



HAL
open science

A counterexample to Thiagarajan's conjecture on regular event structures

Jérémie Chalopin, Victor Chepoi

► To cite this version:

Jérémie Chalopin, Victor Chepoi. A counterexample to Thiagarajan's conjecture on regular event structures. *Journal of Computer and System Sciences*, 2020, 113, pp.76-100. 10.1016/j.jcss.2020.05.001 . hal-03047160

HAL Id: hal-03047160

<https://hal.science/hal-03047160v1>

Submitted on 14 Dec 2020

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

A COUNTEREXAMPLE TO THIAGARAJAN'S CONJECTURE ON REGULAR EVENT STRUCTURES¹

JÉRÉMIE CHALOPIN AND VICTOR CHEPOI

ABSTRACT. We provide a counterexample to a conjecture by Thiagarajan (1996 and 2002) that regular event structures correspond exactly to event structures obtained as unfoldings of finite 1-safe Petri nets. The same counterexample is used to disprove a closely related conjecture by Badouel, Darondeau, and Raoult (1999) that domains of regular event structures with bounded \natural -cliques are recognizable by finite trace automata. Event structures, trace automata, and Petri nets are fundamental models in concurrency theory. There exist nice interpretations of these structures as combinatorial and geometric objects and both conjectures can be reformulated in this framework. Namely, from a graph theoretical point of view, the domains of prime event structures correspond exactly to median graphs; from a geometric point of view, these domains are in bijection with $\text{CAT}(0)$ cube complexes.

A necessary condition for both conjectures to be true is that domains of regular event structures (with bounded \natural -cliques) admit a regular nice labeling (which corresponds to a special coloring of the hyperplanes of the associated $\text{CAT}(0)$ cube complex). To disprove these conjectures, we describe a regular event domain (with bounded \natural -cliques) that does not admit a regular nice labeling. Our counterexample is derived from an example by Wise (1996 and 2007) of a nonpositively curved square complex \mathbf{X} with six squares, whose edges are colored in five colors, and whose universal cover $\tilde{\mathbf{X}}$ is a $\text{CAT}(0)$ square complex containing a particular plane with an aperiodic tiling. We prove that other counterexamples to Thiagarajan's conjecture arise from aperiodic 4-way deterministic tile sets of Kari and Papasoglu (1999) and Lukkarila (2009).

On the positive side, using breakthrough results by Agol (2013) and Haglund and Wise (2008, 2012) from geometric group theory, we prove that Thiagarajan's conjecture is true for regular event structures whose domains occur as principal filters of hyperbolic $\text{CAT}(0)$ cube complexes which are universal covers of finite nonpositively curved cube complexes.

1. INTRODUCTION

Event structures, introduced by Nielsen, Plotkin, and Winskel [34, 47, 49], are a widely recognized abstract model of concurrent computation. An event structure (or more precisely, a prime event structure or an event structure with binary conflict) is a partially ordered set of the occurrences of actions, called events, together with a conflict relation. The partial order captures the causal dependency of events. The conflict relation models incompatibility of events so that two events that are in conflict cannot simultaneously occur in any state of the computation. Consequently, two events that are neither ordered nor in conflict may occur concurrently. More formally, an event structure is a triple $\mathcal{E} = (E, \leq, \#)$, consisting of a set E of events, and two binary relations \leq and $\#$, the causal dependency \leq and the conflict relation $\#$ with the requirement that the conflict is inherited by the partial order \leq . The pairs of events not in $\leq \cup \geq \cup \#$ define the concurrency relation \parallel . The domain of an event structure consists of all computation states, called configurations. Each computation state is a subset of events subject to the constraints that no two conflicting events can occur together in the same computation and if an event occurred in a computation then all events on which it causally depends have occurred too. Therefore, the domain of an event structure \mathcal{E} is the set $\mathcal{D}(\mathcal{E})$ of all finite configurations ordered by inclusion. An event e is said to be enabled by a configuration c if $e \notin c$ and $c \cup \{e\}$ is a configuration. The degree of an event structure \mathcal{E} is the maximum number of

Key words and phrases. Regular event structures, event domains, trace labelings, median graphs, $\text{CAT}(0)$ cube complexes, universal covers, virtually special cube complexes, aperiodic tilings.

¹An extended abstract [16] of this paper appeared in the proceedings of ICALP 2017.

events enabled by a configuration of \mathcal{E} . The future (or the principal filter, or the residual) of a configuration c is the set of all finite configurations c' containing c .

Among other things, the importance of event structures stems from the fact that several fundamental models of concurrent computation lead to event structures. Nielsen, Plotkin, and Winskel [34] proved that every 1-safe Petri net N unfolds into an event structure \mathcal{E}_N . Later results of [35] and [49] show in fact that 1-safe Petri nets and event structures represent each other in a strong sense. In the same vein, Stark [42] established that the domains of configurations of trace automata are exactly the conflict event domains; a presentation of domains of event structures as trace monoids (Mazurkiewicz traces) or as asynchronous transition systems was given in [39] and [10], respectively. In both cases, the events of the resulting event structure are labeled (in the case of trace monoids and trace automata—by the letters of a possibly infinite trace alphabet $M = (\Sigma, I)$) in a such a way that any two events enabled by the same configuration are labeled differently (such a labeling is usually called a nice labeling).

To deal with *finite* 1-safe Petri nets, Thiagarajan [43, 44] introduced the notions of regular event structure and regular trace event structure. A regular event structure \mathcal{E} is an event structure with a finite number of isomorphism types of futures of configurations and finite degree. A regular trace event structure is an event structure \mathcal{E} whose events can be nicely labeled by the letters of a finite trace alphabet $M = (\Sigma, I)$ in a such a way that the labels of any two concurrent events define a pair of I and there exists only a finite number of isomorphism types of labeled futures of configurations. These definitions were motivated by the fact that the event structures \mathcal{E}_N arising from *finite* 1-safe Petri nets N are regular: Thiagarajan [43] proved that event structures of *finite* 1-safe Petri nets correspond to regular trace event structures. This lead Thiagarajan to formulate the following conjecture:

Conjecture 1.1 ([43, 44]). A prime event structure \mathcal{E} is isomorphic to the event structure \mathcal{E}_N arising from a finite 1-safe Petri net N if and only if \mathcal{E} is regular.

Badouel, Darondeau, and Raoult [5] formulated two similar conjectures about conflict event domain that are recognizable by finite trace automata. The first one is equivalent to Conjecture 1.1, while the second one is formulated in a more general setting with an extra condition. We formulate their second conjecture in the particular case of event structures:

Conjecture 1.2 ([5]). A conflict event domain is recognizable if and only if the event structure \mathcal{E} is regular and has bounded \natural -cliques.

In view of previous results, to establish Conjecture 1.1, it is necessary for a regular event structure \mathcal{E} to have a regular nice labeling with letters from some trace alphabet (Σ, I) . Nielsen and Thiagarajan [36] proved in a technically involved but very nice combinatorial way that all regular conflict-free event structures satisfy Conjecture 1.1. In an equally difficult and technical proof, Badouel et al. [5] proved that their conjectures hold for context-free event domains, i.e., for domains whose underlying graph is a context-free graph sensu Müller and Schupp [33]. In this paper, we present a counterexample to Thiagarajan’s Conjecture based on a more geometric and combinatorial view on event structures. We show that our example also provides a counterexample to Conjecture 1.2 of Badouel et al.

We use the striking bijections between the domains of event structures, median graphs, and CAT(0) cube complexes. Median graphs have many nice properties and admit numerous characterizations. They have been investigated in several contexts for more than half a century, and play a central role in metric graph theory; for more detailed information, the interested reader can consult the surveys [6, 7]. On the other hand, CAT(0) cube complexes are central objects in geometric group theory [40, 41, 52]. They have been characterized in a nice combinatorial way by Gromov [22] as simply connected cube complexes in which the links of 0-cubes are simplicial flag complexes. It was proven in [17, 38] that 1-skeleta of CAT(0) cube complexes are exactly the median graphs. Barthélemy and Constantin [9] proved that the Hasse diagrams of domains of event structures are median graphs and every pointed median graph is the domain of an event structure. The bijection between pointed median graphs and event domains established in [9] can be viewed as the classical characterization of prime event domains as prime algebraic

coherent partial orders provided by Nielsen, Plotkin, and Winskel [34]. More recently, this result was rediscovered in [3] in the language of CAT(0) cube complexes. Via these bijections, the events of an event structure \mathcal{E} correspond to the parallelism classes of edges of the domain $D(\mathcal{E})$ viewed as a median graph. We recall these bijections in Section 3.

Since in our paper we deal only with regular event structures, we need to be able to construct regular event domains from CAT(0) cube complexes. By Gromov's characterization, CAT(0) cube complexes are exactly the universal covers of cube complexes satisfying the link condition, i.e., of nonpositively curved cube (NPC) complexes. Of particular importance for us are the CAT(0) cube complexes arising as universal covers of *finite* NPC complexes. In Section 4, we adapt the universal cover construction to directed NPC complexes (Y, o) and show that every principal filter of the directed universal cover (\tilde{Y}, \tilde{o}) is the domain of an event structure. Furthermore, we show that if the NPC complex Y is finite, then this event structure is regular. Motivated by this result, we call an event structure *strongly regular* if its domain is the principal filter of the directed universal cover (\tilde{Y}, \tilde{o}) of a finite directed NPC complex (Y, o) .

Our counterexample to Conjectures 1.1 and 1.2 is a strongly regular event structure derived from Wise's [50,51] nonpositively curved square complex \mathbf{X} obtained from a tile set with six tiles. This counterexample is described in Section 6. In Section 7 we also prove that other counterexamples to Thiagarajan's conjecture arise in a similar way from any aperiodic 4-deterministic tile set, such as the ones constructed by Kari and Papasoglu [29] and Lukkarila [30].

On the positive side, in Section 5 we prove that Thiagarajan's conjecture is true for event structures whose domains arise as principal filters of universal covers of finite special cube complexes. Haglund and Wise [25,26] detected pathologies which may occur in NPC complexes: self-intersecting hyperplanes, one-sided hyperplanes, directly self-osculating hyperplanes, and pairs of hyperplanes, which both intersect and osculate. They called the NPC complexes without such pathologies *special*. The main motivation for introducing and studying special cube complexes was the profound idea of Wise that the famous virtual Haken conjecture for hyperbolic 3-manifolds can be reduced to solving problems about special cube complexes. In a breakthrough result, Agol [1,2] completed this program and solved the virtual Haken conjecture using the deep theory of special cube complexes developed by Haglund and Wise [25,26]. The main ingredient in this proof is Agol's theorem that finite NPC complexes whose universal covers are hyperbolic are virtually special (i.e., they admit finite covers which are special cube complexes). Using this result of Agol, we can specify our previous result and show that Thiagarajan's conjecture is true for strongly regular event structures whose domains occur as principal filters of hyperbolic CAT(0) cube complexes that are universal covers of finite directed NPC complexes. Since context-free domains are hyperbolic, this result can be viewed in some sense as a partial generalization of the result of Badouel et al. [5].

To conclude this introductory section, we briefly describe the construction of our counterexample to Thiagarajan's conjecture. It is based on Wise's [50,51] directed nonpositively curved square complex \mathbf{X} with one vertex and six squares, whose edges are colored in five colors, and whose colored universal cover $\tilde{\mathbf{X}}$ contains a particular directed plane with an aperiodic tiling. The edges of \mathbf{X} are partitioned into two classes (horizontal and vertical edges) and opposite edges of squares are oriented in the same way. As a result, $\tilde{\mathbf{X}}$ is a directed CAT(0) square complex whose edges are colored by the colors of their images in \mathbf{X} and are directed in such a way that all edges dual to the same hyperplane are oriented in the same way. With respect to this orientation, all vertices of $\tilde{\mathbf{X}}$ are equivalent up to automorphism. We modify the complex \mathbf{X} by taking its first barycentric subdivision and by adding to the middles of the edges of \mathbf{X} directed paths of five different lengths (tips) in order to encode the colors of the edges of \mathbf{X} (and $\tilde{\mathbf{X}}$) and to obtain a directed nonpositively curved square complex W . The universal cover \tilde{W} of W is a directed (but no longer colored) CAT(0) square complex. \tilde{W} can be viewed as the first barycentric subdivision of the support \tilde{X} of $\tilde{\mathbf{X}}$ in which to each vertex arising from a middle of an edge of $\tilde{\mathbf{X}}$ a tip encoding the color of the original edge is added. Since \tilde{W} is the universal cover of a finite complex W , \tilde{W} has a finite number of equivalence classes of vertices up to automorphism. From \tilde{W} we derive a domain of a regular event structure $\tilde{W}_{\tilde{\tau}}$ by considering the

future of an arbitrary vertex \tilde{v} of $\tilde{\mathbf{X}}$. Using the fact that $\tilde{\mathbf{X}}$ contains a particular directed plane with an aperiodic tiling, we prove that $\tilde{W}_{\tilde{v}}$ does not admit a regular nice labeling, thus $\tilde{W}_{\tilde{v}}$ is the domain of a regular event structure not having a regular trace labeling.

2. EVENT STRUCTURES

2.1. Event structures and domains. An *event structure* is a triple $\mathcal{E} = (E, \leq, \#)$, where

- E is a set of *events*,
- $\leq \subseteq E \times E$ is a partial order of *causal dependency*,
- $\# \subseteq E \times E$ is a binary, irreflexive, symmetric relation of *conflict*,
- $\downarrow e := \{e' \in E : e' \leq e\}$ is finite for any $e \in E$,
- $e \# e'$ and $e' \leq e''$ imply $e \# e''$.

What we call here an event structure is usually called a *coherent event structure*, an *event structure with a binary conflict*, or a *prime event structure*. Two events e', e'' are *concurrent* (notation $e' \parallel e''$) if they are order-incomparable and they are not in conflict. The conflict $e' \# e''$ between two elements e' and e'' is said to be *minimal* (notation $e' \#_{\mu} e''$) if there is no event $e \neq e', e''$ such that either $e \leq e'$ and $e \# e''$ or $e \leq e''$ and $e \# e'$. We say that e is an *immediate predecessor* of e' (notation $e \triangleleft e'$) if and only if $e \leq e', e \neq e'$, and for every e'' if $e \leq e'' \leq e'$, then $e'' = e$ or $e'' = e'$.

Given two event structures $\mathcal{E}_1 = (E_1, \leq_1, \#_1)$ and $\mathcal{E}_2 = (E_2, \leq_2, \#_2)$, a map $f : E_1 \rightarrow E_2$ is an isomorphism if f is a bijection such that $e \leq_1 e'$ iff $f(e) \leq_2 f(e')$ and $e \#_1 e'$ iff $f(e) \#_2 f(e')$ for every $e, e' \in E_1$. If such an isomorphism exists, then \mathcal{E}_1 and \mathcal{E}_2 are said to be isomorphic; notation $\mathcal{E}_1 \cong \mathcal{E}_2$.

A *configuration* of an event structure $\mathcal{E} = (E, \leq, \#)$ is any finite subset $c \subseteq E$ of events which is *conflict-free* ($e, e' \in c$ implies that e, e' are not in conflict) and *downward-closed* ($e \in c$ and $e' \leq e$ implies that $e' \in c$) [49]. Notice that \emptyset is always a configuration and that $\downarrow e$ and $\downarrow e \setminus \{e\}$ are configurations for any $e \in E$. The *domain* of an event structure is the set $\mathcal{D} := \mathcal{D}(\mathcal{E})$ of all configurations of \mathcal{E} ordered by inclusion; (c', c) is a (directed) edge of the Hasse diagram of the poset $(\mathcal{D}(\mathcal{E}), \subseteq)$ if and only if $c = c' \cup \{e\}$ for an event $e \in E \setminus c$. An event e is said to be *enabled* by a configuration c if $e \notin c$ and $c \cup \{e\}$ is a configuration. Denote by $en(c)$ the set of all events enabled at the configuration c . Two events are called *co-initial* if they are both enabled at some configuration c . Note that if e and e' are co-initial, then either $e \#_{\mu} e'$ or $e \parallel e'$. It is easy to see that two events e and e' are in minimal conflict $e \#_{\mu} e'$ if and only if $e \# e'$ and e and e' are co-initial. The *degree* $\deg(\mathcal{E})$ of an event structure \mathcal{E} is the least positive integer d such that $|en(c)| \leq d$ for any configuration c of \mathcal{E} . We say that \mathcal{E} has *finite degree* if $\deg(\mathcal{E})$ is finite. The *future* (or the *(principal) filter*) $\mathcal{F}(c)$ of a configuration c is the set of all configurations c' containing c : $\mathcal{F}(c) = \uparrow c := \{c' \in \mathcal{D}(\mathcal{E}) : c \subseteq c'\}$, i.e., $\mathcal{F}(c)$ is the principal filter of c in the ordered set $(\mathcal{D}(\mathcal{E}), \subseteq)$.

For an event structure $\mathcal{E} = (E, \leq, \#)$, let \natural be the least irreflexive and symmetric relation on the set of events E such that $e_1 \natural e_2$ if (1) $e_1 \parallel e_2$, or (2) $e_1 \#_{\mu} e_2$, or (3) there exists an event e_3 that is co-initial with e_1 and e_2 at two different configurations such that $e_1 \parallel e_3$ and $e_2 \#_{\mu} e_3$ (see Figure 1 for examples). (If $e_1 \natural e_2$ and this comes from condition (3), then we write $e_1 \natural_{(3)} e_2$.) A \natural -*clique* is any complete subgraph of the graph whose vertices are the events and whose edges are the pairs of events $e_1 e_2$ such that $e_1 \natural e_2$.

A *labeled event structure* $\mathcal{E}^{\lambda} = (\mathcal{E}, \lambda)$ is defined by an *underlying event structure* $\mathcal{E} = (E, \leq, \#)$ and a *labeling* λ that is a map from E to some alphabet Σ . Two labeled event structures $\mathcal{E}_1^{\lambda_1} = (\mathcal{E}_1, \lambda_1)$ and $\mathcal{E}_2^{\lambda_2} = (\mathcal{E}_2, \lambda_2)$ are isomorphic (notation $\mathcal{E}_1^{\lambda_1} \cong \mathcal{E}_2^{\lambda_2}$) if there exists an isomorphism f between the underlying event structures \mathcal{E}_1 and \mathcal{E}_2 such that $\lambda_2(f(e_1)) = \lambda_1(e_1)$ for every $e_1 \in E_1$.

A labeling $\lambda : E \rightarrow \Sigma$ of an event structure \mathcal{E} defines naturally a labeling of the directed edges of the Hasse diagram of its domain $\mathcal{D}(\mathcal{E})$ that we also denote by λ . A labeling $\lambda : E \rightarrow \Sigma$ of an event structure \mathcal{E} is called a *nice labeling* if any two events that are co-initial have different labels [39]. A nice labeling of \mathcal{E} can be reformulated as a labeling of the directed edges of the Hasse diagram of its domain $\mathcal{D}(\mathcal{E})$ subject to the following local conditions:

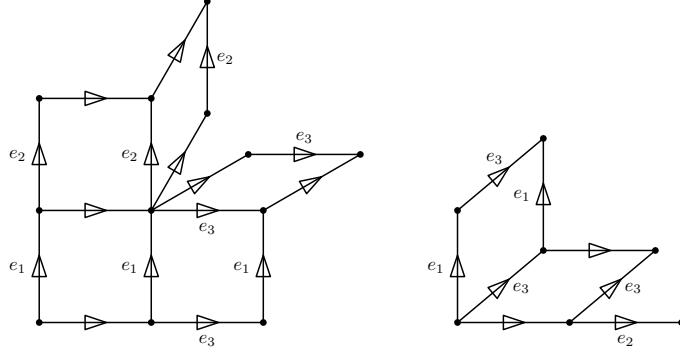


FIGURE 1. Two examples where $e_1 \not\leq e_2$: $e_1 \parallel e_3$ and $e_2 \#_{\mu} e_3$

Determinism: the edges outgoing from the same vertex of $\mathcal{D}(\mathcal{E})$ have different labels;
Concurrency: the opposite edges of each square of $\mathcal{D}(\mathcal{E})$ are labeled with the same labels.

In the following, we use interchangeably the labeling of an event structure and the labeling of the edges of its domain.

2.2. Regular event structures. In this subsection, we recall the definitions of regular event structures, regular trace event structures, and regular nice labelings of event structures. We closely follow the definitions and notations of [36, 43, 44]. Let $\mathcal{E} = (E, \leq, \#)$ be an event structure. Let c be a configuration of \mathcal{E} . Set $\#(c) = \{e' : \exists e \in c, e \#_{\mu} e'\}$. The *event structure rooted at c* is defined to be the triple $\mathcal{E} \setminus c = (E', \leq', \#')$, where $E' = E \setminus (c \cup \#(c))$, \leq' is \leq restricted to $E' \times E'$, and $\#'$ is $\#$ restricted to $E' \times E'$. It can be easily seen that the domain $\mathcal{D}(\mathcal{E} \setminus c)$ of the event structure $\mathcal{E} \setminus c$ is isomorphic to the principal filter $\mathcal{F}(c)$ of c in $\mathcal{D}(\mathcal{E})$ such that any configuration c' of $\mathcal{D}(\mathcal{E})$ corresponds to the configuration $c' \setminus c$ of $\mathcal{D}(\mathcal{E} \setminus c)$.

