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Existence of global strong solution for Korteweg system in one dimension for strongly degenerate viscosity coefficients

Cosmin Burtea [*], Boris Haspot [†]

Abstract

In this paper we prove the existence of global strong solution for the Navier-Stokes Korteweg equations for strongly degenerate viscosity coefficients with initial density far away from vacuum. More precisely, we assume that the viscosity coefficients take the form $\mu(\rho) = \rho^\alpha$ with $\alpha > 1$. The main difficulty of the proof consists in estimating globally in time the L^∞ norm of $\frac{1}{\rho}$. Our method of proof relies on fine algebraic properties of the Navier-Stokes Korteweg system. First we introduce two new effective pressures endowed with weight functions depending both on the viscosity and the capillarity coefficients as some power laws of the density. For these two quantities we show some Oleinik-type estimate which provide the control of the L^∞ norm of $\frac{1}{\rho}$ by applying a maximum principle. It is interesting to point out that the two effective pressure introduced in the present paper depending on the capillary coefficient generalize to the Navier-Stokes Korteweg equations those introduced in [7, 15]. In our proof we make use of additional regularizing effects on the effective velocities which ensure the uniqueness of the solution using a Lagrangian approach.

1 Introduction

We are concerned with compressible fluids endowed with internal capillarity which can be described by the Korteweg-type model (see [43, 34, 19, 42, 1, 9, 23] for its derivation, we refer also to the pioneering work by J.-E. Dunn and J. Serrin in [19]). The conservation of mass and of momentum write:

$$\begin{cases} \dfrac{\partial}{\partial t}\rho + \partial_x(\rho u) = 0, \\ \dfrac{\partial}{\partial t}(\rho u) + \partial_x(\rho u^2) - \partial_x(\mu(\rho)\partial_x u) + \partial_x P(\rho) = \partial_x K \\ (\rho, u)(0, \cdot) = (\rho_0, u_0). \end{cases} \tag{1.1}$$

Here $u = u(t, x) \in \mathbb{R}$ stands for the velocity, $\rho = \rho(t, x) \in \mathbb{R}^+$ is the density, $\mu(\rho) > 0$ is the viscosity coefficient and $P(\rho)$ is the pressure term with P a γ law such that $P(\rho) = \rho^\gamma$ with $\gamma > 1$. The Korteweg tensor reads as:

$$K = \rho\kappa(\rho)\partial_{xx}\rho + \frac{1}{2}(\rho\kappa'(\rho) - \kappa(\rho))(\partial_x\rho)^2 \tag{1.2}$$

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We supplement the problem with initial condition (ρ_0, u_0). We will focus now on the following particular case where $\kappa(\rho)$ and $\mu(\rho)$ are related by the following algebraic relation:

$$\kappa(\rho) = c\frac{\mu(\rho)^2}{\rho^3} \tag{1.3}$$

with $c > 0$. We would like to point out that this specific choice (1.3) on the viscosity and capillary coefficients allows in particular to deal with the so called compressible quantum Navier Stokes system where $\mu(\rho) = \mu\rho$ and $\kappa(\rho) = \frac{\kappa}{\rho}$ with $\mu, \kappa > 0$. This model belongs to the class of quantum fluid models. Such models are used in particular to describe superfluids [39], quantum semiconductors [20], weakly interacting Bose gases [22] and quantum trajectories of Bohmian mechanics [44]. The quantum correction $\frac{\partial_{xx}\sqrt{\rho}}{\sqrt{\rho}}$ can be seen as a quantum potential, the so called Bohm potential. This Bohm potential arises from the fluid dynamical formulation of the single-state Schrödinger equation.
Finally we mention that when $c = 0$ we recover the classical compressible Navier-Stokes equations and when $\mu(\rho) = 0$ and $\kappa(\rho) \neq 0$ we have the so called Euler-Korteweg system (we refer to [3, 4] for the existence of global strong solution with small initial data in dimension $N \geq 3$). As in [7, 26, 28], setting $\varphi'(\rho) = \frac{\mu(\rho)}{\rho^2}$ we can observe that under the condition (1.3) we have:

$$\partial_x K = c\partial_x(\mu(\rho)\partial_{xx}\varphi(\rho)). \tag{1.4}$$

Setting now $v = u + \alpha\partial_x\varphi(\rho)$ as in [2, 21], we have from the mass equation of (1.1):

$$\begin{cases} \partial_t\rho + \partial_x(\rho u) = 0 \\ \rho\partial_t v + \rho u\partial_x v + (\alpha - 1)\partial_x(\mu(\rho)\partial_x u) + \partial_x P(\rho) = \partial_x K. \end{cases} \tag{1.5}$$

Now according to (1.4), we obtain:

$$\rho\partial_t v + \rho u\partial_x v + (\alpha - 1)\partial_x(\mu(\rho)\partial_x u) + \partial_x P(\rho) - c\partial_x(\mu(\rho)\partial_{xx}\varphi(\rho)) = 0. \tag{1.6}$$

If we rewrite the previous equation in terms of v, we get:

$$\begin{aligned} \rho\partial_t v + \rho u\partial_x v + (\alpha - 1)\partial_x(\mu(\rho)\partial_x v) + \partial_x P(\rho) \\ - (\alpha^2 - \alpha + c)\partial_x(\mu(\rho)\partial_{xx}\varphi(\rho)) = 0. \end{aligned} \tag{1.7}$$

We wish now to choose α such that:

$$\alpha^2 - \alpha + c = 0. \tag{1.8}$$

We will restrict in the sequel our attention to the case $0 < c \leq \frac{1}{4}$ (we will explain later why we only consider this case) which ensures the existence of two real α_i with $i \in \{1, 2\}$ satisfying (1.8):

$$\alpha_1 = \frac{1 + \sqrt{1 - 4c}}{2} \text{ or } \alpha_2 = \frac{1 - \sqrt{1 - 4c}}{2}. \tag{1.9}$$

At this level we have then the following equations:

$$\rho\partial_t v_i + \rho u\partial_x v_i + (\alpha_i - 1)\partial_x(\mu(\rho)\partial_x v_i) + \partial_x P(\rho) = 0, \tag{1.10}$$

with $v_i = u + \alpha_i\partial_x\varphi(\rho)$ where $i \in \{1, 2\}$ which will be referred as *effective velocities*. Let us discuss the dissipation of energy, multiplying the equation of momentum conservation

in the system (1.1) by u and integrating by parts over $\mathbb{R}$, we obtain the following natural energy:

$$\begin{aligned}&\int_{\mathbb{R}}\big(\frac{1}{2}\rho u^2+\rho e\left(\rho\right)+\frac{1}{2}\kappa(\rho)(\partial_x\rho)^2\big)(t,x)dx+\int_0^t\int_{\mathbb{R}}\mu(\rho)(\partial_x u)^2(s,x)dsdx\\&\leq\int_{\mathbb{R}}\big(\frac{1}{2}\rho_0 u_0^2+\rho_0 e\left(\rho_0\right)+\frac{1}{2}\kappa(\rho_0)(\partial_x\rho_0)^2\big)(t,x)dx.\end{aligned}\tag{1.11}$$

with $e(\rho)$ defined as follows:

$$e\left(\rho\right)=\frac{\left(\rho^{\gamma-1}-1-\gamma\left(\rho-1\right)\right)}{\left(\gamma-1\right)\rho}=\frac{\rho^{\gamma-1}}{\gamma-1}+\frac{1}{\rho}-\frac{\gamma}{\gamma-1}.\tag{1.12}$$

In the sequel we will deal with the following strongly degenerate viscosity coefficients and with the associated capillary coefficients issued of the formule (1.4):

$$\mu(\rho)=\rho^{\beta}\ \text{ and }\ \kappa(\rho)=c\rho^{2\beta-3},\tag{1.13}$$

with $\beta\geq 1$. With this choice, we can rewrite the energy estimate (1.11) as follows:

$$\int_{\mathbb{R}}\big(\frac{1}{2}\rho u^2+\rho e\left(\rho\right)+\frac{c}{2\left(\beta-\frac{1}{2}\right)^2}\left(\partial_x\rho^{\beta-\frac{1}{2}}\right)^2+\int_0^t\int_{\mathbb{R}}\rho^{\beta}(\partial_x u)^2\leq E_0\left(\rho_0,u_0\right)+cE_{cap}\left(\partial_x\rho_0\right),\tag{1.14}$$

where

$$\begin{cases}E_0\left(\rho_0,u_0\right)=\displaystyle\int_{\mathbb{R}}\tfrac{1}{2}\rho_0 u_0^2+\rho_0 e\left(\rho_0\right),\\ E_{cap}\left(\partial_x\rho_0\right)=\displaystyle\frac{1}{2\left(\beta-\frac{1}{2}\right)^2}\int_{\mathbb{R}}\left(\partial_x\rho_0^{\beta-\frac{1}{2}}\right)^2,\\ E_0\left(\rho_0,u_0\right)+cE_{cap}\left(\partial_x\rho_0\right)\overset{not.}{=}E_{0,tot}.\end{cases}$$

The so-called BD-entropy estimate which is satisfied for the compressible Navier-Stokes system (see [5]) is also verified for the Korteweg system (1.1) and is given by multiplying the equation (1.10) with v_i where $i\in\{1,2\}$:

$$\int_{\mathbb{R}}\big(\frac{1}{2}\rho v_i^2+\rho e\left(\rho\right)\big)(t,x)dx+(1-\alpha_i)\int_0^t\int_{\mathbb{R}}\rho^{\beta}(\partial_x v_i)^2(s,x)dsdx\tag{1.15}$$

$$+\alpha_i\gamma\int_0^t\int_{\mathbb{R}}\rho^{\gamma+\beta-3}(\partial_x\rho)^2(s,x)dsdx\leq\int_{\mathbb{R}}\big(\frac{1}{2}\rho v_{i,0}^2+\rho_0 e\left(\rho_0\right)\big)(x)dx\tag{1.16}$$

$$\begin{aligned}&=\int_{\mathbb{R}}\frac{1}{2}\rho_0 u_0^2+\rho_0 e\left(\rho_0\right)+\frac{\alpha_i}{\beta-\frac{1}{2}}\sqrt{\rho_0}u_0\partial_x\rho_0^{\alpha-\frac{1}{2}}+\alpha_i^2E_{cap}\left(\partial_x\rho_0\right)\\&\leq 2E\left(\rho_0,u_0\right)+2\alpha_i^2E_{cap}\left(\partial_x\rho_0\right).\end{aligned}\tag{1.17}$$

It implies that in the context of the Navier-Stokes-Korteweg system (1.1) with the capillary coefficient satisfying the algebraic relation (1.3), there are two such entropies estimates, one degenerates in the basic energy estimate for compressible Navier-Stokes equations when the capillary coefficient c goes to 0, the other one to the BD-entropy estimate (see [5]). In this paper, we are interested by proving the existence of global

strong solution for the Navier-Stokes Korteweg equation with degenerate viscosity coefficients and capillary coefficients verifying (1.13) (we note that when $c = 0$ we recover the compressible Navier-Stokes system). We briefly mention that the existence of global strong solutions for the system (1.1) with small initial data for $N \geq 2$ is known since the works by Hattori and Li [29] in the case of constant capillary coefficient $\kappa(\rho)$. Danchin and Desjardins in [18] improved this result by working with initial data $(\rho_0 - 1, \rho_0 u_0)$ belonging to the following Besov spaces which are critical for the scaling of the equations $B^{\frac{N}{2}}_{2,1} \times (B^{\frac{N}{2}-1}_{2,1})^N$ (we refer to [24] for the case of the nonisothermal Korteweg system). This result has been extended in [25] and recently in [27] where the second author proves the global existence of strong solution with small initial data in $(B^{\frac{N}{2}}_{2,\infty} \cap L^\infty) \times B^{\frac{N}{2}-1}_{2,\infty}$ generalizing to the Korteweg system the result of Cannone, Meyer, Planchon [10] for Navier-Stokes equation which enables to construct self similar solutions of the Navier-Stokes equations with small initial data. This implies in particular that we can extend to the Navier-Stokes Korteweg system the notion of Oseen solutions in dimension $N = 2$ provided that the vorticity is a Dirac mass $\alpha\delta_0$ with $|\alpha|$ sufficiently small.
The problem of existence of global strong solution for system (1.1) with large initial data and with general viscosity and capillary coefficients remains again largely open even in the one dimensional case. We are going to focus our attention on the one dimensional case, and we wish to start with describing the state of art for the compressible Navier-Stokes equations when $c = 0$ (we will explain after the main difference that one encounters for obtaining similar results for the Navier-Stokes Korteweg system). It is important to explain that for getting such result of global strong solution, the main difficulty is related to the control of the L^∞ norm of $\frac{1}{\rho}$. Indeed it is well known that the strong solution can blow-up in finite time as long as the L^∞ norm of $\frac{1}{\rho}$ does the same (see [15] for viscosity coefficients verifying (1.4) with $\beta > \frac{1}{2}$). Kanel in [33] has been the first to prove the existence of global strong solution for compressible Navier-Stokes equations with arbitrary large initial data in one dimension for constant viscosity coefficients. This result has been extend by Mellet and Vasseur in [40] to the case of viscosity coefficients verifying (1.13) in the case $0 < \beta < \frac{1}{2}$. The main argument of their proof consists in using the Bresch-Dejardins entropy (see [5]) for getting L^∞ estimate of $\frac{1}{\rho}$ that they combine with parabolic regularizing effects on the velocity issued of the momentum equation. We wish to point out that the Bresch-Dejardins entropy gives almost for free the control of $\|\frac{1}{\rho}\|_{L^\infty_{t,x}}$ when $0 \leq \beta < \frac{1}{2}$. In [26], the second author has proved similar results for the case $\frac{1}{2} < \beta \leq 1$ where he used the fact that the effective velocity v satisfies a damped transport equation. It allows to obtain L^∞ estimates on v which is sufficient to obtain L^∞ estimate on $\frac{1}{\rho}$ by using a maximum principle on the mass equation. More recently Constantin *et al* in [15] have generalized the previous results to the case $\beta > 1$ with γ belonging to $[\alpha, \alpha + 1]$ provided that the initial data satisfy: $\partial_x u_0 \leq \rho_0^{\gamma-\alpha}$. The main ingredient of the proof consists in proving that the sign of the so called effective pressure $\mu(\rho)\partial_x u - P(\rho)$ does not change all along the time and to use a maximum principle on the mass equation in order to estimate the L^∞ norm of $\frac{1}{\rho}$. In [7], we have extended the result of [15] when $\beta > 1$ inasmuch as we do not assume any condition of sign on the initial data (furthermore the initial data are less regular, roughly speaking the initial data are only of finite energy). To do this, we have introduced a new effective pressure on which we prove Oleinik type estimate which enables to control the L^∞ norm of $\frac{1}{\rho}$ via a

maximum principle. In conclusion, the problem of the existence of global strong solution in one dimension for the compressible Navier Stokes equations with viscosity coefficient of the form (1.13) is now well understood, however fewer results have been obtained for the Navier-Stokes Korteweg equations which are significantly more complicated because the capillary term of third order derivative on the density.
Charve and the second author in [12] proved the global existence of strong solution for the system (1.1) when $\mu(\rho) = \varepsilon\rho$ and $\kappa(\rho) = \frac{\varepsilon^2}{\rho}$, in addition they show that the global strong solutions converge when ε goes to 0 to a global entropy weak solution of the compressible Euler system with initial data of finite energy. Germain and LeFloch in [21] showed recently the global existence of vacuum and non-vacuum weak solutions for the Korteweg system including the case of viscosity and capillary coefficients of the form $\mu(\rho) = \rho^\beta$, $\kappa(\rho) = \rho^{\beta_1}$ which satisfy in particular a strong coercivity condition which corresponds in the present case to $2\beta - 4 < \beta_1 < 2\beta - 1$ and with $0 \leq \beta < \frac{1}{2}$ or with $\beta_1 < -2$. Furthermore they analyze the zero viscosity-capillarity limit associated with the solutions of the Navier-Stokes-Korteweg system generalizing the results of [12] and recovering at the limit an entropy solution of the Euler system. It is important to point out that they need to impose a tame condition on the viscosity and capillary coefficients which takes the form:

$$\kappa(\rho) \lesssim \frac{\mu(\rho)^2}{\rho^3} \text{ and } \delta(\varepsilon) \lesssim \varepsilon^2, \tag{1.18}$$

if we consider the vanishing viscosity capillary coefficients $\mu_\varepsilon(\rho) = \varepsilon\mu(\rho)$ and $\kappa_\varepsilon(\rho) = \delta(e)\kappa(\rho)$ when $\varepsilon > 0$ goes to 0. Roughly speaking the viscosity tensor involves some parabolic effects on the velocity whereas the capillary tensor generates some dispersive effects (see [3, 4]), the previous tame condition implies in some sense that the parabolic effects will dominates the dispersive effects issue of the capillary tensor. In particular when ε goes to 0 we can expect some strong convergence in suitable functional space whereas the dispersive effects tend to induce strong oscillations which prevent any strong limit but allows only weak limit (see for example the case of the Korteweg de Vries equation [35, 36, 37]). In particular the condition (1.18) implies that the authors in [21] consider the same type of viscosity and capillary coefficients (see (1.13) when they study the zero viscosity-capillarity limit.
Recently Chen et al. in [14] and Chen in [13] have proved for the first time some results of existence of global strong solution for initial density far away from the vacuum in Lagrangian coordinates. More precisely they consider viscosity and capillary coefficients of the form $\mu(\rho) = \rho^{\alpha_1}$ and $\kappa(\rho) = c\rho^{\beta_1}$ with $(\alpha_1, \beta_1) \in \mathbb{R}^2$, it is important to point out that in comparison with the present work there is no relation a priori between α_1 and β_1 (furthermore there is no restriction on $c > 0$). They manage essentially to show such result when $\beta_1 < -2$ which allows in a direct way to control the L^∞ norm of $\frac{1}{\rho}$ by using the energy estimate (indeed roughly speaking the energy estimate ensure that $\partial_x \rho^{\frac{\beta_1}{2}+1}$ is bounded in $L^\infty_T(L^2)$ for any $T > 0$). They deal also with the case $\beta_1 \geq -2$ but in this situation $\alpha_1 < 0$, in particular the viscosity coefficient is not degenerate in this case. The main ideas of the proof is to obtain L^2 estimate on the effective velocity $v = u + \frac{\mu(\rho)}{\rho^2}\partial_x\rho$ by using energy method combine with Sobolev embedding in the spirit of Kanel (see [33]). Furthermore the authors show also the existence of global strong solution when the initial data is a perturbation of a Riemann problem associated to a rarefaction wave

for the compressible Euler problem.

In this article we wish to deal with the case of degenerate viscosity coefficient when $\beta > 1$ (see (1.13)) and with $\beta_1 = 2\beta - 3 > -2$ (the power of the capillary coefficient) in order to extend the results of [13, 14] to these cases. As in [21] when the authors consider the zero viscosity-capillary limit, we assume that the algebraic relation between the viscosity and the capillary coefficient (1.4) is satisfied, furthermore we work also in a regime where the viscosity effects dominates the capillary effects with $0 < c \leq \frac{1}{4}$ (it corresponds to the tame condition introduced in [21]). We would like to explain briefly the main arguments of our proof and the main difficulties which are related to the proof of existence of global strong solution with degenerate viscosity coefficients. First the existence of global strong solution in finite time is well known (see [13, 14]); so in order to show the existence of global strong solution, we start by proving a blow-up criterion for the case $\beta > \frac{1}{2}$ in (1.13) which is relied to estimating the $L^\infty_{t,x}$ norm of $\frac{1}{\rho}$. It implies that the main difficulty for getting the existence of global strong solution with degenerate viscosity coefficient corresponds to control the L^∞ norm of $\frac{1}{\rho}$ all along the time. In the case $\frac{1}{2} < \beta \leq 1$, it is sufficient to estimate the $L^\infty_{t,x}$ norm of each effective velocity v_i with $i \in \{1, 2\}$ to obtain $L^\infty_{t,x}$ estimates on $\frac{1}{\rho}$ using a maximum principle on the mass equation. It is important to precise that we can apply such maximum principle on the equation because the viscosity coefficients are not so degenerate when $\beta \leq \frac{1}{2}$. We refer to [8] for the existence of global strong solution when $\beta \leq 1$.
In the case $\beta > 1$ the previous estimates are not sufficient and it becomes more involved to estimate $\frac{1}{\rho}$. As in [7, 15] we are going to introduce suitable effective pressures w_i for which we can estimate the maximum. This will provide us the control of the $L^\infty_{t,x}$ norm of $\frac{1}{\rho}$ using a maximum principle on the mass equation of (1.10). To do this, we are going to exploit very fine algebraic properties of the Navier-Stokes Korteweg system by setting for $i \in \{1, 2\}$:

$$w_i = f_i(\rho)\partial_x v_i + F_{1,i}(\rho). \tag{1.19}$$

with

$$f_i(\rho) = \rho^{\frac{(\beta-1)\alpha_i - \beta}{2\alpha_i - 1}} \quad \text{and} \quad F_{1,i}(\rho) = \frac{\gamma}{-\alpha_i(\beta+1) + \gamma(2\alpha_i - 1)} \rho^{\frac{-\alpha_i(\beta+1)}{2\alpha_i - 1} + \gamma}. \tag{1.20}$$

We would like to point out that these new effective pressures w_i have weight $f_i(\rho)$ corresponding to some power of the density which depend in a crucial way on the viscosity and capillary coefficients. We show in the Proposition 3.1 that w_i satisfy the following equation:

$$\partial_t w_i + (u + g_{1i})\partial_x w_i + w_i g'_{2i} + (\alpha_i - 1)\partial_x(\frac{\mu(\rho)}{\rho}\partial_x w_i) + g'_{3i} + g_{4i} w_i^2 = 0, \tag{1.21}$$

which is a parabolic equation with damping term. We refer to the Proposition 3.1 for the definition of the terms g_{1i}, g'_{2i}, g'_{3i} and g_{4i}. It is remarkable to note that when $\alpha_1 = 1$ and $\alpha_2 = 0$ which corresponds to the case $c = 0$ of the compressible Navier-Stokes system, then w_1 and w_2 corresponds respectively to the effective pressures $y_1 = \frac{\partial_x v}{\rho} + F_2(\rho)$ and $y_2 = \mu(\rho)\partial_x u - P(\rho)$ respectively introduced in [7] and [15]. In particular in this case w_1 and w_2 satisfy the same equation as the effective pressure defined in [7, 15].

In this sense, we can say that the effective pressure w_i generalize the one defined for compressible Navier-Stokes equations and that in addition w_i converge to the effective pressure of Navier-Stokes equations when c goes to 0. It turns out that we are able to prove an Oleinik type estimate for the effective pressure w_1:

$$w_1(t,x) \leq C_1(t) \quad \forall (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \tag{1.22}$$

with C_1 a continuous increasing function provided that $w_1(0,\cdot) \leq C_0$ with $C_0 \in \mathbb{R}$. Unfortunately in comparison with the case $c = 0$ where $g_{4,1} = 0$, for Navier-Stokes Korteweg equations we observe that $g_{4,1} \leq 0$. In particular, if we consider the equation (1.21), it seems that the sign of $g_{4,1}$ does not allow to apply a maximum principle which provide the global estimate (1.22) as it is the case when $c = 0$. Indeed we have in some sense to deal with a Ricatti equation which can blow up in finite time. However since $g_{4,1}$ depends on $\alpha_1 - 1$ we show that we can prove the estimate (1.22) on a any time interval $[0,T]$ with $T > 0$ fixed provided that the capillary coefficient $c > 0$ is sufficiently small. Using maximum principle for the mass equation of (1.1) allows us to prove that $\frac{1}{\rho}$ is bounded on the time interval $[0,T]$. In order to show the uniqueness of the solutions, we prove regularizing effects on the velocity v_1 and v_2 by extending the Hoff's estimates valid for compressible Navier Stokes system with constant viscosity coefficients (see [30]) to the case of Navier-Stokes Korteweg system with general viscosity coefficients. It enables us in particular to prove that $\partial_x u$ belongs to $L^1_{loc}(L^\infty(\mathbb{R}))$. Passing in Lagrangian formulation we show the uniqueness of the solutions. This result show the existence of almost global strong solution provided that c goes to zero.
In order to prove the existence of global strong solution, we impose a condition on the sign of the initial data. More precisely we assume that $w_1(0,\cdot) \leq 0$ or $w_2(0,\cdot) \leq 0$ which allows to use a maximum principle on the equation (1.21)and to observe that w_1 or w_2 conserve the same sign all along the time. We conclude again by using maximum principle for the mass equation which allows to show that the L^∞ norm of $\frac{1}{\rho}$ can not blow up in finite time what is sufficient to prove the existence of global strong solution.

2 Main results

We now wish to present our main results which concern the existence of global strong solution for the Navier-Stokes Korteweg system with large initial data provided that we impose a condition of sign on the initial data.

Theorem 2.1 *Assume that $\beta > 1$ and $\gamma \in [\beta, \beta+1]$. Let $u_0 \in H^3(\mathbb{R})$, $(\rho_0 - 1) \in H^4(\mathbb{R})$ and $\frac{1}{\rho_0} \in L^\infty(\mathbb{R})$ with the additional following conditions of sign, for any $x \in \mathbb{R}$ we have:*

$$\rho_0^{\frac{(\beta-1)\alpha_2-\beta}{2\alpha_2-1}} \partial_x v_{2,0} + \frac{\gamma}{-\alpha_2(\beta+1)+\gamma(2\alpha_2-1)} \rho_0^{\frac{-\alpha_2(\beta+1)}{2\alpha_2-1}+\gamma} \leq 0 \tag{2.23}$$

or:

$$\rho_0^{\frac{(\beta-1)\alpha_1-\beta}{2\alpha_1-1}} \partial_x v_{1,0} + \frac{\gamma}{-\alpha_1(\beta+1)+\gamma(2\alpha_1-1)} \rho_0^{\frac{-\alpha_1(\beta+1)}{2\alpha_1-1}+\gamma} \leq 0 \tag{2.24}$$

Then there exists a unique global strong solution (ρ, u) for the Navier-Stokes system (1.1) with the following properties. For any given $T > 0$, $L > 0$ there exist a positive constant

C, a positive constant $C(T)$ depending respectively on T and on the initial data such that, if $\sigma(t) = \min(1;t)$, then for $i \in \{1,2\}$:

$$C(T)^{-1} \leq \rho(T,\cdot) \leq C \quad a.e, \tag{2.25}$$

$$\begin{aligned} &\sup_{0<t\leq T} \big(\|\rho(t,\cdot)-1\|_{L^2} + \|u(t,\cdot)\|_{L^2} + \|\partial_x\rho(t,\cdot)\|_{L^2} + \sigma(t)^{\frac{1}{2}}\|\partial_x u(t,\cdot)\|_{L^2} \\ &+ \sigma(t)(\|\dot{u}(t,\cdot)\|_{L^2} + \|\partial_x((1-\alpha_i)\rho^\beta\partial_x v_i(t,\cdot) - P(\rho) + P(1))\|_{L^2}\big) \leq C(T), \end{aligned} \tag{2.26}$$

$$\int_0^T [\|\partial_x u(t,\cdot)\|_{L^2}^2 + \|\partial_x\rho(t,\cdot)\|_{L^2}^2 + \sigma(t)\|\dot{u}(t,\cdot)\|_{L^2}^2 + \sigma^2(t)\|\partial_x\dot{u}(t,\cdot)\|_{L^2}^2]dt \leq C(T), \tag{2.27}$$

$$\int_0^T \sigma^{\frac{1}{2}}(\tau)\,\|\partial_x u(\tau)\|_{L^\infty}^2\,d\tau \leq C(T), \tag{2.28}$$

$$\sup_{0<t\leq T} \sigma(t)\|\partial_x u(t,\cdot)\|_{L^\infty} \leq C(T). \tag{2.29}$$

Remark 1 *We can prove in fact at least on the torus (see [38]) that we control $\frac{1}{\rho}$ in $L^\infty(\mathbb{R}^+, L^\infty)$ and not only in $L^\infty_{loc}(\mathbb{R}^+, L^\infty)$ using the damping on $\partial_x\rho$ provide by the BD entropies (1.17). Indeed (1.17) implies that $\rho^{\frac{\gamma+\beta-3}{2}}\partial_x\rho$ is bounded in $L^2(\mathbb{R}^+, L^2(\mathbb{R}))$, in particular by adapting the same ideas as in [38] where Gagliardo-Niremberg estimate is used in a suitable way we can show that for $T>0$ large enough we have for any $t>T$, $\|\rho(t,\cdot)-1\|_{L^\infty} \leq \frac{1}{2}$ which implies that $\|\frac{1}{\rho}(t,\cdot)\|_{L^\infty} \leq 2$ for $t>T$. Combining this estimate with (2.25) show the result.*

Remark 2 *It is important to note that our result requires to work with c included in the interval $]0,\frac{1}{4}]$. The main reason is that if $c > \frac{1}{4}$ then the α_i with $i \in \{1,2\}$ are complex. We can again obtain the following equation with v_i the effective velocities:*

$$\rho\partial_t v_i + \rho u\partial_x v_i + (\alpha_i - 1)\partial_x(\mu(\rho)\partial_x v_i) + \partial_x P(\rho) = 0, \tag{2.30}$$

however since α_i is complex, we can not apply maximum principle on v_i or on the effectives pressures even if the linearized equation associated to (2.30) is parabolic.

