be a sequence of complex number and assume that for any $\phi, \psi \in \ell^2(\mathbb{Z})$, for any non-constant polynomials P, Q mapping natural numbers to themselves, for any $1 \leq p, q, r \leq +\infty$ such that $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, we have

$$\Big\| \sup_{N\geq 1} \Big|\frac{1}{N}\sum_{n=1}^{N} a_n \phi(j + P(n))\psi(j + Q(n))\Big|\Big\|_{\ell^r(\mathbb{Z})} \leq C.\|\phi\|_{\ell^p(\mathbb{Z})}\|\psi\|_{\ell^q(\mathbb{Z})},$$

where C is an absolutely constant. Then, for any dynamical system $(X, \mathcal{A}, T, \mu)$, for any $f \in L^p(X, \mu)$ and $g \in L^q(X, \mu)$, we have

$$\Big\| \sup_{N\geq 1} \Big|\frac{1}{N}\sum_{n=1}^{N} a_n f(T^{P(n)}x)g(T^{Q(n)}x)\Big|\Big\|_r \leq C\|f\|_p\|g\|_q.$$

We further have

Proposition 4.2. Let (a_n) be a sequence of complex number and assume that for any $\phi, \psi \in \ell^2(\mathbb{Z})$, for any $\lambda > 0$, for any integer $J \geq 2$, for any non-constant polynomials P, Q mapping natural numbers to themselves, for any $1 \leq p, q \leq +\infty$

such that $\frac{1}{p}+\frac{1}{q}=1$, , we have

$$\sup_{\lambda>0}\Big(\lambda\Big|\Big\{1\leq j\leq J\ :\ \sup_{N\geq 1}\Big|\frac{1}{N}\sum_{n=1}^{N}a_n\phi(j+P(n))\psi(j+Q(n))\Big|>\lambda\Big\}\Big|\Big)\leq C\|\phi\|_{\ell^p(\mathbb{Z})}\|\psi\|_{\ell^q(\mathbb{Z})},$$

where C is an absolutely constant. Then, for any dynamical system $(X,\mathcal{A},T,\mu)$, for any $f,g\in L^2(X,\mu)$, we have

$$\sup_{\lambda>0}\Big(\lambda\mu\Big\{x\in X\ :\ \sup_{N\geq 1}\Big|\frac{1}{N}\sum_{n=1}^{N}a_nf(T^{P(n)}x)g(T^{Q(n)}x)\Big|>\lambda\Big\}\Big)\leq C\|f\|_2.\|g\|_2.$$

It is easy to check that Proposition 4.1 and 4.2 hold for the homogeneous k-multilinear ergodic averages, for any $k\geq 3$. Moreover, it is easy to state and to prove the finitary version where $\mathbb{Z}$ is replaced by $\mathbb{Z}/J\mathbb{Z}$ and the functions ϕ and ψ with J-periodic functions.

5. Main result and its proof

The subject of this section is to state and to prove the main result of this paper and its consequences. We begin by stating our main result.

Theorem 5.1. Let $(X,\mathcal{A},\mu,T)$ be an ergodic dynamical system, let $\boldsymbol{\nu}$ be an aperiodic 1-bounded multiplicative function and $a,b\in\mathbb{Z}$. Then, for any $f,g\in L^2(X)$, for almost all $x\in X$,

$$\frac{1}{N}\sum_{n=1}^{N}\boldsymbol{\nu}(n)f(T^{an}x)g(T^{bn}x)\xrightarrow[N\to+\infty]{}0,$$

Consequently, we obtain the following corollary.

Corollary 5.2 ([2])**.** Let $(X,\mathcal{A},\mu,T)$ be an ergodic dynamical system, and T_1,T_2 be powers of T. Then, for any $f,g\in L^2(X)$, for almost all $x\in X$,

$$\frac{1}{N}\sum_{n=1}^{N}\boldsymbol{\nu}(n)f(T_1^nx)g(T_2^nx)\xrightarrow[N\to+\infty]{}0,$$

where $\boldsymbol{\nu}$ is the Liouville function or the Möbius function.

Before proceeding to the proof of Theorem (5.1), we recall the following notations.

Let T be a map acting on a probability space $(X,\mathcal{A},\mu)$ and $\boldsymbol{\nu}$ be an aperiodic 1-bounded multiplicative function. For any any $\rho>1$, we will denote by I_ρ the set $\Big\{(\lfloor\rho^n\rfloor),n\in\mathbb{N}\Big\}$. The maximal functions are defined by

$$M_{N_0,\bar{N}}(f,g)(x)=\sup_{\substack{N_0\leq N\leq\bar{N}\\ N\in I_\rho}}\Big|\frac{1}{N}\sum_{n=1}^{N}\boldsymbol{\nu}(n)f(T^nx)g(T^{-n}x)\Big|,$$

and

$$M_{N_0}(f,g)(x)=\sup_{\substack{N\geq N_0\\ N\in I_\rho}}\Big|\frac{1}{N}\sum_{n=1}^{N}\boldsymbol{\nu}(n)f(T^nx)g(T^{-n}x)\Big|.$$

Obviously,

$$\lim_{\bar{N}\longrightarrow+\infty} M_{N_0,\bar{N}}(f,g)(x) = M_{N_0}(f,g)(x).$$

For the shift $\mathbb{Z}$-action, the maximal functions are denoted by $m_{N_0,\bar{N}}(\phi,\psi)$ and $m_{N_0}(\phi,\psi)$. We denote by $\ell^p(J)$, $p>1$, the subspace of $\ell^p(\mathbb{Z})$, of the observable functions with finite support subset of $[0,J]$. For each $0<\delta<1$, we define a function σ_δ by

$$\sigma_\delta(t) = \theta(\delta,t) - \theta(2\delta,t),$$

where $\theta(\delta,t)$ is the half of bumpfunction with support $(\delta,2\delta]$ and $\theta(\delta,2\delta)=1$. We thus get

$$\sum_{\substack{0<\delta<1\\ \text{dyadic}}} \sigma_\delta(t) = 1, \text{ for each } t\in(0,1].$$

For any $N\in\mathbb{N}^*$, $x\in[1,J]$ and $\theta\in[0,1)$, we put

$$P_{x,N}(\theta) = \frac{1}{N}\sum_{n=x-N}^{x-1} e^{2\pi i n\theta}\psi(n),$$

we further set

$$\boldsymbol{\nu}_x(n) = \boldsymbol{\nu}(n+x),\ \forall n\in\mathbb{N}.$$

For a finite subset E of the torus $\mathbb{T}$ and $\epsilon>0$, we denote its ϵ-neighbourhood by $E(\epsilon)$. We recall that $E(\epsilon)$ is given by

$$E(\epsilon) = \Big\{\lambda\in\mathbb{T}\ :\ \min_{\gamma\in E}\big|\lambda-\gamma\big|<\epsilon\Big\}.$$

We denote by $\mathbb{A}(\mathbb{T})$ the Weiner algebra of the absolutely convergent Fourier series equipped with the norm $|||.|||$. We recall that for any $\varphi\in\mathbb{A}(\mathbb{T})$, we have

$$\big|\big|\big|\varphi\big|\big|\big|_{\mathbb{A}} = \sum_{n\in\mathbb{Z}}\big|\widehat{\varphi}(n)\big|.$$

The Fourier transform on $\ell^2(\mathbb{Z})$ will be denoted by $\mathcal{F}$. It is defined by

$$\mathcal{F}(\phi)(\theta) = \sum_{n\in\mathbb{Z}} f(n)e^{2\pi i n\theta},\ \text{for } \phi\in\ell^2(\mathbb{Z}) \text{ and } \theta\in\mathbb{T}.$$

We need also the following lemma.