For an event structure $\mathcal{E} = (E, \leq, \#)$, define the equivalence relation $R_{\mathcal{E}}$ on its configurations in the following way: for two configurations c and c' set $c R_{\mathcal{E}} c'$ if and only if $\mathcal{E} \setminus c \equiv \mathcal{E} \setminus c'$. The *index* of an event structure \mathcal{E} is the number of equivalence classes of $R_{\mathcal{E}}$, i.e., the number of isomorphism types of futures of configurations of \mathcal{E} . The event structure \mathcal{E} is *regular* [36, 43, 44] if \mathcal{E} has finite index and finite degree.

Now, let $\mathcal{E}^{\lambda} = (\mathcal{E}, \lambda)$ be a labeled event structure. For any configuration c of \mathcal{E} , if we restrict λ to $\mathcal{E} \setminus c$, then we obtain a labeled event structure $(\mathcal{E} \setminus c, \lambda)$ denoted by $\mathcal{E}^{\lambda} \setminus c$. Analogously, define the equivalence relation $R_{\mathcal{E}^{\lambda}}$ on its configurations by setting $c R_{\mathcal{E}^{\lambda}} c'$ if and only if $\mathcal{E}^{\lambda} \setminus c \equiv \mathcal{E}^{\lambda} \setminus c'$. The index of \mathcal{E}^{λ} is the number of equivalence classes of $R_{\mathcal{E}^{\lambda}}$. We say that an event structure \mathcal{E} admits a *regular nice labeling* if there exists a nice labeling λ of \mathcal{E} with a finite alphabet Σ such that \mathcal{E}^{λ} has finite index.

We continue by recalling the definition of regular trace event structures from [43, 44]. A (*Mazurkiewicz*) *trace alphabet* is a pair $M = (\Sigma, I)$, where Σ is a finite non-empty alphabet set and $I \subset \Sigma \times \Sigma$ is an irreflexive and symmetric relation called the *independence relation*. As usual, Σ^* is the set of finite words with letters in Σ . The independence relation I induces the equivalence relation \sim_I , which is the reflexive and transitive closure of the binary relation \leftrightarrow_I : if $\sigma, \sigma' \in \Sigma^*$ and $(a, b) \in I$, then $\sigma ab \sigma' \leftrightarrow_I \sigma ba \sigma'$. The relation $D := (\Sigma \times \Sigma) \setminus I$ is called the *dependence relation*.

An *M-labeled event structure* is a labeled event structure $\mathcal{E}^{\phi} = (\mathcal{E}, \lambda)$, where $\mathcal{E} = (E, \leq, \#)$ is an event structure and $\lambda : E \rightarrow \Sigma$ is a labeling function which satisfies the following conditions:

- (LES1) $e \#_{\mu} e'$ implies $\lambda(e) \neq \lambda(e')$,
- (LES2) if $e \leq e'$ or $e \#_{\mu} e'$, then $(\lambda(e), \lambda(e')) \in D$,
- (LES3) if $(\lambda(e), \lambda(e')) \in D$, then $e \leq e'$ or $e' \leq e$ or $e \#_{\mu} e'$.

We call λ a *trace labeling* of \mathcal{E} with the trace alphabet (Σ, I) . The conditions (LES2) and (LES3) on the labeling function ensures that the concurrency relation \parallel of \mathcal{E} respects the independence relation I of M . In particular, since I is irreflexive, from (LES3) it follows that any two

concurrent events are labeled differently. Since by (LES1) two events in minimal conflict are also labeled differently, this implies that λ is a finite nice labeling of \mathcal{E} .

An M -labeled event structure $\mathcal{E}^\lambda = (\mathcal{E}, \lambda)$ is *regular* if \mathcal{E}^λ has finite index. Finally, an event structure \mathcal{E} is called a *regular trace event structure* [43, 44] iff there exists a trace alphabet $M = (\Sigma, I)$ and a regular M -labeled event structure \mathcal{E}^λ such that \mathcal{E} is isomorphic to the underlying event structure of \mathcal{E}^λ . From the definition immediately follows that every regular trace event structure is also a regular event structure. It turns out that the converse is equivalent to Conjecture 1.1. Namely, [44] establishes the following equivalence (this result dispenses us from giving a formal definition of 1-safe Petri nets; the interested readers can find it in the papers [36, 44]):

Theorem 2.1 ([44, Theorem 1]). *\mathcal{E} is a regular trace event structure if and only if there exists a finite 1-safe Petri net N such that \mathcal{E} and \mathcal{E}_N are isomorphic.*

In view of this theorem, Conjecture 1.1 is equivalent to the following conjecture:

Conjecture 2.2. *\mathcal{E} is a regular event structure if and only if \mathcal{E} is a regular trace event structure.*

Badouel et al. [5] considered recognizable conflict event domains that are more general than the domains of event structures we consider in this paper. Since the domain of an event structure \mathcal{E} is recognizable if and only if \mathcal{E} is a regular trace event structure (see [31, Section 5]), Conjecture 1.2 can be reformulated as follows:

Conjecture 2.3. *\mathcal{E} is a regular event structure iff \mathcal{E} is a regular trace event structure and \mathcal{E} has bounded \natural -cliques.*

Since any regular trace labeling is a regular nice labeling, any regular event structure \mathcal{E} not admitting a regular nice labeling is a counterexample to Conjecture 2.2 (and thus to Conjecture 1.1). If, additionally, \mathcal{E} has bounded \natural -cliques, \mathcal{E} is also a counterexample to Conjecture 2.3 (and thus to Conjecture 1.2).

3. DOMAINS, MEDIAN GRAPHS, AND CAT(0) CUBE COMPLEXES

In this section, we recall the bijections between domains of event structures and median graphs/CAT(0) cube complexes established in [3] and [9], and between median graphs and 1-skeleta of CAT(0) cube complexes established in [17] and [38].

3.1. Median graphs. Let $G = (V, E)$ be a simple, connected, not necessarily finite graph. The *distance* $d_G(u, v)$ between two vertices u and v is the length of a shortest (u, v) -path, and the *interval* $I(u, v)$ between u and v consists of all vertices on shortest (u, v) -paths, that is, of all vertices (metrically) *between* u and v :

$$I(u, v) := \{x \in V : d_G(u, x) + d_G(x, v) = d_G(u, v)\}.$$

An induced subgraph of G (or the corresponding vertex set) is called *convex* if it includes the interval of G between any of its vertices. A graph $G = (V, E)$ is *isometrically embeddable* into a graph $H = (W, F)$ if there exists a mapping $\varphi : V \rightarrow W$ such that $d_H(\varphi(u), \varphi(v)) = d_G(u, v)$ for all vertices $u, v \in V$.

A graph G is called *median* if the interval intersection $I(x, y) \cap I(y, z) \cap I(z, x)$ is a singleton for each triplet x, y, z of vertices. Median graphs are bipartite. Basic examples of median graphs are trees, hypercubes, rectangular grids, and Hasse diagrams of distributive lattices and of median semilattices [6]. With any vertex v of a median graph $G = (V, E)$ is associated a *canonical partial order* \leq_v defined by setting $x \leq_v y$ if and only if $x \in I(v, y)$; v is called the *basepoint* of \leq_v . Since G is bipartite, the Hasse diagram G_v of the partial order (V, \leq_v) is the graph G in which any edge xy is directed from x to y if and only if the inequality $d_G(x, v) < d_G(y, v)$ holds. We call G_v a *pointed median graph*. There is a close relationship between pointed median graphs and median semilattices. A *median semilattice* is a meet semilattice (P, \leq) such that (i) for every x , the *principal ideal* $\downarrow x = \{p \in P : p \leq x\}$ is a distributive lattice, and (ii) any three elements have a least upper bound in P whenever each pair of them does.

Theorem 3.1 ([4]). *The Hasse diagram of any median semilattice is a median graph. Conversely, every median graph defines a median semilattice with respect to any canonical order \leq_v .*

Median graphs can be obtained from hypercubes by amalgams and median graphs are themselves isometric subgraphs of hypercubes [8,32]. The canonical isometric embedding of a median graph G into a (smallest) hypercube can be determined by the so called *Djoković-Winkler* (“parallelism”) relation Θ on the edges of G [21,46]. For median graphs, the equivalence relation Θ can be defined as follows. First say that two edges uv and xy are in relation Θ' if they are opposite edges of a 4-cycle $uvxy$ in G . Then let Θ be the reflexive and transitive closure of Θ' . Any equivalence class of Θ constitutes a cutset of the median graph G , which determines one factor of the canonical hypercube [32]. The cutset (equivalence class) $\Theta(xy)$ containing an edge xy defines a convex split $\{W(x,y), W(y,x)\}$ of G [32], where $W(x,y) = \{z \in V : d_G(z,x) < d_G(z,y)\}$ and $W(y,x) = V \setminus W(x,y)$ (we call the complementary convex sets $W(x,y)$ and $W(y,x)$ *half-spaces*). Conversely, for every convex split of a median graph G there exists at least one edge xy such that $\{W(x,y), W(y,x)\}$ is the given split. We denote by $\{\Theta_i : i \in I\}$ the equivalence classes of the relation Θ (in [9], they were called parallelism classes). For an equivalence class $\Theta_i, i \in I$, we denote by $\{A_i, B_i\}$ the associated convex split. We say that Θ_i *separates* the vertices x and y if $x \in A_i, y \in B_i$ or $x \in B_i, y \in A_i$. The isometric embedding φ of G into a hypercube is obtained by taking a basepoint v , setting $\varphi(v) = \emptyset$ and for any other vertex u , letting $\varphi(u)$ be all parallelism classes of Θ which separate u from v .

We conclude this subsection with the following simple but useful local characterization of convex sets of median graphs (which holds for much more general classes of graphs):

Lemma 3.2. *Let S be a connected subgraph of a median graph G . Then S is a convex subgraph if and only if S is locally-convex, i.e., $I(x,y) \subseteq S$ for any two vertices x, y of S having a common neighbor in S .*

3.2. Nonpositively curved cube complexes. A 0-cube is a single point. A 1-cube is an isometric copy of the segment $[-1,1]$ and has a cell structure consisting of 0-cells $\{\pm 1\}$ and a single 1-cell. An n -cube is an isometric copy of $[-1,1]^n$, and has the product structure, so that each closed cell of $[-1,1]^n$ is obtained by restricting some of the coordinates to $+1$ and some to -1 . A *cube complex* is obtained from a collection of cubes of various dimensions by isometrically identifying certain subcubes. The *dimension* of a cube complex X is the largest value of d for which X contains a d -cube. A *square complex* is a cube complex of dimension 2. The 0-cubes and the 1-cubes of a cube complex X are called *vertices* and *edges* of X and define the graph $X^{(1)}$, the 1-skeleton of X . We denote the vertices of $X^{(1)}$ by $V(X)$ and the edges of $X^{(1)}$ by $E(X)$. For $i \in \mathbb{N}$, we denote by $X^{(i)}$ the i -skeleton of X , i.e., the cube complex consisting of all j -dimensional cubes of X , where $j \leq i$. A *square complex* X is a combinatorial 2-complex whose 2-cells are attached by closed combinatorial paths of length 4. Thus, one can consider each 2-cell as a square attached to the 1-skeleton $X^{(1)}$ of X . The *star* $\text{St}(v, X)$ of a vertex v of X is the subcomplex spanned by all cubes containing v . The *link* of a vertex $x \in X$ is the simplicial complex $\text{Link}(x)$ with a $(d-1)$ -simplex for each d -cube containing x , with simplices attached according to the attachments of the corresponding cubes. The link $\text{Link}(x)$ is said to be a *flag (simplicial) complex* if each $(d+1)$ -clique in $\text{Link}(x)$ spans a d -simplex. This flagness condition of $\text{Link}(x)$ can be restated as follows: whenever three $(k+2)$ -cubes of X share a common k -cube containing x and pairwise share common $(k+1)$ -cubes, then they are contained in a $(k+3)$ -cube of X . A cube complex X is called *simply connected* if it is connected and if every continuous mapping of the 1-dimensional sphere S^1 into X can be extended to a continuous mapping of the disk D^2 with boundary S^1 into X . Note that X is connected iff $G(X) = X^{(1)}$ is connected, and X is simply connected iff $X^{(2)}$ is simply connected. Equivalently, a cube complex X is simply connected if X is connected and every cycle C of its 1-skeleton is null-homotopic, i.e., it can be contracted to a single point by elementary homotopies.

Given two cube complexes X and Y , a *covering (map)* is a surjection $\varphi: Y \rightarrow X$ mapping cubes to cubes and such that $\varphi|_{\text{St}(v,Y)}: \text{St}(v,Y) \rightarrow \text{St}(\varphi(v), X)$ is an isomorphism for every vertex v in Y . The space Y is then called a *covering space* of X . A *universal cover* of X is a

simply connected covering space; it always exists and it is unique up to isomorphism [27, Sections 1.3 and 4.1]. The universal cover of a complex X will be denoted by \tilde{X} . In particular, if X is simply connected, then its universal cover \tilde{X} is X itself.

An important class of cube complexes studied in geometric group theory and combinatorics is the class of nonpositively curved and $\text{CAT}(0)$ cube complexes. We continue by recalling the definition of $\text{CAT}(0)$ spaces. A *geodesic triangle* $\Delta = \Delta(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three points in X (the vertices of Δ) and a geodesic between each pair of vertices (the sides of Δ). A *comparison triangle* for $\Delta(x_1, x_2, x_3)$ is a triangle $\Delta(x'_1, x'_2, x'_3)$ in the Euclidean plane \mathbb{E}^2 such that $d_{\mathbb{E}^2}(x'_i, x'_j) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$. A geodesic metric space (X, d) is defined to be a *CAT(0) space* [22] if all geodesic triangles $\Delta(x_1, x_2, x_3)$ of X satisfy the comparison axiom of Cartan–Alexandrov–Toponogov: *If y is a point on the side of $\Delta(x_1, x_2, x_3)$ with vertices x_1 and x_2 and y' is the unique point on the line segment $[x'_1, x'_2]$ of the comparison triangle $\Delta(x'_1, x'_2, x'_3)$ such that $d_{\mathbb{E}^2}(x'_i, y') = d(x_i, y)$ for $i = 1, 2$, then $d(x_3, y) \leq d_{\mathbb{E}^2}(x'_3, y')$.* A geodesic metric space (X, d) is *nonpositively curved* if it is locally $\text{CAT}(0)$, i.e., any point has a neighborhood inside which the $\text{CAT}(0)$ inequality holds. $\text{CAT}(0)$ spaces can be characterized in several different natural ways and have many strong properties, see for example [14]. In particular, a geodesic metric space (X, d) is $\text{CAT}(0)$ if and only if (X, d) is simply connected and is nonpositively curved. Gromov [22] gave a beautiful combinatorial characterization of $\text{CAT}(0)$ cube complexes, which can be also taken as their definition:

Theorem 3.3 ([22]). *A cube complex X endowed with the ℓ_2 -metric is $\text{CAT}(0)$ if and only if X is simply connected and the links of all vertices of X are flag complexes. If Y is a cube complex in which the links of all vertices are flag complexes, then the universal cover \tilde{Y} of Y is a $\text{CAT}(0)$ cube complex.*

In view of the second assertion of Theorem 3.3, the cube complexes in which the links of vertices are flag complexes are called *nonpositively curved cube complexes* or shortly *NPC complexes*. As a corollary of Gromov’s result, *for any NPC complex X , its universal cover \tilde{X} is $\text{CAT}(0)$.*

A square complex X is a *VH-complex* (*vertical-horizontal complex*) if the 1-cells (edges) of X are partitioned into two sets V and H called *vertical* and *horizontal* edges respectively, and the edges in each square alternate between edges in V and H . Notice that if X is a *VH-complex*, then X satisfies the Gromov’s nonpositive curvature condition since no three squares may pairwise intersect on three edges with a common vertex, thus *VH-complexes* are particular NPC square complexes. A *VH-complex* X is a *complete square complex* (CSC) [51] if any vertical edge and any horizontal edge incident to a common vertex belong to a common square of X . By [51, Theorem 3.8], if X is a complete square complex, then the universal cover \tilde{X} of X is isomorphic to the Cartesian product of two trees. By a *plane* Π in \tilde{X} we will mean a convex subcomplex of \tilde{X} isometric to \mathbb{R}^2 tiled by the grid \mathbb{Z}^2 into unit squares.

We continue with the bijection between $\text{CAT}(0)$ cube complexes and median graphs:

Theorem 3.4 ([17, 38]). *Median graphs are exactly the 1-skeleta of $\text{CAT}(0)$ cube complexes.*

The proof of Theorem 3.4 presented in [17] is based on the following local-to-global characterization of median graphs:

Theorem 3.5 ([17]). *A graph G is a median graph if and only if its cube complex is simply connected and G satisfies the 3-cube condition: if three squares of G pairwise intersect in an edge and all three intersect in a vertex, then they belong to a 3-cube.*

A *midcube* of the d -cube c , with $d \geq 1$, is the isometric subspace obtained by restricting exactly one of the coordinates of d to 0. Note that a midcube is a $(d - 1)$ -cube. The midcubes a and b of X are *adjacent* if they have a common face, and a *hyperplane* H of X is a subspace that is a maximal connected union of midcubes such that, if $a, b \subset H$ are midcubes, either a and b are disjoint or they are adjacent. Equivalently, a hyperplane H is a maximal connected union of midcubes such that, for each cube c , either $H \cap c = \emptyset$ or $H \cap c$ is a single midcube of c .

Theorem 3.6 ([40]). *Each hyperplane H of a $CAT(0)$ cube complex X is a $CAT(0)$ cube complex of dimension at most $\dim X - 1$ and $X \setminus H$ consists of exactly two components, called halfspaces.*

A 1-cube e (an edge) is *dual* to the hyperplane H if the 0-cubes of e lie in distinct halfspaces of $X \setminus H$, i.e., if the midpoint of e is in a midcube contained in H . The relation “dual to the same hyperplane” is an equivalence relation on the set of edges of X ; denote this relation by Θ and denote by $\Theta(H)$ the equivalence class consisting of 1-cubes dual to the hyperplane H (Θ is precisely the parallelism relation on the edges of the median graph $X^{(1)}$).

3.3. Domains versus median graphs/ $CAT(0)$ cube complexes. Theorems 2.2 and 2.3 of Barthélemy and Constantin [9] establish the following bijection between event structures and pointed median graphs (in [9], event structures are called sites):

Theorem 3.7 ([9]). *The (undirected) Hasse diagram of the domain $(\mathcal{D}(\mathcal{E}), \subseteq)$ of any event structure $\mathcal{E} = (E, \leq, \#)$ is a median graph. Conversely, for any median graph G and any basepoint v of G , the pointed median graph G_v is isomorphic to the Hasse diagram of the domain of an event structure.*

The first part of this theorem first establishes that each event domain is a median semilattice (in fact, the conditions (i) and (ii) of a median semilattice are often taken as the definition of a domain, see for example, [5, 48]) and follows from Avann’s Theorem 3.1. The bijection between domains of event structures and median semilattices is equivalent to the bijection between domains of event structures and prime algebraic coherent partial orders established in [34]. With the help of Theorem 3.5, we can provide an alternative proof of the first part of Theorem 3.7, which we hope can be of independent interest. Since we will use it further, we also recall the construction of an event structure from a pointed median graph presented in [9].

Proof of Theorem 3.7. To prove that the square complex of an event domain $\mathcal{D} := \mathcal{D}(\mathcal{E})$ is simply connected one has to show that any cycle σ of the Hasse diagram of \mathcal{D} is 0-homotopic. We proceed by lexicographic induction on the pair $(n_1(\sigma), n_2(\sigma))$, where $n_1(\sigma)$ is the maximum cardinality of a configuration of σ and $n_2(\sigma)$ is the number of configurations (vertices) of σ of size $n_1(\sigma)$. Let c be a configuration of σ of maximum size $n_1(\sigma)$. Then the neighbors c', c'' of c in σ have cardinality $n_1(\sigma) - 1$, say $c' = c \setminus \{e'\}$ and $c'' = c \setminus \{e''\}$. If $e' = e''$, then let σ' be the obtained from σ by removing c . If $e' \neq e''$, then the set $c_0 := c \setminus \{e', e''\}$ is conflict-free and downward closed, thus c_0 is a configuration. As a result, the configurations c, c', c_0, c'' define a square. In this case, let σ' be the cycle obtained from σ by replacing c by c_0 . Note that there is an elementary homotopy from σ to σ' via the square $cc'c_0c''$. In both cases, if $n_2(\sigma) > 1$, then $n_1(\sigma') = n_1(\sigma)$ and $n_2(\sigma') = n_2(\sigma) - 1$. If $n_2(\sigma) = 1$, then $n_1(\sigma') = n_1(\sigma) - 1$. In both cases, by induction hypothesis we may assume that σ' is 0-homotopic. Since there exists an elementary homotopy from σ to σ' in both cases, the cycle σ is also 0-homotopic. To show that the graph of \mathcal{D} satisfies the 3-cube condition, one can see that there exist four possible embeddings of the three squares in \mathcal{D} . In each of these cases one can directly conclude that the vertex v completing them to a 3-cube must be a configuration (see Figure 2). Indeed, in the first three cases, the set $c(v)$ of events corresponding to this vertex is included in a configuration, thus it is conflict-free. It can be also easily seen that in all three cases $c(v)$ is downward-closed, i.e., $c(v)$ is a configuration. In the last case, $c(v) = \sigma \cup \{e_1, e_2, e_3\}$. Each pair of events of $c(v)$ is contained in one of the configurations $\sigma \cup \{e_i, e_j\}, i, j \in \{1, 2, 3\}, i \neq j$, whence $c(v)$ is conflict-free. Pick any $e \in c(v)$. If $e \in \sigma$, then $\downarrow e \subset \sigma$. If $e \in \{e_1, e_2, e_3\}$, say $e = e_1$, then $\downarrow e \subset \sigma \cup \{e_1\}$. In both cases we conclude that $\downarrow e \subset c(v)$, i.e., $c(v)$ is downward-closed, whence $c(v)$ is a configuration.

