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CENTRAL LIMIT THEOREM FOR BIFURCATING MARKOV CHAINS

S. VALÈRE BITSEKI PENDA AND JEAN-FRANÇOIS DELMAS

ABSTRACT. Bifurcating Markov chains (BMC) are Markov chains indexed by a full binary tree representing the evolution of a trait along a population where each individual has two children. We first provide a central limit theorem for general additive functionals of BMC, and prove the existence of three regimes. This corresponds to a competition between the reproducing rate (each individual has two children) and the ergodicity rate for the evolution of the trait. This is in contrast with the work of Guyon (2007), where the considered additive functionals are sums of martingale increments, and only one regime appears. Our first result can be seen as a discrete time version, but with general trait evolution, of results in the time continuous setting of branching particle system from Adamczak and Miłoś (2015), where the evolution of the trait is given by an Ornstein-Uhlenbeck process.

Secondly, motivated by the functional estimation of the density of the invariant probability measure which appears as the asymptotic distribution of the trait, we prove the consistence and the Gaussian fluctuations for a kernel estimator of this density. In this setting, it is interesting to note that the distinction of the three regimes disappears. This second result is a first step to go beyond the threshold condition on the ergodic rate given in previous statistical papers on functional estimation, see *e.g.* Doumic, Hoffmann, Krell and Roberts (2015).

Keywords: Bifurcating Markov chains, bifurcating auto-regressive process, binary trees, fluctuations for tree indexed Markov chain, density estimation.

Mathematics Subject Classification (2020): 60J05, 60F05, 60J80, 62G05, 62F12.

1. INTRODUCTION

Bifurcating Markov chains are a class of stochastic processes indexed by regular binary tree and which satisfy the branching Markov property (see below for a precise definition). This model represents the evolution of a trait along a population where each individual has two children. To the best of our knowledge, the term bifurcating Markov chain (BMC) appears for the first time in the work of Basawa and Zhou [4]. But, it was Guyon who, in [21], highlighted and developed a theory of asymmetric bifurcating Markov chains. Since the works of Guyon, BMC theory has been enriched from probabilistic and statistical point of view and several extensions and models using BMC has been studied; we can cite the works (see also the references therein) of Bercu, de Saporta & Gégout-Petit [7], Delmas & Marsalle [17], Bitseki, Djellout & Guillin [9], Bitseki, Hoffmann & Olivier [10], Doumic, Hoffmann, Krell & Robert [20], Bitseki & Olivier [11, 12] and Hoffmann & Marguet [23].

The recent study of BMC models was motivated by the understanding of the cell division mechanism (where the trait of an individual is given by its growth rate). The first model of BMC, named “symmetric” bifurcating auto-regressive process (BAR), see Section 2.4 for more details in a Gaussian framework, were introduced by Cowan & Staudte [16] in order to analyze cell lineage data. Since the works of Cowan and Staudte, many extensions of their model were studied in Markovian and non-Markovian setting (see for *e.g.* [11] and references therein). In particular, in

[21], Guyon has studied “asymmetric” BAR, see Example 2.3 below, in order to prove statistical evidence of aging in *Escherichia Coli*. *E. Coli* is a rod-shaped bacterium which has two ends called pole. One of the pole is new and the other is older. It reproduces by dividing in two, thus producing two new cells, one has the new pole of the mother and the other one has the old pole of the mother. Using BMC, it was concluded in [21], in accordance with the data and statistical analysis from [31], that on average, the growth rate of old pole cells is slower than that of new pole cells.

Let us also note that bifurcating Markov chains have been used recently in several statistical works to study the estimator of the cell division rate [20, 10, 23]. Moreover, another studies, such as [19], can be generalized using the BMC theory (we refer to the conclusion therein).

In this paper, our objective is first to establish a central limit theorem for additive functionals of BMC and second to apply this for functional estimation of the density of the invariant probability measure associated to the BMC. With respect to our first objective, notice that asymptotic results for BMC have been studied in [21] (law of large numbers and central limit theorem) and in [9] (moderate deviations principle and strong law of large numbers). See [17] for the law of large numbers and central limit theorem for BMC on Galton-Watson tree. Notice also that recently, limit theorems, in particular law of large numbers, has been studied for branching Markov process, see [24] and [15], and that large values of parameters in stable BAR process allows to exhibit two regimes, see [3]. However, the central limit theorems which appear in [21, 7, 17] have been done for additive functionals using increments of martingale, which implies in particular that the functions considered depend on the traits of the mother and its two daughters. The study of the case where the functions depend only on the trait of a single individual has not yet been treated for BMC (in this case it is not useful to solve the Poisson equation and to write additive functional as sums of martingale increments as the error term on the last generation is not negligible in general). For such functions, the central limit theorems have been studied recently for branching Markov processes and for superprocesses [1, 26, 28, 32]. Our first results can be seen as a discrete version of those given in the previous works, but with general ergodic hypothesis on the evolution of the trait. Unlike the results given in [21, 7, 17], we observe three regimes (sub-critical, critical and super-critical), which correspond to a competition between the reproducing rate (here a mother has two daughters) and the ergodicity rate for the evolution of the trait along a lineage taken uniformly at random. This phenomenon already appears in the works of Athreya [2]. For BMC models, we stress that the three regimes already appears for moderate deviations and deviation inequalities in [9, 8, 14].

We present our results in two frameworks. The first one, which corresponds to a point-wise approach, is in the spirit of [21, 17]. It is well adapted, for example, in parametric statistic, where we are interested in estimating “real” parameters of the model. The second one, which corresponds to an $L^2(\mu)$ approach, where μ is the invariant probability measure associated to the BMC, has been motivated by applications in nonparametric statistic, for “functional” estimation of parameters of the model. In those two approaches we provide a different normalization for the fluctuations according to the regime being critical, sub-critical and super-critical, see respectively Corollaries 3.1, 3.2 and 3.4. We shall explicit in a forthcoming paper, that those results allow to recover the one regime result from [21] for additive functionals given by a sum of martingale increments.

Motivated by the functional estimation of the density of the invariant probability measure μ , we develop a kernel estimation in the $L^2(\mu)$ framework under reasonable hypothesis (which are in particular satisfied by the Gaussian symmetric BAR model from Section 2.4). This gives a second family of results on the convergence and Gaussian fluctuations of the estimator. It is interesting,

and surprising as well, to note that the distinction of the three regimes disappears when considering the density estimation and that estimations using different generations provide asymptotically independent fluctuations, see Theorem 3.7 and Remark 3.8. Notice that in the framework of [21], where one is summing martingale increments over different generations (which is a setting different from the density estimation), the asymptotic independence of averages over different generations appears also in [17]. Eventually, we present some simulations on the kernel estimation of the density of μ . We note that in statistical studies which have been done in [20, 10, 13], the ergodicity rate is assumed to be less than $1/2$, which is strictly less than the threshold $1/\sqrt{2}$ for criticality. Moreover, in the latter works, the authors are interested in the non-asymptotic analysis of the estimators. Now, with the new perspective given in our paper, we think that the works in [20, 10, 13] can be extended to the case where the ergodicity rate belongs to $(1/2, 1)$.

The paper is organized as follows. We introduce the BMC model in Section 2.1, and consider two sets of assumptions: the point-wise approach in the spirit of [21] in Section 2.2 and the $L^2(\mu)$ approach in Section 2.3. Those latter are in particular used for the estimation of the density of μ . We check in Section 2.4 that the Gaussian BAR model fulfilled the hypothesis from the point-wise and the $L^2(\mu)$ approaches. The main results are presented in Section 3. See Section 3.1 for results in the sub-critical case, which are particular cases of more general results stated in Sections 5, with technical proofs in Sections 6 (point-wise approach) and 7 ($L^2(\mu)$ approach). See Section 3.2 for results in the critical case, which are particular cases of more general results stated in Sections 9, with technical proofs in Sections 10 (point-wise approach) and 11 ($L^2(\mu)$ approach). See Section 3.3 for results in the super-critical case, which are particular cases of more general results stated in Sections 13, with technical proofs in Section 14 (point-wise approach). After some general fluctuations results for average of different functions over different generations in Section 3.4, which are proved in Sections 8 (sub-critical case), 12 (critical case) and 15 (super-critical case), we present the convergence and fluctuations for the kernel estimation of the density of μ in Section 3.5. We provide some simulations in Section 3.6.

The proof relies essentially on explicit second moments computations and precise upper bounds of fourth moments for BMC which are recalled in Section 4.2.

2. MODELS AND ASSUMPTIONS

2.1. Bifurcating Markov chain: the model. We denote by \mathbb{N} the set of non-negative integers and $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$. If (E, \mathcal{E}) is a measurable space, then $\mathcal{B}(E)$ (resp. $\mathcal{B}_b(E)$, resp. $\mathcal{B}_+(E)$) denotes the set of (resp. bounded, resp. non-negative) \mathbb{R} -valued measurable functions defined on E . For $f \in \mathcal{B}(E)$, we set $\|f\|_\infty = \sup\{|f(x)|, x \in E\}$. For a finite measure λ on (E, \mathcal{E}) and $f \in \mathcal{B}(E)$ we shall write $\langle \lambda, f \rangle$ for $\int f(x) d\lambda(x)$ whenever this integral is well defined. For $n \in \mathbb{N}^*$, the product space E^n is endowed with the product σ -field $\mathcal{E}^{\otimes n}$. If (E, d) is a metric space, then \mathcal{E} will denote its Borel σ -field and the set $\mathcal{C}_b(E)$ (resp. $\mathcal{C}_+(E)$) denotes the set of bounded (resp. non-negative) \mathbb{R} -valued continuous functions defined on E .

Let (S, \mathcal{S}) be a measurable space. Let Q be a probability kernel on $S \times \mathcal{S}$, that is: $Q(\cdot, A)$ is measurable for all $A \in \mathcal{S}$, and $Q(x, \cdot)$ is a probability measure on (S, \mathcal{S}) for all $x \in S$. For any $f \in \mathcal{B}_b(S)$, we set for $x \in S$:

$$(1) \quad (Qf)(x) = \int_S f(y) Q(x, dy).$$

We define (Qf) , or simply Qf , for $f \in \mathcal{B}(S)$ as soon as the integral (1) is well defined, and we have $Qf \in \mathcal{B}(S)$. For $n \in \mathbb{N}$, we denote by Q^n the n -th iterate of Q defined by $Q^0 = I_d$, the identity map on $\mathcal{B}(S)$, and $Q^{n+1}f = Q^n(Qf)$ for $f \in \mathcal{B}_b(S)$.

Let P be a probability kernel on $S \times \mathcal{S}^{\otimes 2}$, that is: $P(\cdot, A)$ is measurable for all $A \in \mathcal{S}^{\otimes 2}$, and $P(x, \cdot)$ is a probability measure on $(S^2, \mathcal{S}^{\otimes 2})$ for all $x \in S$. For any $g \in \mathcal{B}_b(S^3)$ and $h \in \mathcal{B}_b(S^2)$, we set for $x \in S$:

$$(2) \quad (Pg)(x) = \int_{S^2} g(x, y, z) P(x, dy, dz) \quad \text{and} \quad (Ph)(x) = \int_{S^2} h(y, z) P(x, dy, dz).$$

We define (Pg) (resp. (Ph)), or simply Pg for $g \in \mathcal{B}(S^3)$ (resp. Ph for $h \in \mathcal{B}(S^2)$), as soon as the corresponding integral (2) is well defined, and we have that Pg and Ph belong to $\mathcal{B}(S)$.

We now introduce some notations related to the regular binary tree. We set $\mathbb{T}_0 = \mathbb{G}_0 = \{\emptyset\}$, $\mathbb{G}_k = \{0, 1\}^k$ and $\mathbb{T}_k = \bigcup_{0 \leq r \leq k} \mathbb{G}_r$ for $k \in \mathbb{N}^*$, and $\mathbb{T} = \bigcup_{r \in \mathbb{N}} \mathbb{G}_r$. The set \mathbb{G}_k corresponds to the k -th generation, \mathbb{T}_k to the tree up to the k -th generation, and \mathbb{T} the complete binary tree. For $i \in \mathbb{T}$, we denote by $|i|$ the generation of i ($|i| = k$ if and only if $i \in \mathbb{G}_k$) and $iA = \{ij; j \in A\}$ for $A \subset \mathbb{T}$, where ij is the concatenation of the two sequences $i, j \in \mathbb{T}$, with the convention that $\emptyset i = i\emptyset = i$.

We recall the definition of bifurcating Markov chain from [21].

Definition 2.1. *We say a stochastic process indexed by \mathbb{T} , $X = (X_i, i \in \mathbb{T})$, is a bifurcating Markov chain (BMC) on a measurable space (S, \mathcal{S}) with initial probability distribution ν on (S, \mathcal{S}) and probability kernel \mathcal{P} on $S \times \mathcal{S}^{\otimes 2}$ if:*

- (Initial distribution.) The random variable X_\emptyset is distributed as ν .
- (Branching Markov property.) For a sequence $(g_i, i \in \mathbb{T})$ of functions belonging to $\mathcal{B}_b(S^3)$, we have for all $k \geq 0$,

$$\mathbb{E} \left[\prod_{i \in \mathbb{G}_k} g_i(X_i, X_{i0}, X_{i1}) \mid \sigma(X_j; j \in \mathbb{T}_k) \right] = \prod_{i \in \mathbb{G}_k} \mathcal{P}g_i(X_i).$$

Let $X = (X_i, i \in \mathbb{T})$ be a BMC on a measurable space (S, \mathcal{S}) with initial probability distribution ν and probability kernel \mathcal{P} . We define three probability kernels P_0, P_1 and \mathcal{Q} on $S \times \mathcal{S}$ by:

$$P_0(x, A) = \mathcal{P}(x, A \times S), \quad P_1(x, A) = \mathcal{P}(x, S \times A) \quad \text{for } (x, A) \in S \times \mathcal{S}, \quad \text{and} \quad \mathcal{Q} = \frac{1}{2}(P_0 + P_1).$$

Notice that P_0 (resp. P_1) is the restriction of the first (resp. second) marginal of \mathcal{P} to S . Following [21], we introduce an auxiliary Markov chain $Y = (Y_n, n \in \mathbb{N})$ on (S, \mathcal{S}) with Y_0 distributed as X_\emptyset and transition kernel \mathcal{Q} . The distribution of Y_n corresponds to the distribution of X_I , where I is chosen independently from X and uniformly at random in generation \mathbb{G}_n . We shall write \mathbb{E}_x when $X_\emptyset = x$ (i.e. the initial distribution ν is the Dirac mass at $x \in S$).

We end this section with a useful inequality and the Gaussian BAR model.

Remark 2.2. By convention, for $f, g \in \mathcal{B}(S)$, we define the function $f \otimes g \in \mathcal{B}(S^2)$ by $(f \otimes g)(x, y) = f(x)g(y)$ for $x, y \in S$ and introduce the notations:

$$f \otimes_{\text{sym}} g = \frac{1}{2}(f \otimes g + g \otimes f) \quad \text{and} \quad f \otimes^2 = f \otimes f.$$

Notice that $\mathcal{P}(g \otimes_{\text{sym}} \mathbf{1}) = \mathcal{Q}(g)$ for $g \in \mathcal{B}_+(S)$. For $f \in \mathcal{B}_+(S)$, as $f \otimes f \leq f^2 \otimes_{\text{sym}} \mathbf{1}$, we get:

$$(3) \quad \mathcal{P}(f \otimes^2) = \mathcal{P}(f \otimes f) \leq \mathcal{P}(f^2 \otimes_{\text{sym}} \mathbf{1}) = \mathcal{Q}(f^2).$$

Example 2.3 (Gaussian bifurcating autoregressive process). We will consider the real-valued Gaussian bifurcating autoregressive process (BAR) $X = (X_u, u \in \mathbb{T})$ where for all $u \in \mathbb{T}$:

$$\begin{cases} X_{u0} = a_0 X_u + b_0 + \varepsilon_{u0}, \\ X_{u1} = a_1 X_u + b_1 + \varepsilon_{u1}, \end{cases}$$

with $a_0, a_1 \in (-1, 1)$, $b_0, b_1 \in \mathbb{R}$ and $((\varepsilon_{u0}, \varepsilon_{u1}), u \in \mathbb{T})$ an independent sequence of bivariate Gaussian $\mathcal{N}(0, \Gamma)$ random vectors independent of X_\emptyset with covariance matrix, with $\sigma > 0$ and $\rho \in \mathbb{R}$ such that $|\rho| \leq \sigma^2$:

$$\Gamma = \begin{pmatrix} \sigma^2 & \rho \\ \rho & \sigma^2 \end{pmatrix}.$$

Then the process $X = (X_u, u \in \mathbb{T})$ is a BMC with transition probability \mathcal{P} given by:

$$\mathcal{P}(x, dy, dz) = \frac{1}{2\pi\sqrt{\sigma^4 - \rho^2}} \exp\left(-\frac{\sigma^2}{2(\sigma^4 - \rho^2)} g(x, y, z)\right) dydz,$$

with

$$g(x, y, z) = (y - a_0x - b_0)^2 - 2\rho\sigma^{-2}(y - a_0x - b_0)(z - a_1x - b_1) + (z - a_1x - b_1)^2.$$

The transition kernel \mathcal{Q} of the auxiliary Markov chain is defined by:

$$\mathcal{Q}(x, dy) = \frac{1}{2\sqrt{2\pi}\sigma^2} \left(e^{-(y-a_0x-b_0)^2/2\sigma^2} + e^{-(y-a_1x-b_1)^2/2\sigma^2} \right) dy.$$

2.2. Assumptions in the point-wise approach. In this section, we follow the approach of [21] and consider ergodic theorem with respect to the point-wise convergence.

For a set $F \subset \mathcal{B}(S)$ of \mathbb{R} -valued functions, we write $F^2 = \{f^2; f \in F\}$, $F \otimes F = \{f_0 \otimes f_1; f_0, f_1 \in F\}$, and $P(E) = \{Pf; f \in E\}$ whenever a kernel P act on a set of functions E . We state first a structural assumption on the set of functions we shall consider.

Assumption 2.4. *Let $F \subset \mathcal{B}(S)$ be a set of \mathbb{R} -valued functions such that:*

- (i) F is a vector subspace which contains the constants;
- (ii) $F^2 \subset F$;
- (iii) $F \subset L^1(\nu)$;
- (iv) $F \otimes F \subset L^1(\mathcal{P}(x, \cdot))$ for all $x \in S$, and $\mathcal{P}(F \otimes F) \subset F$.

The condition (iv) implies that $P_0(F) \subset F$, $P_1(F) \subset F$ as well as $\mathcal{Q}(F) \subset F$. Notice that if $f \in F$, then even if $|f|$ does not belong to F , using conditions (i) and (ii), we get, with $g = (1 + f^2)/2$, that $|f| \leq g$ and $g \in F$. Typically, when (S, d) is a metric space, the set F can be the set $\mathcal{C}_b(S)$ of bounded real-valued functions, or the set of smooth real-valued functions such that all derivatives have at most polynomials growth.

We consider the following ergodic properties for \mathcal{Q} .

Assumption 2.5. *There exists a probability measure μ on (S, \mathcal{S}) such that $F \subset L^1(\mu)$ and for all $f \in F$, we have the point-wise convergence $\lim_{n \rightarrow \infty} \mathcal{Q}^n f = \langle \mu, f \rangle$ and there exists $g \in F$ with:*

$$(4) \quad |\mathcal{Q}^n(f)| \leq g \quad \text{for all } n \in \mathbb{N}.$$

We consider also the following geometrical ergodicity.

Assumption 2.6. *There exists a probability measure μ on (S, \mathcal{S}) such that $F \subset L^1(\mu)$, and $\alpha \in (0, 1)$ such that for all $f \in F$ there exists $g \in F$ such that:*

$$(5) \quad |\mathcal{Q}^n f - \langle \mu, f \rangle| \leq \alpha^n g \quad \text{for all } n \in \mathbb{N}.$$

A sequence $f = (f_\ell, \ell \in \mathbb{N})$ of elements of F satisfies uniformly (4) and (5) if there is $g \in F$ such that:

$$(6) \quad |\mathcal{Q}^n(f_\ell)| \leq g \quad \text{and} \quad |\mathcal{Q}^n f_\ell - \langle \mu, f_\ell \rangle| \leq \alpha^n g \quad \text{for all } n, \ell \in \mathbb{N}.$$

This implies in particular that $|f_\ell| \leq g$ and $|\langle \mu, f_\ell \rangle| \leq \langle \mu, g \rangle$. Notice that (6) trivially holds if f takes finitely distinct values (*i.e.* the subset $\{f_\ell; \ell \in \mathbb{N}\}$ of F is finite) each satisfying (4) and (5).

We consider the stronger ergodic property based on a second spectral gap.

Assumption 2.7. *There exists a probability measure μ on (S, \mathcal{S}) such that $F \subset L^1(\mu)$, and $\alpha \in (0, 1)$, a finite non-empty set J of indices, distinct complex eigenvalues $\{\alpha_j, j \in J\}$ of the operator \mathcal{Q} with $|\alpha_j| = \alpha$, non-zero complex projectors $\{\mathcal{R}_j, j \in J\}$ defined on $\mathbb{C}F$, the \mathbb{C} -vector space spanned by F , such that $\mathcal{R}_j \circ \mathcal{R}_{j'} = \mathcal{R}_{j'} \circ \mathcal{R}_j = 0$ for all $j \neq j'$ (so that $\sum_{j \in J} \mathcal{R}_j$ is also a projector defined on $\mathbb{C}F$) and a positive sequence $(\beta_n, n \in \mathbb{N})$ converging to 0, such that for all $f \in F$ there exists $g \in F$ and, with $\theta_j = \alpha_j/\alpha$:*

$$(7) \quad \left| \mathcal{Q}^n(f) - \langle \mu, f \rangle - \alpha^n \sum_{j \in J} \theta_j^n \mathcal{R}_j(f) \right| \leq \beta_n \alpha^n g \quad \text{for all } n \in \mathbb{N}.$$

Without loss of generality, we shall assume that the sequence $(\beta_n, n \in \mathbb{N})$ in Assumption 2.7 is non-increasing and bounded from above by 1.

Remark 2.8. In [21], only the ergodic Assumptions 2.4 and 2.5 were assumed. If F contains a set A of bounded functions which is separating (that is two probability measures which coincides on A are equal), then Assumption 2.4 and 2.5 imply in particular that μ is the only invariant measure of \mathcal{Q} . Notice that the geometric ergodicity Assumption 2.6 implies Assumption 2.5, and that Assumption 2.7 implies Assumption 2.6 (with the same α but possibly different function g).

Example 2.9. Let (S, d) be a metric space with \mathcal{S} its Borel σ -field and let Y be a Markov chain uniformly geometrically ergodic, *i.e.* there exists $\alpha \in (0, 1)$ and a finite constant C such that for all $x \in S$:

$$(8) \quad \|\mathcal{Q}^n(x, \cdot) - \mu\|_{TV} \leq C\alpha^n,$$

where, for a signed finite measure π on (S, \mathcal{S}) , its total variation norm is defined by $\|\pi\|_{TV} = \sup_{f \in \mathcal{B}(S), \|f\|_\infty \leq 1} |\langle \pi, f \rangle|$. Then, taking for F the set of \mathbb{R} -valued continuous bounded function $\mathcal{C}_b(S)$, we get that properties (i-iii) from Assumption 2.4 and Assumption 2.6 hold. In particular, Equation (8) implies that (5) holds with $g = C \|f\|_\infty$.

2.3. Assumptions in the $L^2(\mu)$ approach. We assume that μ is an invariant probability measure for \mathcal{Q} . In this section we consider an $L^2(\mu)$ approach, which will be used later on for the estimation of the density of μ .

We state first some regularity assumptions on the kernels \mathcal{P} and \mathcal{Q} and the invariant measure μ we will use later on. Notice first that by Cauchy-Schwartz we have for $f, g \in L^4(\mu)$:

$$|\mathcal{P}(f \otimes g)|^2 \leq \mathcal{P}(f^2 \otimes 1) \mathcal{P}(1 \otimes g^2) \leq 4\mathcal{Q}(f^2) \mathcal{Q}(g^2),$$

so that, as μ is an invariant measure of \mathcal{Q} :

$$(9) \quad \|\mathcal{P}(f \otimes g)\|_{L^2(\mu)} \leq 2 \|\mathcal{Q}(f^2)\|_{L^2(\mu)}^{1/2} \|\mathcal{Q}(g^2)\|_{L^2(\mu)}^{1/2} \leq 2 \|f\|_{L^4(\mu)} \|g\|_{L^4(\mu)},$$

and similarly for $f, g \in L^2(\mu)$:

$$(10) \quad \langle \mu, \mathcal{P}(f \otimes g) \rangle \leq 2 \|f\|_{L^2(\mu)} \|g\|_{L^2(\mu)}.$$

We shall in fact assume that \mathcal{P} (in fact only its symmetrized version) is in a sense an $L^2(\mu)$ operator, see also Remark 2.11 below.

Assumption 2.10. *There exists an invariant probability measure, μ , for the Markov transition kernel \mathcal{Q} .*

(i) *There exists a finite constant M such that for all $f, g, h \in L^2(\mu)$:*

$$(11) \quad \|\mathcal{P}(\mathcal{Q}f \otimes_{\text{sym}} \mathcal{Q}g)\|_{L^2(\mu)} \leq M \|f\|_{L^2(\mu)} \|g\|_{L^2(\mu)},$$

$$(12) \quad \|\mathcal{P}(\mathcal{P}(\mathcal{Q}f \otimes_{\text{sym}} \mathcal{Q}g) \otimes_{\text{sym}} \mathcal{Q}h)\|_{L^2(\mu)} \leq M \|f\|_{L^2(\mu)} \|g\|_{L^2(\mu)} \|h\|_{L^2(\mu)},$$

$$(13) \quad \|\mathcal{P}(f \otimes_{\text{sym}} \mathcal{Q}g)\|_{L^2(\mu)} \leq M \|f\|_{L^4(\mu)} \|g\|_{L^2(\mu)}.$$

(ii) *There exists $k_0 \in \mathbb{N}$, such that the probability measure $\nu \mathcal{Q}^{k_0}$ has a bounded density, say ν_0 , with respect to μ . That is:*

$$\nu \mathcal{Q}^{k_0}(dy) = \nu_0(y) \mu(y) dy \quad \text{and} \quad \|\nu_0\|_{\infty} < +\infty.$$

Remark 2.11. Let μ be an invariant probability measure of \mathcal{Q} . If there exists a finite constant M such that for all $f, g \in L^2(\mu)$:

$$(14) \quad \|\mathcal{P}(f \otimes g)\|_{L^2(\mu)} \leq M \|f\|_{L^2(\mu)} \|g\|_{L^2(\mu)},$$

then we deduce that (11), (12) and (13) hold. Condition (14) is much more natural and simpler than the latter ones, and it allows to give shorter proofs. However Condition (14) appears to be too strong even in the simplest case of the symmetric BAR model developed in Example 2.3 with $a_0 = a_1$ and $b_0 = b_1$. Let a denote the common value of a_0 and a_1 . In fact, according to the value of $a \in (-1, 1)$ in the symmetric BAR model, there exists $k_1 \in \mathbb{N}$ such that for all $f, g \in L^2(\mu)$

$$(15) \quad \|\mathcal{P}(\mathcal{Q}^{k_1} f \otimes \mathcal{Q}^{k_1} g)\|_{L^2(\mu)} \leq M \|f\|_{L^2(\mu)} \|g\|_{L^2(\mu)},$$

with k_1 increasing with $|a|$. As we shall consider only the case $|a| \in [0, 1/\sqrt{2}]$ (corresponding to the sub-critical and critical regime for the $L^2(\mu)$ approach), it will be enough to consider $k_1 = 1$ (but not sufficient to consider $k_1 = 0$). For this reason, we consider (11), that is (15) with $k_1 = 1$. A similar remark holds for (12) and (13). In a sense Condition (15) (and similar extensions of (12) and (13)) is in the same spirit as item (ii) of Assumption 2.10: ones use iterates of \mathcal{Q} to get smoothness on the kernel \mathcal{P} and the initial distribution ν .

Remark 2.12. Let μ be an invariant probability measure of \mathcal{Q} and assume that the transition kernel \mathcal{P} has a density, denoted by p , with respect to the measure $\mu^{\otimes 2}$, that is: $\mathcal{P}(x, dy, dz) = p(x, y, z) \mu(dy) \mu(dz)$ for all $x \in S$. Then the transition kernel \mathcal{Q} has a density, denoted by q , with respect to μ , that is: $\mathcal{Q}(x, dy) = q(x, y) \mu(dy)$ for all $x \in S$ with $q(x, y) = 2^{-1} \int_S (p(x, y, z) + p(x, z, y)) \mu(dz)$. We set:

$$(16) \quad \mathfrak{h}(x) = \left(\int_S q(x, y)^2 \mu(dy) \right)^{1/2}.$$

Assume that:

$$(17) \quad \|\mathcal{P}(\mathfrak{h} \otimes^2)\|_{L^2(\mu)} < +\infty,$$

$$(18) \quad \|\mathcal{P}(\mathcal{P}(\mathfrak{h} \otimes^2) \otimes_{\text{sym}} \mathfrak{h})\|_{L^2(\mu)} < +\infty,$$

and that there exists a finite constant C such that for all $f \in L^4(\mu)$:

$$(19) \quad \|\mathcal{P}(f \otimes_{\text{sym}} \mathfrak{h})\|_{L^2(\mu)} \leq C \|f\|_{L^4(\mu)}.$$

Since $|\mathcal{Q}f| \leq \|f\|_{L^2(\mu)} \mathfrak{h}$, we deduce that (17), (18) and (19) imply respectively (11), (12) and (13).

We consider the following ergodic properties of \mathcal{Q} , which in particular implies that μ is indeed the unique invariant probability measure for \mathcal{Q} . We refer to [18] Section 22 for a detailed account on $L^2(\mu)$ -ergodicity (and in particular Definition 22.2.2 on exponentially convergent Markov kernel).

Assumption 2.13. *The Markov kernel \mathcal{Q} has an (unique) invariant probability measure μ , and \mathcal{Q} is $L^2(\mu)$ exponentially convergent, that is there exists $\alpha \in (0, 1)$ and M finite such that for all $f \in L^2(\mu)$:*

$$(20) \quad \|\mathcal{Q}^n f - \langle \mu, f \rangle\|_{L^2(\mu)} \leq M\alpha^n \|f\|_{L^2(\mu)} \quad \text{for all } n \in \mathbb{N}.$$

We consider the stronger ergodic property based on a second spectral gap (compare with Assumption 2.7). (Notice in particular that Assumption 2.14 implies Assumption 2.13.)

Assumption 2.14. *The Markov kernel \mathcal{Q} has an (unique) invariant probability measure μ , and there exists $\alpha \in (0, 1)$, a finite non-empty set J of indices, distinct complex eigenvalues $\{\alpha_j, j \in J\}$ of the operator \mathcal{Q} with $|\alpha_j| = \alpha$, non-zero complex projectors $\{\mathcal{R}_j, j \in J\}$ defined on $\mathbb{C}L^2(\mu)$, the \mathbb{C} -vector space spanned by $L^2(\mu)$, such that $\mathcal{R}_j \circ \mathcal{R}_{j'} = \mathcal{R}_{j'} \circ \mathcal{R}_j = 0$ for all $j \neq j'$ (so that $\sum_{j \in J} \mathcal{R}_j$ is also a projector defined on $\mathbb{C}L^2(\mu)$) and a positive sequence $(\beta_n, n \in \mathbb{N})$ converging to 0 such that for all $f \in L^2(\mu)$, with $\theta_j = \alpha_j/\alpha$:*

$$(21) \quad \|\mathcal{Q}^n f - \langle \mu, f \rangle - \alpha^n \sum_{j \in J} \theta_j^n \mathcal{R}_j(f)\|_{L^2(\mu)} \leq \beta_n \alpha^n \|f\|_{L^2(\mu)} \quad \text{for all } n \in \mathbb{N}.$$

Remark 2.15. Assume that \mathcal{Q} has a density q with respect to an invariant probability measure μ such that $\mathfrak{h} \in L^2(\mu)$, where \mathfrak{h} is defined in (16), that is:

$$(22) \quad \int_{S^2} q(x, y)^2 \mu(dx) \mu(dy) < +\infty.$$

Then the operator \mathcal{Q} is a non-negative Hilbert-Schmidt operator (and then a compact operator) on $L^2(\mu)$. It is well known that in this case, except for the possible value 0, the spectrum of \mathcal{Q} is equal to the set $\sigma_p(\mathcal{Q})$ of eigenvalues of \mathcal{Q} ; $\sigma_p(\mathcal{Q})$ is a countable set with 0 as the only possible accumulation point and for all $\lambda \in \sigma_p(\mathcal{Q}) \setminus \{0\}$, the eigenspace associated to λ is finite-dimensional (we refer for e.g. to [6, chap. 4] for more details). In particular, if 1 is the only eigenvalue of \mathcal{Q} with modulus 1 and if it has multiplicity 1 (that is the corresponding eigenspace is reduced to the constant functions), then Assumptions 2.13 and 2.14 also hold. Let us mention that $q(x, y) > 0$ $\mu(dx) \otimes \mu(dy)$ -a.s. is a standard condition which implies that 1 is the only eigenvalue of \mathcal{Q} with modulus 1 and that it has multiplicity 1, see for example [5].

We end this section with regularity properties of functions used for kernel density estimators.

Assumption 2.16. *Let $(f_{\ell, n}, n \geq \ell \geq 0)$ be a sequence of real-valued measurable functions defined on S such that:*

- (i) *There exists $\rho \in (0, 1/2)$ such that $\sup_{n \geq \ell \geq 0} 2^{-n\rho} \|f_{\ell, n}\|_{\infty}$ is finite.*
- (ii) *The constants $\mathfrak{c}_2 = \sup_{n \geq \ell \geq 0} \|f_{\ell, n}\|_{L^2(\mu)}$ and $\mathfrak{q}_2 = \sup_{n \geq \ell \geq 0} \|\mathcal{Q}(f_{\ell, n}^2)\|_{\infty}^{1/2}$ are finite.*
- (iii) *There exists a sequence $(\delta_{\ell, n}, n \geq \ell \geq 0)$ of positive numbers such that $\Delta = \sup_{n \geq \ell \geq 0} \delta_{\ell, n}$ is finite, $\lim_{n \rightarrow \infty} \delta_{\ell, n} = 0$ for all $\ell \in \mathbb{N}$, and for all $n \geq \ell \geq 0$:*

$$|\langle \mu, |f_{\ell, n}| \rangle + |\langle \mu, \mathcal{P}(f_{\ell, n} \otimes^2) \rangle| \leq \delta_{\ell, n};$$

and for all $g \in \mathcal{B}_+(S)$:

$$(23) \quad \|\mathcal{P}(|f_{\ell, n}| \otimes_{\text{sym}} \mathcal{Q}g)\|_{L^2(\mu)} \leq \delta_{\ell, n} \|g\|_{L^2(\mu)}.$$

(iv) *The following limit exists and is finite:*

$$(24) \quad \sigma^2 = \lim_{n \rightarrow \infty} \sum_{\ell=0}^n 2^{-\ell} \|f_{\ell,n}\|_{L^2(\mu)}^2 < +\infty.$$

Remark 2.17. Notice that if for all $n \geq \ell \geq 0$ and $g \in \mathcal{B}_+(S)$ we have:

$$\|\mathcal{P}(|f_{\ell,n}| \otimes_{\text{sym}} g)\|_{L^2(\mu)} \leq \delta_{\ell,n} \|g\|_{L^2(\mu)},$$

then we get (23) and $|\langle \mu, \mathcal{P}(f_{\ell,n} \otimes^2) \rangle| \leq \mathfrak{c}_2 \delta_{\ell,n}$ which simplifies (iii) of Assumption 2.16. This hypothesis would be the natural companion to (14).

Remark 2.18. We stress that (i) and (ii) of Assumption 2.16 imply the existence of finite constant C such that for all $n \geq \ell \geq 0$:

$$\langle \mu, f_{\ell,n}^4 \rangle \leq \|f_{\ell,n}\|_{\infty}^2 \langle \mu, f_{\ell,n}^2 \rangle \leq C \mathfrak{c}_2^2 2^{2n\rho} \quad \text{and} \quad \langle \mu, f_{\ell,n}^6 \rangle \leq C \mathfrak{c}_2^2 2^{4n\rho}.$$

For the critical case, we shall assume Assumption 2.16 as well as the following.

Assumption 2.19. *Keeping the same notations as in Assumption 2.16, we further assume that:*

(v)

$$(25) \quad \lim_{n \rightarrow \infty} n \sum_{\ell=0}^n 2^{-\ell/2} \delta_{\ell,n} = 0.$$

(vi) *For all $n \geq \ell \geq 0$:*

$$(26) \quad \|\mathcal{Q}(|f_{\ell,n}|)\|_{\infty} \leq \delta_{\ell,n}.$$

For the super-critical case, we shall assume Assumptions 2.16, 2.19 as well as the following.

Assumption 2.20. *Keeping the same notations as in Assumption 2.19, we further assume that $2\alpha^2 > 1$ and:*

$$(27) \quad \sup_{0 \leq \ell \leq n} (2\alpha^2)^{n-\ell} \delta_{\ell,n}^2 < +\infty \quad \text{and, for all } \ell \in \mathbb{N}, \quad \lim_{n \rightarrow \infty} (2\alpha^2)^{n-\ell} \delta_{\ell,n}^2 = 0.$$

Notice that condition (27) implies (25) when $2\alpha^2 > 1$. The next example gives the framework for the kernel estimation of densities.

Example 2.21. Let $S = \mathbb{R}^d$ with $d \geq 1$. Let γ be such that $d\gamma \in (0, 1)$. Let $f \in \mathcal{B}(\mathbb{R}^d)$ be a kernel function such that

$$(28) \quad \|f\|_{\infty} < +\infty, \quad \|f\|_1 < +\infty, \quad \|f\|_2 = 1 \quad \text{and} \quad \lim_{|x| \rightarrow +\infty} |x|f(x) = 0,$$

where $\|f\|_p = (\int_{\mathbb{R}^d} |f(y)|^p dy)^{1/p}$. Let $(s_\ell, \ell \in \mathbb{N})$ be a sequence of positive bandwidths defined by $s_\ell = 2^{-\ell\gamma}$. For $x \in \mathbb{R}^d$, we consider the sequences of functions $(f_\ell^x, \ell \in \mathbb{N})$ defined by:

$$(29) \quad f_\ell^x(y) = s_\ell^{-d/2} f\left(\frac{x-y}{s_\ell}\right) \quad \text{for } y \in \mathbb{R}^d.$$

Using the notations of Remark 2.12, assume that \mathcal{P} (and thus \mathcal{Q}) has a density, p (resp. q) with respect to $\mu^{\otimes 2}$ (resp. μ) and that μ has a density, still denoted by μ , with respect to the Lebesgue

measure. Recall \mathfrak{h} defined by (16). We assume the following constants are finite:

$$(30) \quad C_0 = \sup_{x,y \in \mathbb{R}^d} (\mu(x) + q(x,y)\mu(y)),$$

$$(31) \quad C_1 = \sup_{y,z \in \mathbb{R}^d} \int_{\mathbb{R}^d} dx \mu(x)\mu(y)\mu(z)p(x,y,z),$$

$$(32) \quad C_2 = \int_{\mathbb{R}^d} dx \mu(x) \sup_{z \in \mathbb{R}^d} \left(\int_{\mathbb{R}^d} dy \mu(y)\mathfrak{h}(y)\mu(z)(p(x,y,z) + p(x,z,y)) \right)^2$$

We consider the sequences of functions $(f_{\ell,n}, n \geq \ell \geq 0)$, $(f_{\ell,n}^{\text{id}}, n \geq \ell \geq 0)$ and $(f_{\ell,n}^0, n \geq \ell \geq 0)$ defined by:

$$(33) \quad f_{\ell,n} = f_{n-\ell}^x, \quad f_{\ell,n}^{\text{id}} = f_n^x \quad \text{and} \quad f_{\ell,n}^0 = f_n^x \mathbf{1}_{\{\ell=0\}}.$$

We first check that they satisfy (i-iii) from Assumption 2.16 and (v-vi) from Assumption 2.19. We consider only the sequence $(f_{\ell,n}, n \geq \ell \geq 0)$, the arguments for the other two being similar. We have $\|f_{\ell,n}\|_{\infty} = s_{n-\ell}^{-d/2} \|f\|_{\infty} = 2^{(n-\ell)d\gamma/2} \|f\|_{\infty}$. Thus property (i) of Assumption 2.16 holds with $\rho = d\gamma/2$. We have $\|f_{\ell,n}\|_2 = 1$ and thus:

$$\|f_{\ell,n}\|_{L^2(\mu)}^2 \leq \|f_{\ell,n}\|_2^2 \|\mu\|_{\infty} \leq C_0 \quad \text{and} \quad \|\mathcal{Q}(f_{\ell,n}^2)\|_{\infty} \leq \|f_{\ell,n}\|_2^2 \sup_{x,y \in \mathbb{R}^d} q(x,y)\mu(y) \leq C_0.$$

We conclude that (ii) of Assumption 2.16 holds with $\mathfrak{c}_2 = \mathfrak{q}_2 = C_0^{1/2}$. We have $\langle \mu, |f_{\ell,n}| \rangle \leq C_0 \|f\|_1 s_{n-\ell}^{d/2}$ and $|\langle \mu, \mathcal{P}(f_{\ell,n} \otimes^2) \rangle| \leq C_1 \|f\|_1^2 s_{n-\ell}^d$. Furthermore, for all $g \in \mathcal{B}_+(\mathbb{R}^d)$, we have $\|\mathcal{P}(|f_{\ell,n}| \otimes_{\text{sym}} \mathcal{Q}g)\|_{L^2(\mu)} \leq C_2 s_{n-\ell}^{d/2} \|f\|_1 \|g\|_{L^2(\mu)}$. We also have $\|\mathcal{Q}(|f_{\ell,n}|)\|_{\infty} \leq C_0 \|f\|_1 s_{n-\ell}^{d/2}$. This implies that (iii) of Assumption 2.16 and (vi) of Assumption 2.19 hold with $\delta_{\ell,n} = c s_{n-\ell}^{d/2} = c 2^{-(n-\ell)d\gamma/2}$ for some finite constant c depending only on C_0, C_1, C_2 and $\|f\|_1$. With this choice of $\delta_{\ell,n}$, notice that (v) of Assumption 2.19 also holds.

Recall that $d\gamma < 1$. Moreover, if we assume that $2^{d\gamma} > 2\alpha^2$, where α is the rate given in Assumption 2.13 (this is restrictive on γ only in the super-critical regime $2\alpha^2 > 1$), then Assumption 2.20 also holds with the latter choice of $\delta_{\ell,n}$.