Lemma 5.3. Let $\phi\in\ell^2(\mathbb{Z})$ and $\varphi\in\mathbb{A}(\mathbb{T})$. Suppose that φ is differentiable with $\varphi'\in L^2(\mathbb{T})$ and $\widehat{\varphi}(0)=0$. Then

$$\|\mathcal{F}^{-1}(\varphi\mathcal{F}(\phi))\|_\infty \le \|\varphi\|_{\mathbb{A}}\big\|f\big\|_\infty, \tag{9}$$

$$\|\varphi\|_{\mathbb{A}} \le C\sqrt{\big\|\varphi\big\|_2}\sqrt{\big\|\varphi'\big\|_2}. \tag{10}$$

Proof. By the convolution theorem, we have

$$\mathcal{F}^{-1}(\varphi\mathcal{F}(\phi)) = \mathcal{F}^{-1}(\varphi)*\phi.$$

Therefore

$$\begin{aligned}\|\mathcal{F}^{-1}(\varphi\mathcal{F}(\phi))\|_\infty &= \sup_{k\in\mathbb{Z}}\Big|\sum_{n\in\mathbb{Z}}(\mathcal{F}^{-1}(\varphi))(n)\phi(k-n)\Big|\\ &\le \Big|\sum_{n\in\mathbb{Z}}(\mathcal{F}^{-1}(\varphi))(n)\Big|\big\|\phi\big\|_\infty\\ &\le \|\varphi\|_{\mathbb{A}}\big\|\phi\big\|_\infty.\end{aligned}$$

We conclude the proof by noticing that $\mathcal{F}^{-1}(\varphi)(n) = \widehat{\varphi}(n)$, for all $n \in \mathbb{Z}$. We proceed now to the proof of (10). By the time differential property of Fourier transform, we have

$$\widehat{\varphi}(n) = \frac{1}{in}\widehat{\varphi'}(n).$$

We refer to [22, Theorem 1.6, p. 4], for the proof. Applying Cauchy-Schwarz inequality combined with Parseval equality, we get

$$\begin{aligned}
\|\varphi\|_{\mathbb{A}} &= \sum_{n\neq 0}\frac{1}{|n|}\big|\widehat{\varphi'}(n)\big| \\
&\leq \Big(\sum_{n\neq 0}\frac{1}{n^2}\Big)^{\frac{1}{2}}\Big(\sum_{n\neq 0}\frac{1}{n^2}\big|\widehat{\varphi'}(n)\big|^2\Big)^{\frac{1}{2}} \\
&\leq \frac{\pi}{\sqrt{3}}\Big(\sum_{n\neq 0}\frac{1}{|n|}\big|\widehat{\varphi'}(n)\big|\big|\widehat{\varphi'}(n)\big|\Big)^{\frac{1}{2}} \\
&\leq \frac{\pi}{\sqrt{3}}\Big(\sum_{n\neq 0}\frac{1}{n^2}\big|\widehat{\varphi'}(n)\big|^2\Big)^{\frac{1}{2}}\Big(\sum_{n\neq 0}\big|\widehat{\varphi'}(n)\big|^2\Big)^{\frac{1}{2}} \\
&\leq \frac{\pi}{\sqrt{3}}\sqrt{\|\varphi\|_2}\sqrt{\|\varphi'\|_2}
\end{aligned}$$

This finish the proof of (10) and the lemma. □

Lemma 5.3 is stated in [9] as equations (3.1) and (3.3).

At this point, we state the key lemma from [9] in the proof of our main result.

Lemma 5.4. Let N_0, N_1, J be a positive integers such that $J >> N_1 > N_0$. Let $\phi, \psi \in \ell^\infty(J)$, and $0 < \delta < 1$ be a dyadic number. Suppose that for any interval $I \subset [0, J]$, with $|I| > N_0$, we have

$$\sup_{\theta\in[0,1)}\left|\frac{1}{|I|}\sum_{n\in I}e^{2\pi in\theta}\psi(n)\right| < \delta. \tag{11}$$

Then,

$$\frac{1}{J}\Bigg|\Big\{x \in [1, J] \ : \ N_x(\text{dyadic}) \in [N_0, N_1], N_x > \frac{1}{\delta},$$

$$\Big|\int \widehat{\phi}(\theta)\Big(P_{x,N_x}(\theta)\sigma_\delta\big(\big|P_{x,N_x}(\theta)\big|\big)\Big)e^{4\pi ix\theta}d\theta\Big| > \delta^{\frac{1}{10^9}}\Big\}\Bigg| < C\delta^{\frac{1}{10^6}}. \tag{12}$$

It follows, by Borel-Cantelli lemma, that under the condition (11), we have

$$\frac{1}{J}\sum_{x=1}^{J} m_{N_0,N_1}(\phi, \psi) < \delta^{\frac{1}{10^9}} \xrightarrow[\delta\to 0]{} 0.$$

Lemma 5.4 is a compilation of the principal result proved by Bourgain in [9]. The assumption (11) is stated as [9, eq (2.10), page 142] and the conclusion is formulated as the last equation [9, eq (2.10), page 161]. For the proof of Lemma 5.4 one may need the following proposition.

Proposition 5.5. Let J be a positive integer and $\delta > 0$. Then, there is a finite subset E_0 of the torus such that

(1) For any $(\lambda,\gamma)\in E_0^2$, $\lambda\neq\gamma$, we have
$$\big|\lambda-\gamma\big|\geq\frac{c}{J},$$
where c is some absolute constant.

(2) $|E_0|<C\delta^{-2}$, for some constant $C>0$.

(3) $\Big\{\lambda\ :\ \Big|\sum_{1\leq n\leq J}\psi(n)e^{2\pi in\lambda}\Big|>\delta J\Big\}\subset E_0\Big(\frac{c}{J}\Big)$, where c is the constant in (1).

Proof. Let us denote by P_0 the analytic polynomial given by
$$P_0(\lambda)=\sum_{1\leq n\leq J}\psi(n)e^{2\pi in\lambda}.$$
By, appealing to Marcinkiewicz-Zygmund interpolation inequalities [35, Theorem 7.5, Chapter X, p.28], if $\lambda_{J,j}, j=0,\cdots,J$ are the $(J+1)$-roots of the unity. Then, we have for $\alpha>1$, and for any analytic polynomial P of degree smaller or equal than J, we have

$$\frac{A_\alpha}{J}\sum_{j=0}^{J}\big|P(e^{2\pi i\lambda_{J,j}})\big|^\alpha\leq\int_{\mathbb{T}}\Big|P(\lambda)\Big|^\alpha dz\leq\frac{B_\alpha}{J}\sum_{j=0}^{n-1}\big|P(e^{2\pi i\lambda_{J,j}})\big|^\alpha,\tag{13}$$