Now, we recall how to define the event structure occurring in the second part of the theorem. Suppose that v is an arbitrary but fixed basepoint of a median graph G . For an equivalence class $\Theta_i, i \in I$, we denote by $\{A_i, B_i\}$ the associated convex split, and suppose without loss of generality that $v \in A_i$. Two equivalence classes Θ_i and Θ_j are said to be *crossing* if there exists a 4-cycle C of G with two opposite edges in Θ_i and two other opposite edges in Θ_j (Θ_i and

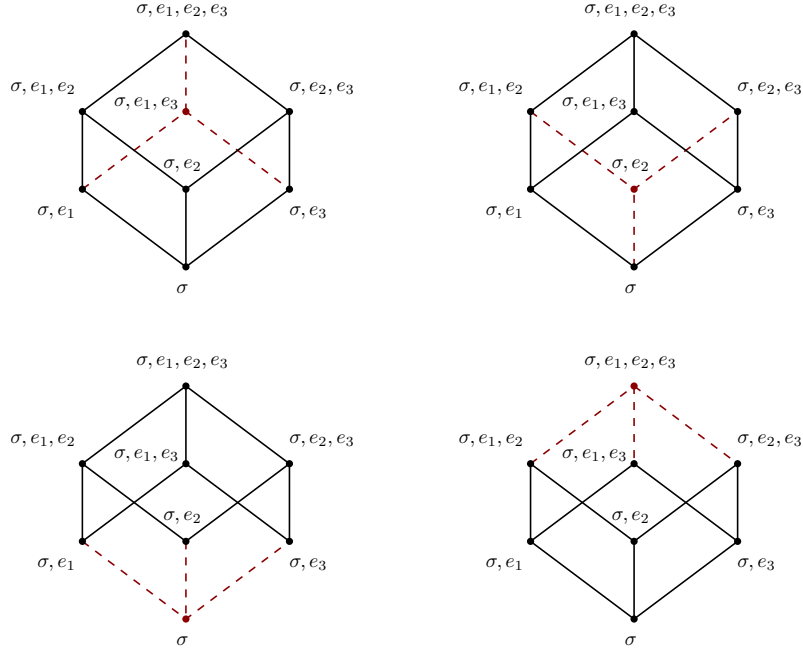


FIGURE 2. The four possible embeddings of the three squares in the 3-cube condition in \mathcal{D} .

Θ_j are called *non-crossing* otherwise). An equivalence class Θ_i *separates* the basepoint v from the equivalence class Θ_j if Θ_i and Θ_j are non-crossing and all edges of Θ_j belong to B_i . The event structure $\mathcal{E}_v = (E, \leq, \#)$ associated with a pointed median graph G_v is defined in the following way. The set E of events is the set $\{\Theta_i : i \in I\}$ of the equivalence classes of Θ . The causal dependency is defined by setting $\Theta_i \leq \Theta_j$ if and only if $\Theta_i = \Theta_j$ or Θ_i separates v from Θ_j . The conflict relation is defined by setting $\Theta_i \# \Theta_j$ if and only if Θ_i and Θ_j are non-crossing, Θ_i does not separate v from Θ_j and Θ_j does not separate v from Θ_i . Finally, the concurrency relation is defined by setting $\Theta_i \parallel \Theta_j$ if and only if Θ_i and Θ_j are crossing. Since each parallelism class Θ_i partitions G into two parts A_i and B_i , it easily follows that $\mathcal{E}_v = (E, \leq, \#)$ satisfies the axiom $\Theta_i \# \Theta_j$ and $\Theta_j \leq \Theta_k$ imply $\Theta_i \# \Theta_k$; consequently \mathcal{E}_v is an event structure. To prove that G_v is the Hasse diagram of the domain $\mathcal{D}(\mathcal{E}_v)$ of the event structure \mathcal{E}_v , consider an isometric embedding of G into a hypercube such that v corresponds to \emptyset . Then any other vertex u of G is encoded by a finite set U consisting of all Θ_i such that Θ_i separates the vertices v and u . Since $\Theta_i \in U$ and $\Theta_j \leq \Theta_i$ implies that Θ_j also separates v from u , and thus Θ_j belongs to U , we conclude that U is downward-closed. If $\Theta_i \# \Theta_j$ and $\Theta_i \in U$, then necessarily Θ_i and v belong to a common halfspace defined by Θ_j . Therefore Θ_j does not separate u from v . This shows that U is conflict-free, i.e., U is a configuration of \mathcal{E}_v . Conversely, any configuration c of \mathcal{E}_v consists of exactly those Θ_i that separate v from the vertex representing c . This concludes the proof of Theorem 3.7. \square

Rephrasing the construction of an event structure from a pointed median graph presented in the proof of Theorem 3.7, to each CAT(0) cube complex X and each vertex v of X one can associate an event structure \mathcal{E}_v such that the domain of \mathcal{E}_v is the 1-skeleton of X pointed at v . The events of \mathcal{E}_v are the hyperplanes of X . Hyperplanes H and H' define concurrent events if and only if they cross, and $H \leq H'$ if and only if $H = H'$ or H separates H' from v . The events defined by H and H' are in conflict if and only if H and H' do not cross and neither separates the other from v .

3.4. Related work. The link between event domains, median graphs, and CAT(0) cube complexes allows a more geometric and combinatorial approach to several questions on event structures (and to work only with CAT(0) cube complexes viewed as event domains). For example, this allowed [18] to disprove the so-called *nice labeling conjecture* of Rozoy and Thiagarajan [39]

asserting that any event structure of finite degree admit a finite nice labeling. The topological dimension $\dim X$ of a CAT(0) cube complex X corresponds to the maximum number of pairwise concurrent events of \mathcal{E}_v and to the clique number of the intersection graph of hyperplanes of X . The degree $\deg(\mathcal{E}_v)$ of the event structure \mathcal{E}_v is equal to the maximum out-degree of a vertex in the canonical order \leq_v of the 1-skeleton of X (and is equal to the clique number of a so-called pointed contact graph of hyperplanes of X [18,20]). In particular, $\dim X \leq \deg(\mathcal{E}_v)$. Notice also that the maximum degree of a vertex of X is upper bounded by $\deg(\mathcal{E}_v) + \dim(X) \leq 2 \deg(\mathcal{E}_v)$ and is equal to the clique number of the contact graph of hyperplanes of X (the intersection graph of the carriers of X) [20,23]. Using this terminology, a nice labeling of the event structure \mathcal{E}_v is equivalent to a coloring of the pointed contact graph of X . Using this combinatorial reformulation and the example of Burling [15] of families of axis-parallel boxes of \mathbb{R}^3 with no three pairwise intersecting boxes and arbitrarily high chromatic number of the intersection graph, [18] describes an example of a CAT(0) 4-dimensional cube complex with maximum degree 12 and infinite chromatic number of the pointed contact graph, thus providing a counterexample to the nice labeling conjecture of Rozoy and Thiagarajan [39]. On the other hand, it is shown in [20] that the nice labeling conjecture is true for event structures whose domains are 2-dimensional (i.e., event structures not containing three pairwise concurrent events).

4. DIRECTED NPC COMPLEXES

Since we can define event structures from their domains, universal covers of NPC complexes represent a rich source of event structures. To obtain regular event structures, it is natural to consider universal covers of finite NPC complexes. Moreover, since domains of event structures are directed, it is natural to consider universal covers of NPC complexes whose edges are directed. However, the resulting directed universal covers are not in general domains of event structures. In particular, the domains corresponding to pointed median graphs given by Theorem 3.7 cannot be obtained in this way. In order to overcome this difficulty, we introduce directed median graphs and directed NPC complexes. Using these notions, one can naturally define regular event structures starting from finite directed NPC complexes.

4.1. Directed median graphs. A *directed median graph* is a pair (G, o) , where G is a median graph and o is an orientation of the edges of G in a such a way that opposite edges of squares of G have the same direction. By transitivity of Θ , all edges from the same parallelism class Θ_i of G have the same direction. Since each Θ_i partitions G into two parts, o defines a partial order \prec_o on the vertex-set of G . For a vertex v of G , let $\mathcal{F}_o(v, G) = \{x \in V : v \prec_o x\}$ be the principal filter of v in the partial order $(V(G), \prec_o)$. For any canonical basepoint order \leq_v of G , (G, \leq_v) is a directed median graph. The converse is obviously not true: the 4-regular tree F_4 directed so that each vertex has two incoming and two outgoing arcs is a directed median graph which is not induced by a basepoint order.

Lemma 4.1. *For any vertex v of a directed median graph (G, o) , the following holds:*

- (i) $\mathcal{F}_o(v, G)$ induces a convex subgraph of G ;
- (ii) the restriction of the partial order \prec_o on $\mathcal{F}_o(v, G)$ coincides with the restriction of the canonical basepoint order \leq_v on $\mathcal{F}_o(v, G)$;
- (iii) $\mathcal{F}_o(v, G)$ together with \prec_o is the domain of an event structure;
- (iv) for any vertex $u \in \mathcal{F}_o(v, G)$, the principal filter $\mathcal{F}_o(u, G)$ is included in $\mathcal{F}_o(v, G)$ and $\mathcal{F}_o(u, G)$ coincides with the principal filter of u with respect to the canonical basepoint order \leq_v on $\mathcal{F}_o(v, G)$.

Proof. To (i): For each parallelism class Θ_i , let A_i, B_i be the two convex subgraphs separated by Θ_i and suppose without loss of generality that all edges of Θ_i are directed from A_i to B_i . Let B be the intersection of all of the B_i s containing the vertex v . We assert that $\mathcal{F}_o(v, G)$ coincides with B . $\mathcal{F}_o(v, G)$ consists of all vertices u of G such that there is a path from v to u in the Hasse diagram of \prec_o . Since each Θ_i is a cutset of G and all edges of Θ_i are directed from A_i to B_i , we conclude that $\mathcal{F}_o(v, G) \subseteq B$. Conversely, let $u \in B$ and pick any shortest path $P(v, u) = (v_0 = v, v_1, \dots, v_{k-1}, v_k = u)$ in G between v and u . We claim that *all edges*

of $P(v, u)$ are directed from v to u , yielding $u \in \mathcal{F}_o(v, G)$. Pick any edge $v_j v_{j+1}$ of $P(v, u)$; suppose that $v_j v_{j+1}$ belongs to the parallelism class Θ_i . By convexity of A_i and B_i , necessarily Θ_i separates the vertices v and u . Since $v, u \in B$, this implies that $v \in A_i$ and $u \in B_i$, i.e., the edge $v_j v_{j+1}$ is directed from v_j to v_{j+1} .

To (ii): First suppose that $u', u \in \mathcal{F}_o(v, G)$ and $u' \leq_v u$. This implies that $u' \in I(v, u)$. Let $P(v, u) = (v_0 = v, v_1, \dots, v_{k-1}, v_k = u)$ be a shortest path of G between v and u passing via u' . By what we have shown in (i), in the Hasse diagram of \prec_o all the edges of $P(v, u)$ are directed from v to u . This implies that all the edges of the subpath of $P(v, u)$ comprised between u' and u are directed from u' to u , yielding $u' \prec_o u$. To prove the converse assertion, suppose by way of contradiction that $\mathcal{F}_o(v, G)$ contains two vertices u', u such that $u' \prec_o u$ however $u' \leq_v u$ is not true, i.e., $u' \notin I(v, u)$. Among all vertices u for which this holds, suppose that u is chosen so that to minimize the length of a shortest directed path from u' to u . Let w be a neighbor of u on a shortest directed path from u' to u . Since $u' \preceq_o w$, $w \in \mathcal{F}_o(v, G)$ and from the choice of u it follows that $u' \leq_v w$, i.e., $u' \in I(v, w)$. Since G is bipartite, either $w \in I(v, u)$ or $u \in I(v, w)$ holds. If $w \in I(v, u)$, since $u' \in I(v, w)$, we conclude that $u' \in I(v, u)$, a contradiction. Therefore $u \in I(v, w)$. Since the edge wu is oriented from w to u and u lies on a shortest path from v to w , we obtain a contradiction with the fact that all the edges of such a shortest path must be directed from v to w . This contradiction establishes that the partial orders \prec_o and \leq_v coincide on $\mathcal{F}_o(v, G)$.

To (iii)&(iv): By (i), the subgraph G' of G induced by $\mathcal{F}_o(v, G)$ is a median graph. By (ii), the partial order \prec_o coincides on G' with the canonical basepoint order \leq_v . By Theorem 3.7, $(V(G'), \prec_o)$ is the domain of an event structure, establishing (iii). Finally, (iv) is an immediate consequence of (ii). \square

4.2. Directed NPC cube complexes. A *directed NPC complex* is a pair (Y, o) , where Y is a NPC complex and o is an orientation of the edges of Y in a such a way that the opposite edges of the same square of Y have the same direction. Such an orientation o of the edges of a NPC complex Y is called an *admissible* orientation of Y . Note that there exist NPC complexes that do not admit any admissible orientation: consider a Möbius band of squares, for example. An admissible orientation o of Y induces in a natural way an orientation \tilde{o} of the edges of its universal cover \tilde{Y} , so that (\tilde{Y}, \tilde{o}) is a directed CAT(0) cube complex and $(\tilde{Y}^{(1)}, \tilde{o})$ is a directed median graph. A *directed plane* in a directed CAT(0) cube complex Y is a plane Π in Y such that for any vertex (i, j) of the grid \mathbb{Z}^2 tiling Π , (i, j) is the source of the edges $(i, j)(i, j + 1)$ and $(i, j)(i + 1, j)$.

In the following, we need to consider directed colored NPC complexes and directed colored median graphs. A coloring ν of a directed NPC complex (Y, o) is an arbitrary map $\nu : E(Y) \rightarrow \Upsilon$ where Υ is a set of colors. Note that a labeling is a coloring, but not the converse: labelings are precisely the colorings in which opposite edges of any square have the same color. In the following, we will denote a directed colored NPC complexes by bold letters like $\mathbf{Y} = (Y, o, \nu)$. Sometimes, we need to forget the colors and the orientations of the edges of these complexes. For a complex \mathbf{Y} , we denote by Y the complex obtained by forgetting the colors and the orientations of the edges of \mathbf{Y} (Y is called the *support* of \mathbf{Y}), and we denote by (Y, o) the directed complex obtained by forgetting the colors of \mathbf{Y} . We also consider directed colored median graphs that will be the 1-skeletons of directed colored CAT(0) cube complexes. Again we will denote such directed colored median graphs by bold letters like $\mathbf{G} = (G, o, \nu)$. Note that (uncolored) directed NPC complexes can be viewed as directed colored NPC complexes where all edges have the same color.

When dealing with directed colored NPC complexes, we consider only homomorphisms that preserve the colors and the directions of the edges. More precisely, $\mathbf{Y}' = (Y', o', \nu')$ is a covering of $\mathbf{Y} = (Y, o, \nu)$ via a covering map φ if Y' is a covering of Y via φ and for any edge $e \in E(Y')$ directed from s to t , $\nu(\varphi(e)) = \nu'(e)$ and $\varphi(e)$ is directed from $\varphi(s)$ to $\varphi(t)$. Since any coloring ν of a directed colored NPC complex Y leads to a coloring of its universal cover \tilde{Y} , one can consider the colored universal cover $\tilde{\mathbf{Y}} = (\tilde{Y}, \tilde{o}, \tilde{\nu})$ of \mathbf{Y} .

When we consider principal filters in directed colored median graphs $\mathbf{G} = (G, o, \nu)$ (in particular, when G is the 1-skeleton of the universal cover $\tilde{\mathbf{Y}}$ of a directed colored NPC complex \mathbf{Y}), we say that two filters are isomorphic if there is an isomorphism between them that preserves the directions and the colors of the edges.

We now formulate the crucial regularity property of directed colored median graphs $(\tilde{Y}^{(1)}, \tilde{o}, \tilde{\nu})$ when (Y, o, ν) is finite.

Lemma 4.2. *If $\mathbf{Y} = (Y, o, \nu)$ is a finite directed colored NPC complex, then $\tilde{\mathbf{Y}}^{(1)} = (\tilde{Y}^{(1)}, \tilde{o}, \tilde{\nu})$ is a directed median graph with at most $|V(Y)|$ isomorphism types of colored principal filters. In particular, if (Y, o) is a finite directed NPC complex, then $(\tilde{Y}^{(1)}, \tilde{o})$ is a directed median graph with at most $|V(Y)|$ isomorphism types of principal filters.*

Proof. Consider a covering map $\varphi : \tilde{\mathbf{Y}} = (\tilde{Y}, \tilde{o}, \tilde{\nu}) \rightarrow \mathbf{Y} = (Y, o, \nu)$. We first show that $(\tilde{Y}^{(1)}, \tilde{o})$ is a directed median graph. By Theorem 3.7, $\tilde{Y}^{(1)}$ is a median graph. Since the image of a square in $\tilde{Y}^{(1)}$ is a square in Y , since φ preserves the direction of the edges, and since two opposite edges of a square of Y have the same direction, any two opposite edges of a square of $\tilde{Y}^{(1)}$ have the same direction. Consequently, $\tilde{Y}^{(1)}$ is a directed median graph.

Consider now two vertices $\tilde{u}, \tilde{u}' \in V(\tilde{Y})$ such that $\varphi(\tilde{u}) = \varphi(\tilde{u}')$. In the following, we show that $\mathcal{F}_o(\tilde{u}, \tilde{\mathbf{Y}}^{(1)})$ and $\mathcal{F}_o(\tilde{u}', \tilde{\mathbf{Y}}^{(1)})$ are isomorphic, which implies that there are at most $|V(Y)|$ different isomorphism types of colored principal filters by Lemma 4.1. The proof is based on the two following claims. The first claim can be easily proved by induction on the length of P .

Claim 1. *For any path $P = (\tilde{u} = \tilde{u}_0, \tilde{u}_1, \dots, \tilde{u}_k)$ in \tilde{Y} , there exists a unique path $P' = (\tilde{u}' = \tilde{u}'_0, \tilde{u}'_1, \dots, \tilde{u}'_k)$ such that $\varphi(\tilde{u}'_i) = \varphi(\tilde{u}_i)$ for all $0 \leq i \leq k$.*

Claim 2. *For any four paths $P_1 = (\tilde{u} = \tilde{u}_0, \tilde{u}_1, \dots, \tilde{u}_k)$, $P'_1 = (\tilde{u}' = \tilde{u}'_0, \tilde{u}'_1, \dots, \tilde{u}'_k)$, $P_2 = (\tilde{u} = \tilde{v}_0, \tilde{v}_1, \dots, \tilde{v}_\ell)$, and $P'_2 = (\tilde{u}' = \tilde{v}'_0, \tilde{v}'_1, \dots, \tilde{v}'_\ell)$ in \tilde{Y} such that $\varphi(\tilde{u}'_i) = \varphi(\tilde{u}_i)$ for all $0 \leq i \leq k$ and $\varphi(\tilde{v}'_j) = \varphi(\tilde{v}_j)$ for all $0 \leq j \leq \ell$, we have $\tilde{u}_k = \tilde{v}_\ell$ if and only if $\tilde{u}'_k = \tilde{v}'_\ell$.*

Proof. For each $0 \leq i \leq k$, let $u_i = \varphi(\tilde{u}_i) = \varphi(\tilde{u}'_i)$ and for each $0 \leq j \leq \ell$, let $v_j = \varphi(\tilde{v}_j) = \varphi(\tilde{v}'_j)$. Suppose that $\tilde{u}_k = \tilde{v}_\ell$ and consider the cycle $C = P_1 \cdot P_2 = (\tilde{u} = \tilde{u}_0, \tilde{u}_1, \dots, \tilde{u}_k = \tilde{v}_\ell, \tilde{v}_{\ell-1}, \dots, \tilde{v}_1, \tilde{v}_0 = \tilde{u})$. Let $n_1(C) = \max\{d(\tilde{u}, \tilde{w}) : \tilde{w} \in C\}$ and $n_2(C) = |\{\tilde{w} : d(\tilde{u}, \tilde{w}) = n_1(C)\}|$. We prove the claim by lexicographic induction on $(n_1(C), n_2(C))$. If $n_1(C) = 0$, then $k = \ell = 0$ and we are done. Suppose now that $n_1(C) \geq 1$.