Let x be in the set of continuity of μ . According to Theorem 1A in [27] (which can be stated in dimension d) and since $\|f\|_2 = 1$, we have that:

$$\lim_{\ell \rightarrow \infty} \|f_{\ell}^x\|_{L^2(\mu)}^2 = \lim_{\ell \rightarrow \infty} \langle \mu, (f_{\ell}^x)^2 \rangle = \mu(x).$$

We deduce that the sequences of functions $(f_{\ell,n}, n \geq \ell \geq 0)$, $(f_{\ell,n}^{\text{id}}, n \geq \ell \geq 0)$ and $(f_{\ell,n}^0, n \geq \ell \geq 0)$ satisfy (iv) of Assumption 2.16 with σ^2 defined by (24) respectively given by:

$$(34) \quad \sigma^2 = 2\mu(x), \quad (\sigma^{\text{id}})^2 = 2\mu(x) \quad \text{and} \quad (\sigma^0)^2 = \mu(x).$$

2.4. Symmetric BAR. We consider a particular case from [16] of the real-valued bifurcating autoregressive process (BAR) from Example 2.3. We keep the same notations. Let $a \in (-1, 1)$ and assume that $a = a_0 = a_1$, $b_0 = b_1 = 0$ and $\rho = 0$. In this particular case the BAR has symmetric kernel as:

$$\mathcal{P}(x, dy, dz) = \mathcal{Q}(x, dy)\mathcal{Q}(x, dz).$$

We have $\mathcal{Q}f(x) = \mathbb{E}[f(ax + \sigma G)]$ and more generally $\mathcal{Q}^n f(x) = \mathbb{E}[f(a^n x + \sqrt{1 - a^{2n}}\sigma_a G)]$, where G is a standard $\mathcal{N}(0, 1)$ Gaussian random variable and $\sigma_a = \sigma(1 - a^2)^{-1/2}$. The kernel \mathcal{Q} admits

a unique invariant probability measure μ , which is $\mathcal{N}(0, \sigma_a^2)$ and whose density, still denoted by μ , with respect to the Lebesgue measure is given by:

$$\mu(x) = \frac{\sqrt{1-a^2}}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(1-a^2)x^2}{2\sigma^2}\right).$$

The density p (resp. q) of the kernel \mathcal{P} (resp. \mathcal{Q}) with respect to $\mu^{\otimes 2}$ (resp. μ) are given by:

$$p(x, y, z) = q(x, y)q(x, z)$$

and

$$q(x, y) = \frac{1}{\sqrt{1-a^2}} \exp\left(-\frac{(y-ax)^2}{2\sigma^2} + \frac{(1-a^2)y^2}{2\sigma^2}\right) = \frac{1}{\sqrt{1-a^2}} e^{-(a^2y^2+a^2x^2-2axy)/2\sigma^2}.$$

Notice that q is symmetric. The operator \mathcal{Q} (in $L^2(\mu)$) is a symmetric integral Hilbert-Schmidt operator whose eigenvalues are given by $\sigma_p(\mathcal{Q}) = (a^n, n \in \mathbb{N})$, their algebraic multiplicity is one and the corresponding eigen-functions $(\bar{g}_n(x), n \in \mathbb{N})$ are defined for $n \in \mathbb{N}$ by :

$$\bar{g}_n(x) = g_n(\sigma_a^{-1}x),$$

where g_n is the Hermite polynomial of degree n ($g_0 = 1$ and $g_1(x) = x$). Let \mathcal{R} be the orthogonal projection on the vector space generated by \bar{g}_1 , that is $\mathcal{R}f = \langle \mu, f\bar{g}_1 \rangle \bar{g}_1$ or equivalently, for $x \in \mathbb{R}$:

$$(35) \quad \mathcal{R}f(x) = \sigma_a^{-1}x \mathbb{E}[Gf(\sigma_a G)].$$

2.4.1. The pointwise approach. Consider F the set of functions $f \in \mathcal{C}^2(\mathbb{R})$ such that f, f' and f'' have at most polynomial growth. And assume that the probability distribution ν has all its moments, which is equivalent to say that $F \subset L^1(\nu)$. Then the set F satisfies Assumption 2.4. We also have that $F \subset L^1(\mu)$. Then, it is not difficult to check directly that Assumption 2.7 also holds with $J = \{j_0\}$, $\alpha_{j_0} = \alpha = a$, $\beta_n = a^n$ and $\mathcal{R}_{j_0} = \mathcal{R}$ (and also Assumptions 2.7 and 2.7 hold).

2.4.2. The $L^2(\mu)$ approach. Recall \mathfrak{h} defined (16). It is not difficult to check that:

$$\mathfrak{h}(x) = (1-a^4)^{-1/4} \exp\left(\frac{a^2(1-a^2)}{1+a^2} \frac{x^2}{2\sigma^2}\right) \quad \text{for } x \in \mathbb{R},$$

and $\mathfrak{h} \in L^2(\mu)$ (that is $\int_{\mathbb{R}^2} q(x, y)^2 \mu(x)\mu(y) dx dy < +\infty$). Using elementary computations, it is possible to check that $\mathcal{Q}\mathfrak{h} \in L^4(\mu)$ if and only if $|a| < 3^{-1/4}$ (whereas $\mathfrak{h} \in L^4(\mu)$ if and only if $|a| < 3^{-1/2}$). As \mathcal{P} is symmetric, we get $\mathcal{P}(\mathfrak{h}^{\otimes 2}) \leq (\mathcal{Q}\mathfrak{h})^2$ and thus (17) holds for $|a| < 3^{-1/4}$. We also get, using Cauchy-Schwartz inequality, that $\|\mathcal{P}(f \otimes_{\text{sym}} \mathfrak{h})\|_{L^2(\mu)} = \|(\mathcal{Q}f)(\mathcal{Q}\mathfrak{h})\|_{L^2(\mu)} \leq \|f\|_{L^4(\mu)} \|\mathcal{Q}(\mathfrak{h})\|_{L^4(\mu)}$, and thus (19) holds for $|a| < 3^{-1/4}$. Some elementary computations give that (18) also holds for $|a| \leq 0.724$ (but (18) fails for $|a| \geq 0.725$). (Notice that $2^{-1/2} < 0.724 < 3^{-1/4}$.) As a consequence of Remark 2.12, if $|a| \leq 0.724$, then (11)-(13) are satisfied and thus (i) of Assumption 2.10 holds.

Notice that $\nu\mathcal{Q}^k$ is the probability distribution of $a^k X_\emptyset + \sigma_a \sqrt{1-a^{2k}} G$, with G a $\mathcal{N}(0, 1)$ random variable independent of X_\emptyset . So property (ii) of Assumption 2.10 holds in particular if ν has compact support (with $k_0 = 1$) or if ν has a density with respect to the Lebesgue measure, which we still denote by ν , such that $\|\nu/\mu\|_\infty$ is finite (with $k_0 \in \mathbb{N}$). Notice that if ν is the probability distribution of $\mathcal{N}(0, \rho_0^2)$, then $\rho_0 > \sigma_a$ (resp. $\rho_0 \leq \sigma_a$) implies that (ii) of Assumption 2.10 fails (is satisfied).

Using that $(\bar{g}_n/\sqrt{n!}, n \in \mathbb{N})$ is an orthonormal basis of $L^2(\mu)$ and Parseval identity, it is easy to check that Assumption 2.14 holds with $J = \{j_0\}$, $\alpha_{j_0} = \alpha = a$, $\beta_n = a^n$ and $\mathcal{R}_{j_0} = \mathcal{R}$.

We end this section, by checking that the constants C_0 , C_1 and C_2 defined in (31), (32) and (30) are finite. The fact that C_0 is finite is clear. Notice that:

$$C_1 = \sup_{y,z \in \mathbb{R}^d} \int_{\mathbb{R}^d} dx \mu(x) \mu(y) \mu(z) p(x, y, z) = \sup_{y,z \in \mathbb{R}^d} \int_{\mathbb{R}^d} dx \mu(x) \mu(y) \mu(z) q(x, y) q(x, z) \leq C_0^2.$$

We also have, using Jensen for the second inequality (and the probability measure $\mu(y)q(x, y) dy$):

$$\begin{aligned} C_2 &= 4 \int_{\mathbb{R}^d} dx \mu(x) \sup_{z \in \mathbb{R}^d} \left(\int_{\mathbb{R}^d} dy \mu(y) \mathfrak{h}(y) \mu(z) q(x, y) q(x, z) \right)^2 \\ &\leq 4C_0^2 \int_{\mathbb{R}^d} dx \mu(x) \left(\int_{\mathbb{R}^d} dy \mu(y) \mathfrak{h}(y) q(x, y) \right)^2 \\ &\leq 4C_0^2 \|\mathfrak{h}\|_{L^2(\mu)}^2 \\ &< +\infty. \end{aligned}$$

Following the conclusion of Example 2.21, we get that if f is a kernel function and $(s_\ell = 2^{-\ell\gamma}, \ell \in \mathbb{N})$ with $\gamma \in (0, 1)$ and $2^\gamma > 2a^2$, a sequence of positive bandwidths, then the sequences of functions $(f_{\ell,n}, n \geq \ell \geq 0)$, $(f_{\ell,n}^{\text{id}}, n \geq \ell \geq 0)$ and $(f_{\ell,n}^0, n \geq \ell \geq 0)$ defined by (33) satisfy Assumptions 2.16 (with σ^2 in (24) given by (34)), 2.19 and 2.20. (Notice that, as $\alpha = a$, if $2\alpha^2 \leq 1$, then $2^\gamma > 2a^2$ holds *a fortiori*.)

3. MAIN RESULTS

Let $X = (X_u, u \in \mathbb{T})$ be a BMC on (S, \mathcal{S}) with initial probability distribution ν , and probability kernel \mathcal{P} . Recall \mathcal{Q} is the induced Markov kernel. We shall assume that μ is an invariant probability measure of \mathcal{Q} . The invariant measure is unique in the $L^2(\mu)$ approach as Assumption 2.13 holds; see Remark 2.8 on the uniqueness of the invariant probability measure in the point-wise approach. For a finite set $A \subset \mathbb{T}$ and a function $f \in \mathcal{B}(S)$, we set:

$$M_A(f) = \sum_{i \in A} f(X_i).$$

We shall be interested in the cases $A = \mathbb{G}_n$ (the n -th generation) and $A = \mathbb{T}_n$ (the tree up to the n -th generation). We recall from [21, Theorem 11 and Corollary 15] that under Assumptions 2.4 and 2.5 (resp. and also Assumption 2.6), we have for $f \in F$ the following convergence in $L^2(\mu)$ (resp. a.s.):

$$(36) \quad \lim_{n \rightarrow \infty} |\mathbb{G}_n|^{-1} M_{\mathbb{G}_n}(f) = \langle \mu, f \rangle \quad \text{and} \quad \lim_{n \rightarrow \infty} |\mathbb{T}_n|^{-1} M_{\mathbb{T}_n}(f) = \langle \mu, f \rangle.$$

Using Lemma 7.1 and the Borel-Cantelli Theorem, one can prove that we also have (36) with the $L^2(\mu)$ and a.s. convergences under Assumptions 2.10-(ii) and 2.13.

We shall now consider the corresponding fluctuations. We will use frequently the following notation:

$$\boxed{\tilde{f} = f - \langle \mu, f \rangle \quad \text{for } f \in L^1(\mu).}$$

3.1. The sub-critical case: $2\alpha^2 < 1$. For $f \in \mathcal{B}(S)$, when it is well defined, we set:

$$(37) \quad \Sigma_{\mathbb{G}}^{\text{sub}}(f) = \langle \mu, \tilde{f}^2 \rangle + \sum_{k \geq 0} 2^k \langle \mu, \mathcal{P} \left(\mathcal{Q}^k \tilde{f} \otimes^2 \right) \rangle \quad \text{and} \quad \Sigma_{\mathbb{T}}^{\text{sub}}(f) = \Sigma_{\mathbb{G}}^{\text{sub}}(f) + 2\Sigma_{\mathbb{T},2}^{\text{sub}}(f),$$

where

$$\Sigma_{\mathbb{T},2}^{\text{sub}}(f) = \sum_{k \geq 1} \langle \mu, \tilde{f} \mathcal{Q}^k \tilde{f} \rangle + \sum_{\substack{k \geq 1 \\ r \geq 0}} 2^r \langle \mu, \mathcal{P} \left(\mathcal{Q}^r \tilde{f} \otimes_{\text{sym}} \mathcal{Q}^{r+k} \tilde{f} \right) \rangle.$$

As a direct consequence of Remarks 5.1 and 4.1, and the more general Theorem 5.2 (point-wise approach) and Corollary 5.3 ($L^2(\mu)$ approach), we get the following result. Its proof is given in Section 5.1 for the point-wise approach, and left to the reader in the $L^2(\mu)$ approach, as it is very similar.

Corollary 3.1. *Let $f \in \mathcal{B}(S)$ and X be a BMC with kernel \mathcal{P} and initial distribution ν such that, with $\alpha \in (0, 1/\sqrt{2})$, either Assumptions 2.4 and 2.6 are in force and $f \in F$, or Assumptions 2.10 and 2.13 are in force and $f \in L^4(\mu)$. Then, we have the following convergence in distribution:*

$$|\mathbb{G}_n|^{-1/2} M_{\mathbb{G}_n}(\tilde{f}) \xrightarrow[n \rightarrow \infty]{(d)} G_1 \quad \text{and} \quad |\mathbb{T}_n|^{-1/2} M_{\mathbb{T}_n}(\tilde{f}) \xrightarrow[n \rightarrow \infty]{(d)} G_2,$$

where G_1 and G_2 are centered Gaussian random variables with respective variances $\Sigma_{\mathbb{G}}^{\text{sub}}(f)$ and $\Sigma_{\mathbb{T}}^{\text{sub}}(f)$ given in (37), which are well defined and finite.

3.2. The critical case $2\alpha^2 = 1$. In the critical case $\alpha = 1/\sqrt{2}$, we shall denote by \mathcal{R}_j the projector on the eigen-space associated to the eigenvalue α_j with $\alpha_j = \theta_j \alpha$, $|\theta_j| = 1$ and for j in the finite set of indices J , see Assumptions 2.7 or 2.14. Since \mathcal{Q} is a real operator, we get that if α_j is a non real eigenvalue, so is $\bar{\alpha}_j$. We shall denote by $\bar{\mathcal{R}}_j$ the projector associated to $\bar{\alpha}_j$. For $f \in \mathcal{B}(S)$, when it is well defined, we set:

$$(38) \quad \Sigma_{\mathbb{G}}^{\text{crit}}(f) = \sum_{j \in J} \langle \mu, \mathcal{P}(\mathcal{R}_j(f) \otimes_{\text{sym}} \bar{\mathcal{R}}_j(f)) \rangle \quad \text{and} \quad \Sigma_{\mathbb{T}}^{\text{crit}}(f) = \Sigma_{\mathbb{G}}^{\text{crit}}(f) + 2\Sigma_{\mathbb{T},2}^{\text{crit}}(f),$$

where

$$\Sigma_{\mathbb{T},2}^{\text{crit}}(f) = \sum_{j \in J} \frac{1}{\sqrt{2}|\theta_j - 1|} \langle \mu, \mathcal{P}(\mathcal{R}_j(f) \otimes_{\text{sym}} \bar{\mathcal{R}}_j(f)) \rangle.$$

As a direct consequence of Remarks 9.1 and 4.1, and the more general Theorem 9.2 (point-wise approach) and Corollary 9.3 ($L^2(\mu)$ approach), we get the following result. The proof which mimic the proof of Corollary 3.1 is left to the reader.

Corollary 3.2. *Let $f \in \mathcal{B}(S)$ and X be a BMC with kernel \mathcal{P} and initial distribution ν such that, with $\alpha = 1/\sqrt{2}$, either Assumptions 2.4 and 2.7 are in force and $f \in F$, or Assumptions 2.10, 2.13 and 2.14 are in force and $f \in L^4(\mu)$. Then, we have the following convergence in distribution:*

$$(n|\mathbb{G}_n|)^{-1/2} M_{\mathbb{G}_n}(\tilde{f}) \xrightarrow[n \rightarrow \infty]{(d)} G_1, \quad \text{and} \quad (n|\mathbb{T}_n|)^{-1/2} M_{\mathbb{T}_n}(\tilde{f}) \xrightarrow[n \rightarrow \infty]{(d)} G_2,$$

where G_1 and G_2 are centered Gaussian real-valued random variables with respective variance $\Sigma_{\mathbb{G}}^{\text{crit}}(f)$ and $\Sigma_{\mathbb{T}}^{\text{crit}}(f)$ given in (38), which are well defined and finite.

Remark 3.3. We stress that the variances $\Sigma_{\mathbb{T}}^{\text{crit}}(f)$ and $\Sigma_{\mathbb{G}}^{\text{crit}}(f)$ can take the value 0. This is the case in particular if the projection of f on the eigenspace corresponding to the eigenvalues α_j equal 0 for all $j \in J$. In the symmetric BAR model developed in Section 2.4, this holds if $\langle \mu, \bar{g}_1 f \rangle = 0$, that is if $\mathbb{E}[G f(\sigma_1 G)] = 0$, with G a standard $\mathcal{N}(0, 1)$ Gaussian random variable (this is in particular the case if f is even).

3.3. The super-critical case $2\alpha^2 > 1$. This case is more delicate. We shall only consider the point-wise approach. The next Corollary is a direct consequence of Corollary 13.5. See also the more general Theorem 13.2, as well as Corollary 13.4 when J is not reduced to a singleton.

Corollary 3.4. *Let X be a BMC with kernel \mathcal{P} and initial distribution ν . Assume that Assumption 2.4 and 2.7 hold with $\alpha \in (1/\sqrt{2}, 1)$ in (7). Assume α is the only eigen-value of \mathcal{Q} with modulus equal to α (and thus J is reduced to a singleton), then we have for $f \in F$:*

$$(2\alpha)^{-n} M_{\mathbb{G}_n}(\tilde{f}) \xrightarrow[n \rightarrow \infty]{\mathbb{P}} M_\infty(f) \quad \text{and} \quad (2\alpha)^{-n} M_{\mathbb{T}_n}(\tilde{f}) \xrightarrow[n \rightarrow \infty]{\mathbb{P}} \frac{2\alpha}{2\alpha - 1} M_\infty(f),$$

where $M_\infty(f)$ is a random variable defined in Corollary 13.5.

Remark 3.5. We assume hypothesis from Corollary 3.4 hold, and let \mathcal{R} be the projector on the eigen-space of \mathcal{Q} associated to the eigenvalue α . The random variable $M_\infty(f)$ defined in 13.5 is $L^2(\mu)$ integrable and we have $\mathbb{E}[M_\infty(f)] = \langle \nu, \mathcal{R}f \rangle$. We stress that if $\mathcal{R}(f) = 0$, then $M_\infty(f) = 0$. In the special case of the symmetric BAR, numerical studies reveal us that when $\mathcal{R}(f) = 0$, the normalisation $(2\alpha)^{-n}$ is not the good one. In fact, our intuition is that if $\mathcal{R}(f) = 0$, we have to find the eigenvalue of highest modulus, say α' , such that the projection of f on the corresponding eigen-space is non-zero; and then if $2(\alpha')^2 > 1$, we are again in a super-critical regime and the correct normalization should be $(2\alpha')^{-n}$ instead of $(2\alpha)^{-n}$. These facts are illustrated in the numerical studies below for the symmetric BAR (see Section 3.6).

Remark 3.6. It is possible to develop an $L^2(\mu)$ approach result similar to Corollary 3.4. But, we stress that Assumption 2.10 may not hold even for symmetric BAR model presented in Section 2.4 if the ergodic rate a is too large, see the comments in Section 2.4.2 (if $|a| \geq 0.725$, then (18) fails and thus (12) might not hold in general). One way to prove the result is either to assume further regularity on the kernel \mathcal{P} (in the spirit of (15)) or assume (11) (which always holds in the symmetric case: $\mathcal{P}(x, dy, dz) = \mathcal{Q}(x, dy)\mathcal{Q}(x, dz)$) and further integrability conditions on the functions f , that is $f \in L^8(\mu)$.

3.4. Preliminary results for the kernel estimation. In view of the density estimation of the invariant probability measure, we shall prove the following result, whose proof is given in Section 8 for the sub-critical case ($\alpha \in (0, 1/\sqrt{2})$), in Section 12 for the critical case ($\alpha = 1/\sqrt{2}$) and in Section 15 for the supercritical case ($\alpha \in (1/\sqrt{2}, 1)$). In this section Assumption 2.10 and 2.13 are in force. Because of condition (ii) in Assumption 2.10 which roughly state that after k_0 generations, the distribution of the induced Markov chain is absolutely continuous with respect to the invariant measure μ , it is better to consider only generations $k \geq k_0$ for some $k_0 \in \mathbb{N}$ (and thus remove the first $k_0 - 1$ generations in the forthcoming quantity $N_{n, \emptyset}(f)$ defined in (44). For this reason, we introduce for $\mathbf{f} = (f_\ell, \ell \in \mathbb{N})$ a sequence of elements of $L^1(\mu)$:

$$(39) \quad N_{n, \emptyset}^{[k_0]}(\mathbf{f}) = |\mathbb{G}_n|^{-1/2} \sum_{\ell=0}^{n-k_0} M_{\mathbb{G}_{n-\ell}}(\tilde{f}_\ell)$$

Theorem 3.7. *Let X be a BMC with kernel \mathcal{P} and initial distribution ν such that Assumption 2.10 (with $k_0 \in \mathbb{N}$) is in force. We suppose also that the following assumption holds: Assumption 2.13 if $\alpha \in (0, 1/\sqrt{2})$ (sub-critical case); Assumption 2.14 if $\alpha = 1/\sqrt{2}$ (critical-case); Assumptions 2.14 and 2.20 if $\alpha \in (1/\sqrt{2}, 1)$ (super-critical case). Let $(f_{\ell, n}, n \geq \ell \geq 0)$ satisfying Assumption 2.16 for $\alpha \in (0, 1)$, Assumption 2.19 for $\alpha \in [1/\sqrt{2}, 1)$ and Assumption 2.20 for $\alpha \in (1/\sqrt{2}, 1)$. Then, we have the following convergence in distribution:*

$$N_{n, \emptyset}^{[k_0]}(\mathbf{f}_n) \xrightarrow[n \rightarrow \infty]{(d)} G,$$

where $\mathfrak{f}_n = (f_{\ell,n}, \ell \in \mathbb{N})$ with the convention that $f_{\ell,n} = 0$ for $\ell > n$ and where G is a centered Gaussian random variable with finite variance σ^2 defined in (24).

Remark 3.8. Assume $\sigma_\ell^2 = \lim_{n \rightarrow \infty} \|f_{\ell,n}\|_{L^2(\mu)}^2$ exists for all $\ell \in \mathbb{N}$; so that σ^2 defined in (24) is also equal to $\sum_{\ell \in \mathbb{N}} 2^{-\ell} \sigma_\ell^2$. According to additive form of the variance σ^2 , we deduce that for fixed $k \in \mathbb{N}$, the random variables $(|\mathbb{G}_n|^{-1/2} M_{\mathbb{G}_n - \ell}(\tilde{f}_{\ell,n}), \ell \in \{0, \dots, k\})$ converges in distribution, as n goes to infinity towards $(G_\ell, \ell \in \{0, \dots, k\})$ which are independent real-valued Gaussian centered random variables with variance $\text{Var}(G_\ell) = 2^{-\ell} \sigma_\ell^2$.

3.5. Statistical applications: kernel estimation for the density of μ . The purpose of this Section is to study asymptotic normality of kernel estimators for the density of the stationary measure of a BMC. We will assume the following set of hypothesis which precise: i) the ergodicity of the BMC; ii) the regularity of the BMC kernel transition and stationary distribution; iii) the regularity of the kernel used for the estimation; and iv) the choice of the bandwidth. For simplicity, we shall consider the framework of Example 2.21.

To be precise, let $X = (X_u, u \in \mathbb{T})$ be a BMC on \mathbb{R}^d , with $d \geq 1$ with initial probability distribution ν , and probability kernel \mathcal{P} . We denote \mathcal{Q} the associated transition kernel on \mathbb{R}^d and μ its stationary probability measure (which will be well defined and unique as we shall assume some ergodicity properties). For the sake of clarity, we summarize below the assumptions which will be in force in this section.

Assumption 3.9.

- (i) **Ergodicity hypothesis:** We assume that the ergodicity Assumption 2.13 holds with some $\alpha \in (0, 1)$.
- (ii) **Regularity of the BMC kernel and initial distribution:** We assume that Assumption 2.10 on the $L^2(\mu)$ regularity of the kernel \mathcal{P} and on the regularity of initial distribution ν holds. The kernel \mathcal{P} (and thus \mathcal{Q}) has a density, say p (resp. q), with respect to $\mu^{\otimes 2}$ (resp. μ) and μ has a density, still denoted by μ , with respect to the Lebesgue measure. We assume that the constants C_0, C_1 and C_2 defined by (30), (31) and (32) are finite. (We mention Remark 2.12 where some integrability condition on p and q implies Assumption 2.10 on the $L^2(\mu)$ regularity of the kernel \mathcal{P} .)
- (iii) **Regularity for the estimation kernel:** We consider an integrable and square integrable kernel $K \in \mathcal{B}(\mathbb{R}^d)$ such that $\int_{\mathbb{R}^d} K(x) dx = 1$ and $f = K / \|K\|_2$ satisfies the conditions (28).
- (iv) **The choice of the bandwidth:** For $n \in \mathbb{N}$, we set $s_n = 2^{-n\gamma}$ for some $\gamma \in (0, 1/d)$ and $2^{d\gamma} > 2\alpha^2$ (which is only restrictive in the super-critical regime $\alpha > 1/\sqrt{2}$).

We define the following kernel estimators of the density μ , for $\mathbb{A}_n \in \{\mathbb{G}_n, \mathbb{T}_n\}$ and $x \in \mathbb{R}^d$:

$$(40) \quad \hat{\mu}_{\mathbb{A}_n}(x) = |\mathbb{A}_n|^{-1} s_n^{-d} \sum_{u \in \mathbb{A}_n} K(s_n^{-1}(x - X_u)).$$

Let us mention that those statistics are strongly inspired from [25, 29, 30]. We have the consistency of the estimator $\hat{\mu}_{\mathbb{A}_n}(x)$ for x in the set of continuity of μ . Its proof is given in Section 16.

Lemma 3.10. *Let X be a BMC with kernel \mathcal{P} and initial distribution ν , K be an estimation kernel and let a bandwidth $(s_n, n \in \mathbb{N})$ such that Assumption 3.9 is in force. Then, we have, for x in the set of continuity of μ and $\mathbb{A}_n \in \{\mathbb{G}_n, \mathbb{T}_n\}$,*

$$\lim_{n \rightarrow \infty} \hat{\mu}_{\mathbb{A}_n}(x) = \mu(x) \quad \text{in probability.}$$

The following assumptions on μ and K will be useful to control de biais term. For $s \in \mathbb{R}_+$, let $\lfloor s \rfloor$ denote its integer part, that is the only integer $n \in \mathbb{N}$ such that $n \leq s < n + 1$ and set $\{s\} = s - \lfloor s \rfloor$ its fractional part.

Assumption 3.11. *We assume that Assumption 3.9 holds and there exists $s > 0$ such that the following holds.*

- (iv) **The density μ belongs to the (isotropic) Hölder class of order $(s, \dots, s) \in \mathbb{R}^d$:** *The density μ admits partial derivatives with respect to x_j , for all $j \in \{1, \dots, d\}$, up to the order $\lfloor s \rfloor$ and there exists a finite constant $L > 0$ such that for all $x = (x_1, \dots, x_d) \in \mathbb{R}^d$, $t \in \mathbb{R}$ and $j \in \{1, \dots, d\}$:*

$$\left| \frac{\partial^{\lfloor s \rfloor} \mu}{\partial x_j^{\lfloor s \rfloor}}(x_{-j}, t) - \frac{\partial^{\lfloor s \rfloor} \mu}{\partial x_j^{\lfloor s \rfloor}}(x) \right| \leq L |x_j - t|^{\{s\}},$$

where (x_{-j}, t) denotes the vector x where we have replaced the j^{th} coordinate x_j by t , with the convention $\partial^0 \mu / \partial x_j^0 = \mu$.

- (v) **The kernel K is of order $(\lfloor s \rfloor, \dots, \lfloor s \rfloor) \in \mathbb{N}^d$:** *We have $\int_{\mathbb{R}^d} |x|^s K(x) dx < \infty$ and $\int_{\mathbb{R}} x_j^k K(x) dx_j = 0$ for all $k \in \{1, \dots, \lfloor s \rfloor\}$ and $j \in \{1, \dots, d\}$.*
- (vi) **Bandwith control:** *We have $\gamma > 1/(2s + d)$, that is $\lim_{n \rightarrow \infty} |\mathbb{G}_n| s_n^{2s+d} = 0$.*

Notice that Assumption 3.11-(iv) implies that μ is at least Hölder continuous as $s > 0$. We now study asymptotic normality of the density kernel estimator. The proof of the next theorem is given in Section 16.

Theorem 3.12. *Assume that Assumption 3.11 holds. Then, for $\mathbb{A}_n \in \{\mathbb{G}_n, \mathbb{T}_n\}$, we have the following convergence in distribution:*

$$|\mathbb{A}_n|^{1/2} s_n^{d/2} (\widehat{\mu}_{\mathbb{A}_n}(x) - \mu(x)) \xrightarrow[n \rightarrow \infty]{(d)} G,$$

where G is a centered Gaussian real-valued random variable with variance $\|K\|_2^2 \mu(x)$.

Remark 3.13. We stress that the asymptotic variance is the same for $\mathbb{A}_n = \mathbb{G}_n$ and $\mathbb{A}_n = \mathbb{T}_n$. This is a consequence of the structure of the asymptotic variance σ^2 in (24) and the fact that $\lim_{n \rightarrow \infty} |\mathbb{T}_n|/|\mathbb{G}_n| = 2$.

Remark 3.14. Using the structure of the asymptotic variance σ^2 in (24) (and see also Remark 3.8), it is easy to deduce that $|\mathbb{G}_n|^{1/2} s_n^{d/2} (\widehat{\mu}_{\mathbb{G}_{n-\ell}}(x) - \mu(x))$ are asymptotically independent for $\ell \in \{0, \dots, k\}$ for any $k \in \mathbb{N}$.

Remark 3.15. In Example 2.21, we have seen that in order to satisfy Assumption 2.20, the bandwidth must be a function of the geometric rate of convergence via the relation $2^{d\gamma} > 2\alpha^2$. This means that for the bandwidth selection problems for the estimation of the density μ , the geometric rate of convergence α could be interpreted as a regularity parameter just like the regularity of the unknown function μ . With this new perspective, we think that the works in [13] can be extended to the cases where $\alpha \in (1/2, 1)$ by studying an adaptive procedure with respect to the unknown geometric rate of convergence α .

3.6. Numerical studies.

3.6.1. *Illustration of phase transitions for the fluctuations in the point-wise and $L^2(\mu)$ cases.* We consider the symmetric BAR model from Section 2.4 with $a = \alpha \in (0, 1)$. Recall α is an eigenvalue with multiplicity one, and we denote by \mathcal{R} the orthogonal projection on the one-dimensional eigenspace associated to α . The expression of \mathcal{R} is given in (35).

In order to illustrate the effects of the geometric rate of convergence α on the fluctuations, we plot for $\mathbb{A}_n \in \{\mathbb{G}_n, \mathbb{T}_n\}$, the slope, say $b_{\alpha,n}$, of the regression line $\log(\text{Var}(|\mathbb{A}_n|^{-1}M_{\mathbb{A}_n}(f)))$ versus $\log(|\mathbb{A}_n|)$ as a function of the geometric rate of convergence α . In the classical cases (e.g. Markov chains), the points are expected to be distributed around the horizontal line $y = -1$. For n large, we have $\log(|\mathbb{A}_n|) \simeq n \log(2)$ and, for the symmetric BAR model, Corollaries 3.1, 3.2 and 3.4 yields that $b_{\alpha,n} \simeq h_1(\alpha)$ with $h_1(\alpha) = \log(\alpha^2 \vee 2^{-1})/\log(2)$ as soon as the limiting Gaussian random variable in Corollaries 3.1 and 3.2 or $M_\infty(f)$ in Corollary 3.4 is non-zero.

For our illustrations, we consider the empirical moments of order $p \in \{1, \dots, 4\}$, that is we use the functions $f(x) = x^p$. As we can see in Figures 1 and 2, these curves present two trends with a phase transition around the rate $\alpha = 1/\sqrt{2}$ for $p \in \{1, 3\}$ and around the rate $\alpha^2 = 1/\sqrt{2}$ for $p \in \{2, 4\}$. For convergence rates $\alpha \in (0, 1/\sqrt{2})$, the trend is similar to that of classic cases. For convergence rates $\alpha \in (1/\sqrt{2}, 1)$, the trend differs to that of classic cases. One can observe that the slope $b_{\alpha,n}$ increases with the value of geometric convergence rate α . We also observe that for $\alpha > 1/\sqrt{2}$, the empirical curves agrees with the graph of $h_1(\alpha) = \log(\alpha^2 \vee 2^{-1})/\log(2)$ for $f(x) = x^p$ when p is odd, see Figure 1. However, the empirical curves does not agree with the graph of h_1 for $f(x) = x^p$ when p is even, see Figure 2, but it agrees with the graph of the function $h_2(\alpha) = \log(\alpha^4 \vee 2^{-1})/\log(2)$. This is due to the fact that for p even, the function $f(x) = x^p$ belongs to the kernel of the projector \mathcal{R} (which is clear from formula (35)), and thus $M_\infty(f) = 0$. In fact, in those two cases, one should take into account the projection on the eigenspace associated to the third eigenvalue, which in this particular case is equal to α^2 . Intuitively, this indeed give a rate of order h_2 . Therefore, the normalization given for $f(x) = x^p$ when p even, is not correct.

3.6.2. *Estimation of the invariant density for symmetric BAR.* In order to illustrate the central limit theorem for the estimator of the invariant density μ , we simulate $n_0 = 500$ samples of a symmetric BAR $X = (X_u^{(a)}, u \in \mathbb{T}_n)$ with different values of the autoregressive coefficient $\alpha = a \in (-1, 1)$. For each sample, we compute the estimator $\hat{\mu}_{\mathbb{A}_n}(x)$ given by (40) and its fluctuation given by

$$(41) \quad \zeta_n = |\mathbb{A}_n|^{1/2} s_n^{d/2} (\hat{\mu}_{\mathbb{A}_n}(x) - \mu(x))$$

for $x \in \mathbb{R}$, the average over $\mathbb{A}_n \in \{\mathbb{G}_n, \mathbb{T}_n\}$, the Gaussian kernel

$$K(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

and the bandwidth $s_n = 2^{-n\gamma}$ with $\gamma \in (0, 1)$. Next, in order to compare theoretical and empirical results, we plot in the same graphic, see Figures 3 and 4:

- The histogram of ζ_n and the density of the centered Gaussian distribution with variance $\mu(x) \|K\|_2^2 = \mu(x)(2\sqrt{\pi})^{-1}$ (see Theorem 3.12).
- The empirical cumulative distribution of ζ_n and the cumulative distribution of the centered Gaussian distribution with variance $\mu(x) \|K\|_2^2 = \mu(x)(2\sqrt{\pi})^{-1}$.

Since the Gaussian kernel is of order $s = 2$ and the dimension is $d = 1$, the bandwidth exponent γ must satisfy the condition $\gamma > 1/5$, so that Assumption 3.11-(vi) holds. Moreover, in the supercritical case, γ must satisfy the supplementary condition $2^\gamma > 2\alpha^2$, that is $\gamma > 1 + \log(\alpha^2)/\log(2)$, so that Assumption 3.9-(iv) holds. In Figure 3, we take $\alpha = 0.5$ and $\alpha = 0.7$ (both of them

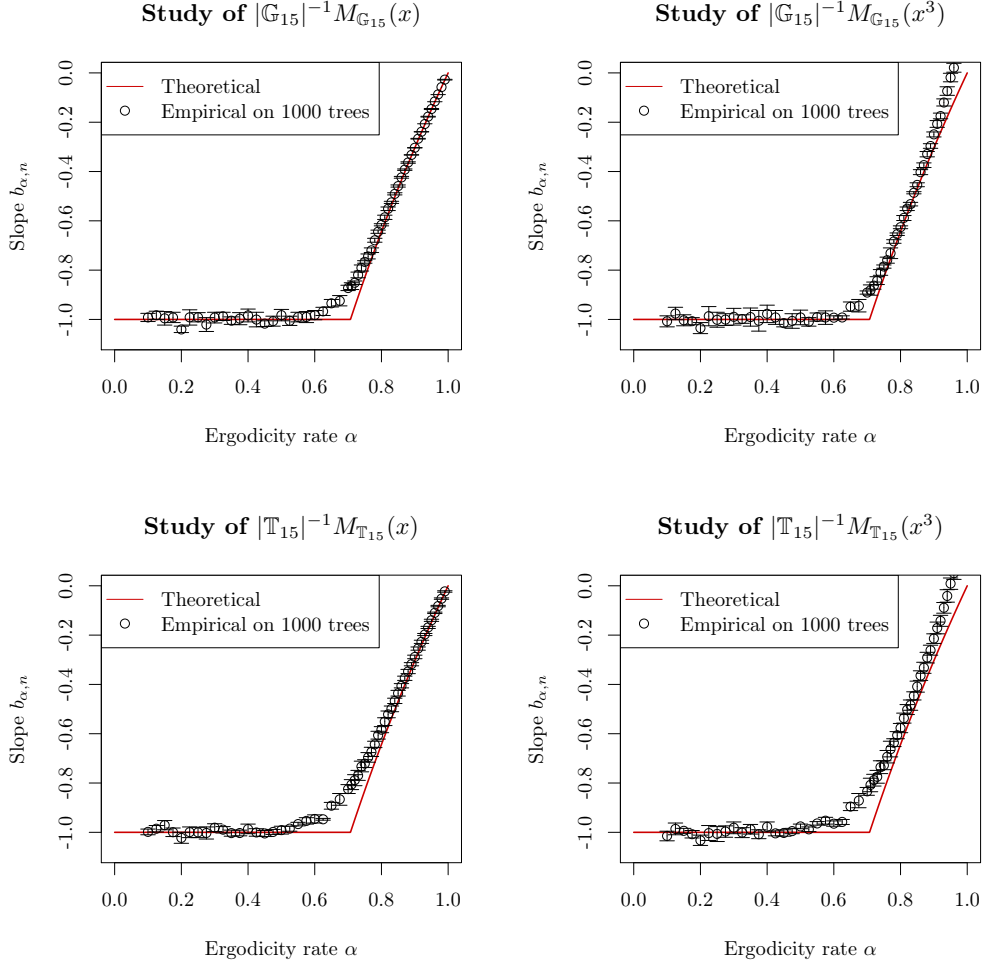


FIGURE 1. Slope $b_{\alpha,n}$ (empirical mean and confidence interval in black) of the regression line $\log(\text{Var}(|\mathbb{A}_n|^{-1}M_{\mathbb{A}_n}(f)))$ versus $\log(|\mathbb{A}_n|)$ as a function of the geometric ergodic rate α , for $n = 15$, $\mathbb{A}_n \in \{\mathbb{G}_n, \mathbb{T}_n\}$ and $f(x) = x^p$ with $p \in \{1, 3\}$. In this case, we have $\mathcal{R}(f) \neq 0$, where \mathcal{R} is the projector defined from formula (35). One can see that the empirical curve (in black) is close to the graph (in red) of the function $h_1(\alpha) = \log(\alpha^2 \vee 2^{-1})/\log(2)$ for $\alpha \in (0, 1)$.

corresponds to the sub-critical case as $2\alpha^2 < 1$) and $\gamma = 1/5 + 10^{-3}$. The simulations agree with results from Theorem 3.12. In Figure 4, we take $\alpha = 0.9$ (super-critical case) and consider $\gamma = 0.696$ and $\gamma = 1/5 + 10^{-3}$. In the former case Assumption 3.9-(iv) is satisfied as $\gamma = 0.696 > 1 + \log((0.9)^2)/\log(2)$, and in the latter case Assumption 3.9-(iv) fails. As one can see in the graphics Figure 4, the estimates agree with the theory in the former case ($\gamma = 0.696$), whereas they are poor in the latter case.

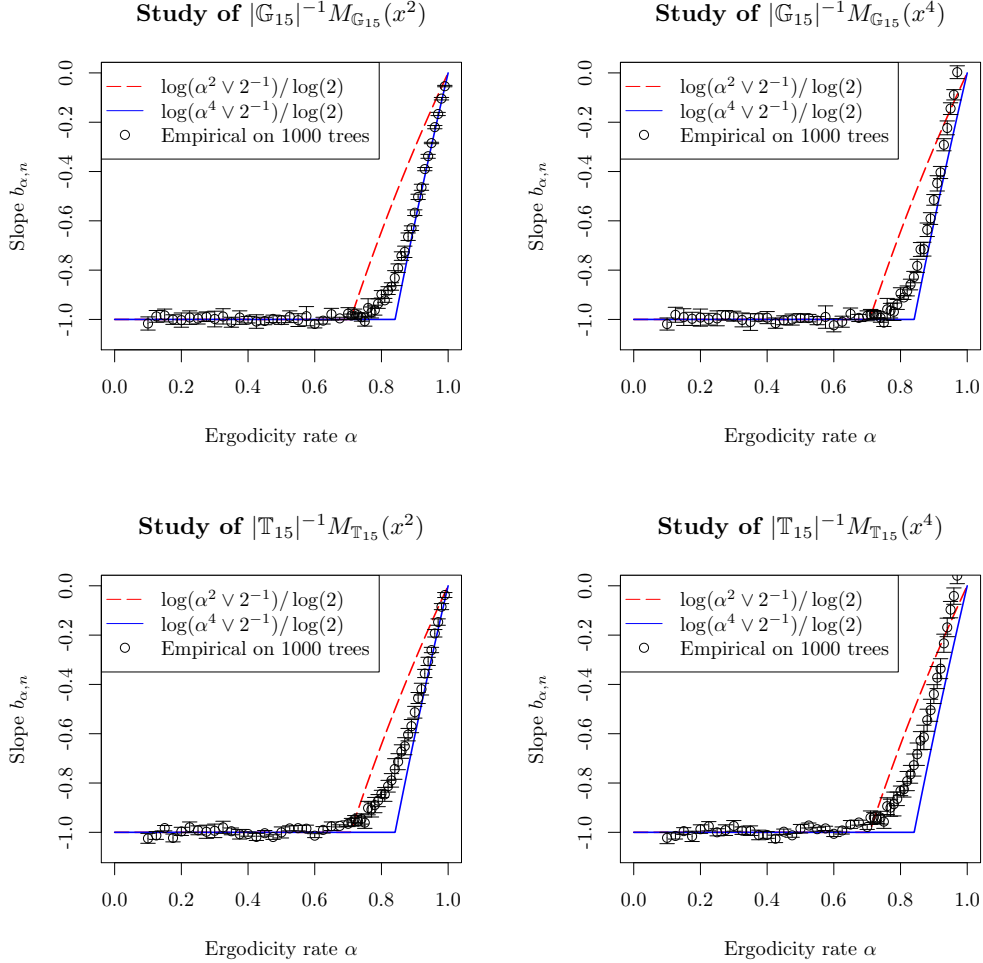


FIGURE 2. Slope $b_{\alpha,n}$ (empirical mean and confidence interval in black) of the regression line $\log(\text{Var}(|\mathbb{A}_n|^{-1}M_{\mathbb{A}_n}(f)))$ versus $\log(|\mathbb{A}_n|)$ as a function of the geometric ergodic rate α , for $n = 15$, $\mathbb{A}_n \in \{\mathbb{G}_n, \mathbb{T}_n\}$ and $f(x) = x^p$ with $p \in \{2, 4\}$. In this case, we have $\mathcal{R}(f) = 0$, where \mathcal{R} is the projector defined from formula (35). One can see that the empirical curve (in black) does not agree with the graph (dash line in red) of the function $h_1(\alpha) = \log(\alpha^2 \vee 2^{-1}) / \log(2)$ for $2\alpha^2 > 1$; but it is close to the graph (in blue) of the function $h_2(\alpha) = \log(\alpha^4 \vee 2^{-1}) / \log(2)$ for $\alpha \in (0, 1)$.

4. FURTHER NOTATIONS AND MOMENTS FORMULA

4.1. **Notations for average of different functions over different generations.** Recall that for $f \in L^1(\mu)$, we set $\tilde{f} = f - \langle \mu, f \rangle$. In order to study the asymptotics of $M_{\mathbb{G}_{n-\ell}}(\tilde{f})$, we shall

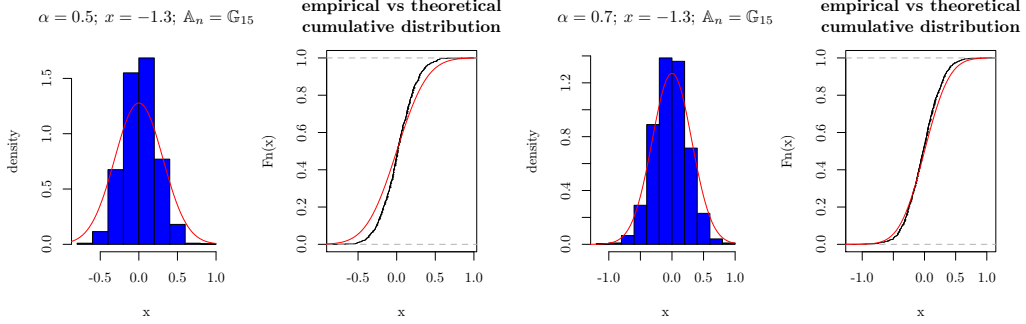
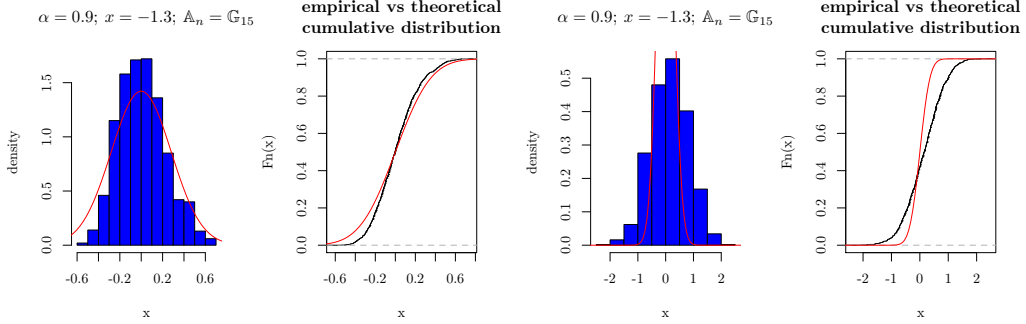


FIGURE 3. Histogram and empirical cumulative distribution of ζ_n given in (41) with $x = -1.3$, $n = 15$, $\mathbb{A}_n = \mathbb{G}_n$ and $\gamma = 1/5 + 10^{-3}$. We consider the ergodicity rate: $\alpha = 0.5$ for the two left graphics and $\alpha = 0.7$ for the two right.