Remark 3 *This result generalizes in particular the works of [15] to the case of the Navier-Stokes Korteweg system. Indeed we have as in [15] a condition of sign on the initial effective pressure $w_{2,0}$ which generalizes the effective pressure of the compressible Navier-Stokes system. We can also observe that the coefficient $\gamma > 1$ is restricted to the interval $[\beta, \beta+1]$ as in [15].*

The second result show the existence of almost global strong solution when c goes to zero without any assumption of sign on the initial data.

Theorem 2.2 *Assume that $\beta > 1$ and $\beta \geq \gamma$. Let $u_0 \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$, $(\rho_0 - 1) \in H^1(\mathbb{R}), \partial_x\rho_0 \in L^\infty(\mathbb{R})$ and $\frac{1}{\rho_0} \in L^\infty(\mathbb{R})$ and that there exists $C \in \mathbb{R}$ such that for any $x > y$ we have:*

$$\frac{v_{1,0}(x) - v_{1,0}(y)}{x-y} \leq C_0. \tag{2.31}$$

Then for any $T > 0$ there exists $c_0 > 0$ sufficiently small (depending on the initial data, T, and the physical coefficients) such that for any $0 < c < c_0$ there exists a unique strong solution (ρ, u) for the Navier-Stokes system (1.1) on a time interval $[0, T]$. Furthermore the solution (ρ, u) satisfies the same regularity assumption as in the Theorem 2.1 on the time interval $[0, T]$. We have in addition for any given $t \in [0, T]$ and for any $x > y$:

$$\frac{v_1(t,x) - v_1(t,y)}{x - y} \leq C(T), \tag{2.32}$$

with $C(T) > 0$.

Remark 4 *It is important to note that when c goes to zero then the maximal time of existence T for a strong solution goes to $+\infty$. It enables us to recover the result of the existence of global strong solution for the compressible Navier-Stokes system as it is proved in [7].*

Remark 5 *In comparison with the Theorem 2.1, we can note that we have no restriction on the size of γ since we assume only that $\gamma \geq \beta$. Furthermore we work with initial data which are less regular. In particular it is not mandatory to assume that $(\rho_0 - 1, u_0)$ is in $H^4(\mathbb{R}) \times H^3(\mathbb{R}))$ as in Theorem 2.1 or in [13, 14].*

In the section 3 and 4 we prove the Theorems 2.1 and 2.2. An appendix is devoted to proof of the Proposition 3.1 which defines new effective pressures, we also give a sketch of the proof of the blow-up criterion of the Theorem 3.3 below.

3 Proof of the Theorem 2.1

In order to prove the existence of global strong solution for the Navier-Stokes Kortexeg system, we start with recalling the following result of existence of strong solution in finite time. In addition we give a blow-up criterion.

Theorem 3.3 *Assume that $\beta > 1$, $\gamma \geq \max(\beta - \frac{1}{2}, 1)$, $s \geq 3$ and $(\rho_0 - 1, u_0) \in H^{s+1} \times H^s(\mathbb{R})$ with $\frac{1}{\rho_0} \in L^\infty(\mathbb{R})$. Then there exists $T^* > 0$ such that there exists a strong solution (ρ, u) of the system (1.1) on $(0, T^*)$ with $\forall T \in (0, T^*)$:*

$$(\rho - 1) \in C(0, T, H^{s+1}(\mathbb{R})) \cap L^2(0, T, H^{s+2}(\mathbb{R})),\ u \in C(0, T, H^s(\mathbb{R})) \cap L^2(0, T, H^{s+1}(\mathbb{R})),$$

and for all $t \in (0, T^*)$:

$$\|\frac{1}{\rho}(t, \cdot)\|_{L^\infty} \leq C(t),$$

where $C(t) < +\infty$ if $t \in (0, T^*)$. In addition, if:

$$\sup_{t \in (0, T^*)} \|\frac{1}{\rho}(t, \cdot)\|_{L^\infty} \leq C < +\infty,$$

then the solution can be continued beyond $(0, T^*)$.

The above result claims that a strong solution in finite time might blow-up is if and only if the L^∞-norm of $\frac{1}{\rho}$ blows-up in T^*. We refer to [13, 14] for the proof of existence of a strong solution in finite time. The blow-up criterion of the Theorem 3.3 is essentially an adaptation to the Korteweg system of the blow-up criterion proved for the compressible Navier-Stokes system in Constantin *et al* (see Theorem 1.1. from [15]) for the torus or in [7] for the whole space. We refer the reader to the Appendix for a sketch of the proof.
Since the assumption on the initial data of the Theorem 2.1 corresponds to the one of the Theorem 3.3, we know that there exists a strong solution (ρ, u) of the system (1.1) on a finite time interval $(0, T^*)$.
We are going to prove that $T^* = +\infty$, by absurd we assume now that $0 < T^* < +\infty$. To do this we wish to apply the blow-up criterion of Theorem 3.3, it suffices then to show that the L^∞ norm of $\frac{1}{\rho}$ can never blow-up in finite time. From (1.10), we recall that we have for $i \in \{1, 2\}$:

$$\begin{cases} \partial_t \rho + v_i \partial_x \rho + \rho \partial_x v_i - \alpha_i \partial_{xx} \varphi(\rho) = 0 \\ \rho \partial_t v_i + \rho u \partial_x v_i + (\alpha_i - 1)\partial_x(\mu(\rho)\partial_x v_i) + \partial_x P(\rho) = 0. \end{cases} \tag{3.33}$$

with $\varphi'(\rho) = \frac{\mu(\rho)}{\rho^2}$. Our main goal will be to apply a maximum principle on the mass equation of (3.33) in order to estimate the L^∞ norm of $\frac{1}{\rho}$. To do this, we are required to obtain an estimate of the type:

$$\partial_x v_i(t, \cdot) \leq C(t), \tag{3.34}$$

for any $t \in (0, T^*)$ with C a continuous bounded function on $(0, T^*)$ and $i \in \{1, 2\}$. Since we wish to control $\partial_x v_i$ we are going to introduce new effectives pressures w_i with $i \in \{1, 2\}$ which generalize the one obtained in the case of compressible Navier-Stokes equations (see [7, 15]) and which are governed by a parabolic equation with a damping term. We state now a crucial Proposition of this paper concerning the effectives pressures w_i (the proof of this Proposition is given in the Appendix).

Proposition 3.1 *We set:*

$$f_i(\rho) = \rho^{\frac{(\beta-1)\alpha_i - \beta}{2\alpha_i - 1}} \quad \textit{and} \quad F_{1,i}(\rho) = \frac{\gamma}{-\alpha_i(\beta+1) + \gamma(2\alpha_i - 1)} \rho^{\frac{-\alpha_i(\beta+1)}{2\alpha_i - 1} + \gamma} \tag{3.35}$$

Furthermore we define w_i as follows with $i \in \{1, 2\}$:

$$w_i = f_i(\rho)\partial_x v_i + F_{1,i}(\rho). \tag{3.36}$$

Then w_i satisfies the following equation:

$$\partial_t w_i + (u + g_{1i})\partial_x w_i + w_i g'_{2i} + (\alpha_i - 1)\partial_x\left(\frac{\mu(\rho)}{\rho}\partial_x w_i\right) + g'_{3i} + g_{4i} w_i^2 = 0, \tag{3.37}$$

with:

$$g_{1i} = \rho^{\beta-2}\partial_x \rho(\alpha_i - 1)\frac{\beta + 2\alpha_i}{2\alpha_i - 1} \tag{3.38}$$

$$\begin{aligned} g'_{2i} = &-(\alpha_i - 1)\alpha_i \rho^{\beta-3}\frac{(\beta-1)\alpha_i - \beta}{2\alpha_i - 1}\frac{\beta+1}{2\alpha_i - 1}(\partial_x \rho)^2 \\ &+ \rho^{\gamma-\beta}\frac{\gamma}{2\alpha_i - 1}\frac{-3\alpha_i\beta - 3\alpha_i + 2\alpha_i\gamma - \gamma + 2\beta + 2}{-\alpha_i(\beta+1) + \gamma(2\alpha_i - 1)}. \end{aligned} \tag{3.39}$$

$$g'_{3i} = \rho^{\frac{-\alpha_i(\beta+1)}{2\alpha_i-1}+2\gamma-\beta}\frac{\gamma^2}{(-\alpha_i(\beta+1)+\gamma(2\alpha_i-1))^2}(\beta+1-\gamma) + (\partial_x\rho)^2\rho^{\frac{-\alpha_i(\beta+1)}{2\alpha_i-1}+\gamma+\beta-3}\frac{\gamma\alpha_i(\gamma-\beta-1)(\gamma-\beta)}{(-\alpha_i(\beta+1)+\gamma(2\alpha_i-1))} \tag{3.40}$$

and:

$$g_{4i} = (\alpha_i-1)\frac{\beta+1}{2\alpha_i-1}\rho^{\frac{-(\beta-1)\alpha_i+\beta}{2\alpha_i-1}}. \tag{3.41}$$

Remark 6 *We can observe that when $\alpha_1 = 1$ and $\alpha_2 = 0$ which corresponds to the compressible Navier-Stokes equations since $c = 0$ in this case, the effective pressure take the following form:*

$$\begin{aligned} w_1 &= \frac{1}{\rho}\partial_x(u+\frac{\mu(\rho)}{\rho^2}\partial_x\rho) + \frac{\gamma}{\gamma-\beta-1}\rho^{\gamma-\beta-1} \\ w_2 &= \rho^\beta\partial_x u - \rho^\gamma \end{aligned} \tag{3.42}$$

We recover in particular the effective pressure which have been defined in [7, 15] for the compressible Navier-Stokes system. In addition in this case w_1 satisfies the following equation :

$$\partial_t w_1 + u\partial_x w_1 + w_1\gamma\rho^{\gamma-\beta} - \frac{\gamma^2}{(\gamma-\beta-1)}\rho^{2\gamma-2\beta-1} + (\partial_x\rho)^2\rho^{\gamma-4}\gamma(\gamma-\beta) = 0, \tag{3.43}$$

We note that w_1 verifies exactly the same equation on the effective pressure as in [7] for the compressible Navier-Stokes system. Concerning w_2 we have:

$$\partial_t w_2+(u+\beta\rho^{\beta-2})\partial_x w_2+w_2(2\beta+2-\gamma)\rho^{\gamma-\beta}-\partial_x(\frac{\mu(\rho)}{\rho}\partial_x w_2)+\rho^{2\gamma-\beta}(\beta+1-\gamma)+(\beta+1)\rho^{-\beta}w_2^2 = 0, \tag{3.44}$$

It corresponds exactly to the equation (6.4) of [15] for the second effective pressure.

It is important now to determinate the sign of g'_{3i} and g_{4i} if we wish to apply a maximum principle.

Proposition 3.2 *We have:*

$$F_{12} \leq 0 \ \ and \ \ F_{11} \leq 0 \ \ if \ \ \gamma \leq \beta+1. \tag{3.45}$$

$$g_{41} \leq 0 \ \ and \ \ g_{42} \geq 0. \tag{3.46}$$

Furthermore we have:

$$g'_{31}, g'_{32} \geq 0 \ \ if \, \gamma \in [\beta, \beta+1] \tag{3.47}$$

Proof of the Proposition 3.2: $F_{1i}(\rho)$ is given by:

$$F_{1,i}(\rho) = \frac{\gamma}{-\alpha_i(\beta+1)+\gamma(2\alpha_i-1)}\rho^{\frac{-\alpha_i(\beta+1)}{2\alpha-1}+\gamma} \tag{3.48}$$

We deduce that $F_{12} \leq 0$ because $-\alpha_2(\beta+1)+\gamma(2\alpha_2-1) \leq 0$. Now when $\gamma \leq \beta+1$ we have $\gamma(2\alpha_1-1)-\alpha_1(\beta+1) \leq \gamma(2\alpha_1-1)-\alpha_1\gamma = \gamma(\alpha_1-1) \leq 0$. it implies then that $F_{11} \leq 0$ if $\gamma \leq \beta+1$.
We recall now that:

$$g_{4i} = (\alpha_i - 1)\frac{\beta+1}{2\alpha_i-1}\rho^{\frac{-(\beta-1)\alpha_i+\beta}{2\alpha_i-1}}. \tag{3.49}$$

From the definition of α_i in (1.9) and since $\alpha_i - 1 \leq 0$ for $i \in \{1,2\}$ and $2\alpha_1 - 1 \geq 0$, $2\alpha_2 - 1 \leq 0$ it implies that:

$$g_{41} \leq 0 \text{ and } g_{42} \geq 0. \tag{3.50}$$

Let us consider now g'_{3i} and we recall using (3.51) that:

$$\begin{aligned} g'_{3i} = \rho^{\frac{-\alpha_i(\beta+1)}{2\alpha_i-1}+2\gamma-\beta}\frac{\gamma^2}{(-\alpha_i(\beta+1)+\gamma(2\alpha_i-1))^2}(\beta+1-\gamma) \\ + (\partial_x\rho)^2\rho^{\frac{-\alpha_i(\beta+1)}{2\alpha_i-1}+\gamma+\beta-3}\frac{\gamma\alpha_i(\gamma-\beta-1)(\gamma-\beta)}{(-\alpha_i(\beta+1)+\gamma(2\alpha_i-1))} \end{aligned} \tag{3.51}$$

First we observe that the first term on the right hand side of (3.51) is always positive if $\gamma \leq \beta+1$ and negative if $\gamma > \beta+1$. Let us deal with the second term on right hand side of (3.51) that we denote:

$$g'_{32i} = (\partial_x\rho)^2\rho^{\frac{-\alpha_i(\beta+1)}{2\alpha_i-1}+\gamma+\beta-3}\frac{\gamma\alpha_i(\gamma-\beta-1)(\gamma-\beta)}{(-\alpha_i(\beta+1)+\gamma(2\alpha_i-1))} \tag{3.52}$$

We deduce that:

- $g'_{322} \geq 0$ if $\beta \leq \gamma \leq \beta+1$ because $-\alpha_2(\beta+1)+\gamma(2\alpha_2-1) \leq 0$ and $\gamma-\beta-1 \leq 0$, inversely $g'_{322} \leq 0$ if $\gamma \leq \beta$ or $\gamma \geq \beta+1$.
- $g'_{321} \geq 0$ if $\beta \leq \gamma \leq \beta+1$ or if $\gamma > \beta+1$ and $\alpha_1 > \frac{\gamma}{2\gamma-\beta-1}$. Indeed if $\beta \leq \gamma \leq \beta+1$ then we have $\gamma(2\alpha_1-1)-\alpha_1(\beta+1) \leq \gamma(2\alpha_1-1)-\alpha_1\gamma = \gamma(\alpha_1-1) \leq 0$. We deduce then that:
$$\frac{(\gamma-\beta-1)(\gamma-\beta)}{(-\alpha_1(\beta+1)+\gamma(2\alpha_1-1))} \geq 0.$$
Now if $\gamma > \beta+1$, we have $\gamma-\beta-1 > 0$ and $(-\alpha_1(\beta+1)+\gamma(2\alpha_1-1)) > 0$ if $\alpha_1 > \frac{\gamma}{2\gamma-\beta-1}$. It is important to observe that: $\frac{1}{2} < \frac{\gamma}{2\gamma-\beta-1} < 1$.

□

3.1 Uniform estimates for $\frac{1}{\rho}$

We are going now to consider the unknown w_2 which satisfies the parabolic equation (3.37), and since we wish to prove some estimate of the form (3.34) it is natural to apply a maximum principle on w_2. Owing to the fact that the solution (ρ, u) is regular we get that w_2 is continuous on $[0,T^*) \times \mathbb{R}$ and in view of $\lim_{x\to\pm\infty} w_2(t,x) = F_{1,2}(1)$, we deduce that for all $t \in [0,T^*)$ we have:

$$\sup_{x\in\mathbb{R}} w_2(t,x) \geq F_{1,2}(1). \tag{3.53}$$

The function

$$t \to \sup_{x\in\mathbb{R}} w_2(t,x)$$

is continuous on $[0,T^*)$, so we deduce that the set

$$D := \left\{ t \in (0,T^*) : \sup_{x\in\mathbb{R}} w_2(t,x) > F_{1,2}(1) \right\}$$

is open in $[0,T^*)$ (with the topology induced from $\mathbb{R}$) we conclude that

$$\left\{ t \in [0,T^*) : \sup_{x\in\mathbb{R}} w_2(t,x) > F_{1,2}(1) \right\} = I_0 \cup \bigcup_{j\in\mathbb{N}^\star} I_j,$$

where $(I_j)_{j\geq 1}$ with $I_j = (a_j, b_j)$ are open disjoint intervals and $I_0 = \emptyset$ if $\sup_{x\in\mathbb{R}} w_2(0,x) = F_{1,2}(1)$ and $I_0 = [0, b_0)$ for some $b_0 \in (0,T^*)$ if $\sup_{x\in\mathbb{R}} w_2(0,x) > F_{1,2}(1)$. From the definition of I_j we have that $\sup_{x\in\mathbb{R}} w_2(a_j,x) = F_{1,2}(1)$ and for all $t \in I_j$ since $w_2(t,\cdot)$ is continuous, it reaches its maximum on $\mathbb{R}$. It implies that for any $j \in \mathbb{N}$ and any $t \in I_j$ there exists a least one point $x_t \in \mathbb{R}$ such that:

$$\sup_{x\in\mathbb{R}} w_2(t,x) \overset{def.}{=} w_M(t) = w_2(t,x_t).$$

For any $t \in (I_0 \cup \cup_{n\in\mathbb{N}*} I_j)^c$, we know that $\sup_{x\in\mathbb{R}} w_2(t,x) = F_{1,2}(1)$. Thus, in order to provide an estimate of w_2 on $[0,T^*)$ we have to show that we can control w_M on $I_0 \cup \cup_{n\in\mathbb{N}*} I_j$ an so, we are going to study the behavior of w_M on all intervals I_j. To fix the ideas let us fix $j_0 \in \mathbb{N}$ and let us analyse what happens on I_{j_0}.
First of all w_M is Lipschitz continuous on any interval I_j and then absolutely continuous, it will be important when we will apply Gronwall Lemma. Indeed from the triangular inequality for the norm $\|f\| = \sup_{x\in\mathbb{R}} |f(x)|$ with f a continuous bounded function on $\mathbb{R}$ we have for $(t_1,t_2) \in I_j$:

$$|w_M(t_1) - w_M(t_2)| \leq \sup_{x\in\mathbb{R}} |w_2(t_1,x) - w_2(t_2,x)| \leq \|\partial_t w_2\|_{L^\infty([t_1,t_2],L^\infty)} |t_1 - t_2|.$$

According to Rademacher's theorem, w_M is differentiable almost everywhere on $I_0 \cup \bigcup_{j\in\mathbb{N}^\star} I_j$,. We are going to verify now that for $t \in I_{j_0}$ (with $j_0 \geq 0$) we have $(w_M)'(t) = \partial_t w_2(t,x_t)$. Indeed we have:

$$\begin{aligned}(w_M)'(t) &= \lim_{h\to 0^+} \frac{w_M(t+h) - w_M(t)}{h} = \lim_{h\to 0^+} \frac{w_2(t+h,x_{t+h}) - w_2(t,x_t)}{h} \\ &\geq \lim_{h\to 0^+} \frac{w_2(t+h,x_t) - w_2(t,x_t)}{h} = \partial_t w_2(t,x_t).\end{aligned}$$

Similarly, we have:

$$\begin{aligned}(w_M)'(t) &= \lim_{h\to 0^+} \frac{w_M(t) - w_M(t-h)}{h} = \lim_{h\to 0^+} \frac{w_2(t,x_t) - w_2(t-h,x_{t-h})}{h} \\ &\leq \lim_{h\to 0^+} \frac{w_2(t,x_t) - w_2(t-h,x_t)}{h} = \partial_t w_2(t,x_t).\end{aligned}$$

We deduce from (3.37) using the fact that $\partial_x w_2(t,x_t) = 0$ and $\partial_{xx} w_2(t,x_t) \leq 0$ since $w_2(t,\cdot)$ reaches its maximum in x_t that for almost everywhere $t \in I_{j_0}$ we have:

$$\partial_t w_M(t) + w_M(t) g'_{22}(t,x_t) + g'_{32}(t,x_t) + g_{42}(t,x_t)(w_M(t))^2 \leq 0. \tag{3.54}$$

Using the Proposition 3.2 we know that $g_{42}(t,x_t) \geq 0$ and $g'_{32}(t,x_t) \geq 0$ for $\beta \in [\gamma, \gamma+1]$, it yields that:

$$\partial_t w_M(t) \leq -w_M(t) g'_{22}(t,x_t). \tag{3.55}$$

Since we know from the proposition 3.2 that $w_M(a_{j_0}) = F_{12}(1) \leq 0$ if $j_0 \geq 1$ when $\gamma \leq \beta+1$ or $w_M(0) = \sup_{x\in\mathbb{R}} w_2(0,x) \leq 0$ from the condition (2.23) in the Theorem 2.1, we deduce from (3.55) that for any $t \in I_{j_0}$ we have:

$$w_M(t) \leq 0. \tag{3.56}$$

It implies finally using the fact that $F_{1,2}(1) \leq 0$ when $\gamma \in [\beta, \beta+1]$ that for any $t \in (0,T^*)$ we have:

$$w_M(t) \leq 0. \tag{3.57}$$

In a similar way, if we assume that the condition (2.24) is satisfied then we can check that for any $t \in (0,T^*)$ we have:

$$w_{M1}(t) \leq 0. \tag{3.58}$$

with:

$$\sup_{x\in\mathbb{R}} w_1(t,x) \overset{def.}{=} w_{M1}(t).$$

Next we recall that we have from the mass equation in (1.1):

$$\partial_t(\frac{1}{\rho}) + u\partial_x(\frac{1}{\rho}) - \frac{1}{\rho}\partial_x u = 0.$$

We can rewrite the equation as follows with $i \in \{1,2\}$:

$$\partial_t(\frac{1}{\rho}) + u\partial_x(\frac{1}{\rho}) - \frac{1}{\rho}\partial_x v_i - \alpha_i \frac{\mu(\rho)}{\rho}\partial_{xx}(\frac{1}{\rho}) - \alpha_i \frac{1}{\rho}\partial_x \mu(\rho)\partial_x(\frac{1}{\rho}) = 0.$$

From the definition of w_i in (3.36), we have for $i \in \{1,2\}$:

$$\partial_t(\frac{1}{\rho}) + u\partial_x(\frac{1}{\rho}) - \frac{1}{\rho f_i(\rho)}(w_i - F_{1i}(\rho)) - \alpha_i \frac{\mu(\rho)}{\rho}\partial_{xx}(\frac{1}{\rho}) - \alpha_i \frac{1}{\rho}\partial_x \mu(\rho)\partial_x(\frac{1}{\rho}) = 0. \tag{3.59}$$

Again, the value of $\frac{1}{\rho}$ is fixed at $\pm\infty$ for al $t \geq 0$ and is equal to 1. We now consider the open set

$$\left\{ t \in [0,T^*) : \sup_{x\in\mathbb{R}} \frac{1}{\rho}(t,x) > 1) \right\} = Q_0 \cup \bigcup_{j\in\mathbb{N}^\star} Q_j,$$

where for $j \geq 1$, Q_j are open disjoint intervals. Following the same arguments as previously, we set now:

$$z(t) = \sup_{x\in\mathbb{R}} \frac{1}{\rho}(t,x),$$

with $t \in (0, T^*)$. We know that in any interval Q_j, there is a point, still denoted x_t such that $z(t) = \frac{1}{\rho(t,x_t)}$. We have then for any $t \in Q_{j_0}$ and $i \in \{1, 2\}$ using (3.59) and the fact that $\partial_x(\frac{1}{\rho})(t, x_t) = 0$, $\partial_{xx}(\frac{1}{\rho})(t, x_t) \leq 0$ (indeed x_t is a point where $\frac{1}{\rho}$ reaches its maximum):

$$\partial_t z(t) \leq \frac{1}{\rho f_i(\rho)}(w_i - F_{1i}(\rho))(t, x_t). \tag{3.60}$$

Let us assume now that (2.23) is satisfied, we have then seen from (3.56) that for any $t \in (0, T^*)$ we have $w_2(t, x_t) \leq 0$. We deduce then from (3.60) and (3.35) that:

$$\begin{aligned}
\partial_t z(t) &\leq \frac{-F_{12}(\rho)}{\rho f_2(\rho)}(t, x_t) \\
&\leq \frac{\gamma}{\alpha_2(\beta+1) - \gamma(2\alpha_2 - 1)} \rho^{\frac{-\alpha_2(\beta+1)}{2\alpha_2-1} + \gamma - 1} \rho^{\frac{\beta-(\beta-1)\alpha_2}{2\alpha_2-1}}(t, x_t) \\
&\leq \frac{\gamma}{\alpha_2(\beta+1) - \gamma(2\alpha_2 - 1)} \rho^{\gamma-1-\beta}(t, x_t) \\
&\leq \frac{\gamma}{\alpha_2(\beta+1) - \gamma(2\alpha_2 - 1)} z(t)^{\beta+1-\gamma}
\end{aligned} \tag{3.61}$$

Since $\gamma \in [\beta, \beta + 1]$, we deduce that $\beta + 1 - \gamma \in [0, 1]$ and applying Gronwall Lemma, we obtain the existence of a continuous function C_2 on $\mathbb{R}^+$ such that for any $t \in Q_{j_0}$ we have:

$$z(t) \leq C_2(t).$$

This implies that for any $t \in (0, T^*)$ we get:

$$\|\frac{1}{\rho}(t, \cdot)\|_{L^\infty} \leq C_3(t). \tag{3.62}$$

with C_3 a continuous function on $[0, T^*]$. Combining the blow-up criterion in Theorem 3.3 and (3.62), it yields that $T^* < +\infty$ is absurd and then $T^* = +\infty$. For for any $t > 0$ we have:

$$\|\frac{1}{\rho}(t, \cdot)\|_{L^\infty} \leq C_3(t), \tag{3.63}$$

with C_3 a continuous function on $\mathbb{R}^+$. Applying the same type of technics we obtain a similar result when we assume that (2.24) is satisfied. We have then proved that the strong solution in finite time of the Theorem 3.3 are in fact global.
We would like to show the estimate (2.26)(2.27), (2.28) and (2.29) of the Theorem 2.1. We simply recall for the moment that our strong solution (ρ, u) satisfy the energy estimates (1.14) and (1.17). Using (1.14), (1.17) and Sobolev embedding we get that for $C > 0$ large enough we have (see [13, 14] for details):

$$\|\rho\|_{L^\infty(\mathbb{R}^+, L^\infty)} \leq C. \tag{3.64}$$

It proves in particular the inequality (5.160) if we combine (4.98) and (3.63).