Suppose that there exists $0 < i < k$ such that $d(\tilde{u}, \tilde{u}_i) = n_1(C)$. Suppose first that $\tilde{u}_{i-1} = \tilde{u}_{i+1}$. Then, since φ is an isomorphism between $\text{St}(\tilde{u}_i, \tilde{Y})$ and $\text{St}(u_i, Y)$ and between $\text{St}(\tilde{u}'_i, \tilde{Y})$ and $\text{St}(u_i, Y)$, necessarily $\tilde{u}'_{i-1} = \tilde{u}'_{i+1}$. By induction hypothesis applied to the paths $P_3 = (\tilde{u} = \tilde{u}_0, \tilde{u}_1, \dots, \tilde{u}_{i-1} = \tilde{u}_{i+1}, \dots, \tilde{u}_k)$, $P'_3 = (\tilde{u}' = \tilde{u}'_0, \tilde{u}'_1, \dots, \tilde{u}'_{i-1} = \tilde{u}'_{i+1}, \dots, \tilde{u}'_k)$, P_2 , and P'_2 , we have $\tilde{u}'_k = \tilde{v}'_\ell$ and we are done. Assume now that $\tilde{u}_{i-1} \neq \tilde{u}_{i+1}$. Since the graph $\tilde{Y}^{(1)}$ is bipartite, we have $d(\tilde{u}, \tilde{u}_{i-1}) = d(\tilde{u}, \tilde{u}_{i+1}) = n_1(C) - 1$ and $d(\tilde{u}_{i-1}, \tilde{u}_{i+1}) = 2$. Since $\tilde{Y}^{(1)}$ is median, there exists \tilde{w}_i such that $d(\tilde{w}_i, \tilde{u}_{i-1}) = d(\tilde{w}_i, \tilde{u}_{i+1}) = 1$ and $d(\tilde{w}_i, \tilde{u}) = n_1(C) - 2$. Note that $\tilde{u}_i \tilde{u}_{i-1} \tilde{w}_i \tilde{u}_{i+1}$ is a square in $\text{St}(\tilde{u}_i, \tilde{Y})$. Consequently, since φ is an isomorphism between $\text{St}(\tilde{u}_i, \tilde{Y})$ and $\text{St}(u_i, Y)$ and between $\text{St}(\tilde{u}'_i, \tilde{Y})$ and $\text{St}(u_i, Y)$, there exists \tilde{w}'_i such that $\tilde{u}'_i \tilde{u}'_{i-1} \tilde{w}'_i \tilde{u}'_{i+1}$ is a square in $\text{St}(\tilde{u}'_i, \tilde{Y})$ and $\varphi(\tilde{w}'_i) = \varphi(\tilde{w}_i)$. By induction hypothesis applied to the paths $P_4 = (\tilde{u} = \tilde{u}_0, \tilde{u}_1, \dots, \tilde{u}_{i-1}, \tilde{w}_i, \tilde{u}_{i+1}, \dots, \tilde{u}_k)$, $P'_4 = (\tilde{u}' = \tilde{u}'_0, \tilde{u}'_1, \dots, \tilde{u}'_{i-1}, \tilde{w}'_i, \tilde{u}'_{i+1}, \dots, \tilde{u}'_k)$, P_2 , and P'_2 , we have $\tilde{u}'_k = \tilde{v}'_\ell$ and we are done. Analogously, if there exists $0 < j < \ell$ such that $d(\tilde{u}, \tilde{v}_j) = n_1(C)$, we can show that $\tilde{u}'_k = \tilde{v}'_\ell$.

Suppose now that $n_1(C) = d(\tilde{u}, \tilde{u}_k = \tilde{v}_\ell)$ and $n_2(C) = 1$. Since $\tilde{Y}^{(1)}$ is bipartite, $d(\tilde{u}, \tilde{u}_{k-1}) = d(\tilde{u}, \tilde{v}_{\ell-1}) = n_1(C) - 1$ and $d(\tilde{u}_{k-1}, \tilde{v}_{\ell-1}) = 2$. Since $\tilde{Y}^{(1)}$ is median, there exists \tilde{w} such that $d(\tilde{w}, \tilde{u}_{k-1}) = d(\tilde{w}, \tilde{v}_{\ell-1}) = 1$ and $d(\tilde{w}, \tilde{u}) = n_1(C) - 2$. Since $\varphi(\tilde{u}_{k-1}) = \varphi(\tilde{u}'_{k-1})$, there exists a unique neighbor \tilde{w}' of \tilde{u}'_{k-1} such that $\varphi(\tilde{w}') = \varphi(\tilde{w})$. Similarly, there exists a unique neighbor \tilde{w}'' of $\tilde{v}'_{\ell-1}$ such that $\varphi(\tilde{w}'') = \varphi(\tilde{w})$. By induction hypothesis applied to the paths $P_5 = (\tilde{u} = \tilde{u}_0, \tilde{u}_1, \dots, \tilde{u}_{k-1}, \tilde{w})$, $P'_5 = (\tilde{u}' = \tilde{u}'_0, \tilde{u}'_1, \dots, \tilde{u}'_{k-1}, \tilde{w}')$, $P_6 = (\tilde{u} = \tilde{v}_0, \tilde{v}_1, \dots, \tilde{v}_{\ell-1}, \tilde{w})$, and $P'_6 = (\tilde{u}' = \tilde{v}'_0, \tilde{v}'_1, \dots, \tilde{v}'_{\ell-1}, \tilde{w}'')$, we have that $\tilde{w}' = \tilde{w}''$. Consequently, since φ induces a bijection

between $\text{St}(\tilde{w}, \tilde{Y})$ and $\text{St}(\varphi(\tilde{w}), Y)$ and between $\text{St}(\tilde{w}', \tilde{Y})$ and $\text{St}(\varphi(\tilde{w}'), Y) = \text{St}(\varphi(\tilde{w}), Y)$, necessarily $\tilde{u}'_k = \tilde{v}'_k$ and we are done. \square

Define now a map $f_{\tilde{u}, \tilde{u}'}$ from $V(\tilde{Y})$ to $V(\tilde{Y})$ such that $f_{\tilde{u}, \tilde{u}'}(\tilde{u}) = \tilde{u}'$. For any vertex $\tilde{v} \in \tilde{Y}$, consider a path $P = (\tilde{u} = \tilde{u}_0, \tilde{u}_1, \dots, \tilde{u}_k = \tilde{v})$ from \tilde{u} to \tilde{v} in \tilde{Y} . By Claim 1, there exists a unique path $P' = (\tilde{u}' = \tilde{u}'_0, \tilde{u}'_1, \dots, \tilde{u}'_k = \tilde{v}')$ such that $\varphi(\tilde{u}'_i) = \varphi(\tilde{u}_i)$ for all $0 \leq i \leq k$ and we let $f_{\tilde{u}, \tilde{u}'}(\tilde{v}) = \tilde{v}'$. Note that $\varphi(f(\tilde{v})) = \varphi(\tilde{v})$ and by Claim 2, $f(\tilde{v})$ is independent of the choice of the path P . Similarly, we can define a map $f_{\tilde{u}', \tilde{u}}$ from \tilde{Y} to \tilde{Y} such that $f_{\tilde{u}', \tilde{u}}(\tilde{u}') = \tilde{u}$ and one can easily see that $f_{\tilde{u}, \tilde{u}'} \circ f_{\tilde{u}', \tilde{u}} = f_{\tilde{u}', \tilde{u}} \circ f_{\tilde{u}, \tilde{u}'} = \text{id}$. Consequently, $f_{\tilde{u}, \tilde{u}'}$ is a bijection from $V(\tilde{Y})$ to $V(\tilde{Y})$. Moreover, from the definition of $f_{\tilde{u}, \tilde{u}'}$ and from Claim 1, if $\tilde{v}_1\tilde{v}_2$ is an edge of \tilde{Y} , then $f_{\tilde{u}, \tilde{u}'}(\tilde{v}_1)f_{\tilde{u}, \tilde{u}'}(\tilde{v}_2)$ is also an edge of \tilde{Y} . Since the orientation of $\tilde{v}_1\tilde{v}_2$ in (\tilde{Y}, \tilde{o}) is the same as the orientation of the edge $v_1v_2 = \varphi(\tilde{v}_1)\varphi(\tilde{v}_2)$ in (Y, o) , it is also the same as the orientation of $f_{\tilde{u}, \tilde{u}'}(\tilde{v}_1)f_{\tilde{u}, \tilde{u}'}(\tilde{v}_2)$ in (\tilde{Y}, \tilde{o}) . Furthermore since $\tilde{\nu}(\tilde{v}_1\tilde{v}_2) = \nu(v_1v_2) = \tilde{\nu}(f_{\tilde{u}, \tilde{u}'}(\tilde{v}_1)f_{\tilde{u}, \tilde{u}'}(\tilde{v}_2))$, $f_{\tilde{u}, \tilde{u}'}$ preserves the colors of the edges of \tilde{Y} . Consequently, $f_{\tilde{u}, \tilde{u}'}$ is an automorphism of $\tilde{\mathbf{Y}} = (\tilde{Y}, \tilde{o}, \tilde{\nu})$ such that $f_{\tilde{u}, \tilde{u}'}(\tilde{u}) = \tilde{u}'$, and thus $\mathcal{F}_o(\tilde{u}, \tilde{\mathbf{Y}}^{(1)})$ and $\mathcal{F}_o(\tilde{u}', \tilde{\mathbf{Y}}^{(1)})$ are isomorphic. \square

Proposition 4.3. *Consider a finite (uncolored) directed NPC complex (Y, o) . Then for any vertex \tilde{v} of the universal cover \tilde{Y} of Y , the principal filter $\mathcal{F}_{\tilde{o}}(\tilde{v}, \tilde{Y}^{(1)})$ with the partial order $\prec_{\tilde{o}}$ is the domain of a regular event structure with at most $|V(Y)|$ different isomorphism types of principal filters.*

Proof. By Theorem 3.3, \tilde{Y} is a CAT(0) cube complex. Combining Lemma 4.1 (iii)-(iv) and Lemma 4.2, we deduce that $(\mathcal{F}_{\tilde{o}}(\tilde{v}, \tilde{Y}^{(1)}), \prec_{\tilde{o}})$ is the domain of a regular event structure with at most $|V(Y)|$ different isomorphism types of principal filters. \square

We will call an event structure $\mathcal{E} = (E, \leq, \#)$ and its domain $\mathcal{D}(\mathcal{E})$ *strongly regular* if $\mathcal{D}(\mathcal{E})$ is isomorphic to a principal filter of the universal cover of some finite directed NPC complex. In view of Proposition 4.3, any strongly regular event structure is regular.

5. THIAGARAJAN'S CONJECTURE AND SPECIAL NPC COMPLEXES

5.1. Special NPC complexes. Consider an NPC complex Y , let \tilde{Y} be its universal cover and let $\varphi : \tilde{Y} \rightarrow Y$ be a covering map. Analogously to CAT(0) cube complexes, one can define the parallelism relation Θ' on the set of edges $E(Y)$ of Y by setting that two edges of Y are in relation Θ' iff they opposite edges of a common 2-cube of Y . Let Θ be the reflexive and transitive closure of Θ' and let $\{\Theta_i : i \in I\}$ denote the equivalence classes of Θ . For an equivalence class Θ_i , the hyperplane H_i associated to Θ_i is the NPC complex consisting of the midcubes of all cubes of Y containing one edge of Θ_i . The edges of Θ_i are *dual* to the hyperplane H_i . Let $\mathcal{H}(Y)$ be the set of hyperplanes of Y .

The hyperplanes of an NPC complex Y do not longer satisfy the nice properties of the hyperplanes of CAT(0) cube complexes: they do not longer partition the complex in exactly two parts, they may self-intersect, self-osculte, two hyperplanes may at the same time cross and osculte, etc. Haglund and Wise [25] detected five types of pathologies which may occur in an NPC complex (see Figure 3):

- (a) self-intersecting hyperplane;
- (b) one-sided hyperplane;
- (c) directly self-oscultating hyperplane;
- (d) indirectly self-oscultating hyperplane;
- (e) a pair of hyperplanes, which both intersect and osculte.

We continue with the definition of each of these pathologies (in which we closely follow [25, Section 3]). Two hyperplanes H_1 and H_2 *intersect* if there exists a cube Q and two distinct midcubes Q_1 and Q_2 of Q such that $Q_1 \subseteq H_1$ and $Q_2 \subseteq H_2$, i.e., there exists a square with two consecutive edges e_1, e_2 such that e_1 is dual to H_1 and e_2 is dual to H_2 .

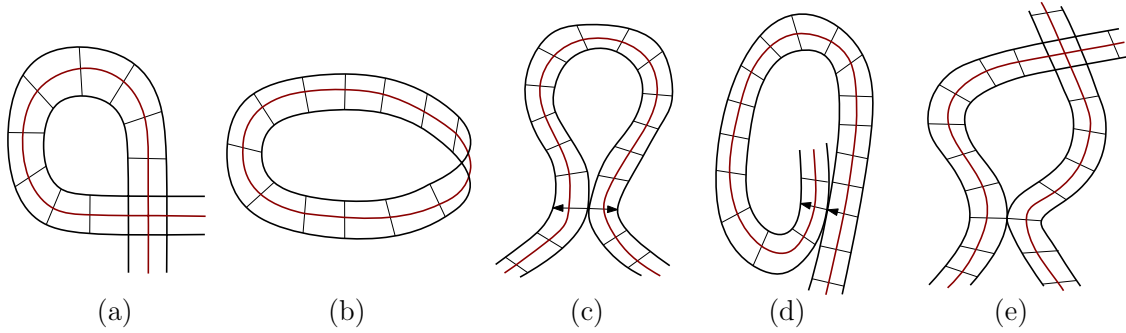


FIGURE 3. A self-intersecting hyperplane (a), a one-sided hyperplane (b), a directly self-intersecting hyperplane (c), an indirectly self-intersecting hyperplane (d), and a pair of hyperplanes that inter-osculate (e).

A hyperplane H of Y *self-intersects* if it contains more than one midcube from the same cube, i.e., there exist two edges e_1, e_2 dual to H that are consecutive in some square of Y (see Figure 3(a)).

A hyperplane H is *two-sided* if $N(H)$ is homeomorphic to the product $H \times (-1, 1)$, and there is a combinatorial map $H \times [-1, 1] \rightarrow X$ mapping $H \times \{0\}$ identically to H . The hyperplane is *one-sided* if it is not two-sided (see Figure 3(b)). As noticed in [25, p.1562], requiring that the hyperplanes of Y are two-sided is equivalent to defining an orientation on the dual edges of H such that all sources of such edges belong to one of the sets $H \times \{-1\}, H \times \{1\}$ and all sinks belong to the other one. This orientation is obtained by taking the equivalence relation generated by elementary parallelism relation: declare two oriented edges e_1 and e_2 of Y *elementary parallel* if there is a square of Y containing e_1 and e_2 as opposite sides and oriented in the same direction. Notice that if (Y, o) is a directed NPC complex, then every hyperplane H of Y is two-sided. Conversely, if every hyperplane H of Y is two-sided, then Y admits admissible orientations (one can choose an admissible orientation for each hyperplane independently).

Let v be a vertex of Y and let e_1, e_2 be two distinct edges incident to v but such that e_1 and e_2 are not consecutive edges in some square containing v . The hyperplanes H_1 and H_2 *osculate* at (v, e_1, e_2) if e_1 is dual to H_1 and e_2 is dual to H_2 . The hyperplane H *self-osculte* at (v, e_1, e_2) if e_1 and e_2 are dual to H . Consider a two-sided hyperplane H and an admissible orientation o of its dual edges. Suppose that H self-osculte at (v, e_1, e_2) . If v is the source of both e_1 and e_2 or the sink of both e_1 and e_2 , then we say that H *directly self-osculte* at (v, e_1, e_2) (see Figure 3(c)). If v is the source of one of e_1, e_2 , and the sink of the other, then we say that H *indirectly self-osculte* at (v, e_1, e_2) (see Figure 3(d)). Note that a self-osculation of a hyperplane H is either direct or indirect, and this is independent of the orientation of the edges dual to H .

Two hyperplanes H_1 and H_2 *inter-osculte* if they both intersect and osculte (see Figure 3(e)).

Haglund and Wise [25, Definition 3.2] called an NPC complex Y *special* if its hyperplanes are two-sided, do not self-intersect, do not directly self-osculte, and no two hyperplanes inter-osculte.

5.2. Trace labelings of special event structures. Consider a finite NPC complex Y and let $\mathcal{H} = \mathcal{H}(Y)$ be the set of hyperplanes of Y . We define a canonical labeling $\lambda_{\mathcal{H}} : E(Y) \rightarrow \mathcal{H}$ by setting $\lambda_{\mathcal{H}}(e) = H$ if the edge e is dual to H . For any covering map $\varphi : \tilde{Y} \rightarrow Y$, $\lambda_{\mathcal{H}}$ is naturally extended to a labeling $\tilde{\lambda}_{\mathcal{H}}$ of $E(\tilde{Y})$ where $\tilde{\lambda}_{\mathcal{H}}(e) = \lambda_{\mathcal{H}}(\varphi(e))$.

We show that strongly regular event structures obtained from finite special cube complexes admit regular trace labellings.

Proposition 5.1. *A finite NPC complex Y with two-sided hyperplanes is special if and only if there exists an independence relation I on $\mathcal{H} = \mathcal{H}(Y)$ such that for any admissible orientation*

o of Y , for any covering map $\varphi : \tilde{Y} \rightarrow Y$, and for any principal filter $\mathcal{D} = (\mathcal{F}_{\tilde{o}}(\tilde{v}, \tilde{Y}^{(1)}), \prec_{\tilde{o}})$ of (\tilde{Y}, \tilde{o}) , the canonical labeling $\tilde{\lambda}_{\mathcal{H}}$ is a regular trace labeling of \mathcal{D} with the trace alphabet (\mathcal{H}, I) .

Proof. Suppose first that there exists an independence relation $I \subseteq \mathcal{H}^2$ such that for any admissible orientation o of Y , for any covering map $\varphi : \tilde{Y} \rightarrow Y$, and any principal filter \mathcal{D} of (\tilde{Y}, \tilde{o}) , $\tilde{\lambda}_{\mathcal{H}}$ is a regular trace labeling of \mathcal{D} with the trace alphabet (\mathcal{H}, I) .

If Y contains a self-intersecting hyperplane H , then there exist a square Q such that the four edges of Q are dual to H . Consider an admissible orientation o of Y and note that there exist two edges e_1, e_2 in Q that have the same source v . In (\tilde{Y}, \tilde{o}) , consider a vertex $\tilde{v} \in \varphi^{-1}(v)$ and note that \tilde{v} has two outgoing edges \tilde{e}_1, \tilde{e}_2 such that $\varphi(\tilde{e}_1) = e_1$ and $\varphi(\tilde{e}_2) = e_2$. Since $\tilde{\lambda}_{\mathcal{H}}(\tilde{e}_1) = \lambda_{\mathcal{H}}(e_1) = \lambda_{\mathcal{H}}(e_2) = \tilde{\lambda}_{\mathcal{H}}(\tilde{e}_2)$, the labeling $\tilde{\lambda}_{\mathcal{H}}$ violates the determinism condition in the principal filter $(\mathcal{F}_{\tilde{o}}(\tilde{v}, \tilde{Y}^{(1)}), \prec_{\tilde{o}})$.

If Y contains a hyperplane H that directly self-osculte at (v, e_1, e_2) , then there exists an orientation o of Y such that e_1 and e_2 have the same source v . In (\tilde{Y}, \tilde{o}) , consider a vertex $\tilde{v} \in \varphi^{-1}(v)$ and note that \tilde{v} has two outgoing edges \tilde{e}_1, \tilde{e}_2 such that $\varphi(\tilde{e}_1) = e_1$ and $\varphi(\tilde{e}_2) = e_2$. Since $\tilde{\lambda}_{\mathcal{H}}(\tilde{e}_1) = \lambda_{\mathcal{H}}(e_1) = \lambda_{\mathcal{H}}(e_2) = \tilde{\lambda}_{\mathcal{H}}(\tilde{e}_2)$, the labeling $\tilde{\lambda}_{\mathcal{H}}$ violates the determinism condition in the principal filter $(\mathcal{F}_{\tilde{o}}(\tilde{v}, \tilde{Y}^{(1)}), \prec_{\tilde{o}})$.

Finally if Y contains two hyperplanes H_1 and H_2 that inter-osculte, then they osculte at (v, e_1, e_2) and they intersect on a square Q . We can choose an orientation o of Y such that v is the source of both e_1 and e_2 . Then there exists a source u in Q that has two outgoing edges e'_1 and e'_2 that are parallel respectively to e_1 and e_2 . Let $\tilde{u} \in \varphi^{-1}(u)$ and $\tilde{v} \in \varphi^{-1}(v)$. Let \tilde{e}_1, \tilde{e}_2 be the respective preimages of the edges e_1, e_2 such that \tilde{v} is the source of \tilde{e}_1, \tilde{e}_2 . Similarly, let $\tilde{e}'_1, \tilde{e}'_2$ be the respective preimages of the edges e_1, e_2 such that \tilde{u} is the source of $\tilde{e}'_1, \tilde{e}'_2$. Note that $\tilde{\lambda}_{\mathcal{H}}(\tilde{e}_1) = \lambda_{\mathcal{H}}(e_1) = \lambda_{\mathcal{H}}(e'_1) = \tilde{\lambda}_{\mathcal{H}}(\tilde{e}'_1)$ and $\tilde{\lambda}_{\mathcal{H}}(\tilde{e}_2) = \lambda_{\mathcal{H}}(e_2) = \lambda_{\mathcal{H}}(e'_2) = \tilde{\lambda}_{\mathcal{H}}(\tilde{e}'_2)$. Consider the principal filters $\mathcal{D}_v = (\mathcal{F}_{\tilde{o}}(\tilde{v}, \tilde{Y}^{(1)}), \prec_{\tilde{o}})$ and $\mathcal{D}_u = (\mathcal{F}_{\tilde{o}}(\tilde{u}, \tilde{Y}^{(1)}), \prec_{\tilde{o}})$. In \mathcal{D}_v , \tilde{e}_1 and \tilde{e}_2 correspond to two events that are in minimal conflict, and thus the pair $(\tilde{\lambda}_{\mathcal{H}}(\tilde{e}_1), \tilde{\lambda}_{\mathcal{H}}(\tilde{e}_2))$ does not belong to the independence relation I . On the other hand, in \mathcal{D}_u , \tilde{e}'_1 and \tilde{e}'_2 belong to a square, and thus they correspond to two concurrent events. Consequently, the pair $(\tilde{\lambda}_{\mathcal{H}}(\tilde{e}'_1), \tilde{\lambda}_{\mathcal{H}}(\tilde{e}'_2))$ belongs to I . Since $\tilde{\lambda}_{\mathcal{H}}(\tilde{e}_1) = \tilde{\lambda}_{\mathcal{H}}(\tilde{e}'_1)$ and $\tilde{\lambda}_{\mathcal{H}}(\tilde{e}_2) = \tilde{\lambda}_{\mathcal{H}}(\tilde{e}'_2)$, we have a contradiction.