(A) $\gamma = 0.696$

(B) $\gamma = 1/5 + 10^{-3}$

FIGURE 4. Histogram and empirical cumulative distribution of ζ_n given in (41) with $x = -1.3$, $n = 15$, $\mathbb{A}_n = \mathbb{G}_n$ and the ergodic rate $\alpha = 0.9$ (super-critical case). We consider the bandwidth exponent $\gamma = 0.696$ (which satisfies Assumption 3.9-(iv)) for the two left graphics and $\gamma = 1/5 + 10^{-3}$ (which does not satisfy Assumption 3.9-(iv)) for the two right.

consider the contribution of the descendants of the individual $i \in \mathbb{T}_{n-\ell}$ for $n \geq \ell \geq 0$:

$$(42) \quad N_{n,i}^\ell(f) = |\mathbb{G}_n|^{-1/2} M_{i\mathbb{G}_{n-|i|-\ell}}(\tilde{f}),$$

where $i\mathbb{G}_{n-|i|-\ell} = \{ij, j \in \mathbb{G}_{n-|i|-\ell}\} \subset \mathbb{G}_{n-\ell}$. For all $k \in \mathbb{N}$ such that $n \geq k + \ell$, we have:

$$M_{\mathbb{G}_{n-\ell}}(\tilde{f}) = \sqrt{|\mathbb{G}_n|} \sum_{i \in \mathbb{G}_k} N_{n,i}^\ell(f) = \sqrt{|\mathbb{G}_n|} N_{n,\emptyset}^\ell(f).$$

Let $\mathbf{f} = (f_\ell, \ell \in \mathbb{N})$ be a sequence of elements of $L^1(\mu)$. We set for $n \in \mathbb{N}$ and $i \in \mathbb{T}_n$:

$$(43) \quad N_{n,i}(\mathbf{f}) = \sum_{\ell=0}^{n-|i|} N_{n,i}^\ell(f_\ell) = |\mathbb{G}_n|^{-1/2} \sum_{\ell=0}^{n-|i|} M_{i\mathbb{G}_{n-|i|-\ell}}(\tilde{f}_\ell).$$

We deduce that $\sum_{i \in \mathbb{G}_k} N_{n,i}(\mathbf{f}) = |\mathbb{G}_n|^{-1/2} \sum_{\ell=0}^{n-k} M_{\mathbb{G}_{n-\ell}}(\tilde{f}_\ell)$ which gives for $k = 0$:

$$(44) \quad \boxed{N_{n,\emptyset}(\mathbf{f}) = |\mathbb{G}_n|^{-1/2} \sum_{\ell=0}^n M_{\mathbb{G}_{n-\ell}}(\tilde{f}_\ell).}$$

The notation $N_{n,\emptyset}$ means that we consider the average from the root \emptyset to the n -th generation. We shall prove the convergence in law of $N_{n,\emptyset}(\mathbf{f})$ for the point-wise and $L^2(\mu)$ approaches in the following sections.

Remark 4.1. We shall consider in particular the following two simple cases. Let $f \in L^1(\mu)$ and consider the sequence $\mathbf{f} = (f_\ell, \ell \in \mathbb{N})$. If $f_0 = f$ and $f_\ell = 0$ for $\ell \in \mathbb{N}^*$, then we get:

$$N_{n,\emptyset}(\mathbf{f}) = |\mathbb{G}_n|^{-1/2} M_{\mathbb{G}_n}(\tilde{f}).$$

If $f_\ell = f$ for $\ell \in \mathbb{N}$, then we shall write $\mathbf{f} = (f, f, \dots)$, and we get, as $|\mathbb{T}_n| = 2^{n+1} - 1$ and $|\mathbb{G}_n| = 2^n$:

$$N_{n,\emptyset}(\mathbf{f}) = |\mathbb{G}_n|^{-1/2} M_{\mathbb{T}_n}(\tilde{f}) = \sqrt{2 - 2^{-n}} |\mathbb{T}_n|^{-1/2} M_{\mathbb{T}_n}(\tilde{f}).$$

Thus, we will easily deduce the fluctuations of $M_{\mathbb{T}_n}(f)$ and $M_{\mathbb{G}_n}(f)$ from the asymptotics of $N_{n,\emptyset}(\mathbf{f})$.

To study the asymptotics of $N_{n,\emptyset}(\mathbf{f})$, it is convenient to write for $n \geq k \geq 1$:

$$(45) \quad N_{n,\emptyset}(\mathbf{f}) = |\mathbb{G}_n|^{-1/2} \sum_{r=0}^{k-1} M_{\mathbb{G}_r}(\tilde{f}_{n-r}) + \sum_{i \in \mathbb{G}_k} N_{n,i}(\mathbf{f}).$$

If $\mathbf{f} = (f, f, \dots)$ is the infinite sequence of the same function f , this becomes:

$$(46) \quad N_{n,\emptyset}(\mathbf{f}) = |\mathbb{G}_n|^{-1/2} M_{\mathbb{T}_n}(\tilde{f}) = |\mathbb{G}_n|^{-1/2} M_{\mathbb{T}_{k-1}}(\tilde{f}) + \sum_{i \in \mathbb{G}_k} N_{n,i}(\mathbf{f}).$$

In the following proofs, we will denote by C any unimportant finite constant which may vary from line to line (in particular C does not depend on n nor on \mathbf{f}).

4.2. Moments formula for BMC. Let $X = (X_i, i \in \mathbb{T})$ be a BMC on (S, \mathcal{S}) with probability kernel \mathcal{P} . Recall that $|\mathbb{G}_n| = 2^n$ and $M_{\mathbb{G}_n}(f) = \sum_{i \in \mathbb{G}_n} f(X_i)$. We also recall that $2\mathcal{Q}(x, A) = \mathcal{P}(x, A \times S) + \mathcal{P}(x, S \times A)$ for $A \in \mathcal{S}$. We use the convention that $\sum_{\emptyset} = 0$.

We recall the following well known and easy to establish many-to-one formulas for BMC.

Lemma 4.2. *Let $f, g \in \mathcal{B}(S)$, $x \in S$ and $n \geq m \geq 0$. Assuming that all the quantities below are well defined, we have:*

$$(47) \quad \mathbb{E}_x [M_{\mathbb{G}_n}(f)] = |\mathbb{G}_n| \mathcal{Q}^n f(x) = 2^n \mathcal{Q}^n f(x),$$

$$(48) \quad \mathbb{E}_x [M_{\mathbb{G}_n}(f)^2] = 2^n \mathcal{Q}^n (f^2)(x) + \sum_{k=0}^{n-1} 2^{n+k} \mathcal{Q}^{n-k-1} (\mathcal{P}(\mathcal{Q}^k f \otimes \mathcal{Q}^k f))(x),$$

$$(49) \quad \mathbb{E}_x [M_{\mathbb{G}_n}(f) M_{\mathbb{G}_m}(g)] = 2^n \mathcal{Q}^m (g \mathcal{Q}^{n-m} f)(x) + \sum_{k=0}^{m-1} 2^{n+k} \mathcal{Q}^{m-k-1} (\mathcal{P}(\mathcal{Q}^k g \otimes_{\text{sym}} \mathcal{Q}^{n-m+k} f))(x).$$

We also give some bounds on $\mathbb{E}_x [M_{\mathbb{G}_n}(f)^4]$, see the proof of Theorem 2.1 in [9]. We will use the notation:

$$g \otimes^2 = g \otimes g.$$

Lemma 4.3. *There exists a finite constant C such that for all $f \in \mathcal{B}(S)$, $n \in \mathbb{N}$ and ν a probability measure on S , assuming that all the quantities below are well defined, there exist functions $\psi_{j,n}$ for $1 \leq j \leq 9$ such that:*

$$\mathbb{E}_\nu [M_{\mathbb{G}_n}(f)^4] = \sum_{j=1}^9 \langle \nu, \psi_{j,n} \rangle,$$

and, with $h_k = \mathcal{Q}^{k-1}(f)$ and (notice that either $|\psi_j|$ or $|\langle \nu, \psi_j \rangle|$ is bounded), writing $\nu g = \langle \nu, g \rangle$:

$$\begin{aligned} |\psi_{1,n}| &\leq C 2^n \mathcal{Q}^n(f^4), \\ |\nu \psi_{2,n}| &\leq C 2^{2n} \sum_{k=0}^{n-1} 2^{-k} |\nu \mathcal{Q}^k \mathcal{P}(\mathcal{Q}^{n-k-1}(|f|^3) \otimes_{\text{sym}} h_{n-k})|, \\ |\psi_{3,n}| &\leq C 2^{2n} \sum_{k=0}^{n-1} 2^{-k} \mathcal{Q}^k \mathcal{P}(\mathcal{Q}^{n-k-1}(f^2) \otimes^2), \\ |\psi_{4,n}| &\leq C 2^{4n} \mathcal{P}(|\mathcal{P}(h_{n-1} \otimes^2) \otimes^2|), \\ |\psi_{5,n}| &\leq C 2^{4n} \sum_{k=2}^{n-1} \sum_{r=0}^{k-1} 2^{-2k-r} \mathcal{Q}^r \mathcal{P}(\mathcal{Q}^{k-r-1} |\mathcal{P}(h_{n-k} \otimes^2)| \otimes^2), \\ |\psi_{6,n}| &\leq C 2^{3n} \sum_{k=1}^{n-1} \sum_{r=0}^{k-1} 2^{-k-r} \mathcal{Q}^r |\mathcal{P}(\mathcal{Q}^{k-r-1} \mathcal{P}(h_{n-k} \otimes^2) \otimes_{\text{sym}} \mathcal{Q}^{n-r-1}(f^2))|, \\ |\nu \psi_{7,n}| &\leq C 2^{3n} \sum_{k=1}^{n-1} \sum_{r=0}^{k-1} 2^{-k-r} |\nu \mathcal{Q}^r \mathcal{P}(\mathcal{Q}^{k-r-1} \mathcal{P}(h_{n-k} \otimes_{\text{sym}} \mathcal{Q}^{n-k-1}(f^2)) \otimes_{\text{sym}} h_{n-r})|, \\ |\psi_{8,n}| &\leq C 2^{4n} \sum_{k=2}^{n-1} \sum_{r=1}^{k-1} \sum_{j=0}^{r-1} 2^{-k-r-j} \mathcal{Q}^j \mathcal{P}(|\mathcal{Q}^{r-j-1} \mathcal{P}(h_{n-r} \otimes^2)| \otimes_{\text{sym}} |\mathcal{Q}^{k-j-1} \mathcal{P}(h_{n-k} \otimes^2)|), \\ |\psi_{9,n}| &\leq C 2^{4n} \sum_{k=2}^{n-1} \sum_{r=1}^{k-1} \sum_{j=0}^{r-1} 2^{-k-r-j} \mathcal{Q}^j |\mathcal{P}(\mathcal{Q}^{r-j-1} \mathcal{P}(h_{n-r} \otimes_{\text{sym}} \mathcal{Q}^{k-r-1} \mathcal{P}(h_{n-k} \otimes^2)) \otimes_{\text{sym}} h_{n-j})|. \end{aligned}$$

We shall use the following lemma in order to bound the term $|\nu \psi_{2,n}|$.

Lemma 4.4. *Let μ be an invariant probability measure on S for \mathcal{Q} . Let $f, g \in L^4(\mu)$. Then we have for all $r \in \mathbb{N}$:*

$$\langle \mu, \mathcal{P}(\mathcal{Q}^r |f|^3 \otimes |g|) \rangle \leq 2 \|f\|_{L^4(\mu)}^3 \|g\|_{L^4(\mu)}.$$

Proof. We have

$$\begin{aligned} \langle \mu, \mathcal{P}(\mathcal{Q}^r |f|^3 \otimes |g|) \rangle &\leq \langle \mu, \mathcal{P}((\mathcal{Q}^r |f|^3)^{4/3} \otimes 1) \rangle^{3/4} \langle \mu, \mathcal{P}(1 \otimes g^4) \rangle^{1/4} \\ &\leq 2 \langle \mu, \mathcal{Q}((\mathcal{Q}^r |f|^3)^{4/3}) \rangle^{3/4} \langle \mu, \mathcal{Q}(g^4) \rangle^{1/4} \\ &\leq 2 \langle \mu, |f|^4 \rangle^{3/4} \langle \mu, |g|^4 \rangle^{1/4}, \end{aligned}$$

where we used Hölder inequality and that $v \otimes w = (v \otimes 1)(1 \otimes w)$ for the first inequality, that $\mathcal{P}(v \otimes 1) \leq 2\mathcal{Q}v$ and $\mathcal{P}(1 \otimes v) \leq 2\mathcal{Q}v$ if v is non-negative for the second inequality, Jensen's inequality and that μ is invariant for \mathcal{Q} for the last. \square

5. THE SUB-CRITICAL CASE: $2\alpha^2 < 1$

We shall consider, when well defined, for a sequence $\mathfrak{f} = (f_\ell, \ell \in \mathbb{N})$ of measurable real-valued functions defined on S , the quantities:

$$(50) \quad \Sigma^{\text{sub}}(\mathfrak{f}) = \Sigma_1^{\text{sub}}(\mathfrak{f}) + 2\Sigma_2^{\text{sub}}(\mathfrak{f}),$$

where:

$$(51) \quad \Sigma_1^{\text{sub}}(\mathfrak{f}) = \sum_{\ell \geq 0} 2^{-\ell} \langle \mu, \tilde{f}_\ell^2 \rangle + \sum_{\ell \geq 0, k \geq 0} 2^{k-\ell} \langle \mu, \mathcal{P} \left((\mathcal{Q}^k \tilde{f}_\ell) \otimes^2 \right) \rangle,$$

$$(52) \quad \Sigma_2^{\text{sub}}(\mathfrak{f}) = \sum_{0 \leq \ell < k} 2^{-\ell} \langle \mu, \tilde{f}_k \mathcal{Q}^{k-\ell} \tilde{f}_\ell \rangle + \sum_{\substack{0 \leq \ell < k \\ r \geq 0}} 2^{r-\ell} \langle \mu, \mathcal{P} \left(\mathcal{Q}^r \tilde{f}_k \otimes_{\text{sym}} \mathcal{Q}^{k-\ell+r} \tilde{f}_\ell \right) \rangle.$$

Remark 5.1. Recall the definitions of $\Sigma_{\mathbb{G}}^{\text{sub}}(f)$ and $\Sigma_{\mathbb{T}}^{\text{sub}}(f)$ given in (37). If we take $\mathfrak{f} = (f, 0, 0, \dots)$, we have $\Sigma^{\text{sub}}(\mathfrak{f}) = \Sigma_{\mathbb{G}}^{\text{sub}}(f)$. If we take $\mathfrak{f} = (f, f, \dots)$, the infinite sequence of the same function f , we have $\Sigma^{\text{sub}}(\mathfrak{f}) = 2\Sigma_{\mathbb{T}}^{\text{sub}}(f)$.

5.1. The point-wise approach. In this section Assumptions 2.4 and 2.6 are in force. We have the following result whose proof is given in Section 6.

Theorem 5.2. *Let X be a BMC with kernel \mathcal{P} and initial distribution ν such that Assumptions 2.4 and 2.6 are in force with $\alpha \in (0, 1/\sqrt{2})$. We have the following convergence in distribution for all sequence $\mathfrak{f} = (f_\ell, \ell \in \mathbb{N})$ of elements of F satisfying Assumptions 2.6 uniformly, that is (6) for some $g \in F$:*

$$N_{n,\emptyset}(\mathfrak{f}) \xrightarrow[n \rightarrow \infty]{(d)} G,$$

where G is a centered Gaussian random variable with variance $\Sigma^{\text{sub}}(\mathfrak{f})$ given by (50), which is well defined and finite.

Convergence in distribution of $N_{n,\emptyset}(\mathfrak{f})$ allows to recover the convergence in distribution of $|\mathbb{G}_n|^{-1/2}(M_{\mathbb{G}_n}(\tilde{f}_0), \dots, M_{\mathbb{G}_{n-k}}(\tilde{f}_k))$. We end this section with the proof of Corollary 3.1.

Proof of Corollary 3.1. Take the infinite sequence $\mathfrak{f} = (f, 0, 0, \dots)$, where only the first component is non-zero, to deduce from Theorem 5.2 the convergence in distribution of $|\mathbb{G}_n|^{-1/2}M_{\mathbb{G}_n}(\tilde{f}) = N_{n,\emptyset}(\mathfrak{f})$. Next, take the infinite sequence $\mathfrak{f} = \mathbf{f} = (f, f, \dots)$ of the same function f in Theorem 5.2 and use (46) and as well as $\lim_{n \rightarrow \infty} |\mathbb{G}_n|/|\mathbb{T}_n| = 1/2$, to get the convergence in distribution for $|\mathbb{T}_n|^{-1/2}M_{\mathbb{T}_n}(\tilde{f}) = (|\mathbb{G}_n|/|\mathbb{T}_n|)^{1/2}N_{n,\emptyset}(\mathbf{f})$. \square

5.2. The $L^2(\mu)$ approach. In this section Assumption 2.10 and 2.13 are in force. Because of condition (ii) in Assumption 2.10 which roughly state that after k_0 generations, the distribution of the induced Markov chain is absolutely continuous with respect to the invariant measure μ , it is better to consider the quantity $N_{n,\emptyset}(\mathfrak{f})$ defined in (44) (see also (45)) without the k_0 first generations. Mimicking the proof of Theorem 5.2 and Corollary 3.1, we get the following result, see a detailed proof in Section 7. Recall $N_{n,\emptyset}^{[k_0]}(\mathfrak{f})$ defined in (39).

Corollary 5.3. *Let X be a BMC with kernel \mathcal{P} and initial distribution ν such that Assumptions 2.10 (with $k_0 \in \mathbb{N}$) and 2.13 are in force with $\alpha \in (0, 1/\sqrt{2})$. We have the following convergence in distribution for all sequence $\mathfrak{f} = (f_\ell, \ell \in \mathbb{N})$ bounded in $L^4(\mu)$ (that is $\sup_{\ell \in \mathbb{N}} \|f_\ell\|_{L^4(\mu)} < +\infty$):*

$$N_{n,\emptyset}^{[k_0]}(\mathfrak{f}) \xrightarrow[n \rightarrow \infty]{(d)} G,$$

where G is centered Gaussian random variable with variance $\Sigma^{\text{sub}}(\mathbf{f})$ given by (50) which is well defined and finite.

6. PROOF OF THEOREM 5.2

Let $(p_n, n \in \mathbb{N})$ be a non-decreasing sequence of elements of \mathbb{N}^* such that, for all $\lambda > 0$:

$$(53) \quad p_n < n, \quad \lim_{n \rightarrow \infty} p_n/n = 1 \quad \text{and} \quad \lim_{n \rightarrow \infty} n - p_n - \lambda \log(n) = +\infty.$$

When there is no ambiguity, we write p for p_n .

Let $i, j \in \mathbb{T}$. We write $i \preceq j$ if $j \in i\mathbb{T}$. We denote by $i \wedge j$ the most recent common ancestor of i and j , which is defined as the only $u \in \mathbb{T}$ such that if $v \in \mathbb{T}$ and $v \preceq i, v \preceq j$ then $v \preceq u$. We also define the lexicographic order $i \leq j$ if either $i \preceq j$ or $v_0 \preceq i$ and $v_1 \preceq j$ for $v = i \wedge j$. Let $X = (X_i, i \in \mathbb{T})$ be a BMC with kernel \mathcal{P} and initial measure ν . For $i \in \mathbb{T}$, we define the σ -field:

$$\mathcal{F}_i = \{X_u; u \in \mathbb{T} \text{ such that } u \leq i\}.$$

By construction, the σ -fields $(\mathcal{F}_i; i \in \mathbb{T})$ are nested as $\mathcal{F}_i \subset \mathcal{F}_j$ for $i \leq j$.

We define for $n \in \mathbb{N}$, $i \in \mathbb{G}_{n-p_n}$ and $\mathbf{f} \in F^{\mathbb{N}}$ the martingale increments:

$$(54) \quad \Delta_{n,i}(\mathbf{f}) = N_{n,i}(\mathbf{f}) - \mathbb{E}[N_{n,i}(\mathbf{f}) | \mathcal{F}_i] \quad \text{and} \quad \Delta_n(\mathbf{f}) = \sum_{i \in \mathbb{G}_{n-p_n}} \Delta_{n,i}(\mathbf{f}).$$

Thanks to (43), we have:

$$\sum_{i \in \mathbb{G}_{n-p_n}} N_{n,i}(\mathbf{f}) = |\mathbb{G}_n|^{-1/2} \sum_{\ell=0}^{p_n} M_{\mathbb{G}_{n-\ell}}(\tilde{f}_\ell) = |\mathbb{G}_n|^{-1/2} \sum_{k=n-p_n}^n M_{\mathbb{G}_k}(\tilde{f}_{n-k}).$$

Using the branching Markov property, and (43), we get for $i \in \mathbb{G}_{n-p_n}$:

$$\mathbb{E}[N_{n,i}(\mathbf{f}) | \mathcal{F}_i] = \mathbb{E}[N_{n,i}(\mathbf{f}) | X_i] = |\mathbb{G}_n|^{-1/2} \sum_{\ell=0}^{p_n} \mathbb{E}_{X_i} [M_{\mathbb{G}_{p_n-\ell}}(\tilde{f}_\ell)].$$

We deduce from (45) with $k = n - p_n$ that:

$$(55) \quad N_{n,\emptyset}(\mathbf{f}) = \Delta_n(\mathbf{f}) + R_0(n) + R_1(n),$$

with

$$(56) \quad R_0(n) = |\mathbb{G}_n|^{-1/2} \sum_{k=0}^{n-p_n-1} M_{\mathbb{G}_k}(\tilde{f}_{n-k}) \quad \text{and} \quad R_1(n) = \sum_{i \in \mathbb{G}_{n-p_n}} \mathbb{E}[N_{n,i}(\mathbf{f}) | \mathcal{F}_i].$$

We have the following elementary lemma.

Lemma 6.1. *Under the assumptions of Theorem 5.2, we have the following convergence:*

$$\lim_{n \rightarrow \infty} \mathbb{E}[R_0(n)^2] = 0.$$

Proof. For all $k \geq 1$, we have:

$$\begin{aligned} \mathbb{E}_x[M_{\mathbb{G}_k}(\tilde{f}_{n-k})^2] &\leq 2^k g_1(x) + \sum_{\ell=0}^{k-1} 2^{k+\ell} \alpha^{2\ell} \mathcal{Q}^{k-\ell-1} (\mathcal{P}(g_2 \otimes g_2))(x) \\ &\leq 2^k g_1(x) + 2^k \sum_{\ell=0}^{k-1} (2\alpha^2)^\ell g_3(x) \\ &\leq 2^k g_4(x), \end{aligned}$$

with $g_1, g_2, g_3, g_4 \in F$ and where we used (48), (6) twice and (4) twice (with f and g replaced by $2(g^2 + \langle \mu, g \rangle^2)$ and g_1 , and with f and g replaced by g and g_2) for the first inequality, (4) (with f and g replaced by $\mathcal{P}(g_2 \otimes g_2)$ and g_3) for the second, and that $2\alpha^2 < 1$ and $g_4 = g_1 + (1 - 2\alpha^2)^{-1} g_3$ for the last. As $g_4 \in F \subset L^1(\nu)$, this implies that $\mathbb{E}[M_{\mathbb{G}_k}(\tilde{f}_{n-k})^2] \leq c^2 2^k$ for some finite constant c which does not depend on n or k . We can take c large enough, so that this upper bound holds also for $k = 0$ and all $n \in \mathbb{N}$, thanks to (6). We deduce that:

$$(57) \quad \mathbb{E}[R_0(n)^2]^{1/2} \leq |\mathbb{G}_n|^{-1/2} \sum_{k=0}^{n-p-1} \mathbb{E}[M_{\mathbb{G}_k}(\tilde{f}_{n-k})^2]^{1/2} \leq c 2^{-n/2} \sum_{k=0}^{n-p-1} 2^{k/2} \leq 3c 2^{-p/2}.$$

Use that $\lim_{n \rightarrow \infty} p = \infty$ to conclude. \square

We have the following lemma.

Lemma 6.2. *Under the assumptions of Theorem 5.2, we have the following convergence:*

$$\lim_{n \rightarrow \infty} \mathbb{E}[R_1(n)^2] = 0.$$

Proof. We set for $p \geq \ell \geq 0$:

$$(58) \quad R_1(\ell, n) = \sum_{i \in \mathbb{G}_{n-p}} \mathbb{E}[N_{n,i}^\ell(f_\ell) | \mathcal{F}_i],$$

so that, thanks to (43), $R_1(n) = \sum_{\ell=0}^p R_1(\ell, n)$. We have for $i \in \mathbb{G}_{n-p}$:

$$(59) \quad |\mathbb{G}_n|^{1/2} \mathbb{E}[N_{n,i}^\ell(f_\ell) | \mathcal{F}_i] = \mathbb{E}[M_{i\mathbb{G}_{p-\ell}}(\tilde{f}_\ell) | X_i] = \mathbb{E}_{X_i}[M_{\mathbb{G}_{p-\ell}}(\tilde{f}_\ell)] = |\mathbb{G}_{p-\ell}| \mathcal{Q}^{p-\ell} \tilde{f}_\ell(X_i),$$

where we used definition (42) of $N_{n,i}^\ell$ for the first equality, the Markov property of X for the second and (47) for the third. We deduce that:

$$R_1(\ell, n) = |\mathbb{G}_n|^{-1/2} |\mathbb{G}_{p-\ell}| M_{\mathbb{G}_{n-p}}(\mathcal{Q}^{p-\ell} \tilde{f}_\ell).$$

Using (48), we get:

$$\begin{aligned} \mathbb{E}_x[R_1(\ell, n)^2] &= |\mathbb{G}_n|^{-1} |\mathbb{G}_{p-\ell}|^2 \mathbb{E}_x \left[\left(M_{\mathbb{G}_{n-p}}(\mathcal{Q}^{p-\ell} \tilde{f}_\ell) \right)^2 \right] \\ &= |\mathbb{G}_n|^{-1} |\mathbb{G}_{p-\ell}|^2 2^{n-p} \mathcal{Q}^{n-p} ((\mathcal{Q}^{p-\ell} \tilde{f}_\ell)^2)(x) \\ &\quad + |\mathbb{G}_n|^{-1} |\mathbb{G}_{p-\ell}|^2 \sum_{k=0}^{n-p-1} 2^{n-p+k} \mathcal{Q}^{n-p-k-1} \left(\mathcal{P} \left(\mathcal{Q}^{k+p-\ell} \tilde{f}_\ell \otimes^2 \right) \right)(x). \end{aligned}$$

We deduce that:

$$\begin{aligned}
(60) \quad \mathbb{E}_x [R_1(\ell, n)^2] &\leq \alpha^{2(p-\ell)} 2^{p-2\ell} \mathcal{Q}^{n-p}(g^2)(x) + 2^{p-2\ell} \sum_{k=0}^{n-p-1} \alpha^{2(k+p-\ell)} 2^k \mathcal{Q}^{n-p-k-1} (\mathcal{P}(g \otimes g)) \\
&\leq \alpha^{2(p-\ell)} 2^{p-2\ell} \left(g_1(x) + \sum_{k=0}^{n-p-1} (2\alpha^2)^k g_2(x) \right) \\
&\leq (2\alpha^2)^p (2\alpha)^{-2\ell} g_3(x),
\end{aligned}$$

with $g_1, g_2, g_3 \in F$ and where we used (6) for the first inequality, (4) twice (with f and g replaced by g^2 and g_1 and by $\mathcal{P}(g \otimes g)$ and g_2) for the second, and that $2\alpha^2 < 1$ for the last. Since $g_3 \in F \subset L^1(\nu)$, this gives that $\mathbb{E} [R_1(\ell, n)^2] \leq (2\alpha^2)^p (2\alpha)^{-2\ell} \langle \nu, g_3 \rangle$. We deduce that:

$$\mathbb{E} [R_1(n)^2]^{1/2} \leq \sum_{\ell=0}^p \mathbb{E} [R_1(\ell, n)^2]^{1/2} \leq a_{1,n} \langle \nu, g_3 \rangle^{1/2},$$

with the sequence $(a_{1,n}, n \in \mathbb{N})$ defined by:

$$(61) \quad a_{1,n} = (2\alpha^2)^{p/2} \sum_{\ell=0}^p (2\alpha)^{-\ell}.$$

Notice the sequence $(a_{1,n}, n \in \mathbb{N})$ converges to 0 since $\lim_{n \rightarrow \infty} p = \infty$, $2\alpha^2 < 1$ and

$$\sum_{\ell=0}^p (2\alpha)^{-\ell} \leq \begin{cases} 2\alpha/(2\alpha-1) & \text{if } 2\alpha > 1, \\ p+1 & \text{if } 2\alpha = 1, \\ (2\alpha)^{-p}/(1-2\alpha) & \text{if } 2\alpha < 1. \end{cases}$$

We conclude that $\lim_{n \rightarrow \infty} \mathbb{E} [R_1(n)^2] = 0$. □

We now study the bracket of Δ_n :

$$(62) \quad V(n) = \sum_{i \in \mathbb{G}_{n-p_n}} \mathbb{E} [\Delta_{n,i}(f)^2 | \mathcal{F}_i].$$

Using (43) and (54), we write:

$$(63) \quad V(n) = |\mathbb{G}_n|^{-1} \sum_{i \in \mathbb{G}_{n-p_n}} \mathbb{E}_{X_i} \left[\left(\sum_{\ell=0}^{p_n} M_{\mathbb{G}_{p_n-\ell}}(\tilde{f}_\ell) \right)^2 \right] - R_2(n) = V_1(n) + 2V_2(n) - R_2(n),$$

with:

$$\begin{aligned}
V_1(n) &= |\mathbb{G}_n|^{-1} \sum_{i \in \mathbb{G}_{n-p_n}} \sum_{\ell=0}^{p_n} \mathbb{E}_{X_i} [M_{\mathbb{G}_{p_n-\ell}}(\tilde{f}_\ell)^2], \\
V_2(n) &= |\mathbb{G}_n|^{-1} \sum_{i \in \mathbb{G}_{n-p_n}} \sum_{0 \leq \ell < k \leq p_n} \mathbb{E}_{X_i} [M_{\mathbb{G}_{p_n-\ell}}(\tilde{f}_\ell) M_{\mathbb{G}_{p_n-k}}(\tilde{f}_k)], \\
R_2(n) &= \sum_{i \in \mathbb{G}_{n-p_n}} \mathbb{E} [N_{n,i}(f) | X_i]^2.
\end{aligned}$$

Lemma 6.3. *Under the assumptions of Theorem 5.2, we have the following convergence:*

$$\lim_{n \rightarrow \infty} \mathbb{E} [R_2(n)] = 0.$$

Proof. We define the sequence $(a_{2,n}, n \in \mathbb{N})$ for $n \in \mathbb{N}$ by:

$$(64) \quad a_{2,n} = 2^{-p} \left(\sum_{\ell=0}^p (2\alpha)^\ell \right)^2.$$

Notice that the sequence $(a_{2,n}, n \in \mathbb{N})$ converges to 0 since $\lim_{n \rightarrow \infty} p = \infty$, $2\alpha^2 < 1$ and

$$\sum_{\ell=0}^p (2\alpha)^\ell \leq \begin{cases} (2\alpha)^{p+1}/(2\alpha-1) & \text{if } 2\alpha > 1, \\ p+1 & \text{if } 2\alpha = 1, \\ 1/(1-2\alpha) & \text{if } 2\alpha < 1. \end{cases}$$

We now compute $\mathbb{E}_x [R_2(n)]$.

$$(65) \quad \begin{aligned} \mathbb{E}_x [R_2(n)] &= |\mathbb{G}_n|^{-1} \sum_{i \in \mathbb{G}_{n-p}} \mathbb{E}_x \left[\mathbb{E}_x \left[\sum_{\ell=0}^p M_{i\mathbb{G}_{p-\ell}}(\tilde{f}_\ell) | X_i \right]^2 \right] \\ &= |\mathbb{G}_n|^{-1} \sum_{i \in \mathbb{G}_{n-p}} \mathbb{E}_x \left[\left(\sum_{\ell=0}^p \mathbb{E}_{X_i} [M_{\mathbb{G}_{p-\ell}}(\tilde{f}_\ell)] \right)^2 \right] \\ &= |\mathbb{G}_n|^{-1} |\mathbb{G}_{n-p}| \mathcal{Q}^{n-p} \left(\left(\sum_{\ell=0}^p |\mathbb{G}_{p-\ell}| \mathcal{Q}^{p-\ell} \tilde{f}_\ell \right)^2 \right) (x) \\ &\leq 2^{-p} \left(\sum_{\ell=0}^p (2\alpha)^{p-\ell} \right)^2 \mathcal{Q}^{n-p}(g^2)(x) \\ &\leq a_{2,n} g_1(x), \end{aligned}$$

with $g_1 \in F$ and where we used the definition of $N_{n,i}(f)$ for the first equality, the Markov property of X for the second, (47) for the third, (6) for the first inequality, and (4) (with f and g replaced by g^2 and g_1) for the last. We conclude that $\lim_{n \rightarrow \infty} \mathbb{E} [R_2(n)] = 0$, using that $\langle \nu, g_1 \rangle$ is finite as $g_1 \in F \subset L^1(\nu)$. \square

We have the following technical lemma.

Lemma 6.4. *Under the assumptions of Theorem 5.2, we have that $\Sigma_2^{\text{sub}}(f)$ defined in (52) is well defined and finite, and that a.s. $\lim_{n \rightarrow \infty} V_2(n) = \Sigma_2^{\text{sub}}(f) < +\infty$.*

Proof. Using (49), we get:

$$(66) \quad V_2(n) = V_5(n) + V_6(n),$$

with

$$\begin{aligned} V_5(n) &= |\mathbb{G}_n|^{-1} \sum_{i \in \mathbb{G}_{n-p}} \sum_{0 \leq \ell < k \leq p} 2^{p-\ell} \mathcal{Q}^{p-k} \left(\tilde{f}_k \mathcal{Q}^{k-\ell} \tilde{f}_\ell \right) (X_i), \\ V_6(n) &= |\mathbb{G}_n|^{-1} \sum_{i \in \mathbb{G}_{n-p}} \sum_{0 \leq \ell < k < p} \sum_{r=0}^{p-k-1} 2^{p-\ell+r} \mathcal{Q}^{p-1-(r+k)} \left(\mathcal{P} \left(\mathcal{Q}^r \tilde{f}_k \otimes_{\text{sym}} \mathcal{Q}^{k-\ell+r} \tilde{f}_\ell \right) \right) (X_i). \end{aligned}$$

We consider the term $V_6(n)$. We have:

$$(67) \quad V_6(n) = |\mathbb{G}_{n-p}|^{-1} M_{\mathbb{G}_{n-p}}(H_{6,n}),$$

with:

$$(68) \quad H_{6,n} = \sum_{\substack{0 \leq \ell < k \\ r \geq 0}} h_{k,\ell,r}^{(n)} \mathbf{1}_{\{r+k < p\}} \quad \text{and} \quad h_{k,\ell,r}^{(n)} = 2^{r-\ell} \mathcal{Q}^{p-1-(r+k)} \left(\mathcal{P} \left(\mathcal{Q}^r \tilde{f}_k \otimes_{\text{sym}} \mathcal{Q}^{k-\ell+r} \tilde{f}_\ell \right) \right).$$

Using (5) and since $\mathcal{P}(\mathcal{Q}^r(F) \otimes \mathcal{Q}^{k-\ell+r}(F)) \subset F$ and $\lim_{n \rightarrow \infty} p = +\infty$, we have that:

$$\lim_{n \rightarrow \infty} h_{k,\ell,r}^{(n)} = h_{k,\ell,r},$$

where the constant $h_{k,\ell,r}$ is equal to $2^{r-\ell} \langle \mu, \mathcal{P}(\mathcal{Q}^r \tilde{f}_k \otimes_{\text{sym}} \mathcal{Q}^{k-\ell+r} \tilde{f}_\ell) \rangle$. Using (5), we also have that:

$$\begin{aligned} |h_{k,\ell,r}^{(n)}| &\leq 2^{r-\ell} \alpha^{k-\ell+2r} \mathcal{Q}^{p-1-(r+k)} (\mathcal{P}(g \otimes g)) \\ &\leq 2^{r-\ell} \alpha^{k-\ell+2r} g_*, \end{aligned}$$

with $g_* \in F$ (which does not depend on n, r, k and ℓ) and where we used (6) for the first inequality and (4) (with f and g replaced by $\mathcal{P}(g \otimes g)$ and g_*). Taking the limit, we also deduce that:

$$|h_{k,\ell,r}| \leq 2^{r-\ell} \alpha^{k-\ell+2r} g_*.$$

Define the constant

$$(69) \quad H_6(\mathbf{f}) = \sum_{\substack{0 \leq \ell < k \\ r \geq 0}} h_{k,\ell,r} = \sum_{\substack{0 \leq \ell < k \\ r \geq 0}} 2^{r-\ell} \langle \mu, \mathcal{P}(\mathcal{Q}^r \tilde{f}_k \otimes_{\text{sym}} \mathcal{Q}^{k-\ell+r} \tilde{f}_\ell) \rangle$$

which is finite as

$$(70) \quad \sum_{0 \leq \ell < k, r \geq 0} 2^{r-\ell} \alpha^{k-\ell+2r} = \frac{2\alpha}{(1-\alpha)(1-2\alpha^2)} < +\infty.$$

Using (5) (with f and g replaced by $\mathcal{P}(\mathcal{Q}^r \tilde{f}_k \otimes_{\text{sym}} \mathcal{Q}^{k-\ell+r} \tilde{f}_\ell)$ and $g_{k,\ell,r}$), we deduce that:

$$|h_{k,\ell,r}^{(n)} - h_{k,\ell,r}| \leq 2^{r-\ell} \alpha^{p-1-(r+k)} g_{k,\ell,r}.$$

Set $r_0 \in \mathbb{N}^*$ and $g_{r_0} = \sum_{0 \leq \ell < k; r \geq 0; k \vee r \leq r_0} g_{k,\ell,r}$. Notice that g_{r_0} belongs to F and is non-negative. Furthermore, we have:

$$\begin{aligned} |H_{6,n} - H_6(\mathbf{f})| &\leq \sum_{\substack{0 \leq \ell < k \\ r \geq 0 \\ k \vee r \leq r_0}} 2^{r-\ell} \alpha^{p-1-(r+k)} g_{r_0} + \sum_{\substack{0 \leq \ell < k \\ r \geq 0 \\ r \vee k > r_0}} \left(|h_{k,\ell,r}^{(n)}| \mathbf{1}_{\{r+k < p\}} + |h_{k,\ell,r}| \right) \\ &\leq (r_0 + 1)^2 2^{r_0+1} \alpha^{p-1-2r_0} g_{r_0} + \gamma_1(r_0) g_*, \end{aligned}$$

with

$$\gamma_1(r_0) = \sum_{\substack{0 \leq \ell < k \\ r \geq 0 \\ r \vee k > r_0}} 2^{r-\ell} \alpha^{k-\ell+2r}.$$

Using (36) with n replaced by $n-p$ and f replaced by g_* and g_{r_0} , and that $\lim_{n \rightarrow \infty} \alpha^p = 0$ as well as $\lim_{n \rightarrow \infty} n-p = \infty$, we deduce that:

$$\limsup_{n \rightarrow \infty} |\mathbb{G}_{n-p}|^{-1} M_{\mathbb{G}_{n-p}}(|H_{6,n} - H_6(\mathbf{f})|) \leq \gamma(r_0) \langle \mu, g_* \rangle.$$

Thanks to (70), we get by dominated convergence that $\lim_{r_0 \rightarrow \infty} \gamma_1(r_0) = 0$. This implies that:

$$\lim_{n \rightarrow \infty} |\mathbb{G}_{n-p}|^{-1} M_{\mathbb{G}_{n-p}}(|H_{6,n} - H_6(\mathbf{f})|) = 0.$$

Since $|\mathbb{G}_{n-p}|^{-1} M_{\mathbb{G}_{n-p}}(\cdot)$ is a probability measure, we deduce from (67) that a.s.:

$$\lim_{n \rightarrow \infty} V_6(n) = \lim_{n \rightarrow \infty} |\mathbb{G}_{n-p}|^{-1} M_{\mathbb{G}_{n-p}}(H_{6,n}) = H_6(f) = \sum_{\substack{0 \leq \ell < k \\ r \geq 0}} 2^{r-\ell} \langle \mu, \mathcal{P} \left(\mathcal{Q}^r \tilde{f}_k \otimes_{\text{sym}} \mathcal{Q}^{k-\ell+r} \tilde{f}_\ell \right) \rangle.$$

Similarly, we get that a.s. $\lim_{n \rightarrow \infty} V_5(n) = H_5(f)$, with the finite constant $H_5(f)$ defined by:

$$(71) \quad H_5(f) = \sum_{0 \leq \ell < k} 2^{-\ell} \langle \mu, \tilde{f}_k \mathcal{Q}^{k-\ell} \tilde{f}_\ell \rangle.$$

Notice that $\Sigma_2^{\text{sub}}(f) = H_5(f) + H_6(f)$ is finite thanks to (6) and (70). This finishes the proof. \square

Using similar arguments as in the proof of Lemma 6.4, we get the following result.