3.2 Estimates à la Hoff

Introducing the convective derivative

$$\dot{v}_i = \partial_t v_i + u\partial_x v_i,$$

with $i \in \{1,2\}$, we rewrite the momentum equation (1.10) as

$$\rho\dot{v}_i - (1-\alpha_i)\partial_x\left(\rho^\beta\partial_x v_i\right) + \partial_x\rho^\gamma = 0.$$

Let us observe that:

$$\begin{aligned}
-\int_{\mathbb{R}}\partial_x\left(\rho^\beta\partial_x v_i\right)\partial_t v_i &= \int_{\mathbb{R}}\rho^\beta\partial_x v_i\partial_{xt}v_i = \frac{1}{2}\int_{\mathbb{R}}\rho^\beta\partial_t\left((\partial_x v_i)^2\right)\\
&= \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}\rho^\beta\left(\partial_x v_i\right)^2 - \frac{1}{2}\int_{\mathbb{R}}\partial_t\rho^\beta(\partial_x v_i)^2. \qquad (3.65)
\end{aligned}$$

Next, we see that:

$$\begin{aligned}
-\int_{\mathbb{R}}\partial_x\left(\rho^\beta\partial_x v_i\right)u\partial_x v_i &= -\int_{\mathbb{R}}u\partial_x\rho^\beta(\partial_x v_i)^2 - \int_{\mathbb{R}}\rho^\beta u\partial_{xx}v_i\partial_x v_i\\
&= -\int_{\mathbb{R}}u\partial_x\rho^\beta(\partial_x v_i)^2 + \frac{1}{2}\int_{\mathbb{R}}\partial_x\left(u\rho^\beta\right)(\partial_x v_i)^2\\
&= -\int_{\mathbb{R}}u\partial_x\rho^\beta(\partial_x v_i)^2 + \frac{1}{2}\int_{\mathbb{R}}\rho^\beta\partial_x u(\partial_x v_i)^2 + \frac{1}{2}\int_{\mathbb{R}}u\partial_x\rho^\beta(\partial_x v_i)^2\\
&= -\frac{1}{2}\int_{\mathbb{R}}u\partial_x\rho^\beta(\partial_x v_i)^2 + \frac{1}{2}\int_{\mathbb{R}}\rho^\beta\partial_x u(\partial_x v_i)^2.
\end{aligned}$$

Thus, we gather that:

$$\begin{aligned}
-\int_{\mathbb{R}}\partial_x\left(\rho^\beta\partial_x v_i\right)\dot{v}_i &= \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}\rho^\beta\left(\partial_x v_i\right)^2 - \frac{1}{2}\int_{\mathbb{R}}\partial_t\rho^\beta(\partial_x v_i)^2 - \frac{1}{2}\int_{\mathbb{R}}u\partial_x\rho^\beta(\partial_x v_i)^2 + \frac{1}{2}\int_{\mathbb{R}}\rho^\beta\partial_x u(\partial_x v_i)^2\\
&= \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}\rho^\beta\left(\partial_x v_i\right)^2 + \frac{1+\beta}{2}\int_{\mathbb{R}}\rho^\beta\partial_x u(\partial_x v_i)^2.
\end{aligned}$$

Moreover, we see that:

$$\begin{aligned}
\int_{\mathbb{R}}\partial_x\rho^\gamma\left(\partial_t v_i + u\partial_x v_i\right) &= -\int_{\mathbb{R}}\rho^\gamma\partial_{tx}v_i + \int_{\mathbb{R}}u\partial_x\rho^\gamma\partial_x v_i\\
&= -\frac{d}{dt}\int_{\mathbb{R}}\rho^\gamma\partial_x v_i + \int_{\mathbb{R}}\partial_t\rho^\gamma\partial_x v_i + \int_{\mathbb{R}}u\partial_x\rho^\gamma\partial_x v_i\\
&= -\frac{d}{dt}\int_{\mathbb{R}}\rho^\gamma\partial_x v_i - \gamma\int_{\mathbb{R}}\rho^\gamma\partial_x v_i\partial_x u.
\end{aligned}$$

Multiplying the momentum equation with $\dot{v}_i$ yields:

$$\begin{aligned}
\int_{\mathbb{R}}\rho\dot{v}_i^2 + \frac{d}{dt}\left\{\frac{1}{2}(1-\alpha_i)\int_{\mathbb{R}}\rho^\beta\left(\partial_x v_i\right)^2 - \int_{\mathbb{R}}\rho^\gamma\partial_x v_i\right\} &= -(1-\alpha_i)\frac{1+\beta}{2}\int_{\mathbb{R}}\rho^\beta(\partial_x v_i)^2\partial_x u\\
&\quad + \gamma\int_{\mathbb{R}}\rho^\gamma\partial_x u\partial_x v_i
\end{aligned} \qquad (3.66)$$

Let us multiply the previous estimate by $\sigma(t) = \min(1,t)$ and integrate in time on $[0,t]$ with $t > 0$, we have then:

$$\begin{aligned}
&\frac{\sigma(t)(1-\alpha_i)}{2}\int_{\mathbb{R}} \rho^{\beta}(t)(\partial_x v_i)^2(t) + \int_0^t \int_{\mathbb{R}} \sigma\rho\dot{v_i}^2 \\
&= \sigma(t)\int_{\mathbb{R}} (\rho^{\gamma}-1)\partial_x v_i + \int_0^{\min\{1,t\}} \int_{\mathbb{R}} \left[\frac{1}{2}(1-\alpha_i)\rho^{\beta}(\partial_x v_i)^2 - (\rho^{\gamma}-1)\partial_x v_i\right] \\
&- (1-\alpha_i)\frac{1+\beta}{2}\int_0^t \int_{\mathbb{R}} \sigma\rho^{\beta}(\partial_x v_i)^2\partial_x u + \gamma\int_0^t \int_{\mathbb{R}} \sigma\rho^{\gamma}\partial_x v_i \partial_x u.
\end{aligned}$$

Let us denote by:

$$A_i(\rho, v_i)(t) = \frac{\sigma(t)(1-\alpha_i)}{2}\int_{\mathbb{R}} \rho^{\beta}(t)(\partial_x v_i)^2(t) + \int_0^t \int_{\mathbb{R}} \sigma\rho\dot{v_i}^2$$

with $i \in \{1,2\}$. Let us observe that using (1.14), (3.64) and (3.63) we have:

$$\begin{aligned}
\sigma(t)\int_{\mathbb{R}} (\rho^{\gamma}-1)\partial_x v_i &\leq \sqrt{\sigma(t)}\left\|\frac{\rho^{\gamma}-1}{\rho^{\frac{\beta}{2}}}\right\|_{L^\infty_t L^2}\left(\int_{\mathbb{R}} \sigma(t)\rho^{\beta}(t)(\partial_x v_i)^2(t)\right)^{\frac{1}{2}} \\
&\leq \frac{1}{1-\alpha_i}C(t)\left\|\frac{\rho^{\gamma}-1}{\rho^{\frac{\beta}{2}}}\right\|^2_{L^\infty_t L^2} + \frac{1}{4}(1-\alpha_i)\int_{\mathbb{R}} \sigma(t)\rho^{\beta}(t)(\partial_x v_i)^2(t) \\
&\leq \frac{1}{1-\alpha_i}C_1(t) + \frac{1}{4}(1-\alpha_i)\int_{\mathbb{R}} \sigma(t)\rho^{\beta}(t)(\partial_x v_i)^2(t),
\end{aligned} \tag{3.67}$$

with C and C_1 positive continuous functions on $\mathbb{R}^+$. Next, we see that owing to the estimate (1.14), (1.17), (3.64) and (3.63), we have that:

$$\int_0^{\min\{1,t\}} \int_{\mathbb{R}} \left[\frac{1}{2}(1-\alpha_i)\rho^{\beta}(\partial_x v_i)^2 - (\rho^{\gamma}-1)\partial_x v_i\right] + \gamma\int_0^t \int_{\mathbb{R}} \sigma\rho^{\gamma}\partial_x v_i \partial_x u \leq (1+\frac{1}{\sqrt{1-\alpha_i}})C_2(t), \tag{3.68}$$

with C_2 a continuous positive function on $\mathbb{R}^+$. Combining (3.66), (3.67) and (3.68) , we thus get for all $t \geq 0$:

$$\begin{aligned}
A_i(\rho, v_i)(t) &\leq C(t)(1+\frac{1}{1-\alpha_i}) + \frac{1}{4}(1-\alpha_i)\int_{\mathbb{R}} \sigma(t)\rho^{\beta}(t)(\partial_x v_i)^2(t) \\
&\qquad - (1-\alpha_i)\frac{1+\beta}{2}\int_0^t \int_{\mathbb{R}} \sigma\rho^{\beta}\partial_x u(\partial_x v_i)^2 \\
&\leq C_3(t)(1+\frac{1}{1-\alpha_i}) + \frac{1}{2}A_i(\rho, u)(t) - (1-\alpha_i)\frac{1+\beta}{2}\int_0^t \int_{\mathbb{R}} \sigma\rho^{\beta}\partial_x u(\partial_x v_i)^2
\end{aligned}$$

with C_3 a continuous positive fonction on $\mathbb{R}^+$. Consequently it yields:

$$A_i(\rho, v_i)(t) \leq C(t)(1+\frac{1}{1-\alpha_i}) - (1-\alpha_i)(1+\beta)\int_0^t \int_{\mathbb{R}} \sigma\rho^{\alpha}(\partial_x v_i)^2\partial_x u$$

which also implies that:

$$\sup_{\tau\in[0,t]} A_i(\rho, v_i)(\tau) \leq C_{\frac{1}{1-\alpha_i}}(t) - (1-\alpha_i)(1+\beta)\int_0^t \int_{\mathbb{R}} \sigma\rho^{\beta}(\partial_x v_i)^2\partial_x u \tag{3.69}$$

with $C_{\frac{1}{1-\alpha_i}}$ an increasing positive continuous function on $\mathbb{R}^+$. Let us observe that for all $\varepsilon > 0$ we have using Gagliardo-Nirenberg inequality (1.14), (1.17) and (3.64):

$$\begin{aligned}
&\int_0^t \sigma^{\frac{1}{2}}(\tau) \left\| \left((1-\alpha_i)\rho^\beta \partial_x v_i - \rho^\gamma \right)(\tau) \right\|_{L^\infty}^2 \\
&\leq 2\int_0^t \sigma^{\frac{1}{2}}(\tau) \left\| \left((1-\alpha_i)\rho^\beta \partial_x v_i - (\rho^\gamma - 1) \right)(\tau) \right\|_{L^\infty}^2 + 2t \qquad (3.70)\\
&\leq 2C\int_0^t \sigma^{\frac{1}{2}}(\tau) \left\| \left((1-\alpha_i)\rho^\beta \partial_x v_i - (\rho^\gamma - 1) \right)(\tau) \right\|_{L^2} \left\| \partial_x \left((1-\alpha_i)\rho^\beta \partial_x v_i - \rho^\gamma \right)(\tau) \right\|_{L^2} + 2t \\
&\leq C_\varepsilon \int_0^t \left\| \left((1-\alpha_i)\rho^\beta \partial_x v_i - (\rho^\gamma - 1) \right)(\tau) \right\|_{L^2}^2 + \varepsilon \int_0^t \sigma(\tau) \left\| \partial_x \left((1-\alpha_i)\rho^\beta \partial_x v_i - \rho^\gamma \right)(\tau) \right\|_{L^2}^2 + 2t \\
&\leq C_\varepsilon \int_0^t \left\| \left((1-\alpha_i)\rho^\beta \partial_x v_i - (\rho^\gamma - 1) \right)(\tau) \right\|_{L^2}^2 + \varepsilon \int_0^t \sigma(\tau) \left\| \rho \dot{v}_i(\tau) \right\|_{L^2}^2 + 2t \\
&\leq C(t,\varepsilon) + \varepsilon \|\rho\|_{L^\infty([0,t],L^\infty)} A_i(\rho, v_i)(t) \qquad (3.71)\\
&\leq C(t,\varepsilon) + \varepsilon C_0 A_i(\rho, v_i)(t), \qquad (3.72)
\end{aligned}$$

with $C(\cdot,\varepsilon)$ a continuous positive function on $\mathbb{R}^+$ and $C, C_\varepsilon, C_0 > 0$ large enough. We are going now to estimate the last term of (3.69) and using (1.14), (1.17), (3.64), (3.63), (3.72) with $\varepsilon = \frac{\sqrt{1-4c}}{(MC_0)}$ (with $M > 0$ sufficiently large that we will determinate later) and the fact that $u = \frac{1}{\sqrt{1-4c}}((1-\alpha_2)v_2 - (1-\alpha_1)v_1)$ we obtain that for $C > 0$ large enough:

$$\begin{aligned}
&|\int_0^t \int_{\mathbb{R}} \sigma\rho^\beta (\partial_x v_i)^2 \partial_x u| = \frac{1}{\sqrt{1-4c}} |\int_0^t \int_{\mathbb{R}} \sigma (\partial_x v_i)^2 \rho^\beta ((1-\alpha_2)\partial_x v_2 - (1-\alpha_1)\partial_x v_1)| \\
&\leq C\int_0^t \left(\sigma^{\frac{1}{4}} \big(\|(1-\alpha_2)\rho^\beta \partial_x v_2 - \rho^\gamma\|_{L^\infty} + \|(1-\alpha_1)\rho^\beta \partial_x v_1 - \rho^\gamma\|_{L^\infty} \big) \sigma^{\frac{3}{4}} \int_{\mathbb{R}} (\partial_x v_i)^2(\tau, x)\, dx \right) d\tau \\
&\leq C\int_0^t \sigma^{\frac{1}{2}}(\tau) \left(\left\| \left((1-\alpha_2)\rho^\beta \partial_x v_2 - \rho^\gamma \right)(\tau) \right\|_{L^\infty}^2 + \left\| \left((1-\alpha_1)\rho^\beta \partial_x v_1 - \rho^\gamma \right)(\tau) \right\|_{L^\infty}^2 \right) \\
&+ 2\int_0^t \sigma^{\frac{3}{2}}(\tau) \left(\int_{\mathbb{R}} (\partial_x v_i)^2(\tau)\, dx \right)^2 \\
&\leq C(t) + \frac{1}{M}\big(A_1(\rho, v_1)(t) + A_2(\rho, v_2)(t) \big) + C\int_0^t \left\| \frac{1}{\rho(\tau)} \right\|_{L^\infty}^{2\beta} \sigma^{\frac{3}{2}}(\tau) \big(\int_{\mathbb{R}} \rho^\beta (\partial_x v_i)^2(\tau)\, dx \big)^2 \\
&\leq C(t) + \frac{1}{M}\big(A_1(\rho, v_1)(t) + A_2(\rho, v_2)(t) \big) + C_1(t) \int_0^t \sigma(\tau) \big(\int_{\mathbb{R}} \rho^\beta (\partial_x v_i)^2(\tau)\, dx \big)^2 d\tau \\
&\leq C(t) + \frac{1}{M}\big(A_1(\rho, v_1)(t) + A_2(\rho, v_2)(t) \big) + \frac{2C_1(t)}{1-\alpha_i} \int_0^t A_i(\rho, v_i)(\tau) \int_{\mathbb{R}} (\rho^\beta \partial_x v_i)^2(\tau)\, d\tau, \qquad (3.73)
\end{aligned}$$

with C and C_1 continuous increasing positive functions. Finally, putting together (3.69) and (3.73) we get that for $i \in \{1, 2\}$

$$\begin{aligned}
\sup_{\tau\in[0,t]} A_i(\rho, v_i)(\tau) &\leq C_{\frac{1}{1-\alpha_i}}(t) + \frac{(1-\alpha_i)(1+\beta)}{M} \big(\sup_{\tau\in[0,t]} A_1(\rho, v_1)(\tau) + \sup_{\tau\in[0,t]} A_2(\rho, v_1)(\tau) \big) \\
&+ C_3(t) \int_0^t A_i(\rho, v_i)(\tau) \int_{\mathbb{R}} (\rho^\beta \partial_x v_i)^2(\tau)\, d\tau, \qquad (3.74)
\end{aligned}$$

with C_3 an increasing continuous function. Using Gronwall's lemma, (1.17) and taking M large enough leads to:

$$\sup_{\tau\in[0,t]} A_i\left(\rho, v_i\right)(\tau) \leq C_{\frac{1}{1-\alpha_1}}(t), \tag{3.75}$$

with $C_{\frac{1}{1-\alpha_1}}$ an increasing continuous function depending on $\frac{1}{1-\alpha_1}$. The control over $A_i\left(\rho, v_i\right)$ and (3.72) yields

$$\int_0^t \sigma^{\frac{1}{2}}(\tau) \left\| \left((1-\alpha_i)\rho^\beta \partial_x v_i - \rho^\gamma \right)(\tau) \right\|_{L^\infty}^2 d\tau \leq C_{\frac{1}{1-\alpha_1}}(t),$$

with $C_{\frac{1}{1-\alpha_1}}$ an increasing continuous function depending on $\frac{1}{1-\alpha_1}$ and consequently we get using in addition (3.64) for $i \in \{1,2\}$:

$$\int_0^t \sigma^{\frac{1}{2}}(\tau) \left\|\partial_x v_i(\tau)\right\|_{L^\infty}^2 d\tau \leq C_{\frac{1}{1-\alpha_1}}(t). \tag{3.76}$$

The last inequality also provides an estimate in $L_t^1\left(L^\infty\right)$ of $\partial_x v_i$ for any $t > 0$ with $i \in \{1,2\}$ using Cauchy-Schwarz inequality:

$$\int_0^t \left\|\partial_x v_i(\tau)\right\|_{L^\infty} d\tau \leq \left(\int_0^t \sigma^{-\frac{1}{2}}(\tau)\, d\tau\right)^{\frac{1}{2}} \left(\int_0^t \sigma^{\frac{1}{2}}(\tau) \left\|\partial_x v_i(\tau)\right\|_{L^\infty}^2\right)^{\frac{1}{2}} \leq C_{\frac{1}{1-\alpha_1}}(t).$$

In particular it implies that $\partial_x u$ belongs to $L^1_{loc}(L^\infty(\mathbb{R}))$. Next, we aim at obtaining estimate for the L^2-norm of $\partial_x \dot{v}_i$, to do this we apply the operator $\partial_t + u\partial_x$ to the momentum equations (1.10):

$$\left(\partial_t + u\partial_x\right)(\rho\dot{v}_i) - (1-\alpha_i)(\partial_t + u\partial_x)\partial_x\left(\rho^\beta \partial_x v_i\right) + \left(\partial_t\partial_x P(\rho) + u\partial_{xx}P(\rho)\right) = 0. \tag{3.77}$$

and we test the previous equation with $\dot{v}_i$ for $i \in \{1,2\}$. Next we observe that:

$$\int_{\mathbb{R}} (\rho\dot{v}_i)_t\, \dot{v}_i = \int_{\mathbb{R}} \rho_t \dot{v}_i^2 + \frac{1}{2}\int_{\mathbb{R}} \rho \frac{d\dot{v}_i^2}{dt} = \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} \rho\dot{v}_i^2 + \frac{1}{2}\int_{\mathbb{R}} \rho_t \dot{v}_i^2.$$

We have in addition:

$$\int_{\mathbb{R}} u\partial_x\left(\rho\dot{v}_i\right)\dot{v}_i = -\int_{\mathbb{R}} \rho\dot{v}_i\partial_x\left(u\dot{v}_i\right) = -\int_{\mathbb{R}} \partial_x u\rho\dot{v}_i^2 + \frac{1}{2}\int_{\mathbb{R}} (\rho u)_x\, \dot{v}_i^2.$$

Summing the above two relations gives:

$$\int_{\mathbb{R}} \left(\partial_t + u\partial_x\right)(\rho\dot{v}_i)\dot{v}_i = \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} \rho\dot{v}_i^2 - \int_{\mathbb{R}} \partial_x u\rho\dot{v}_i^2. \tag{3.78}$$

Next, we focus on the second term of (3.77):

$$-\int_{\mathbb{R}} (\partial_t + u\partial_x)\partial_x\left(\rho^\beta\partial_x v_i\right)\dot{v}_i = \int_{\mathbb{R}} \partial_t\rho^\beta\partial_x v_i\partial_x\dot{v}_i + \int_{\mathbb{R}} \rho^\beta\partial_x\partial_t v_i\partial_x\dot{v}_i + \int_{\mathbb{R}} \partial_x(\rho^\beta\partial_x v_i)\partial_x(u\dot{v}_i) \tag{3.79}$$

Let us deal with the last term appearing in the above inequality :

$$
\begin{aligned}
&\int_{\mathbb{R}} \partial_x(\rho^\beta \partial_x v_i)\partial_x(u\dot{v}_i) \\
&= \int_{\mathbb{R}} \partial_x\rho^\beta \partial_x v_i \partial_x u\dot{v}_i + \int_{\mathbb{R}} u\partial_x\rho^\beta \partial_x v_i \partial_x \dot{v}_i + \int_{\mathbb{R}} \rho^\beta \partial^2_{xx} v_i \partial_x u\dot{v}_i + \int_{\mathbb{R}} \rho^\beta u\partial^2_{xx} v_i \partial_x \dot{v}_i \\
&= \int_{\mathbb{R}} \partial_x\rho^\beta \partial_x v_i \partial_x u\dot{v}_i + \int_{\mathbb{R}} u\partial_x\rho^\beta \partial_x v_i \partial_x \dot{v}_i - \int_{\mathbb{R}} \partial_x v_i \partial_x u\partial_x(\rho^\beta \dot{v}_i) - \int_{\mathbb{R}} \partial_x v_i \partial_{xx} u\rho^\beta \dot{v}_i \\
&+ \int_{\mathbb{R}} \rho^\beta \partial_x(u\partial_x v_i)\partial_x \dot{v}_i - \int_{\mathbb{R}} \partial_x u\partial_x v_i \rho^\beta \partial_x \dot{v}_i
\end{aligned}
\tag{3.80}
$$

Combining the two identities (3.79) and (3.80) we obtain:

$$
\begin{aligned}
&-\int_{\mathbb{R}} (\partial_t + u\partial_x)\partial_x\left(\rho^\alpha \partial_x u\right)\dot{v}_i = \int_{\mathbb{R}} \partial_t\rho^\beta \partial_x v_i \partial_x \dot{v}_i + \int_{\mathbb{R}} u\partial_x\rho^\beta \partial_x v_i \partial_x \dot{v}_i \\
&+ \int_{\mathbb{R}} \rho^\beta \partial_x\partial_t v_i \partial_x \dot{v}_i + \int_{\mathbb{R}} \rho^\beta \partial_x(u\partial_x v_i)\partial_x \dot{v}_i + \int_{\mathbb{R}} \partial_x\rho^\beta \partial_x v_i \partial_x u\dot{v}_i - \int_{\mathbb{R}} \partial_x v_i \partial_x u\partial_x(\rho^\beta \dot{v}_i) \\
&- \int_{\mathbb{R}} \partial_x v_i \partial_{xx} u\rho^\beta \dot{v}_i - \int_{\mathbb{R}} \partial_x u\partial_x v_i \rho^\beta \partial_x \dot{v}_i \\
&= -\beta \int_{\mathbb{R}} \rho^\beta \partial_x u\partial_x v_i \partial_x \dot{v}_i + \int_{\mathbb{R}} \rho^\beta (\partial_x \dot{v}_i)^2 + \int_{\mathbb{R}} \partial_x\rho^\beta \partial_x v_i \partial_x u\dot{v}_i - \int_{\mathbb{R}} \partial_x v_i \partial_x u\partial_x(\rho^\beta \dot{v}_i) \\
&- \int_{\mathbb{R}} \partial_x v_i \partial_{xx} u\rho^\beta \dot{v}_i - \int_{\mathbb{R}} \partial_x u\partial_x v_i \rho^\beta \partial_x \dot{v}_i \\
&= -\beta \int_{\mathbb{R}} \rho^\beta \partial_x u\partial_x v_i \partial_x \dot{v}_i + \int_{\mathbb{R}} \rho^\beta (\partial_x \dot{v}_i)^2 - \int_{\mathbb{R}} \partial_x v_i \partial_{xx} u\rho^\beta \dot{v}_i - 2\int_{\mathbb{R}} \partial_x u\partial_x v_i \rho^\beta \partial_x \dot{v}_i
\end{aligned}
\tag{3.81}
$$

Let us observe that:

$$
\begin{aligned}
&\int_{\mathbb{R}} (\partial_x \rho^\gamma_t + u\partial_{xx}\rho^\gamma)\dot{v}_i = -\int_{\mathbb{R}} \rho^\gamma_t \partial_x \dot{v}_i + \int_{\mathbb{R}} u\partial_{xx}\rho^\gamma \dot{v}_i \\
&= \int_{\mathbb{R}} u\partial_x\rho^\gamma \partial_x \dot{v}_i + \gamma \int_{\mathbb{R}} \rho^\gamma \partial_x u\partial_x \dot{v}_i + \int_{\mathbb{R}} u\partial_{xx}\rho^\gamma \dot{v}_i = -\int_{\mathbb{R}} \partial_x u\partial_x \rho^\gamma \dot{v}_i + \gamma \int_{\mathbb{R}} \rho^\gamma \partial_x u\partial_x \dot{v}_i \\
&= \int_{\mathbb{R}} \partial_x u\rho \dot{v}_i{}^2 - (1-\alpha_i)\int_{\mathbb{R}} \partial_x u\partial_x\left(\rho^\beta \partial_x v_i\right)\dot{v}_i + \gamma \int_{\mathbb{R}} \rho^\gamma \partial_x u\partial_x \dot{v}_i, \\
&= \int_{\mathbb{R}} \partial_x u\rho \dot{v}_i{}^2 + (1-\alpha_i)\int_{\mathbb{R}} \rho^\beta \partial_x v_i \partial_x\left(\dot{v}_i \partial_x u\right) + \gamma \int_{\mathbb{R}} \rho^\gamma \partial_x u\partial_x \dot{v}_i, \\
&= \int_{\mathbb{R}} \partial_x u\rho \dot{v}_i{}^2 + (1-\alpha_i)\int_{\mathbb{R}} \rho^\beta \partial_x u\partial_x v_i \partial_x \dot{v}_i + (1-\alpha_i)\int_{\mathbb{R}} \dot{v}_i \rho^\beta \partial_x v_i \partial^2_{xx} u + \gamma \int_{\mathbb{R}} \rho^\gamma \partial_x u\partial_x \dot{v}_i,
\end{aligned}
\tag{3.82}
$$

where we have used the equation of the momentum to replace $\partial_x\rho^\gamma$ by:

$$
-\partial_x\rho^\gamma = \rho\dot{v}_i - (1-\alpha_i)\partial_x(\rho^\beta \partial_x v_i).
$$

Adding the equalities (3.78), (3.81) and (3.82) we get:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}\rho\dot{v_i}^2-\int_{\mathbb{R}}\partial_x u\rho\dot{v_i}^2-(1-\alpha_i)\beta\int_{\mathbb{R}}\rho^\beta\partial_x u\partial_x v_i\partial_x\dot{v_i}+(1-\alpha_i)\int_{\mathbb{R}}\rho^\beta(\partial_x\dot{v_i})^2\\
&-(1-\alpha_i)\int_{\mathbb{R}}\partial_x v_i\partial_{xx}u\rho^\beta\dot{v_i}-2(1-\alpha_i)\int_{\mathbb{R}}\partial_x u\partial_x v_i\rho^\beta\partial_x\dot{v_i}+\int_{\mathbb{R}}\partial_x u\rho\dot{v_i}^2\\
&+(1-\alpha_i)\int_{\mathbb{R}}\rho^\beta\partial_x u\partial_x v_i\partial_x\dot{v_i}+(1-\alpha_i)\int_{\mathbb{R}}\dot{v_i}\rho^\beta\partial_x v_i\partial_{xx}u+\gamma\int_{\mathbb{R}}\rho^\gamma\partial_x u\partial_x\dot{v_i}=0.
\end{aligned}$$

We have then obtained:

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}\rho\dot{v_i}^2+(1-\alpha_i)\int_{\mathbb{R}}\rho^\beta(\partial_x\dot{v_i})^2=(1-\alpha_i)(1+\beta)\int_{\mathbb{R}}\rho^\beta\partial_x u\partial_x v_i\partial_x\dot{v_i}-\gamma\int_{\mathbb{R}}\rho^\gamma\partial_x u\partial_x\dot{v_i}.$$

Multiplying with $\sigma^2(t)$ and integrating in time on $[0,t]$ with $t>0$ yields:

$$\begin{aligned}
&B_i(\rho,v_i)(t)=\frac{1}{2}\int_{\mathbb{R}}\sigma^2(t)\rho\dot{v_i}^2(t)+(1-\alpha_i)\int_0^t\int_{\mathbb{R}}\sigma^2(t)\rho^\beta(\partial_x\dot{v_i})^2\\
&=\int_0^{\min(1,t)}\int_{\mathbb{R}}\sigma\rho\dot{v_i}^2+(1-\alpha_i)(\beta+1)\int_0^t\int_{\mathbb{R}}\sigma^2\rho^\beta\partial_x u\partial_x v_i\partial_x\dot{v_i}-\gamma\int_0^t\int_{\mathbb{R}}\sigma^2\rho^\gamma\partial_x u\partial_x\dot{v_i}.
\end{aligned}\tag{3.83}$$

From (3.75) we deduce that,

$$\int_0^{\min(1,t)}\int_{\mathbb{R}}\sigma\rho\dot{v_i}^2\leq A_i(\rho,v_i)(1)\leq C,\tag{3.84}$$

for all $t>0$. Next using (3.64) we get:

$$\begin{aligned}
|\gamma\int_0^t\int_{\mathbb{R}}\sigma^2\rho^\gamma\partial_x u\partial_x\dot{v_i}|&\leq\gamma\left\|\rho^{\gamma-\beta}\right\|_{L_t^\infty L^\infty}\left(\int_0^t\int_{\mathbb{R}}\sigma^2\rho^\beta(\partial_x u)^2\right)^{\frac{1}{2}}\left(\int_0^t\int_{\mathbb{R}}\sigma^2\rho^\beta(\partial_x\dot{v_i})^2\right)^{\frac{1}{2}}\\
&\leq C(t)(1+\frac{1}{1-\alpha_i})+\frac{1}{8}B_i(\rho,v_i)(t),
\end{aligned}\tag{3.85}$$

with C a continuous increasing function. Finally, using again (3.75), (1.14), (1.17), (3.64) and (3.63), we get:

$$\begin{aligned}
&(\beta+1)|\int_0^t\int_{\mathbb{R}}\sigma^2\rho^\beta\partial_x u\partial_x v_i\partial_x\dot{v_i}|\leq\frac{1}{4}\int_0^t\int_{\mathbb{R}}\sigma^2\rho^\beta(\partial_x\dot{v_i})^2+(\beta+1)^2\int_0^t\int_{\mathbb{R}}\sigma^2\rho^\beta(\partial_x u)^2(\partial_x v_i)^2\\
&\leq\frac{1}{4(1-\alpha_i)}B_i(\rho,v_i)(t)+(\beta+1)^2\left\|\frac{1}{\rho}\right\|^{2\beta}_{L_t^\infty(L^\infty)}\int_0^t\int_{\mathbb{R}}\sigma^2\rho^{3\beta}(\partial_x u)^2(\partial_x v_i)^2\\
&\leq\frac{1}{4(1-\alpha_i)}B_i(\rho,v_i)(t)+C(t)\int_0^t\sigma^2\left\|\rho^\beta\partial_x u\right\|^2_{L^\infty}\int_{\mathbb{R}}\rho^\beta(\partial_x v_i)^2\\
&\leq\frac{1}{4(1-\alpha_i)}B_i(\rho,v_i)(t)+\frac{C(t)}{1-\alpha_i}\sup_{\tau\in[0,t]}\sigma^2(\tau)\left\|(\rho^\beta\partial_x u)(\tau)\right\|^2_{L^\infty}.
\end{aligned}\tag{3.86}$$