where A_α and B_α are independent of J and P. According, a finite family of $(\lambda_{J,j})_{j=0,\cdots,J}$, is said to be a Marcinkiewicz-Zygmund family (*MZ family) if 13 holds for it. Chui and Zhong [12] proved that* $(\lambda_{J,j})_{j=0,\cdots,J}$, *is a MZ family if and only if the sample points* $(\lambda_{J,j})_{j=0,\cdots,J}$, *satisfies*
$$\min_{0\leq j<k\leq J}\Big|\lambda_{J,j}-\lambda_{J,k}\Big|\geq\frac{c}{J},$$
for some positive constant c.
Let $(\lambda_{J,j})_{j=0,\cdots,J}$, *be a MZ family and* E_0 *be the maximal subset of it such that*
$$E_0\subset\Big\{\lambda\ :\ \Big|P_0(\lambda)\Big|>\delta J\Big\}.$$
Then, obviously E_0 *satisfy the condition* (1). *To check that the condition* (2) *is fulfilled, it suffices to notice that, by* (13) *for* $\alpha=2$, *we have*
$$\delta^2J^2|E_0|\leq\sum_{j\in E_0}|P_0(\lambda_{J,j})||^2\leq B_2J\Big\|P_0\Big\|_2^2.$$
Moreover, by Parseval identity, $\big\|P_0\big\|_2^2\leq J$ *since* ψ *is* 1*-bounded. Whence,*
$$|E_0|\leq B_2\delta^{-2}.$$
Finally, it is easy to check the last condition is satisfied by the maximality of E_0. *Indeed, assume by contradiction that* (3) *does not holds. Then, there is* λ *such that* $\lambda\in\Big\{\lambda\ :\ \Big|P_0(\lambda)\Big|>\delta J\Big\}$ *and* $\big|\lambda-\lambda_{J,j}\big|\geq\frac{c}{J}$, *for each* $j\in E_0$. *Therefore, either* $\big|\lambda-\lambda_{J,j}\big|\geq\frac{c}{J}$, *for each* $j\in E_0^c$ *and we can thus add* λ *to the family or for some for each* $j\in E_0^c$, *we have* $\big|\lambda-\lambda_{J,j}\big|<\frac{c}{J}$. *In this case, we replace* $\lambda_{J,j}\in E_0^c$ *by* λ *in the family* $(\lambda_{J,j})_{j=0,\cdots,J}$. *In both cases we reach a contradiction since* E_0 *is maximal.* □

Proposition 5.5 is stated in [9] as equation (3.4) and (6.1). The proof of Lemma 5.4 is based also on the following Local Partition Lemma. Its proof is based on the nice properties of de la Vallée-Poussin Kernel. We recall that de la Vallée-Poussin Kernel $V_{(n,p)}$ of order $n+p$ is given by

$$V_{(n,p)}(x) = \frac{n+p}{p} K_{n+p}(x) - \frac{n}{p} K_n(x),$$

where K_n is Fejèr kernel. This later kernel is defined by

$$K_n(x) = \sum_{j=-n}^{n} \Big(1 - \frac{|j|}{n+1}\Big) e^{2\pi i j x} = \frac{1}{n+1} \left\{ \frac{\sin\big(\pi(n+1)x\big)}{\sin(\pi x)} \right\}^2. \tag{14}$$

Therefore

$$V_{(n,p)}(x) = \frac{1}{p} \frac{\big(\sin\big(\pi(n+p)x\big)\big)^2 - \big(\sin\big(\pi n x\big)\big)^2}{(\sin(\pi x))^2}. \tag{15}$$

We further have $\widehat{V_{n,p}}(j) = 1$, if $|j| \leq n$. We can associate to $V_{n,p}$ a compactly supported piecewise linear function $v_{n,p}$ on $\mathbb{R}$ given by

$$v_{n,p}(t) = \begin{cases} 1 & \text{if } |t| \leq n \\ \frac{n+p-|t|}{p} & \text{if } n \leq |t| \leq n+p \\ 0 & \text{if } |t| \geq n+p \end{cases}$$

The connection between those two families of functions is given by

$$V_{(n,p)}(x) = V_{(rm,(s-r)m)}(x) \tag{16}$$

$$= \sum_{j=-sm+1}^{sm-1} v_{r,s-r}\Big(\frac{j}{N}\Big) e^{2\pi i j x}, \tag{17}$$

where $m = \gcd(n, n+p)$, $n = rm$ and $n+p = sm$. Notice that the classical de la Vallée-Poussin kernel is given by

$$V_{(n,n)}(x) = 2K_{2n}(x) - K_n(x) \tag{18}$$

$$= \sum_{|j|\leq n} e^{2\pi i j x} + \sum_{n<|j|<2n} \Big(2 - \frac{|j|}{N}\Big) e^{2\pi i j x}. \tag{19}$$

Moreover, as suggested by Bourgain in [9, eq. (5.5), p.149], for any $\gamma > 0$ and a family of the de la Vallée-Poussin $V_{(n,\lfloor \gamma.n \rfloor)}$ kernel, the family of functions $v_{r,s-r}$ can be made smooth such that for any $M > \gamma^{-1}$, there is $D > 1$ such that

$$\int_{\left\{|x| > \frac{M}{2.n+1}\right\}} |V_{(n,\lfloor \gamma.n \rfloor)}(x)| dx < C\gamma^{-1}\big(\gamma.M\big)^{-D}. \tag{20}$$

In connection with Bourgain's observation we have the following lemma.

Lemma 5.6. Let n, D be a positive integers with $D > 2$, $\gamma > 0$ and $M > \gamma^{-1}$. Let I be a finite interval of integers and $V_{(n,p)}$ be the associated de la Vallée-Poussin kernel with $p > \gamma^{D+1} M^{D-2} |I|^2$ and $\mathrm{supp}(v_{n,p}) \subset I$. Then, we have

$$\int_{\left\{|x| > \frac{M}{|I|}\right\}} |V_{(n,p)}(x)| dx < C\gamma^{-1}\big(\gamma.M\big)^{-D}, \tag{21}$$

where C is an absolute constant.

Proof. By (15), we have

$$\int_{\left\{|x|>\frac{M}{|I|}\right\}} |V_{(n,p)}|dx \tag{22}$$

$$\leq \int_{\left\{|x|>\frac{M}{|I|}\right\}} \frac{1}{p}\frac{\left|\left(\sin\left(\pi(n+p)x\right)\right)^2-\left(\sin\left(\pi nx\right)\right)^2\right|}{(\sin(\pi x))^2}dx$$

$$\leq \int_{\left\{|x|>\frac{M}{|I|}\right\}} \frac{1}{p}\frac{2}{(\sin(\pi x))^2}dx \tag{23}$$

Now, remembering that $\sin(\frac{\tau}{2}) > \frac{\tau}{\pi}$ for any $0 < \tau < \pi$, we obtain

$$\int_{\left\{|x|>\frac{M}{|I|}\right\}} |V_{(n,p)}(x)|dx \leq \frac{1}{p}.\frac{1}{\frac{2.M^2}{|I|^2}} \tag{24}$$

$$\leq C\gamma^{-1}\big(\gamma.M\big)^{-D}. \tag{25}$$

The last inequality is due to the fact that $p > \gamma^{D+1}M^{D-2}|I|^2$, and the proof of the lemma is complete. □

We state now the Local Partition Lemma of Bourgain and we present its proof.

Lemma 5.7 (Local Partition Lemma of Bourgain (LP))**.** Let I be a finite interval of integers and $P(x)$ be a trigonometric polynomial such that, $P(x) = \sum_{n\in I} g_n e^{2\pi inx}$ with $|g_n| \leq 1$, for each $n \in I$. Let $\left\{I_\alpha\right\}_{\alpha\in A}$ be a partition of I in subintervales. Put

$$P_\alpha(x) = \sum_{n\in I_\alpha} g_n e^{2\pi inx},$$

and let E be a finite set in the torus identify with $(0, 1]$. Then, for any $R > 1$, we have

$$\int_{E(|I|^{-1})} \big|P(x)\big|^2 dx$$

$$\leq \sum_\alpha \int_{E(R.|I_\alpha|^{-1})} \big|P_\alpha(x)\big|^2 dx + CR^{-D}|E||I| + CR^{-\frac{1}{4}}|I|, \tag{26}$$

where D is an arbitrarily chosen exponent and the constant C depends on D.

Proof. By Riesz representation theorem, there exist $\phi \in L^2(\mathbb{T})$ such that

(1) $\big\|\phi\big\|_2 = 1$,

(2) $\mathrm{supp}(\phi) \subset E(|I|^{-1})$,

(3) $\int_{E(|I|^{-1})} P(x)\overline{\phi}(x)dx = \Big\|P|_{E(|I|^{-1})}\Big\|_2$.