Conversely, suppose that Y is a finite special NPC complex. We define the independence relation $I \subseteq \mathcal{H} \times \mathcal{H}$ as follows: $(H_1, H_2) \in I$ if and only if the hyperplanes H_1 and H_2 intersect. From its definition, the binary relation I is symmetric. Since no hyperplane of Y self-intersects, I is also irreflexive, and thus (\mathcal{H}, I) is a finite trace alphabet.

Consider an admissible orientation o of Y , a vertex $\tilde{v} \in V(\tilde{Y})$, a covering map $\varphi : \tilde{Y} \rightarrow Y$ and consider the principal filter $\mathcal{D} = (\mathcal{F}_{\tilde{o}}(\tilde{v}, \tilde{Y}^{(1)}), \prec_{\tilde{o}})$. By Proposition 4.3, \mathcal{D} is the domain of a regular event structure \mathcal{E} . As explained in Subsection 3.3, the events of \mathcal{E} are the hyperplanes of \mathcal{D} . Hyperplanes \tilde{H} and \tilde{H}' are concurrent if and only if they cross, and $\tilde{H} \leq \tilde{H}'$ if and only if $\tilde{H} = \tilde{H}'$ or \tilde{H} separates \tilde{H}' from v . The events \tilde{H} and \tilde{H}' are in conflict iff \tilde{H} and \tilde{H}' do not cross and neither separates the other from v . Note that this implies that $\tilde{H} \prec \tilde{H}'$ iff \tilde{H} separate \tilde{H}' from v and \tilde{H} and \tilde{H}' osculte, and $\tilde{H} \#_{\mu} \tilde{H}'$ iff \tilde{H} and \tilde{H}' osculte and neither of \tilde{H} and \tilde{H}' separates the other from v . Notice also that each hyperplane \tilde{H}' of \mathcal{D} is the intersection of a hyperplane \tilde{H} of \tilde{Y} with \mathcal{D} .

We show that $\tilde{\lambda}_{\mathcal{H}}$ is a regular trace labeling of \mathcal{D} with the trace alphabet (\mathcal{H}, I) . First note that if \tilde{e}_1, \tilde{e}_2 are opposite edges of a square of \mathcal{D} , then $e_1 = \varphi(\tilde{e}_1)$ and $e_2 = \varphi(\tilde{e}_2)$ are opposite edges of a square of Y and thus $\tilde{\lambda}_{\mathcal{H}}(\tilde{e}_1) = \lambda_{\mathcal{H}}(e_1) = \lambda_{\mathcal{H}}(e_2) = \tilde{\lambda}_{\mathcal{H}}(\tilde{e}_2)$. Consequently, $\tilde{\lambda}_{\mathcal{H}}$ is a labeling of the edges of \mathcal{D} . From Lemma 4.2, \mathcal{D} has at most $|V(Y)|$ isomorphism types of colored principal filters. Therefore, in order to show that $\tilde{\lambda}_{\mathcal{H}}$ is a regular trace labeling of \mathcal{D} , we just need to show that $\tilde{\lambda}_{\mathcal{H}}$ satisfies the conditions (LES1), (LES2), and (LES3).

For any two hyperplanes \tilde{H}_1, \tilde{H}_2 in minimal conflict in \mathcal{D} , there exist an edge \tilde{e}_1 dual to \tilde{H}_1 and an edge \tilde{e}_2 dual to \tilde{H}_2 such that \tilde{e}_1 and \tilde{e}_2 have the same source \tilde{u} . Note that since \tilde{H}_1 and \tilde{H}_2 are in conflict, \tilde{e}_1 and \tilde{e}_2 do not belong to a common square of \mathcal{D} . Moreover, if \tilde{e}_1 and \tilde{e}_2

are in a square \tilde{Q} in \tilde{Y} , then since there is a directed path from \tilde{v} to \tilde{u} , and since \tilde{u} is the source of \tilde{Q} , all vertices of \tilde{Q} are in $(\mathcal{F}_{\tilde{o}}(\tilde{v}, \tilde{Y}^{(1)}), \prec_{\tilde{o}}) = \mathcal{D}$. Consequently, the hyperplane \tilde{H}_1 and \tilde{H}_2 osculate at $(\tilde{u}, \tilde{e}_1, \tilde{e}_2)$ in \tilde{Y} . Let $u = \varphi(\tilde{u})$, $e_1 = \varphi(\tilde{e}_1)$, and $e_2 = \varphi(\tilde{e}_2)$, and note that u is the source of e_1 and e_2 . Let H_1 and H_2 be the hyperplanes of Y that are respectively dual to e_1 and e_2 . Since φ is a covering map, e_1 and e_2 do not belong to a common square. Consequently, H_1 and H_2 osculate at (u, e_1, e_2) . If $H_1 = H_2$, H_1 directly self-oscultates at (u, e_1, e_2) , which is impossible because Y is special. Consequently, $\tilde{\lambda}_{\mathcal{H}}(\tilde{e}_1) = \lambda_{\mathcal{H}}(e_1) = H_1$ is different from $\tilde{\lambda}_{\mathcal{H}}(\tilde{e}_2) = \lambda_{\mathcal{H}}(e_2) = H_2$, establishing (LES1). Moreover, since no two hyperplanes of Y inter-oscultate, we know that H_1 and H_2 do not intersect, and thus $(H_1, H_2) \notin I$, establishing (LES2) when $\tilde{H}_1 \#_{\mu} \tilde{H}_2$.

Suppose now that $\tilde{H}_1 < \tilde{H}_2$ in \mathcal{D} . There exist an edge \tilde{e}_1 dual to \tilde{H}_1 and an edge \tilde{e}_2 dual to \tilde{H}_2 such that the sink \tilde{u} of \tilde{e}_1 is the source of \tilde{e}_2 . Since \tilde{H}_1 separates \tilde{H}_2 from \tilde{v} in \mathcal{D} , \tilde{H}_1 also separates \tilde{H}_2 from \tilde{v} in \tilde{Y} . Consequently, \tilde{e}_1 and \tilde{e}_2 do not belong to a common square of \tilde{Y} and the hyperplanes \tilde{H}_1 and \tilde{H}_2 osculate at $(\tilde{u}, \tilde{e}_1, \tilde{e}_2)$. Let $u = \varphi(\tilde{u})$, $e_1 = \varphi(\tilde{e}_1)$, and $e_2 = \varphi(\tilde{e}_2)$, and note that u is the sink of e_1 and the source of e_2 . Let H_1 and H_2 be the hyperplanes of Y that are respectively dual to e_1 and e_2 . Since φ is a covering map, e_1 and e_2 do not belong to a common square. Consequently, H_1 and H_2 osculate at (u, e_1, e_2) . If $H_1 = H_2$, then since I is irreflexive, $(H_1, H_1) \notin I$. If $H_1 \neq H_2$, since no two hyperplanes of Y inter-oscultate, we know that H_1 and H_2 do not intersect, and thus $(H_1, H_2) \notin I$, establishing (LES2) when $\tilde{H}_1 < \tilde{H}_2$.

We prove (LES3) by contraposition. Consider two hyperplanes \tilde{H}_1, \tilde{H}_2 that are concurrent, i.e., they intersect in \mathcal{D} . Since \tilde{H}_1 and \tilde{H}_2 intersect in \tilde{Y} , there exists a square \tilde{Q} containing two consecutive edges \tilde{e}_1, \tilde{e}_2 that are respectively dual to \tilde{H}_1, \tilde{H}_2 . Let H_1 and H_2 be the hyperplanes of Y that are respectively dual to $e_1 = \varphi(\tilde{e}_1)$ and $e_2 = \varphi(\tilde{e}_2)$. Note that $\tilde{\lambda}_{\mathcal{H}}(\tilde{e}_1) = H_1$ and $\tilde{\lambda}_{\mathcal{H}}(\tilde{e}_2) = H_2$. Since φ is a covering map, e_1 and e_2 belong to a square in Y . Then H_1 and H_2 intersect, and therefore $(H_1, H_2) \in I$, establishing (LES3). \square

A finite NPC complex X is called *virtually special* [25, 26] if X admits a finite special cover, i.e., there exists a finite special NPC complex Y and a covering map $\varphi : Y \rightarrow X$. We will call a strongly regular event structure $\mathcal{E} = (E, \leq, \#)$ and its domain $\mathcal{D}(\mathcal{E})$ *cover-special* if $\mathcal{D}(\mathcal{E})$ is isomorphic to a principal filter of the universal cover of some virtually special complex with an admissible orientation.

Theorem 5.2. *Any cover-special event structure \mathcal{E} admits a regular trace labelling, i.e., Thiagarajan's conjecture is true for cover-special event structures.*

Proof. Let $\mathcal{D} = \mathcal{D}(\mathcal{E})$ be the domain of \mathcal{E} and suppose that \mathcal{D} is the principal filter $\mathcal{D} = (\mathcal{F}_{\tilde{o}}(\tilde{v}, \tilde{X}^{(1)}), \prec_{\tilde{o}})$ of (\tilde{X}, \tilde{o}) for a virtually special complex X and an admissible orientation o of its edges. Let Y be a finite special cover of X and let $\varphi : Y \rightarrow X$ be a covering map. Let o' be the orientation of the edges of Y obtained from o via φ . Note that (X, o) and (Y, o') have the same universal cover $(\tilde{X}, \tilde{o}) = (\tilde{Y}, \tilde{o}')$

In particular, the principal filter $\mathcal{D} = (\mathcal{F}_{\tilde{o}}(\tilde{v}, \tilde{X}^{(1)}), \prec_{\tilde{o}})$ of (\tilde{X}, \tilde{o}) is the principal filter $(\mathcal{F}_{\tilde{o}'}(\tilde{v}, \tilde{Y}^{(1)}), \prec_{\tilde{o}'})$ of (\tilde{Y}, \tilde{o}') . Since Y is finite and special, by Proposition 5.1 there exists an independence relation I on the hyperplanes $\mathcal{H} = \mathcal{H}(Y)$ of Y such that the canonical labeling $\tilde{\lambda}_{\mathcal{H}}$ of $\mathcal{D} = (\mathcal{F}_{\tilde{o}'}(\tilde{v}, \tilde{Y}^{(1)}), \prec_{\tilde{o}'})$ is a regular trace labeling with the trace alphabet (\mathcal{H}, I) . Therefore, Thiagarajan's conjecture holds for the event domain \mathcal{D} . \square

5.3. Strongly hyperbolic regular event structures. In this subsection, we show that Thiagarajan's conjecture holds for a large and natural class of strongly regular event structures, namely those arising from hyperbolic CAT(0) cube complexes. It turns out that strongly hyperbolic regular event structures are cover-special. This is a consequence of the solution by Agol [1] of the virtual Haken conjecture for hyperbolic 3-manifolds. This breakthrough result of Agol is based on the theory of special cube complexes developed by Haglund and Wise [25, 26].

Similarly to nonpositive curvature, Gromov hyperbolicity is defined in metric terms. However, as for the CAT(0) property, the hyperbolicity of a CAT(0) cube complex can be expressed

in a purely combinatorial way. A metric space (X, d) is δ -hyperbolic [14, 22] if for any four points v, w, x, y of X , the two largest of the distance sums $d(v, w) + d(x, y)$, $d(v, x) + d(w, y)$, $d(v, y) + d(w, x)$ differ by at most $2\delta \geq 0$. A graph $G = (X, E)$ endowed with its standard graph-distance d_G is δ -hyperbolic if the metric space (X, d_G) is δ -hyperbolic. In case of geodesic metric spaces and graphs, δ -hyperbolicity can be defined in other equivalent ways, e.g., via thin or slim geodesic triangles. For example, a geodesic metric space (X, d) is 2δ -hyperbolic, if all geodesic triangles $\Delta(x, y, z)$ of (X, d) are δ -slim, i.e., for any point u on the side $[x, y]$ the distance from u to $[x, z] \cup [z, y]$ is at most δ . This definition expresses the negative curvature of a geodesic metric space. A metric space (X, d) is hyperbolic if there exists $\delta < \infty$ such that (X, d) is δ -hyperbolic. In case of median graphs, i.e., of 1-skeletons of CAT(0) cube complexes, the hyperbolicity can be characterized in the following way:

Lemma 5.3 ([19, 23]). *Let X be a CAT(0) cube complex. Then its 1-skeleton $X^{(1)}$ is hyperbolic if and only if all isometrically embedded square grids are uniformly bounded.*

We call an event structure $\mathcal{E} = (E, \leq, \#)$ and its domain $\mathcal{D}(\mathcal{E})$ hyperbolic if $\mathcal{D}(\mathcal{E})$ is isomorphic to a principal filter of a directed CAT(0) cube complex, whose 1-skeleton is hyperbolic. We call an event structure $\mathcal{E} = (E, \leq, \#)$ and its domain $\mathcal{D}(\mathcal{E})$ strongly hyperbolic regular if there exists a finite directed NPC complex (X, o) such that \tilde{X} is hyperbolic and \mathcal{D} is a principal filter of $(\tilde{X}^{(1)}, \tilde{o})$. Note that an event structure can be strongly regular and hyperbolic without being strongly regular hyperbolic (see Remark 7.2).

Hyperbolic CAT(0) cube complexes with uniformly bounded degrees have several strong and nice properties. It was shown in [24] that such CAT(0) cube complexes can be isometrically embedded into the Cartesian product of finitely many trees. Analogously to the nice labeling conjecture of [39], a similar result does not hold for general CAT(0) cube complexes of uniformly bounded degrees [20]. Modifying the arguments of [24] it can be shown that hyperbolic event structures with bounded degrees admit finite nice labelings (these labelings are not necessarily regular). Again this does not hold for general event structures (see Subsection 3.4).

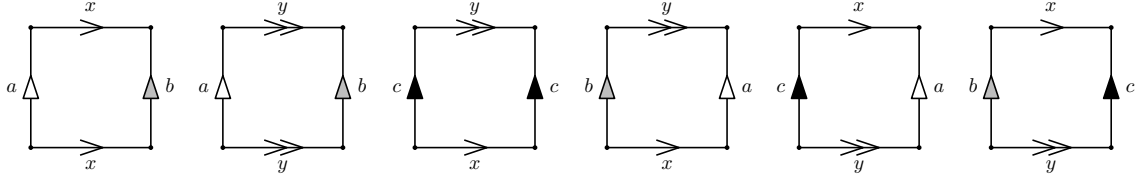
The main result of this section is based on the following very deep and important result of Agol [1], following much work of Haglund and Wise [25, 26]. Agol's original result is formulated in group-theoretical terms. Its following reformulation (see, for example, [12, Theorem 6.7]) in the particular case of finite NPC complexes is particularly appropriate for our purposes:

Theorem 5.4 ([1]). *Let X be a finite nonpositively curved cube complex. If the fundamental group $\pi_1(X)$ of X is hyperbolic, then X is virtually special.*

The condition that $\pi_1(X)$ is hyperbolic is equivalent to the fact that the universal cover \tilde{X} of X is hyperbolic. Indeed, it is well-known that $\pi_1(X)$ acts properly by deck transformations on \tilde{X} ; see [27] and [14, Remark 8.3(2)]. Since X is finite, this action of $\pi_1(X)$ on \tilde{X} is cocompact. Consequently, $\pi_1(X)$ acts properly and cocompactly by isometries on \tilde{X} . By Švarc-Milnor lemma [14, Proposition 8.19], the Cayley graph of $\pi_1(X)$ is quasi-isometric to \tilde{X} . Since hyperbolicity is an invariant of quasi-isometry [14, Theorem 1.9], $\pi_1(X)$ is hyperbolic if and only if \tilde{X} is hyperbolic. Therefore, any finite NPC complex X that has a hyperbolic universal cover is virtually special.

Theorem 5.5. *Any strongly hyperbolic regular event structure admits a regular trace labeling, i.e., Thiagarajan's conjecture is true for strongly hyperbolic regular event structures.*

Proof. Let $\mathcal{D} = \mathcal{D}(\mathcal{E})$ be the domain of a strongly hyperbolic regular event structure \mathcal{E} . Consider a finite NPC complex (X, o) such that \tilde{X} is hyperbolic and \mathcal{D} is the principal filter $(\mathcal{F}_{\tilde{o}}(\tilde{v}, \tilde{X}^{(1)}), \prec_{\tilde{o}})$ for some $\tilde{v} \in \tilde{X}$. By Theorem 5.4 of Agol, finite NPC complexes with hyperbolic universal covers are virtually special, thus \mathcal{E} is a cover-special event structure. By Theorem 5.2, Thiagarajan's conjecture is true for \mathcal{E} . \square


FIGURE 4. The 6 squares defining the complex \mathbf{X}

6. WISE'S EVENT DOMAIN $(\widetilde{W}_{\tilde{v}}, \prec_{\tilde{\sigma}^*})$

In this section, we construct the domain $(\widetilde{W}_{\tilde{v}}, \prec_{\tilde{\sigma}^*})$ of a regular event structure (with bounded \natural -cliques) that does not admit a regular nice labeling. To do so, we start with a directed colored CSC (complete square complex) \mathbf{X} introduced by Wise [51]. Recall that in such complexes, the edges are classified vertical or horizontal, each edge has an orientation and a color, and any two incident edges belong to a square.

6.1. Wise's square complex \mathbf{X} and its universal cover $\widetilde{\mathbf{X}}$. The complex \mathbf{X} consists of six squares as indicated in Figure 4 (reproducing Figure 3 of [51]). Each square has two vertical and two horizontal edges. The horizontal edges are oriented from left to right and vertical edges from bottom to top. Denote this orientation of edges by o . The vertical edges of squares are colored white, grey, and black and denoted a, b , and c , respectively. The horizontal edges of squares are colored by single or double arrow, and denoted x and y , respectively. The six squares are glued together by identifying edges of the same color and respecting the directions to obtain the square complex \mathbf{X} . Note that \mathbf{X} has a unique vertex, five edges, and six squares. It can be directly checked that \mathbf{X} is a complete square complex, and consequently (X, o) is a directed NPC complex. Let H_X denote the subcomplex of X consisting of the 2 horizontal edges and let V_X denote the subcomplex of X consisting of the 3 vertical edges.

The universal cover \widetilde{H}_X of H_X is the 4-regular infinite tree F_4 . Its edges inherit the orientations from their images in H_X : each vertex of \widetilde{H}_X has two incoming and two outgoing arcs. Analogously, the universal cover \widetilde{V}_X of V_X is the 6-regular infinite tree F_6 where each vertex has three incoming and three outgoing arcs. Let \tilde{v}_1 be any vertex of \widetilde{H}_X . Then the principal filter of \tilde{v}_1 is the infinite binary tree T_2 rooted at \tilde{v}_1 : all its vertices except \tilde{v}_1 have one incoming and two outgoing arcs, while \tilde{v}_1 has two outgoing arcs and no incoming arc. Analogously, the principal filter of any vertex \tilde{v}_2 in the ordered set \widetilde{V}_X is the infinite ternary tree T_3 rooted at \tilde{v}_2 .

Let $\widetilde{\mathbf{X}}$ be the universal cover of \mathbf{X} and let $\varphi: \widetilde{\mathbf{X}} \rightarrow \mathbf{X}$ be a covering map. Let \widetilde{X} denote the support of $\widetilde{\mathbf{X}}$. Since \mathbf{X} is a CSC, by [51, Theorem 3.8], \widetilde{X} is the Cartesian product $F_4 \times F_6$ of the trees F_4 and F_6 . The edges of $\widetilde{\mathbf{X}}$ are colored and oriented as their images in \mathbf{X} , and are also classified as horizontal or vertical edges. The squares of $\widetilde{\mathbf{X}}$ are oriented as their images in \mathbf{X} , thus two opposite edges of the same square of $\widetilde{\mathbf{X}}$ have the same direction. This implies that all classes of parallel edges of $\widetilde{\mathbf{X}}$ are oriented in the same direction. Denote this orientation of the edges of $\widetilde{\mathbf{X}}$ by \tilde{o} . The 1-skeleton $\widetilde{X}^{(1)}$ of $\widetilde{\mathbf{X}}$ together with \tilde{o} is a directed median graph. Let $\tilde{v} = (\tilde{v}_1, \tilde{v}_2)$ be any vertex of \widetilde{X} , where \tilde{v}_1 and \tilde{v}_2 are the coordinates of \tilde{v} in the trees F_4 and F_6 . Then the principal filter $\mathcal{F}_{\tilde{o}}(\tilde{v}, \widetilde{X}^{(1)})$ of \tilde{v} is the Cartesian product of the principal filters of \tilde{v}_1 in F_4 and of \tilde{v}_2 in F_6 , i.e., is isomorphic to $T_2 \times T_3$.