Lemma 6.5. *Under the assumptions of Theorem 5.2, we have that $\Sigma_1^{\text{sub}}(f)$ in (51) is well defined and finite, and that a.s. $\lim_{n \rightarrow \infty} V_1(n) = \Sigma_1^{\text{sub}}(f)$.*

Proof. Using (48), we get:

$$(72) \quad V_1(n) = V_3(n) + V_4(n),$$

with

$$V_3(n) = |\mathbb{G}_n|^{-1} \sum_{i \in \mathbb{G}_{n-p}} \sum_{\ell=0}^p 2^{p-\ell} \mathcal{Q}^{p-\ell}(\tilde{f}_\ell^2)(X_i),$$

$$V_4(n) = |\mathbb{G}_n|^{-1} \sum_{i \in \mathbb{G}_{n-p}} \sum_{\ell=0}^{p-1} \sum_{k=0}^{p-\ell-1} 2^{p-\ell+k} \mathcal{Q}^{p-1-(\ell+k)} \left(\mathcal{P} \left(\mathcal{Q}^k \tilde{f}_\ell \otimes^2 \right) \right) (X_i).$$

We consider the term $V_4(n)$. We have:

$$(73) \quad V_4(n) = |\mathbb{G}_{n-p}|^{-1} M_{\mathbb{G}_{n-p}}(H_{4,n}),$$

with:

$$(74) \quad H_{4,n} = \sum_{\ell \geq 0, k \geq 0} h_{\ell,k}^{(n)} \mathbf{1}_{\{\ell+k < p\}} \quad \text{and} \quad h_{\ell,k}^{(n)} = 2^{k-\ell} \mathcal{Q}^{p-1-(\ell+k)} \left(\mathcal{P} \left(\mathcal{Q}^k \tilde{f}_\ell \otimes^2 \right) \right).$$

Using (5), we have that:

$$\lim_{n \rightarrow \infty} h_{\ell,k}^{(n)} = h_{\ell,k},$$

where the constant $h_{\ell,k}$ is equal to $2^{k-\ell} \langle \mu, \mathcal{P} \left(\mathcal{Q}^k \tilde{f}_\ell \otimes^2 \right) \rangle$. We also have that:

$$\begin{aligned} |h_{\ell,k}^{(n)}| &\leq 2^{k-\ell} \alpha^{2k} \mathcal{Q}^{p-1-(\ell+k)} (\mathcal{P}(g \otimes g)) \\ &\leq 2^{k-\ell} \alpha^{2k} g_*, \end{aligned}$$

with $g_* \in F$ (which does not depend on n, ℓ and k) and where we used (6) for the first inequality and (4) (with f and g replaced by $\mathcal{P}(g \otimes g)$ and g_*). Taking the limit, we also deduce that:

$$|h_{\ell,k}| \leq 2^{k-\ell} \alpha^{2k} g_*.$$

Define the constant

$$(75) \quad H_4(f) = \sum_{\ell \geq 0, k \geq 0} h_{\ell,k},$$

which is finite as:

$$(76) \quad \sum_{\ell \geq 0, k \geq 0} 2^{k-\ell} \alpha^{2k} = 2/(1-2\alpha^2) < +\infty.$$

Using (5) (with f and g replaced by $\mathcal{P}(\mathcal{Q}^k \tilde{f}_\ell \otimes^2)$ and $g_{\ell,k}$), we deduce that:

$$|h_{\ell,k}^{(n)} - h_{\ell,k}| \leq 2^{k-\ell} \alpha^{p-1-(\ell+k)} g_{\ell,k},$$

Set $r_0 \in \mathbb{N}$ and $g_{r_0} = \sum_{\ell \vee k \leq r_0} g_{\ell,k}$. Notice that g_{r_0} belongs to F . Furthermore, we have:

$$\begin{aligned} |H_{4,n} - H_4(f)| &\leq \sum_{\ell \vee k \leq r_0} 2^{k-\ell} \alpha^{p-1-(\ell+k)} g_{r_0} + \sum_{\ell \vee k > r_0} \left(|h_{\ell,k}^{(n)}| \mathbf{1}_{\{\ell+k \leq p-1\}} + |h_{\ell,k}| \right) \\ &\leq (r_0 + 1)^2 2^{r_0} \alpha^{p-1-2r_0} g_{r_0} + \gamma_2(r_0) g_*, \end{aligned}$$

with $\gamma_2(r_0) = 2 \sum_{\ell \vee k > r_0} 2^{k-\ell} \alpha^{2k}$. Using (36) with n replaced by $n-p$ and f replaced by g_* and g_{r_0} , and that $\lim_{n \rightarrow \infty} \alpha^p = 0$ as well as $\lim_{n \rightarrow \infty} n-p = \infty$, we deduce that:

$$\limsup_{n \rightarrow \infty} |\mathbb{G}_{n-p}|^{-1} M_{\mathbb{G}_{n-p}}(|H_{4,n} - H_4(f)|) \leq \gamma_2(r_0) \langle \mu, g_* \rangle.$$

Thanks to (76), we get by dominated convergence that $\lim_{r_0 \rightarrow \infty} \gamma_2(r_0) = 0$. We deduce that:

$$\lim_{n \rightarrow \infty} |\mathbb{G}_{n-p}|^{-1} M_{\mathbb{G}_{n-p}}(|H_{4,n} - H_4(f)|) = 0.$$

Since $|\mathbb{G}_{n-p}|^{-1} M_{\mathbb{G}_{n-p}}(\cdot)$ is a probability measure, we deduce from (73) that a.s.:

$$\lim_{n \rightarrow \infty} V_4(n) = \lim_{n \rightarrow \infty} |\mathbb{G}_{n-p}|^{-1} M_{\mathbb{G}_{n-p}}(H_{4,n}) = H_4(f) = \sum_{\ell \geq 0, k \geq 0} 2^{k-\ell} \langle \mu, \mathcal{P}(\mathcal{Q}^k \tilde{f}_\ell \otimes^2) \rangle.$$

Similarly, we get that a.s. $\lim_{n \rightarrow \infty} V_3(n) = H_3(f)$ with the finite constant $H_3(f)$ defined by

$$(77) \quad H_3(f) = \sum_{\ell \geq 0} 2^{-\ell} \langle \mu, \tilde{f}_\ell^2 \rangle.$$

Notice that $\Sigma_1^{\text{sub}}(f) = H_3(f) + H_4(f)$ is finite thanks to (6) and (76). This finishes the proof. \square

The next Lemma is a direct consequence of (63) and Lemmas 6.3, 6.4 and 6.5.

Lemma 6.6. *Under the assumptions of Theorem 5.2, we have the following convergence in probability $\lim_{n \rightarrow \infty} V(n) = \Sigma^{\text{sub}}(f)$, where, with $\Sigma_1^{\text{sub}}(f)$ and $\Sigma_2^{\text{sub}}(f)$ defined by (51) and (52):*

$$\Sigma^{\text{sub}}(f) = \Sigma^{\text{sub}}(f) = \Sigma_1^{\text{sub}}(f) + 2\Sigma_2^{\text{sub}}(f).$$

We now check the Lindeberg condition using a fourth moment condition. We set:

$$(78) \quad R_3(n) = \sum_{i \in \mathbb{G}_{n-p_n}} \mathbb{E} [\Delta_{n,i}(f)^4].$$

Lemma 6.7. *Under the assumptions of Theorem 5.2, we have that $\lim_{n \rightarrow \infty} R_3(n) = 0$.*

Proof. We have:

$$(79) \quad \begin{aligned} R_3(n) &\leq 16 \sum_{i \in \mathbb{G}_{n-p}} \mathbb{E} [N_{n,i}(f)^4] \\ &\leq 16(p+1)^3 \sum_{\ell=0}^p \sum_{i \in \mathbb{G}_{n-p}} \mathbb{E} [N_{n,i}^\ell(\tilde{f}_\ell)^4], \end{aligned}$$

where we used that $(\sum_{k=0}^r a_k)^4 \leq (r+1)^3 \sum_{k=0}^r a_k^4$ for the two inequalities (resp. with $r=1$ and $r=p$) and also Jensen inequality and (54) for the first and (43) for the last. Using (42), we get:

$$(80) \quad \mathbb{E} \left[N_{n,i}^\ell (\tilde{f}_\ell)^4 \right] = |\mathbb{G}_n|^{-2} \mathbb{E} [h_{n,\ell}(X_i)], \quad \text{with} \quad h_{n,\ell}(x) = \mathbb{E}_x \left[M_{\mathbb{G}_{p-\ell}}(\tilde{f}_\ell)^4 \right].$$

Thanks to Lemma 4.3, we get there exists $g_1 \in F$ such that for all $n \geq p \geq \ell \geq 0$:

$$(81) \quad |h_{n,\ell}| \leq 2^{2(p-\ell)} g_1.$$

We deduce that:

$$\begin{aligned} R_3(n) &\leq 16n^3 \sum_{\ell=0}^p \sum_{i \in \mathbb{G}_{n-p}} |\mathbb{G}_n|^{-2} 2^{2(p-\ell)} \mathbb{E} [g_1(X_i)] \\ &\leq 16n^3 2^{-2(n-p)} \mathbb{E} [M_{\mathbb{G}_{n-p}}(g_1)] \\ &\leq 16n^3 2^{-(n-p)} \langle \nu, \mathcal{Q}^{n-p} g_1 \rangle, \end{aligned}$$

where we used (47) for the third inequality. Since g_1 belongs to F , we deduce from (4) that $\mathcal{Q}^{n-p} g_1 \leq g_2$ for some $g_2 \in F$ and all $n \geq p \geq 0$. This gives that:

$$R_3(n) \leq 16n^3 2^{-(n-p)} \langle \nu, g_2 \rangle.$$

This ends the proof as $\lim_{n \rightarrow \infty} p = \infty$ and $\lim_{n \rightarrow \infty} n - p - \lambda \log(n) = +\infty$ for all $\lambda > 0$. \square

We can now use Theorem 3.2 and Corollary 3.1, p. 58, and the remark p. 59 from [22] to deduce from Lemmas 6.6 and 6.7 that $\Delta_n(f)$ converges in distribution towards a Gaussian real-valued random variable with deterministic variance $\Sigma^{\text{sub}}(f)$ given by (50). Using (55) and Lemmas 6.1 and 6.2, we then deduce Theorem 5.2.

7. PROOF OF COROLLARY 5.3

We first state a very useful Lemma which holds in sub-critical, critical and super-critical cases.

Lemma 7.1. *Let X be a BMC with kernel \mathcal{P} and initial distribution ν such that (ii) from Assumption 2.10 (with $k_0 \in \mathbb{N}$) is in force. There exists a finite constant C , such that for all $f \in \mathcal{B}_+(S)$ all $n \geq k_0$, we have:*

$$(82) \quad |\mathbb{G}_n|^{-1} \mathbb{E}[M_{\mathbb{G}_n}(f)] \leq C \|f\|_{L^1(\mu)} \quad \text{and} \quad |\mathbb{G}_n|^{-1} \mathbb{E} [M_{\mathbb{G}_n}(f)^2] \leq C \sum_{k=0}^n 2^k \|Q^k f\|_{L^2(\mu)}^2.$$

Proof. Using the first moment formula (47), (ii) from Assumption 2.10 and the fact that μ is invariant for \mathcal{Q} , we get that:

$$|\mathbb{G}_n|^{-1} \mathbb{E}[M_{\mathbb{G}_n}(f)] = \langle \nu, \mathcal{Q}^n f \rangle \leq \|\nu_0\|_\infty \langle \mu, \mathcal{Q}^{n-k_0} f \rangle = \|\nu_0\|_\infty \langle \mu, f \rangle.$$

We also have:

$$\begin{aligned}
|\mathbb{G}_n|^{-1} \mathbb{E} [M_{\mathbb{G}_n}(f)^2] &= \langle \nu, \mathcal{Q}^n(f^2) \rangle + \sum_{k=0}^{n-1} 2^k \langle \nu, \mathcal{Q}^{n-k-1} (\mathcal{P}(\mathcal{Q}^k f \otimes^2)) \rangle \\
&\leq \langle \nu, \mathcal{Q}^n(f^2) \rangle + \sum_{k=0}^{n-1} 2^k \langle \nu, \mathcal{Q}^{n-k} ((\mathcal{Q}^k f)^2) \rangle \\
&\leq \langle \nu, \mathcal{Q}^n(f^2) \rangle + \sum_{k=0}^{n-k_0} 2^k \langle \nu, \mathcal{Q}^{n-k} ((\mathcal{Q}^k f)^2) \rangle + \sum_{k=n-k_0+1}^{n-1} 2^k \langle \nu, \mathcal{Q}^{k_0} ((\mathcal{Q}^{n-k_0} f)^2) \rangle \\
&\leq C \sum_{k=0}^{n-k_0} 2^k \|\mathcal{Q}^k f\|_{L^2(\mu)}^2,
\end{aligned}$$

where we used the second moment formula (48) for the equality, (3) for the first inequality, Jensen inequality for the second, and (ii) from Assumption 2.10 and the fact that μ is invariant for \mathcal{Q} for the last. \square

We set for $k \in \mathbb{N}^*$:

$$(83) \quad c_k(\mathbf{f}) = \sup_{n \in \mathbb{N}} \|f_n\|_{L^k(\mu)} \quad \text{and} \quad q_k(\mathbf{f}) = \sup_{n \in \mathbb{N}} \|\mathcal{Q}(f_n^k)\|_{\infty}^{1/k}.$$

We now follow very closely the proof of theorem 5.2 and we keep notations from this section. We will denote by C any unimportant finite constant which may vary from line to line (but in particular C does not depend on n nor on \mathbf{f} , but may depends on k_0 and $\|\nu_0\|_{\infty}$). Recall $(p_n, n \in \mathbb{N})$ is such that (53) holds. Assume that n is large enough so that $n - p_n - 1 \geq k_0$. We have:

$$N_{n, \emptyset}^{[k_0]}(\mathbf{f}) = \Delta_n(\mathbf{f}) + R_0^{k_0}(n) + R_1(n),$$

where $\Delta_n(\mathbf{f})$ and $R_1(n)$ are defined in (54) and (56), and :

$$R_0^{k_0}(n) = |\mathbb{G}_n|^{-1/2} \sum_{k=k_0}^{n-p_n-1} M_{\mathbb{G}_k}(\tilde{f}_{n-k}).$$

Lemma 7.2. *Under the assumptions of Corollary 5.3, we have the following convergence:*

$$\lim_{n \rightarrow \infty} \mathbb{E}[R_0^{k_0}(n)^2] = 0.$$

Proof. Assume $n - p \geq k_0$. We write:

$$R_0^{k_0}(n) = |\mathbb{G}_n|^{-1/2} \sum_{k=k_0}^{n-p-1} \sum_{i \in \mathbb{G}_{k_0}} M_{i\mathbb{G}_{k-k_0}}(\tilde{f}_{n-k}).$$

We have that $\sum_{i \in \mathbb{G}_{k_0}} \mathbb{E}[M_{i\mathbb{G}_{k-k_0}}(\tilde{f}_{n-k})^2] = \mathbb{E}[M_{\mathbb{G}_{k_0}}(h_{k,n})]$, where:

$$h_{k,n}(x) = \mathbb{E}_x[M_{\mathbb{G}_{k-k_0}}(\tilde{f}_{n-k})^2].$$

We deduce from (ii) from Assumption 2.10, see (82), that $\mathbb{E}[M_{\mathbb{G}_{k_0}}(h_{k,n})] \leq C \langle \mu, h_{k,n} \rangle$. We have also that:

$$\langle \mu, h_{k,n} \rangle = \mathbb{E}_{\mu}[M_{\mathbb{G}_{k-k_0}}(\tilde{f}_{n-k})^2] \leq C 2^k \sum_{\ell=0}^k 2^{\ell} \|\mathcal{Q}^{\ell} \tilde{f}_{n-k}\|_{L^2(\mu)}^2 \leq C 2^k c_2^2(\mathbf{f}) \sum_{\ell=0}^k 2^{\ell} \alpha^{2\ell} \leq C 2^k c_2^2(\mathbf{f}),$$

where we used (82) for the first inequality (notice one can take $k_0 = 0$ in this case as we consider the expectation \mathbb{E}_μ), (20) in the second, and $2\alpha^2 < 1$ in the last. We deduce that:

$$(84) \quad \mathbb{E}[R_0^{k_0}(n)^2]^{1/2} \leq |\mathbb{G}_n|^{-1/2} \sum_{k=k_0}^{n-p-1} (2^{k_0} \mathbb{E}[M_{\mathbb{G}_{k_0}}(h_{k,n})])^{1/2} \leq C 2^{-p/2} c_2(\mathfrak{f}),$$

where we used that the sequence \mathfrak{f} is bounded in $L^2(\mu)$. Use that $\lim_{n \rightarrow \infty} p = \infty$ to conclude. \square

We have the following lemma.

Lemma 7.3. *Under the assumptions of Corollary 5.3, we have the following convergence:*

$$\lim_{n \rightarrow \infty} \mathbb{E}[R_1(n)^2] = 0.$$

Proof. We set for $p \geq \ell \geq 0$, $n - p \geq k_0$ and $j \in \mathbb{G}_{k_0}$:

$$R_{1,j}(\ell, n) = \sum_{i \in j\mathbb{G}_{n-p-k_0}} \mathbb{E}[N_{n,i}^\ell(f_\ell) | \mathcal{F}_i],$$

so that $R_1(n) = \sum_{\ell=0}^p \sum_{j \in \mathbb{G}_{k_0}} R_{1,j}(\ell, n)$. Using (59), we get for $j \in \mathbb{G}_{k_0}$:

$$R_{1,j}(\ell, n) = |\mathbb{G}_n|^{-1/2} |\mathbb{G}_{p-\ell}| M_{j\mathbb{G}_{n-p-k_0}}(Q^{p-\ell} \tilde{f}_\ell).$$

We deduce from the Markov property of X that $\mathbb{E}[R_{1,j}(\ell, n)^2 | \mathcal{F}_j] = 2^{-n+2(p-\ell)} h_{\ell,n}(X_j)$ with $h_{\ell,n}(x) = \mathbb{E}_x[M_{\mathbb{G}_{n-p-k_0}}(Q^{p-\ell} \tilde{f}_\ell)^2]$. We have, thanks to (ii) from Assumption 2.10, see (82), that:

$$\sum_{j \in \mathbb{G}_{k_0}} \mathbb{E}[R_{1,j}(\ell, n)^2] = 2^{-n+2(p-\ell)} \mathbb{E}[M_{\mathbb{G}_{k_0}}(h_{\ell,n})] \leq C 2^{-n+2(p-\ell)} \langle \mu, h_{\ell,n} \rangle.$$

We have:

$$(85) \quad \langle \mu, h_{\ell,n} \rangle = \mathbb{E}_\mu[M_{\mathbb{G}_{n-p-k_0}}(Q^{p-\ell} \tilde{f}_\ell)^2] \leq C 2^{n-p} \sum_{k=0}^{n-p-k_0} 2^k \|Q^k Q^{p-\ell} \tilde{f}_\ell\|_{L^2(\mu)}^2 \\ \leq C 2^{n-p} \alpha^{2(p-\ell)} c_2^2(\mathfrak{f}),$$

where we used (82) for the first inequality (notice one can take $k_0 = 0$ in this case as we consider the expectation \mathbb{E}_μ), (20) in the second, and $2\alpha^2 < 1$ in the last. We deduce that:

$$\sum_{j \in \mathbb{G}_{k_0}} \mathbb{E}[R_{1,j}(\ell, n)^2] \leq C \alpha^{2(p-\ell)} 2^{p-2\ell} c_2^2(\mathfrak{f}).$$

We get that:

$$(86) \quad \mathbb{E}[R_1(n)^2]^{1/2} \leq \sum_{\ell=0}^p \left(2^{k_0} \sum_{j \in \mathbb{G}_{k_0}} \mathbb{E}[R_{1,j}(\ell, n)^2] \right)^{1/2} \leq C c_2(\mathfrak{f}) a_{1,n},$$

where the sequence $(a_{1,n}, n \in \mathbb{N})$ defined in (61) does not depend on \mathfrak{f} and converges to 0. Then use that \mathfrak{f} is bounded in $L^2(\mu)$ to conclude. \square

Remark 7.4. From the proof of Lemma 7.2, see in particular (84) and of Lemma 7.3, see in particular (86), we deduce that

$$\mathbb{E}\left[\left(N_{n,\emptyset}^{[k_0]}(\mathfrak{f}) - \Delta_n(\mathfrak{f})\right)^2\right] \leq a_{0,n} c_2^2(\mathfrak{f}),$$

where the sequence $(a_{0,n}, n \in \mathbb{N})$ converges to 0 and does not depend on \mathfrak{f} .

We now study the bracket of $\Delta_n(\mathfrak{f})$, $V(n) = \sum_{i \in \mathbb{G}_{n-p_n}} \mathbb{E} [\Delta_{n,i}(\mathfrak{f})^2 | \mathcal{F}_i]$. Recall that $V(n) = V_1(n) + 2V_2(n) - R_2(n)$ with $V_1(n)$, $V_2(n)$ and $R_2(n)$ defined after (63).

Lemma 7.5. *Under the assumptions of Theorem 5.2, we have the following convergence:*

$$\lim_{n \rightarrow \infty} \mathbb{E} [R_2(n)] = 0.$$

Proof. From the proof of Lemma 6.3, we have using (ii) from Assumption 2.10:

$$\begin{aligned} \mathbb{E} [R_2(n)] &= |\mathbb{G}_n|^{-1} |\mathbb{G}_{n-p}| \langle \nu, \mathcal{Q}^{n-p} \left(\left(\sum_{\ell=0}^p |\mathbb{G}_{p-\ell}| \mathcal{Q}^{p-\ell} \tilde{f}_\ell \right)^2 \right) \rangle \\ (87) \quad &\leq C 2^{-p} \left(\sum_{\ell=0}^p |\mathbb{G}_{p-\ell}| \|\mathcal{Q}^{p-\ell} \tilde{f}_\ell\|_{L^2(\mu)} \right)^2. \end{aligned}$$

We deduce that:

$$(88) \quad \mathbb{E} [R_2(n)] \leq C c_2^2(\mathfrak{f}) a_{2,n},$$

where the sequence $(a_{2,n}, n \in \mathbb{N})$ defined in (64) does not depend on \mathfrak{f} and converges to 0. Then use that \mathfrak{f} is bounded in $L^2(\mu)$ to conclude. \square

The proof of the following lemmas are written in such a way that part of their arguments will be used in the proof of Theorem 3.7 in the sub-critical case. We first study the limit of $V_2(n)$.

Lemma 7.6. *Under the assumptions of Corollary 5.3, we have that in probability $\lim_{n \rightarrow \infty} V_2(n) = \Sigma_2^{\text{sub}}(\mathfrak{f})$ with $\Sigma_2^{\text{sub}}(\mathfrak{f})$ finite and defined in (52).*

Proof. We recall $V_5(n)$ and $V_6(n)$ defined from (66). We consider the term $V_6(n)$, and we recall, see (67), that:

$$V_6(n) = |\mathbb{G}_{n-p}|^{-1} M_{\mathbb{G}_{n-p}}(H_{6,n}),$$

with, see (68):

$$H_{6,n} = \sum_{\substack{0 \leq \ell < k \\ r \geq 0}} h_{k,\ell,r}^{(n)} \mathbf{1}_{\{r+k < p\}} \quad \text{and} \quad h_{k,\ell,r}^{(n)} = 2^{r-\ell} \mathcal{Q}^{p-1-(r+k)} \left(\mathcal{P} \left(\mathcal{Q}^r \tilde{f}_k \otimes_{\text{sym}} \mathcal{Q}^{k-\ell+r} \tilde{f}_\ell \right) \right).$$

Recall $H_6(\mathfrak{f}) = \sum_{0 \leq \ell < k; r \geq 0} h_{k,\ell,r}$ with $h_{k,\ell,r} = 2^{r-\ell} \langle \mu, \mathcal{P} \left(\mathcal{Q}^r \tilde{f}_k \otimes_{\text{sym}} \mathcal{Q}^{k-\ell+r} \tilde{f}_\ell \right) \rangle = \langle \mu, h_{k,\ell,r}^{(n)} \rangle$ from (69). Thanks to (10) and (20), we get that:

$$(89) \quad |h_{k,\ell,r}| \leq C 2^{r-\ell} \|\mathcal{Q}^r \tilde{f}_k\|_{L^2(\mu)} \|\mathcal{Q}^{k-\ell+r} \tilde{f}_\ell\|_{L^2(\mu)} \leq C 2^{r-\ell} \alpha^{k-\ell+2r} \|f_\ell\|_{L^2(\mu)} \|f_k\|_{L^2(\mu)}.$$

We deduce that $|h_{k,\ell,r}| \leq C 2^{r-\ell} \alpha^{k-\ell+2r} c_2^2(\mathfrak{f})$ and, as the sum $\sum_{0 \leq \ell < k, r \geq 0} 2^{r-\ell} \alpha^{k-\ell+2r}$ is finite:

$$(90) \quad |H_6(\mathfrak{f})| \leq C c_2^2(\mathfrak{f}).$$

We write $H_6(\mathfrak{f}) = H_6^{[n]}(\mathfrak{f}) + B_{6,n}(\mathfrak{f})$, with

$$(91) \quad H_6^{[n]}(\mathfrak{f}) = \sum_{\substack{0 \leq \ell < k \\ r \geq 0}} h_{k,\ell,r} \mathbf{1}_{\{r+k < p\}} \quad \text{and} \quad B_{6,n}(\mathfrak{f}) = \sum_{\substack{0 \leq \ell < k \\ r \geq 0}} h_{k,\ell,r} \mathbf{1}_{\{r+k \geq p\}}.$$

As $\lim_{n \rightarrow \infty} \mathbf{1}_{\{r+k \geq p\}} = 0$, we get from (89), (90) and dominated convergence that $\lim_{n \rightarrow \infty} B_{6,n}(\mathbf{f}) = 0$ and thus:

$$(92) \quad \lim_{n \rightarrow \infty} H_6^{[n]}(\mathbf{f}) = H_6(\mathbf{f}).$$

We set $A_{6,n}(\mathbf{f}) = H_{6,n} - H_6^{[n]}(\mathbf{f}) = \sum_{\substack{0 \leq \ell < k \\ r \geq 0}} h_{k,\ell,r}^{(n)} - h_{k,\ell,r} \mathbf{1}_{\{r+k < p\}}$, so that from the definition of $V_6(n)$, we get that:

$$V_6(n) - H_6^{[n]}(\mathbf{f}) = |\mathbb{G}_{n-p}|^{-1} M_{\mathbb{G}_{n-p}}(A_{6,n}(\mathbf{f})).$$

We now study the second moment of $|\mathbb{G}_{n-p}|^{-1} M_{\mathbb{G}_{n-p}}(A_{6,n}(\mathbf{f}))$. Using (82), we get for $n-p \geq k_0$:

$$|\mathbb{G}_{n-p}|^{-2} \mathbb{E} [M_{\mathbb{G}_{n-p}}(A_{6,n}(\mathbf{f}))^2] \leq C |\mathbb{G}_{n-p}|^{-1} \sum_{j=0}^{n-p} 2^j \|\mathcal{Q}^j(A_{6,n}(\mathbf{f}))\|_{L^2(\mu)}^2.$$

Recall $c_k(\mathbf{f})$ and $q_k(\mathbf{f})$ from (83). We deduce that

$$(93) \quad \begin{aligned} \|\mathcal{Q}^j(A_{6,n}(\mathbf{f}))\|_{L^2(\mu)} &\leq \sum_{\substack{0 \leq \ell < k \\ r \geq 0}} \|\mathcal{Q}^j h_{k,\ell,r}^{(n)} - h_{k,\ell,r}\|_{L^2(\mu)} \mathbf{1}_{\{r+k < p\}} \\ &\leq C \sum_{\substack{0 \leq \ell < k \\ r \geq 0}} 2^{r-\ell} \alpha^{p-1-(r+k)+j} \|\mathcal{P}(\mathcal{Q}^r \tilde{f}_k \otimes_{\text{sym}} \mathcal{Q}^{k-\ell+r} \tilde{f}_\ell)\|_{L^2(\mu)} \mathbf{1}_{\{r+k < p\}} \\ &\leq C c_2^2(\mathbf{f}) \alpha^j \sum_{\substack{0 \leq \ell < k \\ r \geq 1}} 2^{r-\ell} \alpha^{p-(r+k)} \alpha^{k-\ell+2r} \mathbf{1}_{\{r+k < p\}} \\ &\quad + C \alpha^j \sum_{0 \leq \ell < k} 2^{-\ell} \alpha^{p-k} \|\mathcal{P}(\tilde{f}_k \otimes_{\text{sym}} \mathcal{Q}^{k-\ell} \tilde{f}_\ell)\|_{L^2(\mu)} \mathbf{1}_{\{k < p\}} \\ &\leq C c_2(\mathbf{f}) c_4(\mathbf{f}) \alpha^j \sum_{\substack{0 \leq \ell < k \\ r \geq 0}} 2^{r-\ell} \alpha^{p-(r+k)} \alpha^{k-\ell+2r} \mathbf{1}_{\{r+k < p\}} \\ (94) \quad &\leq C c_2(\mathbf{f}) c_4(\mathbf{f}) \alpha^j, \end{aligned}$$

where we used the triangular inequality for the first inequality; (20) for the second; (11) for $r \geq 1$ and (20) again for the third; (13) for $r = 0$ to get the $c_4(\mathbf{f})$ term and $c_2(\mathbf{f}) \leq c_4(\mathbf{f})$ for the fourth; and that $\sum_{0 \leq \ell < k, r \geq 0} 2^{r-\ell} \alpha^{k-\ell+2r}$ is finite for the last. As $\sum_{j=0}^{\infty} (2\alpha^2)^j$ is finite, we deduce that:

$$(96) \quad \mathbb{E} \left[\left(V_6(n) - H_6^{[n]}(\mathbf{f}) \right)^2 \right] = |\mathbb{G}_{n-p}|^{-2} \mathbb{E} [M_{\mathbb{G}_{n-p}}(A_{6,n}(\mathbf{f}))^2] \leq C c_2^2(\mathbf{f}) c_4^2(\mathbf{f}) 2^{-(n-p)}.$$

We now consider the term $V_5(n)$ defined just after (66):

$$V_5(n) = |\mathbb{G}_{n-p}|^{-1} M_{\mathbb{G}_{n-p}}(H_{5,n}),$$

with

$$H_{5,n} = \sum_{0 \leq \ell < k} h_{k,\ell}^{(n)} \mathbf{1}_{\{k \leq p\}} \quad \text{and} \quad h_{k,\ell}^{(n)} = 2^{-\ell} \mathcal{Q}^{p-k} \left(\tilde{f}_k \mathcal{Q}^{k-\ell} \tilde{f}_\ell \right).$$

Recall the constant $H_5(\mathbf{f}) = \sum_{0 \leq \ell < k} h_{k,\ell}$ with $h_{k,\ell} = 2^{-\ell} \langle \mu, \tilde{f}_k \mathcal{Q}^{k-\ell} \tilde{f}_\ell \rangle$ from (71). We have using Cauchy-Schwartz inequality and (20) that:

$$(97) \quad |h_{k,\ell}| \leq C 2^{-\ell} \alpha^{k-\ell} \|f_\ell\|_{L^2(\mu)} \|f_k\|_{L^2(\mu)} \leq C 2^{-\ell} \alpha^{k-\ell} c_2^2(\mathbf{f}).$$

As the sum $\sum_{0 \leq \ell < k} 2^{-\ell} \alpha^{k-\ell}$ is finite, we deduce that:

$$(98) \quad |H_5(\mathbf{f})| \leq C c_2^2(\mathbf{f}).$$

We write $H_5(\mathbf{f}) = H_5^{[n]}(\mathbf{f}) + B_{5,n}(\mathbf{f})$, with

$$(99) \quad H_5^{[n]}(\mathbf{f}) = \sum_{0 \leq \ell < k} h_{k,\ell} \mathbf{1}_{\{k \leq p\}} = \sum_{0 \leq \ell < k} 2^{-\ell} \langle \mu, \tilde{f}_k \mathcal{Q}^{k-\ell} \tilde{f}_\ell \rangle \mathbf{1}_{\{k \leq p\}} \quad \text{and} \quad B_{5,n}(\mathbf{f}) = \sum_{0 \leq \ell < k} h_{k,\ell} \mathbf{1}_{\{k > p\}}.$$

As $\lim_{n \rightarrow \infty} \mathbf{1}_{\{k > p\}} = 0$, we deduce from (97) and (98) that $\lim_{n \rightarrow \infty} B_{5,n}(\mathbf{f}) = 0$ by dominated convergence and thus:

$$(100) \quad \lim_{n \rightarrow \infty} H_5^{[n]}(\mathbf{f}) = H_5(\mathbf{f}).$$

We set $A_{5,n}(\mathbf{f}) = H_{5,n} - H_5^{[n]}(\mathbf{f}) = \sum_{0 \leq \ell < k} (h_{k,\ell}^{(n)} - h_{k,\ell}) \mathbf{1}_{\{k \leq p\}}$, so that from the definition of $V_5(n)$, we get that:

$$(101) \quad V_5(n) - H_5^{[n]}(\mathbf{f}) = |\mathbb{G}_{n-p}|^{-1} M_{\mathbb{G}_{n-p}}(A_{5,n}(\mathbf{f})).$$

We now study the second moment of $|\mathbb{G}_{n-p}|^{-1} M_{\mathbb{G}_{n-p}}(A_{5,n}(\mathbf{f}))$. Using (82), we get for $n-p \geq k_0$:

$$|\mathbb{G}_{n-p}|^{-2} \mathbb{E} [M_{\mathbb{G}_{n-p}}(A_{5,n}(\mathbf{f}))^2] \leq C |\mathbb{G}_{n-p}|^{-1} \sum_{j=0}^{n-p} 2^j \|\mathcal{Q}^j(A_{5,n}(\mathbf{f}))\|_{L^2(\mu)}^2.$$

We also have that:

$$(102) \quad \begin{aligned} \|\mathcal{Q}^j(A_{5,n}(\mathbf{f}))\|_{L^2(\mu)} &\leq \sum_{0 \leq \ell < k} \|\mathcal{Q}^j h_{k,\ell}^{(n)} - h_{k,\ell}\|_{L^2(\mu)} \mathbf{1}_{\{k \leq p\}} \\ &\leq C \sum_{0 \leq \ell < k} 2^{-\ell} \alpha^{p-k+j} \|\tilde{f}_k \mathcal{Q}^{k-\ell} \tilde{f}_\ell\|_{L^2(\mu)} \mathbf{1}_{\{k \leq p\}} \end{aligned}$$

$$(103) \quad \leq C c_4^2(\mathbf{f}) \alpha^j,$$

where we used the triangular inequality for the first inequality, (20) for the second, and Cauchy-Schwartz inequality for the last. As $\sum_{j=0}^{\infty} (2\alpha^2)^j$ is finite, we deduce that:

$$(104) \quad \mathbb{E} \left[\left(V_5(n) - H_5^{[n]}(\mathbf{f}) \right)^2 \right] = |\mathbb{G}_{n-p}|^{-2} \mathbb{E} [M_{\mathbb{G}_{n-p}}(A_{5,n}(\mathbf{f}))^2] \leq C c_4^4(\mathbf{f}) 2^{-(n-p)}.$$

Since $c_2(\mathbf{f}) \leq c_4(\mathbf{f})$, we deduce from (96) and (104), as $V_2(n) = V_5(n) + V_6(n)$ (see (66)), that:

$$(105) \quad \mathbb{E} \left[\left(V_2(n) - H_2^{[n]}(\mathbf{f}) \right)^2 \right] \leq C c_4^4(\mathbf{f}) 2^{-(n-p)} \quad \text{with} \quad H_2^{[n]}(\mathbf{f}) = H_6^{[n]}(\mathbf{f}) + H_5^{[n]}(\mathbf{f}).$$

Since, according to (92) and (100) and $\Sigma_2^{\text{sub}}(\mathbf{f}) = H_6(\mathbf{f}) + H_5(\mathbf{f})$ (see (52)), we get $\lim_{n \rightarrow \infty} H_2^{[n]}(\mathbf{f}) = \Sigma_2^{\text{sub}}(\mathbf{f})$. This implies that $\lim_{n \rightarrow \infty} V_2(n) = \Sigma_2^{\text{sub}}(\mathbf{f})$ in probability. \square

We now study the limit of $V_1(n)$.

Lemma 7.7. *Under the assumptions of Corollary 5.3, we have that in probability $\lim_{n \rightarrow \infty} V_1(n) = \Sigma_1^{\text{sub}}(\mathbf{f}) < +\infty$ with $\Sigma_1^{\text{sub}}(\mathbf{f})$ finite and defined in (51).*

Proof. We recall $V_1(n) = V_3(n) + V_4(n)$, see (72) and thereafter for the definition of $V_3(n)$ and $V_4(n)$. We first consider the term $V_4(n)$, and we recall, see (73), that:

$$V_4(n) = |\mathbb{G}_{n-p}|^{-1} M_{\mathbb{G}_{n-p}}(H_{4,n}),$$

with, see (74):

$$H_{4,n} = \sum_{\ell \geq 0, k \geq 0} h_{\ell,k}^{(n)} \mathbf{1}_{\{\ell+k < p\}} \quad \text{and} \quad h_{\ell,k}^{(n)} = 2^{k-\ell} \mathcal{Q}^{p-1-(\ell+k)} \left(\mathcal{P} \left(\mathcal{Q}^k \tilde{f}_\ell \otimes^2 \right) \right).$$

Recall the constant $H_4(\mathbf{f}) = \sum_{\ell \geq 0, k \geq 0} h_{\ell,k}$ with $h_{\ell,k} = 2^{k-\ell} \langle \mu, \mathcal{P} \left(\mathcal{Q}^k \tilde{f}_\ell \otimes^2 \right) \rangle$ from (75). Thanks to (3) and (20), we have:

$$(106) \quad |h_{\ell,k}| \leq 2^{k-\ell} \|\mathcal{Q}^k \tilde{f}_\ell\|_{L^2(\mu)}^2 \leq C 2^{k-\ell} \alpha^{2k} \|f_\ell\|_{L^2(\mu)}^2 \leq C 2^{k-\ell} \alpha^{2k} c_2^2(\mathbf{f}),$$

and thus, as the sum $\sum_{\ell \geq 0, k \geq 0} 2^{k-\ell} \alpha^{2k}$ is finite:

$$(107) \quad |H_4(\mathbf{f})| \leq C c_2^2(\mathbf{f}).$$

We write $H_4(\mathbf{f}) = H_4^{[n]}(\mathbf{f}) + B_{4,n}(\mathbf{f})$, with

$$(108) \quad H_4^{[n]}(\mathbf{f}) = \sum_{\ell \geq 0, k \geq 0} h_{\ell,k} \mathbf{1}_{\{\ell+k < p\}} \quad \text{and} \quad B_{4,n}(\mathbf{f}) = \sum_{\ell \geq 0, k \geq 0} h_{\ell,k} \mathbf{1}_{\{\ell+k \geq p\}}.$$

Using that $\lim_{n \rightarrow \infty} \mathbf{1}_{\{\ell+k \geq p\}} = 0$, we deduce from (106), (107) and dominated convergence that $\lim_{n \rightarrow \infty} B_{4,n}(\mathbf{f}) = 0$, and thus:

$$(109) \quad \lim_{n \rightarrow \infty} H_4^{[n]}(\mathbf{f}) = H_4(\mathbf{f}).$$

We set $A_{4,n}(\mathbf{f}) = H_{4,n} - H_4^{[n]}(\mathbf{f}) = \sum_{\ell \geq 0, k \geq 0} (h_{\ell,k}^{(n)} - h_{\ell,k}) \mathbf{1}_{\{\ell+k < p\}}$, so that from the definition of $V_4(n)$, we get that:

$$V_4(n) - H_4^{[n]}(\mathbf{f}) = |\mathbb{G}_{n-p}|^{-1} M_{\mathbb{G}_{n-p}}(A_{4,n}(\mathbf{f})).$$

We now study the second moment of $|\mathbb{G}_{n-p}|^{-1} M_{\mathbb{G}_{n-p}}(A_{4,n}(\mathbf{f}))$. Using (82), we get for $n-p \geq k_0$:

$$|\mathbb{G}_{n-p}|^{-2} \mathbb{E} [M_{\mathbb{G}_{n-p}}(A_{4,n}(\mathbf{f}))^2] \leq C |\mathbb{G}_{n-p}|^{-1} \sum_{j=0}^{n-p} 2^j \|\mathcal{Q}^j(A_{4,n}(\mathbf{f}))\|_{L^2(\mu)}^2.$$

Using (3), we obtain that $\|\mathcal{P}(\tilde{f}_\ell \otimes \tilde{f}_\ell)\|_{L^2(\mu)} \leq c_4^2(\mathbf{f})$. We deduce that:

$$\begin{aligned} \|\mathcal{Q}^j(A_{4,n}(\mathbf{f}))\|_{L^2(\mu)} &\leq \sum_{\ell \geq 0, k \geq 0} \|\mathcal{Q}^j h_{\ell,k}^{(n)} - h_{\ell,k}\|_{L^2(\mu)} \mathbf{1}_{\{\ell+k < p\}} \\ &\leq C \sum_{\ell \geq 0, k \geq 0} 2^{k-\ell} \alpha^{p-1-(\ell+k)+j} \|\mathcal{P} \left(\mathcal{Q}^k \tilde{f}_\ell \otimes^2 \right)\|_{L^2(\mu)} \mathbf{1}_{\{\ell+k < p\}} \\ &\leq C c_2^2(\mathbf{f}) \alpha^j \sum_{\ell \geq 0, k > 0} 2^{k-\ell} \alpha^{p-(\ell+k)} \alpha^{2k} \mathbf{1}_{\{\ell+k < p\}} \\ &\quad + C \alpha^j \sum_{\ell \geq 0} 2^{-\ell} \alpha^{p-\ell} \|\mathcal{P} \left(\tilde{f}_\ell \otimes^2 \right)\|_{L^2(\mu)} \mathbf{1}_{\{\ell < p\}} \end{aligned}$$

$$(111) \quad \leq C c_4^2(\mathbf{f}) \alpha^j,$$

where we used the triangular inequality for the first inequality; (20) for the second; (11) for $k \geq 1$ and (20) again for the third; and (3) as well as $c_2(f) \leq c_4(f)$ for the last. As $\sum_{j=0}^{\infty} (2\alpha^2)^j$ is finite, we deduce that:

$$(112) \quad \mathbb{E} \left[\left(V_4(n) - H_4^{[n]}(f) \right)^2 \right] = |\mathbb{G}_{n-p}|^{-2} \mathbb{E} \left[M_{\mathbb{G}_{n-p}}(A_{4,n}(f))^2 \right] \leq C c_4^4(f) 2^{-(n-p)}.$$

We now consider the term $V_3(n)$ defined just after (72):

$$V_3(n) = |\mathbb{G}_{n-p}|^{-1} M_{\mathbb{G}_{n-p}}(H_{3,n}),$$

with

$$H_{3,n} = \sum_{\ell \geq 0} h_\ell^{(n)} \mathbf{1}_{\{\ell \leq p\}} \quad \text{and} \quad h_\ell^{(n)} = 2^{-\ell} \mathcal{Q}^{p-\ell} \left(\tilde{f}_\ell^2 \right).$$

Recall the constant $H_3(f) = \sum_{\ell \geq 0} h_\ell$ with $h_\ell = 2^{-\ell} \langle \mu, \tilde{f}_\ell^2 \rangle = \langle \mu, h_\ell^{(n)} \rangle$ from (77). As $h_\ell \leq \|f_\ell\|_{L^2(\mu)}^2 \leq c_2^2(f)$, we get that $H_3(f) \leq 2c_2^2(f)$. We write $H_3(f) = H_3^{[n]}(f) + B_{3,n}(f)$, with

$$(113) \quad H_3^{[n]}(f) = \sum_{\ell \geq 0} h_\ell \mathbf{1}_{\{\ell \leq p\}} \quad \text{and} \quad B_{3,n}(f) = \sum_{\ell \geq 0} h_\ell \mathbf{1}_{\{\ell > p\}}.$$

As $\lim_{n \rightarrow \infty} \mathbf{1}_{\{\ell > p\}} = 0$, we get from dominated convergence that $\lim_{n \rightarrow \infty} B_{3,n}(f) = 0$ and thus:

$$(114) \quad \lim_{n \rightarrow \infty} H_3^{[n]}(f) = H_3(f).$$

We set $A_{3,n}(f) = H_{3,n} - H_3^{[n]}(f) = \sum_{\ell \geq 0} (h_\ell^{(n)} - h_\ell) \mathbf{1}_{\{\ell \leq p\}}$, so that from the definition of $V_3(n)$, we get that:

$$(115) \quad V_3(n) - H_3^{[n]}(f) = |\mathbb{G}_{n-p}|^{-1} M_{\mathbb{G}_{n-p}}(A_{3,n}(f)).$$

We now study the second moment of $|\mathbb{G}_{n-p}|^{-1} M_{\mathbb{G}_{n-p}}(A_{3,n}(f))$. Using (82), we get for $n-p \geq k_0$:

$$(116) \quad |\mathbb{G}_{n-p}|^{-2} \mathbb{E} \left[M_{\mathbb{G}_{n-p}}(A_{3,n}(f))^2 \right] \leq C |\mathbb{G}_{n-p}|^{-1} \sum_{j=0}^{n-p} 2^j \| \mathcal{Q}^j(A_{3,n}(f)) \|_{L^2(\mu)}^2.$$

We have that

$$(117) \quad \begin{aligned} \| \mathcal{Q}^j(A_{3,n}(f)) \|_{L^2(\mu)} &\leq \sum_{\ell \geq 0} \| \mathcal{Q}^j h_\ell^{(n)} - h_\ell \|_{L^2(\mu)} \mathbf{1}_{\{\ell \leq p\}} \\ &\leq C \sum_{\ell \geq 0} 2^{-\ell} \| \mathcal{Q}^{j+p-\ell} \tilde{g} \|_{L^2(\mu)} \mathbf{1}_{\{\ell \leq p\}} \quad \text{with} \quad g = \tilde{f}_\ell^2 \end{aligned}$$

$$(118) \quad \begin{aligned} &\leq C \sum_{\ell \geq 0} 2^{-\ell} \alpha^{j+p-\ell} \| \tilde{f}_\ell^2 \|_{L^2(\mu)} \mathbf{1}_{\{\ell \leq p\}} \\ &\leq C c_4^2(f) \alpha^j, \end{aligned}$$

where we used the triangular inequality for the first inequality; and (20) for the third. As $\sum_{j=0}^{\infty} (2\alpha^2)^j$ is finite, we deduce that:

$$(119) \quad \mathbb{E} \left[\left(V_3(n) - H_3^{[n]}(f) \right)^2 \right] = |\mathbb{G}_{n-p}|^{-2} \mathbb{E} \left[M_{\mathbb{G}_{n-p}}(A_{3,n}(f))^2 \right] \leq C c_4^4(f) 2^{-(n-p)}.$$

Since $c_2(\mathbf{f}) \leq c_4(\mathbf{f})$, we deduce from (112) and (119) that:

$$(120) \quad \mathbb{E} \left[\left(V_1(n) - H_1^{[n]}(\mathbf{f}) \right)^2 \right] \leq C c_4^4(\mathbf{f}) 2^{-(n-p)} \quad \text{with} \quad H_1^{[n]}(\mathbf{f}) = H_4^{[n]}(\mathbf{f}) + H_3^{[n]}(\mathbf{f}).$$

Since, according to (109) and (114) $\Sigma_1^{\text{sub}}(\mathbf{f}) = H_4(\mathbf{f}) + H_3(\mathbf{f})$ (see (51)), we get $\lim_{n \rightarrow \infty} H_1^{[n]}(\mathbf{f}) = \Sigma_1^{\text{sub}}(\mathbf{f})$. This implies that $\lim_{n \rightarrow \infty} V_1(n) = \Sigma_1^{\text{sub}}(\mathbf{f})$ in probability. \square

The next Lemma is a direct consequence of (63) and Lemmas 7.5, 7.6 and 7.7.