Let us observe that for all $t>0$ we have using Gagliardo-Nirenberg inequality with $C>0$ large enough, (4.98), (3.63) and the fact that $u:=\frac{1}{\sqrt{1-4c}}((1-\alpha_2)v_2-(1-\alpha_1)v_1)$

$$\begin{aligned}
&\sigma^2(t)\left\|\rho^\beta\partial_x u(t)\right\|_{L^\infty}^2=\frac{1}{1-4c}\sigma^2(t)\left\|(1-\alpha_2)\rho^\beta\partial_x v_2-(\rho^\gamma-1)-(1-\alpha_1)\rho^\beta\partial_x v_1+(\rho^\gamma-1)(t)\right\|_{L^\infty}^2\\
&\leq\frac{C\sigma^2(t)}{1-4c}\Big(\left\|\Big((1-\alpha_2)\rho^\beta\partial_x v_2-(\rho^\gamma-1)\Big)(t)\right\|_{L^2}\left\|\partial_x\Big((1-\alpha_2)\rho^\beta\partial_x v_2-(\rho^\gamma-1)\Big)(t)\right\|_{L^2}\\
&+\left\|\Big((1-\alpha_1)\rho^\beta\partial_x v_1-(\rho^\gamma-1)\Big)(t)\right\|_{L^2}\left\|\partial_x\Big((1-\alpha_1)\rho^\beta\partial_x v_1-(\rho^\gamma-1)\Big)(t)\right\|_{L^2}\Big)\\
&\leq\frac{C\sigma^2(t)}{1-4c}\left(\left\|(1-\alpha_2)\rho^\beta\partial_x v_2\right\|_{L^2}+C(t)\|\rho-1\|_{L^2}\right)\|\rho\dot{v}_2\|_{L^2}\\
&\qquad+\frac{C\sigma^2(t)}{1-4c}\left(\left\|(1-\alpha_1)\rho^\beta\partial_x v_1\right\|_{L^2}+C(t)\|\rho-1\|_{L^2}\right)\|\rho\dot{v}_1\|_{L^2}\\
&\leq C(t)\sigma^{\frac12}(t)\left(\sigma^{\frac12}\left\|(1-\alpha_2)\rho^\beta\partial_x v_2\right\|_{L^2}+\sigma^{\frac12}(t)C(t)\right)\sigma(t)\left\|\rho^{\frac12}\dot{v}_2\right\|_{L^2} \qquad (3.87)\\
&\qquad+C(t)\sigma^{\frac12}(t)\left(\sigma^{\frac12}(t)\left\|(1-\alpha_1)\rho^\beta\partial_x v_1\right\|_{L^2}+\sigma^{\frac12}(t)C(t)\right)\sigma(t)\left\|\rho^{\frac12}\dot{v}_1\right\|_{L^2}\\
&\leq C_1(t)\left(A_2^{\frac12}(\rho,v_2)(t)+C_1(t)\right)B_2^{\frac12}(\rho,v_2)(t)+C_1(t)\left(A_1^{\frac12}(\rho,v_1)(t)+C_1(t)\right)B_1^{\frac12}(\rho,v_1)(t), \qquad (3.88)
\end{aligned}$$

with C,C_1 continuous functions on $\mathbb{R}^+$. Thus, we get from (3.86), (3.88), (3.75) and Young inequality:

$$\begin{aligned}
&(1-\alpha_i)(\beta+1)\left|\int_0^t\int_{\mathbb{R}}\sigma\rho^\beta\partial_x u\partial v_i\partial_x\dot{v}_i\right|\\
&\leq\frac14 B_i(\rho,v_i)(t)+C(t)\left(A_2^{\frac12}(\rho,v_2)(t)+C(t)\right)B_2^{\frac12}(\rho,v_2)(t)\\
&\qquad+C(t)\left(A_1^{\frac12}(\rho,v_1)(t)+C(t)\right)B_1^{\frac12}(\rho,v_1)(t) \qquad (3.89)\\
&\leq C_1(t)+\frac12 B_i(\rho,v_i)(t)+\frac14 B_j(\rho,v_j)(t).
\end{aligned}$$

with $j\neq i$ and $j\in\{1,2\}$ and with C_1 a continuous function on $\mathbb{R}^+$. Gathering (3.84), (3.85) and (3.89) yields the fact that B_i is also bounded:

$$B_i(\rho,v_i)(t)\leq C(t), \qquad (3.90)$$

with C a continuous increasing function. The control over $\left\|\frac{1}{\rho}\right\|_{L^\infty}$, $A_i(\rho,u)$ and $B_i(\rho,u)$ gives us, via the estimate (3.88) the following

$$\sigma(t)\|\partial_x u(t)\|_{L^\infty}\leq C(t), \qquad (3.91)$$

for any $t\geq 0$. It concludes the proof of the Theorem 2.1.

4 Proof of the Theorem 2.2

Since we deal with initial data which are less regular as in the Theorem 2.1, we can not directly used the Theorem 3.3 for getting strong solution in finite time. In order to

overcome this difficulty we start by regularizing the initial data as follows:

$$\begin{cases} \rho_0^n = j_n * \rho_0, \\ v_{01}^n = j_n * v_{10}, \\ u_0^n = v_{10}^n - \alpha_1 \partial_x \varphi\,(\rho_0^n) \\ v_{02}^n = u_0^n + \alpha_2 \partial_x \varphi(\rho_0^n). \end{cases} \tag{4.92}$$

with j_n a regularizing kernel, $j_n(y) = nj(ny)$ with $0 \leq j \leq 1$, $\int_{\mathbb{R}} j(y)dy = 1$, $j \in C^\infty(\mathbb{R})$ and supp$j \subset [-2, 2]$. Here $v_{10} = u_0 + \alpha_1 \partial_x \varphi(\rho_0)$ and $v_{20} = u_0 + \alpha_2 \partial_x \varphi(\rho_0)$. In particular since $u_0, \partial_x \rho_0, \rho$ and $\frac{1}{\rho_0}$ are in L^∞, we deduce easily that there exists $C > 0$ independent on n such that for any $n \in \mathbb{N}$ we have:

$$\|v_{01}^n\|_{L^\infty} + \|v_{02}^n\|_{L^\infty} \leq C. \tag{4.93}$$

We deduce that $(\rho_0^n - 1, v_{0,1}^n)$ belong to all Sobolev spaces $H^s(\mathbb{R})$ with $s \geq 0$ and that:

$$0 < \left\| \frac{1}{\rho_0} \right\|_{L^\infty} \leq \rho_0^n \leq \|\rho_0\|_{L^\infty} < +\infty. \tag{4.94}$$

By composition theorem for Sobolev spaces we can prove that $\varphi(\rho_0^n) - \varphi(1)$ belongs to $H^k(\mathbb{R})$ for any $k \geq 0$ and consequently we obtain that $u_0^n \in H^k(\mathbb{R})$ for $k \geq 3$. Finally we have for $x > y$ and using (2.31):

$$\frac{v_{01}^n(x) - v_{01}^n(y)}{x - y} = \int_{\mathbb{R}} (\frac{v_{01}(x - z) - v_{01}(y - z)}{x - y}) j_n(z) dz \leq C_0$$

and in particular we deduce that for any $x \in \mathbb{R}$, we have:

$$\partial_x v_{01}^n(x) \leq C_0. \tag{4.95}$$

where C_0 is the constant appearing in (2.31) . From (4.94) and (4.95) we deduce also that:

$$w_{01}^n(x) \leq C_1 \tag{4.96}$$

with$C_1 > 0$ large enough and from Proposition 3.1 we have set $w_{01}^n = f_1(\rho_0^n)\partial_x v_{01}^n + F_{1,1}(\rho^n)$.

Next, Theorem 3.3 gives the existence of strong solutions (ρ_n, u_n) of the system ((1.1) on a finite time interval $(0, T_n)$ with $T_n > 0$. Our main goal now is to prove that for any n, we have $T_n = +\infty$. To do this, we are going again to use the blow-up criterion of Theorem 3.3. More precisely we wish to show that it exists a continuous fonction on $\mathbb{R}^+$ such that for any $n \in \mathbb{N}$ we have for any $t \in (0, \min(T_n, T^*))$ with $T^* > 0$ independent on n and depending in a suitable way of the initial data (ρ_0, u_0) and of α_2:

$$\|\frac{1}{\rho_n}(t, \cdot)\|_{L^\infty} \leq C(t), \tag{4.97}$$

with C a continuous function on $\mathbb{R}^+$. We now simply recall that the strong solution (ρ_n, u_n) satisfy the energy estimates (1.14) and (1.17) for any $t \in (0, T_n)$. Using (1.14), (1.17) and Sobolev embedding we get that for $C > 0$ large enough we have for any $t \in (0, T_n)$ (see [32, 7, 13, 14] for details):

$$\|\rho_n(t, \cdot)\|_{L^\infty} \leq C. \tag{4.98}$$

4.1 Estimate of the L^∞ norm of $(\frac{1}{\rho_n})_{n\in\mathbb{N}}$

We are going now to proceed as in the section 3 by considering the effective pressure $w_{1,n}$ which satisfies the parabolic equation (3.37). Since $\lim_{x\to\pm\infty} w_{1,n}(t,x) = F_{1,1}(1)$, we deduce that for all $t \in [0, T_n)$ we have:

$$\sup_{x\in\mathbb{R}} w_{1,n}(t,x) \geq F_{1,1}(1). \tag{4.99}$$

We set:

$$D_n := \left\{ t \in (0, T_n) : \sup_{x\in\mathbb{R}} w_{1,n}(t,x) > F_{1,1}(1) \right\}$$

which is open in $[0, T_n)$ and we have:

$$D_n = I_0^n \cup \bigcup_{j\in\mathbb{N}^\star} I_j^n,$$

where $\left(I_j^n\right)_{j\geq 1}$ with $I_j^n = \left(a_j^n, b_j^n\right)$ are open disjoint intervals and $I_0 = \emptyset$ if $\sup_{x\in\mathbb{R}} w_{1,n}(0,x) = F_{1,1}(1)$ and $I_0^n = [0, b_0^n)$ for some $b_0^n \in (0, T_n)$ if $\sup_{x\in\mathbb{R}} w_{1,n}(0,x) > F_{1,1}(1)$. From the definition of I_j we have that $\sup_{x\in\mathbb{R}} w_{1,n}(a_j, x) = F_{1,1}(1)$ and for all $t \in I_j^n$ since $w_{1,n}(t,\cdot)$ is continuous, it reaches its maximum on $\mathbb{R}$. It implies that for any $j \in \mathbb{N}$ and any $t \in I_j$ there exists a point $x_t^n \in \mathbb{R}$ such that:

$$\sup_{x\in\mathbb{R}} w_{1,n}(t,x) \overset{def.}{=} w_M^n(t) = w_{1,n}(t, x_t^n).$$

As previously we are going simply evaluate $w_{1,n}$ on an interval $I_{j_0}^n$, it is important to note for the sequel that we have from (4.96):

$$w_{01}^n(x) \leq C_1 \tag{4.100}$$

Proceeding as in the previous section, we recall that w_M^n is differentiable almost everywhere on D_n and for any $t \in I_{j_0}^n$, we deduce from (3.37) that:

$$\partial_t w_M^n(t) + w_M^n(t)(g_{21}')^n(t) + (g_{31}')^n(t) + (g_{41})^n(t)(w_M^n)(t)^2 \leq 0, \tag{4.101}$$

with:

$$\begin{aligned}(g_{21}')^n(t) = &-(\alpha_1 - 1)\alpha_1 \rho_n(t, x_t^n)^{\beta-3} \frac{(\beta-1)\alpha_1 - \beta}{2\alpha_1 - 1} \frac{\beta+1}{2\alpha_1 - 1} (\partial_x \rho_n)(t, x_t^n)^2 \\ &+ \rho_n(t, x_t^n)^{\gamma-\beta} \frac{\gamma}{2\alpha_1 - 1} \frac{-3\alpha_1\beta - 3\alpha_1 + 2\alpha_1\gamma - \gamma + 2\beta + 2}{-\alpha_1(\beta+1) + \gamma(2\alpha_1 - 1)}.\end{aligned} \tag{4.102}$$

$$\begin{aligned}(g_{31}')^n(t) = &\rho_n(t, x_t^n)^{\frac{-\alpha_1(\beta+1)}{2\alpha_1-1} + 2\gamma - \beta} \frac{\gamma^2}{(-\alpha_1(\beta+1) + \gamma(2\alpha_1 - 1))^2} (\beta + 1 - \gamma) \\ &+ (\partial_x \rho_n)^2(t, x_t^n) \rho_n(t, x_t^n)^{\frac{-\alpha_1(\beta+1)}{2\alpha_1-1} + \gamma + \beta - 3} \frac{\gamma\alpha_1(\gamma - \beta - 1)(\gamma - \beta)}{(-\alpha_1(\beta+1) + \gamma(2\alpha_1 - 1))}\end{aligned} \tag{4.103}$$

and:

$$g_{41}^n(t) = (\alpha_1 - 1)\frac{\beta+1}{2\alpha_1 - 1}\rho_n(t, x_t^n)^{\frac{-(\beta-1)\alpha_1+\beta}{2\alpha_1-1}}. \tag{4.104}$$

First case, $\gamma \in [\beta, \beta+1[$

First we are going to study the case $\gamma \in [\beta, \beta+1[$, we know from the proposition 3.2 that $(g'_{31})^n(t) \geq 0$ then from (4.101) we deduce that w_M^n satisfies the following equation on I_j^n with $j \geq 0$:

$$\partial_t w_M^n(t) + w_M^n(t)(g'_{21})^n(t) + (g_{41})^n(t)(w_M^n)(t)^2 \leq 0, \tag{4.105}$$

It implies that w_M^n is a subsolution of a Bernoulli equation. Now we can consider the behavior of $w_M^n(t)$ on $I_j^n = (a_j^n, b_j^n)$ when $j \geq 1$ and where we know that $w_M^n(a_j^n) = F_{1,1}(1) \leq 0$ from Proposition 3.2. From (4.105) using the fact that w_n^M is absolutely continuous and that $(g'_{21})^n$, $(g_{41})^n$ are in $L^1(0, T_n)$ we have for any $t \in I_j^n$ with $j \geq 1$:

$$\begin{aligned} &\partial_t(w_M^n(t)e^{\int_{a_j^n}^t \left((g'_{21})^n(s)+(g_{41})^n(s)w_M^n(s)\right)ds} \leq 0, \\ &w_M^n(t)e^{\int_{a_j^n}^t \left((g'_{21})^n(s)+(g_{41})^n(s)w_M^n(s)\right)ds} \leq w_M^n(a_j^n) \leq 0. \end{aligned} \tag{4.106}$$

It implies in particular that for any t in I_j^n we have for $j \geq 1$:

$$w_M^n(t) \leq 0. \tag{4.107}$$

We are in a similar situation if we consider I_0^n and that we assume $w_M^n(0) \leq 0$. We are then reduced to study the behavior of w_M^n on I_0^n when $w_M^n(0) > 0$. Now as previously since w_n^M is continuous we deduce that:

$$I_0^n \cap \{t \in \mathbb{R}^+,\ w_n^M(t) > \frac{1}{2}\} = \cup_j K_j^n,$$

with $K_0^n = [0, c_0^n[$ and $K_j^n =]c_j^n, d_j^n[$ for $j \geq 1$ with $w_M^n(c_j^n) = \frac{1}{2}$ for $j \geq 1$. We are then reduced to study the behavior of w_M^n on each K_j^n with $j \geq 0$. Now we can observe that for any $t \in (0, T_n)$ we have:

$$\begin{aligned} &-(\alpha_1 - 1)\alpha_1\rho_n(t, x_t^n)^{\beta-3}\frac{(\beta-1)\alpha_1 - \beta}{2\alpha_1 - 1}\frac{\beta+1}{2\alpha_1 - 1}(\partial_x\rho_n)(t, x_t^n)^2 \leq 0 \\ &\rho_n(t, x_t^n)^{\gamma-\beta}\frac{\gamma}{2\alpha_1 - 1}\frac{-3\alpha_1\beta - 3\alpha_1 + 2\alpha_1\gamma - \gamma + 2\beta + 2}{-\alpha_1(\beta+1) + \gamma(2\alpha_1 - 1)} \geq 0. \end{aligned} \tag{4.108}$$

The second inequality is true if α_1 is sufficiently close from 1 (it depends in particular of γ and β), in other words if $c > 0$ the capillary coefficient is sufficiently small. It is exactly the case that we consider in the Theorem 2.2. From (4.105) we deduce then that for any $t \in K_j^n$:

$$\partial_t w_M^n(t) \leq w_M^n(t)(g'_{211})^n(t) - (g_{41})^n(t)(w_M^n)(t)^2. \tag{4.109}$$

with:

$$(g'_{211})^n(t) = (\alpha_1 - 1)\alpha_1\rho_n(t, x_t^n)^{\beta-3}\frac{(\beta-1)\alpha_1 - \beta}{2\alpha_1 - 1}\frac{\beta+1}{2\alpha_1 - 1}(\partial_x\rho_n)(t, x_t^n)^2 \tag{4.110}$$

From (4.104) we have always $g_{41}^n(t) \leq 0$ and for any $t \in K_j^n$ we get using (4.98):

$$|g_{41}^n(t)| \leq (1-\alpha_1)\frac{\beta+1}{2\alpha_1-1}\|\rho_n\|_{L^\infty((0,T_n),L^\infty)}^{\frac{-(\beta-1)\alpha_1+\beta}{2\alpha_1-1}} \leq (1-\alpha_1)\frac{\beta+1}{2\alpha_1-1}C^{\frac{-(\beta-1)\alpha_1+\beta}{2\alpha_1-1}} \tag{4.111}$$

From (4.102), we have for any $t \in K_j^n$ and using (4.98), there exists $C > 0$ large enough and independent on n such that:

$$|(g'_{211})^n(t)| \leq C(1-\alpha_1)\|\rho_n(t,\cdot)^{1-\beta}\|_{L^\infty}\|\partial_x\varphi(\rho_n)(t,\cdot)\|_{L^\infty}^2. \tag{4.112}$$

We recall now that $\partial_x\varphi(\rho_n) = \frac{1}{2\alpha_1-1}(v_{1,n}-v_{2,n})$, it implies then that there exists $C > 0$ large enough such that for $t \in K_j^n$:

$$|(g'_{211})^n(t)| \leq C(1-\alpha_1)\|(v_{1,n}-v_{2,n})(t,\cdot)\|_{L^\infty}^2\|\frac{1}{\rho_n(t,\cdot)}\|^{\beta-1}. \tag{4.113}$$

We must now estimate the L^∞ norm of $v_{1,n}$ and $v_{2,n}$, we recall that $v_{1,n}$ and $v_{2,n}$ satisfy the equations (1.10) that we can rewrite as follows using the fact that $\partial_x\rho_n^\gamma = \frac{\gamma}{2\alpha_1-1}\rho_n^{\gamma+1-\beta}(v_{1,n}-v_{2,n})$:

$$\begin{cases} \rho_n\partial_t v_{1,n} + \rho_n u_n\partial_x v_{1,n} - (1-\alpha_1)\partial_x(\mu(\rho_n)\partial_x v_{1,n}) + \dfrac{\gamma}{2\alpha_1-1}\rho_n^{\gamma+1-\beta}(v_{1,n}-v_{2,n}) = 0 \\ \rho_n\partial_t v_{2,n} + \rho_n u_n\partial_x v_{2,n} - (1-\alpha_2)\partial_x(\mu(\rho_n)\partial_x v_{2,n}) + \dfrac{\gamma}{2\alpha_1-1}\rho_n^{\gamma+1-\beta}(v_{1,n}-v_{2,n}) = 0 \end{cases} \tag{4.114}$$

Applying again a maximum principle, we get using (4.93) for any $t \in (0,T_n)$ and for $C > 0$ large enough:

$$\|v_{1,n}(t,\cdot)\|_{L^\infty} + \|v_{2,n}(t,\cdot)\|_{L^\infty} \leq C + C\int_0^t \|\rho_n(s,\cdot)\|_{L^\infty}^{\gamma+1-\beta}(\|v_{1,n}(s,\cdot)\|_{L^\infty} + \|v_{2,n}(s,\cdot)\|_{L^\infty})ds. \tag{4.115}$$

From (4.98) and using Gronwall inequality, there exists $C > 0$ large enough such that for any $t \in (0,T_n)$ we have:

$$\|v_{1,n}(t,\cdot)\|_{L^\infty} + \|v_{2,n}(t,\cdot)\|_{L^\infty} \leq Ce^{Ct}. \tag{4.116}$$

Combining now (4.116) , (4.98)and (4.113), it yields that for any $t \in K_j^n$ and $C > 0$ large enough:

$$|(g'_{211})^n(t)| \leq C(1-\alpha_1)e^{Ct}\|\frac{1}{\rho_n(t,\cdot)}\|_{L^\infty}^{\beta-1}. \tag{4.117}$$

Combining (4.117), (4.111) and (4.119), we have for $C > 0$ large enough and $t \in K_j^n$ with $j \geq 0$:

$$\partial_t w_M^n(t) \leq C(1-\alpha_1)e^{Ct}\|\frac{1}{\rho_n(t,\cdot)}\|_{L^\infty}^{\beta-1}w_M^n(t) + (1-\alpha_1)C(w_M^n)(t)^2. \tag{4.118}$$

In particular on K_j^n since w_M^n is strictly positive and $\frac{1}{w_M^n}$ is Lipschitz on K_j^n then absolutely continuous. It will be possible in particular to apply Gronwall Lemma. More precisely dividing (4.119) by $(w_M^n)^2$ we have:

$$\partial_t(-\frac{1}{w_M^n(t)}) \leq C(1-\alpha_1)e^{Ct}\|\frac{1}{\rho_n(t,\cdot)}\|_{L^\infty}^{\beta-1}\frac{1}{w_M^n(t)} + (1-\alpha_1)C. \tag{4.119}$$

It implies in particular that for $t \in K_j^n$ we have:

$$\partial_t(-\frac{1}{w_M^n(t)}e^{\int_{c_j^n}^t C(1-\alpha_1)e^{Cs}\|\frac{1}{\rho_n(s,\cdot)}\|_{L^\infty}^{\beta-1}ds}) \leq (1-\alpha_1)Ce^{\int_{c_j^n}^t C(1-\alpha_1)e^{Cs}\|\frac{1}{\rho_n(s,\cdot)}\|_{L^\infty}^{\beta-1}ds}. \quad (4.120)$$

We can now integrate since we work with absolutely continuous function, and we have:

$$-\frac{1}{w_M^n(t)}e^{\int_{c_j^n}^t C(1-\alpha_1)e^{Cs}\|\frac{1}{\rho_n(s,\cdot)}\|_{L^\infty}^{\beta-1}ds}+\frac{1}{w_M^n(c_j^n)} \leq (1-\alpha_1)C\int_{c_j^n}^t e^{\int_{c_j^n}^u C(1-\alpha_1)e^{Cs}\|\frac{1}{\rho_n(s,\cdot)}\|_{L^\infty}^{\beta-1}ds}du. \quad (4.121)$$

It gives in particular using the fact that $w_M^n(c_j^n) = \frac{1}{2}$ for $j \geq 1$ that for any $t \in K_j^n$ with $j \geq 1$:

$$\frac{1}{w_M^n(t)} \geq e^{-\int_{c_j^n}^t C(1-\alpha_1)e^{Cs}\|\frac{1}{\rho_n(s,\cdot)}\|_{L^\infty}^{\beta-1}ds}(2-(1-\alpha_1)C\int_{c_j^n}^t e^{\int_{c_j^n}^u C(1-\alpha_1)e^{Cs}\|\frac{1}{\rho_n(s,\cdot)}\|_{L^\infty}^{\beta-1}ds}du). \quad (4.122)$$

We deduce now that for any $t \in K_j^n$ with $j \geq 1$ we have:

$$w_M^n(t) \leq e^{\int_{c_j^n}^t C(1-\alpha_1)e^{Cs}\|\frac{1}{\rho_n(s,\cdot)}\|_{L^\infty}^{\beta-1}ds}\frac{1}{2-(1-\alpha_1)C\int_{c_j^n}^t e^{\int_{c_j^n}^u C(1-\alpha_1)e^{Cs}\|\frac{1}{\rho_n(s,\cdot)}\|_{L^\infty}^{\beta-1}ds}du}. \quad (4.123)$$

provided that we have:

$$2-(1-\alpha_1)C\int_{c_j^n}^t e^{\int_{c_j^n}^u C(1-\alpha_1)e^{Cs}\|\frac{1}{\rho_n(s,\cdot)}\|_{L^\infty}^{\beta-1}ds}du > 0. \quad (4.124)$$

We have similarly for $t \in K_0^n$:

$$w_M^n(t) \leq e^{\int_{c_j^n}^t C(1-\alpha_1)e^{Cs}\|\frac{1}{\rho_n(s,\cdot)}\|_{L^\infty}^{\beta-1}ds}\frac{1}{\frac{1}{w_n^M(0)}-(1-\alpha_1)C\int_{c_j^n}^t e^{\int_{c_j^n}^u C(1-\alpha_1)e^{Cs}\|\frac{1}{\rho_n(s,\cdot)}\|_{L^\infty}^{\beta-1}ds}du}. \quad (4.125)$$

provided that:

$$\frac{1}{w_n^M(0)}-(1-\alpha_1)C\int_{c_j^n}^t e^{\int_{c_j^n}^u C(1-\alpha_1)e^{Cs}\|\frac{1}{\rho_n(s,\cdot)}\|_{L^\infty}^{\beta-1}ds}du > 0. \quad (4.126)$$

Now we are interested in estimating the L^∞ norm of $\frac{1}{\rho_n(t,\cdot)}$ for $t \in (0, T_n)$. Proceeding as in the section 3.1, setting $z_n(t) = \sup_{x\in\mathbb{R}}\frac{1}{\rho_n(t,x)}$ with:

$$\{t \in [0, T_n), \sup_{x\in\mathbb{R}}\frac{1}{\rho_n(t,x)} > 1\} = Q_0^n \cup \cup_{j\geq 1}Q_j^n,$$

with Q_j^n open intervals. From (3.60), (3.35) we have for any $t \in Q_j^n$ with $j \geq 0$:

$$\begin{aligned}\partial_t z_n(t) &\leq \frac{1}{\rho_n f_1(\rho_n)}(w_n - F_{11}(\rho_n))(t, x_t^n)\\ &\leq \rho_n^{\frac{(1-\alpha_1)(\beta+1)}{2\alpha_1-1}}(t, x_t^n)w_n(t, x_t^n) + \frac{\gamma}{\alpha_1(\beta+1)-\gamma(2\alpha_1-1)}\rho_n^{\gamma-\beta-1}(t, x_t^n)\end{aligned} \quad (4.127)$$

Now using (4.98), (4.127) and the fact that $\gamma \in [\beta, \beta+1[$ we deduce that for $C > 0$ large enough we have:

$$\partial_t z_n(t) \leq C w_n(t, x_t^n) + C(1 + z_n(t)). \tag{4.128}$$

Now we are going to fix $T > 0$ and we are going to prove that for any $n \in \mathbb{N}$ we have $T_n \geq T$ provided that $c > 0$ the capillary coefficient is sufficiently small. To do this we set:

$$T_{1,n} = \sup\{t \in [0, \min(T, T_n)[, \forall s \in (0,t) \ \ \|\frac{1}{\rho_n(s,\cdot)}\|_{L^\infty} < M(\|\frac{1}{\rho_{0,n}}\|_{L^\infty} + CT)e^{CT}\}$$

with C defined in (4.128) and $M > 2$ sufficiently large that we will fix later. We wish now to prove that for any $n \in \mathbb{N}$, we have

$$T_{1,n} = \min(T, T_n) \tag{4.129}$$

provided that $c > 0$ is sufficiently small. If we prove this claim, we deduce then that for any $n \in \mathbb{N}$, we have

$$\|\frac{1}{\rho_n(s,\cdot)}\|_{L^\infty([0,T_{1,n}],L^\infty)} \leq M(\|\frac{1}{\rho_{0,n}}\|_{L^\infty} + CT)e^{CT} < +\infty$$

Using the blow-up criterion of Theorem 3.3, we deduce that necessarily we have for any $n \in \mathbb{N}$, $T_n > T$ and in addition for any $n \in \mathbb{N}$ we have:

$$\|\frac{1}{\rho_n(s,\cdot)}\|_{L^\infty([0,T],L^\infty)} \leq M(\|\frac{1}{\rho_{0,n}}\|_{L^\infty} + CT)e^{CT} \tag{4.130}$$

Let us prove now that (4.129) is satisfied provided that $c > 0$ is small enough. First by contin uity of the fonction $z_n(t) = \|\frac{1}{\rho_n(t,\cdot)}\|_{L^\infty}$ we deduce that $T_{1,n} > 0$ and that $E_n = \{t \in [0, \min(T, T_n)], \forall s \in (0,t) \ \ \|\frac{1}{\rho_n(s,\cdot)}\|_{L^\infty} < 2(\|\frac{1}{\rho_{0,n}}\|_{L^\infty} + CT)e^{CT}\} = [0, T_{1,n}[$. We are now going to assume by absurd that $T_{1,n} < \min(T, T_n)$.
From (4.123) we deduce that for any $t \in K_j^n \cap [0, T_{1,n}[$ with $j \geq 1$ we have:

$$\begin{aligned} w_M^n(t) \leq\ & e^{\int_{c_j^n}^t C(1-\alpha_1)e^{Cs}(M(\|\frac{1}{\rho_{0,n}}\|_{L^\infty}+CT)e^{CT})^{\beta-1}ds} \\ & \times \frac{1}{2-(1-\alpha_1)C\int_{c_j^n}^t e^{\int_{c_j^n}^u C(1-\alpha_1)e^{Cs}(M(\|\frac{1}{\rho_{0,n}}\|_{L^\infty}+CT)e^{CT})^{\beta-1}ds}du}. \end{aligned} \tag{4.131}$$