By Lemma 4.1, the orientation of the edges of $\mathcal{F}_{\tilde{o}}(\tilde{v}, \widetilde{X}^{(1)})$ corresponds to the canonical basepoint orientation of $\mathcal{F}_{\tilde{o}}(\tilde{v}, \widetilde{X}^{(1)})$ with \tilde{v} as the basepoint. Moreover, by Proposition 4.3, $\mathcal{F}_{\tilde{o}}(\tilde{v}, \widetilde{X}^{(1)})$ is the domain of a regular event structure with one isomorphism type of principal filters. We summarize this in the following result:

Lemma 6.1. *For any vertex \tilde{v} of \widetilde{X} , $\mathcal{F}_{\tilde{o}}(\tilde{v}, \widetilde{X}^{(1)})$ is the domain of a regular event structure with one isomorphism class of futures.*

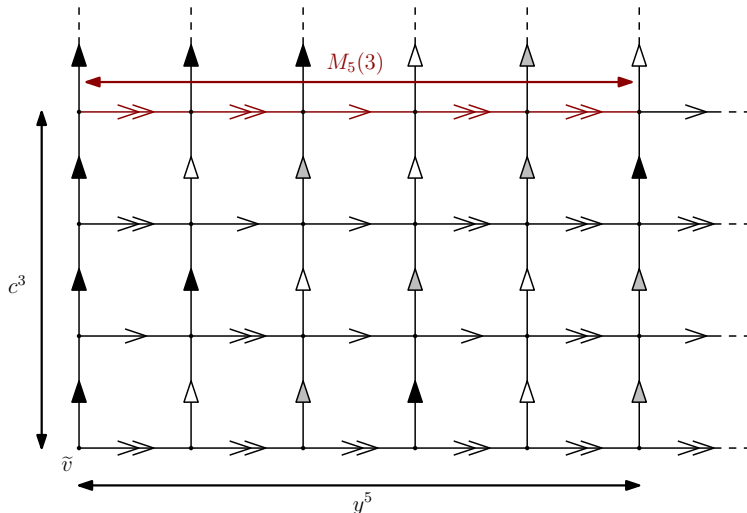


FIGURE 5. Part of the plane Π_{yc}^{++} appearing in $\tilde{\mathbf{X}}$

6.2. Aperiodicity of $\tilde{\mathbf{X}}$. We recall here the main properties of $\tilde{\mathbf{X}}$ established in [51, Section 5]. Let $\tilde{v} = (\tilde{v}_1, \tilde{v}_2)$ be an arbitrary vertex of $\tilde{\mathbf{X}}$, where \tilde{v}_1 and \tilde{v}_2 are defined as before. From the definition of the covering map, the loop of \mathbf{X} colored y gives rise to a bi-infinite horizontal path P_y of $\tilde{\mathbf{X}}^{(1)}$ passing via \tilde{v} and whose all edges are colored y and are directed from left to right. Analogously, there exists a bi-infinite vertical path P_c of $\tilde{\mathbf{X}}^{(1)}$ passing via \tilde{v} and whose all edges are colored c and are directed from bottom to top.

The projection of P_y on the horizontal factor F_4 is a bi-infinite path P^h of F_4 passing via \tilde{v}_1 . Analogously, the projection of P_c on the vertical factor F_6 is a bi-infinite path P^v of F_6 passing via \tilde{v}_2 . Consequently, the convex hull $\text{conv}(P_y \cup P_c)$ of $P_y \cup P_c$ in the graph $\tilde{\mathbf{X}}^{(1)}$ is isomorphic to the Cartesian product of $P^h \times P^v$ of the paths P^h and P^v . Therefore the subcomplex of $\tilde{\mathbf{X}}$ spanned by $\text{conv}(P_y \cup P_c)$ is a directed plane Π_{yc} tiled into squares (recall that each square is of one of 6 types and its sides are colored by the letters a, b, c, x, y), see Figure 5. Wise showed that the plane Π_{yc} is not tiled periodically by the preimages of the squares of \mathbf{X} .

Theorem 6.2 ([51, Theorem 5.3]). *The plane Π_{yc} tiled into squares is not doubly periodic.*

In our counterexample we will use the following result of [51] that was used to show that the plane Π_{yc} is not tiled periodically by the preimages of the squares of \mathbf{X} . Denote by P_y^+ the (directed) subpath of P_y having \tilde{v} as a source (this is a one-infinite horizontal path). Analogously, let P_c^+ be the (directed) subpath of P_c having \tilde{v} as a source. The convex hull of $P_y^+ \cup P_c^+$ is a quarter of the plane Π_{yc} , which we denote by Π_{yc}^{++} . Any shortest path in $\tilde{\mathbf{X}}^{(1)}$ from \tilde{v} to a vertex $\tilde{u} \in \Pi_{yc}^{++}$ can be viewed as a word in the alphabet $A = \{a, b, c, x, y\}$. For an integer $n \geq 0$, denote by y^n the horizontal subpath of P_y^+ beginning at \tilde{v} and having length n . Analogously, for an integer $m \geq 0$, denote by c^m the vertical subpath of P_c^+ beginning at \tilde{v} and having length m . Let $M_n(m)$ denote the horizontal path of Π_{yc}^{++} of length n beginning at the endpoint of the vertical path c^m . $M_n(m)$ determines a word which is the label of the side opposite to y^n in the rectangle which is the convex hull of y^n and c^m (see Figure 5). Let $M_n(m)$ also denote this corresponding word.

Proposition 6.3 ([51, Proposition 5.9]). *For each n , the words $\{M_n(m) : 0 \leq m \leq 2^n - 1\}$ are all distinct, and thus, every positive word in x and y of length n is $M_n(m)$ for some m .*

This proposition is called in [51] “period doubling”. It immediately establishes Theorem 6.2 because it shows that the period of the infinite vertical strip of Π_{yc}^{++} of width n and bounded on the left by the path P_c^+ has period 2^n . Alternatively, every positive word in x and y appears in Π_{yc}^{++} , and thus Π_{yc} cannot be periodic.

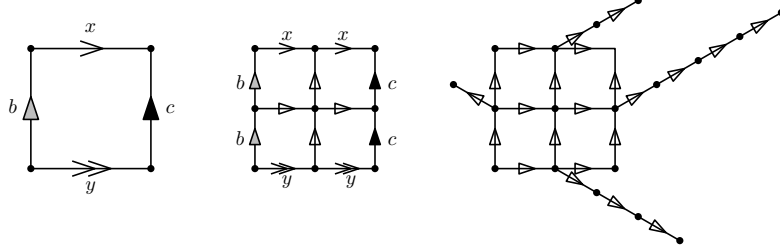


FIGURE 6. A square of \mathbf{X} and the corresponding subcomplexes in $(\beta\mathbf{X}, o')$ and (W, o^*)

6.3. The square complex W and its universal cover \widetilde{W} . Let $\beta\mathbf{X}$ denote the first barycentric subdivision of \mathbf{X} : each square C of \mathbf{X} is subdivided into four squares C_1, C_2, C_3, C_4 by adding a middle vertex to each edge of C and connecting it to the center of C by an edge. This way each edge e of C is subdivided into two edges e_1, e_2 , which inherit the orientation and the color of e . The four edges connecting the middle vertices of the edges of C to the center of C are oriented from left to right and from bottom to top (see the middle figure of Figure 6). Denote the resulting orientation by o' . This way, $(\beta\mathbf{X}, o')$ is a directed and colored square complex. Again, denote by βX the support of $\beta\mathbf{X}$. The universal cover $\widetilde{\beta X}$ of βX is the Cartesian product $\beta F_4 \times \beta F_6$ of the trees βF_4 and βF_6 , where βF_4 is the first barycentric subdivision of F_4 and βF_6 is the first barycentric subdivision of F_6 . Additionally, $(\widetilde{\beta X}, \widetilde{o'})$ is a directed CAT(0) square complex. We assign a *type* to each vertex of $\widetilde{\beta X}$: the preimage of the unique vertex of \mathbf{X} is of type 0 and is called a *0-vertex*, the preimages of the middles of edges of \mathbf{X} are of type 1 and are called *1-vertices*, and the preimages of centers of squares of \mathbf{X} are of type 2 and are called *2-vertices*.

To encode the colors of the edges of \mathbf{X} , we introduce our central object, the square complex W (whose edges are no longer colored). Let $A = \{a, b, c, x, y\}$ and let $r : A \rightarrow \{1, 2, 3, 4, 5\}$ be a bijective map. The complex W is obtained from βX by adding to each 1-vertex z of βX a path R_z of length $r(\alpha)$ if z is the middle of an edge colored $\alpha \in A$ in \mathbf{X} . The path R_z has one end at z (called the *root* of R_z) and z is the unique common vertex of R_z and βX (we call such added paths R_z *tips*).

The square complex W has 27 vertices: the unique vertex of \mathbf{X} , the 6 vertices which are the barycenters of the original squares, 5 vertices which are the barycenters of the original edges of \mathbf{X} , and 15 vertices which are new vertices lying on tips. The complex W has 49 edges: 10 corresponding to the 5 original edges that have been subdivided, 24 connecting the barycenters of the original squares to the barycenters of the original edges and 15 forming the tips. The complex W has 24 squares: 4 for each original square.

Denote by o^* the orientation of the edges of W defined as follows: the edges of βX are oriented as in $(\beta X, o')$ and the edges of tips are oriented away from their roots (see the rightmost figure of Figure 6 for the encoding of the last square of Figure 4). As a result, we obtain a finite directed NPC square complex (W, o^*) .

Consider the universal cover \widetilde{W} of W . It can be viewed as the complex $\widetilde{\beta X}$ with a path of length $r(\alpha)$ added to each 1-vertex which encodes an edge of $\widetilde{\mathbf{X}}$ of color $\alpha \in A$. We say that the vertices of \widetilde{W} lying only on tips are of type 3 and they are called *3-vertices*. Let \widetilde{o}^* denote the orientation of the edges of \widetilde{W} induced by the orientation o^* of W . Then $(\widetilde{W}, \widetilde{o}^*)$ is a directed CAT(0) square complex. Since W is finite, by Proposition 4.3, the directed median graph $(\widetilde{W}^{(1)}, \widetilde{o}^*)$ has a finite number of isomorphism types of principal filters $\mathcal{F}_{\widetilde{o}^*}(z, \widetilde{W}^{(1)})$.

Let \widetilde{v} be any 0-vertex of \widetilde{W} . Denote by $\widetilde{W}_{\widetilde{v}}$ the principal filter $\mathcal{F}_{\widetilde{o}^*}(\widetilde{v}, \widetilde{W}^{(1)})$ of \widetilde{v} in $(\widetilde{W}^{(1)}, \widetilde{o}^*)$. By Proposition 4.3, $\widetilde{W}_{\widetilde{v}}$ together with the partial order $\prec_{\widetilde{o}^*}$ is the domain of a regular event structure, which we call *Wise's event domain*. Since vertices of different types of \widetilde{W} are incident to a different number of outgoing squares, any isomorphism between two filters of $(\widetilde{W}_{\widetilde{v}}, \prec_{\widetilde{o}^*})$ preserves the types of vertices. We summarize all this in the following:

Proposition 6.4. $(\widetilde{W}_{\tilde{v}}, \prec_{\tilde{\sigma}^*})$ is the domain of a regular event structure. Any isomorphism between any two filters of $(\widetilde{W}_{\tilde{v}}, \prec_{\tilde{\sigma}^*})$ preserves the types of vertices.

6.4. $(\widetilde{W}_{\tilde{v}}, \prec_{\tilde{\sigma}^*})$ **does not have a regular nice labeling.** In this subsection we prove that the event structure associated with Wise's regular event domain is a counterexample to Thiagarajan's conjecture.

Theorem 6.5. $(\widetilde{W}_{\tilde{v}}, \prec_{\tilde{\sigma}^*})$ does not admit a regular nice labeling. Consequently, Conjectures 1.1 and 2.2 are false.

Proof. Since $\widetilde{W}_{\tilde{v}}$ is the principal filter of a 0-vertex \tilde{v} , $\widetilde{W}_{\tilde{v}}$ contains all vertices of $\widetilde{\mathbf{X}}$ located in the quarter of plane Π_{yc}^{++} of $\widetilde{\mathbf{X}}$, in particular it contains the vertices of the paths P_c^+ and P_y^+ . Notice also that $\widetilde{W}_{\tilde{v}}$ contains the barycenters and the tips corresponding to the edges of Π_{yc}^{++} .

Suppose by way of contradiction that $\widetilde{W}_{\tilde{v}}$ has a regular nice labeling λ . Since $\widetilde{W}_{\tilde{v}}$ has only a finite number of isomorphism types of labeled filters, the vertical path P_c^+ contains two 0-vertices, \tilde{z}' and \tilde{z}'' , which have isomorphic labeled principal filters. Let \tilde{z}' be the end of the vertical subpath c^k of P_c^+ and \tilde{z}'' be the end of the vertical subpath c^m of P_c^+ , and suppose without loss of generality that $k < m$. Let $n > 0$ be a positive integer such that $m \leq 2^n - 1$. Consider the horizontal convex paths $M_n(k)$ and $M_n(m)$ of Π_{yc}^{++} of length n beginning at the vertices \tilde{z}' and \tilde{z}'' , respectively. For any $0 \leq i \leq n$, denote by $\tilde{z}_{k,i}$ the i th vertex of $M_n(k)$ (in particular, $\tilde{z}_{k,0} = \tilde{z}'$). Analogously, denote by $\tilde{z}_{m,i}$ the i th vertex of $M_n(m)$ (in particular, $\tilde{z}_{m,0} = \tilde{z}''$). In $\widetilde{W}_{\tilde{v}}$, the paths $M_n(k)$ and $M_n(m)$ give rise to two convex horizontal paths $M_n^*(k)$ and $M_n^*(m)$ obtained from $M_n(k)$ and $M_n(m)$ by subdividing their edges. Denote by $\tilde{u}_{k,i}$ the unique common neighbor of $\tilde{z}_{k,i}$ and $\tilde{z}_{k,i+1}$, $0 \leq i < n$, in $M_n^*(k)$ (and in $\widetilde{W}^{(1)}$). Analogously, denote by $\tilde{u}_{m,i}$ the unique common neighbor of $\tilde{z}_{m,i}$ and $\tilde{z}_{m,i+1}$, $0 \leq i < n$ (see Figure 7). The paths $M_n^*(k)$ and $M_n^*(m)$ belong to the principal filters $\mathcal{F}_{\tilde{\sigma}^*}(\tilde{z}', \widetilde{W}^{(1)})$ and $\mathcal{F}_{\tilde{\sigma}^*}(\tilde{z}'', \widetilde{W}^{(1)})$, respectively.

By Proposition 6.3, the words $M_n(k)$ and $M_n(m)$ are different. Let f be an isomorphism between the filters $\mathcal{F}_{\tilde{\sigma}^*}(\tilde{z}_{k,0}, \widetilde{W}^{(1)})$ and $\mathcal{F}_{\tilde{\sigma}^*}(\tilde{z}_{m,0}, \widetilde{W}^{(1)})$. Since the words $M_n(k)$ and $M_n(m)$ are different, from the choice of the lengths of tips in the complexes W and \widetilde{W} it follows that f cannot map the path $M_n^*(k)$ to the path $M_n^*(m)$ by a vertical translation, i.e., there exists an index $0 \leq j < n$ such that $f(\tilde{z}_{k,j+1}) \neq \tilde{z}_{m,j+1}$; let i be the smallest such index. Set $\tilde{z} := f(\tilde{z}_{k,i+1})$ and $\tilde{u} := f(\tilde{u}_{k,i})$. Since f preserves the types of vertices, \tilde{z} is a 0-vertex and \tilde{u} is a 1-vertex. Since f maps a convex path $M_n^*(k)$ to a convex path, \tilde{u} is the unique common neighbor of $\tilde{z}_{m,i}$ and \tilde{z} . Since each 1-vertex is the barycenter of a unique edge of $\widetilde{\mathbf{X}}$ and $\tilde{z} \neq \tilde{z}_{m,i+1}$, we deduce that $\tilde{u} \neq \tilde{u}_{m,i}$. The edge $\tilde{z}_{k,i}\tilde{u}_{k,i}$ is directed from $\tilde{z}_{k,i}$ to $\tilde{u}_{k,i}$. Analogously the edges $\tilde{z}_{m,i}\tilde{u}_{m,i}$ and $\tilde{z}_{m,i}\tilde{u}$ are directed from $\tilde{z}_{m,i}$ to $\tilde{u}_{m,i}$ and \tilde{u} , respectively. Since $\tilde{z}_{k,i}\tilde{u}_{k,i}$ and $\tilde{z}_{m,i}\tilde{u}_{m,i}$ are parallel edges, they define the same event and therefore $\lambda(\tilde{z}_{k,i}\tilde{u}_{k,i}) = \lambda(\tilde{z}_{m,i}\tilde{u}_{m,i})$. On the other hand, since f maps the edge $\tilde{z}_{k,i}\tilde{u}_{k,i}$ to the edge $\tilde{z}_{m,i}\tilde{u}$ and since the map f preserves the labels, we have $\lambda(\tilde{z}_{k,i}\tilde{u}_{k,i}) = \lambda(\tilde{z}_{m,i}\tilde{u})$. As a result, $\tilde{z}_{m,i}$ has two outgoing edges, $\tilde{z}_{m,i}\tilde{u}_{m,i}$ and $\tilde{z}_{m,i}\tilde{u}$, having the same label, contrary to the assumption that λ is a nice labeling. This contradiction shows that $(\widetilde{W}_{\tilde{v}}, \prec_{\tilde{\sigma}^*})$ does not admit a regular nice labeling. By Proposition 4.3, $(\widetilde{W}_{\tilde{v}}, \prec_{\tilde{\sigma}^*})$ is the domain of a regular event structure, establishing that Conjectures 1.1 and 2.2 are false. This concludes the proof of the theorem. \square

6.5. $(\widetilde{W}_{\tilde{v}}, \prec_{\tilde{\sigma}^*})$ **has bounded \natural -cliques.** In this section, we show that our counterexample to Thiagarajan's conjecture also provides a counterexample to Conjecture 2.3 (and thus to Conjecture 1.2) of Badouel et al [5]. In [5], the conjecture was stated for conflict event domains that are more general than the domain of event structures we consider in this paper. However, we show in the next proposition that their conjecture does not hold even for the domains of event structures.

Proposition 6.6. Wise's event domain $(\widetilde{W}_{\tilde{v}}, \prec_{\tilde{\sigma}^*})$ has bounded \natural -cliques. Consequently, $(\widetilde{W}_{\tilde{v}}, \prec_{\tilde{\sigma}^*})$ is a counterexample to Conjectures 1.2 and 2.3.

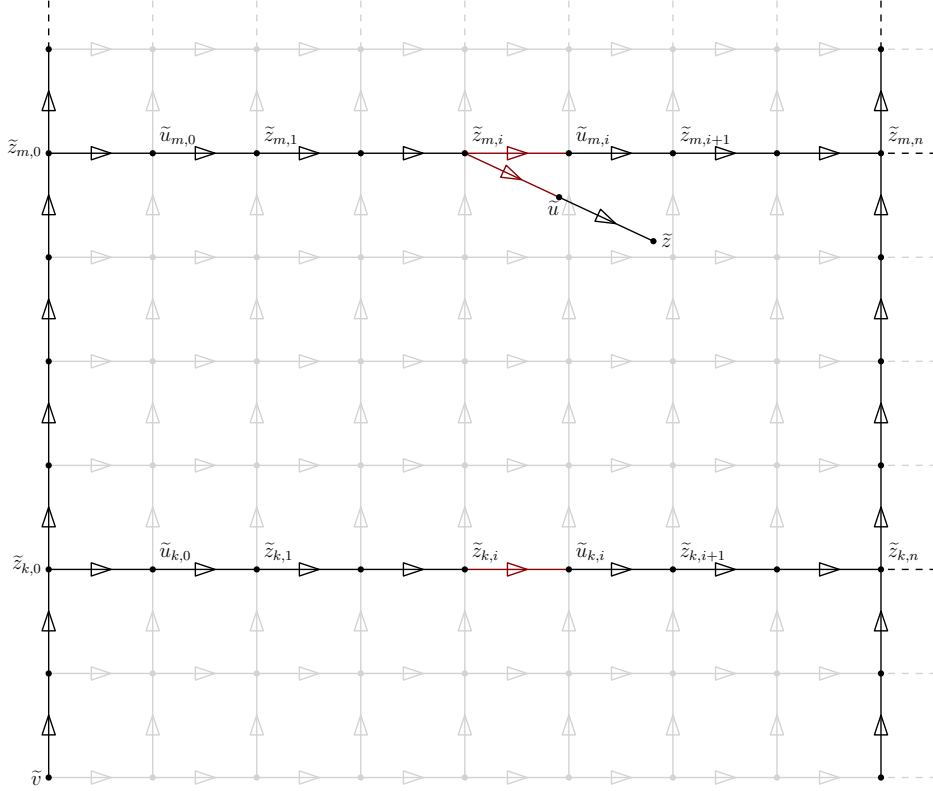


FIGURE 7. To the proof of Theorem 6.5

Proof. By Proposition 4.3, $(\widetilde{W}_{\tilde{v}}, \prec_{\tilde{\sigma}^*})$ is the domain of a regular event structure. Recall that each event corresponds to a class of parallel edges of $\widetilde{W}_{\tilde{v}}^{(1)}$. We refer to the events of $(\widetilde{W}_{\tilde{v}}, \prec_{\tilde{\sigma}^*})$ as *vertical*, *horizontal*, and *tip-events* depending of the type of edges from their parallelism class.