Let $(V_\alpha)_{\alpha\in A}$ be a family of the smooth de la Vallée-Poussin kernel which satisfy for each $\alpha \in A$,

(1) $\mathrm{supp}(\widehat{V_\alpha}) \subset I_\alpha$,

(2) (20) with $\gamma \in (0, \frac{1}{10})$.

Put

$$Q_\alpha = P_\alpha * V_\alpha \text{ and } \psi_\alpha = \phi_\alpha * V_\alpha,$$

where $\phi_\alpha = \mathcal{F}(\mathcal{F}^{-1}(\phi)|_{I_\alpha})$. Notice that we have $\psi_\alpha = \phi * V_\alpha$, $\left\|\phi_\alpha\right\|_2 \leq 1$, and by the convolution theorem $\left\|\psi_\alpha\right\|_2 \leq \left\|\phi\right\|_2 = 1$. Moreover, by proprieties 3 of ϕ and the definition of P_α and ϕ_α, we have

$$\begin{aligned}
\left\|P|_{E(|I|^{-1})}\right\|_2 &=< P(x), \phi > \\
&= \sum_{\alpha\in A}\sum_{n\in I_\alpha} g_n\widehat{\phi}(n) \\
&= \sum_{\alpha\in A}\sum_{n\in \mathbb{Z}} g_n\widehat{\phi}(n)1_{I_\alpha}(n) \\
&= \sum_{\alpha\in A}\sum_{n\in \mathbb{Z}} \widehat{P_\alpha}(n)\widehat{\phi}|_{I_\alpha}(n) \\
&= \sum_{\alpha\in A} < P_\alpha, \phi_\alpha > \\
&\leq \sum_{\alpha\in A} \Big| < P_\alpha, \phi_\alpha > \Big| \qquad (27)
\end{aligned}$$

For th estimation of $\left\|P|_{E(|I|^{-1})}\right\|_2$, we are going to replace up to some errors $\Big| < P_\alpha, \phi_\alpha > \Big|$ with $\Big| < Q_\alpha, \phi_\alpha > \Big|$. We start with the estimation of the error.

Applying Parserval identity combined with the definition and the properties of P_α and Q_α, we get

$$\begin{aligned}
\left\|P_\alpha - Q_\alpha\right\|_2^2 &= \left\|P_\alpha - P_\alpha * V_\alpha\right\|_2^2 \\
&= \sum_{n\in I_\alpha} \Big|\widehat{\Big(P_\alpha - P_\alpha * V_\alpha\Big)}(n)\Big|^2 \\
&= \sum_{n\in I_\alpha} \Big|\widehat{P_\alpha}(n)\big(1 - \widehat{V_\alpha}(n)\big)\Big|^2 \qquad (28)
\end{aligned}$$

But, by our assumption, we can write $I_\alpha = I_\alpha^1 \bigcup I_\alpha^2$, with $I_\alpha^1 \bigcap I_\alpha = \emptyset$ and $|I_\alpha^2| = \gamma.|I_\alpha|$. This combined with the nice proprieties of V_α and the 1-boundness og (g_n) gives

$$\left\|P_\alpha - Q_\alpha\right\|_2^2 = \qquad = \sum_{n\in I_\alpha^2} \Big|\widehat{P_\alpha}(n)\big(1 - \widehat{V_\alpha}(n)\big)\Big|^2 < 4\gamma|I_\alpha|. \qquad (29)$$

Whence

$$\left\|P_\alpha - Q_\alpha\right\|_2 < 2\sqrt{\gamma|I_\alpha|}. \qquad (30)$$

This combined with Cauchy-Schwarz inequality and 1-boundness of $\|\phi_\alpha\|_2$ yields

$$\begin{aligned}
\Big|\langle P_\alpha, \phi_\alpha\rangle - \langle Q_\alpha, \phi_\alpha\rangle\Big| &\leq \left\|P_\alpha - Q_\alpha\right\|_2\left\|\phi_\alpha\right\|_2 \\
&\leq 2\sqrt{\gamma|I_\alpha|}. \qquad (31)
\end{aligned}$$

Now, we proceed to the estimation of $\langle Q_\alpha, \phi_\alpha\rangle$. By the definition of Q_α, ϕ_α and ψ_α, we have

$$\begin{aligned}\langle Q_\alpha, \phi_\alpha\rangle &= \langle P_\alpha * V_\alpha, \phi_\alpha\rangle \\ &= \sum_{n\in I_\alpha} \widehat{P_\alpha}(n)\widehat{V_\alpha}(n)\overline{\widehat{\phi_\alpha}(n)} \\ &= \langle P_\alpha, \phi_\alpha * V_\alpha\rangle \\ &= \langle P_\alpha, \psi_\alpha\rangle. \end{aligned} \tag{32}$$

We can thus write

$$\begin{aligned}\langle Q_\alpha, \phi_\alpha\rangle &= \langle P_\alpha(\mathbb{1}_{E(|I|^{-1})+\mathbb{1}_{E(|I|^{-1})^c}}), \psi_\alpha\rangle \\ &= \langle P_\alpha\, \mathbb{1}_{E(|I|^{-1})}, \psi_\alpha\rangle + \langle P_\alpha\, \mathbb{1}_{E(|I|^{-1})^c}, \psi_\alpha\rangle \end{aligned} \tag{33}$$

This combined with Cauchy-Schwarz inequality gives

$$\Big|\langle Q_\alpha, \phi_\alpha\rangle\Big| \leq \big\|P_\alpha\, \mathbb{1}_{E\left(\frac{R}{|I_\alpha|}\right)}\big\|_2 \big\|\psi_\alpha\big\|_2 + \big\|P_\alpha\big\|_2 \big\|\psi_\alpha\, \mathbb{1}_{E\left(\frac{R}{|I_\alpha|}\right)^c}\big\|_2. \tag{34}$$

Applying again Cauchy-Schwarz inequality combined with (1) and (2) we obtain, for any $\theta \in (0,1]$

$$\begin{aligned}\big|\psi(\theta)\big| &= \Big|\int_{E(|I|^{-1})} \phi(t)V_\alpha(\theta - t)dt\Big| \\ &\leq \Big(\int_{E(|I|^{-1})} \big|V_\alpha(\theta-t)\big|^2 dt\Big)^{\frac{1}{2}} \\ &\leq \big\|V_\alpha\big\|_2 = 1. \end{aligned} \tag{35}$$

Therefore

$$\begin{aligned}\big\|\psi_\alpha\, \mathbb{1}_{E\left(\frac{R}{|I_\alpha|}\right)^c}\big\|_2^2 &= \int \big|\psi_\alpha(\theta)\big|^2\, \mathbb{1}_{E\left(\frac{R}{|I_\alpha|}\right)^c}(\theta)d\theta \\ &\leq \int \big|\psi_\alpha(\theta)\big|\, \mathbb{1}_{E\left(\frac{R}{|I_\alpha|}\right)^c}(\theta)d\theta \\ &\leq \int \mathbb{1}_{E\left(\frac{R}{|I_\alpha|}\right)^c}(\theta) \int_{E(|I|^{-1})} |\phi(t)||V_\alpha(\theta-t)|dt \end{aligned} \tag{36}$$

But, assuming $t \in E(|I|^{-1})$, we get that $\theta \in E\left(\frac{R}{|I_\alpha|}\right)^c$ implies $|\theta - t| > \frac{R}{2|I_\alpha|}$. If not, since for some $\lambda \in E$, we have $|\lambda - t| < \frac{1}{|I|} < \frac{R}{2|I_\alpha|}$, by the triangle inequality, we get $|\lambda - \theta| < \frac{R}{2|I_\alpha|}$ which which gives a contradiction. We can thus rewrite (35) as follows

$$\big\|\psi_\alpha\, \mathbb{1}_{E\left(\frac{R}{|I_\alpha|}\right)^c}\big\|_2^2 \leq \int_{E(|I|^{-1})} |\phi(t)| \int_{\{|\theta-t|>\frac{R}{2|I_\alpha|}\}} |V_\alpha(\theta-t)|dt \tag{37}$$