Lemma 7.8. *Under the assumptions of Corollary 5.3, we have $\lim_{n \rightarrow \infty} V(n) = \Sigma^{\text{sub}}(\mathbf{f})$ in probability, where, with $\Sigma_1^{\text{sub}}(\mathbf{f})$ and $\Sigma_2^{\text{sub}}(\mathbf{f})$ defined by (51) and (52), we have:*

$$\Sigma^{\text{sub}}(\mathbf{f}) = \Sigma_1^{\text{sub}}(\mathbf{f}) + 2\Sigma_2^{\text{sub}}(\mathbf{f}).$$

We now check the Lindeberg condition using a fourth moment condition. Recall $R_3(n) = \sum_{i \in \mathbb{G}_{n-pn}} \mathbb{E} [\Delta_{n,i}(\mathbf{f})^4]$ defined in (78).

Lemma 7.9. *Under the assumptions of Corollary 5.3, we have that $\lim_{n \rightarrow \infty} R_3(n) = 0$.*

Proof. Recall (79) and (80) from the proof of Lemma 6.7, so that:

$$R_3(n) \leq C n^3 \sum_{\ell=0}^p \sum_{i \in \mathbb{G}_{n-p}} |\mathbb{G}_n|^{-2} \mathbb{E} [h_{n,\ell}(X_i)], \quad \text{with} \quad h_{n,\ell}(x) = \mathbb{E}_x \left[M_{\mathbb{G}_{p-\ell}}(\tilde{f}_\ell)^4 \right].$$

Using (82) (with f and n replaced by $h_{n,\ell}$ and $n-p$), we get that:

$$(121) \quad R_3(n) \leq C n^3 2^{-n-p} \sum_{\ell=0}^p \mathbb{E}_\mu \left[M_{\mathbb{G}_{p-\ell}}(\tilde{f}_\ell)^4 \right].$$

Now we give the main steps to get an upper bound of $\mathbb{E}_\mu \left[M_{\mathbb{G}_{p-\ell}}(\tilde{f}_\ell)^4 \right]$. Recall that:

$$\|\tilde{f}_\ell\|_{L^4(\mu)} \leq C c_4(\mathbf{f}).$$

We have:

$$(122) \quad \mathbb{E}_\mu \left[M_{\mathbb{G}_{p-\ell}}(\tilde{f}_\ell)^4 \right] \leq C c_4^4(\mathbf{f}) \quad \text{for } \ell \in \{p-2, p-1, p\}.$$

Now we consider the case $0 \leq \ell \leq p-3$. Let the functions $\psi_{j,p-\ell}$, with $1 \leq j \leq 9$, from Lemma 4.3, with f replaced by \tilde{f}_ℓ so that for $\ell \in \{0, \dots, p-3\}$

$$(123) \quad \mathbb{E}_\mu \left[M_{\mathbb{G}_{p-\ell}}(\tilde{f}_\ell)^4 \right] = \sum_{j=1}^9 \langle \mu, \psi_{j,p-\ell} \rangle.$$

We now assume that $p-\ell-1 \geq 2$. We shall give bounds on $\langle \mu, \psi_{j,p-\ell} \rangle$ based on computations similar to those in the second step in the proof of Theorem 2.1 in [9]. We set $h_k = Q^{k-1} \tilde{f}_\ell$ so that for $k \in \mathbb{N}^*$:

$$(124) \quad \|h_k\|_{L^2(\mu)} \leq C \alpha^k c_2(\mathbf{f}) \quad \text{and} \quad \|h_k\|_{L^4(\mu)} \leq C c_4(\mathbf{f}).$$

We recall the notation $f \otimes f = f \otimes^2$. We deduce for $k \geq 2$ from (11) applied with $h_k = Q h_{k-1}$ and for $k=1$ from (9) and (124) that:

$$(125) \quad \|\mathcal{P}(h_k \otimes^2)\|_{L^2(\mu)} \leq \begin{cases} C \alpha^{2k} c_2^2(\mathbf{f}) & \text{for } k \geq 2, \\ C c_4^2(\mathbf{f}) & \text{for } k = 1. \end{cases}$$

Upper bound of $\langle \mu, |\psi_{1,p-\ell}| \rangle$. We have:

$$(126) \quad \langle \mu, |\psi_{1,p-\ell}| \rangle \leq C 2^{p-\ell} \langle \mu, \mathcal{Q}^{p-\ell}(\tilde{f}_\ell^4) \rangle \leq C 2^{p-\ell} c_4^4(\mathbf{f}).$$

Upper bound of $|\langle \mu, \psi_{2,p-\ell} \rangle|$. Using Lemma 4.4 for the second inequality and (124) for the third, we get:

$$(127) \quad \begin{aligned} |\langle \mu, \psi_{2,p-\ell} \rangle| &\leq C 2^{2(p-\ell)} \sum_{k=0}^{p-\ell-1} 2^{-k} |\langle \mu, \mathcal{Q}^k \mathcal{P} \left(\mathcal{Q}^{p-\ell-k-1} (|\tilde{f}_\ell|^3) \otimes_{\text{sym}} h_{p-\ell-k} \right) \rangle| \\ &\leq C 2^{2(p-\ell)} \sum_{k=0}^{p-\ell-1} 2^{-k} c_4^3(\mathbf{f}) \|h_{p-\ell-k}\|_{L^4(\mu)} \\ &\leq C 2^{2(p-\ell)} c_4^4(\mathbf{f}). \end{aligned}$$

Upper bound of $\langle \mu, |\psi_{3,p-\ell}| \rangle$. Using (10), we easily get:

$$(128) \quad \langle \mu, |\psi_{3,p-\ell}| \rangle \leq C 2^{2(p-\ell)} \sum_{k=0}^{p-\ell-1} 2^{-k} \langle \mu, \mathcal{Q}^k \mathcal{P} \left(\mathcal{Q}^{p-\ell-k-1} (\tilde{f}_\ell^2 \otimes^2) \right) \rangle \leq C 2^{2(p-\ell)} c_4^4(\mathbf{f}).$$

Upper bound of $\langle \mu, |\psi_{4,p-\ell}| \rangle$. Using (10) and then (125) with $p-\ell-1 \geq 2$, we get:

$$(129) \quad \begin{aligned} \langle \mu, |\psi_{4,p-\ell}| \rangle &\leq C 2^{4(p-\ell)} \langle \mu, \mathcal{P} (|\mathcal{P}(h_{p-\ell-1} \otimes^2) \otimes^2|) \rangle \\ &\leq C 2^{4(p-\ell)} \|\mathcal{P}(h_{p-\ell-1} \otimes^2)\|_{L^2(\mu)}^2 \\ &\leq C 2^{4(p-\ell)} \alpha^{4(p-\ell)} c_2^4(\mathbf{f}) \end{aligned}$$

$$(130) \quad \leq C 2^{2(p-\ell)} c_2^4(\mathbf{f}).$$

Upper bound of $\langle \mu, |\psi_{5,p-\ell}| \rangle$. We have:

$$\langle \mu, |\psi_{5,p-\ell}| \rangle \leq C 2^{4(p-\ell)} \sum_{k=2}^{p-\ell-1} \sum_{r=0}^{k-1} 2^{-r} \Gamma_{k,r}^{[5]},$$

with

$$\Gamma_{k,r}^{[5]} = 2^{-2k} \langle \mu, \mathcal{P} (\mathcal{Q}^{k-r-1} |\mathcal{P}(h_{p-\ell-k} \otimes^2)| \otimes^2) \rangle.$$

Using (10) and then (125), we get:

$$(131) \quad \Gamma_{k,r}^{[5]} \leq C 2^{-2k} \|\mathcal{P}(h_{p-\ell-k} \otimes^2)\|_{L^2(\mu)}^2$$

$$(132) \quad \leq C 2^{-2(p-\ell)} c_4^4(\mathbf{f}) \mathbf{1}_{\{k=p-\ell-1\}} + C 2^{-2k} \alpha^{4(p-\ell-k)} c_2^4(\mathbf{f}) \mathbf{1}_{\{k \leq p-\ell-2\}}.$$

We deduce that $\langle \mu, |\psi_{5,p-\ell}| \rangle \leq C 2^{2(p-\ell)} c_4^4(\mathbf{f})$.

Upper bound of $\langle \mu, |\psi_{6,p-\ell}| \rangle$. We have:

$$\langle \mu, |\psi_{6,p-\ell}| \rangle \leq C 2^{3(p-\ell)} \sum_{k=1}^{p-\ell-1} \sum_{r=0}^{k-1} 2^{-r} \Gamma_{k,r}^{[6]},$$

with

$$\Gamma_{k,r}^{[6]} = 2^{-k} \langle \mu, \mathcal{Q}^r \mathcal{P} \left(\mathcal{Q}^{k-r-1} |\mathcal{P}(h_{p-\ell-k} \otimes^2)| \otimes_{\text{sym}} \mathcal{Q}^{p-\ell-r-1} (\tilde{f}_\ell^2) \right) \rangle.$$

Using (10) and then (125), we get:

$$(133) \quad \Gamma_{k,r}^{[6]} \leq C 2^{-k} \|\mathcal{P}(h_{p-\ell-k} \otimes^2)\|_{L^2(\mu)} \|\mathcal{Q}^{p-\ell-r-1}(\tilde{f}_\ell^2)\|_{L^2(\mu)}$$

$$(134) \quad \leq C 2^{-(p-\ell)} c_4^4(\mathbf{f}) \mathbf{1}_{\{k=p-\ell-1\}} + C 2^{-k} \alpha^{2(p-\ell-k)} c_2^2(\mathbf{f}) c_4^2(\mathbf{f}) \mathbf{1}_{\{k \leq p-\ell-2\}}$$

We deduce that $\langle \mu, |\psi_{6,p-\ell}| \rangle \leq C 2^{2(p-\ell)} c_4^4(\mathbf{f})$.

Upper bound of $|\langle \mu, \psi_{7,p-\ell} \rangle|$. We have:

$$(135) \quad |\langle \mu, \psi_{7,p-\ell} \rangle| \leq C 2^{3(p-\ell)} \sum_{k=1}^{p-\ell-1} \sum_{r=0}^{k-1} 2^{-r} \Gamma_{k,r}^{[7]},$$

with

$$(136) \quad \Gamma_{k,r}^{[7]} = 2^{-k} |\langle \mu, \mathcal{Q}^r \mathcal{P} \left(\mathcal{Q}^{k-r-1} \mathcal{P} \left(h_{p-\ell-k} \otimes_{\text{sym}} \mathcal{Q}^{p-\ell-k-1}(\tilde{f}_\ell^2) \right) \otimes_{\text{sym}} h_{p-\ell-r} \right) \rangle|.$$

For $k \leq p - \ell - 2$, we have:

$$(137) \quad \begin{aligned} \Gamma_{k,r}^{[7]} &\leq C 2^{-k} \|\mathcal{P}(h_{p-\ell-k} \otimes_{\text{sym}} \mathcal{Q}^{p-\ell-k-1}(\tilde{f}_\ell^2))\|_{L^2(\mu)} \|h_{p-\ell-r}\|_{L^2(\mu)} \\ &\leq C 2^{-k} \|h_{p-\ell-k-1}\|_{L^2(\mu)} \|\mathcal{Q}^{p-\ell-k-2}(\tilde{f}_\ell^2)\|_{L^2(\mu)} \alpha^{p-\ell-r} c_2(\mathbf{f}) \mathbf{1}_{\{k \leq p-\ell-2\}} \\ &\leq C 2^{-k} \alpha^{2(p-\ell-k)} c_2^2(\mathbf{f}) c_4^2(\mathbf{f}) \mathbf{1}_{\{k \leq p-\ell-2\}}, \end{aligned}$$

where we used (10) for the first inequality; (11) for the second; and (124) for the third. We now consider the case $k = p - \ell - 1$. Let $g \in \mathcal{B}_+(S)$. As $2ba^2 \leq b^3 + a^3$ for a, b non-negative, we get that $g \otimes g^2 \leq g^3 \otimes_{\text{sym}} \mathbf{1}$ and thus:

$$(138) \quad \mathcal{P}(g \otimes_{\text{sym}} g^2) \leq 2\mathcal{Q}(g^3).$$

Writing $A_r = \Gamma_{p-\ell-1,r}^{[7]}$, we get using (138) for the first inequality and Lemma 4.4 for the second:

$$\begin{aligned} A_r &= 2^{-p-\ell-1} |\langle \mu, \mathcal{P} \left(\mathcal{Q}^{p-\ell-2-r} \mathcal{P} \left(\tilde{f}_\ell \otimes_{\text{sym}} \tilde{f}_\ell^2 \right) \otimes_{\text{sym}} h_{p-\ell-r} \right) \rangle| \\ &\leq C 2^{-(p-\ell)} \langle \mu, \mathcal{P} \left(\mathcal{Q}^{p-\ell-1-r} |\tilde{f}_\ell^3| \otimes_{\text{sym}} |\mathcal{Q}^{p-\ell-1-r} \tilde{f}_\ell| \right) \rangle \\ &\leq C 2^{-(p-\ell)} c_4^4(\mathbf{f}). \end{aligned}$$

Since $c_2(\mathbf{f}) \leq c_4(\mathbf{f})$, we deduce that $|\langle \mu, \psi_{7,p-\ell} \rangle| \leq C 2^{2(p-\ell)} c_4^4(\mathbf{f})$.

Upper bound of $\langle \mu, |\psi_{8,p-\ell}| \rangle$. We have:

$$(139) \quad \langle \mu, |\psi_{8,p-\ell}| \rangle \leq C 2^{4(p-\ell)} \sum_{k=2}^{p-\ell-1} \sum_{r=1}^{k-1} \sum_{j=0}^{r-1} 2^{-j} \Gamma_{k,r,j}^{[8]},$$

with

$$\Gamma_{k,r,j}^{[8]} \leq 2^{-k-r} \langle \mu, \mathcal{Q}^j \mathcal{P} \left(|\mathcal{Q}^{r-j-1} \mathcal{P}(h_{p-\ell-r} \otimes^2)| \otimes_{\text{sym}} |\mathcal{Q}^{k-j-1} \mathcal{P}(h_{p-\ell-k} \otimes^2)| \right) \rangle.$$

Using (10) and then (125) (twice and noticing that $p - \ell - r \geq 2$), we get:

$$(140) \quad \Gamma_{k,r,j}^{[8]} \leq C 2^{-k-r} \|\mathcal{P}(h_{p-\ell-r} \otimes^2)\|_{L^2(\mu)} \|\mathcal{P}(h_{p-\ell-k} \otimes^2)\|_{L^2(\mu)}$$

$$(141) \quad \leq C 2^{-k-r} \alpha^{2(p-\ell-r)} c_2^2(\mathbf{f}) \left(\alpha^{2(p-\ell-k)} c_2^2(\mathbf{f}) + c_4^2(\mathbf{f}) \mathbf{1}_{\{k=p-\ell-1\}} \right).$$

We deduce that $\langle \mu, |\psi_{8,p-\ell}| \rangle \leq C 2^{2(p-\ell)} c_4^4(\mathbf{f})$.

Upper bound of $\langle \mu, |\psi_{g,p-\ell}| \rangle$. We have:

$$(142) \quad \langle \mu, |\psi_{g,p-\ell}| \rangle \leq C 2^{4(p-\ell)} \sum_{k=2}^{p-\ell-1} \sum_{r=1}^{k-1} \sum_{j=0}^{r-1} 2^{-j} \Gamma_{k,r,j}^{[9]},$$

with

$$\Gamma_{k,r,j}^{[9]} \leq 2^{-k-r} \langle \mu, \mathcal{Q}^j \mathcal{P} (\mathcal{Q}^{r-j-1} |\mathcal{P} (h_{p-\ell-r} \otimes_{\text{sym}} \mathcal{Q}^{k-r-1} \mathcal{P} (h_{p-\ell-k} \otimes^2)) | \otimes_{\text{sym}} |h_{p-\ell-j}|) \rangle.$$

For $r \leq k-2$, we have:

$$(143) \quad \begin{aligned} \Gamma_{k,r,j}^{[9]} &\leq C 2^{-k-r} \|\mathcal{P} (h_{p-\ell-r} \otimes_{\text{sym}} \mathcal{Q}^{k-r-1} \mathcal{P} (h_{p-\ell-k} \otimes^2))\|_{L^2(\mu)} \|h_{p-\ell-j}\|_{L^2(\mu)} \\ &\leq C 2^{-k-r} \|h_{p-\ell-r-1}\|_{L^2(\mu)} \|\mathcal{P} (h_{p-\ell-k} \otimes^2)\|_{L^2(\mu)} \|h_{p-\ell-j}\|_{L^2(\mu)} \end{aligned}$$

$$(144) \quad \leq C 2^{-k-r} \alpha^{2(p-\ell-r)} c_2^2(\mathbf{f}) \left(\alpha^{2(p-\ell-k)} c_2^2(\mathbf{f}) \mathbf{1}_{\{k \leq p-\ell-2\}} + c_4^2(\mathbf{f}) \mathbf{1}_{\{k=p-\ell-1\}} \right),$$

where we used (10) for the first inequality; (11) as $p-\ell-r \geq 2$ and $k-r-1 \geq 1$ for the second; and (124) (two times) and (125) (one time) for the last. For $r = k-1$ and $k \leq p-\ell-2$, we have:

$$(145) \quad \begin{aligned} \Gamma_{k,r,j}^{[9]} &\leq C 2^{-2k} \|\mathcal{P} (h_{p-\ell-k+1} \otimes_{\text{sym}} \mathcal{P} (h_{p-\ell-k} \otimes^2))\|_{L^2(\mu)} \|h_{p-\ell-j}\|_{L^2(\mu)} \\ &\leq C 2^{-2k} \|h_{p-\ell-k}\|_{L^2(\mu)} \|h_{p-\ell-k-1}\|_{L^2(\mu)}^2 \|h_{p-\ell-j}\|_{L^2(\mu)} \end{aligned}$$

$$(146) \quad \leq C 2^{-2k} \alpha^{4(p-\ell-k)} c_2^4(\mathbf{f}),$$

where we used (10) for the first inequality; (12)¹ as $p-\ell-k \geq 2$ for the second; and (124) (three times) for the last. For $r = k-1 = p-\ell-2$, we have:

$$(147) \quad \begin{aligned} \Gamma_{k,r,j}^{[9]} &\leq C 2^{-2(p-\ell)} \|\mathcal{P} (\mathcal{Q} \tilde{f}_\ell \otimes_{\text{sym}} \mathcal{P} (\tilde{f}_\ell \otimes^2))\|_{L^2(\mu)} \|h_{p-\ell-j}\|_{L^2(\mu)} \\ &\leq C 2^{-2(p-\ell)} \|\mathcal{P} (\mathcal{Q} \tilde{f}_\ell \otimes_{\text{sym}} \mathcal{Q}(\tilde{f}_\ell^2))\|_{L^2(\mu)} \|h_{p-\ell-j}\|_{L^2(\mu)} \end{aligned}$$

$$(148) \quad \leq C 2^{-2(p-\ell)} c_4^2(\mathbf{f}) \alpha^{p-\ell-j} c_2^2(\mathbf{f}),$$

where we used (10) for the first inequality, (3) (with f replaced by f_ℓ) for the second and (11) as well as (125) (with $p-\ell-j \geq 2$) for the last. Taking all together, we deduce that $\langle \mu, |\psi_{g,p-\ell}| \rangle \leq C 2^{2(p-\ell)} c_4^2(\mathbf{f}) c_2^2(\mathbf{f})$.

Wrapping all the upper bounds with (123) we deduce that for $\ell \in \{0, \dots, p-3\}$

$$\mathbb{E}_\mu \left[M_{\mathbb{G}_{p-\ell}}(\tilde{f}_\ell)^4 \right] \leq C 2^{2(p-\ell)} c_4^4(\mathbf{f}).$$

Thanks to (122), this equality holds for $\ell \in \{0, \dots, p\}$. We deduce from (121) that:

$$(149) \quad R_3(n) \leq C n^3 2^{-(n-p)} c_4^4(\mathbf{f}).$$

This proves that $\lim_{n \rightarrow \infty} R_3(n) = 0$. □

The proof of Corollary 5.3 uses then the same arguments as the proof of Theorem 5.2.

¹Notice this is the only place in the proof of Corollary 5.3 where we use (12).

8. PROOF OF THEOREM 3.7 IN THE SUB-CRITICAL CASE ($2\alpha^2 < 1$)

We keep notations from Section 7 and assume that Assumptions 2.10 and 2.13 hold with $\alpha \in (0, 1/\sqrt{2})$. Let $(f_{\ell,n}, n \geq \ell \geq 0)$ be a sequence of function satisfying Assumption 2.16. We set $f_{\ell,n} = 0$ for $\ell > n \geq 0$ and $\mathbf{f}_n = (f_{\ell,n}, \ell \in \mathbb{N})$. Recall the definition of $c_k(\mathbf{f})$ and $q_k(\mathbf{f})$ in (83). Assumption 2.16 (ii) gives that $\mathbf{c}_2 = \sup_{n \in \mathbb{N}} c_2(\mathbf{f}_n)$ and $\mathbf{q}_2 = \sup_{n \in \mathbb{N}} q_2(\mathbf{f}_n)$ are finite. In particular we have for all $\ell, n \in \mathbb{N}$ that:

$$\|\tilde{f}_{\ell,n}\|_{L^2(\mu)} \leq \mathbf{c}_2.$$

We deduce from Remark 7.4 that $\mathbb{E} \left[\left(N_{n,\emptyset}^{[k_0]}(\mathbf{f}_n) - \Delta_n(\mathbf{f}_n) \right)^2 \right] \leq a_{0,n} \mathbf{c}_2^2$ for a sequence $(a_{0,n}, n \in \mathbb{N})$ which converges to 0 and does not depend on the sequences \mathbf{f}_n . We consider the bracket of the martingale $\Delta_n(\mathbf{f}_n)$ given by $V(n) = \sum_{i \in \mathbb{G}_{n-p_n}} \mathbb{E} [\Delta_{n,i}(\mathbf{f}_n)^2 | \mathcal{F}_i]$.

Lemma 8.1. *Under the assumptions of Theorem 3.7 ($2\alpha^2 < 1$), we have that $V(n)$ converges in probability towards σ^2 defined by (24).*

Proof. Let $f \in L^2(\mu)$ and recall that $\tilde{f} = f - \langle \mu, f \rangle$. We deduce from $\langle \mu, f \rangle = \langle \mu, \mathcal{Q}f \rangle \leq \|\mathcal{Q}f\|_\infty \leq \|\mathcal{Q}(f^2)\|_\infty^{1/2}$ that:

$$(150) \quad \|\mathcal{Q}\tilde{f}\|_\infty \leq 2 \|\mathcal{Q}(f^2)\|_\infty^{1/2} \quad \text{and} \quad \|\mathcal{Q}(\tilde{f}^2)\|_\infty \leq 4 \|\mathcal{Q}(f^2)\|_\infty.$$

Note that thanks to Assumption 2.16 we have, for all $k, \ell, r \in \mathbb{N}$, and $j > 0$:

$$(151) \quad \lim_{n \rightarrow \infty} |\langle \mu, \tilde{f}_{k,n} \mathcal{Q}^j \tilde{f}_{\ell,n} \rangle| = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} |\langle \mu, \mathcal{P} \left(\mathcal{Q}^r \tilde{f}_{k,n} \otimes_{\text{sym}} \mathcal{Q}^j \tilde{f}_{\ell,n} \right) \rangle| = 0.$$

Indeed, we have thanks to Assumption 2.16 (iii):

$$|\langle \mu, \tilde{f}_{k,n} \mathcal{Q}^j \tilde{f}_{\ell,n} \rangle| \leq \|\mathcal{Q}\tilde{f}_{\ell,n}\|_\infty \langle \mu, |\tilde{f}_{k,n}| \rangle \leq 4 \|\mathcal{Q}f_{\ell,n}^2\|_\infty^{1/2} \langle \mu, |f_{k,n}| \rangle \leq 4\mathbf{q}_2 \delta_{k,n}.$$

We also have thanks to Assumption 2.16 (iii), for $g = \mathcal{Q}^{j-1}|\tilde{f}_{\ell,n}|$ and $r = 0$:

$$\begin{aligned} |\langle \mu, \mathcal{P} \left(\mathcal{Q}^r \tilde{f}_{k,n} \otimes_{\text{sym}} \mathcal{Q}^j \tilde{f}_{\ell,n} \right) \rangle| &\leq \langle \mu, \mathcal{P} \left(|\tilde{f}_{k,n}| \otimes_{\text{sym}} \mathcal{Q}g \right) \rangle \\ &\leq \langle \mu, \mathcal{P}(\mathbf{1} \otimes_{\text{sym}} \mathcal{Q}g) \rangle \langle \mu, |f_{k,n}| \rangle + \|\mathcal{P}(|f_{k,n}| \otimes_{\text{sym}} \mathcal{Q}g)\|_{L^2(\mu)} \\ &\leq 2 \|g\|_{L^2(\mu)} \delta_{k,n} \\ &\leq 2\mathbf{c}_2 \delta_{k,n}, \end{aligned}$$

and for $r \geq 1$ using (150) and that $\langle \mu, \mathcal{P}(\mathbf{1} \otimes_{\text{sym}} h) \rangle = \langle \mu, h \rangle$:

$$|\langle \mu, \mathcal{P} \left(\mathcal{Q}^r \tilde{f}_{k,n} \otimes_{\text{sym}} \mathcal{Q}^j \tilde{f}_{\ell,n} \right) \rangle| \leq \langle \mu, \mathcal{P}(\mathbf{1} \otimes_{\text{sym}} \mathcal{Q}g) \rangle \|\mathcal{Q}^r \tilde{f}_{k,n}\|_\infty \leq 2\mathbf{q}_2 \delta_{\ell,n}.$$

Then use that for all $k \in \mathbb{N}$ fixed, we have $\lim_{n \rightarrow \infty} \delta_{k,n} = 0$ to conclude that (151) holds.

Recall that $V = V_1 + 2V_2 - R_2$ with V_1, V_2 and R_2 defined after (63) with \mathbf{f} replaced by \mathbf{f}_n . According to the proof of Lemma 7.8, see (88), we have $\lim_{n \rightarrow \infty} R_2(n) = 0$ in probability.

We now prove that $\lim_{n \rightarrow \infty} V_2(n) = 0$ in probability. For this we give a second look at the proof of (105). Recall the definition of c_k and q_k given in (83). More precisely, the term (94) can be bound from above using (150) and $\|\mathcal{P}(\tilde{f}_k \otimes_{\text{sym}} \mathcal{Q}^{k-\ell} \tilde{f}_\ell)\|_{L^2(\mu)} \leq \|\mathcal{Q}\tilde{f}_\ell\|_\infty \|\mathcal{P}(\tilde{f}_k \otimes_{\text{sym}} \mathbf{1})\|_{L^2(\mu)} \leq 2q_2(\mathbf{f}) c_2(\mathbf{f})$ as $k > \ell$, and thus (95) can be replaced by $C c_2(\mathbf{f}) (c_2(\mathbf{f}) + q_2(\mathbf{f})) \alpha^j$. Therefore the upper bound in (96) can be replaced by $C c_2^2(\mathbf{f}) (c_2(\mathbf{f}) + q_2(\mathbf{f}))^2 2^{-(n-p)}$. The term (102) can be

bound from above using $\|\tilde{f}_k \mathcal{Q}^{k-\ell} \tilde{f}_\ell\|_{L^2(\mu)} \leq \|\tilde{f}_k\|_{L^2(\mu)} \|\mathcal{Q}^{k-\ell} \tilde{f}_\ell\|_\infty \leq c_2(\mathbf{f}) q_2(\mathbf{f})$ as $k > \ell$, and thus (103) can be replaced by $C c_2(\mathbf{f}) q_2(\mathbf{f}) \alpha^j$. Therefore the upper bound in (104) can be replaced by $C c_2^2(\mathbf{f}) q_2^2(\mathbf{f}) 2^{-(n-p)}$. As $V_2 = V_6 + V_5$, we deduce that (compare with (105) and replace \mathbf{f} by \mathbf{f}_n):

$$\mathbb{E} \left[\left(V_2(n) - H_2^{[n]}(\mathbf{f}_n) \right)^2 \right] \leq C \left(c_2^4(\mathbf{f}_n) + c_2^2(\mathbf{f}_n) q_2^2(\mathbf{f}_n) \right) 2^{-(n-p)} \leq C \left(\mathbf{c}_2^4 + \mathbf{c}_2^2 \mathbf{q}_2^2 \right) 2^{-(n-p)},$$

with $H_2^{[n]}(\mathbf{f}) = H_5^{[n]}(\mathbf{f}) + H_6^{[n]}(\mathbf{f})$. Since according to (ii) in Assumption 2.16 \mathbf{c}_2 and \mathbf{q}_2 are finite, we deduce that $\lim_{n \rightarrow \infty} V_2(n) - H_2^{[n]}(\mathbf{f}_n) = 0$ in probability. We now check that $\lim_{n \rightarrow \infty} H_2^{[n]}(\mathbf{f}_n) = 0$. Using (91) and (99), we get that:

$$|H_2^{[n]}(\mathbf{f}_n)| \leq \sum_{k > \ell \geq 0} 2^{-\ell} |\langle \mu, \tilde{f}_{k,n} \mathcal{Q}^{k-\ell} \tilde{f}_{\ell,n} \rangle| + \sum_{\substack{k > \ell \geq 0 \\ r \geq 0}} 2^{r-\ell} |\langle \mu, \mathcal{P} \left(\mathcal{Q}^r \tilde{f}_{k,n} \otimes_{\text{sym}} \mathcal{Q}^{k-\ell+r} \tilde{f}_{\ell,n} \right) \rangle|.$$

Recall the definition of Δ in Assumption 2.16 (iii). Thanks to (20) and (10) we have:

$$(152) \quad \begin{aligned} |\langle \mu, \tilde{f}_{k,n} \mathcal{Q}^{k-\ell} \tilde{f}_{\ell,n} \rangle| &\leq \mathbf{c}_2^2 \alpha^{k-\ell}, \\ |\langle \mu, \mathcal{P} \left(\mathcal{Q}^r \tilde{f}_{k,n} \otimes_{\text{sym}} \mathcal{Q}^{k-\ell+r} \tilde{f}_{\ell,n} \right) \rangle| &\leq C \mathbf{c}_2^2 \alpha^{k-\ell+2r}. \end{aligned}$$

Since $\sum_{0 \leq \ell < k} 2^{-\ell} \alpha^{k-\ell} + \sum_{\substack{0 \leq \ell < k \\ r \geq 0}} 2^{r-\ell} \alpha^{k-\ell+2r}$ is finite, we deduce from (91), (99), (151) and dominated convergence that $\lim_{n \rightarrow \infty} H_2^{[n]}(\mathbf{f}_n) = 0$. This implies that $\lim_{n \rightarrow \infty} V_2(n) = 0$ in probability.

We now prove that $\lim_{n \rightarrow \infty} V_1(n) = \sigma^2$ in probability. For this we give a second look at the proof of (120). More precisely, concerning $V_4(n)$, the term (110) can be bound from above using $\|\mathcal{P}(\tilde{f}_\ell \otimes \tilde{f}_\ell)\|_{L^2(\mu)} \leq \|\mathcal{Q}(\tilde{f}_\ell^2)\|_{L^2(\mu)} \leq q_2^2(\mathbf{f})$ (thanks to (3) for the first inequality) and thus (111) can be replaced by $C (c_2^2(\mathbf{f}) + q_2^2(\mathbf{f})) \alpha^j$. Then the upper bound in (112) can be replaced by $C (c_2^4(\mathbf{f}) + q_2^4(\mathbf{f})) 2^{-(n-p)}$. Concerning $V_3(n)$, the more delicate term (117) can be bound from above using (with $g = \tilde{f}_\ell^2$):

$$\begin{aligned} \sum_{\ell \geq 0} 2^{-\ell} \|\mathcal{Q}^{j+p-\ell} \tilde{g}\|_{L^2(\mu)} \mathbf{1}_{\{\ell \leq p\}} &= 2^{-p} \|\tilde{g}\|_{L^2(\mu)} \mathbf{1}_{\{j=0\}} + \sum_{\ell=0}^p 2^{-\ell} \|\mathcal{Q}^{j+p-\ell-1} \mathcal{Q} \tilde{g}\|_{L^2(\mu)} \mathbf{1}_{\{j+p-\ell > 0\}} \\ &\leq C c_4^2(\mathbf{f}) 2^{-p} \mathbf{1}_{\{j=0\}} + C \sum_{\ell \geq 0} 2^{-\ell} \alpha^{j+p-\ell} \|\mathcal{Q} \tilde{g}\|_{L^2(\mu)} \\ &\leq C c_4^2(\mathbf{f}) 2^{-p} \mathbf{1}_{\{j=0\}} + C q_2^2(\mathbf{f}) \alpha^j, \end{aligned}$$

where we used (150) for the last inequality. Therefore the upper bound in (119) can be replaced by $C c_4^4(\mathbf{f}) 2^{-n} + C q_2^4(\mathbf{f}) 2^{-(n-p)}$. As $V_1 = V_4 + V_3$, we deduce that (compare with (120) and replace \mathbf{f} by \mathbf{f}_n):

$$\mathbb{E} \left[\left(V_1(n) - H_1^{[n]}(\mathbf{f}_n) \right)^2 \right] \leq C \left((c_4^4(\mathbf{f}_n) + q_2^4(\mathbf{f}_n)) 2^{-(n-p)} + c_4^4(\mathbf{f}_n) 2^{-n} \right),$$

with $H_1^{[n]}(\mathbf{f}) = H_3^{[n]}(\mathbf{f}) + H_4^{[n]}(\mathbf{f})$. Since $c_4^4(\mathbf{f}_n) \leq c_2^2(\mathbf{f}_n) c_\infty^2(\mathbf{f}_n) \leq C_\rho c_2^2(\mathbf{f}_n) 2^{2n\rho}$ with $\rho \in (0, 1/2)$ and some finite constant C_ρ according to (i) in Assumption 2.16, and since $\lim_{n \rightarrow \infty} p/n = 1$ so that $2^{-n(1-2\rho)} \leq 2^{-(n-p)}$ (at least for n large enough), we deduce from (ii) in Assumption 2.16 that:

$$\mathbb{E} \left[\left(V_1(n) - H_1^{[n]}(\mathbf{f}_n) \right)^2 \right] \leq C \left(\mathbf{c}_2^4 + \mathbf{q}_2^4 + C_\rho \mathbf{c}_2^2 \right) 2^{-(n-p)}$$

and thus $\lim_{n \rightarrow \infty} V_1(n) - H_1^{[n]}(\mathbf{f}_n) = 0$ in probability. We check that $\lim_{n \rightarrow \infty} H_1^{[n]}(\mathbf{f}_n) = \sigma^2$. Recall (see (113) and (108)) that:

$$H_3^{[n]}(\mathbf{f}_n) = \sum_{\ell \geq 0} 2^{-\ell} \langle \mu, \tilde{f}_{\ell,n}^2 \rangle \mathbf{1}_{\{\ell \leq p\}} \quad \text{and} \quad |H_4^{[n]}(\mathbf{f}_n)| \leq \sum_{\ell \geq 0, k \geq 0} 2^{k-\ell} |\langle \mu, \mathcal{P}(\mathcal{Q}^k \tilde{f}_{\ell,n} \otimes^2) \rangle|.$$

Thanks to (3) and (20) (see also (106)) we have that $|\langle \mu, \mathcal{P}(\mathcal{Q}^k \tilde{f}_{\ell,n} \otimes^2) \rangle| \leq C \mathbf{c}_2^2 \alpha^{2k}$. Using Assumption 2.16 (iii), we get that

$$(153) \quad |\langle \mu, \mathcal{P}(\tilde{f}_{\ell,n} \otimes^2) \rangle| \leq |\langle \mu, \mathcal{P}(f_{\ell,n} \otimes^2) \rangle| + \langle \mu, f_{\ell,n} \rangle^2 \leq (1 + \Delta) \delta_{\ell,n}.$$

We deduce from (151) (for $k \geq 1$) and the previous upper-bound (for $k = 0$) and dominated convergence that $\lim_{n \rightarrow \infty} H_4^{[n]}(\mathbf{f}_n) = 0$.

We now prove that $\lim_{n \rightarrow \infty} H_3^{[n]}(\mathbf{f}_n) = \sigma^2$. We define $\sigma_n^2 = \sum_{\ell=0}^n 2^{-\ell} \|f_{\ell,n}\|_{L^2(\mu)}^2$, so that by Assumption 2.16 (iv), $\lim_{n \rightarrow \infty} \sigma_n^2 = \sigma^2$. We have:

$$|H_3^{[n]}(\mathbf{f}_n) - \sigma_n^2| \leq \sum_{\ell=p+1}^n 2^{-\ell} \langle \mu, f_{\ell,n}^2 \rangle + \sum_{\ell=0}^p 2^{-\ell} \langle \mu, f_{\ell,n} \rangle^2 \leq \mathbf{c}_2^2 2^{-p} + \Delta \sum_{\ell=0}^p 2^{-\ell} \delta_{\ell,n}.$$

Then use dominated convergence to deduce that $\lim_{n \rightarrow \infty} |H_3^{[n]}(\mathbf{f}_n) - \sigma_n^2| = 0$. This implies that $\lim_{n \rightarrow \infty} V_1(n) = \sigma^2$ in probability.

Use that $V = V_1 + 2V_2 - R_2$ and the previous convergences to deduce the result. \square

We now check the Lindeberg condition using a fourth moment condition. We set $R_3(n) = \sum_{i \in \mathbb{G}_{n-p,n}} \mathbb{E}[\Delta_{n,i}(\mathbf{f}_n)^4]$, see (78). The proof of the next lemma is a delicate adaptation of the proof of Lemma 7.9.

Lemma 8.2. *Under the assumptions of Theorem 3.7 ($2\alpha^2 < 1$), we get $\lim_{n \rightarrow \infty} R_3(n) = 0$.*

Then, we end the proof of Theorem 3.7 in the sub-critical case by arguing as in the (end of the) proof of Corollary 5.3.

Proof of Lemma 8.2. We keep notations from the proof of Lemma 7.9. In this setting, (121), (122) and (123) hold with f_ℓ replaced by $f_{\ell,n}$ and $c_4(\mathbf{f})$ by $c_4(\mathbf{f}_n)$. We now look precisely at the terms in (123). Set $h_k = \mathcal{Q}^{k-1} \tilde{f}_{\ell,n}$ which is consistent with notations from the proof of Lemma 7.9. From (126), we get:

$$\langle \mu, |\psi_{1,p-\ell}| \rangle \leq C 2^{p-\ell} c_4^4(\mathbf{f}_n).$$

From (127), we get, with $g = |\tilde{f}_{\ell,n}|^3$:

$$\begin{aligned}
|\langle \mu, \psi_{2,p-\ell} \rangle| &\leq C 2^{2(p-\ell)} \sum_{k=0}^{p-\ell-1} 2^{-k} |\langle \mu, \mathcal{Q}^k \mathcal{P} \left(\mathcal{Q}^{p-\ell-k-1} (|\tilde{f}_{\ell,n}|^3) \otimes_{\text{sym}} h_{p-\ell-k} \right) \rangle| \\
&= C 2^{2(p-\ell)} \sum_{k=0}^{p-\ell-1} 2^{-k} |\langle \mu, \mathcal{P} \left(\mathcal{Q}^{p-\ell-k-1} (\tilde{g}) \otimes_{\text{sym}} h_{p-\ell-k} \right) \rangle| \\
(154) \quad &\leq C 2^{2(p-\ell)} \sum_{k=0}^{p-\ell-1} 2^{-k} \|\mathcal{Q}^{p-\ell-k-1} \tilde{g}\|_{L^2(\mu)} \|h_{p-\ell-k}\|_{L^2(\mu)} \\
&\leq C 2^{2(p-\ell)} \sum_{k=0}^{p-\ell-1} 2^{-k} \alpha^{2(p-k-\ell)} \|g\|_{L^2(\mu)} \|\tilde{f}_{\ell,n}\|_{L^2(\mu)} \\
&\leq C 2^{p-\ell} c_6^3(\mathbf{f}_n) \mathbf{c}_2,
\end{aligned}$$

where we used that $\langle \mu, \mathcal{P}(\mathbf{1} \otimes_{\text{sym}} h_{p-\ell-k}) \rangle = 2\langle \mu, \mathcal{Q}h_{p-\ell-k} \rangle = 0$ for the equality, (10) for the second inequality, (20) and (124) for the third. We easily deduce from (128), distinguishing according to $k = p - \ell - 1$ (then use (10)) and $k \leq p - \ell - 2$ (then use $|\mathcal{Q}(\tilde{f}_{\ell,n}^2)| \leq 4\mathfrak{q}_2^2$, see (150)) that:

$$(155) \quad \langle \mu, |\psi_{3,p-\ell}| \rangle \leq C 2^{p-\ell} c_4^4(\mathbf{f}_n) + C 2^{2(p-\ell)} \mathfrak{q}_2^2 \mathbf{c}_2^2.$$

From (130), we get:

$$\langle \mu, |\psi_{4,p-\ell}| \rangle \leq C 2^{2(p-\ell)} \mathbf{c}_2^4.$$

As $\mathcal{P}(\tilde{f}_{\ell,n} \otimes^2) \leq \mathcal{Q}(\tilde{f}_{\ell,n}^2) \leq \mathfrak{q}_2^2$, we deduce that

$$(156) \quad \|\mathcal{P}(\tilde{f}_{\ell,n} \otimes^2)\|_{L^2(\mu)} \leq C \mathfrak{q}_2 \mathbf{c}_2$$

and thus that $C 2^{-2(p-\ell)} c_4^4(\mathbf{f})$ in (132) (corresponding to $k = p - \ell - 1$) can be replaced by $C 2^{-2(p-\ell)} \mathfrak{q}_2^2 \mathbf{c}_2^2$, so that:

$$\langle \mu, |\psi_{5,p-\ell}| \rangle \leq C 2^{2(p-\ell)} \mathbf{c}_2^2 (\mathfrak{q}_2^2 + \mathbf{c}_2^2).$$

The same trick in (134) and using that $\|\mathcal{Q}^j(\tilde{f}_{\ell}^2)\|_{L^2(\mu)} \leq 4\mathfrak{q}_2 \min(\mathfrak{q}_2, \mathbf{c}_2)$ for all $j \in \mathbb{N}^*$ (see (150)) lead to:

$$\langle \mu, |\psi_{6,p-\ell}| \rangle \leq C 2^{2(p-\ell)} \mathbf{c}_2^2 \mathfrak{q}_2^2.$$

The term $|\langle \mu, \psi_{7,p-\ell} \rangle|$ is more delicate. We first consider $\Gamma_{k,r}^{[7]}$ defined in (136) when $k = p - \ell - 1$.