Claim. *If $e_1 \not\sharp e_2$ and e_1 and e_2 are either both vertical or both horizontal, then $e_1 \#_{\mu} e_2$.*

Proof. Without loss of generality, assume that both events e_1 and e_2 are vertical, and note that e_1 and e_2 cannot be concurrent. Suppose by way of contradiction that $e_1 \not\sharp_{(3)} e_2$. Then there exists an event e_3 such that $e_1 \parallel e_3$, $e_2 \#_{\mu} e_3$ and e_3 is co-initial with e_1 and e_2 at two different configurations. Since $e_1 \parallel e_3$ and e_1 is vertical, the event e_3 cannot be vertical or a tip-event. Hence e_3 is horizontal. From the definition of $\widetilde{W}_{\tilde{v}}$ it follows that the horizontal and vertical edges come from the Cartesian product of two trees. Therefore any pair of horizontal and vertical events defines a square of $\widetilde{W}_{\tilde{v}}$, thus they are concurrent. This contradicts the fact that $e_3 \#_{\mu} e_2$ and establishes the claim. \square

Let Q be a \sharp -clique of $\widetilde{W}_{\tilde{v}}$. We asserts that the size of Q is at most 11. Suppose that $|Q| \geq 12$. From the definition of $(\widetilde{W}, \tilde{\sigma}^*)$ it follows that $(\widetilde{W}_{\tilde{v}}, \prec_{\tilde{\sigma}^*})$ has degree 5: the out-degree of any 0-vertex is 5, the out-degree of any 1-vertex is either 4 or 5, the out-degree of any 2-vertex is 2, and the out-degree of any 3-vertex is either 0 or 1. This implies that the maximum number of events of Q that are pairwise concurrent or in minimal conflict is 5. From the definition of $(\widetilde{W}_{\tilde{v}}, \prec_{\tilde{\sigma}^*})$ it also follows that two tip-events cannot be concurrent or in minimal conflict. Also from condition (3) in the definition of \sharp it immediately follows that Q cannot contain two tip-events e_1 and e_2 such that $e_1 \not\sharp_{(3)} e_2$. Indeed, if this happen, then there exists an event e_3 such that $e_1 \parallel e_3$, thus e_1 and e_3 cannot be tip-events. Consequently, the \sharp -clique Q contains at most one tip-event. Since $|Q| \geq 12$, Q contains at least 6 vertical or horizontal events, say Q contains a subset Q' of 6 vertical events. Since all events of Q' are vertical, they are not pairwise concurrent. Since Q' is a \sharp -clique and at most 5 events of Q' can be pairwise in minimal conflict,

this implies that Q' must contain two events e_1, e_2 such that $e_1 \natural_{(3)} e_2$. But this is impossible by the claim. Therefore $(\widetilde{W}_{\tilde{v}}, \prec_{\tilde{\sigma}^*})$ is a regular conflict event domain with bounded \natural -cliques and bounded degree. Since by Theorem 6.5 $(\widetilde{W}_{\tilde{v}}, \prec_{\tilde{\sigma}^*})$ does not admit a regular nice labeling, this shows that Conjecture 2.3 is false. \square

Remark 6.7. In the proof of Proposition 6.6, we use the fact that any pair of horizontal and vertical events are concurrent. This property holds because \mathbf{X} is a CSC (complete square complex). Note that the fact that \mathbf{X} is a CSC is not an essential property of \mathbf{X} in the proof of Theorem 6.5.

Consequently, if we want to adapt the proof of Theorem 6.5 to other square complexes to find other counterexamples to Thiagarajan's Conjecture 1.1, it may be sufficient to consider VH -complexes (see Section 7), but in order to use the arguments in the proof of Proposition 6.6 to find other counterexamples to Badouel et al.'s Conjecture 1.2, we need to consider complete square complexes.

7. APERIODIC TILINGS AND REGULAR EVENT STRUCTURES

Our counterexample $(\widetilde{W}_{\tilde{v}}, \prec_{\tilde{\sigma}^*})$ of a regular 2-dimensional event domain without a regular labeling heavily uses the fact that the universal cover $\widetilde{\mathbf{X}}$ of Wise's complex \mathbf{X} [51] contains a particular aperiodic tiled plane (that is called *antitorus* by Wise). In this section, we show that the relationship between the existence of aperiodic planes and nonexistence of regular labelings is more general. Namely, we explain how to obtain other counterexamples from 4-way deterministic aperiodic tile sets.

Tiles (or *Wang-tiles*) are unit squares with colored edges. The edges of a Wang tile are called *top* (or *North*), *right* (or *East*), *bottom* (or *South*) and *left* (or *West*) edges in a natural way. A *tile set* T is a finite collection of Wang-tiles, placed with their edges horizontal and vertical. A *tiling* is a mapping $f : \mathbb{Z}^2 \rightarrow T$ that assigns a tile to each integer lattice point of the plane. A tiling f is *valid* if every two adjacent tiles have the same color on their common edge. Note that a tile may not be rotated or flipped, i.e., each tile has a bottom-top and left-right orientation. A tiling f is *periodic* with period $(a, b) \in \mathbb{Z}^2 \setminus \{(0, 0)\}$ if for every $(x, y) \in \mathbb{Z}^2$, $f(x, y) = f(x + a, y + b)$. If there exists a valid periodic tiling with tiles of T , then there exists a valid *doubly periodic* tiling with tiles of T [37], i.e., a valid tiling f and two integers $a, b > 0$ such that $f(x, y) = f(x + a, y) = f(x, y + b)$ for every $(x, y) \in \mathbb{Z}^2$. A tile set T is called *aperiodic* if there exists a valid tiling with tiles of T , and there does not exist any periodic valid tiling with tiles of T .

Let $T = \{t_1, \dots, t_n\}$ be a tile set. We consider each tile t_i as a unit square whose edges are directed and colored. Suppose that each square t_i has two vertical and two horizontal edges and suppose that the horizontal and the vertical edges of all squares are colored differently, i.e., the set of colors can be partitioned into horizontal colors and vertical colors. The horizontal edges are directed from left to right and the vertical edges are directed from bottom to top.

A Wang tile set is said to be *NW-deterministic* [29], if within the tile set there does not exist two different tiles that have the same colors on their top and left edges. *NE-deterministic*, *SW-deterministic*, and *SE-deterministic* tile sets are defined analogously. A Wang tile set is *4-way deterministic* [29] if it is NW-, NE-, SW-, and SE-deterministic. Kari and Papasoglu [29] presented a 4-way deterministic aperiodic tile set T_{KP} .

Given a 4-way deterministic set of tiles T , let $\mathbf{X}(T) = (X(T), o, \nu)$ be the finite square complex obtained by identifying all the vertices and gluing together the squares of T along the sides which have the same color respecting their orientation. Then $\mathbf{X}(T)$ is a VH -complex that has a unique vertex. Consequently, the universal cover $\widetilde{\mathbf{X}}(T)$ of $\mathbf{X}(T)$ is a $CAT(0)$ VH -complex. Denote by $W(T)$ the finite directed NPC complex derived from $\mathbf{X}(T)$ in the same way as the complex W was derived from Wise's complex \mathbf{X} in Subsection 6.3 (taking the first barycentric subdivision and adding tips of different lengths to encode the different colors). Let $(\widetilde{W}(T)_{\tilde{v}}, \prec_{\tilde{\sigma}^*})$ denote the 2-dimensional event domain derived from $\widetilde{\mathbf{X}}(T)$ in the same way as $(\widetilde{W}_{\tilde{v}}, \prec_{\tilde{\sigma}^*})$ was derived from $\widetilde{\mathbf{X}}$. Since $(\widetilde{W}(T)_{\tilde{v}}, \prec_{\tilde{\sigma}^*})$ comes from the universal cover of the finite

directed NPC complex $W(T)$, $(\widetilde{W}(T)_{\tilde{v}}, \prec_{\tilde{\sigma}^*})$ is a strongly regular event structure. The following lemma establishes a connection between the existence of valid tilings for 4-way deterministic tile sets and the existence of directed planes in the universal covers of the derived VH -complexes.

Lemma 7.1. *For a 4-way deterministic tile set T , the following conditions are equivalent:*

- (i) *there exists a valid tiling with the tiles of T ;*
- (ii) *the universal cover $\widetilde{\mathbf{X}}(T)$ of the square complex $\mathbf{X}(T)$ contains directed planes;*
- (iii) *the strongly regular domain $(\widetilde{W}(T)_{\tilde{v}}, \prec_{\tilde{\sigma}^*})$ is not hyperbolic.*

Proof. The implication (i) \Rightarrow (ii) is trivial and the implication (ii) \Rightarrow (iii) follows from Lemma 5.3. Suppose now that $(\widetilde{W}(T)_{\tilde{v}}, \prec_{\tilde{\sigma}^*})$ is not hyperbolic. Then by Lemma 5.3, for any integer k , the VH -complex $\widetilde{W}(T)_{\tilde{v}}$ contains a square grid of size $2k \times 2k$. The following claim implies that in such a grid, we can find a $k \times k$ directed square grid in the directed VH -complex $(\widetilde{W}(T)_{\tilde{v}}, \tilde{\sigma}^*)$.

Claim. *For any vertical (respectively, horizontal) edge \tilde{e} going from \tilde{u} to \tilde{w} and for any two squares Q_1, Q_2 in $(\widetilde{W}(T)_{\tilde{v}}, \tilde{\sigma}^*)$ intersecting on \tilde{e} , \tilde{u} cannot be the sink of both horizontal (respectively, vertical) edges of Q_1 and Q_2 incident to \tilde{u} .*

Proof. By way of contradiction, assume that u is the sink of the horizontal edges $\tilde{e}_1 = \tilde{u}_1\tilde{u}$ of Q_1 and $\tilde{e}_2 = \tilde{u}_2\tilde{u}$ of Q_2 . By Lemma 4.1, $\tilde{u}_1, \tilde{u}_2 \in I(\tilde{v}, \tilde{u})$ and the median \tilde{m} of \tilde{u}_1, \tilde{u}_2 , and \tilde{v} is adjacent to \tilde{u}_1, \tilde{u}_2 and at distance 2 from \tilde{u} . Consequently, $\tilde{u}\tilde{u}_1\tilde{m}\tilde{u}_2$ is a square of $\widetilde{W}(T)_{\tilde{v}}$ and thus of $\widetilde{W}(T)$ but since $\tilde{u}\tilde{u}_1$ and $\tilde{u}\tilde{u}_2$ are horizontal edges, this contradicts the fact that $\widetilde{\mathbf{X}}(T)$ is a VH -complex. \square

Consequently, we can tile arbitrary large squares of the plane with the tiles of T . By a folklore compactness result from tiling theory, this implies that we can find a valid tiling of the plane with the tiles of T , concluding the proof of (iii) \Rightarrow (i). \square

Note that if T is a 4-way deterministic aperiodic tile set, all the directed planes of $\widetilde{\mathbf{X}}(T)$ are tiled in an aperiodic way. In the case of the tile set of Wise [51] from Figure 4, the $\text{CAT}(0)$ square complex $\widetilde{\mathbf{X}}$ contains aperiodic directed planes but it also contains some periodic directed planes.

Remark 7.2. As explained in [29, Section 4], the universal cover $\widetilde{\mathbf{X}}(T)$ of the complex $\mathbf{X}(T)$ derived from a tile set T can contain periodic planes that are not directed. This may happen even if T does not tile the plane or if T is an aperiodic tile set.

For these reasons, if T does not tile the plane, the directed $\text{CAT}(0)$ complexes $\widetilde{\mathbf{X}}(T)$ and $\widetilde{W}(T)$ are not necessarily hyperbolic, even if all principal filters $(\widetilde{W}_{\tilde{v}}, \prec_{\tilde{\sigma}^*})$ are hyperbolic domains.

We now explain how to derive a counterexample to Thiagarajan's conjectures from any 4-way deterministic aperiodic tile set.

Theorem 7.3. *For any 4-way deterministic aperiodic tile set T , the NPC square complex $W(T)$ is not virtually special and the 2-dimensional event domain $(\widetilde{W}(T)_{\tilde{v}}, \prec_{\tilde{\sigma}^*})$ does not admit a regular nice labeling.*

Proof. Consider a 4-way deterministic aperiodic tile set T and the associated NPC square complexes $\mathbf{X}(T)$ and $W(T)$. Since T tiles the plane, every vertex $\tilde{v} \in \widetilde{\mathbf{X}}(T)$ is contained in a directed colored plane Π of $\widetilde{\mathbf{X}}(T)$. Note that the support Π of Π is the product of a directed path containing only horizontal edges and of a directed path containing only vertical edges. Consequently, in the directed $\text{CAT}(0)$ complex $(\widetilde{W}(T), \tilde{\sigma}^*)$, every 0-vertex \tilde{v} is contained in a directed plane Π^* where Π^* is the first barycentric subdivision of Π . Consequently, the directed $\text{CAT}(0)$ complex $(\widetilde{W}(T)_{\tilde{v}}, \tilde{\sigma}^*)$ contains a quarter of the directed plane Π^* that we denote by Π^{*++} . Note that Π^{*++} is the barycentric subdivision of a quarter of plane Π^{++} of the directed plane Π . Let P_H^* be the horizontal path of Π^{*++} containing \tilde{v} .

Suppose that $(\widetilde{W}(T)_{\widetilde{v}}, \prec_{\widetilde{\sigma}^*})$ admits a regular nice labeling λ . This implies that there exist two 0-vertices $\widetilde{y}, \widetilde{y}' \in V(P_H^*)$ that have isomorphic labeled principal filters. Let P_V^* and $P_V'^*$ be the vertical paths of Π^{*++} containing respectively \widetilde{y} and \widetilde{y}' . Let $P_V^* = (\widetilde{y} = \widetilde{y}_0, \widetilde{u}_0, \widetilde{y}_1, \widetilde{u}_1, \dots, \widetilde{y}_j, \widetilde{u}_j, \dots)$ and $P_V'^* = (\widetilde{y}' = \widetilde{y}'_0, \widetilde{u}'_0, \widetilde{y}'_1, \widetilde{u}'_1, \dots, \widetilde{y}'_j, \widetilde{u}'_j, \dots)$. Note that for every j , \widetilde{y}_j and \widetilde{y}'_j are 0-vertices while \widetilde{u}_j and \widetilde{u}'_j are 1-vertices. Note that $P_V = (\widetilde{y} = \widetilde{y}_0, \widetilde{y}_1, \dots, \widetilde{y}_j, \dots)$ and $P_V' = (\widetilde{y}' = \widetilde{y}'_0, \widetilde{y}'_1, \dots, \widetilde{y}'_j, \dots)$ are paths of Π^{*++} .

Note that for any j , $\widetilde{y}_j \widetilde{u}_j$ and $\widetilde{y}'_j \widetilde{u}'_j$ are parallel edges as well as $\widetilde{u}_j \widetilde{y}_{j+1}$ and $\widetilde{u}'_j \widetilde{y}'_{j+1}$. Consequently, $\lambda(\widetilde{y}_j \widetilde{u}_j) = \lambda(\widetilde{y}'_j \widetilde{u}'_j)$ and $\lambda(\widetilde{u}_j \widetilde{y}_{j+1}) = \lambda(\widetilde{u}'_j \widetilde{y}'_{j+1})$. Since λ is a nice labeling (and thus is deterministic), and since \widetilde{y} and \widetilde{y}' have isomorphic labeled principal filters, one can easily show by induction on j that for any j , \widetilde{y}_j and \widetilde{y}'_j (respectively, \widetilde{u}_j and \widetilde{u}'_j) have isomorphic labeled principal filters. Consequently, for any j , the tips attached to \widetilde{u}_j and \widetilde{u}'_j have the same length, i.e., the edges $\widetilde{y}_j \widetilde{y}_{j+1}$ and $\widetilde{y}'_j \widetilde{y}'_{j+1}$ have the same color $\nu(\widetilde{y}_j \widetilde{y}_{j+1}) = \nu(\widetilde{y}'_j \widetilde{y}'_{j+1})$ in $\widetilde{\mathbf{X}}(T)$.

Since λ is a regular nice labeling of $(\widetilde{W}(T)_{\widetilde{v}}, \prec_{\widetilde{\sigma}^*})$, there exists $0 \leq k < m$ such that \widetilde{y}_k and \widetilde{y}_m have isomorphic labeled principal filters. Let P_k^* and P_m^* be the horizontal paths of Π^{*++} going respectively from \widetilde{y}_k to \widetilde{y}'_k and from \widetilde{y}_m to \widetilde{y}'_m . Let ℓ be the distance from \widetilde{y}_k to \widetilde{y}'_k in $\widetilde{\mathbf{X}}(T)$ and let $P_k^* = (\widetilde{y}_k = \widetilde{y}_{k,0}, \widetilde{u}_{k,0}, \widetilde{y}_{k,1}, \widetilde{u}_{k,1}, \dots, \widetilde{u}_{k,\ell-1}, \widetilde{y}_{k,\ell} = \widetilde{y}'_k)$ and $P_m^* = (\widetilde{y}_m = \widetilde{y}_{m,0}, \widetilde{u}_{m,0}, \widetilde{y}_{m,1}, \widetilde{u}_{m,1}, \dots, \widetilde{u}_{m,\ell-1}, \widetilde{y}_{m,\ell} = \widetilde{y}'_m)$. Note that $P_k = (\widetilde{y}_k = \widetilde{y}_{k,0}, \widetilde{y}_{k,1}, \dots, \widetilde{y}_{k,\ell} = \widetilde{y}'_k)$ and $P_m = (\widetilde{y}_m = \widetilde{y}_{m,0}, \widetilde{y}_{m,1}, \dots, \widetilde{y}_{m,\ell} = \widetilde{y}'_m)$ are paths of the plane Π . Using the same arguments as for P_V^* and $P_V'^*$, one can show that for any $0 \leq i \leq \ell - 1$, the edges $\widetilde{y}_{k,j} \widetilde{y}_{k,j+1}$ and $\widetilde{y}_{m,j} \widetilde{y}_{m,j+1}$ have the same color $\nu(\widetilde{y}_{k,j} \widetilde{y}_{k,j+1}) = \nu(\widetilde{y}_{m,j} \widetilde{y}_{m,j+1})$ in $\widetilde{\mathbf{X}}(T)$.

Consider the rectangle R of Π with corners $\widetilde{y}_k, \widetilde{y}_m, \widetilde{y}'_m,$ and \widetilde{y}'_k . For any $k \leq j < m$, $\nu(\widetilde{y}_j \widetilde{y}_{j+1}) = \nu(\widetilde{y}'_j \widetilde{y}'_{j+1})$ in $\widetilde{\mathbf{X}}(T)$, i.e., the same sequence of colors appears on both vertical sides of R . Similarly, the same sequence of colors appears on both horizontal sides of the rectangle R . Since we can tile the plane by using copies of R , it is possible to find a periodic tiling of the plane using tiles of T . But this is impossible, since T is an aperiodic tile set. Consequently, the 2-dimensional event domain $(\widetilde{W}(T)_{\widetilde{v}}, \prec_{\widetilde{\sigma}^*})$ does not admit a regular nice labeling, and by Theorem 5.2, $W(T)$ is not virtually special. \square

Using the tile set T_{KP} of [29], Lukkarila [30] proved that for 4-way deterministic tile sets the tiling problem is undecidable. An immediate consequence of this result and of Theorem 7.3 is that there exists an infinite number of counterexamples to Conjecture 1.1.

Remark 7.4. Note that the VH -complex $W(T)$ derived from a 4-way deterministic tile set T is not necessarily a CSC complex. Consequently, we cannot directly generalize the proof of Proposition 6.6 to show that if T is aperiodic, then $(\widetilde{W}(T)_{\widetilde{v}}, \prec_{\widetilde{\sigma}^*})$ is a counterexample to Conjecture 1.2 (see Remark 6.7).

8. CONCLUSIONS AND OPEN QUESTIONS

8.1. Conclusions. In this paper, we presented an example of a regular event domain $(\widetilde{W}_{\widetilde{v}}, \prec_{\widetilde{\sigma}^*})$ with bounded degree and bounded \natural -cliques which does not admit a regular nice labeling, providing a counterexample to Conjecture 1.1 of Thiagarajan [43, 44] and Conjecture 1.2 of Badouel, Darondeau, and Raoult [5]. Furthermore, we show that this counterexample is not singular and that, in fact, there exists an infinite number of counterexamples to Conjecture 1.1 arising from the 4-way deterministic aperiodic tile sets constructed by Kari-Papasoglu [29] and Lukkarilla [30].

The event domain $(\widetilde{W}_{\widetilde{v}}, \prec_{\widetilde{\sigma}^*})$ is a principal filter of a directed 2-dimensional CAT(0) cube complex which is the universal cover of a finite directed colored CSC. At first, one can think that after trees, such cube complexes are the next simplest event domains on which Conjectures 1.1 or 1.2 must be true. Moreover, it was shown in [20] that any 2-dimensional CAT(0) cube complex of bounded degree admits a finite nice labeling. A finite nice labeling of $\widetilde{W}_{\widetilde{v}}$ can be also directly derived from the fact that \widetilde{W} is a product of two trees with attached tips of various

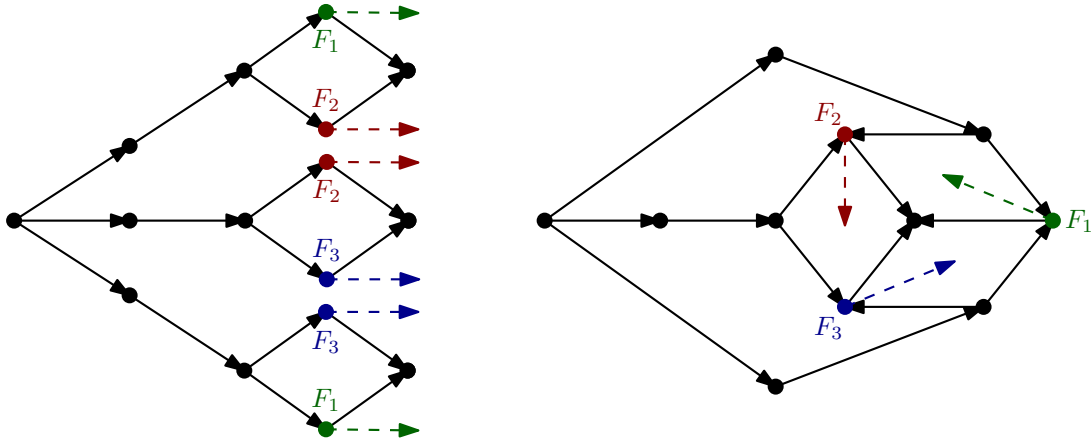


FIGURE 8. If we factorize the domain on the left over the equivalence classes of futures, we obtain the square complex on the right that is not an NPC square complex

lengths at 1-vertices. However, it turned out that finding a *regular* nice labeling is not always possible even in the case of 2-dimensional event domains (even those arising from CSC).