Applying (20) and Cauchy-Schwarz inequality to get

$$\begin{aligned}\big\|\psi_\alpha\, \mathbb{1}_{E\left(\frac{R}{|I_\alpha|}\right)^c}\big\|_2^2 &\leq \int_{E(|I|^{-1})} |\phi(t)| \int_{\{|\theta-t|>\frac{R}{2|I_\alpha|}\}} |V_\alpha(\theta-t)|dt \\ &\leq \int_{E(|I|^{-1})} |\phi(t)|dt\Big(C.\gamma^{-1}(\gamma.R)^{-D}\Big) \\ &\leq \Big(\mathrm{Leb}(E(|I|^{-1}))\Big)^{\frac{1}{2}}\Big(C.\gamma^{-1}(\gamma.R)^{-D}\Big) \\ &\leq \Big(|E|.|I|^{-1}\Big)^{\frac{1}{2}}\Big(C.\gamma^{-1}(\gamma.R)^{-D}\Big), \end{aligned} \tag{38}$$

where Leb stand for Lebesgue measure on the circle. Summarizing, we get

$$\begin{aligned}\Big\|P|_{E(|I|^{-1})}\Big\|_2 &\leq \sum_{\alpha\in A}\Big|\langle Q_\alpha,\phi_\alpha\rangle\Big| + 2\sqrt{\gamma}\sum_{\alpha\in A}\sqrt{|I_\alpha|}\\ &\leq \sum_{\alpha\in A}\big\|P_\alpha\,\mathbb{1}_{E\left(\frac{R}{|I_\alpha|}\right)}\big\|_2\|\phi_\alpha\|_2\\ &+\Big(|E|.|I|^{-1}\Big)^{\frac12}\Big(C.\gamma^{-1}(\gamma.R)^{-D}\Big)^{\frac12}\sum_{\alpha\in A}\|P_\alpha\|_2\\ &+2\sqrt{\gamma}\sum_{\alpha\in A}\sqrt{|I_\alpha|}\end{aligned}\tag{39}$$

by (27), (31),(34) and (38). Applying again Cauchy-Schwarz inequality we see that

$$\sum_{\alpha\in A}\sqrt{|I_\alpha|}\leq\sqrt{|I|}$$

$$\sum_{\alpha\in A}\big\|P_\alpha\,\mathbb{1}_{E\left(\frac{R}{|I_\alpha|}\right)}\big\|_2\|\phi_\alpha\|_2\leq\Big(\sum_{\alpha\in A}\big\|P_\alpha\,\mathbb{1}_{E\left(\frac{R}{|I_\alpha|}\right)}\big\|^2\Big)^{\frac12}$$

$$\sum_{\alpha\in A}\|P_\alpha\|_2\leq\sqrt{|I|},$$

since, by our assumption, $\sum_{\alpha\in A}\|\phi_\alpha\|_2^2\leq 1$ and $\|P_\alpha\|_2^2\leq|I_\alpha|$, for each $\alpha\in A$. Consequently, we can rewrite (39) as follows

$$\begin{aligned}\Big\|P|_{E(|I|^{-1})}\Big\|_2 &\leq \sum_{\alpha\in A}\Big|\langle Q_\alpha,\phi_\alpha\rangle\Big| + 2\sqrt{\gamma}\sum_{\alpha\in A}\sqrt{|I_\alpha|}\\ &\leq\Big(\sum_{\alpha\in A}\big\|P_\alpha\,\mathbb{1}_{E\left(\frac{R}{|I_\alpha|}\right)}\big\|_2^2\Big)^{\frac12}\\ &+\sqrt{|E|}\Big(C.\gamma^{-1}(\gamma.R)^{-D}\Big)^{\frac12}+2\sqrt{\gamma}\sqrt{|I|}\end{aligned}\tag{40}$$

Taking $\gamma=\frac{1}{\sqrt[4]{R}}$, we obtain

$$\begin{aligned}\Big\|P|_{E(|I|^{-1})}\Big\|_2 &\leq\Big(\sum_{\alpha\in A}\big\|P_\alpha\,\mathbb{1}_{E\left(\frac{R}{|I_\alpha|}\right)}\big\|_2^2\Big)^{\frac12}\\ &+\sqrt{|E|}\Big(\sqrt{C}.R^{\frac{1-3D}{8}}\Big)+2R^{-\frac18}\sqrt{|I|}\end{aligned}\tag{41}$$

Squaring, we get

$$\begin{aligned}\Big\|P|_{E(|I|^{-1})}\Big\|_2^2 &\leq\sum_{\alpha\in A}\big\|P_\alpha\,\mathbb{1}_{E\left(\frac{R}{|I_\alpha|}\right)}\big\|_2^2+C|E|R^{\frac{1-3D}{4}}+4R^{-\frac14}|I|\\ &+2\Big(\sum_{\alpha\in A}\big\|P_\alpha\,\mathbb{1}_{E\left(\frac{R}{|I_\alpha|}\right)}\big\|_2^2\Big)^{\frac12}\sqrt{|E|}\Big(\sqrt{C}.R^{\frac{1-3D}{4}}\Big)\\ &+4\Big(\sum_{\alpha\in A}\big\|P_\alpha\,\mathbb{1}_{E\left(\frac{R}{|I_\alpha|}\right)}\big\|_2^2\Big)^{\frac12}R^{-\frac18}\sqrt{|I|}\\ &+4\sqrt{|E|}\Big(\sqrt{C}.R^{\frac{1-3D}{8}}\Big)R^{-\frac18}\sqrt{|I|}\end{aligned}\tag{42}$$

by the virtue of the simple identity

$$(a+b+c)^2=a^2+b^2+c^2+2ab+2ac+2bc,\ \forall a,b,c\in\mathbb{R}.$$

Now, remembering that

$$\sum_{\alpha\in A}\big\|P_\alpha\,\mathbb{1}_{E\left(\frac{R}{|I_\alpha|}\right)}\big\|_2^2\leq\sum_{\alpha\in A}\big\|P_\alpha\big\|_2^2\leq\sum_{\alpha\in A}\big|I_\alpha\big|\leq|I|.$$

We can rewrite (42) as follows

$$\begin{aligned}\Big\|P|_{E(|I|^{-1})}\Big\|_2^2&\leq\sum_{\alpha\in A}\big\|P_\alpha\,\mathbb{1}_{E\left(\frac{R}{|I_\alpha|}\right)}\big\|_2^2+C|E|R^{\frac{1-3D}{4}}+4R^{-\frac{1}{4}}|I|\\&+2\sqrt{|I|}\sqrt{|E|}\Big(\sqrt{C}.R^{\frac{1-3D}{4}}\Big)\\&+4\sqrt{|I|}R^{-\frac{1}{8}}\sqrt{|I|}\\&+4\sqrt{|E|}\Big(\sqrt{C}.R^{\frac{1-3D}{8}}\Big)R^{-\frac{1}{8}}\sqrt{|I|}\end{aligned}\tag{43}$$

We thus conclude that

$$\Big\|P|_{E(|I|^{-1})}\Big\|_2^2\leq\sum_{\alpha\in A}\big\|P_\alpha\,\mathbb{1}_{E\left(\frac{R}{|I_\alpha|}\right)}\big\|_2^2+KR^{-D'}|E||I|+8R^{-\frac{1}{4}}|I|\tag{44}$$

for some absolutely positive constant K and an arbitrary $D'>0$ since $D>1$ is arbitrary and $D'=\frac{1-3D}{8}$. This complete the proof of the lemma. □

Remark 5.8. Despite the fact that our proof follows Bourgain's proof, it is slightly different.