It is now clear that choosing $c > 0$ sufficiently small, then $1-\alpha_1$ is sufficiently small such that (4.124) is satisfied for any $t \in K_j^n \cap [0, T_{1,n}[$ with $j \geq 1$ and we have in addition:

$$w_M^n(t) \leq 1. \tag{4.132}$$

We have a similar result for $t \in K_0^n \cap [0, T_{1,n}[$ taking $c > 0$ sufficiently small which can written as follows:

$$w_M^n(t) \leq 2w_M^n(0). \tag{4.133}$$

From (4.132),(4.133) and from the definition of K_j^n with $j \geq 0$, we deduce that for any $t \in [0, T_{1,n}[$ we have:

$$w_M^n(t) \leq \max(1, 2w_M^n(0)). \tag{4.134}$$

From (4.128), (4.134) and for any $t \in [0, T_{1,n}[\cap Q_j^n$ with $j \geq 0$ we have:

$$\partial_t z_n(t) \leq C \max(1, 2w_M^n(0)) + C(1 + z_n(t)). \tag{4.135}$$

In particular using Gronwall Lemma, it implies that for any $t \in [0, T_{1,n}[\cap Q_j^n$ with $j \geq 1$ and using the fact that if $Q_j^n =]e_j^n, f_j^n[$ we have $z_n(e_j^n) = 1$:

$$z_n(t) \leq (1 + C(1 + \max(1, 2w_M^n(0))T)e^{CT} \tag{4.136}$$

For $t \in [0, T_{1,n}[\cap Q_j^n$ we have:

$$z_n(t) \leq (\|\frac{1}{\rho_{0,n}}\|_{L^\infty} + C(1 + \max(1, 2w_M^n(0))T)e^{CT} \tag{4.137}$$

And finally when $t \in [0, T_{1,n}[\backslash(Q_0^n \cup \cup_{j\geq 1} Q_j^n)$, we know that

$$z_n(t) = 1 \tag{4.138}$$

From (4.139), (4.137) and (4.138) we deduce that for any $t \in [0, T_{1,n}[$ we have:

$$z_n(t) \leq (\|\frac{1}{\rho_{0,n}}\|_{L^\infty} + C(1 + \max(1, 2w_M^n(0))T)e^{CT} \tag{4.139}$$

Now taking $M = 2(1 + \max(1, 2w_M^n(0)))$ we have proved that for any $t \in [0, T_{1,n}[$ we have:

$$\|\frac{1}{\rho_n(t,\cdot)}\|_{L^\infty} \leq \frac{M}{2}(\|\frac{1}{\rho_{0,n}}\|_{L^\infty} + CT)e^{CT}\} \tag{4.140}$$

It contradicts the definition of $T_{1,n}$ and it implies that $T_{1,n} < \min(T, T_n)$ is absurd. In conclusion we have proved that for any $n \in \mathbb{N}$ we have $T_n > T$ and from (4.130):

$$\|\frac{1}{\rho_n(s,\cdot)}\|_{L^\infty([0,T],L^\infty)} \leq M(\|\frac{1}{\rho_0}\|_{L^\infty} + CT)e^{CT} \tag{4.141}$$

Second case, $\gamma \geq \beta + 1$

The only point we change when we consider the case $\gamma \geq \beta + 1$ is that the term $(g'_{31})^n(t)$ is not necessary positive when $\gamma \geq \beta + 1$. In particular we recall that:

$$\begin{aligned}(g'_{31})^n(t) = \rho_n(t, x_t^n)^{\frac{-\alpha_1(\beta+1)}{2\alpha_1-1}+2\gamma-\beta} \frac{\gamma^2}{(-\alpha_1(\beta+1) + \gamma(2\alpha_1 - 1))^2}(\beta + 1 - \gamma) \\ + (\partial_x \rho_n)^2(t, x_t^n)\rho_n(t, x_t^n)^{\frac{-\alpha_1(\beta+1)}{2\alpha_1-1}+\gamma+\beta-3} \frac{\gamma\alpha_1(\gamma-\beta-1)(\gamma-\beta)}{(-\alpha_1(\beta+1) + \gamma(2\alpha_1 - 1))}\end{aligned} \tag{4.142}$$

We can observe that the term $(\partial_x \rho_n)^2(t, x_t^n)\rho_n(t, x_t^n)^{\frac{-\alpha_1(\beta+1)}{2\alpha_1-1}+\gamma+\beta-3} \frac{\gamma\alpha_1(\gamma-\beta-1)(\gamma-\beta)}{(-\alpha_1(\beta+1)+\gamma(2\alpha_1-1))}$ is always positive provided that $c > 0$ is sufficiently small. Indeed when $\gamma = \beta + 1$ this term is null and when $\gamma > \beta + 1$ it requires that $\alpha_1 \geq \frac{\gamma}{\gamma+(\gamma-\beta-1)}$.In opposite the term $\rho_n(t, x_t^n)^{\frac{-\alpha_1(\beta+1)}{2\alpha_1-1}+2\gamma-\beta} \frac{\gamma^2}{(-\alpha_1(\beta+1)+\gamma(2\alpha_1-1))^2}(\beta + 1 - \gamma)$ is always negative, we deduce then from (4.101) that for any $t \in D_n$ we have:

$$\partial_t w_M^n(t) + w_M^n(t)(g'_{21})^n(t) + (g_{41})^n(t)(w_M^n)(t)^2 + g_5^n(t) \leq 0, \tag{4.143}$$

with:

$$g_5^n(t) = \rho_n(t, x_t^n)^{\frac{-\alpha_1(\beta+1)}{2\alpha_1-1}+2\gamma-\beta} \frac{\gamma^2}{(-\alpha_1(\beta+1)+\gamma(2\alpha_1-1))^2}(\beta+1-\gamma). \tag{4.144}$$

It is now easy to observe that for $c > 0$ the capillary coefficient we have $\frac{-\alpha_1(\beta+1)}{2\alpha_1-1}+2\gamma-\beta > 0$ and then we can bound the term g_5^n using the L^∞ estimate on ρ_n (4.98). We can then proceed as in the previous section and show that for any $T > 0$ it exists $c_0 > 0$ sufficiently small such that for any $c \leq c_0$ we have $T_n > T$ for any $n \in \mathbb{N}$ and for any $t \leq T$ we have for any $n \in \mathbb{N}$:

$$\|\frac{1}{\rho_n}\|_{L^\infty([0,T]\times\mathbb{R})} \leq C(T) < +\infty \tag{4.145}$$

with $C(T)$ sufficiently large. In addition we have seen that for $\gamma \geq \beta$ we have always for any $n \in \mathbb{N}$, $t \in [0, T]$ and $x \in \mathbb{R}$:

$$w_{1,n}(t, x) \leq C(T), \tag{4.146}$$

with $C(T) > 0$ large enough. From (4.146), (4.98), (4.141) and (4.145) we get for any $n \in \mathbb{N}$, $t \in [0, T]$ and $x \in \mathbb{R}$:

$$\partial_x v_{1,n}(t, x) \leq C(T), \tag{4.147}$$

4.2 Compactness

Let us fix now $T > 0$ and we wish now to prove that the approximate solution $(\rho_n, u_n)_{n\in\mathbb{N}}$ converge up to a subsequence to a unique solution (ρ, u) of the system (1.1) on $[0, T]$ provided that $0 < c \leq c_0$ with c_0 sufficiently small. We recall the previous estimates that we have obtained with $C(T) > 0$ depending on the initial data of the Theorem 2.2, $c > 0$ and of T, we have for any $n \in \mathbb{N}$, $i \in \{1, 2\}$ and $0 \leq t \leq T$:

$$C(T)^{-1} \leq \rho_n(t, \cdot) \leq C(T), \tag{4.148}$$

$$\begin{aligned}\sup_{0<t\leq T} \big(&\|\rho_n(t,\cdot)-1\|_{L^2} + \|v_{i,n}(t,\cdot)\|_{L^2} + \|\partial_x\rho_n(t,\cdot)\|_{L^2} + \sigma(t)^{\frac{1}{2}}\|\partial_x v_{i,n}(t,\cdot)\|_{L^2} \\ &+ \sigma(t)(\|\dot{v_{i,n}}(t,\cdot)\|_{L^2} + \|(1-\alpha_i)\partial_x(\rho_n^\beta\partial_x v_{i,n}(t,\cdot) - P(\rho_n) + P(1))\|_{L^2}\big) \leq C(T),\end{aligned} \tag{4.149}$$

$$\int_0^T [\|\partial_x v_{i,n}(t,\cdot)\|_{L^2}^2 + \|\partial_x\rho_n(t,\cdot)\|_{L^2}^2 + \sigma(t)\|\dot{v_{i,n}}(t,\cdot)\|_{L^2}^2 + \sigma^2(t)\|\partial_x\dot{v_{i,n}}(t,\cdot)\|_{L^2}^2]dt \leq C(T), \tag{4.150}$$

$$\int_0^T \sigma^{\frac{1}{2}}(\tau)\,\|\partial_x u_n(\tau)\|_{L^\infty}^2\,d\tau \leq C(T). \tag{4.151}$$

$$\sup_{0<t\leq T} \sigma(t)^{\frac{1}{2}}\|\partial_x u_n(t,\cdot)\|_{L^\infty} \leq C(T). \tag{4.152}$$

We mention that (4.149) and (4.150) are also true if we replace $v_{i,n}$ by u_n since $u_n = \frac{1}{\sqrt{1-4c}}((1-\alpha_2)v_{2,n} - (1-\alpha_1)v_{1,n})$. Using classical arguments (see [32, 40]), we prove that up to a subsequence, $(\rho_n, u_n)_{n\in\mathbb{N}}$ converges in the sense of distributions to (ρ, u) a weak solution of (1.1) on $[0, T]$. Furthermore the limit functions ρ, u ihave all the bounds (4.148), (4.149), (4.150), (4.151) and (4.152) via Fatou type-lemmas for the weak

topology.
We wish now to prove (2.32). Using classical arguments of thpe Aubin-Lions Lemma, we can prove that up to a subsequence $(v_{1,n})_{n\in\mathbb{N}}$ converges almost everywhere to v_1 in $[0,T]\times\mathbb{R}$. Using now (4.147) and the fact that $v_{1,n}$ converges almost everywhere to v_1 on $[0,T]\times\mathbb{R}$ implies (2.32) since for all $x>y$ and $t\in[0,T]$ we have:

$$\frac{v_{1,n}(t,x)-v_{1,n}(t,y)}{x-y}=\frac{1}{x-y}\int_y^x \partial_z v_{1,n}(t,z)dz\leq C(T).$$

It concludes the proof of (2.32).

4.3 Uniqueness

The uniqueness is a consequence of the fact that on $[0,T]$, $\partial_x u$ belongs to $L^1([0,T_1],L^\infty)$. We refer to [7] for the details of the proof where we use a Lagrangian formulation.

5 Appendix

5.1 Proof of the Proposition 3.1

In this Appendix, we start by giving a proof of the Proposition 3.1. We work in fact in a more general case as $\mu(\rho)=\rho^\beta$ since we deal with general viscosity coefficient $\mu(\rho)>0$, pressure term $P(\rho)$ and a capillary coefficient $\kappa(\rho)$ satisfying (1.3). It enables us in particular to extend the notion of effective pressures for compressible Navier-Stokes equations (see [7, 15]) to a general framework for the Navier-Stokes Korteweg system.

Proposition 5.3 *Setting $w_i=f_i(\rho)\partial_x v_i+F_{1,i}(\rho)$ with $i\in\{1,2\}$ and f_i, $F_{1,i}$ such that:*

$$f_i(\rho)=\frac{(\mu(\rho))^{\frac{\alpha_i-1}{2\alpha_i-1}}}{\rho^{\frac{\alpha_i}{2\alpha_i-1}}}\quad \textit{and}\quad F'_{1,i}(\rho)=\frac{1}{2\alpha_i-1}(\rho\mu(\rho))^{\frac{-\alpha_i}{2\alpha_i-1}}P'(\rho). \tag{5.153}$$

If (ρ,u) is a classical solution of the system (1.1) then w_i with $i\in\{1,2\}$ satisfy the following parabolic equation:

$$\partial_t w_i+(u+g_{1,i})\partial_x w_i+w_i g'_{2,i}+(\alpha_i-1)\partial_x(\frac{\mu(\rho)}{\rho}\partial_x w_i)+g'_{3,i}+g_{4,i}w^2=0, \tag{5.154}$$

with setting $F'(\rho)=\frac{P'(\rho)}{\rho}$:

$$\begin{aligned}
g_{1,i}&=-2(\alpha_i-1)\partial_x f(\rho)\frac{\mu(\rho)}{\rho}\frac{1}{f_i(\rho)}+(\alpha_i-1)\frac{\partial_x\mu(\rho)}{\rho},\\
g_{4,i}&=\frac{\alpha_i-1}{\alpha_i}\frac{\rho}{f_i^2(\rho)}\big(\frac{\mu'(\rho)}{\mu(\rho)}f_i(\rho)-f'_i(\rho)\big),\\
g'_{2,i}&=g_{2,i}-2\frac{\alpha_i-1}{\alpha_i}\frac{\rho}{f_i^2(\rho)}\big(\frac{\mu'(\rho)}{\mu(\rho)}f_i(\rho)-f'_i(\rho)\big)F_{1,i}(\rho),
\end{aligned}$$

$$
\begin{aligned}
g_{2,i} &= 2(\alpha_i-1)\partial_x f_i(\rho)\frac{\mu(\rho)}{\rho}\frac{f_i'(\rho)}{f_i(\rho)^2}\partial_x\rho + \frac{(\alpha_i-1)}{\alpha_i^2}\mu'(\rho)\rho^2\frac{2\mu(\rho)-\mu'(\rho)\rho}{\mu(\rho)^3}(v_i-u)^2 \\
&+ (\alpha_i-1)\frac{(\partial_x\rho)^2(\mu''(\rho)\rho-\mu'(\rho))}{\rho^2} - (\alpha_i-1)\frac{\partial_x\mu(\rho)}{\rho}\frac{f_i'(\rho)}{f_i(\rho)}\partial_x\rho + \frac{F'(\rho)}{\alpha_i}\frac{\rho^2}{\mu(\rho)} \\
&- (\alpha_i-1)\frac{\mu(\rho)}{\rho}F_{1,i}'(\rho)\frac{\rho^2}{\alpha_i f_i(\rho)\mu(\rho)} - \frac{(\alpha_i-1)}{f_i(\rho)}\big((\partial_x\rho)^2\big(f_i'(\rho)\frac{\mu'(\rho)\rho-\mu(\rho)}{\rho^2} + \frac{\mu(\rho)}{\rho}f_i''(\rho)\big) \\
&+ \frac{1}{\alpha_i^2}\rho f_i'(\rho)(\frac{2\rho\mu(\rho)-\rho^2\mu'(\rho)}{\mu(\rho)^2})(v_i-u)^2\big)
\end{aligned}
\tag{5.155}
$$

and:

$$
\begin{aligned}
g_{3,i}' &= g_{32,i} + g_{33,i}', \\
g_{33,i}' &= g_{33,i} + \frac{\alpha_i-1}{\alpha_i}\frac{\rho}{f_i^2(\rho)}\big(\frac{\mu'(\rho)}{\mu(\rho)}f_i(\rho) - f_i'(\rho)\big)F_{1,i}(\rho)^2, \\
g_{33,i} &= -F_{1,i}(\rho)\big(\frac{P'(\rho)\rho}{\alpha_i}\frac{1}{\mu(\rho)} - (\alpha_i-1)F_{1,i}'(\rho)\frac{\rho}{\alpha_i f_i(\rho)}\big), \\
g_{32,i} &= -\partial_x F_1(\rho)\big(-2(\alpha_i-1)\partial_x f_i(\rho)\frac{\mu(\rho)}{\rho}\frac{1}{f_i(\rho)} + (\alpha_i-1)\frac{\partial_x\mu(\rho)}{\rho}\big) \\
&- F_{1,i}(\rho)\Bigg(2(\alpha_i-1)\partial_x f_i(\rho)\frac{\mu(\rho)}{\rho}\frac{f_i'(\rho)}{f_i(\rho)^2}\partial_x\rho + \frac{(\alpha_i-1)}{\alpha_i^2}\mu'(\rho)\rho^2\frac{2\mu(\rho)-\mu'(\rho)\rho}{\mu(\rho)^3}(v_i-u)^2 \\
&+ (\alpha_i-1)\frac{(\partial_x\rho)^2(\mu''(\rho)\rho-\mu'(\rho))}{\rho^2} - (\alpha_i-1)\frac{\partial_x\mu(\rho)}{\rho}\frac{f_i'(\rho)}{f_i(\rho)}\partial_x\rho \\
&- \frac{(\alpha_i-1)}{f_i(\rho)}\big((\partial_x\rho)^2\big(f_i'(\rho)\frac{\mu'(\rho)\rho-\mu(\rho)}{\rho^2} + \frac{\mu(\rho)}{\rho}f_i''(\rho)\big) + \frac{1}{\alpha_i^2}\rho f_i'(\rho)(\frac{2\rho\mu(\rho)-\rho^2\mu'(\rho)}{\mu(\rho)^2})(v_i-u)^2\big)\Bigg) \\
&- (\alpha_i-1)\big(\partial_x(\frac{\mu(\rho)}{\rho})\partial_x F_{1,i}(\rho) + \frac{\mu(\rho)}{\rho}F_{1,i}''(\rho)(\partial_x\rho)^2 + \frac{\mu(\rho)}{\rho}F_{1,i}'(\rho)\frac{\rho^3}{\alpha_i^2}\frac{2\mu(\rho)-\rho\mu'(\rho)}{\mu(\rho)^3}(v_i-u)^2\big) \\
&+ f_i(\rho)F''(\rho)(\partial_x\rho)^2 + \frac{f_i(\rho)F'(\rho)\rho^3}{\alpha_i^2}\frac{2\mu(\rho)-\rho\mu'(\rho)}{\mu(\rho)^3}(v_i-u)^2.
\end{aligned}
\tag{5.156}
$$

Proof of the Proposition 5.3: In the sequel for simplicity in the notation, we forget the subscript $i \in \{1,2\}$. Derivating (1.10) we have with $F'(\rho) = \frac{P'(\rho)}{\rho}$:

$$
\begin{aligned}
&\partial_t\partial_x v + u\partial_x\partial_x v + \partial_x u\partial_x v + (\alpha-1)\partial_x(\frac{\mu(\rho)}{\rho}\partial_x\partial_x v) + (\alpha-1)\partial_x(\frac{\partial_x\mu(\rho)}{\rho})\partial_x v \\
&+ (\alpha-1)\frac{\partial_x\mu(\rho)}{\rho}\partial_x\partial_x v + \partial_{xx}F(\rho) = 0
\end{aligned}
\tag{5.157}
$$

We recall now that:

$$
\begin{cases}
\partial_x\rho = \dfrac{\rho^2}{\alpha\mu(\rho)}(v-u) \\
\partial_{xx}\rho = \dfrac{\rho^3}{\alpha^2}\dfrac{2\mu(\rho)-\rho\mu'(\rho)}{\mu(\rho)^3}(v-u)^2 + \dfrac{\rho^2}{\alpha\mu(\rho)}(\partial_x v - \partial_x u)
\end{cases}
\tag{5.158}
$$

We have now:

$$\partial_x(\frac{\partial_x\mu(\rho)}{\rho}) = \frac{\mu'(\rho)\partial_{xx}\rho}{\rho} + \frac{\mu''(\rho)(\partial_x\rho)^2}{\rho} - \frac{\mu'(\rho)(\partial_x\rho)^2}{\rho^2}$$

Using again (5.158), we obtain:

$$\begin{aligned}\partial_x(\frac{\partial_x\mu(\rho)}{\rho}) = \frac{1}{\alpha^2}\mu'(\rho)\rho^2\frac{2\mu(\rho)-\mu'(\rho)\rho}{\mu(\rho)^3}(v-u)^2 + \frac{1}{\alpha}\frac{\rho\mu'(\rho)}{\mu(\rho)}(\partial_x v - \partial_x u)\\ + \frac{(\partial_x\rho)^2(\mu''(\rho)\rho-\mu'(\rho))}{\rho^2}\end{aligned}$$

Plugging this equality in (5.157), we get:

$$\begin{aligned}&\partial_t\partial_x v + u\partial_x\partial_x v + \partial_x u\partial_x v - \frac{\alpha-1}{\alpha}\frac{\rho\mu'(\rho)}{\mu(\rho)}\partial_x v\partial_x u + (\alpha-1)\partial_x(\frac{\mu(\rho)}{\rho}\partial_x\partial_x v)\\ &+ \frac{(\alpha-1)}{\alpha^2}\mu'(\rho)\rho^2\frac{2\mu(\rho)-\mu'(\rho)\rho}{\mu(\rho)^3}(v-u)^2\partial_x v + \frac{\alpha-1}{\alpha}\frac{\rho\mu'(\rho)}{\mu(\rho)}(\partial_x v)^2\\ &+ (\alpha-1)\frac{(\partial_x\rho)^2(\mu''(\rho)\rho-\mu'(\rho))}{\rho^2}\partial_x v + (\alpha-1)\frac{\partial_x\mu(\rho)}{\rho}\partial_x\partial_x v + \partial_{xx}F(\rho) = 0\end{aligned} \tag{5.159}$$

Next we multiply the previous equation by a regular function $f(\rho)$ in order to eliminate the terms in $\partial_x u\partial_x v$ (we will fix later the function $f(\rho)$), we have then:

$$\begin{aligned}&\partial_t(f(\rho)\partial_x v) + u\partial_x(f(\rho)\partial_x v) - (\partial_t f(\rho) + u\partial_x f(\rho))\partial_x v\\ &+ f(\rho)\partial_x u\partial_x v(1 - \frac{\alpha-1}{\alpha}\frac{\rho\mu'(\rho)}{\mu(\rho)}) + (\alpha-1)f(\rho)\partial_x(\frac{\mu(\rho)}{\rho}\partial_x\partial_x v)\\ &+ \frac{(\alpha-1)}{\alpha^2}\mu'(\rho)\rho^2\frac{2\mu(\rho)-\mu'(\rho)\rho}{\mu(\rho)^3}(v-u)^2 f(\rho)\partial_x v\\ &+ \frac{\alpha-1}{\alpha}\frac{\rho\mu'(\rho)}{\mu(\rho)}f(\rho)(\partial_x v)^2 + (\alpha-1)\frac{(\partial_x\rho)^2(\mu''(\rho)\rho-\mu'(\rho))}{\rho^2}f(\rho)\partial_x v\\ &+ (\alpha-1)\frac{\partial_x\mu(\rho)}{\rho}f(\rho)\partial_x\partial_x v + f(\rho)\partial_{xx}F(\rho) = 0\end{aligned} \tag{5.160}$$

Next we have:

$$f(\rho)\partial_x\partial_x v = \partial_x(f(\rho)\partial_x v) - \partial_x f(\rho)\partial_x v. \tag{5.161}$$

and:

$$\begin{aligned}&f(\rho)\partial_x(\frac{\mu(\rho)}{\rho}\partial_x\partial_x v) = \partial_x(\frac{\mu(\rho)f(\rho)}{\rho}\partial_x\partial_x v) - \partial_x f(\rho)\frac{\mu(\rho)}{\rho}\partial_x\partial_x v\\ &= \partial_x(\frac{\mu(\rho)}{\rho}\partial_x(f(\rho)\partial_x v)) - \partial_x(\frac{\mu(\rho)}{\rho}\partial_x f(\rho)\partial_x v) - \partial_x f(\rho)\frac{\mu(\rho)}{\rho}\partial_x\partial_x v\\ &= \partial_x(\frac{\mu(\rho)}{\rho}\partial_x(f(\rho)\partial_x v)) - \partial_x(\frac{\mu(\rho)}{\rho}\partial_x f(\rho))\partial_x v - \frac{\mu(\rho)}{\rho}\partial_x f(\rho)\partial_x\partial_x v - \partial_x f(\rho)\frac{\mu(\rho)}{\rho}\partial_x\partial_x v\\ &= \partial_x(\frac{\mu(\rho)}{\rho}\partial_x(f(\rho)\partial_x v)) - \partial_x(\frac{\mu(\rho)}{\rho}\partial_x f(\rho))\partial_x v - 2\partial_x f(\rho)\frac{\mu(\rho)}{\rho}\partial_x\partial_x v\end{aligned} \tag{5.162}$$

Next from (5.161) we have:

$$\partial_{xx}v = \frac{1}{f(\rho)}\partial_x(f(\rho)\partial_x v) - \frac{f'(\rho)}{f(\rho)}\partial_x\rho\partial_x v \tag{5.163}$$

and from (5.158):

$$\begin{aligned}
&\partial_x(\frac{\mu(\rho)}{\rho}\partial_x f(\rho)) = \partial_x(\frac{\mu(\rho)}{\rho})\partial_x f(\rho) + \frac{\mu(\rho)}{\rho}(f'(\rho)\partial_{xx}\rho + f''(\rho)(\partial_x\rho)^2)\\
&= f'(\rho)\frac{\mu'(\rho)\rho - \mu(\rho)}{\rho^2}(\partial_x\rho)^2 + \frac{\mu(\rho)}{\rho}f''(\rho)(\partial_x\rho)^2 + \frac{\mu(\rho)f'(\rho)}{\rho}\big(\frac{\rho^3}{\alpha^2}\frac{2\mu(\rho) - \rho\mu'(\rho)}{\mu(\rho)^3}(v-u)^2\\
&\qquad\qquad + \frac{\rho^2}{\alpha\mu(\rho)}(\partial_x v - \partial_x u)\big)\\
&= (\partial_x\rho)^2\big(f'(\rho)\frac{\mu'(\rho)\rho - \mu(\rho)}{\rho^2} + \frac{\mu(\rho)}{\rho}f''(\rho)\big) + \frac{1}{\alpha^2}\rho f'(\rho)(\frac{2\rho\mu(\rho) - \rho^2\mu'(\rho)}{\mu(\rho)^2})(v-u)^2\\
&+ \frac{1}{\alpha}\rho f'(\rho)(\partial_x v - \partial_x u)
\end{aligned} \tag{5.164}$$

We obtain then combining (5.162), (5.163) and (5.164):

$$\begin{aligned}
&f(\rho)\partial_x(\frac{\mu(\rho)}{\rho}\partial_x\partial_x v) = \partial_x(\frac{\mu(\rho)}{\rho}\partial_x(f(\rho)\partial_x v)) - 2\partial_x f(\rho)\frac{\mu(\rho)}{\rho}\big(\frac{1}{f(\rho)}\partial_x(f(\rho)\partial_x v) - \frac{f'(\rho)}{f(\rho)}\partial_x\rho\partial_x v\big)\\
&- \partial_x v\big((\partial_x\rho)^2\big(f'(\rho)\frac{\mu'(\rho)\rho - \mu(\rho)}{\rho^2} + \frac{\mu(\rho)}{\rho}f''(\rho)\big) + \frac{1}{\alpha^2}\rho f'(\rho)(\frac{2\rho\mu(\rho) - \rho^2\mu'(\rho)}{\mu(\rho)^2})(v-u)^2\\
&+ \frac{1}{\alpha}\rho f'(\rho)(\partial_x v - \partial_x u)\big).
\end{aligned} \tag{5.165}$$

Now combining (5.165), (5.163) and (5.160), we have:

$$\begin{aligned}
&\partial_t(f(\rho)\partial_x v) + u\partial_x(f(\rho)\partial_x v) - (\partial_t f(\rho) + u\partial_x f(\rho))\partial_x v + f(\rho)\partial_x u\partial_x v(1 - \frac{\alpha-1}{\alpha}\frac{\rho\mu'(\rho)}{\mu(\rho)})\\
&+ (\alpha-1)\partial_x(\frac{\mu(\rho)}{\rho}\partial_x(f(\rho)\partial_x v)) - 2(\alpha-1)\partial_x f(\rho)\frac{\mu(\rho)}{\rho}\big(\frac{1}{f(\rho)}\partial_x(f(\rho)\partial_x v) - \frac{f'(\rho)}{f(\rho)}\partial_x\rho\partial_x v\big)\\
&- (\alpha-1)\partial_x v\big((\partial_x\rho)^2\big(f'(\rho)\frac{\mu'(\rho)\rho - \mu(\rho)}{\rho^2} + \frac{\mu(\rho)}{\rho}f''(\rho)\big) + \frac{1}{\alpha^2}\rho f'(\rho)(\frac{2\rho\mu(\rho) - \rho^2\mu'(\rho)}{\mu(\rho)^2})(v-u)^2\\
&+ \frac{1}{\alpha}\rho f'(\rho)(\partial_x v - \partial_x u)\big)\\
&+ \frac{(\alpha-1)}{\alpha^2}\mu'(\rho)\rho^2\frac{2\mu(\rho) - \mu'(\rho)\rho}{\mu(\rho)^3}(v-u)^2 f(\rho)\partial_x v\\
&+ \frac{\alpha-1}{\alpha}\frac{\rho\mu'(\rho)}{\mu(\rho)}f(\rho)(\partial_x v)^2 + (\alpha-1)\frac{(\partial_x\rho)^2(\mu''(\rho)\rho - \mu'(\rho))}{\rho^2}f(\rho)\partial_x v\\
&+ (\alpha-1)\frac{\partial_x\mu(\rho)}{\rho}f(\rho)(\frac{1}{f(\rho)}\partial_x(f(\rho)\partial_x v) - \frac{f'(\rho)}{f(\rho)}\partial_x\rho\partial_x v) + f(\rho)\partial_{xx}F(\rho) = 0
\end{aligned} \tag{5.166}$$