On the positive side, we proved that Thiagarajan's conjecture is true for cover-special event structures (recall that Nielsen and Thiagarajan established this conjecture for conflict-free event structures and Badouel et al. proved it for context-free event structures). As a consequence of deep results from geometric group theory by Agol and Haglund-Wise, we deduce that strongly hyperbolic regular event structures are cover-special, showing that Thiagarajan's conjecture holds for a large and natural class of event domains.

We conclude the paper with a list of open problems, and we hope that some of these problems will be solved positively.

8.2. Regular versus strongly regular event structures. In view of Proposition 4.3, any strongly regular event structure is regular. One can ask if the converse holds (this was also mentioned by a referee of a preliminary version of this paper [16]):

Question 8.1. Is any regular event structure strongly regular?

A natural way to derive a finite directed NPC complex from the domain \mathcal{D} of a regular event structure \mathcal{E} is to factorize \mathcal{D} over all equivalence classes of futures (i.e., to identify in a single vertex all configurations having the same principal filter up to isomorphism). Unfortunately, this construction does not preserve the non-positive curvature of \mathcal{D} . For example, consider a domain \mathcal{D} as described on the left of Figure 8. In the figure, only a part of the domain is described: one has to imagine that the dashed arrows lead to the remaining part of the domain with the assumption that two nodes that have the same label have isomorphic principal filters. When we factorize the domain \mathcal{D} over the equivalence classes of futures, we obtain the square complex on the left of Figure 8. Note that this square complex is not an NPC square complex as it contains three squares that intersect in a vertex and that pairwise intersect on edges and these three squares do not belong to a 3-cube.

This phenomenon does not arise if we consider VH -complexes and isomorphisms that preserve vertical and horizontal edges. More formally, the domain $\mathcal{D} = \mathcal{D}(\mathcal{E})$ of an event structure \mathcal{E} is a VH -domain if \mathcal{D} is a VH -complex. In this case, \mathcal{E} is called a VH -event structure and the events of \mathcal{E} are partitioned into vertical and horizontal events. A VH -event structure \mathcal{E} is VH -regular if \mathcal{E} has finite degree and has a finite number of principal filters up to isomorphism preserving vertical and horizontal events. In this case, the domain $\mathcal{D}(\mathcal{E})$ is called a *regular VH -domain*.

Even in this case, we do not know how to define formally a directed NPC square complex according to the factorization mentioned above such that the original domain is a principal filter of the universal cover of this complex.

Question 8.2. Does any regular VH -domain occur as a principal filter of the universal cover of some finite directed VH -complex?

8.3. Hyperbolic event domains. There are several natural reasons to investigate hyperbolic event domains. Similarly to $CAT(0)$ and NPC spaces, Gromov hyperbolicity is defined by a metric condition. However, similarly to the $CAT(0)$ property, the hyperbolicity of a $CAT(0)$ cube complex can be expressed in purely combinatorial way, by requiring that all isometric square grids have bounded size. Theorem 5.5 establishes that Thiagarajan's conjecture is true for strongly hyperbolic regular event structures. We conjecture that this result can be generalized in the following way:

Conjecture 8.3. Any strongly regular event structure with a hyperbolic domain admits a regular nice labeling.

Conjecture 1.2 was positively solved by Badouel et al. [5] for context-free domains, which are particular hyperbolic domains:

Lemma 8.4. *Any context-free graph $G = (V, E)$ is hyperbolic.*

Proof. Let $G = (V, E)$ be a graph of uniformly bounded degree and v be an arbitrary root (basepoint) of G . Let $S_i = \{x \in V : d_G(v, x) = i\}$ denote the sphere of radius i centered at v . A connected component F of the subgraph of G induced by $V \setminus S_i(v)$ is called an *end* of G . The vertices of $F \cap S_{i+1}(v)$ are called *frontier points* and this set is denoted by $\Delta(F)$ [33] and called a *cluster*. Let $\Phi(G)$ denote the set of all ends of G , i.e., the connected subgraphs of $G(V \setminus S_i(v))$, when i ranges over the natural numbers. An *end-isomorphism* between two ends F and F' of G is a mapping f between F and F' such that f is a graph isomorphism and f maps $\Delta(F)$ to $\Delta(F')$. Then G is called a *context-free graph* [33] if $\Phi(G)$ has only finitely many isomorphism classes under end-isomorphisms. Since G has uniformly bounded degree, each cluster $\Delta(F)$ is finite. Moreover, from the definition of context-free graphs follows that a context-free graph G has only finitely many isomorphism classes of clusters, thus there exists a constant $\delta < \infty$ such that the diameter of any cluster of G is bounded by δ . By [19, Proposition 12] any graph G whose diameters of clusters is uniformly bounded by δ is δ -hyperbolic (in fact, G is quasi-isometric to a tree). \square

The following conjecture generalizes Theorem 5.5, the results of [5] in the case of event structures considered in this paper, and Conjecture 8.3.

Conjecture 8.5. Conjectures 1.1 and 1.2 are true for hyperbolic event domains.

By Lemma 5.3, the 1-skeleton $X^{(1)}$ of a $CAT(0)$ cube complex is hyperbolic if and only if all isometrically embedded square grids are uniformly bounded. In the language of event structures, an isometrically embedded $n \times n$ grid H corresponds to a conflict-free event structure defined by $2n$ distinct events $e_1, \dots, e_n, f_1, \dots, f_n$ such that any two events e_i, f_j are concurrent and any two events e_i, e_j or f_i, f_j are either causally dependent or concurrent. The isometricity follows from the fact that the events $e_1, \dots, e_n, f_1, \dots, f_n$ are pairwise distinct. If this grid is embedded in a hypercube, then any two events e_i, e_j or f_i, f_j are concurrent. On the other hand, if $e_1 < e_2 < \dots < e_n$ and $f_1 < f_2 < \dots < f_n$, then this grid is isometrically embedded as a directed flat square grid. A (*directed*) *flat square grid* of side n (respectively, a (*directed*) *flat plane*) of a median graph G is a (directed) $n \times n$ -grid H (respectively, $\mathbb{Z} \times \mathbb{Z}$ -grid) isometrically embedded in G such that any two squares of H sharing a common edge do not belong to a common 3-cube of G . Note that if H is a flat square grid or a flat plane of a median graph G , then H is a locally-convex subgraph of G , and by Lemma 3.2, H is a convex subgraph of G . This shows that if G contains a flat square grid of size n , then the graph Γ_{\parallel} of the concurrent relation \parallel contains an induced complete bipartite subgraph $K_{n,n}$. In a median graph not containing 3-cubes (i.e., 1-skeletons of 2-dimensional $CAT(0)$ cube complexes), each embedded grid or plane is flat. We continue with a stronger version of Conjecture 8.5.

Conjecture 8.6. Conjectures 1.1 and 1.2 are true for event domains with uniformly bounded sizes of directed flat square grids.

A first step to solve this question could be to consider event structures such that the graph Γ_{\parallel} does not admit induced complete bipartite subgraphs $K_{n,n}$ with arbitrarily large n .

8.4. Confusion-free domains. As we noticed already, Conjecture 1.1 was positively solved by Nielsen and Thiagarajan [36] for conflict-free event structures. A possible way to generalize this result is to consider confusion-free domains.

Conflict-free event structures can be viewed as the event structures for which the minimal-conflict graph $\Gamma_{\#_{\mu}}$ is edgeless, i.e., each event of \mathcal{E} is a connected component of $\Gamma_{\#_{\mu}}$. (Notice that conflict-free domains are not hyperbolic because they may contain \mathbb{Z}^n for any n .) Therefore, one way to extend the result of [36] is to consider more complex minimal-conflict graphs $\Gamma_{\#_{\mu}}$. One possible such extension is to consider the event structures whose minimal-conflict graphs $\Gamma_{\#_{\mu}}$ are disjoint unions of cliques. Such event structures can be viewed as an extension of confusion-free event structures. An event structure \mathcal{E} is *confusion-free* [34] if the reflexive closure of minimal conflict is transitive and $e\#_{\mu}e'$ implies $\downarrow e \setminus \{e\} = \downarrow e' \setminus \{e'\}$ (we use the definition from [45, Proposition 2.4]). From the first condition it follows that for a confusion-free event structure the graph $\Gamma_{\#_{\mu}}$ is a disjoint union of cliques. Confusion-free event structures correspond to deterministic concrete data structures [28] and to confusion-free occurrence nets [34].

Question 8.7. Do Conjectures 1.1 and 1.2 hold for confusion-free event structures? More generally, do they hold for event structures whose minimal-conflict graph $\Gamma_{\#_{\mu}}$ is a disjoint unions of cliques?

8.5. Undecidability questions. We think that the relationship between the existence of aperiodic tile sets and the nonexistence of regular nice labelings of the associated event structures may help to prove some undecidability results. We conjecture that one cannot decide if a regular event structure satisfies Thiagarajan's conjecture:

Conjecture 8.8. There does not exist an algorithm that, given a strongly regular event domain \mathcal{D} , can determine whether or not \mathcal{D} admits a regular nice labeling.

The intuition behind is that one can use Lukkarilla's construction [30] to prove this conjecture. As in the proof of undecidability of the classical tiling problem [11, 37], the undecidability proof of Lukkarila is based on a reduction from the *Turing machine halting problem*. More precisely, for any Turing machine \mathcal{M} , Lukkarila constructs a 4-way deterministic tile set $T_{\mathcal{M}}$ such that either $T_{\mathcal{M}}$ is an aperiodic tile set (this corresponds to the case when the Turing machine \mathcal{M} does not halt), or $T_{\mathcal{M}}$ does not tile the plane (this corresponds to the case when the Turing machine \mathcal{M} halts). In the first case, by Theorem 7.3, the domain $(\widetilde{W}(T_{\mathcal{M}})_{\bar{v}}, \prec_{\bar{\sigma}^*})$ does not admit a regular nice labeling. In the second case, by Lemma 7.1, $(\widetilde{W}(T_{\mathcal{M}})_{\bar{v}}, \prec_{\bar{\sigma}^*})$ is a strongly regular domain that is hyperbolic. Consequently, if Conjecture 8.3 was true, $(\widetilde{W}(T_{\mathcal{M}})_{\bar{v}}, \prec_{\bar{\sigma}^*})$ would admit a regular nice labeling. This would prove Conjecture 8.8.

Another possible way to prove Conjecture 8.8 would be to answer the following question in a positive way and use Theorem 5.2.

Question 8.9. Given a 4-way deterministic tile set T such that there is no valid tiling with the tiles of T , is it true that the VH -complex $W(T)$ is virtually special?

Note that if there was a positive answer to this question, this would answer a question of Agol [2, Question 3] and confirm the following conjecture of Bridson and Wilton [13]:

Conjecture 8.10 ([13, Conjecture 1.2]). There does not exist an algorithm that, given a finite NPC square complex Y , can determine whether or not Y is virtually special.

Indeed, in Lukkarila's construction, if the Turing machine \mathcal{M} does not halt, then by Theorem 7.3 $W(T_{\mathcal{M}})$ is not virtually special. On the other hand, if the Turing machine \mathcal{M} halts, then if the answer to Question 8.9 was positive, $W(T_{\mathcal{M}})$ would be virtually special.

Acknowledgements. We are grateful to P.S. Thiagarajan for some email exchanges on Conjecture 1 and paper [36] and to our colleague R. Morin for several useful discussions. The work on the results not presented in the preliminary version [16] (Sections 5 and 7) was supported by ANR project DISTANCIA (ANR-17-CE40-0015).

REFERENCES

- [1] I. Agol. The virtual Haken conjecture. *Doc. Math.*, 18:1045–1087, 2013. With an appendix by Ian Agol, Daniel Groves, and Jason Manning.
- [2] I. Agol. Virtual properties of 3-manifolds. *Proceedings of the International Congress of Mathematicians, Seoul*, pages 157–186, 2014.
- [3] F. Ardila, M. Owen, and S. Sullivant. Geodesics in CAT(0) cubical complexes. *Adv. Appl. Math.*, 48(1):142–163, 2012.
- [4] S.P. Avann. Metric ternary distributive semi-lattices. *Proc. Amer. Math. Soc.*, 12(3):407–414, 1961.
- [5] E. Badouel, Ph. Darondeau, and J.-C. Raoult. Context-free event domains are recognizable. *Inf. Comput.*, 149(2):134–172, 1999.
- [6] H.-J. Bandelt and V. Chepoi. Metric graph theory and geometry: a survey. In J. E. Goodman, J. Pach, and R. Pollack, editors, *Surveys on Discrete and Computational Geometry: Twenty Years Later*, volume 453 of *Contemp. Math.*, pages 49–86. AMS, Providence, RI, 2008.
- [7] H.-J. Bandelt and J. Hedlíková. Median algebras. *Discr. Math.*, 45(1):1–30, 1983.
- [8] H.-J. Bandelt and M. van de Vel. Embedding topological median algebras in products of dendrons. *Proc. London Math. Soc.*, s3-58(3):439–453, 1989.
- [9] J.-P. Barthélemy and J. Constantin. Median graphs, parallelism and posets. *Discr. Math.*, 111(1-3):49–63, 1993.
- [10] M.A. Bednarczyk. *Categories of Asynchronous Systems*. PhD thesis, University of Sussex, 1987.
- [11] R. Berger. The undecidability of the domino problem. *Mem. Amer. Math. Soc.*, 66, 1966.
- [12] M. Bridson. Cube complexes, subgroups of mapping class groups, and nilpotent genus. In M. Bestvina, M. Sageev, and K. Vogtmann, editors, *Geometric Group Theory*, volume 21 of *IAS/Park City Mathematics Series*, pages 381–399. AMS, Institute for Advanced Study, 2012.
- [13] M. Bridson and H. Wilton. On the recognition problem for vitrually special cube complexes. *arXiv preprint*, 2017. [arXiv:1408.2325v2](https://arxiv.org/abs/1408.2325v2).
- [14] M.R. Bridson and A. Haefliger. *Metric Spaces of Non-Positive Curvature*, volume 319 of *Grundlehren der mathematischen Wissenschaften*. Springer-Verlag, Berlin, 1999.
- [15] J.P. Burling. *On coloring problems of prototypes*. PhD thesis, Univ. of Colorado, 1965.
- [16] J. Chalopin and V. Chepoi. A counterexample to Thiagarajan’s conjecture on regular event structures. In *ICALP*, volume 80 of *LIPICs*, pages 101:1–101:14. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2017.
- [17] V. Chepoi. Graphs of some CAT(0) complexes. *Adv. Appl. Math.*, 24(2):125–179, 2000.
- [18] V. Chepoi. Nice labeling problem for event structures: a counterexample. *SIAM J. Comput.*, 41(4):715–727, 2012.
- [19] V. Chepoi, F.F. Dragan, B. Estellon, M. Habib, and Y Vaxès. Diameters, centers, and approximating trees of delta-hyperbolic geodesic spaces and graphs. In *Symposium on Computational Geometry*, pages 59–68. ACM, 2008.
- [20] V. Chepoi and M. F. Hagen. On embeddings of CAT(0) cube complexes into products of trees via colouring their hyperplanes. *J. Comb. Theory, Ser. B*, 103(4):428–467, 2013.
- [21] D.Ž. Djoković. Distance-preserving subgraphs of hypercubes. *J. Comb. Theory, Ser. B*, 14(3):263–267, 1973.
- [22] M. Gromov. Hyperbolic groups. In S. M. Gersten, editor, *Essays in group theory*, volume 8 of *Math. Sci. Res. Inst. Publ.*, pages 75–263. Springer, New York, 1987.
- [23] M. F. Hagen. Weak hyperbolicity of cube complexes and quasi-arboreal groups. *J. Topology*, 7(2):385–418, 2014.
- [24] F. Haglund. *Aspects combinatoires de la théorie géométrique des groupes*. HDR, U. Paris Sud, 2008.
- [25] F. Haglund and D.T. Wise. Special cube complexes. *Geom. Funct. Anal.*, 17(5):1551–1620, 2008.
- [26] F. Haglund and D.T. Wise. A combination theorem for special cube complexes. *Annals Math.*, 176(3):1427–1482, 2012.
- [27] A. Hatcher. *Algebraic Topology*. Cambridge University Press, Cambridge, 2002.
- [28] G. Kahn and D.G. Plotkin. Concrete domains. *Theor. Comput. Sci.*, 121(1&2):187–277, 1993.
- [29] J. Kari and P. Papasoglu. Deterministic aperiodic tile sets. *GFAA, Geom. Funct. Anal.*, 9(2):353–369, 1999.
- [30] V. Lukkarila. The 4-way deterministic tiling problem is undecidable. *Theor. Comput. Sci.*, 410(16):1516–1533, 2009.
- [31] R. Morin. Concurrent automata vs. asynchronous systems. In *MFCS 2005*, volume 3618 of *LNCS*, pages 686–698. Springer, 2005.
- [32] H.M. Mulder. *The Interval Function of a Graph*, volume 132 of *Mathematical Centre tracts*. Mathematisch Centrum, Amsterdam, 1980.
- [33] D.E. Muller and P.E. Schupp. The theory of ends, pushdown automata, and second-order logic. *Theor. Comput. Sci.*, 37:51–75, 1985.
- [34] M. Nielsen, G.D. Plotkin, and G. Winskel. Petri nets, event structures and domains, part I. *Theor. Comput. Sci.*, 13:85–108, 1981.
- [35] M. Nielsen, G. Rozenberg, and P.S. Thiagarajan. Transition systems, event structures and unfoldings. *Inf. Comput.*, 118(2):191–207, 1995.

- [36] M. Nielsen and P.S. Thiagarajan. Regular event structures and finite Petri nets: the conflict-free case. In *ICATPN 2002*, volume 2360 of *LNCS*, pages 335–351. Springer, 2002.
- [37] R. Robinson. Undecidability and nonperiodicity for tilings of the plane. *Inventiones Mathematicae*, 12(3):177–209, 1971.
- [38] M. Roller. Poc sets, median algebras and group actions. Technical report, Univ. of Southampton, 1998.
- [39] B. Rozoy and P.S. Thiagarajan. Event structures and trace monoids. *Theor. Comput. Sci.*, 91(2):285–313, 1991.
- [40] M. Sageev. Ends of group pairs and non-positively curved cube complexes. *Proc. London Math. Soc.*, s3-71(2):585–617, 1995.
- [41] M. Sageev. CAT(0) cube complexes and groups. In M. Bestvina, M. Sageev, and K. Vogtmann, editors, *Geometric Group Theory*, volume 21 of *IAS/Park City Mathematics Series*, pages 6–53. AMS, Institute for Advanced Study, 2012.
- [42] E.W. Stark. Connections between a concrete and an abstract model of concurrent systems. In *Mathematical Foundations of Programming Semantics 1989*, volume 442 of *LNCS*, pages 53–79. Springer, 1989.
- [43] P.S. Thiagarajan. Regular trace event structures. Technical Report BRICS RS-96-32, Computer Science Department, Aarhus University, Aarhus, Denmark, 1996.
- [44] P.S. Thiagarajan. Regular event structures and finite Petri nets: A conjecture. In *Formal and Natural Computing*, volume 2300 of *LNCS*, pages 244–256. Springer, 2002.
- [45] D. Varacca, H. Völzer, and G. Winskel. Probabilistic event structures and domains. *Theor. Comput. Sci.*, 358(2-3):173–199, 2006.
- [46] P.M. Winkler. Isometric embedding in products of complete graphs. *Discr. Appl. Math.*, 7(2):221–225, 1984.
- [47] G. Winskel. *Events in computation*. PhD thesis, Edinburgh Univ., 1980.
- [48] G. Winskel. An introduction to event structures. In *REX Workshop*, volume 354 of *Lecture Notes in Computer Science*, pages 364–397. Springer, 1988.
- [49] G. Winskel and M. Nielsen. Models for concurrency. In S. Abramsky, Dov M. Gabbay, and T. S. E. Maibaum, editors, *Handbook of Logic in Computer Science (Vol. 4)*, pages 1–148. Oxford University Press, 1995.
- [50] D.T. Wise. *Non-positively curved squared complexes, aperiodic tilings, and non-residually finite groups*. PhD thesis, Princeton University, 1996.
- [51] D.T. Wise. Complete square complexes. *Comment. Math. Helv.*, 82(4):683–724, 2007.
- [52] D.T. Wise. *From Riches to Raags: 3-manifolds, Right-angled Artin Groups, and Cubical Geometry*, volume 117 of *CBMS Regional Conference Series in Mathematics*. AMS, Providence, RI, 2012.

LABORATOIRE D'INFORMATIQUE ET SYSTÈMES, AIX-MARSEILLE UNIVERSITÉ AND CNRS
E-mail address: jeremie.chalopin@lif.univ-mrs.fr

LABORATOIRE D'INFORMATIQUE ET SYSTÈMES, AIX-MARSEILLE UNIVERSITÉ AND CNRS
E-mail address: victor.chepoi@lif.univ-mrs.fr