Writing $A_r = \Gamma_{p-\ell-1,r}^{[7]}$ and setting $g = \mathcal{P}(\tilde{f}_{\ell} \otimes_{\text{sym}} \tilde{f}_{\ell}^2)$, we get that:

$$\begin{aligned}
A_r &= 2^{-(p-\ell-1)} |\langle \mu, \mathcal{P}(\mathcal{Q}^{p-\ell-r-2} g \otimes_{\text{sym}} h_{p-\ell-r}) \rangle| \\
&= 2^{-(p-\ell-1)} |\langle \mu, \mathcal{P}(\mathcal{Q}^{p-\ell-r-2} \tilde{g} \otimes_{\text{sym}} h_{p-\ell-r}) \rangle| \\
&\leq C 2^{-(p-\ell)} \|\mathcal{Q}^{p-\ell-r-2} \tilde{g}\|_{L^2(\mu)} \|h_{p-\ell-r}\|_{L^2(\mu)} \\
&\leq C 2^{-(p-\ell)} \alpha^{2(p-\ell-r)} c_6^3(\mathbf{f}_n) \mathbf{c}_2,
\end{aligned}$$

where we used for $\langle \mu, \mathcal{P}(\mathbf{1} \otimes_{\text{sym}} h_{p-\ell-r}) \rangle = 0$ for the second equality; (10) for the first inequality; and (20) and that (see (138) with $g = \tilde{f}_{\ell,n}$) $\|g\|_{L^2(\mu)} \leq c_6^3(\mathbf{f}_n)$ for second. We now consider $k \leq p - \ell - 2$.

We deduce from (137) that:

$$\begin{aligned} \Gamma_{k,r}^{[7]} &\leq C 2^{-k} \|\mathcal{P}\left(h_{p-\ell-k} \otimes_{\text{sym}} \mathcal{Q}^{p-\ell-k-1}(\tilde{f}_\ell^2)\right)\|_{L^2(\mu)} \|h_{p-\ell-r}\|_{L^2(\mu)} \\ &\leq C 2^{-k} \|h_{p-\ell-k}\|_{L^2(\mu)} \mathfrak{q}_2^2 \|h_{p-\ell-r}\|_{L^2(\mu)} \\ &\leq C 2^{-k} \alpha^{2(p-\ell-k)} \mathfrak{c}_2^2 \mathfrak{q}_2^2. \end{aligned}$$

We deduce from (135) that:

$$|\langle \mu, \psi_{7,p-\ell} \rangle| \leq C 2^{p-\ell} \mathfrak{c}_6^3(\mathfrak{f}_n) \mathfrak{c}_2 + C 2^{2(p-\ell)} \mathfrak{c}_2^2 \mathfrak{q}_2^2.$$

According to (3) and (150), we have:

$$\mathcal{P}(h_{p-\ell-k} \otimes^2) \leq \mathcal{Q}(h_{p-\ell-k}^2) \leq 4\mathfrak{q}_2^2 \quad \text{for } k = p - \ell - 1.$$

So we can replace the term $\mathfrak{c}_4^2(\mathfrak{f})$ in (141) by \mathfrak{q}_2^2 and deduce from (139) that:

$$\langle \mu, |\psi_{8,p-\ell}| \rangle \leq C 2^{2(p-\ell)} \mathfrak{c}_2^2 (\mathfrak{c}_2^2 + \mathfrak{q}_2^2).$$

Similarly the term $\mathfrak{c}_4^2(\mathfrak{f})$ in (144) can be replaced by \mathfrak{q}_2^2 . Notice that

$$\|\mathcal{P}\left(\mathcal{Q}\tilde{f}_\ell \otimes_{\text{sym}} \mathcal{Q}(\tilde{f}_\ell^2)\right)\|_{L^2(\mu)} \leq C \mathfrak{q}_2^2 \mathfrak{c}_2,$$

so the term $\mathfrak{c}_4^2(\mathfrak{f})$ in (148) can be replaced by \mathfrak{q}_2^2 . We then deduce from (142) and the computations thereafter, that:

$$\langle \mu, |\psi_{9,p-\ell}| \rangle \leq C 2^{2(p-\ell)} \mathfrak{c}_2^2 (\mathfrak{c}_2^2 + \mathfrak{q}_2^2).$$

In conclusion, we get that:

$$\begin{aligned} R_3(n) &\leq C n^3 2^{-n-p} \left(\mathfrak{c}_4^4(\mathfrak{f}_n) + \sum_{\ell=0}^{p-3} \sum_{j=1}^9 \langle \mu, \psi_{j,p-\ell} \rangle \right) \\ &\leq C n^3 2^{-n-p} \left(\mathfrak{c}_4^4(\mathfrak{f}_n) + \sum_{\ell=0}^{p-3} \left[2^{p-\ell} (\mathfrak{c}_4^4(\mathfrak{f}_n) + \mathfrak{c}_6^3(\mathfrak{f}_n) \mathfrak{c}_2) + 2^{2(p-\ell)} \mathfrak{c}_2^2 (\mathfrak{c}_2^2 + \mathfrak{q}_2^2) \right] \right) \\ &\leq C n^3 \left(2^{-n(1-2\rho)} + 2^{(n-p)} (\mathfrak{c}_2^2 + \mathfrak{q}_2^2) \right) \mathfrak{c}_2^2, \end{aligned}$$

where, we used (121), (122) and (123) for the first inequality; and $\mathfrak{c}_4^4(\mathfrak{f}_n) \leq C \mathfrak{c}_2^2 2^{2n\rho}$ and $\mathfrak{c}_6^3(\mathfrak{f}_n) \leq C \mathfrak{c}_2 2^{2n\rho}$ with $\rho \in (0, 1/2)$ thanks to Remark 2.18 and (i) from Assumption 2.16 for the last one. As $\rho \in (0, 1/2)$ by Assumption 2.16 (i), we deduce that $\lim_{n \rightarrow \infty} R_3(n) = 0$. \square

9. THE CRITICAL CASE: $2\alpha^2 = 1$

In the critical case $\alpha = 1/\sqrt{2}$, we shall denote by \mathcal{R}_j the projector on the eigen-space associated to the eigenvalue α_j with $\alpha_j = \theta_j \alpha$, $|\theta_j| = 1$ and for j in the finite set of indices J . Since \mathcal{Q} is a real operator, we get that if α_j is a non real eigenvalue, so is $\bar{\alpha}_j$. We shall denote by $\bar{\mathcal{R}}_j$ the projector associated to $\bar{\alpha}_j$. Recall that the sequence $(\beta_n, n \in \mathbb{N})$ in Assumptions 2.7 or 2.14 is non-increasing and bounded from above by 1. For all measurable real-valued function f defined on S , we set, when this is well defined:

$$(157) \quad \boxed{\hat{f} = \tilde{f} - \sum_{j \in J} \mathcal{R}_j(f) \quad \text{with} \quad \tilde{f} = f - \langle \mu, f \rangle.}$$

We shall consider, when well defined, for a sequence $\mathbf{f} = (f_\ell, \ell \in \mathbb{N})$ of measurable real-valued functions defined on S , the quantities:

$$(158) \quad \Sigma^{\text{crit}}(\mathbf{f}) = \Sigma_1^{\text{crit}}(\mathbf{f}) + 2\Sigma_2^{\text{crit}}(\mathbf{f}),$$

where:

$$(159) \quad \Sigma_1^{\text{crit}}(\mathbf{f}) = \sum_{k \geq 0} 2^{-k} \langle \mu, \mathcal{P}f_{k,k}^* \rangle = \sum_{k \geq 0} 2^{-k} \sum_{j \in J} \langle \mu, \mathcal{P}(\mathcal{R}_j(f_k) \otimes_{\text{sym}} \overline{\mathcal{R}}_j(f_k)) \rangle,$$

$$(160) \quad \Sigma_2^{\text{crit}}(\mathbf{f}) = \sum_{0 \leq \ell < k} 2^{-(k+\ell)/2} \langle \mu, \mathcal{P}f_{k,\ell}^* \rangle,$$

with, for $k, \ell \in \mathbb{N}$:

$$(161) \quad f_{k,\ell}^* = \sum_{j \in J} \theta_j^{\ell-k} \mathcal{R}_j(f_k) \otimes_{\text{sym}} \overline{\mathcal{R}}_j(f_\ell).$$

Notice that $f_{k,\ell}^* = f_{\ell,k}^*$ and that $f_{k,\ell}^*$ is real-valued as $\overline{\theta_j^{\ell-k} \mathcal{R}_j(f_k) \otimes \overline{\mathcal{R}}_j(f_\ell)} = \theta_{j'}^{\ell-k} \mathcal{R}_{j'}(f_k) \otimes \overline{\mathcal{R}}_{j'}(f_\ell)$ for j' such that $\alpha_{j'} = \overline{\alpha_j}$ and thus $\mathcal{R}_{j'} = \overline{\mathcal{R}}_j$.

Remark 9.1. Recall the definitions of $\Sigma_{\mathbb{G}}^{\text{crit}}(f)$ and $\Sigma_{\mathbb{T}}^{\text{crit}}(f)$ given in (38). If we take $\mathbf{f} = (f, 0, 0, \dots)$, we have $\Sigma^{\text{crit}}(\mathbf{f}) = \Sigma_{\mathbb{G}}^{\text{crit}}(f)$. If we take $\mathbf{f} = (f, f, \dots)$, the infinite sequence of the same function f , we have $\Sigma^{\text{crit}}(\mathbf{f}) = 2\Sigma_{\mathbb{T}}^{\text{crit}}(f)$.

9.1. The point-wise approach. We shall consider sequences $\mathbf{f} = (f_\ell, \ell \in \mathbb{N})$ of elements of F which satisfies Assumption 2.7 uniformly, that is such that there exists $g \in F$ with:

$$(162) \quad |\mathcal{Q}^n(f_\ell)| \leq g, \quad |\mathcal{Q}^n(\tilde{f}_\ell)| \leq \alpha^n g \quad \text{and} \quad |\mathcal{Q}^n(\hat{f}_\ell)| \leq \beta_n \alpha^n g \quad \text{for all } n, \ell \in \mathbb{N}.$$

We deduce that there exists a finite constant c_J depending only on $\{\alpha_j, j \in J\}$ such that for all $\ell \in \mathbb{N}, n \in \mathbb{N}, j_0 \in J$:

$$(163) \quad |f_\ell| \leq g, \quad |\tilde{f}_\ell| \leq g, \quad |\langle \mu, f_\ell \rangle| \leq \langle \mu, g \rangle, \quad \left| \sum_{j \in J} \theta_j^n \mathcal{R}_j(f_\ell) \right| \leq 2g \quad \text{and} \quad |\mathcal{R}_{j_0}(f_\ell)| \leq c_J g,$$

where for the last inequality, we used that the Vandermonde matrix $(\theta_j^n; j \in J, n \in \{0, \dots, |J| - 1\})$ is invertible. Notice that (162) holds in particular if (7) holds for all $f \in F$ and $\mathbf{f} = (f_n, n \in \mathbb{N})$ takes finitely distinct values in F (i.e. the set $\{f_\ell; \ell \in \mathbb{N}\} \subset F$ is finite). The proof of the following result is given in Section 10.

Theorem 9.2. *Let X be a BMC with kernel \mathcal{P} and initial distribution ν . Assume that Assumptions 2.4 and 2.7 hold with $\alpha = 1/\sqrt{2}$. We have the following convergence in distribution for all sequence $\mathbf{f} = (f_\ell, \ell \in \mathbb{N})$ of elements of F satisfying Assumptions 2.7 uniformly, that is (162) for some $g \in F$:*

$$n^{-1/2} N_{n,0}(\mathbf{f}) \xrightarrow[n \rightarrow \infty]{(d)} G,$$

where G is a Gaussian real-valued random variable with variance $\Sigma^{\text{crit}}(\mathbf{f})$ given by (158), which is well defined and finite.

9.2. The $L^2(\mu)$ approach. In this section we use Assumptions 2.10, 2.13 and 2.14. We recall $N_{n,\emptyset}^{[k_0]}(\mathbf{f})$ defined in (39). We shall consider sequences $\mathbf{f} = (f_\ell, \ell \in \mathbb{N})$ which satisfies Assumption 2.14. We have the following result whose proof, given in Section 11, mimics that of Theorem 9.2 and Corollary 3.2.

Corollary 9.3. *Let X be a BMC with kernel \mathcal{P} and initial distribution ν such that Assumptions 2.10 (with $k_0 \in \mathbb{N}$), 2.13 and 2.14 are in force with $\alpha = 1/\sqrt{2}$. We have the following convergence in distribution for all sequence $\mathbf{f} = (f_\ell, \ell \in \mathbb{N})$ bounded in $L^4(\mu)$ (that is $\sup_{\ell \in \mathbb{N}} \|f_\ell\|_{L^4(\mu)} < +\infty$):*

$$n^{-1/2} N_{n,\emptyset}^{[k_0]}(\mathbf{f}) \xrightarrow[n \rightarrow \infty]{(d)} G,$$

where G is centered Gaussian random variable with variance $\Sigma^{\text{crit}}(\mathbf{f})$ given by (158), which is well defined and finite.

10. PROOF OF THEOREM 9.2

We keep notation from Section 6. Let $(p_n, n \in \mathbb{N})$ be an increasing sequence of elements of \mathbb{N} such that (53) holds. When there is no ambiguity, we write p for p_n . Recall the definitions of $\Delta_n(\mathbf{f})$ and $N_{n,\emptyset}(\mathbf{f})$ from (54) and (55), as well as $R_0(n)$ and $R_1(n)$ from (56). We have the following elementary lemma.

Lemma 10.1. *Under the assumptions of Theorem 9.2, we have the following convergence:*

$$\lim_{n \rightarrow \infty} n^{-1} \mathbb{E}[R_0(n)^2] = 0.$$

Proof. Following the proof of Lemma 6.1, and using that $2\alpha^2 = 1$ so that $\sum_{\ell=0}^{k-1} (2\alpha^2)^\ell = k$, we get there exists some finite constant c depending on \mathbf{f} such that $\mathbb{E}[M_{\mathbb{G}_k}(\tilde{f}_{n-k})^2] \leq c^2 (k+1)2^k$ for all $k \geq 0$. This implies that:

$$\mathbb{E}[R_0(n)^2]^{1/2} \leq |\mathbb{G}_n|^{-1/2} \sum_{k=0}^{n-p-1} \mathbb{E}[M_{\mathbb{G}_k}(\tilde{f}_{n-k})^2]^{1/2} \leq c 2^{-n/2} \sum_{k=0}^{n-p-1} \sqrt{k+1} 2^{k/2} \leq Cc \sqrt{n} 2^{-p/2}.$$

Then use that $\lim_{n \rightarrow \infty} p/n = 1$ to conclude. \square

We have the following lemma.

Lemma 10.2. *Under the assumptions of Theorem 9.2, we have the following convergence:*

$$\lim_{n \rightarrow \infty} n^{-1} \mathbb{E}[R_1(n)^2] = 0.$$

Proof. Following the proof of Lemma 6.2 with the same notations, and using that $2\alpha^2 = 1$ so that $\sum_{k=0}^{n-p-1} (2\alpha^2)^k = n-p$ in (60), we get that there exists $g_3 \in F$ such that $\mathbb{E}[R_1(\ell, n)^2] \leq (n-p+1)(2\alpha)^{-2\ell} \langle \nu, g_3 \rangle$, where $R_1(\ell, n)$ is defined in (58). As $2\alpha = \sqrt{2}$ and $R_1(n) = \sum_{\ell=0}^p R_1(\ell, n)$, we deduce that:

$$\mathbb{E}[R_1(n)^2]^{1/2} \leq \sum_{\ell=0}^p \mathbb{E}[R_1(\ell, n)^2]^{1/2} \leq 4\sqrt{n-p+1} \langle \nu, g_3 \rangle^{1/2}.$$

Use that $\lim_{n \rightarrow \infty} p/n = 1$ to conclude. \square

Recall $\Delta_n(\mathbf{f})$ defined in (54), and its bracket defined by $V(n) = \sum_{i \in \mathbb{G}_{n-p_n}} \mathbb{E}[\Delta_{n,i}(\mathbf{f})^2 | \mathcal{F}_i]$ defined in (62). Recall, see (63), that $V(n) = V_1(n) + 2V_2(n) - R_2(n)$. We study the convergence of each term of the latter right hand side.

Lemma 10.3. *Under the assumptions of Theorem 9.2, we have the following convergence:*

$$\lim_{n \rightarrow \infty} n^{-1/2} \mathbb{E}[R_2(n)] = 0.$$

Proof. Following the proof of Lemma 6.3 with the same notations and using that $2\alpha^2 = 1$ so that $\sum_{\ell=0}^p (2\alpha)^\ell \leq C2^{p/2}$ in (65), we get that $\mathbb{E}[R_2(n)] \leq C \langle \nu, g_1 \rangle$, with $g_1 \in F$. This gives the result. \square

Recall $f_{k,\ell}^*$ defined in (161). For $k, \ell, r \in \mathbb{N}$, we will consider the \mathbb{C} -valued functions on S^2 :

$$(164) \quad f_{k,\ell,r} = \left(\sum_{j \in J} \theta_j^r \mathcal{R}_j(f_k) \right) \otimes_{\text{sym}} \left(\sum_{j \in J} \theta_j^{r+k-\ell} \mathcal{R}_j(f_\ell) \right) \quad \text{and} \quad f_{k,\ell,r}^\circ = f_{k,\ell,r} - f_{k,\ell}^*.$$

Lemma 10.4. *Under the assumptions of Theorem 9.2, we have that a.s. $\lim_{n \rightarrow \infty} n^{-1} V_2(n) = \Sigma_2^{\text{crit}}(f)$ with $\Sigma_2^{\text{crit}}(f)$ defined by (160) which is well defined and finite.*

Proof. We keep the decomposition (66) of $V_2(n) = V_5(n) + V_6(n)$ given in the proof of Lemma 6.4. We first consider the term $V_6(n)$ given in (67) by:

$$(165) \quad V_6(n) = |\mathbb{G}_{n-p}|^{-1} M_{\mathbb{G}_{n-p}}(H_{6,n}),$$

with:

$$(166) \quad H_{6,n} = \sum_{\substack{0 \leq \ell < k \leq p \\ r \geq 0}} h_{k,\ell,r}^{(n)} \mathbf{1}_{\{r+k < p\}} \quad \text{and} \quad h_{k,\ell,r}^{(n)} = 2^{r-\ell} \mathbb{Q}^{p-1-(r+k)} \left(\mathcal{P} \left(\mathbb{Q}^r \tilde{f}_k \otimes_{\text{sym}} \mathbb{Q}^{k-\ell+r} \tilde{f}_\ell \right) \right).$$

We set

$$(167) \quad \bar{H}_{6,n} = \sum_{0 \leq \ell < k \leq p; r \geq 0} \bar{h}_{k,\ell,r}^{(n)} \mathbf{1}_{\{r+k < p\}}$$

where for $0 \leq \ell < k \leq p$ and $0 \leq r < p - k$:

$$\bar{h}_{k,\ell,r}^{(n)} = 2^{r-\ell} \alpha^{k-\ell+2r} \mathbb{Q}^{p-1-(r+k)} (\mathcal{P} f_{k,\ell,r}) = 2^{-(k+\ell)/2} \mathbb{Q}^{p-1-(r+k)} (\mathcal{P} f_{k,\ell,r}),$$

where we used that $2\alpha^2 = 1$. We have:

$$\begin{aligned} |h_{k,\ell,r}^{(n)} - \bar{h}_{k,\ell,r}^{(n)}| &\leq 2^{r-\ell} \mathbb{Q}^{p-1-(r+k)} \left(\mathcal{P} \left(\left| \mathbb{Q}^r \tilde{f}_k \otimes_{\text{sym}} \mathbb{Q}^{k-\ell+r} \tilde{f}_\ell - \alpha^{k-\ell+2r} f_{k,\ell,r} \right| \right) \right) \\ &\leq C 2^{r-\ell} \beta_r \alpha^{k-\ell+2r} \mathbb{Q}^{p-1-(r+k)} (\mathcal{P}(g \otimes g)) \\ &\leq C \beta_r 2^{-(k+\ell)/2} g_1^*, \end{aligned}$$

where we wrote (with r' and f replaced by r and f_k and by $k - \ell + r$ and f_ℓ) that

$$(168) \quad \mathbb{Q}^{r'} \tilde{f} = \mathbb{Q}^{r'} \hat{f} + \alpha^{r'} \sum_{j \in J} \theta_j^{r'} \mathcal{R}_j(f)$$

and used (162), (163) and that $(\beta_n, n \in \mathbb{N})$ is non-decreasing for the second inequality and used (4) (with f and g replaced by $\mathcal{P}(g \otimes g)$ and g_1^*) for the last. We deduce that:

$$|H_{6,n} - \bar{H}_{6,n}| \leq \sum_{0 \leq \ell < k \leq p, r \geq 0} |h_{k,\ell,r}^{(n)} - \bar{h}_{k,\ell,r}^{(n)}| \mathbf{1}_{\{r+k < p\}} \leq C \left(\sum_{r=0}^n \beta_r \right) g_1^*.$$

As $\lim_{n \rightarrow \infty} \beta_n = 0$, we get that $\lim_{n \rightarrow \infty} n^{-1} \sum_{r=0}^n \beta_r = 0$. We deduce from (36) that a.s.:

$$(169) \quad \lim_{n \rightarrow \infty} n^{-1} |\mathbb{G}_{n-p}|^{-1} M_{\mathbb{G}_{n-p}}(|H_{6,n} - \bar{H}_{6,n}|) = 0.$$

We set $H_6^{[n]} = \sum_{0 \leq \ell < k \leq p; r \geq 0} h_{k,\ell,r} \mathbf{1}_{\{r+k < p\}}$ with for $0 \leq \ell < k \leq p$ and $0 \leq r < p - k$:

$$(170) \quad h_{k,\ell,r} = 2^{-(k+\ell)/2} \langle \mu, \mathcal{P}f_{k,\ell,r} \rangle = \langle \mu, \bar{h}_{k,\ell,r}^{(n)} \rangle.$$

Notice that:

$$\begin{aligned} |\bar{h}_{k,\ell,r}^{(n)} - h_{k,\ell,r}| &\leq 2^{-(k+\ell)/2} \sum_{j,j' \in J} \left| \mathbb{Q}^{p-1-(r+k)} (\mathcal{P}(\mathcal{R}_j f_k \otimes_{\text{sym}} \mathcal{R}_{j'} f_\ell)) - \langle \mu, \mathcal{P}(\mathcal{R}_j f_k \otimes_{\text{sym}} \mathcal{R}_{j'} f_\ell) \rangle \right| \\ &\leq 2^{-(k+\ell)/2} \alpha^{p-1-(r+k)} \sum_{j,j' \in J} g_{k,\ell,j,j'} \\ &= 2^{-(k+\ell)/2} \alpha^{p-1-(r+k)} g_{k,\ell}, \end{aligned}$$

where we used (5) (with f and g replaced by $\mathcal{P}(\mathcal{R}_j f_k \otimes_{\text{sym}} \mathcal{R}_{j'} f_\ell)$ and $g_{k,\ell,j,j'}$) for the second inequality and $g_{k,\ell} = \sum_{j,j' \in J} g_{k,\ell,j,j'}$ for the equality. We have that $g_{k,\ell}$ belongs to F . Since $|\mathcal{P}f_{k,\ell,r}| \leq \mathcal{P}|f_{k,\ell,r}| \leq 4\mathcal{P}(g \otimes g)$, thanks to the fourth inequality in (163), we deduce from (4) (with f and g replaced by $4\mathcal{P}(g \otimes g)$ and g_2^*) that for all $0 \leq \ell < k$ and $0 \leq r < p - k$:

$$|\bar{h}_{k,\ell,r}^{(n)}| \leq 2^{-(k+\ell)/2} g_2^* \quad \text{and} \quad |h_{k,\ell,r}| \leq 2^{-(k+\ell)/2} \langle \mu, g_2^* \rangle.$$

Set $r_0 \in \mathbb{N}$ and $g_{r_0} = \sum_{0 \leq \ell < k \leq r_0} g_{k,\ell}$. Notice that g_{r_0} belongs to F and is non-negative. Furthermore, we have for n large enough so that $p > 2r_0$:

$$\begin{aligned} |\bar{H}_{6,n} - H_6^{[n]}| &\leq \sum_{\substack{0 \leq \ell < k \leq p \\ r \geq 0}} |\bar{h}_{k,\ell,r}^{(n)} - h_{k,\ell,r}| \mathbf{1}_{\{r+k < p\}} \\ &\leq \sum_{0 \leq \ell < k \leq r_0} \sum_{r=0}^{p-k-1} 2^{-(k+\ell)/2} \alpha^{p-1-(r+k)} g_{r_0} + \sum_{\substack{0 \leq \ell < k \leq p \\ k > r_0}} \sum_{r=0}^{p-k-1} \left(|\bar{h}_{k,\ell,r}^{(n)}| + |h_{k,\ell,r}| \right) \mathbf{1}_{\{r+k < p\}} \\ &\leq C g_{r_0} + \sum_{0 \leq \ell < k \leq p, k > r_0} (p-k) 2^{-(k+\ell)/2} (g_2^* + \langle \mu, g_2^* \rangle) \\ &\leq C g_{r_0} + Cn 2^{-r_0/2} (g_2^* + \langle \mu, g_2^* \rangle). \end{aligned}$$

We deduce that:

$$\limsup_{n \rightarrow \infty} n^{-1} |\mathbb{G}_{n-p}|^{-1} M_{\mathbb{G}_{n-p}} (|\bar{H}_{6,n} - H_6^{[n]}|) \leq C 2^{-r_0/2} \langle \mu, g_2^* \rangle.$$

Since r_0 can be arbitrary large, we get that:

$$(171) \quad \lim_{n \rightarrow \infty} n^{-1} |\mathbb{G}_{n-p}|^{-1} M_{\mathbb{G}_{n-p}} (|\bar{H}_{6,n} - H_6^{[n]}|) = 0.$$

We set for $k, \ell \in \mathbb{N}$:

$$(172) \quad h_{k,\ell}^* = 2^{-(k+\ell)/2} \langle \mu, \mathcal{P}(f_{k,\ell}^*) \rangle.$$

Using the last inequality in (163) and the definition (161) of $f_{k,\ell}^*$, we deduce there exists a finite constant c independent of n such that, for all $k, \ell \in \mathbb{N}$, $|h_{k,\ell}^*| \leq c 2^{-(k+\ell)/2}$. This implies that $H_0^* = \sum_{0 \leq \ell < k} (k+1) |h_{k,\ell}^*|$ is finite and (see (160)) the sum:

$$(173) \quad H_6^*(f) = \sum_{0 \leq \ell < k} h_{k,\ell}^* = \Sigma_2^{\text{crit}}(f)$$

is well defined and finite. We write:

$$h_{k,\ell,r} = h_{k,\ell}^* + h_{k,\ell,r}^\circ,$$

with

$$h_{k,\ell,r}^\circ = 2^{-(k+\ell)/2} \langle \mu, \mathcal{P} f_{k,\ell,r}^\circ \rangle,$$

where we recall that $f_{k,\ell,r}^\circ = f_{k,\ell,r} - f_{k,\ell}^*$, and

$$(174) \quad H_6^{[n]} = H_6^{[n],*} + H_6^{[n],\circ}$$

with

$$H_6^{[n],*} = \sum_{0 \leq \ell < k \leq p} (p-k) h_{k,\ell}^* \quad \text{and} \quad H_6^{[n],\circ} = \sum_{0 \leq \ell < k \leq p; r \geq 0} h_{k,\ell,r}^\circ \mathbf{1}_{\{r+k < p\}}.$$

Recall $\lim_{n \rightarrow \infty} p/n = 1$. We have:

$$|n^{-1} H_6^{[n],*} - H_6^*(f)| \leq |n^{-1} p - 1| |H_6^*(f)| + n^{-1} H_0^* + \sum_{\substack{0 \leq \ell < k \\ k > p}} |h_{k,\ell}^*|,$$

so that $\lim_{n \rightarrow \infty} |n^{-1} H_6^{[n],*} - H_6^*(f)| = 0$ and thus:

$$(175) \quad \lim_{n \rightarrow \infty} n^{-1} H_6^{[n],*} = H_6^*(f).$$

We now prove that $n^{-1} H_6^{[n],\circ}$ converges towards 0. We have:

$$(176) \quad f_{k,\ell,r}^\circ = \sum_{j,j' \in J, \theta_j \theta_{j'} \neq 1} (\theta_{j'} \theta_j)^r \theta_{j'}^{k-\ell} \mathcal{R}_j f_k \otimes_{\text{sym}} \mathcal{R}_{j'} f_\ell.$$

This gives:

$$(177) \quad \begin{aligned} |H_6^{[n],\circ}| &= \left| \sum_{0 \leq \ell < k \leq p, r \geq 0} 2^{-(k+\ell)/2} \langle \mu, \mathcal{P} f_{k,\ell,r}^\circ \rangle \mathbf{1}_{\{r+k < p\}} \right| \\ &\leq \sum_{0 \leq \ell < k \leq p} 2^{-(k+\ell)/2} \sum_{j,j' \in J, \theta_j \theta_{j'} \neq 1} \left| \langle \mu, \mathcal{P}(\mathcal{R}_j f_k \otimes_{\text{sym}} \mathcal{R}_{j'} f_\ell) \rangle \right| \left| \sum_{r=0}^{p-k-1} (\theta_{j'} \theta_j)^r \right| \\ &\leq c, \end{aligned}$$

with $c = c_J^2 \langle \mu, \mathcal{P}(g \otimes g) \rangle \sum_{0 \leq \ell < k \leq p} 2^{-(k+\ell)/2} \sum_{j,j' \in J, \theta_j \theta_{j'} \neq 1} |1 - \theta_{j'} \theta_j|^{-1}$, and where we used (176) for the first inequality, the last inequality of (163) for the second. Since J is finite, we deduce that c is finite. This gives that $\lim_{n \rightarrow \infty} n^{-1} H_6^{[n],\circ} = 0$. Recall that $H_6^{[n]}$ and $H_6^*(f)$ are complex numbers (*i.e.* constant functions). Use (174) and (175) to get that:

$$(178) \quad \lim_{n \rightarrow \infty} n^{-1} H_6^{[n]} = H_6^*(f)$$

so that, as $|\mathbb{G}_{n-p}|^{-1} M_{\mathbb{G}_{n-p}}(\cdot)$ is a probability measure, a.s.:

$$(179) \quad \lim_{n \rightarrow \infty} n^{-1} |\mathbb{G}_{n-p}|^{-1} M_{\mathbb{G}_{n-p}}(H_6^{[n]}) = H_6^*(f).$$

In conclusion, use (169), (171), (179) and the definition (165) of $V_6(n)$ to deduce that a.s.:

$$\lim_{n \rightarrow \infty} n^{-1} V_6(n) = H_6^*(f) = \sum_{0 \leq \ell < k} 2^{-(k+\ell)/2} \langle \mu, \mathcal{P} f_{k,\ell}^* \rangle = \Sigma_2^{\text{crit}}(f),$$

where $f_{k,\ell}^*$ is defined in (161) and $\Sigma_2^{\text{crit}}(f)$ in (160). Recall that:

$$V_5(n) = |\mathbb{G}_n|^{-1} \sum_{i \in \mathbb{G}_{n-p}} \sum_{0 \leq \ell < k \leq p} 2^{p-\ell} Q^{p-k} \left(\tilde{f}_k Q^{k-\ell} \tilde{f}_\ell \right) (X_i) = |\mathbb{G}_{n-p}|^{-1} M_{\mathbb{G}_{n-p}}(\Phi_n),$$

where

$$\Phi_n = \sum_{0 \leq \ell < k \leq p} 2^{-\ell} Q^{p-k} \left(\tilde{f}_k Q^{k-\ell} \tilde{f}_\ell \right).$$

We have:

$$|\Phi_n| \leq \sum_{0 \leq \ell < k \leq p} 2^{-\ell} \alpha^{k-\ell} Q^{p-k}(g^2) \leq \sum_{0 \leq \ell < k \leq p} 2^{-(k+\ell)/2} g_1 \leq C g_1,$$

where we used (5) for the first inequality and (4) (with f and g replaced by g^2 and g_1) in the second. Then, use (36) to conclude that a.s.:

$$\lim_{n \rightarrow \infty} n^{-1} V_5(n) = 0.$$

This ends the proof of the Lemma. \square

Using similar arguments as in the proof of Lemma 10.4, we get the following result.

Lemma 10.5. *Under the assumptions of Theorem 9.2, we have that a.s. $\lim_{n \rightarrow \infty} n^{-1} V_1(n) = \Sigma_1^{\text{crit}}(f)$ with $\Sigma_1^{\text{crit}}(f)$ defined by (159) which is well defined and finite.*

Proof. We recall $V_1(n) = V_3(n) + V_4(n)$, see (72) and thereafter for the definition of $V_3(n)$ and $V_4(n)$. We first consider the term $V_3(n)$. Recall that $V_3(n) = |\mathbb{G}_{n-p}| M_{\mathbb{G}_{n-p}}(\Phi_n)$ with $\Phi_n = \sum_{\ell=0}^p 2^{-\ell} Q^{p-\ell}(\tilde{f}_\ell^2)$. We have $\tilde{f}_\ell^2 \leq g^2$ and $Q^{p-\ell}(g^2) \leq g_1$ for some $g_1 \in F$ and thus $|\Phi_n| \leq 2g_1$. We therefore deduce that a.s. $\lim_{n \rightarrow \infty} n^{-1} V_3(n) = 0$.

We consider the term $V_4(n) = |\mathbb{G}_{n-p}|^{-1} M_{\mathbb{G}_{n-p}}(H_{4,n})$ (see (73)) with $H_{4,n}$ given by (74):

$$H_{4,n} = \sum_{\ell \geq 0, k \geq 0} h_{\ell,k}^{(n)} \mathbf{1}_{\{\ell+k < p\}} \quad \text{and} \quad h_{\ell,k}^{(n)} = 2^{k-\ell} Q^{p-1-(\ell+k)} \left(\mathcal{P} \left(Q^k \tilde{f}_\ell \otimes^2 \right) \right).$$

Recall $f_{\ell,\ell,k}$ defined in (164). We set $\bar{H}_{4,n} = \sum_{\ell \geq 0, k \geq 0} \bar{h}_{\ell,k}^{(n)} \mathbf{1}_{\{\ell+k < p\}}$ with

$$\bar{h}_{\ell,k}^{(n)} = 2^{k-\ell} \alpha^{2k} Q^{p-1-(\ell+k)} (\mathcal{P} f_{\ell,\ell,k}) = 2^{-\ell} Q^{p-1-(\ell+k)} (\mathcal{P} f_{\ell,\ell,k}),$$

where we used that $2\alpha^2 = 1$. We have:

$$\begin{aligned} |h_{\ell,k}^{(n)} - \bar{h}_{\ell,k}^{(n)}| &\leq 2^{k-\ell} Q^{p-1-(\ell+k)} \left(\mathcal{P} \left(\left| Q^k \tilde{f}_\ell \otimes Q^k \tilde{f}_\ell - \alpha^{2k} f_{\ell,\ell,k} \right| \right) \right) \\ &\leq C 2^{k-\ell} \beta_k \alpha^{2k} Q^{p-1-(\ell+k)} (\mathcal{P} (g \otimes g)) \\ &\leq \beta_k 2^{-\ell} g_1^*, \end{aligned}$$

with $g_1^* \in F$, where we used (162), with the representation $Q^k \tilde{f}_\ell = Q^k \hat{f}_\ell + \alpha^k \sum_{j \in J} \theta_j^k \mathcal{R}_j(f_\ell)$, (163) for the second inequality and (4) for the last. We deduce that:

$$|H_{4,n} - \bar{H}_{4,n}| \leq \sum_{\ell \geq 0, k \geq 0} |h_{\ell,k}^{(n)} - \bar{h}_{\ell,k}^{(n)}| \mathbf{1}_{\{\ell+k < p\}} \leq 2 \left(\sum_{k=0}^n \beta_k \right) g_1^*.$$

As $\lim_{n \rightarrow \infty} \beta_n = 0$, we get that $\lim_{n \rightarrow \infty} n^{-1} \sum_{k=0}^n \beta_k = 0$. We deduce from (36) that a.s.:

$$(180) \quad \lim_{n \rightarrow \infty} n^{-1} |\mathbb{G}_{n-p}|^{-1} M_{\mathbb{G}_{n-p}}(|H_{4,n} - \bar{H}_{4,n}|) = 0.$$

We set $H_4^{[n]} = \sum_{\ell \geq 0, k \geq 0} h_{\ell, k} \mathbf{1}_{\{\ell+k < p\}}$ with:

$$h_{\ell, k} = 2^{-\ell} \langle \mu, \mathcal{P}f_{\ell, \ell, k} \rangle.$$

Notice that:

$$\begin{aligned} |\bar{h}_{\ell, k}^{(n)} - h_{\ell, k}| &\leq 2^{-\ell} \sum_{j, j' \in J} \left| \mathcal{Q}^{p-1-(\ell+k)} (\mathcal{P}(\mathcal{R}_j f_\ell \otimes_{\text{sym}} \mathcal{R}_{j'} f_\ell)) - \langle \mu, \mathcal{P}(\mathcal{R}_j f_\ell \otimes_{\text{sym}} \mathcal{R}_{j'} f_\ell) \rangle \right| \\ &\leq 2^{-\ell} \alpha^{p-1-(\ell+k)} \sum_{j, j' \in J} g_{\ell, j, j'} \\ &= 2^{-\ell} \alpha^{p-1-(\ell+k)} g_\ell, \end{aligned}$$

where we used (5) (with f and g replaced by $\mathcal{P}(\mathcal{R}_j f_\ell \otimes_{\text{sym}} \mathcal{R}_{j'} f_\ell)$ and $g_{\ell, j, j'}$) for the second inequality and $g_\ell = \sum_{j, j' \in J} g_{\ell, j, j'}$ for the equality. We have that g_ℓ belongs to F . Since $|\mathcal{P}f_{\ell, \ell, k}| \leq \mathcal{P}|f_{\ell, \ell, k}| \leq 4\mathcal{P}(g \otimes g)$, thanks to the fourth inequality in (163), we deduce from (4) (with f and g replaced by $4\mathcal{P}(g \otimes g)$ and g_2^*) that:

$$|\bar{h}_{\ell, k}^{(n)}| \leq 2^{-\ell} g_2^* \quad \text{and} \quad |h_{\ell, k}| \leq 2^{-\ell} \langle \mu, g_2^* \rangle.$$

Set $r_0 \in \mathbb{N}$ and $g_{r_0} = \sum_{0 \leq \ell \leq r_0} g_\ell$. Notice that g_{r_0} belongs to F and is non-negative. Furthermore, we have for n large enough so that $p > 2r_0$:

$$\begin{aligned} |\bar{H}_{4, n} - H_4^{[n]}| &\leq \sum_{\ell \geq 0, k \geq 0} |\bar{h}_{\ell, k}^{(n)} - h_{\ell, k}| \mathbf{1}_{\{\ell+k < p\}} \\ &\leq \sum_{0 \leq \ell \leq r_0, k \geq 0} 2^{-\ell} \alpha^{p-1-(\ell+k)} g_{r_0} \mathbf{1}_{\{\ell+k < p\}} + \sum_{\ell > r_0, k \geq 0} \left(|\bar{h}_{\ell, k}^{(n)}| + |h_{\ell, k}| \right) \mathbf{1}_{\{\ell+k < p\}} \\ &\leq C g_{r_0} + \sum_{\ell > r_0} (p - \ell) 2^{-\ell} (g_2^* + \langle \mu, g_2^* \rangle) \mathbf{1}_{\{\ell < p\}} \\ &\leq C g_{r_0} + n 2^{-r_0} (g_2^* + \langle \mu, g_2^* \rangle). \end{aligned}$$

We deduce that:

$$\limsup_{n \rightarrow \infty} n^{-1} |\mathbb{G}_{n-p}|^{-1} M_{\mathbb{G}_{n-p}} (|\bar{H}_{4, n} - H_4^{[n]}|) \leq 2^{1-r_0} \langle \mu, g_2^* \rangle.$$

Since r_0 can be arbitrary large, we get that a.s.:

$$(181) \quad \lim_{n \rightarrow \infty} n^{-1} |\mathbb{G}_{n-p}|^{-1} M_{\mathbb{G}_{n-p}} (|\bar{H}_{4, n} - H_4^{[n]}|) = 0.$$

Now, we study the limit of $H_4^{[n]}$. We set for $k, \ell \in \mathbb{N}$:

$$h_\ell^* = 2^{-\ell} \langle \mu, \mathcal{P}f_{\ell, \ell}^* \rangle.$$

Using the last inequality in (163) and the definition (161) of $f_{\ell, \ell}^*$, we deduce there exists a finite constant c independent of n (but depending on \mathfrak{f}) such that, for all $\ell \in \mathbb{N}$, $|h_\ell^*| \leq c2^{-\ell}$. This implies that $H_0^* = \sum_{\ell \geq 0} (\ell + 1) |h_\ell^*|$ is finite and the sum

$$H_4^*(\mathfrak{f}) = \sum_{\ell \geq 0} h_\ell^*$$

is well defined and finite. We write:

$$h_{\ell, k} = h_\ell^* + h_{\ell, k}^\circ,$$

with $h_{\ell,k}^\circ = 2^{-\ell} \langle \mu, \mathcal{P} f_{\ell,\ell,k}^\circ \rangle$, where $f_{\ell,\ell,k}^\circ = f_{\ell,\ell,k} - f_{\ell,\ell}^*$ is defined in (164), and

$$H_4^{[n]} = H_4^{[n],*} + H_4^{[n],\circ},$$

with $H_4^{[n],*} = \sum_{\ell \geq 0} (p - \ell) h_\ell^*$ and $H_4^{[n],\circ} = \sum_{\ell \geq 0, k \geq 0} h_{\ell,k}^\circ \mathbf{1}_{\{\ell+k < p\}}$. We have:

$$|n^{-1} H_4^{[n],*} - H_4^*(f)| \leq |n^{-1} p - 1| H_4^*(f) + n^{-1} H_0^* + \sum_{\ell > p} |h_\ell^*|,$$

so that $\lim_{n \rightarrow \infty} |n^{-1} H_4^{[n],*} - H_4^*(f)| = 0$ and thus:

$$(182) \quad \lim_{n \rightarrow \infty} n^{-1} H_4^{[n],*} = H_4^*(f).$$

We now prove that $n^{-1} H_4^{[n],\circ}$ converges towards 0. We have:

$$(183) \quad f_{\ell,\ell,k}^\circ = \sum_{j,j' \in J, \theta_j \theta_{j'} \neq 1} (\theta_{j'} \theta_j)^k \mathcal{R}_{j'} f_\ell \otimes_{\text{sym}} \mathcal{R}_j f_\ell.$$

This gives:

$$\begin{aligned} |H_4^{[n],\circ}| &= \left| \sum_{\ell \geq 0, k \geq 0} 2^{-\ell} \langle \mu, \mathcal{P} f_{\ell,\ell,k}^\circ \rangle \mathbf{1}_{\{\ell+k < p\}} \right| \\ &\leq \sum_{\ell \geq 0} 2^{-\ell} \sum_{j,j' \in J, \theta_j \theta_{j'} \neq 1} \left| \langle \mu, \mathcal{P}(\mathcal{R}_{j'} f_\ell \otimes \mathcal{R}_j f_\ell) \rangle \right| \left| \sum_{k=0}^{p-\ell-1} (\theta_{j'} \theta_j)^k \right| \\ &\leq c, \end{aligned}$$

with $c = c_J^2 \langle \mu, \mathcal{P}(g \otimes g) \rangle \sum_{\ell \geq 0} 2^{-\ell} \sum_{j,j' \in J, \theta_j \theta_{j'} \neq 1} |1 - \theta_{j'} \theta_j|^{-1}$, and where we used (183) for the first inequality, the last inequality of (163) for the second. Since J is finite, we deduce that c is finite. This gives that $\lim_{n \rightarrow \infty} n^{-1} H_4^{[n],\circ} = 0$. Recall that $H_4^{[n]}$ and $H_4^*(f)$ are complex numbers (*i.e.* constant functions). Use (182) to get that:

$$\lim_{n \rightarrow \infty} n^{-1} H_4^{[n]} = H_4^*(f)$$

so that, as $|\mathbb{G}_{n-p}|^{-1} M_{\mathbb{G}_{n-p}}(\cdot)$ is a probability measure, a.s.:

$$(184) \quad \lim_{n \rightarrow \infty} n^{-1} |\mathbb{G}_{n-p}|^{-1} M_{\mathbb{G}_{n-p}}(H_4^{[n]}) = H_4^*(f).$$

In conclusion, use (180), (181), (184) and the definition (73) of $V_4(n)$ to deduce that a.s.:

$$\lim_{n \rightarrow \infty} n^{-1} V_4(n) = H_4^*(f) = \sum_{\ell \geq 0} 2^{-\ell} \langle \mu, \mathcal{P} f_{\ell,\ell}^* \rangle = \Sigma_1^{\text{crit}}(f),$$

where $f_{\ell,\ell}^*$ is defined in (161) and $\Sigma_1^{\text{crit}}(f)$ in (159). \square

The proof of the next Lemma is a direct consequence of (63) and Lemmas 10.3, 10.5 and 10.4.