The LP Lemma of Bourgain is useful in the following form

Lemma 5.9 (The ϵ-localization of LP Lemma)**.** Let I be a finite interval of integers and $P(x)$ be a trigonometric polynomial such that, $P(x)=\sum_{n\in I}g_ne^{2\pi inx}$ with $|g_n|\leq1$, for each $n\in I$. Let $\Big\{I_\alpha\Big\}_{\alpha\in A}$ be a partition of I in subintervales such that $|I_\alpha|<\epsilon|I|$ with $0<\epsilon<\frac{1}{10}$. Put

$$P_\alpha(x)=\sum_{n\in I_\alpha}g_ne^{2\pi inx},$$

and let E be a finite set in the torus identify with $(0,1]$. Then, for any $R>1$, we have

$$\begin{aligned}&\int_{E(R|I|^{-1})}\big|P(x)\big|^2dx\\&\leq\sum_\alpha\int_{E(R.|I_\alpha|^{-1})}\big|P_\alpha(x)\big|^2dx+CR^{-D}|E||I|+CR^{-\frac{1}{4}}|I|,\end{aligned}\tag{45}$$

where D is an arbitrarily chosen exponent and the constant C depends on D.

Proof. We are going to apply Lemma 5.7. For that let E' be a subset of $E\big(\frac{R}{|I|}\big)$ such that $|E'|<2R|E|$ and $E\big(\frac{R}{|I|}\big)\subset E'\big(\frac{1}{|I|}\big)$. We thus have, for each $\alpha\in A$,

$$E'\Big(\frac{R}{2|I_\alpha|}\Big)\subset E\Big(\frac{R}{|I_\alpha|}\Big).$$

Indeed, if $\lambda\in E'\big(\frac{R}{2|I_\alpha|}\big)$, then, for some $\xi\in E'$, we have

$$\Big|\lambda-\xi\Big|<\frac{R}{2|I_\alpha|}.$$

But E' is a a subset of $E\big(\frac{R}{|I|}\big)$. Therefore, for some $\eta \in E$, we have

$$\Big|\xi - \eta\Big| < \frac{R}{|I|} < \frac{R}{2|I_\alpha|}.$$

Applying the triangle inequality, we get

$$\Big|\lambda - \eta\Big| < \frac{R}{|I_\alpha|}.$$

Now, by appealing to Lemma 5.7, we obtain

$$\Big\|P|_{E'(|I|^{-1})}\Big\|_2^2 \leq \sum_{\alpha\in A} \big\|P_\alpha\, \mathbb{1}_{E'\left(\frac{R}{2|I_\alpha|}\right)}\big\|_2^2 + K\Big(\frac{R}{2}\Big)^{-D}|E'||I| + 8\Big(\frac{R}{2}\Big)^{-\frac{1}{4}}|I|$$

Whence

$$\Big\|P|_{E\left(\frac{R}{|I|}\right)}\Big\|_2^2 \leq \sum_{\alpha\in A} \big\|P_\alpha\, \mathbb{1}_{E\left(\frac{R}{|I_\alpha|}\right)}\big\|_2^2 + CR^{-D}|E||I| + 16R^{-\frac{1}{4}}|I|,$$

and this achieve the proof of the lemma. □

The proof of the fundamental lemma (Lemma 5.4) is based also on the famous λ-separated lemmas due to Bourgain. These lemmas are in the heart of Bourgain's method. Its proof involve Lépingle inequalities (also called Lébingle Lemma). We refer to [11, Lemma 3.3] (see also [31]). For this later inequalities, one need to introduce the variation norms. Let $(a_n)_{n\in\mathbb{N}}$ be a complex sequence and s_0 a positive integer. The variation norm of order s is given by

$$\big\|(a_n)\big\|_{v_s} = \sup_{\substack{J\\ n_1<n_2<\cdots<n_J}} \Big(\sum_{k=1}^{J-1} |a_{n_{k+1}} - a_{n_k}|^s\Big)^{\frac{1}{s}}.$$

At this point, we are able to recall the λ-separated Lemmas.

Lemma 5.10 (λ-separated Lemma)**.** Let $\lambda_1, \cdots, \lambda_K$ be a K points on the circle such that $|\lambda_i - \lambda_j| \geq \frac{1}{2^{s-1}}$, $\forall i \neq j$, with $s > 0$. Let $f \in L^2[0,1)$. Then, there is an absolute constant $C > 0$ such that

$$\Big\| \sup_{j>s} \Big| \int_{V_j} e^{2\pi i n\alpha} f(\alpha) d\alpha\Big|\Big\|_{\ell^2(\mathbb{Z})} \leq C(\log(K))^2 \big\|f\big\|_2,$$

where V_j is $\frac{1}{2^j}$-neighborhood of $\{\lambda_1, \cdots, \lambda_K\}$ given by

$$V_j = \Big\{\lambda \in [0,1)/ \min_{1\leq r\leq K} |\lambda - \lambda_r| < \frac{1}{2^j}\Big\}.$$

As a consequence of Lépingle Lemma, we have the following lemma needed here [9, Lemma 3.23].

Lemma 5.11 (The entropic λ-separated Lemma)**.** Let $\lambda_1, \cdots, \lambda_K$ be a K points on the circle such that $|\lambda_i - \lambda_j| \geq \tau$, $\forall i \neq j$. Let $f \in \ell^2(\mathbb{Z}))$, $\chi = \mathbb{1}_{[0,1]}$ and $\chi_N = \frac{1}{N}\,\mathbb{1}_{[0,N]}$ and consider, for $x \in \mathbb{Z}$,

$$\Gamma_x = \Big\{ \int_0^1 \widehat{f}(\lambda)\widehat{\chi_N}(\lambda - \lambda_k\big)_{1\leq k\leq K}, N(\text{dyadic}) > \frac{1}{\tau}\Big\} \tag{46}$$

as a subset of K-dimensional Hilbert space. Then, for any $t > 0$

$$\int N(\Gamma_x, t)dx \leq \frac{c}{t^2}\big\|f\big\|_2^2, \tag{47}$$

where, for a subset A of a Hilbert space H, $N(A,\epsilon), \epsilon > 0$ stand for the metrical entropy numbers, that is, the minimal number of balls of radius ϵ needed to cover A.

More details on Lemma 5.4 will be given in the revised version of [1].

We proceed now to the proof of our main result (Theorem 5.1).

Proof of Theorem 5.1. Without loss of generality, we assume that the map T is ergodic. We further assume that f, g are in $L^\infty(X,\mu)$ with $\left|\left|\left|g\right|\right|\right|_{U^2} = 0$ (notice that we can interchange the role of f and g). Therefore, by Lemma 3.4 combined with Theorem 3.3, for any prime $p \neq q$, we have

$$\frac{1}{N}\sum_{n=1}^{N} z^{(p-q)n} g(T^{-pn}x)g(T^{-qn}x) \xrightarrow[N\to+\infty]{} 0, \text{ for almost all } x \tag{48}$$

We thus get, by Proposition 3.1, the following

$$\frac{1}{N}\sum_{n=1}^{N} \boldsymbol{\nu}(n) z^{-n} g(T^{-n}x) \xrightarrow[N\to+\infty]{} 0, \text{ for almost all } x. \tag{49}$$

It follows that if f is an eigenfunction, then the convergence holds.