We recall now that using the mass equation, we have:

$$-(\partial_t f(\rho) + u\partial_x f(\rho))\partial_x v = f'(\rho)\rho\partial_x u\partial_x v.$$

From (5.166) we have then:

$$
\begin{aligned}
&\partial_t(f(\rho)\partial_x v) + u\partial_x(f(\rho)\partial_x v) + \partial_x u\partial_x v\big(f(\rho)(1-\frac{\alpha-1}{\alpha}\frac{\rho\mu'(\rho)}{\mu(\rho)}) + (2-\frac{1}{\alpha})f'(\rho)\rho\big)\\
&+(\alpha-1)\partial_x(\frac{\mu(\rho)}{\rho}\partial_x(f(\rho)\partial_x v)) + f(\rho)\partial_{xx}F(\rho)\\
&+\partial_x(f(\rho)\partial_x v)\big(-2(\alpha-1)\partial_x f(\rho)\frac{\mu(\rho)}{\rho}\frac{1}{f(\rho)} + (\alpha-1)\frac{\partial_x\mu(\rho)}{\rho}\big)\\
&+\partial_x v\Bigg(2(\alpha-1)\partial_x f(\rho)\frac{\mu(\rho)}{\rho}\frac{f'(\rho)}{f(\rho)}\partial_x\rho + \frac{(\alpha-1)}{\alpha^2}\mu'(\rho)\rho^2\frac{2\mu(\rho)-\mu'(\rho)\rho}{\mu(\rho)^3}(v-u)^2 f(\rho)\\
&+(\alpha-1)\frac{(\partial_x\rho)^2(\mu''(\rho)\rho-\mu'(\rho))}{\rho^2}f(\rho) - (\alpha-1)\frac{\partial_x\mu(\rho)}{\rho}f'(\rho)\partial_x\rho\\
&-(\alpha-1)\big((\partial_x\rho)^2\big(f'(\rho)\frac{\mu'(\rho)\rho-\mu(\rho)}{\rho^2} + \frac{\mu(\rho)}{\rho}f''(\rho)\big)\\
&+\frac{1}{\alpha^2}\rho f'(\rho)(\frac{2\rho\mu(\rho)-\rho^2\mu'(\rho)}{\mu(\rho)^2})(v-u)^2\big)\Bigg) + \frac{\alpha-1}{\alpha}\rho\big(\frac{\mu'(\rho)}{\mu(\rho)}f(\rho) - f'(\rho)\big)(\partial_x v)^2 = 0.
\end{aligned}
\tag{5.167}
$$

We note now that using (5.158) we have:

$$
\begin{aligned}
&f(\rho)\partial_{xx}F(\rho) = f(\rho)(F'(\rho)\partial_{xx}\rho + F''(\rho)(\partial_x\rho)^2)\\
&= f(\rho)F'(\rho)\big(\frac{\rho^3}{\alpha^2}\frac{2\mu(\rho)-\rho\mu'(\rho)}{\mu(\rho)^3}(v-u)^2 + \frac{\rho^2}{\alpha\mu(\rho)}(\partial_x v - \partial_x u)\big) + f(\rho)F''(\rho)(\partial_x\rho)^2\\
&= \frac{f(\rho)F'(\rho)\rho^3}{\alpha^2}\frac{2\mu(\rho)-\rho\mu'(\rho)}{\mu(\rho)^3}(v-u)^2 + \frac{f(\rho)F'(\rho)}{\alpha}\frac{\rho^2}{\mu(\rho)}(\partial_x v - \partial_x u) + f(\rho)F''(\rho)(\partial_x\rho)^2
\end{aligned}
\tag{5.168}
$$

Plugging (5.168) in (5.166) it yields:

$$
\begin{aligned}
&\partial_t(f(\rho)\partial_x v) + u\partial_x(f(\rho)\partial_x v) + \partial_x u\partial_x v\big(f(\rho)(1-\frac{\alpha-1}{\alpha}\frac{\rho\mu'(\rho)}{\mu(\rho)}) + (2-\frac{1}{\alpha})f'(\rho)\rho\big)\\
&+(\alpha-1)\partial_x(\frac{\mu(\rho)}{\rho}\partial_x(f(\rho)\partial_x v))\\
&+\partial_x(f(\rho)\partial_x v)\big(-2(\alpha-1)\partial_x f(\rho)\frac{\mu(\rho)}{\rho}\frac{1}{f(\rho)} + (\alpha-1)\frac{\partial_x\mu(\rho)}{\rho}\big)\\
&+\partial_x v\Bigg(2(\alpha-1)\partial_x f(\rho)\frac{\mu(\rho)}{\rho}\frac{f'(\rho)}{f(\rho)}\partial_x\rho + \frac{(\alpha-1)}{\alpha^2}\mu'(\rho)\rho^2\frac{2\mu(\rho)-\mu'(\rho)\rho}{\mu(\rho)^3}(v-u)^2 f(\rho)\\
&+(\alpha-1)\frac{(\partial_x\rho)^2(\mu''(\rho)\rho-\mu'(\rho))}{\rho^2}f(\rho) - (\alpha-1)\frac{\partial_x\mu(\rho)}{\rho}f'(\rho)\partial_x\rho + \frac{f(\rho)F'(\rho)}{\alpha}\frac{\rho^2}{\mu(\rho)}\\
&-(\alpha-1)\big((\partial_x\rho)^2\big(f'(\rho)\frac{\mu'(\rho)\rho-\mu(\rho)}{\rho^2} + \frac{\mu(\rho)}{\rho}f''(\rho)\big) + \frac{1}{\alpha^2}\rho f'(\rho)(\frac{2\rho\mu(\rho)-\rho^2\mu'(\rho)}{\mu(\rho)^2})(v-u)^2\big)\Bigg)\\
&+\frac{\alpha-1}{\alpha}\rho\big(\frac{\mu'(\rho)}{\mu(\rho)}f(\rho) - f'(\rho)\big)(\partial_x v)^2 + \frac{f(\rho)F'(\rho)\rho^3}{\alpha^2}\frac{2\mu(\rho)-\rho\mu'(\rho)}{\mu(\rho)^3}(v-u)^2\\
&-\frac{f(\rho)F'(\rho)}{\alpha}\frac{\rho^2}{\mu(\rho)}\partial_x u + f(\rho)F''(\rho)(\partial_x\rho)^2 = 0
\end{aligned}
\tag{5.169}
$$

Now we are going to chose the function $f(\rho)$ such that:

$$f(\rho)(1-\frac{\alpha-1}{\alpha}\frac{\rho\mu'(\rho)}{\mu(\rho)})+(2-\frac{1}{\alpha})f'(\rho)\rho=0 \tag{5.170}$$

It implies that there exists $C\geq 0$ such that:

$$f(\rho)=C\frac{(\mu(\rho))^{\frac{\alpha-1}{2\alpha-1}}}{\rho^{\frac{\alpha}{2\alpha-1}}}. \tag{5.171}$$

In the sequel we will take $C=1$. We can now rewrite the previous system (5.169) as follows with $w=f(\rho)\partial_x v+F_1(\rho)$ and with F_1 that we will define later, it gives using (5.169):

$$\begin{aligned}
&\partial_t w+u\partial_x w+F_1'(\rho)\rho\partial_x u+(\alpha-1)\partial_x(\frac{\mu(\rho)}{\rho}\partial_x w)-(\alpha-1)\partial_x(\frac{\mu(\rho)}{\rho}\partial_x F_1(\rho))\\
&+\partial_x w\big(-2(\alpha-1)\partial_x f(\rho)\frac{\mu(\rho)}{\rho}\frac{1}{f(\rho)}+(\alpha-1)\frac{\partial_x\mu(\rho)}{\rho}\big)\\
&-\partial_x F_1(\rho)\big(-2(\alpha-1)\partial_x f(\rho)\frac{\mu(\rho)}{\rho}\frac{1}{f(\rho)}+(\alpha-1)\frac{\partial_x\mu(\rho)}{\rho}\big)\\
&+\partial_x v\Bigg(2(\alpha-1)\partial_x f(\rho)\frac{\mu(\rho)}{\rho}\frac{f'(\rho)}{f(\rho)}\partial_x\rho+\frac{(\alpha-1)}{\alpha^2}\mu'(\rho)\rho^2\frac{2\mu(\rho)-\mu'(\rho)\rho}{\mu(\rho)^3}(v-u)^2f(\rho)\\
&+(\alpha-1)\frac{(\partial_x\rho)^2(\mu''(\rho)\rho-\mu'(\rho))}{\rho^2}f(\rho)-(\alpha-1)\frac{\partial_x\mu(\rho)}{\rho}f'(\rho)\partial_x\rho+\frac{f(\rho)F'(\rho)}{\alpha}\frac{\rho^2}{\mu(\rho)}\\
&-(\alpha-1)\big((\partial_x\rho)^2\big(f'(\rho)\frac{\mu'(\rho)\rho-\mu(\rho)}{\rho^2}+\frac{\mu(\rho)}{\rho}f''(\rho)\big)+\frac{1}{\alpha^2}\rho f'(\rho)(\frac{2\rho\mu(\rho)-\rho^2\mu'(\rho)}{\mu(\rho)^2})(v-u)^2\big)\Bigg)\\
&+\frac{\alpha-1}{\alpha}\rho\big(\frac{\mu'(\rho)}{\mu(\rho)}f(\rho)-f'(\rho)\big)(\partial_x v)^2+\frac{f(\rho)F'(\rho)\rho^3}{\alpha^2}\frac{2\mu(\rho)-\rho\mu'(\rho)}{\mu(\rho)^3}(v-u)^2\\
&-\frac{f(\rho)F'(\rho)}{\alpha}\frac{\rho^2}{\mu(\rho)}\partial_x u+f(\rho)F''(\rho)(\partial_x\rho)^2=0
\end{aligned} \tag{5.172}$$

Now basic computation gives using (5.158):

$$\begin{aligned}
&\partial_x(\frac{\mu(\rho)}{\rho}\partial_x F_1(\rho))=\partial_x(\frac{\mu(\rho)}{\rho})\partial_x F_1(\rho)+\frac{\mu(\rho)}{\rho}F_1''(\rho)(\partial_x\rho)^2+\frac{\mu(\rho)}{\rho}F_1'(\rho)\frac{\rho^3}{\alpha^2}\frac{2\mu(\rho)-\rho\mu'(\rho)}{\mu(\rho)^3}(v-u)^2\\
&+\frac{\mu(\rho)}{\rho}F_1'(\rho)\frac{\rho^2}{\alpha\mu(\rho)}(\partial_x v-\partial_x u)
\end{aligned} \tag{5.173}$$

Plugging (5.173) in (5.172), we get:

$$\begin{aligned}
&\partial_t w+u\partial_x w+(\alpha-1)\partial_x(\frac{\mu(\rho)}{\rho}\partial_x w)+\partial_x w\big(-2(\alpha-1)\partial_x f(\rho)\frac{\mu(\rho)}{\rho}\frac{1}{f(\rho)}+(\alpha-1)\frac{\partial_x\mu(\rho)}{\rho}\big)\\
&-\partial_x F_1(\rho)\big(-2(\alpha-1)\partial_x f(\rho)\frac{\mu(\rho)}{\rho}\frac{1}{f(\rho)}+(\alpha-1)\frac{\partial_x\mu(\rho)}{\rho}\big)\\
&+\partial_x v\Bigg(2(\alpha-1)\partial_x f(\rho)\frac{\mu(\rho)}{\rho}\frac{f'(\rho)}{f(\rho)}\partial_x\rho+\frac{(\alpha-1)}{\alpha^2}\mu'(\rho)\rho^2\frac{2\mu(\rho)-\mu'(\rho)\rho}{\mu(\rho)^3}(v-u)^2f(\rho)
\end{aligned}$$

$$
\begin{aligned}
&+(\alpha-1)\frac{(\partial_x\rho)^2(\mu''(\rho)\rho-\mu'(\rho))}{\rho^2}f(\rho)-(\alpha-1)\frac{\partial_x\mu(\rho)}{\rho}f'(\rho)\partial_x\rho+\frac{f(\rho)F'(\rho)}{\alpha}\frac{\rho^2}{\mu(\rho)}\\
&-(\alpha-1)\frac{\rho F_1'(\rho)}{\alpha}-(\alpha-1)\big((\partial_x\rho)^2\big(f'(\rho)\frac{\mu'(\rho)\rho-\mu(\rho)}{\rho^2}+\frac{\mu(\rho)}{\rho}f''(\rho)\big)\\
&+\frac{1}{\alpha^2}\rho f'(\rho)(\frac{2\rho\mu(\rho)-\rho^2\mu'(\rho)}{\mu(\rho)^2})(v-u)^2\big)\Big)+\frac{\alpha-1}{\alpha}\rho\big(\frac{\mu'(\rho)}{\mu(\rho)}f(\rho)-f'(\rho)\big)(\partial_x v)^2\\
&+\frac{f(\rho)F'(\rho)\rho^3}{\alpha^2}\frac{2\mu(\rho)-\rho\mu'(\rho)}{\mu(\rho)^3}(v-u)^2\\
&-(\alpha-1)\big(\partial_x(\frac{\mu(\rho)}{\rho})\partial_x F_1(\rho)+\frac{\mu(\rho)}{\rho}F_1''(\rho)(\partial_x\rho)^2+\frac{\mu(\rho)}{\rho}F_1'(\rho)\frac{\rho^3}{\alpha^2}\frac{2\mu(\rho)-\rho\mu'(\rho)}{\mu(\rho)^3}(v-u)^2\big)\\
&+\partial_x u\big(F_1'(\rho)\rho-\frac{f(\rho)F'(\rho)}{\alpha}\frac{\rho^2}{\mu(\rho)}+(\alpha-1)\frac{\rho F_1'(\rho)}{\alpha}\big)+f(\rho)F''(\rho)(\partial_x\rho)^2=0
\end{aligned}
\tag{5.174}
$$

We choose now $F_1(\rho)$ such that:

$$
(2-\frac{1}{\alpha})F_1'(\rho)\rho-\frac{f(\rho)F'(\rho)}{\alpha}\frac{\rho^2}{\mu(\rho)}=0. \tag{5.175}
$$

With this choice, it yields:

$$
\begin{aligned}
&\partial_t w+u\partial_x w+(\alpha-1)\partial_x(\frac{\mu(\rho)}{\rho}\partial_x w)+\partial_x w\big(-2(\alpha-1)\partial_x f(\rho)\frac{\mu(\rho)}{\rho}\frac{1}{f(\rho)}+(\alpha-1)\frac{\partial_x\mu(\rho)}{\rho}\big)\\
&-\partial_x F_1(\rho)\big(-2(\alpha-1)\partial_x f(\rho)\frac{\mu(\rho)}{\rho}\frac{1}{f(\rho)}+(\alpha-1)\frac{\partial_x\mu(\rho)}{\rho}\big)\\
&+f(\rho)\partial_x v\Big(2(\alpha-1)\partial_x f(\rho)\frac{\mu(\rho)}{\rho}\frac{f'(\rho)}{f(\rho)^2}\partial_x\rho+\frac{(\alpha-1)}{\alpha^2}\mu'(\rho)\rho^2\frac{2\mu(\rho)-\mu'(\rho)\rho}{\mu(\rho)^3}(v-u)^2\\
&+(\alpha-1)\frac{(\partial_x\rho)^2(\mu''(\rho)\rho-\mu'(\rho))}{\rho^2}-(\alpha-1)\frac{\partial_x\mu(\rho)}{\rho}\frac{f'(\rho)}{f(\rho)}\partial_x\rho+\frac{F'(\rho)}{\alpha}\frac{\rho^2}{\mu(\rho)}\\
&-(\alpha-1)F_1'(\rho)\frac{\rho}{\alpha f(\rho)}-\frac{(\alpha-1)}{f(\rho)}\big((\partial_x\rho)^2\big(f'(\rho)\frac{\mu'(\rho)\rho-\mu(\rho)}{\rho^2}+\frac{\mu(\rho)}{\rho}f''(\rho)\big)\\
&+\frac{1}{\alpha^2}\rho f'(\rho)(\frac{2\rho\mu(\rho)-\rho^2\mu'(\rho)}{\mu(\rho)^2})(v-u)^2\big)\Big)+\frac{\alpha-1}{\alpha}\rho\big(\frac{\mu'(\rho)}{\mu(\rho)}f(\rho)-f'(\rho)\big)(\partial_x v)^2\\
&+f(\rho)F''(\rho)(\partial_x\rho)^2+\frac{f(\rho)F'(\rho)\rho^3}{\alpha^2}\frac{2\mu(\rho)-\rho\mu'(\rho)}{\mu(\rho)^3}(v-u)^2\\
&-(\alpha-1)\big(\partial_x(\frac{\mu(\rho)}{\rho})\partial_x F_1(\rho)+\frac{\mu(\rho)}{\rho}F_1''(\rho)(\partial_x\rho)^2+\frac{\mu(\rho)}{\rho}F_1'(\rho)\frac{\rho^3}{\alpha^2}\frac{2\mu(\rho)-\rho\mu'(\rho)}{\mu(\rho)^3}(v-u)^2\big)=0
\end{aligned}
\tag{5.176}
$$

Using (5.171), we note that (5.175) is satisfied if:

$$
F_1'(\rho)=\frac{1}{2\alpha-1}(\rho\mu(\rho))^{\frac{-\alpha}{2\alpha-1}}P'(\rho). \tag{5.177}
$$

We can now rewrite (5.176) as follows:

$$
\begin{aligned}
&\partial_t w + u\partial_x w + (\alpha-1)\partial_x(\frac{\mu(\rho)}{\rho}\partial_x w) + f(\rho)F''(\rho)(\partial_x\rho)^2\\
&+\partial_x w\big(-2(\alpha-1)\partial_x f(\rho)\frac{\mu(\rho)}{\rho}\frac{1}{f(\rho)} + (\alpha-1)\frac{\partial_x\mu(\rho)}{\rho}\big)\\
&-\partial_x F_1(\rho)\big(-2(\alpha-1)\partial_x f(\rho)\frac{\mu(\rho)}{\rho}\frac{1}{f(\rho)} + (\alpha-1)\frac{\partial_x\mu(\rho)}{\rho}\big)\\
&+w\Bigg(2(\alpha-1)\partial_x f(\rho)\frac{\mu(\rho)}{\rho}\frac{f'(\rho)}{f(\rho)^2}\partial_x\rho + \frac{(\alpha-1)}{\alpha^2}\mu'(\rho)\rho^2\frac{2\mu(\rho)-\mu'(\rho)\rho}{\mu(\rho)^3}(v-u)^2\\
&+(\alpha-1)\frac{(\partial_x\rho)^2(\mu''(\rho)\rho-\mu'(\rho))}{\rho^2} - (\alpha-1)\frac{\partial_x\mu(\rho)}{\rho}\frac{f'(\rho)}{f(\rho)}\partial_x\rho + \frac{F'(\rho)}{\alpha}\frac{\rho^2}{\mu(\rho)}\\
&-(\alpha-1)F_1'(\rho)\frac{\rho}{\alpha f(\rho)} - \frac{(\alpha-1)}{f(\rho)}\big((\partial_x\rho)^2\big(f'(\rho)\frac{\mu'(\rho)\rho-\mu(\rho)}{\rho^2} + \frac{\mu(\rho)}{\rho}f''(\rho)\big)\\
&+\frac{1}{\alpha^2}\rho f'(\rho)(\frac{2\rho\mu(\rho)-\rho^2\mu'(\rho)}{\mu(\rho)^2})(v-u)^2\big)\Bigg)\\
&-F_1(\rho)\Bigg(2(\alpha-1)\partial_x f(\rho)\frac{\mu(\rho)}{\rho}\frac{f'(\rho)}{f(\rho)^2}\partial_x\rho + \frac{(\alpha-1)}{\alpha^2}\mu'(\rho)\rho^2\frac{2\mu(\rho)-\mu'(\rho)\rho}{\mu(\rho)^3}(v-u)^2\\
&+(\alpha-1)\frac{(\partial_x\rho)^2(\mu''(\rho)\rho-\mu'(\rho))}{\rho^2} - (\alpha-1)\frac{\partial_x\mu(\rho)}{\rho}\frac{f'(\rho)}{f(\rho)}\partial_x\rho + \frac{F'(\rho)}{\alpha}\frac{\rho^2}{\mu(\rho)}\\
&-(\alpha-1)F_1'(\rho)\frac{\rho}{\alpha f(\rho)} - \frac{(\alpha-1)}{f(\rho)}\big((\partial_x\rho)^2\big(f'(\rho)\frac{\mu'(\rho)\rho-\mu(\rho)}{\rho^2} + \frac{\mu(\rho)}{\rho}f''(\rho)\big)\\
&+\frac{1}{\alpha^2}\rho f'(\rho)(\frac{2\rho\mu(\rho)-\rho^2\mu'(\rho)}{\mu(\rho)^2})(v-u)^2\big)\Bigg)\\
&+\frac{\alpha-1}{\alpha}\rho\big(\frac{\mu'(\rho)}{\mu(\rho)}f(\rho) - f'(\rho)\big)(\partial_x v)^2 + \frac{f(\rho)F'(\rho)\rho^3}{\alpha^2}\frac{2\mu(\rho)-\rho\mu'(\rho)}{\mu(\rho)^3}(v-u)^2\\
&-(\alpha-1)\big(\partial_x(\frac{\mu(\rho)}{\rho})\partial_x F_1(\rho) + \frac{\mu(\rho)}{\rho}F_1''(\rho)(\partial_x\rho)^2 + \frac{\mu(\rho)}{\rho}F_1'(\rho)\frac{\rho^3}{\alpha^2}\frac{2\mu(\rho)-\rho\mu'(\rho)}{\mu(\rho)^3}(v-u)^2\big) = 0.
\end{aligned}
\tag{5.178}
$$

It means that w satisfies the following equation:

$$
\partial_t w + u\partial_x w + \partial_x w g_1 + w g_2 + (\alpha-1)\partial_x(\frac{\mu(\rho)}{\rho}\partial_x w) + g_3 = 0 \tag{5.179}
$$

with:

$$
g_1 = -2(\alpha-1)\partial_x f(\rho)\frac{\mu(\rho)}{\rho}\frac{1}{f(\rho)} + (\alpha-1)\frac{\partial_x\mu(\rho)}{\rho}. \tag{5.180}
$$

$$
\begin{aligned}
g_2 &= 2(\alpha-1)\partial_x f(\rho)\frac{\mu(\rho)}{\rho}\frac{f'(\rho)}{f(\rho)^2}\partial_x\rho + \frac{(\alpha-1)}{\alpha^2}\mu'(\rho)\rho^2\frac{2\mu(\rho)-\mu'(\rho)\rho}{\mu(\rho)^3}(v-u)^2\\
&+(\alpha-1)\frac{(\partial_x\rho)^2(\mu''(\rho)\rho-\mu'(\rho))}{\rho^2} - (\alpha-1)\frac{\partial_x\mu(\rho)}{\rho}\frac{f'(\rho)}{f(\rho)}\partial_x\rho + \frac{F'(\rho)}{\alpha}\frac{\rho^2}{\mu(\rho)}\\
&-(\alpha-1)F_1'(\rho)\frac{\rho}{\alpha f(\rho)} - \frac{(\alpha-1)}{f(\rho)}\big((\partial_x\rho)^2\big(f'(\rho)\frac{\mu'(\rho)\rho-\mu(\rho)}{\rho^2} + \frac{\mu(\rho)}{\rho}f''(\rho)\big)\\
&+\frac{1}{\alpha^2}\rho f'(\rho)(\frac{2\rho\mu(\rho)-\rho^2\mu'(\rho)}{\mu(\rho)^2})(v-u)^2\big)
\end{aligned}
\tag{5.181}
$$

and:

$$
\begin{aligned}
g_3 = &-\partial_x F_1(\rho)\big(-2(\alpha-1)\partial_x f(\rho)\frac{\mu(\rho)}{\rho}\frac{1}{f(\rho)} + (\alpha-1)\frac{\partial_x \mu(\rho)}{\rho}\big) \\
&- F_1(\rho)\Bigg(2(\alpha-1)\partial_x f(\rho)\frac{\mu(\rho)}{\rho}\frac{f'(\rho)}{f(\rho)^2}\partial_x\rho + \frac{(\alpha-1)}{\alpha^2}\mu'(\rho)\rho^2\frac{2\mu(\rho)-\mu'(\rho)\rho}{\mu(\rho)^3}(v-u)^2 \\
&+ (\alpha-1)\frac{(\partial_x\rho)^2(\mu''(\rho)\rho-\mu'(\rho))}{\rho^2} - (\alpha-1)\frac{\partial_x\mu(\rho)}{\rho}\frac{f'(\rho)}{f(\rho)}\partial_x\rho + \frac{F'(\rho)}{\alpha}\frac{\rho^2}{\mu(\rho)} \\
&- (\alpha-1)F_1'(\rho)\frac{\rho}{\alpha f(\rho)} - \frac{(\alpha-1)}{f(\rho)}\big((\partial_x\rho)^2\big(f'(\rho)\frac{\mu'(\rho)\rho-\mu(\rho)}{\rho^2} + \frac{\mu(\rho)}{\rho}f''(\rho)\big) \\
&+ \frac{1}{\alpha^2}\rho f'(\rho)(\frac{2\rho\mu(\rho)-\rho^2\mu'(\rho)}{\mu(\rho)^2})(v-u)^2\big)\Bigg) \\
&+ \frac{\alpha-1}{\alpha}\rho\big(\frac{\mu'(\rho)}{\mu(\rho)}f(\rho) - f'(\rho)\big)(\partial_x v)^2 + \frac{f(\rho)F'(\rho)\rho^3}{\alpha^2}\frac{2\mu(\rho)-\rho\mu'(\rho)}{\mu(\rho)^3}(v-u)^2 \\
&- (\alpha-1)\big(\partial_x(\frac{\mu(\rho)}{\rho})\partial_x F_1(\rho) + \frac{\mu(\rho)}{\rho}F_1''(\rho)(\partial_x\rho)^2 + \frac{\mu(\rho)}{\rho}F_1'(\rho)\frac{\rho^3}{\alpha^2}\frac{2\mu(\rho)-\rho\mu'(\rho)}{\mu(\rho)^3}(v-u)^2\big) \\
&+ f(\rho)F''(\rho)(\partial_x\rho)^2
\end{aligned}
\tag{5.182}
$$

Furthermore we can rewrite g_3 as follows:

$$
g_3 = g_{31} + g_{32} + g_{33},
$$

with:

$$
\begin{aligned}
g_{31} &= \frac{\alpha-1}{\alpha}\rho\big(\frac{\mu'(\rho)}{\mu(\rho)}f(\rho) - f'(\rho)\big)(\partial_x v)^2, \\
g_{33} &= -F_1(\rho)\big(\frac{P'(\rho)\rho}{\alpha}\frac{1}{\mu(\rho)} - (\alpha-1)F_1'(\rho)\frac{\rho}{\alpha f(\rho)}\big).
\end{aligned}
\tag{5.183}
$$

and:

$$
\begin{aligned}
g_{32} = &-\partial_x F_1(\rho)\big(-2(\alpha-1)\partial_x f(\rho)\frac{\mu(\rho)}{\rho}\frac{1}{f(\rho)} + (\alpha-1)\frac{\partial_x \mu(\rho)}{\rho}\big) \\
&- F_1(\rho)\Bigg(2(\alpha-1)\partial_x f(\rho)\frac{\mu(\rho)}{\rho}\frac{f'(\rho)}{f(\rho)^2}\partial_x\rho + \frac{(\alpha-1)}{\alpha^2}\mu'(\rho)\rho^2\frac{2\mu(\rho)-\mu'(\rho)\rho}{\mu(\rho)^3}(v-u)^2 \\
&+ (\alpha-1)\frac{(\partial_x\rho)^2(\mu''(\rho)\rho-\mu'(\rho))}{\rho^2} - (\alpha-1)\frac{\partial_x\mu(\rho)}{\rho}\frac{f'(\rho)}{f(\rho)}\partial_x\rho \\
&- \frac{(\alpha-1)}{f(\rho)}\big((\partial_x\rho)^2\big(f'(\rho)\frac{\mu'(\rho)\rho-\mu(\rho)}{\rho^2} + \frac{\mu(\rho)}{\rho}f''(\rho)\big) + \frac{1}{\alpha^2}\rho f'(\rho)(\frac{2\rho\mu(\rho)-\rho^2\mu'(\rho)}{\mu(\rho)^2})(v-u)^2\big)\Bigg) \\
&- (\alpha-1)\big(\partial_x(\frac{\mu(\rho)}{\rho})\partial_x F_1(\rho) + \frac{\mu(\rho)}{\rho}F_1''(\rho)(\partial_x\rho)^2 + \frac{\mu(\rho)}{\rho}F_1'(\rho)\frac{\rho^3}{\alpha^2}\frac{2\mu(\rho)-\rho\mu'(\rho)}{\mu(\rho)^3}(v-u)^2\big) \\
&+ f(\rho)F''(\rho)(\partial_x\rho)^2 + \frac{f(\rho)F'(\rho)\rho^3}{\alpha^2}\frac{2\mu(\rho)-\rho\mu'(\rho)}{\mu(\rho)^3}(v-u)^2
\end{aligned}
\tag{5.184}
$$

We note that g_{32} has terms only in $(\partial_x\rho)^2$ and $(v-u)^2$. We are now going to rewrite g_{31} as follows:

$$\begin{aligned}
g_{31} &= \frac{\alpha-1}{\alpha}\rho\big(\frac{\mu'(\rho)}{\mu(\rho)f(\rho)} - \frac{f'(\rho)}{f(\rho)^2}\big)(f(\rho)\partial_x v)^2 \\
&= \frac{\alpha-1}{\alpha}\rho\big(\frac{\mu'(\rho)}{\mu(\rho)f(\rho)} - \frac{f'(\rho)}{f(\rho)^2}\big)(w-F_1(\rho))^2 \\
&= \frac{\alpha-1}{\alpha}\frac{\rho}{f^2(\rho)}\big(\frac{\mu'(\rho)}{\mu(\rho)}f(\rho)-f'(\rho)\big)w^2 + \frac{\alpha-1}{\alpha}\frac{\rho}{f^2(\rho)}\big(\frac{\mu'(\rho)}{\mu(\rho)}f(\rho)-f'(\rho)\big)F_1(\rho)^2 \\
&- 2\frac{\alpha-1}{\alpha}\frac{\rho}{f^2(\rho)}\big(\frac{\mu'(\rho)}{\mu(\rho)}f(\rho)-f'(\rho)\big)F_1(\rho)w
\end{aligned} \tag{5.185}$$

We finally rewrite the equation (5.186) as follows:

$$\partial_t w + (u+g_1)\partial_x w + wg_2' + (\alpha-1)\partial_x\big(\frac{\mu(\rho)}{\rho}\partial_x w\big) + g_3' + g_4 w^2 = 0, \tag{5.186}$$

with:

$$\begin{aligned}
g_4 &= \frac{\alpha-1}{\alpha}\frac{\rho}{f^2(\rho)}\big(\frac{\mu'(\rho)}{\mu(\rho)}f(\rho)-f'(\rho)\big), \\
g_3' &= g_{32} + g_{33}', \\
g_2' &= g_2 - 2\frac{\alpha-1}{\alpha}\frac{\rho}{f^2(\rho)}\big(\frac{\mu'(\rho)}{\mu(\rho)}f(\rho)-f'(\rho)\big)F_1(\rho),
\end{aligned} \tag{5.187}$$

and:

$$g_{33}' = g_{33} + \frac{\alpha-1}{\alpha}\frac{\rho}{f^2(\rho)}\big(\frac{\mu'(\rho)}{\mu(\rho)}f(\rho)-f'(\rho)\big)F_1(\rho)^2. \tag{5.188}$$

It concludes the proof of the Proposition 5.3. □

We recall now that we are interested in dealing with the following viscosity and capillary coefficients (with $\beta \geq 0$):

$$\mu(\rho) = \rho^\beta,\ \kappa(\rho) = c\rho^{2\beta-3} \ \text{ and } \ P(\rho) = \rho^\gamma. \tag{5.189}$$

From (5.171) and (5.177) we obtain:

$$f(\rho) = \rho^{\frac{(\beta-1)\alpha-\beta}{2\alpha-1}} \ \text{ and } \ F_1'(\rho) = \frac{\gamma}{2\alpha-1}\rho^{\frac{-\alpha(\beta+1)}{2\alpha-1}}\rho^{\gamma-1} \tag{5.190}$$

and:

$$F_1(\rho) = \frac{\gamma}{-\alpha(\beta+1)+\gamma(2\alpha-1)}\rho^{\frac{-\alpha(\beta+1)}{2\alpha-1}+\gamma} \tag{5.191}$$

We are now interested in computing g_1, g_2', g_3' and g_4 with the choice (5.189) of viscosity and capillary coefficients.