Lemma 10.6. *Under the assumptions of Theorem 9.2, we have the following convergence in probability:*

$$\lim_{n \rightarrow \infty} n^{-1} V(n) = \Sigma_1^{\text{crit}}(f) + 2\Sigma_2^{\text{crit}}(f),$$

where $\Sigma_1^{\text{crit}}(f)$ and $\Sigma_2^{\text{crit}}(f)$, defined by (159) and (160), are well defined and finite.

We now check the Lindeberg condition. Recall $R_3(n)$ defined in (78).

Lemma 10.7. *Under the assumptions of Theorem 9.2, we have that $\lim_{n \rightarrow \infty} n^{-2} R_3(n) = 0$.*

Proof. Keeping the notation of Lemma 6.7, using Lemma 4.3 (with the main contribution coming from $\psi_{8,n}$ and $\psi_{9,n}$ therein), we get (compare with (81)) that for $n \geq p \geq \ell \geq 0$:

$$|h_{n,\ell}| \leq (p - \ell)^2 2^{2(p-\ell)} g_1,$$

with $h_{n,\ell}(x) = \mathbb{E}_x \left[M_{\mathbb{G}_{p-\ell}}(\tilde{f}_\ell)^4 \right]$ and $g_1 \in F$. Following the proof of Lemma 6.7, we get that:

$$n^{-2} R_3(n) \leq 16 n^3 2^{-(n-p)} \langle \nu, g_2 \rangle.$$

This ends the proof as $\lim_{n \rightarrow \infty} p = \infty$ and $\lim_{n \rightarrow \infty} n - p - \lambda \log(p) = +\infty$ for all $\lambda > 0$. \square

The proof of Theorem 9.2 mimics then the proof of Theorem 5.2.

11. PROOF OF COROLLARY 9.3

We keep notations from Section 7 on the sub-critical case, and adapt very closely the arguments of this section. We recall that $c_k(\mathbf{f}) = \sup\{\|f_n\|_{L^k(\mu)}, n \in \mathbb{N}\}$ for all $k \in \mathbb{N}$. We recall that C denotes any unimportant finite constant which may vary from line to line, which does not depend on n or \mathbf{f} .

Lemma 11.1. *Under the assumptions of Corollary 9.3, we have that $\lim_{n \rightarrow \infty} \mathbb{E}[n^{-1} R_0^{k_0}(n)^2] = 0$.*

Proof. Mimicking the proof of Lemma 7.2, we get:

$$\lim_{n \rightarrow \infty} \mathbb{E}[R_0^{k_0}(n)^2]^{1/2} \leq \lim_{n \rightarrow \infty} C c_2(\mathbf{f}) \sqrt{n} 2^{-p/2} = 0.$$

This trivially implies the result. \square

Lemma 11.2. *Under the assumptions of Corollary 9.3, we have that $\lim_{n \rightarrow \infty} \mathbb{E}[n^{-1} R_1(n)^2] = 0$.*

Proof. Mimicking the proof of Lemma 7.3, we get $\mathbb{E}[R_1(n)^2]^{1/2} \leq C c_2(\mathbf{f}) \sqrt{n-p}$. As $\lim_{n \rightarrow \infty} p/n = 1$, this implies that $\lim_{n \rightarrow \infty} \mathbb{E}[n^{-1} R_1(n)^2] = 0$. \square

Similarly to Lemma 7.5, we get the following result on $R_2(n)$.

Lemma 11.3. *Under the assumptions of Corollary 9.3, we have that $\lim_{n \rightarrow \infty} \mathbb{E}[n^{-1/2} R_2(n)] = 0$.*

We now consider the asymptotics of $V_2(n)$.

Lemma 11.4. *Under the assumptions of Corollary 9.3, we have that $\lim_{n \rightarrow \infty} n^{-1} V_2(n) = \Sigma_2^{\text{crit}}(\mathbf{f})$ in probability, where $\Sigma_2^{\text{crit}}(\mathbf{f})$, defined in (160), is well defined and finite.*

In the proof, we shall use the analogue of (13) with f replaced by \hat{f} in the left hand-side, whereas $f \in L^4(\mu)$ does imply that $\tilde{f} \in L^4(\mu)$ but does not imply that $\hat{f} \in L^4(\mu)$. Thanks to (13), we get for $f \in L^4(\mu)$ and $g \in L^2(\mu)$, as $\mathcal{R}_j f = \alpha_j^{-1} \mathcal{Q} \mathcal{R}_j f$ and $|\alpha_j| = \alpha$, that:

$$\begin{aligned} \|\mathcal{P}(\hat{f} \otimes_{\text{sym}} \mathcal{Q}g)\|_{L^2(\mu)} &\leq \|\mathcal{P}(\tilde{f} \otimes_{\text{sym}} \mathcal{Q}g)\|_{L^2(\mu)} + \alpha^{-1} \sum_{j \in J} \|\mathcal{P}(\mathcal{Q}(\mathcal{R}_j f) \otimes_{\text{sym}} \mathcal{Q}g)\|_{L^2(\mu)} \\ &\leq C \left(\|f\|_{L^4(\mu)} + \|f\|_{L^2(\mu)} \right) \|g\|_{L^2(\mu)} \\ (185) \quad &\leq C \|f\|_{L^4(\mu)} \|g\|_{L^2(\mu)}. \end{aligned}$$

Proof. We follow the proof of Lemma 10.4. We keep the decomposition (66) of $V_2(n) = V_5(n) + V_6(n)$ given in the proof of Lemma 6.4. We recall $V_6(n) = |\mathbb{G}_{n-p}|^{-1} M_{\mathbb{G}_{n-p}}(H_{6,n})$ with $H_{6,n}$ defined in (166). We set

$$\bar{V}_6(n) = |\mathbb{G}_{n-p}|^{-1} M_{\mathbb{G}_{n-p}}(\bar{H}_{6,n})$$

with $\bar{H}_{6,n}$ defined in (167). For $f \in L^2(\mu)$, we recall \hat{f} defined in (157). We set:

$$\begin{aligned} h_{k,\ell,r}^{(n,1)} &= 2^{r-\ell} \mathcal{Q}^{p-1-(r+k)} (\mathcal{P}(\mathcal{Q}^r(\hat{f}_k) \otimes_{\text{sym}} \mathcal{Q}^{k-\ell+r}(\hat{f}_\ell))), \\ h_{k,\ell,r}^{(n,2)} &= 2^{r-\ell} \mathcal{Q}^{p-1-(r+k)} (\mathcal{P}(\mathcal{Q}^r(\hat{f}_k) \otimes_{\text{sym}} \mathcal{Q}^{k-\ell+r}(\sum_{j \in J} \mathcal{R}_j(f_\ell)))), \\ h_{k,\ell,r}^{(n,3)} &= 2^{r-\ell} \mathcal{Q}^{p-1-(r+k)} (\mathcal{P}(\mathcal{Q}^r(\sum_{j \in J} \mathcal{R}_j(f_k)) \otimes_{\text{sym}} \mathcal{Q}^{k-\ell+r}(\hat{f}_\ell))), \end{aligned}$$

so that $h_{k,\ell,r}^{(n)} = \bar{h}_{k,\ell,r}^{(n)} + \sum_{i=1}^3 h_{k,\ell,r}^{(n,i)}$. Thanks to (11) for $r \geq 1$ and (185) for $r = 0$, we have using Jensen's inequality, (21) and the fact that the sequence $(\beta_r, r \in \mathbb{N})$ is nonincreasing:

$$\|h_{k,\ell,r}^{(n,1)}\|_{L^2(\mu)} \leq C 2^{-(k+\ell)/2} \beta_r \|f_\ell\|_{L^2(\mu)} \begin{cases} \|f_k\|_{L^2(\mu)} & \text{for } r \geq 1, \\ \|f_k\|_{L^4(\mu)} & \text{for } r = 0. \end{cases}$$

Using the same arguments, that $\langle \mu, \mathcal{R}_j(g) \rangle = 0$ for $g \in L^2(\mu)$ (as $\mathcal{R}_j(g)$ is an eigen-vector of \mathcal{Q} associated to α_j) and that $\|\sum_{j \in J} \mathcal{R}_j(f_\ell)\|_{L^2(\mu)} \leq C \|f_\ell\|_{L^2(\mu)}$ (as \mathcal{R}_j are bounded operators on $L^2(\mu)$), we get:

$$\|h_{k,\ell,r}^{(n,2)}\|_{L^2(\mu)} + \|h_{k,\ell,r}^{(n,3)}\|_{L^2(\mu)} \leq C 2^{-(k+\ell)/2} \beta_r \|f_\ell\|_{L^2(\mu)} \begin{cases} \|f_k\|_{L^2(\mu)} & \text{for } r \geq 1, \\ \|f_k\|_{L^4(\mu)} & \text{for } r = 0. \end{cases}$$

We deduce that

$$(186) \quad \sum_{i=1}^3 \|h_{k,\ell,r}^{(n,i)}\|_{L^2(\mu)} \leq C c_2(f) c_4(f) 2^{-(k+\ell)/2} \beta_r.$$

Using (82) for the first inequality, Jensen's inequality for the second inequality, the triangular inequality for the third inequality and (186) for the last inequality, we get:

$$\begin{aligned} (187) \quad \mathbb{E} \left[(V_6(n) - \bar{V}_6(n))^2 \right] &= |\mathbb{G}_{n-p}|^{-2} \mathbb{E} [M_{\mathbb{G}_{n-p}}(H_6(n) - \bar{H}_6(n))^2] \\ &\leq C |\mathbb{G}_{n-p}|^{-1} \sum_{m=0}^{n-p} 2^m \|\mathcal{Q}^m(H_6(n) - \bar{H}_6(n))\|_{L^2(\mu)}^2 \\ &\leq C \|H_6(n) - \bar{H}_6(n)\|_{L^2(\mu)}^2 \\ &\leq C \left(\sum_{0 \leq \ell < k < p} \sum_{r=0}^{p-k-1} \sum_{i=1}^3 \|h_{n,k,\ell,r}^{(n,i)}\|_{L^2(\mu)} \right)^2 \\ &\leq C c_2(f)^2 c_4(f)^2 \left(\sum_{r=0}^p \beta_r \right)^2. \end{aligned}$$

We deduce that

$$\mathbb{E}[(V_6(n) - \bar{V}_6(n))^2] \leq C c_2(f)^2 c_4(f)^2 \left(\sum_{r=0}^p \beta_r \right)^2,$$

and then that

$$(188) \quad \lim_{n \rightarrow \infty} \mathbb{E}[n^{-2}(V_6(n) - \bar{V}_6(n))^2] = 0.$$

We recall $h_{k,\ell,r}$ defined in (170) and that:

$$H_6^{[n]} = \sum_{0 \leq \ell < k < p} \sum_{r=0}^{p-k-1} h_{k,\ell,r} = \langle \mu, \bar{H}_{6,n} \rangle.$$

We have:

$$\begin{aligned} \mathbb{E}[(\bar{V}_6(n) - H_6^{[n]})^2] &\leq C |\mathbb{G}_{n-p}|^{-1} \sum_{m=0}^{n-p} 2^m \|\mathcal{Q}^m(\bar{H}_{6,n} - H_6^{[n]})\|_{L^2(\mu)}^2 \\ &\leq C |\mathbb{G}_{n-p}|^{-1} \sum_{m=0}^{n-p} 2^m \left(\sum_{0 \leq \ell < k \leq p} \sum_{r=0}^{p-k-1} \alpha^{m+p-r-k} 2^{-(k+\ell)/2} \|\mathcal{P}f_{k,\ell,r}\|_{L^2(\mu)} \right)^2 \\ &\leq C(n-p) |\mathbb{G}_{n-p}|^{-1} \left(\sum_{0 \leq \ell < k \leq p} \sum_{r=0}^{p-k-1} 2^{-(p+\ell-r)/2} \|\mathcal{P}(f_{k,\ell,r})\|_{L^2(\mu)} \right)^2 \\ &\leq C(n-p) |\mathbb{G}_{n-p}|^{-1} \left(\sum_{0 \leq \ell < k < p} 2^{-(\ell+k)/2} \|\sum_{j \in J} \mathcal{R}_j(f_k)\|_{L^2(\mu)} \|\sum_{j \in J} \mathcal{R}_j(f_\ell)\|_{L^2(\mu)} \right)^2 \\ &\leq C(n-p) |\mathbb{G}_{n-p}|^{-1} c_2^4(\mathbf{f}), \end{aligned}$$

where we used (82) for the first inequality, (20) for the second, $\alpha = 1/\sqrt{2}$ for the third, (11) and the fact that $\mathcal{Q}(\sum_{j \in J} \mathcal{R}_j f) = \sum_{j \in J} \alpha_j \mathcal{R}_j(f)$, with $|\alpha_j| = 1/\sqrt{2}$, for the fourth, $\|\sum_{j \in J} \mathcal{R}_j(f)\|_{L^2(\mu)} \leq \|f\|_{L^2(\mu)}$ for the last. From the latter inequality we conclude that:

$$(189) \quad \lim_{n \rightarrow \infty} \mathbb{E}[n^{-2}(\bar{V}_6(n) - H_6^{[n]})^2] = 0.$$

Recall $h_{k,\ell}^*$ defined in (172) and $H_6^*(\mathbf{f})$ defined in (173). Using (10), we have:

$$|h_{k,\ell}^*| \leq C 2^{-(k+\ell)/2} \sum_{j \in J} \|\mathcal{R}_j(f_k)\|_{L^2(\mu)} \|\mathcal{R}_j(f_\ell)\|_{L^2(\mu)} \leq C 2^{-(k+\ell)/2} c_2^2(\mathbf{f}).$$

This implies that $H_6^*(\mathbf{f}) \leq C c_2^2(\mathbf{f})$ and then that $H_6^*(\mathbf{f})$ is well defined. Following the proof of (178), and using (10) in the upper bound (177) so that:

$$\left| \langle \mu, \mathcal{P}(\mathcal{R}_{j'} f_k \otimes_{\text{sym}} \mathcal{R}_j f_\ell) \rangle \right| \leq 2 \|\mathcal{R}_{j'}(f_k)\|_{L^2(\mu)} \|\mathcal{R}_j(f_\ell)\|_{L^2(\mu)} \leq C \|f_k\|_{L^2(\mu)} \|f_\ell\|_{L^2(\mu)},$$

we prove that:

$$(190) \quad \lim_{n \rightarrow \infty} n^{-1} H_6^{[n]} = H_6^*(\mathbf{f}).$$

It follows from (188), (189) and (190) that:

$$(191) \quad \lim_{n \rightarrow \infty} \mathbb{E}[(n^{-1} V_6(n) - H_6^*(\mathbf{f}))^2] = 0.$$

We recall $H_5^{[n]}(\mathbf{f})$ defined in (99). From (101), we have:

$$\mathbb{E}[n^{-2} V_5(n)^2] \leq 2n^{-2} |\mathbb{G}_{n-p}|^{-2} \mathbb{E}[M_{\mathbb{G}_{n-p}}(A_{5,n}(\mathbf{f}))^2] + 2n^{-2} H_5^{[n]}(\mathbf{f})^2.$$

Using (97) with $\alpha = 1/\sqrt{2}$, we get $|H_5^{[n]}(\mathbf{f})| \leq C c_2^2(\mathbf{f})$ and thus:

$$\lim_{n \rightarrow \infty} n^{-2} H_5^{[n]}(\mathbf{f})^2 = 0.$$

Next, as (103) holds for $\alpha = 1/\sqrt{2}$, we get (104) with the right hand-side replaced by $C c_4^4(\mathbf{f}) (n-p)2^{-(n-p)}$, and thus:

$$\lim_{n \rightarrow \infty} n^{-2} |\mathbb{G}_{n-p}|^{-2} \mathbb{E} [M_{\mathbb{G}_{n-p}}(A_{5,n}(\mathbf{f}))^2] = 0.$$

It then follows that:

$$\lim_{n \rightarrow \infty} \mathbb{E}[n^{-2} V_5(n)^2] = 0.$$

Finally, since $V_2(n) = V_5(n) + V_6(n)$, we get thanks to (173) that in probability $\lim_{n \rightarrow \infty} n^{-1} V_2(n) = H_6^*(\mathbf{f}) = \Sigma_2^{\text{crit}}(\mathbf{f})$. \square

Lemma 11.5. *Under the assumptions of Corollary 9.3, we have that in probability $\lim_{n \rightarrow \infty} V_1(n) = \Sigma_1^{\text{crit}}(\mathbf{f})$, where $\Sigma_1^{\text{crit}}(\mathbf{f})$, defined in (159), is well defined and finite.*

Proof. We recall the decomposition (72): $V_1(n) = V_3(n) + V_4(n)$. First, following the proof of (191) in the spirit of the proof of (112), we get:

$$\lim_{n \rightarrow \infty} \mathbb{E}[(n^{-1} V_4(n) - H_4^*(\mathbf{f}))^2] = 0 \quad \text{with} \quad H_4^*(\mathbf{f}) = \sum_{\ell \geq 0} 2^{-\ell} \langle \mu, \mathcal{P}(\sum_{j \in J} \mathcal{R}_j(f_\ell) \otimes_{\text{sym}} \bar{\mathcal{R}}_j(f_\ell)) \rangle = \Sigma_1^{\text{crit}}(\mathbf{f}).$$

Let us stress that the proof requires to use (9). Since $\sum_{\ell \geq 0} 2^{-\ell} |\langle \mu, \mathcal{P}(\sum_{j \in J} \mathcal{R}_j(f_\ell) \otimes_{\text{sym}} \bar{\mathcal{R}}_j(f_\ell)) \rangle| \leq \sum_{\ell \geq 0} 2^{-\ell} c_2^2(\mathbf{f})$, we deduce that $\Sigma_1^{\text{crit}}(\mathbf{f})$ is well defined and finite. Next, from (115) we have

$$\mathbb{E}[n^{-2} V_3(n)^2] \leq 2n^{-2} |\mathbb{G}_{n-p}|^{-2} \mathbb{E} [M_{\mathbb{G}_{n-p}}(A_{3,n}(\mathbf{f}))^2] + 2n^{-2} H_3^{[n]}(\mathbf{f})^2.$$

It follows from (119) (with an extra term $n-p$ as $2\alpha^2 = 1$ in the right hand side) and (114) that $\lim_{n \rightarrow \infty} \mathbb{E}[n^{-2} V_3(n)^2] = 0$. Finally the result of the lemma follows as $V_1 = V_3 + V_4$. \square

We now check the Lindeberg condition using a fourth moment condition. Recall $R_3(n) = \sum_{i \in \mathbb{G}_{n-p_n}} \mathbb{E} [\Delta_{n,i}(\mathbf{f})^4]$ defined in (78).

Lemma 11.6. *Under the assumptions of Corollary 9.3, we have that $\lim_{n \rightarrow \infty} n^{-2} R_3(n) = 0$.*

Proof. Following line by line the proof of Lemma 7.9 with the same notations and taking $\alpha = 1/\sqrt{2}$, we get that concerning $|\langle \mu, \psi_{i,p-\ell} \rangle|$ or $\langle \mu, |\psi_{i,p-\ell}| \rangle$, the bounds for $i \in \{1, 2, 3, 4\}$ are the same; the bounds for $i \in \{5, 6, 7\}$ have an extra $(p-\ell)$ term, the bounds for $i \in \{8, 9\}$ have an extra $(p-\ell)^2$ term. This leads to (compare with (149)):

$$R_3(n) \leq C n^5 2^{-(n-p)} c_4^4(\mathbf{f})$$

which implies that $\lim_{n \rightarrow \infty} n^{-2} R_3(n) = 0$. \square

The proof of Corollary 9.3 then follows the proof of Corollary 5.3.

12. PROOF OF THEOREM 3.7 IN THE CRITICAL CASE ($2\alpha^2 = 1$)

We keep notations from Sections 8 and 11. We assume that Assumptions 2.10 and 2.14 hold with $\alpha = 1/\sqrt{2}$. Let $(f_{\ell,n}, n \geq \ell \geq 0)$ be a sequence of function satisfying Assumptions 2.16 and 2.19. We set $f_{\ell,n} = 0$ for $\ell > n \geq 0$ and $\mathbf{f}_n = (f_{\ell,n}, \ell \in \mathbb{N})$. Recall the definition of $c_k(\mathbf{f})$ and $q_k(\mathbf{f})$ in (83). Assumption 2.16 (ii) gives that $\mathbf{c}_2 = \sup_{n \in \mathbb{N}} c_2(\mathbf{f}_n)$ and $\mathbf{q}_2 = \sup_{n \in \mathbb{N}} q_2(\mathbf{f}_n)$ are finite.

Lemma 12.1. *Under the assumptions of Theorem 3.7 ($2\alpha^2 = 1$), we get $\lim_{n \rightarrow \infty} \mathbb{E}[R_0^{k_0}(n)^2] = 0$.*

Proof. Mimicking the proof of Lemma 7.2, we get, as $\lim_{n \rightarrow \infty} p/n = 1$:

$$\lim_{n \rightarrow \infty} \mathbb{E}[R_0^{k_0}(n)^2] \leq C \lim_{n \rightarrow \infty} c_2^2(\mathfrak{f}_n) n 2^{-p} \leq C c_2^2 \lim_{n \rightarrow \infty} n 2^{-p} = 0.$$

□

The proof of the next lemma differs from Lemma 11.2, because of the missing term n^{-1} in front of $R_1(n)^2$.

Lemma 12.2. *Under the assumptions of Theorem 3.7 ($2\alpha^2 = 1$), we get $\lim_{n \rightarrow \infty} \mathbb{E}[R_1(n)^2] = 0$.*

Proof. We keep notation from the proof of Lemma 7.3. Notice that (26) implies that:

$$(192) \quad \|\mathcal{Q}\tilde{f}_{\ell,n}\|_{L^2(\mu)} \leq \|\mathcal{Q}f_{\ell,n}\|_{L^2(\mu)} \leq \|\mathcal{Q}(|f_{\ell,n}|)\|_{L^2(\mu)} \leq \delta_{\ell,n}.$$

We deduce that for $k \in \mathbb{N}$:

$$(193) \quad \|\mathcal{Q}^k \tilde{f}_{\ell,n}\|_{L^2(\mu)} \leq \alpha^{k-1} \delta_{\ell,n} \mathbf{1}_{\{k \geq 1\}} + c_2 \mathbf{1}_{\{k=0\}}.$$

Using the first inequality of (85) with \mathfrak{f}_n instead of \mathfrak{f} , (20) and (193), we get:

$$\begin{aligned} \langle \mu, h_{\ell,n} \rangle &\leq C 2^{n-p} \sum_{k=0}^{n-p-k_0} 2^k \|\mathcal{Q}^k \mathcal{Q}^{p-\ell} \tilde{f}_{\ell,n}\|_{L^2(\mu)}^2 \\ &\leq C 2^{n-p} \alpha^{2(p-\ell)} \sum_{k=0}^{n-p-k_0} 2^k \alpha^{2k-2} \|\mathcal{Q} \tilde{f}_{\ell,n}\|_{L^2(\mu)}^2 \mathbf{1}_{\{k+p-\ell \geq 1\}} + C 2^{n-p} \|f_{\ell,n}\|_{L^2(\mu)}^2 \mathbf{1}_{\{\ell=p\}} \\ &\leq C (n-p) 2^{n-2p+\ell} \delta_{\ell,n}^2 + C 2^{n-p} c_2^2 \mathbf{1}_{\{\ell=p\}}. \end{aligned}$$

From the first inequality of (86), we have

$$\begin{aligned} \mathbb{E}[R_1(n)^2]^{1/2} &\leq C \sum_{\ell=0}^p ((n-p)2^{-\ell} \delta_{\ell,n}^2 + \mathbf{1}_{\{\ell=p\}} 2^{-p} c_2^2)^{1/2} \\ &\leq C \left(2^{-p/2} c_2 + \sqrt{n} \sum_{\ell=0}^n 2^{-\ell/2} \delta_{\ell,n} \right). \end{aligned}$$

Then use (25) to conclude. □

Following the proof of Lemma 7.5, we get the following result.

Lemma 12.3. *Under the assumptions of Theorem 3.7 ($2\alpha^2 = 1$), we get $\lim_{n \rightarrow \infty} \mathbb{E}[R_2(n)] = 0$.*

We now consider the limit of $V_2(n)$.

Lemma 12.4. *Under the assumptions of Theorem 3.7 ($2\alpha^2 = 1$), we get $\lim_{n \rightarrow \infty} V_2(n) = 0$ in probability.*

Proof. To prove that $\lim_{n \rightarrow \infty} V_2(n) = 0$ in probability, we give a closer look at the proof of (105). Recall the definition of c_k and q_k given in (83). More precisely, the term (94) can be bound from above using (150) and $\|\mathcal{P}(\tilde{f}_k \otimes_{\text{sym}} \mathcal{Q}^{k-\ell} \tilde{f}_\ell)\|_{L^2(\mu)} \leq \|\mathcal{Q}\tilde{f}_\ell\|_\infty \|\mathcal{P}(\tilde{f}_k \otimes_{\text{sym}} \mathbf{1})\|_{L^2(\mu)} \leq 2q_2(\mathfrak{f}) c_2(\mathfrak{f})$ as $k > \ell$, and thus (95) can be replaced by $C c_2(\mathfrak{f}) (c_2(\mathfrak{f}) + q_2(\mathfrak{f})) \alpha^j$. Therefore the upper bound in (96) can be replaced by $C c_2^2(\mathfrak{f}) (c_2(\mathfrak{f}) + q_2(\mathfrak{f}))^2 (n-p) 2^{-(n-p)}$. The term (102) can be bound from above using $\|\tilde{f}_k \mathcal{Q}^{k-\ell} \tilde{f}_\ell\|_{L^2(\mu)} \leq \|\tilde{f}_k\|_{L^2(\mu)} \|\mathcal{Q}^{k-\ell} \tilde{f}_\ell\|_\infty \leq c_2(\mathfrak{f}) q_2(\mathfrak{f})$ as $k > \ell$, and thus (103) can be replaced by $C c_2(\mathfrak{f}) q_2(\mathfrak{f}) \alpha^j$. Therefore the upper bound in (104) can be replaced by

$C c_2^2(\mathbf{f}) q_2^2(\mathbf{f}) (n-p) 2^{-(n-p)}$. As $V_2 = V_6 + V_5$, we deduce that (compare with (105) and replace \mathbf{f} by \mathbf{f}_n):

$$\begin{aligned} \mathbb{E} \left[\left(V_2(n) - H_2^{[n]}(\mathbf{f}_n) \right)^2 \right] &\leq C \left(c_2^4(\mathbf{f}_n) + c_2^2(\mathbf{f}_n) q_2^2(\mathbf{f}_n) \right) (n-p) 2^{-(n-p)} \\ &\leq C \left(\mathbf{c}_2^4 + \mathbf{c}_2^2 \mathbf{q}_2^2 \right) (n-p) 2^{-(n-p)}, \end{aligned}$$

with $H_2^{[n]}(\mathbf{f}_n) = H_5^{[n]}(\mathbf{f}_n) + H_6^{[n]}(\mathbf{f}_n)$. Since according to (ii) in Assumption, 2.16 \mathbf{c}_2 and \mathbf{q}_2 are finite, we deduce that $\lim_{n \rightarrow \infty} V_2(n) - H_2^{[n]}(\mathbf{f}_n) = 0$ in probability. We now check that $\lim_{n \rightarrow \infty} H_2^{[n]}(\mathbf{f}_n) = 0$. From (99), we get that $|H_5^{[n]}(\mathbf{f}_n)| \leq \sum_{k>\ell \geq 0} 2^{-\ell} |\langle \mu, \tilde{f}_{k,n} \mathcal{Q}^{k-\ell} \tilde{f}_{\ell,n} \rangle|$, and using (151) and (152) which are a consequence of Assumption 2.16, and the fact that $\sum_{k>\ell \geq 0} 2^{-\ell} \alpha^{k-\ell}$ is finite, we get by dominated convergence that $\lim_{n \rightarrow \infty} H_5^{[n]}(\mathbf{f}_n) = 0$.

Using (91), we get that:

$$(194) \quad |H_6^{[n]}(\mathbf{f}_n)| \leq \sum_{k>\ell \geq 0} \sum_{r=0}^{p-k-1} 2^{r-\ell} |\langle \mu, \mathcal{P} \left(\mathcal{Q}^r \tilde{f}_{k,n} \otimes_{\text{sym}} \mathcal{Q}^{k-\ell+r} \tilde{f}_{\ell,n} \right) \rangle|.$$

Using (10) and (26) (or more precisely (192)), we obtain:

$$\begin{aligned} |H_6^{[n]}(\mathbf{f}_n)| &\leq \sum_{k>\ell \geq 0} \sum_{r=0}^{p-k-1} 2^{r-\ell} \|\mathcal{Q}^r \tilde{f}_{k,n}\|_{L^2(\mu)} \|\mathcal{Q}^{k-\ell+r} \tilde{f}_{\ell,n}\|_{L^2(\mu)} \\ &\leq \sum_{k>\ell \geq 0} \sum_{r=0}^{p-k-1} 2^{r-\ell} \alpha^{k-\ell+2r} \mathbf{c}_2 \|\mathcal{Q} \tilde{f}_{\ell,n}\|_{L^2(\mu)} \\ &\leq n \sum_{\ell=0}^n 2^{-\ell} \delta_{\ell,n}. \end{aligned}$$

Then, use (25) to conclude. \square

Lemma 12.5. *Under the assumptions of Theorem 3.7 ($2\alpha^2 = 1$), we get $\lim_{n \rightarrow \infty} V_1(n) = \sigma^2$ in probability.*

Proof. To prove that $\lim_{n \rightarrow \infty} V_1(n) = \sigma^2$ in probability, we give a closer look at the proof of (120). More precisely, the term (110) can be bound from above using (150) and thus (111) can be replaced by $C (c_2^2(\mathbf{f}) + q_2^2(\mathbf{f})) \alpha^j$. Therefore the upper bound in (112) can be replaced by $C (c_2^4(\mathbf{f}) + q_2^4(\mathbf{f})) (n-p) 2^{-(n-p)}$. The term (118) can be bound from above by $C c_4^2(\mathbf{f}) \alpha^j 2^{-p/2}$, as $\alpha = 1/\sqrt{2}$, and the upper bound in (119) can then be replaced by $C c_4^4(\mathbf{f}) n 2^{-n}$. As $V_1 = V_4 + V_3$, using (i) from Assumption 2.16, we deduce that (compare with (120) and replace \mathbf{f} by \mathbf{f}_n):

$$\begin{aligned} \mathbb{E} \left[\left(V_1(n) - H_1^{[n]}(\mathbf{f}_n) \right)^2 \right] &\leq C \left(c_2^4(\mathbf{f}_n) + q_2^4(\mathbf{f}_n) \right) (n-p) 2^{-(n-p)} + C c_4^4(\mathbf{f}_n) n 2^{-n} \\ &\leq C \left(\mathbf{c}_2^4 + \mathbf{q}_2^4 \right) (n-p) 2^{-(n-p)} + C n 2^{-n(1-2\rho)}, \end{aligned}$$

with $H_1^{[n]}(\mathbf{f}_n) = H_4^{[n]}(\mathbf{f}_n) + H_3^{[n]}(\mathbf{f}_n)$. This implies that $\lim_{n \rightarrow \infty} V_1(n) - H_1^{[n]}(\mathbf{f}_n) = 0$ in probability. See the proof of Lemma 8.1 to get that $\lim_{n \rightarrow \infty} H_3^{[n]}(\mathbf{f}_n) = \sigma^2$. Recall (108) for the definition of

$H_4^{[n]}(f)$. We have:

$$\begin{aligned} |H_4^{[n]}(f_n)| &\leq \sum_{\ell \geq 0; k \geq 0} 2^{k-\ell} |\langle \mu, \mathcal{P}(\mathcal{Q}^k \tilde{f}_{\ell,n} \otimes^2) \rangle| \mathbf{1}_{\{\ell+k < p\}} \\ &\leq \sum_{\ell \geq 0; k \geq 1} 2^{k-\ell} \alpha^{2k} \|Q|\tilde{f}_{\ell,n}|\|_{\infty} \mathbf{1}_{\{\ell+k < p\}} + \sum_{\ell=0}^{p-1} 2^{-\ell} |\langle \mu, \mathcal{P}(\tilde{f}_{\ell,n} \otimes^2) \rangle| \\ &\leq Cn \sum_{\ell=0}^n 2^{-\ell} \delta_{\ell,n} + C(1 + \Delta) \sum_{\ell=0}^n 2^{-\ell} \delta_{\ell,n}. \end{aligned}$$

Thanks to (25) from Assumption 2.19, we get $\lim_{n \rightarrow \infty} H_4^{[n]}(f_n) = 0$, and thus $\lim_{n \rightarrow \infty} H_1^{[n]}(f_n) = \sigma^2$. This finishes the proof. \square

As a conclusion of Lemmas 12.3, 12.4 and 12.5 and since $V(n) = V_1(n) + 2V_2(n) - R_2(n)$ (see (63)), we deduce that $\lim_{n \rightarrow \infty} V(n) = \sigma^2$ in probability.

We now check the Lindeberg condition using a fourth moment condition. Recall $R_3(n) = \sum_{i \in \mathbb{G}_{n-p_n}} \mathbb{E}[\Delta_{n,i}(f)^4]$ defined in (78).

Lemma 12.6. *Under the assumptions of Theorem 3.7 ($2\alpha^2 = 1$), we get $\lim_{n \rightarrow \infty} n^{-2} R_3(n) = 0$.*

Proof. Following line by line the proof of Lemma 8.2 with the same notations and taking $\alpha = 1/\sqrt{2}$, we get that concerning $|\langle \mu, \psi_{i,p-\ell} \rangle|$ or $\langle \mu, |\psi_{i,p-\ell}| \rangle$, the bounds for $i \in \{1, 3, 4\}$ are the same; the bounds for $i \in \{2, 5, 6\}$ have an extra $(p-\ell)$ term, the bounds for $i \in \{7, 8, 9\}$ have an extra $(p-\ell)^2$ term. This leads to:

$$R_3(n) \leq C n^5 \left(2^{-n} c_4^4(f_n) + 2^{-n} c_6^3(f_n) c_2 + 2^{-(n-p)} c_2^2 (c_2^2 + q_2^2) \right).$$

Then conclude as in the proof of Lemma 8.2. \square

Then, we end the proof of Theorem 3.7 with $2\alpha^2 = 1$ by arguing as in the (end of the) proof of Corollary 9.3.

13. THE SUPER-CRITICAL CASE: $2\alpha^2 > 1$ FOR THE POINT WISE APPROACH

We consider the super-critical case $\alpha \in (1/\sqrt{2}, 1)$ in the point-wise approach. We shall assume that Assumption 2.7 holds. Recall (7) and (21) with the eigenvalues $\{\alpha_j = \theta_j \alpha, j \in J\}$ of \mathcal{Q} , with modulus equal to α (i.e. $|\theta_j| = 1$) and the projector \mathcal{R}_j on the eigen-space associated to eigenvalue α_j . Recall that the sequence $(\beta_n, n \in \mathbb{N})$ in Assumption 2.7 can be (and will be) chosen non-increasing and bounded from above by 1.

We shall consider the filtration $\mathcal{H} = (\mathcal{H}_n, n \in \mathbb{N})$ defined by $\mathcal{H}_n = \sigma(X_i, i \in \mathbb{T}_n)$. The next lemma exhibits martingales related to the projector \mathcal{R}_j .

Lemma 13.1. *Let X be a BMC with kernel \mathcal{P} and initial distribution ν . Assume that Assumption 2.4 and 2.7 hold with $\alpha \in (1/\sqrt{2}, 1)$ in (7). Then, for all $j \in J$ and $f \in F$, the sequence $M_j(f) = (M_{n,j}(f), n \in \mathbb{N})$, with*

$$M_{n,j}(f) = (2\alpha_j)^{-n} M_{\mathbb{G}_n}(\mathcal{R}_j(f)),$$

is a \mathcal{H} -martingale which converges a.s. and in L^2 to a random variable, say $M_{\infty,j}(f)$.

Proof. Let $f \in F$ and $j \in J$. Use that $\mathcal{R}_j(F) \subset \mathbb{C}F$ to deduce that $\mathbb{E}[|M_{n,j}(f)|^2]$ is finite. We have for $n \in \mathbb{N}^*$:

$$\begin{aligned} \mathbb{E}[M_{n,j}(f)|\mathcal{H}_{n-1}] &= (2\alpha_j)^{-n} \sum_{i \in \mathbb{G}_{n-1}} \mathbb{E}[\mathcal{R}_j f(X_{i0}) + \mathcal{R}_j f(X_{i1})|\mathcal{H}_{n-1}] \\ &= (2\alpha_j)^{-n} \sum_{i \in \mathbb{G}_{n-1}} 2\mathcal{Q}\mathcal{R}_j f(X_i) \\ &= (2\alpha_j)^{-(n-1)} \sum_{i \in \mathbb{G}_{n-1}} \mathcal{R}_j f(X_i) \\ &= M_{n-1,j}(f), \end{aligned}$$

where the second equality follows from branching Markov property and the third follows from the fact that \mathcal{R}_j is the projection on the eigen-space associated to the eigen-value α_j of \mathcal{Q} . This gives that $M_j(f)$ is a \mathcal{H} -martingale.

We also have, writing f_j for $\mathcal{R}_j(f)$:

$$\begin{aligned} \mathbb{E}[|M_{n,j}(f)|^2] &= (2\alpha)^{-2n} \mathbb{E}[M_{\mathbb{G}_n}(f_j)M_{\mathbb{G}_n}(\bar{f}_j)] \\ &= (2\alpha^2)^{-n} \langle \nu, \mathcal{Q}^n(|f_j|^2) \rangle + (2\alpha)^{-2n} \sum_{k=0}^{n-1} 2^{n+k} \langle \nu, \mathcal{Q}^{n-k-1} \mathcal{P}(\mathcal{Q}^k f_j \otimes_{\text{sym}} \mathcal{Q}^k \bar{f}_j) \rangle \\ (195) \quad &= (2\alpha^2)^{-n} \langle \nu, \mathcal{Q}^n(|f_j|^2) \rangle + (2\alpha^2)^{-n} \sum_{k=0}^{n-1} (2\alpha^2)^k \langle \nu, \mathcal{Q}^{n-k-1} \mathcal{P}(f_j \otimes_{\text{sym}} \bar{f}_j) \rangle \\ &\leq (2\alpha^2)^{-n} \langle \nu, g_1 \rangle + (2\alpha^2)^{-n} \sum_{k=0}^{n-1} (2\alpha^2)^k \langle \nu, g_2 \rangle \\ &\leq \langle \nu, g_3 \rangle, \end{aligned}$$

where we used the definition of $M_{n,j}$ for the first equality, (49) with $m = n$ for the second equality, the fact that f_j (resp. \bar{f}_j) is an eigen-function associated to the eigenvalue α_j (resp. $\bar{\alpha}_j$) for the third equality, (4) twice (with f and g replaced by $|f_j|^2$ and g_1 and by $\mathcal{P}(f_j \otimes_{\text{sym}} \bar{f}_j)$ and g_2) for the first inequality and $2\alpha^2 > 1$ as well as $g_3 = g_1 + g_2/(2\alpha^2 - 1)$ for the last inequality. Since g_3 belongs to F and does not depend on n , this implies that $\sup_{n \in \mathbb{N}} \mathbb{E}[|M_{n,j}(f)|^2] < +\infty$. Thus the martingale $M_j(f)$ converges a.s. and in L^2 towards a limit. \square

Now, we state the main result of this section. Recall that $\theta_j = \alpha_j/\alpha$ and $|\theta_j| = 1$ and $M_{\infty,j}$ is defined in Lemma 13.1.

Theorem 13.2. *Let X be a BMC with kernel \mathcal{P} and initial distribution ν . Assume that Assumptions 2.4 and 2.7 hold with $\alpha \in (1/\sqrt{2}, 1)$ in (7). We have the following convergence in probability for all sequence $\mathfrak{f} = (f_\ell, \ell \in \mathbb{N})$ of elements of F satisfying Assumptions 2.7 uniformly, that is (162) holds for some $g \in F$:*

$$(2\alpha^2)^{-n/2} N_{n,\emptyset}(\mathfrak{f}) - \sum_{\ell \in \mathbb{N}} (2\alpha)^{-\ell} \sum_{j \in J} \theta_j^{n-\ell} M_{\infty,j}(f_\ell) \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0.$$

Remark 13.3. We stress that if for all $\ell \in \mathbb{N}$, the orthogonal projection of f_ℓ on the eigen-spaces corresponding to the eigenvalues 1 and α_j , $j \in J$, equal 0, then $M_{\infty,j}(f_\ell) = 0$ for all $j \in J$ and in this case, we have

$$(2\alpha^2)^{-n/2} N_{n,\emptyset}(\mathfrak{f}) \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0.$$

As a direct consequence of the previous Theorem and Remark 4.1, we deduce the following results. Recall that $\tilde{f} = f - \langle \mu, f \rangle$.

Corollary 13.4. *Under the assumptions of Theorem 13.2, we have for all $f \in F$:*

$$(2\alpha)^{-n} M_{\mathbb{T}_n}(\tilde{f}) - \sum_{j \in J} \theta_j^n (1 - (2\alpha\theta_j)^{-1})^{-1} M_{\infty, j}(f) \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0$$

$$(2\alpha)^{-n} M_{\mathbb{G}_n}(\tilde{f}) - \sum_{j \in J} \theta_j^n M_{\infty, j}(f) \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0.$$

Proof. The proof follows easily if, in Theorem 13.2, we first take $\mathbf{f} = (f, f, \dots)$, the infinite sequence of the same function f , and next, $\mathbf{f} = (f, 0, \dots)$, the infinite sequence where only the first component is non-zero, and then use (44). \square

We directly deduce the following two Corollaries.

Corollary 13.5. *Under the hypothesis of Theorem 13.2, if α is the only eigen-value of \mathcal{Q} with modulus equal to α (and thus J is reduced to a singleton), then we have:*

$$(2\alpha^2)^{-n/2} N_{n, \emptyset}(\mathbf{f}) \xrightarrow[n \rightarrow \infty]{\mathbb{P}} \sum_{\ell \in \mathbb{N}} (2\alpha)^{-\ell} M_{\infty}(f_{\ell}),$$

where, for $f \in F$, $M_{\infty}(f) = \lim_{n \rightarrow \infty} (2\alpha)^{-n} M_{\mathbb{G}_n}(\mathcal{R}(f))$, and \mathcal{R} is the projection on the eigen-space associated to the eigen-value α .

14. PROOF OF THEOREM 13.2

Recall the sequence $(\beta_n, n \in \mathbb{N})$ defined in Assumption 2.7 and the σ -field $\mathcal{H}_n = \sigma\{X_u, u \in \mathbb{T}_n\}$. Let $(\hat{p}_n, n \in \mathbb{N})$ be a sequence of integers such that \hat{p}_n is even and (for $n \geq 3$):

$$(196) \quad \frac{2n}{3} < \hat{p}_n < n, \quad \lim_{n \rightarrow \infty} (n - \hat{p}_n) = \infty \quad \text{and} \quad \lim_{n \rightarrow \infty} \alpha^{-(n - \hat{p}_n)} \beta_{2^{-1}\hat{p}_n} = 0.$$

Notice such sequences exist. When there is no ambiguity, we shall write \hat{p} for \hat{p}_n . We deduce from (45) that:

$$(197) \quad N_{n, \emptyset}(\mathbf{f}) = R_0(n) + R_4(n) + T_n(\mathbf{f}),$$

with notations from (55) and (56):

$$R_0(n) = |\mathbb{G}_n|^{-1/2} \sum_{k=0}^{n - \hat{p}_n - 1} M_{\mathbb{G}_k}(\tilde{f}_{n-k}),$$

$$T_n(\mathbf{f}) = R_1(n) = \sum_{i \in \mathbb{G}_{n - \hat{p}_n}} \mathbb{E}[N_{n, i}(\mathbf{f}) | \mathcal{H}_{n - \hat{p}_n}],$$

$$R_4(n) = \Delta_n = \sum_{i \in \mathbb{G}_{n - \hat{p}_n}} (N_{n, i}(\mathbf{f}) - \mathbb{E}[N_{n, i}(\mathbf{f}) | \mathcal{H}_{n - \hat{p}_n}]).$$

Furthermore, using the branching Markov property, we get for all $i \in \mathbb{G}_{n - \hat{p}_n}$:

$$(198) \quad \mathbb{E}[N_{n, i}(\mathbf{f}) | \mathcal{H}_{n - \hat{p}_n}] = \mathbb{E}[N_{n, i}(\mathbf{f}) | X_i].$$

We have the following elementary lemma.