Now, for f and g in the orthocompelment of eigenfunctions, following Bourgain's approach, we will use the finitary method. Therefore, by (49), we can write

$$\frac{1}{N}\sum_{x-N\leq n<x} \boldsymbol{\nu}_x(n) z^{-n}\psi(n) \xrightarrow[N\to+\infty]{} 0, \tag{50}$$

where $\psi \in \ell^\infty(\mathbb{Z})$ with $|||\psi|||_{U^2} = 0$. We set

$$Q_{x,N}(\theta) = \frac{1}{N}\sum_{x-N\leq n<x} \boldsymbol{\nu}_x(n) e^{-2\pi i n\theta}\psi(n).$$

Therefore, for $\delta_0 > 0$, there exist $N_0 \in \mathbb{N}$ such that for any interval $I \subset [1, J)$ with $|I| > N_0$ we have

$$\left|\frac{1}{|I|}\sum_{n\in I} \boldsymbol{\nu}_{-x_0}(n) e^{-2\pi i n\theta}\psi(n)\right| < \delta_0, \tag{51}$$

where $x_0 = \min(I)$.

We proceed now to the application of the fundamental lemma of Bourgain (Lemma 5.4). We stress that this is not a direct application of the lemma. In fact, one may need to check that the condition ((3.6) to (3.9)) of Lemma 3.5 from [9] are satisfied. For that, we first notice that, by (50), the condition (3.6) is fulfilled, and a straightforward computation yields that $\left\|Q_{x,N}\right\|_2$ is less than $N^{-1/2}$ since $\boldsymbol{\nu}$ and ψ are 1-bounded. We further have that the condition (3.8) and (3.9) are fulfilled, since the map $z \mapsto z\sigma_\delta(|z|)$ is a Lipschitz function and the constant is independent

of δ. Indeed,

$$\begin{aligned}
&\Big\|Q_{x,N}(\theta)-Q_{x',N}(\theta)\Big\|_2^2\\
&=\Big\|\frac{1}{N}\sum_{n=1}^{N}\boldsymbol{\nu}(n)\psi(x-n)e^{2\pi in\theta}-\frac{1}{N}\sum_{n=1}^{N}\boldsymbol{\nu}(n)\psi(x'-n)e^{2\pi in\theta}\Big\|_2\\
&=\frac{1}{N^2}\sum_{n=1}^{N}|\boldsymbol{\nu}(n)|^2\big|\psi(x-n)-\psi(x'-n)\big|^2 \qquad (52)\\
&\leq\frac{1}{N^2}\sum_{n=1}^{N}\big|\psi(x-n)-\psi(x'-n)\big|^2
\end{aligned}$$

The last inequality follows from the 1-boundedness of $\boldsymbol{\nu}$. We further have

$$\begin{aligned}
&\frac{1}{N^2}\sum_{n=1}^{N}\big|\psi(x-n)-\psi(x'-n)\big|^2\\
&=\Big\|\frac{1}{N}\sum_{n=1}^{N}\psi(x-n)e^{2\pi in\theta}-\frac{1}{N}\sum_{n=1}^{N}\psi(x'-n)e^{2\pi in\theta}\Big\|_2^2\\
&<\frac{|x-x'|}{N^2}. \qquad (53)
\end{aligned}$$

It is still to check that the condition (3.9) is fulfilled. This can be done, by applying Bernstein-Zygmund inequalities [35, Theorem 3.13, Chapter X, p. 11] combined with the 1-boundedness of $\boldsymbol{\nu}$ and ψ. We thus get

$$\Big\|\partial_\theta\Big(e^{2\pi ix\theta}Q_{x,N}(\theta)\Big)\Big\|_2\leq N.\Big\|\frac{1}{N}\sum_{n=1}^{N}\boldsymbol{\nu}(n)\psi(x-n)e^{2\pi in\theta}\Big\|_2<\sqrt{N} \qquad (54)$$

Following the path of Bourgain's proof, the rest of the proof is accomplished by applying the LP lemma (Lemma 5.7) (see also [9, Lemma 5.1]) and Lemma 5.10 (see also [9, Lemma 3.23] and [11, Lemma 4.1])). We thus get, for any $\phi\in\ell^\infty(\mathbb{Z})$,

$$\begin{aligned}
&\frac{1}{J}\Big|\Big\{x\in[1,J]\ :\ \max_{\substack{N_0<N<N_1,\\ N(\text{dyadic})>\frac{1}{\delta}}}\Big|\frac{1}{N}\sum_{n=1}^{N}\boldsymbol{\nu}(n)\phi(n+x)\psi(n-x)\Big|>\delta^{\frac{1}{10^9}}\Big\}\Big|\\
&<C\delta^{\frac{1}{10^6}}.
\end{aligned}$$

Since, by Fourier transform transfer, we have

$$\begin{aligned}
&\frac{1}{N}\sum_{n=1}^{N}\boldsymbol{\nu}(n)\phi(n)\psi(n)\\
&=\int_0^1\widehat{\phi}(\theta)\Big(\frac{1}{N}\sum_{n=1}^{N}\boldsymbol{\nu}(n)e^{-2\pi i(x-n)\theta}\psi(x-n)\Big)e^{4i\pi x\theta}d\theta.
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
&\Big|\int_0^1\widehat{\phi}(\theta)\Big(\frac{1}{N}\sum_{n=1}^{N}\boldsymbol{\nu}(n)e^{-2\pi i(x-n)\theta}\psi(x-n)\Big)e^{4i\pi x\theta}d\theta\Big|\\
&\leq\sum_{\substack{\delta_0>\delta>N_x^{-1}\\ \text{dyadic}}}\Big|\int_0^1\widehat{\phi}(\theta)\Big(Q_{x,N}(\theta)\sigma_\delta(|Q_{x,N}(\theta)|)\Big)\Big)e^{4i\pi x\theta}d\theta\Big|+\frac{C}{\sqrt[4]{N_x}}. \qquad (55)
\end{aligned}$$

This is due to the following fact

$$\Big|\int_0^1 \widehat{\phi}(\theta)\Big(\frac{1}{N}\sum_{n=1}^{N}\boldsymbol{\nu}(n)e^{-2\pi i(x-n)\theta}\psi(x-n)(\theta)\Big)e^{4i\pi x\theta}d\theta\Big|$$

$$= \Big|\int_0^1 \widehat{\phi}(\theta)\Big(\sum_{\substack{0<\delta<1\\ \text{dyadic}}} Q_{x,N}(\theta)\sigma_\delta(|Q_{x,N}(\theta)|)\Big)e^{4i\pi x\theta}d\theta\Big| \tag{56}$$

$$\leq \sum_{\substack{\delta_0>\delta>N_x^{-1}\\ \text{dyadic}}}\Big|\int_0^1 \widehat{\phi}(\theta)\Big(Q_{x,N}(\theta)\sigma_\delta(\big|Q_{x,N}(\theta)\big|)\Big)\Big)e^{4i\pi x\theta}d\theta\Big| + \Big|\sum_{\substack{\delta\notin[\frac{1}{N_x},\delta_0]\\ \text{dyadic}}}\cdots\Big|. \tag{57}$$

$$\leq \sum_{\substack{\delta_0>\delta>N_x^{-1}\\ \text{dyadic}}}\Big|\int_0^1 \widehat{\phi}(\theta)\Big(Q_{x,N}(\theta)\sigma_\delta(\big|Q_{x,N}(\theta)\big|)\Big)\Big)e^{4i\pi x\theta}d\theta\Big| + \frac{C}{\sqrt[4]{N_x}}. \tag{58}$$