Value of g_1

We have then using (5.180) and (5.190):

$$g_1 = -2(\alpha-1)\partial_x f(\rho)\frac{\mu(\rho)}{\rho}\frac{1}{f(\rho)} + (\alpha-1)\frac{\partial_x\mu(\rho)}{\rho}$$

$$
\begin{aligned}
&= \partial_x\rho(-2(\alpha-1)\frac{\mu(\rho)}{\rho}\frac{f'(\rho)}{f(\rho)} + (\alpha-1)\frac{\mu'(\rho)}{\rho}) \\
&= \partial_x\rho(\alpha-1)(-2\frac{(\beta-1)\alpha-\beta}{2\alpha-1}\rho^{\beta-2} + \beta\rho^{\beta-2}) \\
&= \rho^{\beta-2}\partial_x\rho(\alpha-1)\frac{\beta+2\alpha}{2\alpha-1}
\end{aligned}
\tag{5.192}
$$

Value of g_4

We are now interested in looking at the value of g_4. First we have from (5.187) and (5.190):

$$
\begin{aligned}
g_4 &= \frac{\alpha-1}{\alpha}\frac{\rho}{f(\rho)^2}\big(\frac{\mu'(\rho)}{\mu(\rho)}f(\rho) - f'(\rho)\big) \\
&= \frac{\alpha-1}{\alpha}\big(\beta - \frac{(\beta-1)\alpha-\beta}{2\alpha-1}\big)\rho^{\frac{-(\beta-1)\alpha+\beta}{2\alpha-1}} \\
&= (\alpha-1)\frac{\beta+1}{2\alpha-1}\rho^{\frac{-(\beta-1)\alpha+\beta}{2\alpha-1}}.
\end{aligned}
\tag{5.193}
$$

Value of g'_{33}

We recall from (5.188) that:

$$
g'_{33} = g_{33} + \frac{\alpha-1}{\alpha}\frac{\rho}{f^2(\rho)}\big(\frac{\mu'(\rho)}{\mu(\rho)}f(\rho) - f'(\rho)\big)F_1(\rho)^2. \tag{5.194}
$$

with:

$$
g_{33} = -F_1(\rho)\big(\frac{P'(\rho)\rho}{\alpha}\frac{1}{\mu(\rho)} - (\alpha-1)F_1'(\rho)\frac{\rho}{\alpha f(\rho)}\big). \tag{5.195}
$$

From (5.190), (5.194) and (5.195) we obtain then:

$$
\begin{aligned}
g'_{33} &= -F_1(\rho)\big(\frac{P'(\rho)\rho}{\alpha}\frac{1}{\mu(\rho)} - (\alpha-1)F_1'(\rho)\frac{\rho}{\alpha f(\rho)}\big) + \frac{\alpha-1}{\alpha}\frac{\rho}{f^2(\rho)}\big(\frac{\mu'(\rho)}{\mu(\rho)}f(\rho) - f'(\rho)\big)F_1(\rho)^2 \\
&= \rho^{\frac{-\alpha(\beta+1)}{2\alpha-1}+2\gamma-\beta}\frac{-\gamma^2}{-\alpha(\beta+1)+\gamma(2\alpha-1)}\frac{1}{2\alpha-1} + (\alpha-1)\frac{\beta+1}{2\alpha-1}\rho^{\frac{-\alpha(\beta+1)}{2\alpha-1}+2\gamma-\beta}\big(\frac{\gamma}{-\alpha(\beta+1)+\gamma(2\alpha-1)}\big)^2 \\
&= \rho^{\frac{-\alpha(\beta+1)}{2\alpha-1}+2\gamma-\beta}\frac{\gamma^2}{(-\alpha(\beta+1)+\gamma(2\alpha-1))^2}(\beta+1-\gamma).
\end{aligned}
\tag{5.196}
$$

Value of g'_3

We start by estimating g_{32}. From (5.184) we recall that:

$$
\begin{aligned}
g_{32} &= -\partial_x F_1(\rho)\big(-2(\alpha-1)\partial_x f(\rho)\frac{\mu(\rho)}{\rho}\frac{1}{f(\rho)} + (\alpha-1)\frac{\partial_x\mu(\rho)}{\rho}\big) \\
&- F_1(\rho)\Bigg(2(\alpha-1)\partial_x f(\rho)\frac{\mu(\rho)}{\rho}\frac{f'(\rho)}{f(\rho)^2}\partial_x\rho + \frac{(\alpha-1)}{\alpha^2}\mu'(\rho)\rho^2\frac{2\mu(\rho)-\mu'(\rho)\rho}{\mu(\rho)^3}(v-u)^2
\end{aligned}
$$

$$
\begin{aligned}
&+(\alpha-1)\frac{(\partial_x\rho)^2(\mu''(\rho)\rho-\mu'(\rho))}{\rho^2}-(\alpha-1)\frac{\partial_x\mu(\rho)}{\rho}\frac{f'(\rho)}{f(\rho)}\partial_x\rho\\
&-\frac{(\alpha-1)}{f(\rho)}\big((\partial_x\rho)^2\big(f'(\rho)\frac{\mu'(\rho)\rho-\mu(\rho)}{\rho^2}+\frac{\mu(\rho)}{\rho}f''(\rho)\big)\\
&+\frac{1}{\alpha^2}\rho f'(\rho)(\frac{2\rho\mu(\rho)-\rho^2\mu'(\rho)}{\mu(\rho)^2})(v-u)^2\big)\Big)\\
&+\frac{f(\rho)F'(\rho)\rho^3}{\alpha^2}\frac{2\mu(\rho)-\rho\mu'(\rho)}{\mu(\rho)^3}(v-u)^2+f(\rho)F''(\rho)(\partial_x\rho)^2\\
&-(\alpha-1)\big(\partial_x(\frac{\mu(\rho)}{\rho})\partial_xF_1(\rho)+\frac{\mu(\rho)}{\rho}F_1''(\rho)(\partial_x\rho)^2+\frac{\mu(\rho)}{\rho}F_1'(\rho)\frac{\rho^3}{\alpha^2}\frac{2\mu(\rho)-\rho\mu'(\rho)}{\mu(\rho)^3}(v-u)^2\big).
\end{aligned}
\tag{5.197}
$$

It gives then using the fact that $(v-u)^2=\alpha^2\frac{\mu(\rho)^2}{\rho^4}(\partial_x\rho)^2$:

$$
\begin{aligned}
g_{32}=&-(\partial_x\rho)^2\big(-2(\alpha-1)F_1'(\rho)f'(\rho)\frac{\mu(\rho)}{\rho}\frac{1}{f(\rho)}+(\alpha-1)F_1'(\rho)\frac{\mu'(\rho)}{\rho}\big)\\
&-(\partial_x\rho)^2\Big(2(\alpha-1)F_1(\rho)f'(\rho)\frac{\mu(\rho)}{\rho}\frac{f'(\rho)}{f(\rho)^2}+F_1(\rho)(\alpha-1)\mu'(\rho)\rho^2\frac{2\mu(\rho)-\mu'(\rho)\rho}{\mu(\rho)^3}\frac{\mu(\rho)^2}{\rho^4}\\
&+(\alpha-1)\frac{F_1(\rho)(\mu''(\rho)\rho-\mu'(\rho))}{\rho^2}-(\alpha-1)\frac{\mu'(\rho)F_1(\rho)}{\rho}\frac{f'(\rho)}{f(\rho)}\\
&-\frac{(\alpha-1)}{f(\rho)}\big(F_1(\rho)\big(f'(\rho)\frac{\mu'(\rho)\rho-\mu(\rho)}{\rho^2}+\frac{\mu(\rho)}{\rho}f''(\rho)\big)\\
&+F_1(\rho)\rho f'(\rho)(\frac{2\rho\mu(\rho)-\rho^2\mu'(\rho)}{\mu(\rho)^2})\frac{\mu(\rho)^2}{\rho^4}\big)\Big)\\
&+f(\rho)F'(\rho)\rho^3\frac{2\mu(\rho)-\rho\mu'(\rho)}{\mu(\rho)^3}\frac{\mu(\rho)^2}{\rho^4}(\partial_x\rho)^2+f(\rho)F''(\rho)(\partial_x\rho)^2\\
&-(\alpha-1)(\partial_x\rho)^2\big(\frac{\mu'(\rho)\rho-\mu(\rho)}{\rho^2}F_1'(\rho)+\frac{\mu(\rho)}{\rho}F_1''(\rho)+\frac{\mu(\rho)}{\rho}F_1'(\rho)\rho^3\frac{2\mu(\rho)-\rho\mu'(\rho)}{\mu(\rho)^3}\frac{\mu(\rho)^2}{\rho^4}\big).
\end{aligned}
\tag{5.198}
$$

We have then obtained:

$$
\begin{aligned}
g_{32}=&(\partial_x\rho)^2\Big(2(\alpha-1)F_1'(\rho)f'(\rho)\frac{\mu(\rho)}{\rho}\frac{1}{f(\rho)}-(\alpha-1)F_1'(\rho)\frac{\mu'(\rho)}{\rho}\\
&-2(\alpha-1)F_1(\rho)f'(\rho)\frac{\mu(\rho)}{\rho}\frac{f'(\rho)}{f(\rho)^2}-F_1(\rho)(\alpha-1)\mu'(\rho)\frac{2\mu(\rho)-\mu'(\rho)\rho}{\mu(\rho)\rho^2}\\
&-(\alpha-1)\frac{F_1(\rho)(\mu''(\rho)\rho-\mu'(\rho))}{\rho^2}+(\alpha-1)\frac{\mu'(\rho)F_1(\rho)}{\rho}\frac{f'(\rho)}{f(\rho)}\\
&+\frac{(\alpha-1)}{f(\rho)}\big(F_1(\rho)\big(f'(\rho)\frac{\mu'(\rho)\rho-\mu(\rho)}{\rho^2}+\frac{\mu(\rho)}{\rho}f''(\rho)\big)+F_1(\rho)f'(\rho)(\frac{2\rho\mu(\rho)-\rho^2\mu'(\rho)}{\rho^3})\big)\\
&+f(\rho)F'(\rho)\frac{2\mu(\rho)-\rho\mu'(\rho)}{\mu(\rho)\rho}-(\alpha-1)\big(\frac{\mu'(\rho)\rho-\mu(\rho)}{\rho^2}F_1'(\rho)\\
&+\frac{\mu(\rho)}{\rho}F_1''(\rho)+\frac{\mu(\rho)}{\rho}F_1'(\rho)\frac{2\mu(\rho)-\rho\mu'(\rho)}{\mu(\rho)\rho}\big)+f(\rho)F''(\rho)\Big)
\end{aligned}
\tag{5.199}
$$

We have now:

$$g_{3,2} = k_1 + k_2 + k_3 + k_4, \tag{5.200}$$

with:

$$\begin{aligned}
k_1 &= (\partial_x\rho)^2(\alpha-1)F_1'(\rho)\big(2\frac{f'(\rho)}{f(\rho)}\frac{\mu(\rho)}{\rho} - \frac{\mu'(\rho)}{\rho} - \frac{\mu(\rho)}{\rho^2}\big)\\
k_2 &= (\partial_x\rho)^2(\alpha-1)F_1(\rho)\big(-2(\frac{f'(\rho)}{f(\rho)})^2\frac{\mu(\rho)}{\rho} - \frac{\mu'(\rho)}{\rho^2} + \frac{(\mu'(\rho))^2}{\mu(\rho)\rho} - \frac{\mu''(\rho)}{\rho} + \frac{\mu'(\rho)}{\rho}\frac{f'(\rho)}{f(\rho)}\\
&\quad + \frac{f'(\rho)}{f(\rho)}\frac{\mu(\rho)}{\rho^2} + \frac{f''(\rho)}{f(\rho)}\frac{\mu(\rho)}{\rho}\big)\\
k_3 &= -(\partial_x\rho)^2(\alpha-1)\frac{\mu(\rho)}{\rho}F_1''(\rho)\\
k_4 &= (\partial_x\rho)^2\big(f(\rho)F'(\rho)\frac{2\mu(\rho)-\rho\mu'(\rho)}{\mu(\rho)\rho} + f(\rho)F''(\rho)\big).
\end{aligned}$$

We start by computing k_4, using (5.190) we have:

$$f(\rho)F'(\rho)\frac{2\mu(\rho)-\rho\mu'(\rho)}{\mu(\rho)\rho} = \rho^{\frac{(\beta-1)\alpha-\beta}{2\alpha-1}+\gamma-3}\gamma(2-\beta)$$

$$f(\rho)F''(\rho) = \gamma(\gamma-2)\rho^{\frac{-\alpha(\beta+1)}{2\alpha-1}+\gamma+\beta-3}$$

and:

$$k_4 = \gamma(\gamma-\beta)\rho^{\frac{-\alpha(\beta+1)}{2\alpha-1}+\gamma+\beta-3}(\partial_x\rho)^2. \tag{5.201}$$

Similarly we get:

$$\begin{aligned}
k_1 &= (\partial_x\rho)^2(\alpha-1)F_1'(\rho)\big(2\frac{f'(\rho)}{f(\rho)}\frac{\mu(\rho)}{\rho} - \frac{\mu'(\rho)}{\rho} - \frac{\mu(\rho)}{\rho^2}\big)\\
&= (\partial_x\rho)^2(\alpha-1)\frac{\gamma}{2\alpha-1}\rho^{\frac{-\alpha(\beta+1)}{2\alpha-1}+\gamma+\beta-3}\big(2\frac{(\beta-1)\alpha-\beta}{2\alpha-1} - \beta - 1\big)\\
&= (\partial_x\rho)^2(\alpha-1)\frac{\gamma}{2\alpha-1}\rho^{\frac{-\alpha(\beta+1)}{2\alpha-1}+\gamma+\beta-3}\frac{-4\alpha-\beta+1}{2\alpha-1}.
\end{aligned} \tag{5.202}$$

We have now:

$$\begin{aligned}
k_2 &= (\partial_x\rho)^2(\alpha-1)F_1(\rho)\big(-2(\frac{f'(\rho)}{f(\rho)})^2\frac{\mu(\rho)}{\rho} - \frac{\mu'(\rho)}{\rho^2} + \frac{(\mu'(\rho))^2}{\mu(\rho)\rho} - \frac{\mu''(\rho)}{\rho} + \frac{\mu'(\rho)}{\rho}\frac{f'(\rho)}{f(\rho)}\\
&\quad + \frac{f'(\rho)}{f(\rho)}\frac{\mu(\rho)}{\rho^2} + \frac{f''(\rho)}{f(\rho)}\frac{\mu(\rho)}{\rho}\big)\\
&= (\partial_x\rho)^2(\alpha-1)\frac{\gamma}{-\alpha(\beta+1)+\gamma(2\alpha-1)}\rho^{\frac{-\alpha(\beta+1)}{2\alpha-1}+\gamma+\beta-3}\big(-2(\frac{(\beta-1)\alpha-\beta}{2\alpha-1})^2\\
&\quad - \beta + \beta^2 - \beta(\beta-1) + (\frac{(\beta-1)\alpha-\beta}{2\alpha-1})(\beta+1) + (\frac{(\beta-1)\alpha-\beta}{2\alpha-1})(\frac{(\beta-1)\alpha-\beta}{2\alpha-1} - 1)\big)\\
&= (\partial_x\rho)^2(\alpha-1)\frac{\gamma}{-\alpha(\beta+1)+\gamma(2\alpha-1)}\rho^{\frac{-\alpha(\beta+1)}{2\alpha-1}+\gamma+\beta-3}\big(-2(\frac{(\beta-1)\alpha-\beta}{2\alpha-1})^2
\end{aligned}$$

$$
\begin{aligned}
&+ (\frac{(\beta-1)\alpha-\beta}{2\alpha-1})(\beta+1) + (\frac{(\beta-1)\alpha-\beta}{2\alpha-1})(\frac{(\beta-1)\alpha-\beta}{2\alpha-1}-1)))\\
&= (\partial_x\rho)^2(\alpha-1)\frac{\gamma}{-\alpha(\beta+1)+\gamma(2\alpha-1)}\frac{(\beta-1)\alpha-\beta}{2\alpha-1}\rho^{\frac{-\alpha(\beta+1)}{2\alpha-1}+\gamma+\beta-3}\frac{\alpha(\beta+1)}{2\alpha-1}
\end{aligned}
\tag{5.203}
$$

And:

$$
\begin{aligned}
k_3 &= -(\partial_x\rho)^2(\alpha-1)\frac{\mu(\rho)}{\rho}F_1''(\rho)\\
&= -(\partial_x\rho)^2(\alpha-1)\rho^{\beta-1}\frac{\gamma}{2\alpha-1}(\frac{-\alpha(\beta+1)}{2\alpha-1}+\gamma-1)\rho^{\frac{-\alpha(\beta+1)}{2\alpha-1}}\rho^{\gamma-2}
\end{aligned}
\tag{5.204}
$$

Finally combining (5.200), (5.202), (5.203), (5.204) and (5.201), we have:

$$
\begin{aligned}
g_{32} &= \gamma(\gamma-\beta)\rho^{\frac{-\alpha(\beta+1)}{2\alpha-1}+\gamma+\beta-3}(\partial_x\rho)^2 + (\partial_x\rho)^2(\alpha-1)\frac{\gamma}{2\alpha-1}\rho^{\frac{-\alpha(\beta+1)}{2\alpha-1}+\gamma+\beta-3}\frac{-4\alpha-\beta+1}{2\alpha-1}\\
&+ (\partial_x\rho)^2(\alpha-1)\frac{\gamma}{-\alpha(\beta+1)+\gamma(2\alpha-1)}\frac{(\beta-1)\alpha-\beta}{2\alpha-1}\rho^{\frac{-\alpha(\beta+1)}{2\alpha-1}+\gamma+\beta-3}\frac{\alpha(\beta+1)}{2\alpha-1}\\
&- (\partial_x\rho)^2(\alpha-1)\rho^{\beta-1}\frac{\gamma}{2\alpha-1}(\frac{-\alpha(\beta+1)}{2\alpha-1}+\gamma-1)\rho^{\frac{-\alpha(\beta+1)}{2\alpha-1}}\rho^{\gamma-2}\\
&= (\partial_x\rho)^2\rho^{\frac{-\alpha(\beta+1)}{2\alpha-1}+\gamma+\beta-3}\Bigg(\gamma(\gamma-\beta) + (\alpha-1)\frac{\gamma}{2\alpha-1}\frac{-4\alpha-\beta+1}{2\alpha-1}\\
&+ (\alpha-1)\frac{\gamma}{-\alpha(\beta+1)+\gamma(2\alpha-1)}\frac{(\beta-1)\alpha-\beta}{2\alpha-1}\frac{\alpha(\beta+1)}{2\alpha-1} - (\alpha-1)(\frac{-\alpha(\beta+1)}{2\alpha-1}+\gamma-1)\frac{\gamma}{2\alpha-1}\Bigg)
\end{aligned}
\tag{5.205}
$$

We have now:

$$
(\alpha-1)\frac{\gamma}{2\alpha-1}\frac{-4\alpha-\beta+1}{2\alpha-1} + (\alpha-1)\frac{\alpha(\beta+1)}{2\alpha-1}\frac{\gamma}{2\alpha-1} = \frac{(\alpha-1)\gamma}{(2\alpha-1)^2}(-3\alpha-\beta+1+\alpha\beta)
\tag{5.206}
$$

$$
\begin{aligned}
&(\alpha-1)\frac{\gamma}{2\alpha-1}\frac{-4\alpha-\beta+1}{2\alpha-1} - (\alpha-1)(\frac{-\alpha(\beta+1)}{2\alpha-1}+\gamma-1)\frac{\gamma}{2\alpha-1}\\
&\qquad = \frac{(\alpha-1)\gamma}{(2\alpha-1)^2}(\gamma-\beta+\alpha(\beta-2\gamma-1)).
\end{aligned}
\tag{5.207}
$$

Next we have:

$$
\begin{aligned}
&(\alpha-1)\frac{\gamma}{2\alpha-1}\frac{-4\alpha-\beta+1}{2\alpha-1} + (\alpha-1)\frac{\gamma}{-\alpha(\beta+1)+\gamma(2\alpha-1)}\frac{(\beta-1)\alpha-\beta}{2\alpha-1}\frac{\alpha(\beta+1)}{2\alpha-1}\\
&- (\alpha-1)(\frac{-\alpha(\beta+1)}{2\alpha-1}+\gamma-1)\frac{\gamma}{2\alpha-1}\\
&= \frac{\gamma(\alpha-1)(\gamma-\beta)}{(2\alpha-1)^2(-\alpha(\beta+1)+\gamma(2\alpha-1))}(4\alpha\gamma-\gamma-4\gamma\alpha^2).
\end{aligned}
\tag{5.208}
$$

And finally we get:

$$
\begin{aligned}
&\gamma(\gamma-\beta) + (\alpha-1)\frac{\gamma}{2\alpha-1}\frac{-4\alpha-\beta+1}{2\alpha-1}\\
&+ (\alpha-1)\frac{\gamma}{-\alpha(\beta+1)+\gamma(2\alpha-1)}\frac{(\beta-1)\alpha-\beta}{2\alpha-1}\frac{\alpha(\beta+1)}{2\alpha-1} - (\alpha-1)(\frac{-\alpha(\beta+1)}{2\alpha-1}+\gamma-1)\frac{\gamma}{2\alpha-1}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\gamma(\gamma-\beta)}{(2\alpha-1)^2(-\alpha(\beta+1)+\gamma(2\alpha-1))}(\alpha(\gamma-\beta-1)+4\alpha^2(\beta+1-\gamma)+4\alpha^3(\gamma-\beta-1)) \\
&= \frac{\gamma(\gamma-\beta-1)(\gamma-\beta)}{(2\alpha-1)^2(-\alpha(\beta+1)+\gamma(2\alpha-1))}(\alpha-4\alpha^2+4\alpha^3).
\end{aligned} \tag{5.209}
$$

It implies from (5.209) and (5.205) that:

$$
g_{32} = (\partial_x\rho)^2\rho^{\frac{-\alpha(\beta+1)}{2\alpha-1}+\gamma+\beta-3}\frac{\gamma\alpha(\gamma-\beta-1)(\gamma-\beta)}{(-\alpha(\beta+1)+\gamma(2\alpha-1))}. \tag{5.210}
$$

Combining (5.187), (5.196) and (5.210) we have obtained:

$$
\begin{aligned}
g_3' = \rho^{\frac{-\alpha(\beta+1)}{2\alpha-1}+2\gamma-\beta}&\frac{\gamma^2}{(-\alpha(\beta+1)+\gamma(2\alpha-1))^2}(\beta+1-\gamma) \\
&+ (\partial_x\rho)^2\rho^{\frac{-\alpha(\beta+1)}{2\alpha-1}+\gamma+\beta-3}\frac{\gamma\alpha(\gamma-\beta-1)(\gamma-\beta)}{(-\alpha(\beta+1)+\gamma(2\alpha-1))}
\end{aligned} \tag{5.211}
$$

Value of g_2'

From (5.187) we have:

$$
g_2' = g_2 - 2\frac{\alpha-1}{\alpha}\frac{\rho}{f^2(\rho)}\big(\frac{\mu'(\rho)}{\mu(\rho)}f(\rho)-f'(\rho)\big)F_1(\rho). \tag{5.212}
$$

with from (5.181):

$$
\begin{aligned}
g_2 &= 2(\alpha-1)\partial_x f(\rho)\frac{\mu(\rho)}{\rho}\frac{f'(\rho)}{f(\rho)^2}\partial_x\rho + \frac{(\alpha-1)}{\alpha^2}\mu'(\rho)\rho^2\frac{2\mu(\rho)-\mu'(\rho)\rho}{\mu(\rho)^3}(v-u)^2 \\
&+ (\alpha-1)\frac{(\partial_x\rho)^2(\mu''(\rho)\rho-\mu'(\rho))}{\rho^2} - (\alpha-1)\frac{\partial_x\mu(\rho)}{\rho}\frac{f'(\rho)}{f(\rho)}\partial_x\rho + \frac{F'(\rho)}{\alpha}\frac{\rho^2}{\mu(\rho)} \\
&- (\alpha-1)\frac{\mu(\rho)}{\rho}F_1'(\rho)\frac{\rho^2}{\alpha f(\rho)\mu(\rho)} - \frac{(\alpha-1)}{f(\rho)}\big((\partial_x\rho)^2\big(f'(\rho)\frac{\mu'(\rho)\rho-\mu(\rho)}{\rho^2}+\frac{\mu(\rho)}{\rho}f''(\rho)\big) \\
&+ \frac{1}{\alpha^2}\rho f'(\rho)(\frac{2\rho\mu(\rho)-\rho^2\mu'(\rho)}{\mu(\rho)^2})(v-u)^2\big).
\end{aligned} \tag{5.213}
$$