Lemma 14.1. *Under the assumptions of Theorem 13.2, we have the following convergence:*

$$\lim_{n \rightarrow \infty} (2\alpha^2)^{-n} \mathbb{E}[R_0(n)^2] = 0.$$

Proof. We follow the proof of Lemma 6.1. As $2\alpha^2 > 1$, we get that $\mathbb{E}[M_{\mathbb{G}_k}(\tilde{f}_{n-k})^2] \leq 2^k(2\alpha^2)^k \langle \nu, g \rangle$ for some $g \in F$ and all $n \geq k \geq 0$. This implies, see (57), that for some constant C which does not depend on n or \hat{p} :

$$\mathbb{E} [R_0(n)^2]^{1/2} \leq C 2^{-\hat{p}/2} (2\alpha^2)^{(n-\hat{p})/2}.$$

It follows from the previous inequality that $(2\alpha^2)^{-n} \mathbb{E} [R_0(n)^2] \leq C(2\alpha)^{-2\hat{p}}$. Then use $2\alpha > 1$ and $\lim_{n \rightarrow \infty} \hat{p} = \infty$ to conclude. \square

Next, we have the following lemma.

Lemma 14.2. *Under the assumptions of Theorem 13.2, we have the following convergence:*

$$\lim_{n \rightarrow \infty} (2\alpha^2)^{-n} \mathbb{E} [R_4(n)^2] = 0.$$

Proof. First, we have:

$$\begin{aligned} \mathbb{E}[R_4(n)^2] &= \mathbb{E} \left[\left(\sum_{i \in \mathbb{G}_{n-\hat{p}}} (N_{n,i}(\mathbf{f}) - \mathbb{E}[N_{n,i}(\mathbf{f})|X_i]) \right)^2 \right] \\ &= \mathbb{E} \left[\sum_{i \in \mathbb{G}_{n-\hat{p}}} \mathbb{E}[(N_{n,i}(\mathbf{f}) - \mathbb{E}[N_{n,i}(\mathbf{f})|X_i])^2 | \mathcal{H}_{n-\hat{p}}] \right] \\ (199) \quad &\leq \mathbb{E} \left[\sum_{i \in \mathbb{G}_{n-\hat{p}}} \mathbb{E}[N_{n,i}(\mathbf{f})^2 | X_i] \right], \end{aligned}$$

where we used (198) for the first equality and the branching Markov chain property for the second and the last inequality. Note that for all $i \in \mathbb{G}_{n-\hat{p}}$ we have

$$\begin{aligned} \mathbb{E} [\mathbb{E}[N_{n,i}(\mathbf{f})^2 | X_i]] &= |\mathbb{G}_n|^{-1} \mathbb{E} \left[\left(\sum_{\ell=0}^{\hat{p}} M_{i\mathbb{G}_{\hat{p}-k}}(\tilde{f}_\ell) \right)^2 \middle| X_i \right] \\ &\leq |\mathbb{G}_n|^{-1} \left(\sum_{\ell=0}^{\hat{p}} \mathbb{E}_{X_i} [M_{\mathbb{G}_{\hat{p}-\ell}}(\tilde{f}_\ell)^2]^{1/2} \right)^2, \end{aligned}$$

where we used the definition of $N_{n,i}(\mathbf{f})$ for the equality and the Minkowski's inequality for the last inequality. We have:

$$\begin{aligned} \mathbb{E}_{X_i} [M_{\mathbb{G}_{\hat{p}-\ell}}(\tilde{f}_\ell)^2] &= 2^{\hat{p}-\ell} Q^{\hat{p}-\ell}(\tilde{f}_\ell^2)(X_i) + \sum_{k=0}^{\hat{p}-\ell-1} 2^{\hat{p}-\ell+k} Q^{\hat{p}-\ell-k-1}(\mathcal{P}(Q^k \tilde{f}_\ell \otimes Q^k \tilde{f}_\ell))(X_i) \\ &\leq 2^{\hat{p}-\ell} g_2(X_i) + \sum_{k=0}^{\hat{p}-\ell-1} 2^{\hat{p}-\ell+k} \alpha^{2k} Q^{\hat{p}-\ell-k-1}(\mathcal{P}(g_1 \otimes g_1))(X_i) \\ &\leq 2^{\hat{p}-\ell} g_2(X_i) + \sum_{k=0}^{\hat{p}-\ell-1} 2^{\hat{p}-\ell} (2\alpha^2)^k g_3(X_i) \\ &\leq (2\alpha)^{2(\hat{p}-\ell)} g_4(X_i), \end{aligned}$$

where we used (48) for the first equality, (ii) of Assumption 2.4 and (6) for the first inequality, (4) and (iv) of Assumption 2.4 for the second, and $2\alpha^2 > 1$ for the last. The latter inequality implies that, with g_5 equal to g_4 up to a finite multiplicative constant:

$$(200) \quad \mathbb{E}[N_{n,i}(f)^2 | X_i] \leq |\mathbb{G}_n|^{-1} \left(\sum_{\ell=0}^{\hat{p}} (2\alpha)^{(\hat{p}-\ell)} \right)^2 g_4(X_i) = 2^{-n} (2\alpha)^{2\hat{p}} g_5(X_i).$$

Using (199), (200) and (47) as well as Assumption 2.5 with $g_6 \in F$, we get:

$$(2\alpha^2)^{-n} \mathbb{E}[R_4(n)^2] \leq (2\alpha^2)^{-n} 2^{n-\hat{p}} 2^{-n} (2\alpha)^{2\hat{p}} \langle \nu, \mathcal{Q}^n g_5 \rangle \leq (2\alpha^2)^{-(n-\hat{p})} \langle \nu, g_6 \rangle.$$

We then conclude using that $2\alpha^2 > 1$ and $\lim_{n \rightarrow \infty} (n - \hat{p}) = \infty$. \square

Now, we study the third term of the right hand side of (197). First, note that:

$$\begin{aligned} T_n(f) &= \sum_{i \in \mathbb{G}_{n-\hat{p}}} \mathbb{E}[N_{n,i}(f) | X_i] \\ &= \sum_{i \in \mathbb{G}_{n-\hat{p}}} |\mathbb{G}_n|^{-1/2} \sum_{\ell=0}^{\hat{p}} \mathbb{E}_{X_i}[M_{\mathbb{G}_{\hat{p}-\ell}}(\tilde{f}_\ell)] \\ &= |\mathbb{G}_n|^{-1/2} \sum_{i \in \mathbb{G}_{n-\hat{p}}} \sum_{\ell=0}^{\hat{p}} 2^{\hat{p}-\ell} \mathcal{Q}^{\hat{p}-\ell}(\tilde{f}_\ell)(X_i), \end{aligned}$$

where we used (198) for the first equality, the definition (43) of $N_n(f)$ for the second equality and (47) for the last equality. Next, projecting in the eigen-space associated to the eigenvalue α_j , we get

$$T_n(f) = T_n^{(1)}(f) + T_n^{(2)}(f),$$

where, with $\hat{f} = f - \langle \mu, f \rangle - \sum_{j \in J} \mathcal{R}_j(f)$ defined in (157):

$$\begin{aligned} T_n^{(1)}(f) &= |\mathbb{G}_n|^{-1/2} \sum_{i \in \mathbb{G}_{n-\hat{p}}} \sum_{\ell=0}^{\hat{p}} 2^{\hat{p}-\ell} \left(\mathcal{Q}^{\hat{p}-\ell}(\hat{f}_\ell) \right) (X_i), \\ T_n^{(2)}(f) &= |\mathbb{G}_n|^{-1/2} \sum_{i \in \mathbb{G}_{n-\hat{p}}} \sum_{\ell=0}^{\hat{p}} 2^{\hat{p}-\ell} \alpha^{\hat{p}-\ell} \sum_{j \in J} \theta_j^{\hat{p}-\ell} \mathcal{R}_j(f_\ell)(X_i). \end{aligned}$$

We have the following lemma.

Lemma 14.3. *Under the assumptions of Theorem 13.2, we have the following convergence:*

$$\lim_{n \rightarrow \infty} (2\alpha^2)^{-n/2} \mathbb{E}[|T_n^{(1)}(f)|] = 0.$$

Proof. We have:

$$\begin{aligned}
(2\alpha^2)^{-n/2} \mathbb{E}[|T_n^{(1)}(\mathbf{f})|] &\leq (2\alpha)^{-n} \mathbb{E} \left[\sum_{i \in \mathbb{G}_{n-\hat{p}}} \sum_{\ell=0}^{\hat{p}} 2^{\hat{p}-\ell} |\mathcal{Q}^{\hat{p}-\ell}(\hat{f}_\ell)(X_i)| \right] \\
&\leq (2\alpha)^{-n} \mathbb{E} \left[\sum_{i \in \mathbb{G}_{n-\hat{p}}} \sum_{\ell=0}^{\hat{p}} 2^{\hat{p}-\ell} \alpha^{\hat{p}-\ell} \beta_{\hat{p}-\ell} g(X_i) \right] \\
&= \sum_{\ell=0}^{\hat{p}} 2^{-\ell} \alpha^{-(n-\hat{p}+\ell)} \beta_{\hat{p}-\ell} \langle \nu, \mathcal{Q}^{n-\hat{p}} g \rangle,
\end{aligned}$$

where we used the definition of $T_n^{(1)}(\mathbf{f})$ for the first inequality, (162) for the second and (47) for the equality. Using (4) and the property (iii), we get that there is a finite positive constant C independent of n and \hat{p} such that $\langle \nu, \mathcal{Q}^{n-\hat{p}} g \rangle < C$. Using the first and third Conditions in (196), we get that $\lim_{n \rightarrow \infty} \sup_{\ell \in \{0, \dots, \hat{p}\}} \alpha^{-(n-\hat{p}+\ell)} \beta_{\hat{p}-\ell} = 0$. Thus, we deduce by dominated convergence, as $2\alpha^2 > 1$, that $\lim_{n \rightarrow \infty} (2\alpha^2)^{-n/2} \mathbb{E}[|T_n^{(1)}(\mathbf{f})|] = 0$. \square

Now, we deal with the term $T_n^{(2)}(\mathbf{f})$ in the following result. Recall $M_{\infty,j}$ defined in Lemma 13.1.

Lemma 14.4. *Under the assumptions of Theorem 13.2, we have the following convergence:*

$$(2\alpha^2)^{-n/2} T_n^{(2)}(\mathbf{f}) - \sum_{\ell \in \mathbb{N}} (2\alpha)^{-\ell} \sum_{j \in J} \theta_j^{n-\ell} M_{\infty,j}(f_\ell) \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0.$$

Proof. By definition of $T_n^{(2)}(\mathbf{f})$, we have $T_n^{(2)}(\mathbf{f}) = 2^{-n/2} \sum_{\ell=0}^{\hat{p}} (2\alpha)^{n-\ell} \sum_{j \in J} \theta_j^{n-\ell} M_{n,j}(f_\ell)$ and thus:

$$\begin{aligned}
(201) \quad (2\alpha^2)^{-n/2} T_n^{(2)}(\mathbf{f}) - \sum_{\ell \in \mathbb{N}} (2\alpha)^{-\ell} \sum_{j \in J} \theta_j^{n-\ell} M_{\infty,j}(f_\ell) \\
= \sum_{\ell=0}^{\hat{p}} (2\alpha)^{-\ell} \sum_{j \in J} \theta_j^{n-\ell} (M_{n,j}(f_\ell) - M_{\infty,j}(f_\ell)) - \sum_{\ell=\hat{p}+1}^{\infty} (2\alpha)^{-\ell} \sum_{j \in J} \theta_j^{n-\ell} M_{\infty,j}(f_\ell).
\end{aligned}$$

Using that $|\theta_j| = 1$, we get:

$$\mathbb{E} \left[\left| \sum_{\ell=0}^{\hat{p}} (2\alpha)^{-\ell} \sum_{j \in J} \theta_j^{n+\hat{p}-\ell} (M_{n,j}(f_\ell) - M_{\infty,j}(f_\ell)) \right| \right] \leq \sum_{\ell=0}^{\hat{p}} (2\alpha)^{-\ell} \sum_{j \in J} \mathbb{E} [|M_{n,j}(f_\ell) - M_{\infty,j}(f_\ell)|].$$

Now, using (6), a close inspection of the proof of Lemma 13.1, see (195), reveals us that there exists a finite constant C (depending on \mathbf{f}) such that for all $j \in J$, we have:

$$(202) \quad \sup_{\ell \in \mathbb{N}} \sup_{n \in \mathbb{N}} \mathbb{E} [|M_{n,j}(f_\ell)|^2] \leq C.$$

The $L^2(\nu)$ convergence in Lemma 13.1 yields that:

$$(203) \quad \sup_{\ell \in \mathbb{N}} \mathbb{E} [|M_{\infty,j}(f_\ell)|^2] \leq C \quad \text{and} \quad \sup_{\ell \in \mathbb{N}} \sup_{n \in \mathbb{N}} \sum_{j \in J} \mathbb{E} [|M_{n,j}(f_\ell) - M_{\infty,j}(f_\ell)|] < 2|J|\sqrt{C}.$$

Since Lemma 13.1 implies that $\lim_{n \rightarrow \infty} \mathbb{E} [|M_{n,j}(f_\ell) - M_{\infty,j}(f_\ell)|] = 0$, we deduce, as $2\alpha > 1$ by the dominated convergence theorem that:

$$(204) \quad \lim_{n \rightarrow +\infty} \mathbb{E} \left[\left| \sum_{\ell=0}^{\hat{p}} (2\alpha)^{-\ell} \sum_{j \in J} \theta_j^{n+\hat{p}-\ell} (M_{n,j}(f_\ell) - M_{\infty,j}(f_\ell)) \right| \right] = 0.$$

On the other hand, we have

$$(205) \quad \mathbb{E}\left[\left|\sum_{\ell=\hat{p}+1}^{\infty} (2\alpha)^{-\ell} \sum_{j \in J} \theta_j^{n-\ell} M_{\infty,j}(f_\ell)\right|\right] \leq \sum_{\ell=\hat{p}+1}^{\infty} (2\alpha)^{-\ell} \sum_{j \in J} \mathbb{E}[|M_{\infty,j}(f_\ell)|] \leq |J| \sqrt{C} \sum_{\ell=\hat{p}+1}^{\infty} (2\alpha)^{-\ell},$$

where we used $|\theta_j| = 1$ for the first inequality and the Cauchy-Schwarz inequality and (203) for the second inequality. Finally, from (201), (204) and (205) (with $\lim_{n \rightarrow \infty} \sum_{\ell=\hat{p}+1}^{\infty} (2\alpha)^{-\ell} = 0$), we get the result of the lemma. \square

15. PROOF OF THEOREM 3.7 IN THE SUPER-CRITICAL CASE ($2\alpha^2 > 1$)

We assume $\alpha \in (1/\sqrt{2}, 1)$. We follow line by line the proof of Theorem 3.7 in Section 12 with $\alpha > 1/\sqrt{2}$ instead of $\alpha = 1/\sqrt{2}$, and use notations from Sections 7, 8 and 11. We recall that $\mathbf{c}_2 = \sup\{c_2(\mathbf{f}_n), n \in \mathbb{N}\}$ and $\mathbf{q}_2 = \sup_{n \in \mathbb{N}} q_2(\mathbf{f}_n)$ are finite thanks to Assumption 2.16 (ii). We will denote C any unimportant finite constant which may vary line to line, independent on n and \mathbf{f}_n . Let $(p_n, n \in \mathbb{N})$ be an increasing sequence of elements of \mathbb{N} such that (53) holds. When there is no ambiguity, we write p for p_n .

Lemma 15.1. *Under the assumptions of Theorem 3.7 ($2\alpha^2 > 1$), we get $\lim_{n \rightarrow \infty} \mathbb{E}[R_0^{k_0}(n)^2] = 0$.*

Proof. Mimicking the proof of Lemma 7.2 or Lemma 12.1, we get, as $\lim_{n \rightarrow \infty} p/n = 1$:

$$\lim_{n \rightarrow \infty} \mathbb{E}[R_0^{k_0}(n)^2] \leq C \lim_{n \rightarrow \infty} c_2^2(\mathbf{f}_n) (2\alpha^2)^{n-p} 2^{-p} \leq C \mathbf{c}_2^2 \lim_{n \rightarrow \infty} (2\alpha^2)^{n-p} 2^{-p} = 0.$$

\square

Lemma 15.2. *Under the assumptions of Theorem 3.7 ($2\alpha^2 > 1$), we get $\lim_{n \rightarrow \infty} \mathbb{E}[R_1(n)^2] = 0$.*

Proof. Following the proof of Lemma 12.2 with $\alpha^2 > 1/2$, we get:

$$\begin{aligned} \mathbb{E}[R_1(n)^2]^{1/2} &\leq C \sum_{\ell=0}^p (2^{-\ell} (2\alpha^2)^{n-\ell} \delta_{\ell,n}^2 + \mathbf{1}_{\{\ell=p\}} 2^{-p} \mathbf{c}_2^2)^{1/2} \\ &\leq C \left(2^{-p/2} \mathbf{c}_2 + \sum_{\ell=0}^n 2^{-\ell/2} (2\alpha^2)^{(n-\ell)/2} \delta_{\ell,n} \right). \end{aligned}$$

Then use (27) and dominated convergence theorem to conclude. \square

From Lemmas 15.1 and 15.2, it follows that

$$\lim_{n \rightarrow \infty} \mathbb{E}[(N_{n,\emptyset}^{[k_0]}(\mathbf{f}_n) - \Delta_n(\mathbf{f}_n))^2] = 0.$$

Lemma 15.3. *Under the assumptions of Theorem 3.7 ($2\alpha^2 > 1$), we get $\lim_{n \rightarrow \infty} \mathbb{E}[R_2(n)] = 0$.*

Proof. From (87) and (192) we get

$$\mathbb{E}[R_2(n)] \leq C \mathbf{c}_2^2 2^{-p} + \left(\sum_{\ell=0}^n 2^{-\ell/2} (2\alpha^2)^{(p-\ell)/2} \delta_{\ell,n} \right)^2.$$

Then use (27), $2\alpha^2 > 1$ and dominated convergence theorem to conclude. \square

We now consider the limit of $V_2(n)$.

Lemma 15.4. *Under the assumptions of Theorem 3.7 ($2\alpha^2 > 1$), we get $\lim_{n \rightarrow \infty} V_2(n) = 0$ in probability.*

Proof. Following the proof of Lemma 12.4 with $\alpha > 1/\sqrt{2}$, we get that the upper-bound in (104) can be replaced by $Cc_2^2(\mathfrak{f})q_2^2(\mathfrak{f})\alpha^{2(n-p)}$. We get that for $r \geq 1$:

$$\|\mathcal{P}(\mathcal{Q}^r \tilde{f}_{k,n} \otimes_{\text{sym}} \mathcal{Q}^{k-\ell+r} \tilde{f}_{\ell,n})\|_{L^2(\mu)} \leq C\alpha^{2r+k-\ell}\delta_{k,n}\delta_{\ell,n} \leq C(2\alpha^2)^{-n}2^{(\ell+k)/2}\alpha^{2(r+k)},$$

where we used Assumption 2.19 (vi) for the first inequality and Assumption 2.20 for the second. Thus the bound (93) can be replaced by $C(2\alpha^2)^{-(n-p)}\alpha^j$. The term (94) is handled as in the proof of Lemma 12.4. This gives that (95) can be replaced by $C\alpha^j$. Therefore the upper bound in (96) can be replaced by $C\alpha^{2(n-p)}$. As $V_2 = V_6 + V_5$, we deduce that $\mathbb{E}[(V_2(n) - H_2^{[n]}(\mathfrak{f}_n))^2] \leq C\alpha^{2(n-p)}$. (Compare with (105) and replace \mathfrak{f} by \mathfrak{f}_n .) It follows that $\lim_{n \rightarrow \infty} V_2(n) - H_2^{[n]}(\mathfrak{f}_n) = 0$ in probability. As in the proof of Lemma 12.4 we also have $\lim_{n \rightarrow \infty} H_5^{[n]}(\mathfrak{f}_n) = 0$. Using (10) and (192), we deduce from (194) and Assumption 2.20 that:

$$\begin{aligned} |H_6^{[n]}(\mathfrak{f}_n)| &\leq C \sum_{0 \leq \ell < k \leq p} 2^{-\ell}\alpha^{k-\ell}\delta_{\ell,n} + C \sum_{0 \leq \ell < k \leq p} \sum_{r=1}^{p-k-1} 2^{r-\ell}\delta_{k,n}\delta_{\ell,n}\alpha^{2r+k-\ell} \\ &\leq C \sum_{0 \leq \ell < k \leq p} 2^{-\ell}\alpha^{k-\ell}(2\alpha^2)^{-(n-\ell)/2} + C \sum_{0 \leq \ell < k \leq n} (2\alpha^2)^{-(n-p)}2^{-(\ell+k)/2} \\ &\leq C(2\alpha^2)^{-(n-p)}. \end{aligned}$$

Since $H_2^{[n]}(\mathfrak{f}_n) = H_5^{[n]}(\mathfrak{f}_n) + H_6^{[n]}(\mathfrak{f}_n)$, it follows that $\lim_{n \rightarrow \infty} |H_2^{[n]}(\mathfrak{f}_n)| = 0$. We deduce that $\lim_{n \rightarrow \infty} V_2(n) = 0$ in probability. \square

Lemma 15.5. *Under the assumptions of Theorem 3.7 ($2\alpha^2 > 1$), we get $\lim_{n \rightarrow \infty} V_1(n) = \sigma^2$ in probability.*

Proof. We follow the proof of Lemma 12.5 with $\alpha > 1/\sqrt{2}$ and use the same trick as in the proof of Lemma 15.4 based on Assumption 2.20. We get, with the details left to the reader:

$$\mathbb{E}[(V_4(n) - H_4^{[n]}(\mathfrak{f}_n))^2] \leq C\alpha^{2(n-p)}.$$

We set $g_{\ell,n} = \tilde{f}_{\ell,n}^2$. From (117), we have for $j \in \{0, \dots, n-p\}$:

$$\begin{aligned}
\|\mathcal{Q}^j(A_{3,n}(\mathbf{f}))\|_{L^2(\mu)} &\leq C \sum_{\ell=0}^p 2^{-\ell} \|\mathcal{Q}^{p-\ell} \tilde{g}_{\ell,n}\|_{L^2(\mu)} \mathbf{1}_{\{j=0\}} + C \sum_{\ell=0}^p 2^{-\ell} \|\mathcal{Q}^{j+p-\ell} \tilde{g}_{\ell,n}\|_{L^2(\mu)} \mathbf{1}_{\{j \geq 1\}} \\
&= C 2^{-p} \|\tilde{g}_{p,n}\|_{L^2(\mu)} \mathbf{1}_{\{j=0\}} + C \sum_{\ell=0}^{p-1} 2^{-\ell} \|\mathcal{Q}^{p-\ell-1}(\mathcal{Q} \tilde{g}_{\ell,n})\|_{L^2(\mu)} \mathbf{1}_{\{j=0\}} \\
&\quad + C \sum_{\ell=0}^p 2^{-\ell} \|\mathcal{Q}^{j+p-\ell-1}(\mathcal{Q} \tilde{g}_{\ell,n})\|_{L^2(\mu)} \mathbf{1}_{\{j \geq 1\}} \\
&\leq C 2^{-p} \|f_{\ell,n}^2\|_{L^2(\mu)} \mathbf{1}_{\{j=0\}} + C \sum_{\ell=0}^{p-1} 2^{-\ell} \alpha^{p-\ell-1} \|\mathcal{Q} \tilde{g}_{\ell,n}\|_{L^2(\mu)} \mathbf{1}_{\{j=0\}} \\
&\quad + C \sum_{\ell=0}^p 2^{-\ell} \alpha^{j+p-\ell-1} \|\mathcal{Q} \tilde{g}_{\ell,n}\|_{L^2(\mu)} \mathbf{1}_{\{j \geq 1\}} \\
&\leq C 2^{-p} \|f_{\ell,n}\|_{L^4(\mu)}^2 \mathbf{1}_{\{j=0\}} + C (\mathfrak{q}_2^2 + \mathfrak{c}_2^2) \sum_{\ell=0}^p 2^{-\ell} \alpha^{j+p-\ell} \\
&\leq C \mathfrak{c}_2^2 2^{-p} 2^{n\rho} \mathbf{1}_{\{j=0\}} + C (\mathfrak{q}_2^2 + \mathfrak{c}_2^2) \alpha^{j+p},
\end{aligned}$$

where we used Remark 2.18, (ii) of Assumption 2.16, (20) and (150). From the latter inequality, we get using (115) and (116):

$$\mathbb{E}[(V_3(n) - H_3^{[n]}(\mathbf{f}_n))^2] \leq C(2^{-(1-2\rho)n} + \alpha^{2n}).$$

The latter inequalities imply that $\lim_{n \rightarrow \infty} \mathbb{E}[(V_1(n) - H_1^{[n]}(\mathbf{f}_n))^2] = 0$, with $H_1^{[n]}(\mathbf{f}_n) = H_4^{[n]}(\mathbf{f}_n) + H_3^{[n]}(\mathbf{f}_n)$. From the proof of Lemma 8.1 we have $\lim_{n \rightarrow \infty} H_3^{[n]}(\mathbf{f}_n) = \sigma^2$. Next, we have

$$\begin{aligned}
|H_4^{[n]}(\mathbf{f}_n)| &\leq \sum_{\ell \geq 0, k \geq 0} 2^{k-\ell} |\langle \mu, \mathcal{P}(\mathcal{Q}^k \tilde{f}_{\ell,n} \otimes^2) \rangle| \mathbf{1}_{\{\ell+k < p\}} \\
&\leq C \sum_{\ell \geq 0, k \geq 1} 2^{k-\ell} \alpha^{2k} \|Q|\tilde{f}_{\ell,n}|\|_{\infty}^2 \mathbf{1}_{\{\ell+k < p\}} + C \sum_{\ell=0}^{p-1} 2^{-\ell} |\langle \mu, \mathcal{P}(\tilde{f}_{\ell,n} \otimes^2) \rangle| \\
&\leq C \sum_{\ell \geq 0, k \geq 1} 2^{k-\ell} \alpha^{2k} (2\alpha^2)^{-(n-\ell)} \mathbf{1}_{\{\ell+k < p\}} + C(1 + \Delta) \sum_{\ell=0}^{p-1} 2^{-\ell} (2\alpha^2)^{-(n-\ell)/2} \\
&\leq C (2\alpha^2)^{-(n-p)} + C(1 + \Delta) (2\alpha^2)^{-n/2},
\end{aligned}$$

where we used (108) and the definition of $h_{\ell,k}$ therein for the first inequality; (10) for the second; Assumption 2.16 (iii), Assumption 2.19 (vi), (153) and Assumption 2.20 (twice) for the third. We deduce that $\lim_{n \rightarrow \infty} |H_4^{[n]}(\mathbf{f}_n)| = 0$. This ends the proof. \square

We now check the Lindeberg condition. For that purpose, we have the following result.

Lemma 15.6. *Under the assumptions of Theorem 3.7 ($2\alpha^2 > 1$), we have $\lim_{n \rightarrow \infty} n^{-2} R_3(n) = 0$.*

Proof. From (121), (122) and (123), we have

$$R_3(n) \leq C n^3 2^{-n-p} c_4^A(\mathbf{f}_n) + n^3 2^{-n-p} \sum_{\ell=0}^p \sum_{j=1}^9 \langle \mu, \psi_{j,p-\ell} \rangle.$$

Now, we will bound above each term in the latter sum. For that purpose, we will follow line by line the proof of Lemma 7.9 with \mathfrak{f}_n instead of \mathfrak{f} and we will intensively use (26) and (192). We will also use the fact that for all nonnegative sequence $(a_\ell, \ell \in \mathbb{N})$ such that $\sum_{\ell \geq 0} a_\ell < \infty$, the sequence $(\sum_{\ell=0}^n a_\ell (2\alpha^2)^{n-\ell} \delta_{\ell,n}^2, n \in \mathbb{N})$ is bounded as a consequence of the first part of (27) from Assumption 2.20. (Notice that by the second part of (27) and the dominated convergence theorem, the latter sequence converges towards 0; but we shall not need this.). Recall from Assumption 2.16 that $\rho \in (0, 1/2)$.

The term $n^3 2^{-n-p} c_4^A(\mathfrak{f}_n)$. From the first inequality in Remark 2.18, we have

$$n^3 2^{-n-p} c_4^A(\mathfrak{f}_n) \leq C n^3 2^{-(1-2\rho)n-p}.$$

The term $n^3 2^{-n-p} \sum_{\ell=0}^{p-3} \langle \mu, |\psi_{1,p-\ell}| \rangle$. Using (126) and Remark 2.18, we get:

$$n^3 2^{-n-p} \sum_{\ell=0}^{p-3} \langle \mu, |\psi_{1,p-\ell}| \rangle \leq C c_2^2 n^3 2^{-(1-2\rho)n}.$$

The term $n^3 2^{-n-p} \sum_{\ell=0}^{p-3} |\langle \mu, \psi_{2,p-\ell} \rangle|$. From (127) and distinguishing the case $k = p - \ell - 1$ and $k \leq p - \ell - 2$, we get following the approach given in (154) for the latter case and using (26) for the latter case:

$$|\langle \mu, \psi_{2,p-\ell} \rangle| \leq C 2^{p-\ell} 2^{2n\rho} + C (2\alpha)^{2(p-\ell)} 2^{2n\rho} \delta_{\ell,n}^2.$$

This implies that

$$\begin{aligned} n^3 2^{-n-p} \sum_{\ell=0}^{p-3} \langle \mu, |\psi_{2,p-\ell}| \rangle &\leq C n^3 2^{-n(1-2\rho)} \sum_{\ell=0}^{p-3} 2^{-\ell} + C n^3 2^{-n(1-2\rho)} \sum_{\ell=0}^{p-3} 2^{-\ell} (2\alpha^2)^{p-\ell} \delta_{\ell,n}^2 \\ &\leq C n^3 2^{-n(1-2\rho)}. \end{aligned}$$

The term $n^3 2^{-n-p} \sum_{\ell=0}^{p-3} \langle \mu, |\psi_{3,p-\ell}| \rangle$. From (155) we have

$$n^3 2^{-n-p} \sum_{\ell=0}^{p-3} \langle \mu, |\psi_{3,p-\ell}| \rangle \leq C n^3 2^{-(1-2\rho)n} + C n^3 2^{-n+p}$$

The term $n^3 2^{-n-p} \sum_{\ell=0}^{p-3} \langle \mu, |\psi_{4,p-\ell}| \rangle$. From (129) we have

$$\langle \mu, |\psi_{4,p-\ell}| \rangle \leq C 2^{2(p-\ell)} ((2\alpha^2)^{p-\ell} \delta_{\ell,n}^2)^2 \leq C 2^{2(p-\ell)}$$

and thus

$$n^3 2^{-n-p} \sum_{\ell=0}^{p-3} \langle \mu, |\psi_{4,p-\ell}| \rangle \leq C n^3 2^{-n+p}.$$

The term $n^3 2^{-n-p} \sum_{\ell=0}^{p-3} \langle \mu, |\psi_{5,p-\ell}| \rangle$. From (131) and distinguishing the case $k = p - \ell - 1$ (and using (iii) of Assumption 2.16), $k = p - \ell - 2$ and $k \leq p - \ell - 3$ (and using (27) of Assumption 2.20), we get $\langle \mu, |\psi_{5,p-\ell}| \rangle \leq C 2^{2(p-\ell)}$ and thus

$$n^3 2^{-n-p} \sum_{\ell=0}^{p-3} \langle \mu, |\psi_{5,p-\ell}| \rangle \leq C n^3 2^{-n+p}.$$

The term $n^3 2^{-n-p} \sum_{\ell=0}^{p-3} \langle \mu, |\psi_{6,p-\ell}| \rangle$. Very similarly, from (133), we have

$$n^3 2^{-n-p} \sum_{\ell=0}^{p-3} \langle \mu, |\psi_{6,p-\ell}| \rangle \leq C n^3 2^{-n+p}.$$

The term $n^3 2^{-n-p} \sum_{\ell=0}^{p-3} |\langle \mu, \psi_{7,p-\ell} \rangle|$. We set $g_{k,n} = \mathcal{P} \left(h_{p-\ell-k} \otimes_{\text{sym}} \mathcal{Q}^{p-\ell-k-1}(\tilde{f}_\ell^2) \right)$. Using that $\langle \mu, \mathcal{P}(\mathbf{1} \otimes_{\text{sym}} h_{p-\ell-r}) \rangle = 0$, (136) and (10), we obtain

$$\begin{aligned} \Gamma_{k,r}^{[7]} &= 2^{-k} |\langle \mu, \mathcal{P}(\mathcal{Q}^{k-r-1}(\tilde{g}_{k,n}) \otimes_{\text{sym}} \mathcal{Q}^{p-\ell-r-1}(\tilde{f}_{\ell,n})) \rangle| \\ &\leq C 2^{-k} \|\mathcal{Q}^{k-r-1}(\tilde{g}_{k,n})\|_{L^2(\mu)} \|\mathcal{Q}^{p-\ell-2-r}(\mathcal{Q}\tilde{f}_{\ell,n})\|_{L^2(\mu)}. \end{aligned}$$

For $k = p - \ell - 1$, we have

$$\Gamma_{p-\ell-1,r}^{[7]} \leq C 2^{-(p-\ell)} \|\mathcal{Q}^{k-r-1}(\tilde{g}_{k,n})\|_{L^2(\mu)} \|\mathcal{Q}^{p-\ell-2-r}(\mathcal{Q}\tilde{f}_{\ell,n})\|_{L^2(\mu)} \leq C 2^{-(p-\ell)} \alpha^{2(p-\ell-r)} \delta_{\ell,n}^2 2^{2\rho n},$$

where we used (20), (192) and the following inequality:

$$\|\mathcal{P}(\tilde{f}_{\ell,n} \otimes_{\text{sym}} (\tilde{f}_{\ell,n}^2))\|_{L^2(\mu)} \leq C \delta_{\ell,n} 2^{2\rho n}$$

which is a consequence of (i) and (iii) of Assumption 2.16, (26) from Assumption 2.19 and

$$\|\mathcal{P}(|f_{\ell,n}| \otimes_{\text{sym}} f_{\ell,n}^2)\|_{L^2(\mu)} \leq C \|\mathcal{Q}(|f_{\ell,n}|^3)^{1/3} \mathcal{Q}(|f_{\ell,n}|^3)^{2/3}\|_{L^2(\mu)} = C \|\mathcal{Q}(|f_{\ell,n}|^3)\|_{L^2(\mu)} \leq C \delta_{\ell,n} 2^{2\rho n}.$$

Next, for $k \leq p - \ell - 2$, using (20), (192), (26) and (ii) of Assumption 2.16, we obtain

$$\begin{aligned} \Gamma_{k,r}^{[7]} &\leq C 2^{-k} \|\mathcal{Q}^{k-r-1}(\tilde{g}_{k,n})\|_{L^2(\mu)} \|\mathcal{Q}^{p-\ell-2-r}(\mathcal{Q}\tilde{f}_{\ell,n})\|_{L^2(\mu)} \\ &\leq C 2^{-k} \alpha^{k-r+(p-\ell-r)} \|h_{p-\ell-k}\|_{L^2(\mu)} \|\mathcal{Q}(\tilde{f}_\ell)^2\|_{L^2(\mu)} \delta_{\ell,n} \\ &\leq C 2^{-k} \alpha^{2(p-\ell-r)} \delta_{\ell,n}^2. \end{aligned}$$

Thanks to Assumption 2.20, it follows from the foregoing that

$$|\langle \mu, \psi_{7,p-\ell} \rangle| \leq C 2^{3(p-\ell)} \sum_{k=1}^{p-\ell-1} \sum_{r=0}^{k-1} 2^{-r} \Gamma_{k,r}^{[7]} \leq C 2^{2(p-\ell)} (\alpha^{2(p-\ell)} \delta_{\ell,n}^2 2^{2\rho n} + (2\alpha^2)^{p-\ell} \delta_{\ell,n}^2),$$

and thus, we obtain

$$n^3 2^{-n-p} \sum_{\ell=0}^{p-3} |\langle \mu, \psi_{7,p-\ell} \rangle| \leq C n^3 (2^{-(1-2\rho)n} + 2^{-n+p}).$$

The term $n^3 2^{-n-p} \sum_{\ell=0}^{p-3} \langle \mu, |\psi_{8,p-\ell}| \rangle$. From (140) and distinguishing the cases $k = p - \ell - 1$ and $k \leq p - \ell - 2$, we have,

$$\Gamma_{k,r,j}^{[8]} \leq C 2^{-k-r} \alpha^{4(p-\ell)-2(k+r)} \delta_{\ell,n}^2 (\mathbf{1}_{\{k=p-\ell-1\}} + \delta_{\ell,n}^2 \mathbf{1}_{\{k \leq p-\ell-2\}}),$$

where we use (11), (20) and (192) for the case $k \leq p - \ell - 2$, and we used in addition (156) for the case $k = p - \ell - 1$. From (139), the latter inequality implies that

$$n^3 2^{-n-p} \sum_{\ell=0}^{p-3} \langle \mu, |\psi_{8,p-\ell}| \rangle \leq C n^3 2^{-n+p} \sum_{\ell=0}^{p-3} 2^{-2\ell} (2\alpha^2)^{p-\ell} \delta_{\ell,n}^2 (1 + (2\alpha^2)^{p-\ell} \delta_{\ell,n}^2) \leq C n^3 2^{-n+p}.$$

The term $n^3 2^{-n-p} \sum_{\ell=0}^{p-3} \langle \mu, |\psi_{9,p-\ell}| \rangle$. From (143), (145) and (147) with \mathbf{f}_n instead of \mathbf{f} , using (11), (20), (192) and (156), we obtain

$$\langle \mu, |\psi_{9,p-\ell}| \rangle \leq C 2^{2(p-\ell)} (2\alpha^2)^{p-\ell} \delta_{\ell,n}^2 (1 + (2\alpha^2)^{p-\ell} \delta_{\ell,n}^2).$$

It then follows that

$$n^3 2^{-n-p} \sum_{\ell=0}^{p-3} \langle \mu, |\psi_{9,p-\ell}| \rangle \leq C n^3 2^{-n+p} \sum_{\ell=0}^{p-3} 2^{-\ell} (2\alpha^2)^{p-\ell} \delta_{\ell,n}^2 \leq C n^3 2^{-n+p}.$$

From the previous bounds, we deduce that $\lim_{n \rightarrow \infty} R_3(n) = 0$. \square

Finally, arguing as in the (end of the) proof of Corollary 9.3, we end the proof of Theorem 3.7 in the super-critical case.

16. PROOF OF THE STATISTICAL RESULTS

We recall the following result due to Bochner (see [27] for more details).

Lemma 16.1. *Let $(s_n, n \in \mathbb{N})$ be a sequence of positive numbers converging to 0 as n goes to infinity. Let $g : \mathbb{R}^d \rightarrow \mathbb{R}$ be a measurable function such that $\int_{\mathbb{R}^d} |g(x)| dx < +\infty$. Let $f : \mathbb{R}^d \rightarrow \mathbb{R}$ be a measurable function such that $\|f\|_\infty < +\infty$, $\int_{\mathbb{R}^d} |f(y)| dy < +\infty$ and $\lim_{|x| \rightarrow +\infty} |x|f(x) = 0$. Define*

$$g_n(x) = s_n^{-d} \int_{\mathbb{R}^d} f(s_n^{-1}(x-y)) g(y) dy.$$

Then, we have at every point x of continuity of g ,

$$\lim_{n \rightarrow +\infty} g_n(x) = g(x) \int_{\mathbb{R}} f(y) dy.$$

Proof of Lemma 3.10. We begin the proof with $\mathbb{A}_n = \mathbb{T}_n$. We have the following decomposition:

$$(206) \quad \widehat{\mu}_{\mathbb{T}_n}(x) - \mu(x) = \frac{\sqrt{|\mathbb{G}_n|}}{|\mathbb{T}_n| s_n^{d/2}} N_{n,\emptyset}(\mathbf{f}_n) + B_n(x),$$

where $\mathbf{f}_n = (f_{\ell,n}, \ell \in \mathbb{N})$ with the functions $f_{\ell,n}$ defined by $f_{\ell,n}(y) = s_n^{-d/2} K(s_n^{-1}(x-y)) \mathbf{1}_{\{\ell \leq n\}}$ for $y \in \mathbb{R}^d$; $N_{n,\emptyset}$ is defined in (44) with \mathbf{f} replaced by \mathbf{f}_n ; and the bias term:

$$B_n(x) = \frac{1}{|\mathbb{T}_n| s_n^{d/2}} \sum_{\ell=0}^n 2^{n-\ell} \langle \mu, f_{\ell,n} \rangle - \mu(x) = \langle \mu, s_n^{-d} K(s_n^{-1}(x-\cdot)) \rangle - \mu(x).$$

Thanks to Remark 2.21, with this choice of functions ($f_{\ell,n}, n \geq \ell \geq 0$), we have that Assumptions 2.16, 2.19 and 2.20 hold. First, since $\lim_{n \rightarrow \infty} |\mathbb{G}_n| s_n^d = \infty$ as $\gamma < 1$, we get, as a direct consequence of Theorems 3.7:

$$\lim_{n \rightarrow \infty} \frac{\sqrt{|\mathbb{G}_n|}}{|\mathbb{T}_n| s_n^{d/2}} N_{n,\emptyset}(\mathbf{f}_n) = 0 \quad \text{in probability.}$$

Next, it follows from Lemma 16.1 that $\lim_{n \rightarrow \infty} B_n(x) = 0$. By considering the functions $f_{\ell,n}(y) = s_n^{-d/2} K(s_n^{-1}(x-y)) \mathbf{1}_{\{\ell=n\}}$ for $y \in \mathbb{R}^d$, we similarly get the result for the case $\mathbb{A}_n = \mathbb{G}_n$. \square

Proof of Theorem 3.12. The sub-critical case and $\mathbb{A}_n = \mathbb{T}_n$. We keep notations from the proof of Lemma 3.10. From Theorem 3.7 and from the decomposition (206), it suffices to prove that

$$(207) \quad \lim_{n \rightarrow \infty} |\mathbb{T}_n|^{1/2} s_n^{d/2} B_n(x) = 0.$$

Using that

$$\begin{aligned} \mu(x - s_n y) - \mu(x) &= \sum_{j=1}^d (\mu(x_1 - s_n y_1, \dots, x_j - s_n y_j, x_{j+1}, \dots, x_d) \\ &\quad - \mu(x_1 - s_n y_1, \dots, x_{j-1} - s_n y_{j-1}, x_j, x_{j+1}, \dots, x_d)), \end{aligned}$$

the Taylor expansion, Assumption 3.11, we get, for some finite constant $C > 0$,

$$\begin{aligned} |\mathbb{T}_n|^{1/2} s_n^{d/2} B_n(x) &= \sqrt{|\mathbb{T}_n| s_n^d} \left| \int_{\mathbb{R}^d} s_n^{-d} K(s_n^{-1}(x - y)) \mu(y) dy - \mu(x) \right| \\ &= \sqrt{|\mathbb{T}_n| s_n^d} \left| \int_{\mathbb{R}^d} K(y) (\mu(x - s_n y) - \mu(x)) dy \right| \\ &\leq C \sqrt{|\mathbb{T}_n| s_n^d} \sum_{j=1}^d \int_{\mathbb{R}^d} K(y) \frac{(s_n |y_j|)^s}{[s]!} dy \\ &\leq C \sqrt{|\mathbb{T}_n| s_n^{2s+d}}. \end{aligned}$$

Then Equation (207) follows, since $\lim_{n \rightarrow \infty} |\mathbb{G}_n| s_n^{2s+d} = 0$. Finally, the result follows from Theorem 3.7 and (207). This ends the proof for $\mathbb{A}_n = \mathbb{T}_n$.

The sub-critical case and $\mathbb{A}_n = \mathbb{G}_n$. The proof is similar.

The critical and super-critical cases. The proof follows the same lines, using Theorem 3.7 in the critical and super-critical cases and the decomposition (206). \square

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