The last inequality can be obtained by applying Lemma 5.3 to the functions $\varphi(\theta) = \sum_{\substack{\delta\notin[\frac{1}{N_x},\delta_0]\\ \text{dyadic}}} Q_{x,N}(\theta)\sigma_\delta(\big|Q_{x,N}(\theta)\big|)$ and ϕ. We further notice that

$$\Big\|Q_{x,N}\Big\|_2 \leq \frac{1}{\sqrt{N_x}}$$

and

$$\big\|\phi\big\|_\infty \leq 1.$$

To finish the proof, we need to point out that any bounded aperiodic mutilplicative function is statistically orthogonal to any nilsequence, by the generalized Daboussi-Delange theorem [17, Theorem 2.5]. We further notice that for any $f, g \in L^2(X,\mu)$, and any $\varepsilon > 0$, there exist $f_1, g_1 \in L^\infty(X,\mu)$ such that

$$\Big\|f - f_1\Big\|_2 < \sqrt{\varepsilon}, \text{ and } \Big\|g - g_1\Big\|_2 < \sqrt{\varepsilon}.$$

Moreover, by Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&\Big|\frac{1}{N}\sum_{n=1}^{N}\boldsymbol{\nu}(n)(f-f_1)(T^nx)(g-g_1)(T^{-n}x)\Big|\\
\leq\ &\frac{1}{N}\sum_{n=1}^{N}\big|(f-f_1)(T^nx)\big|\big|(g-g_1)(T^{-n}x)\big|\\
\leq\ &\Big(\frac{1}{N}\sum_{n=1}^{N}\big|(f-f_1)(T^nx)\big|^2\Big)^{\frac{1}{2}}\Big(\frac{1}{N}\sum_{n=1}^{N}\big|(g-g_1)(T^{-n}x)\big|^2\Big)^{\frac{1}{2}}
\end{aligned}$$

Applying the ergodic theorem, it follows that for almost all $x \in X$, we have

$$\limsup_{N\longrightarrow+\infty}\Big|\frac{1}{N}\sum_{n=1}^{N}\boldsymbol{\nu}(n)(f-f_1)(T^nx)(g-g_1)(T^{-n}x)\Big| < \varepsilon.$$

Whence, we can write

$$\begin{aligned}
&\limsup_{N\longrightarrow+\infty}\Big|\frac{1}{N}\sum_{n=1}^{N}\boldsymbol{\nu}(n)f(T^nx)g(T^{-n}x)\Big|\\
\leq\ &\limsup_{N\longrightarrow+\infty}\Big|\frac{1}{N}\sum_{n=1}^{N}\boldsymbol{\nu}(n)f_1(T^nx)g(T^{-n}x)\Big|\\
+\ &\limsup_{N\longrightarrow+\infty}\Big|\frac{1}{N}\sum_{n=1}^{N}\boldsymbol{\nu}(n)f(T^nx)g_1(T^{-n}x)\Big|\\
+\ &\limsup_{N\longrightarrow+\infty}\Big|\frac{1}{N}\sum_{n=1}^{N}\boldsymbol{\nu}(n)f_1(T^nx)g_1(T^{-n}x)\Big|\\
\leq\ &\varepsilon+\limsup_{N\longrightarrow+\infty}\Big|\frac{1}{N}\sum_{n=1}^{N}\boldsymbol{\nu}(n)f_1(T^nx)g(T^{-n}x)\Big|\\
+\ &\limsup_{N\longrightarrow+\infty}\Big|\frac{1}{N}\sum_{n=1}^{N}\boldsymbol{\nu}(n)f(T^nx)g_1(T^{-n}x)\Big|.
\end{aligned}$$

We thus need to estimate

$$\limsup_{N\longrightarrow+\infty}\Big|\frac{1}{N}\sum_{n=1}^{N}\boldsymbol{\nu}(n)f_1(T^nx)g(T^{-n}x)\Big|,$$

and

$$\limsup_{N\longrightarrow+\infty}\Big|\frac{1}{N}\sum_{n=1}^{N}\boldsymbol{\nu}(n)f(T^nx)g_1(T^{-n}x)\Big|.$$

In the same manner we can see that

$$\begin{aligned}
&\limsup_{N\longrightarrow+\infty}\Big|\frac{1}{N}\sum_{n=1}^{N}\boldsymbol{\nu}(n)f_1(T^nx)(g-g_1)(T^{-n}x)\Big|\\
\leq\ &\limsup_{N\longrightarrow+\infty}\Big(\frac{1}{N}\sum_{n=1}^{N}|f_1(T^nx)|^2\Big)^{\frac{1}{2}}\limsup_{N\longrightarrow+\infty}\Big(\frac{1}{N}\sum_{n=1}^{N}|(g-g_1)(T^{-n}x)|^2\Big)^{\frac{1}{2}}\\
\leq\ &\big\|f_1\big\|_2\big\|g-g_1\big\|_2\\
\leq\ &\Big(\big\|f\big\|_2+\sqrt{\varepsilon}\Big).\sqrt{\varepsilon}
\end{aligned}$$

This gives

$$\begin{aligned}
&\limsup_{N\longrightarrow+\infty}\Big|\frac{1}{N}\sum_{n=1}^{N}\boldsymbol{\nu}(n)f_1(T^nx)g(T^{-n}x)\Big|\\
\leq\ &\Big(\Big\|f\Big\|_2+\sqrt{\varepsilon}\Big).\sqrt{\varepsilon}+\limsup_{N\longrightarrow+\infty}\Big|\frac{1}{N}\sum_{n=1}^{N}\boldsymbol{\nu}(n)f_1(T^nx)g_1(T^{-n}x)\Big|\\
\leq\ &\Big(\big\|f\big\|_2+\sqrt{\varepsilon}\Big).\sqrt{\varepsilon}+0
\end{aligned}$$

Summarizing, we obtain the following estimates

$$\begin{aligned}
&\limsup_{N\longrightarrow+\infty}\Big|\frac{1}{N}\sum_{n=1}^{N}\boldsymbol{\nu}(n)f(T^nx)g(T^{-n}x)\Big|\\
\leq\ &\varepsilon+\Big(\big\|f\big\|_2+\sqrt{\varepsilon}\Big).\sqrt{\varepsilon}+\Big(\big\|g\big\|_2+\sqrt{\varepsilon}\Big).\sqrt{\varepsilon}
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we conclude that for almost every $x \in X$,

$$\frac{1}{N}\sum_{n=1}^{N} \boldsymbol{\nu}(n) f(T^n x) g(T^{-n}x) \xrightarrow[N\to+\infty]{} 0.$$

This complete the proof of the theorem. □

It is noticed in [2] that the convergence almost sure holds for the short interval can be obtained for the Liouville and Möbius functions by applying the following Zhan's estimation [32]: for each $A > 0$, for any $\varepsilon > 0$, we have

$$\max_{z\in\mathbb{T}} \left| \sum_{N\le n\le N+M} z^n \boldsymbol{\lambda}(n) \right| \le C_{A,\varepsilon} \frac{M}{\log^A(M)} \text{ for some } C_{A,\varepsilon} > 0, \tag{59}$$

provided that $M \geq N^{\frac{5}{8}+\varepsilon}$. Here,

Question. we ask on the convergence almost sure in the short interval for the bilinear ergodic bilinear averages with bounded aperiodic multiplicative weight.

Remark 5.12. In the forthcoming revised version of [1]. The author will present the strategy of the proof of Bourgain double ergodic theorem and its adaptation to prove the polynomials Bourgain bilinear ergodic theorem as it is stated in that paper. [1]

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[1] The paper [1] was posted on Arxiv on 5 August 2019, and very recently (3 August 2020), the authors in [25] provided a proof of a partial Bourgain bilinear ergodic theorem for polynomials for σ-finite measure space. It is seems that therein, the authors extended some ideas introduced by the author based on the Caldéron principal and Bourgain's method [11] combined with the duality argument due to F. Riesz (precisely, ℓ^1-ℓ^∞-duality) and the result of Peluse-Pendiville in additive combinatorics with some tools from Adelic harmonic analysis. In the revised version [1], the author will take into account their remarks.

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