We have now since$(v-u)^2 = \frac{\alpha^2\mu(\rho)^2}{\rho^4}(\partial_x\rho)^2$:

$$
\begin{aligned}
g_2' &= (\alpha-1)(\partial_x\rho)^2\Bigg(2\frac{\mu(\rho)}{\rho}(\frac{f'(\rho)}{f(\rho)})^2 + \mu'(\rho)\frac{2\mu(\rho)-\mu'(\rho)\rho}{\rho^2\mu(\rho)} \\
&+ \frac{\mu''(\rho)\rho-\mu'(\rho)}{\rho^2} - \frac{\mu'(\rho)}{\rho}\frac{f'(\rho)}{f(\rho)} - \frac{1}{f(\rho)}\big(f'(\rho)\frac{\mu'(\rho)\rho-\mu(\rho)}{\rho^2}+\frac{\mu(\rho)}{\rho}f''(\rho) \\
&+ f'(\rho)\frac{2\mu(\rho)-\rho\mu'(\rho)}{\rho^2})\Bigg) \\
&+ \frac{F'(\rho)}{\alpha}\frac{\rho^2}{\mu(\rho)} - (\alpha-1)\frac{\mu(\rho)}{\rho}F_1'(\rho)\frac{\rho^2}{\alpha f(\rho)\mu(\rho)} - 2\frac{\alpha-1}{\alpha}\frac{\rho}{f^2(\rho)}\big(\frac{\mu'(\rho)}{\mu(\rho)}f(\rho)-f'(\rho)\big)F_1(\rho)
\end{aligned} \tag{5.214}
$$

and we get:

$$
\begin{aligned}
g_2' &= (\alpha-1)(\partial_x\rho)^2\Big(2\frac{\mu(\rho)}{\rho}\frac{(f'(\rho))^2}{f(\rho)^2}+\frac{\mu'(\rho)}{\rho^2}-\frac{(\mu'(\rho))^2}{\rho\mu(\rho)}+\frac{\mu''(\rho)}{\rho}-\frac{f'(\rho)}{f(\rho)}\frac{\mu'(\rho)}{\rho}\\
&-\frac{f'(\rho)}{f(\rho)}\frac{\mu(\rho)}{\rho^2}-\frac{f''(\rho)}{f(\rho)}\frac{\mu(\rho)}{\rho}\Big)\\
&+\frac{F'(\rho)}{\alpha}\frac{\rho^2}{\mu(\rho)}-(\alpha-1)\frac{\mu(\rho)}{\rho}F_1'(\rho)\frac{\rho^2}{\alpha f(\rho)\mu(\rho)}-2\frac{\alpha-1}{\alpha}\frac{\rho}{f^2(\rho)}\big(\frac{\mu'(\rho)}{\mu(\rho)}f(\rho)-f'(\rho)\big)F_1(\rho)
\end{aligned}
\tag{5.215}
$$

Using (5.190), it yields:

$$
\begin{aligned}
&2\frac{\mu(\rho)}{\rho}\frac{(f'(\rho))^2}{f(\rho)^2}+\frac{\mu'(\rho)}{\rho^2}-\frac{(\mu'(\rho))^2}{\rho\mu(\rho)}+\frac{\mu''(\rho)}{\rho}-\frac{f'(\rho)}{f(\rho)}\frac{\mu'(\rho)}{\rho}-\frac{f'(\rho)}{f(\rho)}\frac{\mu(\rho)}{\rho^2}-\frac{f''(\rho)}{f(\rho)}\frac{\mu(\rho)}{\rho}\\
&=\rho^{\beta-3}\Big(2(\frac{(\beta-1)\alpha-\beta}{2\alpha-1})^2+\beta-\beta^2+(\beta-1)\beta-(\beta+1)(\frac{(\beta-1)\alpha-\beta}{2\alpha-1})\\
&-(\frac{(\beta-1)\alpha-\beta}{2\alpha-1})(\frac{(\beta-1)\alpha-\beta}{2\alpha-1}-1)\Big)\\
&=-\alpha\rho^{\beta-3}\frac{(\beta-1)\alpha-\beta}{2\alpha-1}\frac{\beta+1}{2\alpha-1}
\end{aligned}
\tag{5.216}
$$

From (5.190), we have:

$$
\begin{aligned}
&\frac{F'(\rho)}{\alpha}\frac{\rho^2}{\mu(\rho)}-(\alpha-1)\frac{\mu(\rho)}{\rho}F_1'(\rho)\frac{\rho^2}{\alpha f(\rho)\mu(\rho)}-2\frac{\alpha-1}{\alpha}\frac{\rho}{f^2(\rho)}\big(\frac{\mu'(\rho)}{\mu(\rho)}f(\rho)-f'(\rho)\big)F_1(\rho)\\
&=\rho^{\gamma-\beta}(\frac{\gamma}{\alpha}-\frac{\alpha-1}{\alpha}\frac{\gamma}{2\alpha-1}-2\frac{\alpha-1}{\alpha}(\beta-\frac{(\beta-1)\alpha-\beta}{2\alpha-1})\frac{\gamma}{-\alpha(\beta+1)+\gamma(2\alpha-1)})\\
&=\rho^{\gamma-\beta}(\frac{\gamma}{2\alpha-1}-2\frac{(\alpha-1)(\beta+1)}{2\alpha-1}\frac{\gamma}{-\alpha(\beta+1)+\gamma(2\alpha-1)})\\
&=\rho^{\gamma-\beta}\frac{\gamma}{2\alpha-1}\frac{-3\alpha\beta-3\alpha+2\alpha\gamma-\gamma+2\beta+2}{-\alpha(\beta+1)+\gamma(2\alpha-1)}.
\end{aligned}
\tag{5.217}
$$

From (5.215), (5.216) and (5.217) we have finally:

$$
\begin{aligned}
g_2' &= -(\alpha-1)\alpha\rho^{\beta-3}\frac{(\beta-1)\alpha-\beta}{2\alpha-1}\frac{\beta+1}{2\alpha-1}(\partial_x\rho)^2\\
&+\rho^{\gamma-\beta}\frac{\gamma}{2\alpha-1}\frac{-3\alpha\beta-3\alpha+2\alpha\gamma-\gamma+2\beta+2}{-\alpha(\beta+1)+\gamma(2\alpha-1)}.
\end{aligned}
\tag{5.218}
$$

Combining now (5.186), (5.187), (5.188), (5.192),(5.193), (5.211) and (5.218) we can show the Proposition 3.1.

5.2 Proof of the blow-up criterion in the Theorem 3.3

We are now giving a sketch of the proof of the blow-up criterion in the Theorem 3.3. The part concerning the existence of strong solution in finite time is classical (see [13, 14]). We begin by observing that the Korteweg system (1.10) can be written under the following

form using the fact that $\rho^{\beta-2}\partial_x\rho = \frac{1}{\sqrt{1-4c}}(v_1 - v_2)$:

$$\begin{cases} \partial_t v_1 + u\partial_x v_1 - (1-\alpha_1)\partial_x\left(\rho^{\beta-1}\partial_x v_1\right) = -\frac{\gamma}{\sqrt{1-4c}}\rho^{\gamma-\beta}\left(v_1 - v_2\right) + \frac{1-\alpha_1}{\sqrt{1-4c}}(v_1 - v_2)\partial_x v_1, \\ \partial_t v_2 + u\partial_x v_2 - (1-\alpha_2)\partial_x\left(\rho^{\beta-1}\partial_x v_2\right) = -\frac{\gamma}{\sqrt{1-4c}}\rho^{\gamma-\beta}\left(v_1 - v_2\right) + \frac{1-\alpha_2}{\sqrt{1-4c}}(v_1 - v_2)\partial_x v_2 \end{cases} \tag{5.219}$$

Let us recall now classical Kato-Ponce comutator estimates in Sobolev spaces.

Lemma 1 (Kato-Ponce) *The following estimates holds true for $s > 0$ with $\mathcal{F}\Lambda_s f(\xi) = |\xi|^s \mathcal{F}f(\xi)$ for f a temperated distribution and $C > 0$ large enough:*

$$\|\Lambda_s\left(fg\right)\|_{L^2} \leq C(\|f\|_{L^\infty}\|\Lambda_s g\|_{L^2} + \|g\|_{L^\infty}\|\Lambda_s f\|_{L^2}), \tag{5.220}$$

$$\|\Lambda_s\left(f\partial_x g\right) - f\Lambda_s\partial_x g\|_{L^2} \leq C\left(\|\partial_x f\|_{L^\infty}\|\Lambda_s g\|_{L^2} + \|\Lambda_s f\|_{L^2}\|\partial_x g\|_{L^\infty}\right). \tag{5.221}$$

In the sequel we wish to describe how to preserve all along the time the H^s norm of v_1, v_2 and $\rho - 1$ for $s > \frac{3}{2}$. We apply the operator Λ_s to the system (5.219):

$$\begin{cases} \partial_t\Lambda_s v_1 + u\partial_x\Lambda_s v_1 - (1-\alpha_1)\partial_x\left(\rho^{\beta-1}\partial_x\Lambda_s v_1\right) = \Lambda_s\left(\frac{1-\alpha_1}{\sqrt{1-4c}}(v_1 - v_2)\partial_x v_1\right) - \Lambda_s\left(\frac{\gamma}{\sqrt{1-4c}}\rho^{\gamma-\beta}\left(v_1 - v_2\right)\right) \\ -\left[\Lambda_s, u\right]\partial_x v_1 - (1-\alpha_1)\partial_x\left(\left[\rho^{\beta-1}, \Lambda_s\right]\partial_x v_1\right), \\ \partial_t\Lambda_s v_2 + u\partial_x\Lambda_s v_2 - (1-\alpha_2)\partial_x\left(\rho^{\beta-1}\partial_x\Lambda_s v_2\right) = \Lambda_s\left(\frac{1-\alpha_2}{\sqrt{1-4c}}(v_1 - v_2)\partial_x v_2\right) - \Lambda_s\left(\frac{\gamma}{\sqrt{1-4c}}\rho^{\gamma-\beta}\left(v_1 - v_2\right)\right) \\ -\left[\Lambda_s, u\right]\partial_x v_2 - (1-\alpha_2)\partial_x\left(\left[\rho^{\beta-1}, \Lambda_s\right]\partial_x v_2\right), \end{cases} \tag{5.222}$$

Multiply the first equation with $\Lambda_s v_1$ and integrate over $\mathbb{R}$, we get that:

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}|\Lambda_s v_1|^2 + (1-\alpha_1)\int_{\mathbb{R}}\rho^{\beta-1}|\partial_x\Lambda_s v_1|^2 = \frac{1}{2}\int_{\mathbb{R}}\partial_x u\,|\Lambda_s v_1|^2 + \int_{\mathbb{R}}\Lambda_s\left(\frac{1-\alpha_1}{\sqrt{1-4c}}(v_1 - v_2)\partial_x v_1\right)\Lambda_s v_1$$
$$-\int_{\mathbb{R}}\Lambda_s\left(\frac{\gamma}{\sqrt{1-4c}}\rho^{\gamma-\beta}\left(v_1 - v_2\right)\right)\Lambda_s v_1 - \int_{\mathbb{R}}\left[\Lambda_s, u\right]\partial_x v_1\Lambda_s v_1 - (1-\alpha_1)\int_{\mathbb{R}}\partial_x\left(\left[\rho^{\beta-1}, \Lambda_s\right]\partial_x v_1\right)\Lambda_s v_1. \tag{5.223}$$

Multiplyng the second equation of (5.222) with $\Lambda_s v$ we obtain that:

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}|\Lambda_s v_2|^2 + (1-\alpha_2)\int_{\mathbb{R}}\rho^{\beta-1}|\partial_x\Lambda_s v_2|^2 = \frac{1}{2}\int_{\mathbb{R}}\partial_x u\,|\Lambda_s v_2|^2 + \int_{\mathbb{R}}\Lambda_s\left(\frac{1-\alpha_2}{\sqrt{1-4c}}(v_1 - v_2)\partial_x v_2\right)\Lambda_s v_2$$
$$-\int_{\mathbb{R}}\Lambda_s\left(\frac{\gamma}{\sqrt{1-4c}}\rho^{\gamma-\beta}\left(v_1 - v_2\right)\right)\Lambda_s v_2 - \int_{\mathbb{R}}\left[\Lambda_s, u\right]\partial_x v_2\Lambda_s v_2 - (1-\alpha_2)\int_{\mathbb{R}}\partial_x\left(\left[\rho^{\beta-1}, \Lambda_s\right]\partial_x v_2\right)\Lambda_s v_2. \tag{5.224}$$

If we add up (5.223) and (5.224), it yields that

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}|\Lambda_s v_1|^2 + (1-\alpha_1)\int_{\mathbb{R}}\rho^{\beta-1}|\partial_x\Lambda_s v_1|^2 + \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}|\Lambda_s v_2|^2 + (1-\alpha_2)\int_{\mathbb{R}}\rho^{\beta-1}|\partial_x\Lambda_s v_2|^2$$
$$= \frac{1}{2}\int_{\mathbb{R}}\partial_x u\,|\Lambda_s v_1|^2 + \int_{\mathbb{R}}\Lambda_s\left(\frac{1-\alpha_1}{\sqrt{1-4c}}(v_1 - v_2)\partial_x v_1\right)\Lambda_s v_1 + \frac{1}{2}\int_{\mathbb{R}}\partial_x u\,|\Lambda_s v_2|^2$$
$$+\int_{\mathbb{R}}\Lambda_s\left(\frac{1-\alpha_2}{\sqrt{1-4c}}(v_1 - v_2)\partial_x v_2\right)\Lambda_s v_2 - \int_{\mathbb{R}}\Lambda_s\left(\frac{\gamma}{\sqrt{1-4c}}\rho^{\gamma-\beta}\left(v_1 - v_2\right)\right)\Lambda_s v_1$$
$$-\int_{\mathbb{R}}\left[\Lambda_s, u\right]\partial_x v_1\Lambda_s v_1 - (1-\alpha_1)\int_{\mathbb{R}}\partial_x\left(\left[\rho^{\beta-1}, \Lambda_s\right]\partial_x v_1\right)\Lambda_s v_1 - \int_{\mathbb{R}}\Lambda_s\left(\frac{\gamma}{\sqrt{1-4c}}\rho^{\gamma-\beta}\left(v_1 - v_2\right)\right)\Lambda_s v_2$$
$$-\int_{\mathbb{R}}\left[\Lambda_s, u\right]\partial_x v_2\Lambda_s v_2 - (1-\alpha_2)\int_{\mathbb{R}}\partial_x\left(\left[\rho^{\beta-1}, \Lambda_s\right]\partial_x v_2\right)\Lambda_s v_2. \tag{5.225}$$

In the following lines, we analyse the different terms appearing in the left hand side of (5.225). The first four terms are treated in the following manner using Lemma 1 and for any $\varepsilon > 0$ with $C, C_\varepsilon > 0$ sufficiently large we have using Young inequality:

$$
\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}} \partial_x u\,|\Lambda_s v_1|^2 + \int_{\mathbb{R}} \Lambda_s\left(\frac{1-\alpha_1}{\sqrt{1-4c}}(v_1-v_2)\partial_x v_1\right)\Lambda_s v_1 + \frac{1}{2}\int_{\mathbb{R}} \partial_x u\,|\Lambda_s v_2|^2 \\
&+ \int_{\mathbb{R}} \Lambda_s\left(\frac{1-\alpha_2}{\sqrt{1-4c}}(v_1-v_2)\partial_x v_2\right)\Lambda_s v_2 \\
&\leq C\Big(\|\partial_x u\|_{L^\infty}\left(\|\Lambda_s v_1\|_{L^2}^2 + \|\Lambda_s v_2\|_{L^2}^2\right) + \|\Lambda_s v_1\|_{L^2}\big(\|\partial_x v_1\|_{L^\infty}\|\Lambda_s(v_1-v_2)\|_{L^2} \\
&+ \|\partial_x\Lambda_s v_1\|_{L^2}\|v_1-v_2\|_{L^\infty}\big) + \|\Lambda_s v_2\|_{L^2}\big(\|\partial_x v_2\|_{L^\infty}\|\Lambda_s(v_1-v_2)\|_{L^2} + \|\partial_x\Lambda_s v_2\|_{L^2}\|v_1-v_2\|_{L^\infty}\|\big)\Big), \\
&\leq C_\varepsilon\Big(\|\partial_x u\|_{L^\infty}\left(\|\Lambda_s v_1\|_{L^2}^2 + \|\Lambda_s v_2\|_{L^2}^2\right) + \|\Lambda_s(v_1-v_2)\|_{L^2}\big(\|\Lambda_s v_1\|_{L^2}\|\partial_x v_1\|_{L^\infty} + \|\Lambda_s v_2\|_{L^2}\|\partial_x v_2\|_{L^\infty}\big) \\
&+ \|v_1-v_2\|_{L^\infty}^2\|\rho^{1-\beta}\|_{L^\infty}\|\Lambda_s v_1\|_{L^2}^2 + \|v_1-v_2\|_{L^\infty}^2\|\rho^{1-\beta}\|_{L^\infty}\|\Lambda_s v_2\|_{L^2}^2\Big) \\
&+ \varepsilon\|\rho^{\frac{\beta-1}{2}}\partial_x\Lambda_s v_2\|_{L^2} + \varepsilon\|\rho^{\frac{\beta-1}{2}}\partial_x\Lambda_s v_1\|_{L^2}.
\end{aligned}
\tag{5.226}
$$

We have now for $C > 0$ large enough:

$$
\begin{aligned}
&-\int_{\mathbb{R}} \Lambda_s\left(\frac{\gamma}{\sqrt{1-4c}}\rho^{\gamma-\beta}\,(v_1-v_2)\right)\Lambda_s v_1 - \int_{\mathbb{R}} \Lambda_s\left(\frac{\gamma}{\sqrt{1-4c}}\rho^{\gamma-\beta}\,(v_1-v_2)\right)\Lambda_s v_2 \\
&\leq C(\left\|\rho^{\gamma-\beta}\right\|_{L^\infty} + 1)\left(\|\Lambda_s v_1\|_{L^2} + \|\Lambda_s v_2\|_{L^2}\right)\left(\|\Lambda_s v_1\|_{L^2} + \|\Lambda_s v_2\|_{L^2}\right) \\
&\qquad + \|(v_1-v_2)\|_{L^\infty}\left\|\Lambda_s\left(\rho^{\gamma-\beta}-1\right)\right\|_{L^2}\left(\|\Lambda_s v_1\|_{L^2} + \|\Lambda_s v_2\|_{L^2}\right).
\end{aligned}
\tag{5.227}
$$

In a similar way using Lemma 1, we get for $C > 0$ large enough:

$$
\begin{aligned}
&-\int_{\mathbb{R}} [\Lambda_s, u]\,\partial_x v_1\Lambda_s v_1 - \int_{\mathbb{R}} [\Lambda_s, u]\,\partial_x v_2\Lambda_s v_2 \leq C\|\Lambda_s v_1\|_{L^2}(\|\Lambda_s v_1\|_{L^2}\|\partial_x u\|_{L^\infty} \\
&+ \|\Lambda_s u\|_{L^2}\|\partial_x v_1\|_{L^\infty}) + C\|\Lambda_s v_2\|_{L^2}(\|\Lambda_s v_2\|_{L^2}\|\partial_x u\|_{L^\infty} + \|\Lambda_s u\|_{L^2}\|\partial_x v_2\|_{L^\infty}).
\end{aligned}
\tag{5.228}
$$

The last term are treated as follows using the fact that $\rho^{\beta-2}\partial_x\rho = \frac{1}{\sqrt{1-4c}}(v_1-v_2)$ for any $\varepsilon > 0$ and $C_\varepsilon; C_1 > 0$ large enough we get:

$$
\begin{aligned}
&-(1-\alpha_1)\int_{\mathbb{R}} \partial_x\left(\left[\rho^{\beta-1}, \Lambda_s\right]\partial_x v_1\right)\Lambda_s v_1 - (1-\alpha_2)\int_{\mathbb{R}} \partial_x\left(\left[\rho^{\beta-1}, \Lambda_s\right]\partial_x v_2\right)\Lambda_s v_2. \\
&\leq C(\left\|\left[\rho^{\beta-1}, \Lambda_s\right]\partial_x v_1\right\|_{L^2}\|\partial_x\Lambda_s v_1\|_{L^2} + \left\|\left[\rho^{\beta-1}, \Lambda_s\right]\partial_x v_2\right\|_{L^2}\|\partial_x\Lambda_s v_2\|_{L^2}) \\
&\leq C_1\|\partial_x\Lambda_s v_1\|_{L^2}\big(\|\partial_x\rho^{\beta-1}\|_{L^\infty}\|\Lambda_s v_1\|_{L^2} + \|\Lambda_s(\rho^{\beta-1}-1)\|_{L^2}\|\partial_x v_1\|_{L^\infty}\big) \\
&+ C_1\|\partial_x\Lambda_s v_2\|_{L^2}\big(\|\partial_x\rho^{\beta-1}\|_{L^\infty}\|\Lambda_s v_2\|_{L^2} + \|\Lambda_s(\rho^{\beta-1}-1)\|_{L^2}\|\partial_x v_2\|_{L^\infty}\big) \\
&\leq C_\varepsilon\left\|\rho^{1-\beta}\right\|_{L^\infty}\left(\left\|\partial_x\rho^{\beta-1}\right\|_{L^\infty}\|\Lambda_s v_1\|_{L^2} + \|\partial_x v_1\|_{L^\infty}\left\|\Lambda_s\left(\rho^{\beta-1}-1\right)\right\|_{L^2}\right)^2 + \varepsilon\left\|\rho^{\frac{\beta-1}{2}}\partial_x\Lambda_s v_1\right\|_{L^2}^2
\end{aligned}
$$

$$
\begin{aligned}
&+C_\varepsilon \left\|\rho^{1-\beta}\right\|_{L^\infty}\left(\left\|\partial_x\rho^{\beta-1}\right\|_{L^\infty}\|\Lambda_s v_2\|_{L^2}+\|\partial_x v_2\|_{L^\infty}\left\|\Lambda_s\left(\rho^{\beta-1}-1\right)\right\|_{L^2}\right)^2+\varepsilon\left\|\rho^{\frac{\beta-1}{2}}\partial_x\Lambda_s v_2\right\|_{L^2}^2\\
&\leq C_\varepsilon \left\|\rho^{1-\beta}\right\|_{L^\infty}\left(\|v_1-v_2\|_{L^\infty}\|\Lambda_s v_1\|_{L^2}+\|\partial_x v_1\|_{L^\infty}\left\|\Lambda_s\left(\rho^{\beta-1}-1\right)\right\|_{L^2}\right)^2+\varepsilon\left\|\rho^{\frac{\beta-1}{2}}\partial_x\Lambda_s v_1\right\|_{L^2}^2\\
&+C_\varepsilon \left\|\rho^{1-\beta}\right\|_{L^\infty}\left(\|v_1-v_2\|_{L^\infty}\|\Lambda_s v_2\|_{L^2}+\|\partial_x v_2\|_{L^\infty}\left\|\Lambda_s\left(\rho^{\beta-1}-1\right)\right\|_{L^2}\right)^2+\varepsilon\left\|\rho^{\frac{\beta-1}{2}}\partial_x\Lambda_s v_2\right\|_{L^2}^2.
\end{aligned}
\tag{5.229}
$$

Let us observe that in the estimates (5.227) (5.229) we have to treat the H^s-norm of $\rho^{\gamma-\beta}-1$ and $\rho^{\beta-1}-1$. This is now our goal. For each $\theta\neq 0$, we may write that

$$
\partial_t\rho^\theta+u\partial_x\rho^\theta=-\theta\rho^\theta\partial_x u.
$$

and consequently applying Λ_s to the previous equality we have:

$$
\partial_t\Lambda_s\left(\rho^\theta-1\right)+u\partial_x\Lambda_s\left(\rho^\theta-1\right)=-\theta\Lambda_s\left(\rho^\theta\partial_x u\right)-\left[\Lambda_s,u\right]\partial_x\left(\rho^\theta-1\right).
$$

We get then for any $\varepsilon>0$ and $C,C_\varepsilon>0$ large enough:

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}\left|\Lambda_s\left(\rho^\theta-1\right)\right|^2\leq\frac{1}{2}\|\partial_x u\|_{L^\infty}\left\|\Lambda_s\left(\rho^\theta-1\right)\right\|_{L^2}^2\\
&+C\theta\left(\|\partial_x u\|_{L^\infty}\left\|\Lambda_s\left(\rho^\theta-1\right)\right\|_{L^2}+(\left\|\left(\rho^\theta-1\right)\right\|_{L^\infty}+1)\|\Lambda_s\partial_x u\|_{L^2}\right)\left\|\Lambda_s\left(\rho^\theta-1\right)\right\|_{L^2}\\
&+C\left(\|\partial_x u\|_{L^\infty}\left\|\Lambda_s\left(\rho^\theta-1\right)\right\|_{L^2}+\left\|\partial_x\rho^\theta\right\|_{L^\infty}\|\Lambda_s u\|_{L^2}\right)\left\|\Lambda_s\left(\rho^\theta-1\right)\right\|_{L^2}\\
&\leq C_\varepsilon\left(\|\partial_x u\|_{L^\infty}+\left\|\rho^{1-\beta}\right\|_{L^\infty}(\left\|\left(\rho^\theta-1\right)\right\|_{L^\infty}^2+1)+\left\|\rho^{\theta+1-\beta}\right\|_{L^\infty}^2\|v_1-v_2\|_{L^\infty}^2\right)\left\|\Lambda_s\left(\rho^\theta-1\right)\right\|_{L^2}^2\\
&+\varepsilon\|\Lambda_s u\|_{L^2}^2+\varepsilon\left\|\rho^{\frac{\beta-1}{2}}\partial_x\Lambda_s u\right\|_{L^2}^2
\end{aligned}
\tag{5.230}
$$

Thus putting together the estimates (5.225), (5.226),(5.227),(5.228),(5.229) and (5.230) for $\theta=\beta-1$, $\gamma-\beta$ we get choosing $\varepsilon>0$ small enough and applying Gronwall Lemma, there exists $C>0$ large enough such that:

$$
\begin{aligned}
&\int_{\mathbb{R}}\frac{1}{2}(|\Lambda_s v_1|^2+|\Lambda_s v_2|^2+|\Lambda_s(\rho^{\beta-1}-1)|^2+|\Lambda_s(\rho^{\gamma-\beta}-1)|^2)\\
&+\frac{(1-\alpha_1)}{2}\int_0^t\int_{\mathbb{R}}\rho^{\beta-1}\left|\partial_x\Lambda_s v_1\right|^2(s,x)dsdx+(1-\alpha_2)2\int_0^t\int_{\mathbb{R}}\rho^{\beta-1}\left|\partial_x\Lambda_s v_2\right|^2(s,x)dsdx\\
&\leq C\left(u_0,\rho_0\right)\exp\left(C\int_0^t\left(1+\left\|\left(\rho,\frac{1}{\rho}\right)\right\|_{L^\infty}\right)^\delta\left(1+\|(v_1,v_2,\partial_x v_1,\partial v_2)\|_{L^\infty}^2\right)\right)
\end{aligned}
\tag{5.231}
$$

with δ depending on β and γ. We mention also that $C\left(u_0,\rho_0\right)$ depends on $\|u_0\|_{H^s}$, $\|\rho_0-1\|_{H^s}$, $\|\rho_0\|_{L^\infty}$ and $\|\frac{1}{\rho_0}\|_{L^\infty}$. Let us denote now by:

$$
\widetilde{A}_i\left(\rho,v_i\right)(t)=\frac{(1-\alpha_i)}{2}\int_{\mathbb{R}}\rho^\beta(t)\left(\partial_x v_i\right)^2(t)+\int_0^t\int_{\mathbb{R}}\rho\dot{v_i}^2.
\tag{5.232}
$$

with $i\in\{1,2\}$. Using the same techniques as in the section on the Hoff estimates but with $\sigma(t)=1$ now, we may show that for $\delta_2,C>0$ sufficiently large

$$
\widetilde{A_i}\left(\rho,v_i\right)(t)\leq C_0\exp\left(Ct\left(1+\left\|(\rho,\frac{1}{\rho})\right\|_{L_t^\infty(L^\infty)}\right)^{\delta_2}\right)
\tag{5.233}
$$

which ensures a control on $\|\partial_x v_i\|_{L^2_t(L^\infty)}$ for $i \in \{1,2\}$ as in (3.76) provided that we control $\|(\rho, \frac{1}{\rho})\|_{L^\infty_t(L^\infty)}$ To summarize:

- The Bresch-Desjardins entropy provides control on $\|\rho\|_{L^\infty_t(L^\infty_x)}$ for any $t > 0$,
- The Hoff-type estimates ensure that $\|(\partial_x v_1, \partial_x v_2)\|_{L^2_t(L^\infty_x)}$ is controlled by $\left\|\left(\rho, \frac{1}{\rho}\right)\right\|_{L^\infty_t(L^\infty)}$
- The energy estimates (1.17) yield using Sobolev embedding that $\|(v_1, v_2)\|_{L^2_t(L^\infty_x)}$ is controlled by $\left\|\left(\rho, \frac{1}{\rho}\right)\right\|_{L^\infty_t(L^\infty)}$.

Taking into account the estimate (5.231) we get that for any $T > 0$ and any $s > \frac{3}{2}$ the H^s-Sobolev norm of $(u, v_1, v_2, \rho - 1)$ is uniformly controlled by $\left\|\frac{1}{\rho}\right\|_{L^\infty_t(L^\infty)}$. It explains why we get a blow-uo criterion depending on the quantity $\left\|\frac{1}{\rho}\right\|_{L^\infty_{T^*}(L^\infty)}$ when T^* is the maximal time interval for a strong solution.

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