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To cite this version:

Sadek Alharbat, Corinne Blondel. POINCARÉ POLYNOMIAL FOR FULLY COMMUTATIVE ELEMENTS IN THE SYMMETRIC GROUP. 2020. hal-03046972

HAL Id: hal-03046972

https://hal.science/hal-03046972

Preprint submitted on 8 Dec 2020

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arXiv:2010.03417v1 [math.CO] 7 Oct 2020

POINCARÉ POLYNOMIAL FOR FULLY COMMUTATIVE ELEMENTS IN THE SYMMETRIC GROUP

SADEK AL HARBAT AND CORINNE BLONDEL

S.Rogers: We need a plan of attack!

T.Stark: I have a plan: ... attack!!
AVENGERS 2012, M.C.U.

ABSTRACT. Let $W^c(A_n)$ be the set of fully commutative elements of the Coxeter group $W(A_n)$. Let

$$a_n(q) = \sum_{w \in W^c(A_n)} q^{l(w)}.$$

We compute $a_n(q)$.

CONTENTS

1. Introduction 2
2. Main recurrence relation and its general solution 4
2.1. Fully commutative elements of type A 4
2.2. Groundwork: two recurrence relations 5
2.3. Recurrence relation for the Poincaré polynomial 8
2.4. Value at 1 and Catalan number 8
3. A family of polynomials 8
3.1. First step 8
3.2. The polynomials Π and $\Sigma\Pi$ 9
3.3. The polynomials b_j^k 10
4. Conclusion and further questions 15
4.1. A general formula 15
4.2. Matrix interpretation 16
4.3. The Poincaré polynomial for $W^c(A_n)$ 17
4.4. A slightly shorter formula 17
4.5. Extensive formulas 20
REFERENCES 21

Date: October 8, 2020.
Sadek Al Harbat was supported by Fondecyt Postdoctoral grant 3170544.

1. Introduction

Full-commutativity. In a Coxeter system (W, S) a fully commutative element (say an FC element) is an element of which any reduced expression can be arrived to from any other by commutation relations.

We focus here on the Coxeter system $(W(A_n), S)$, $S = \{\sigma_1, \cdots, \sigma_n\}$, that can be viewed as the symmetric group $\mathfrak{S}_{n+1}$ of permutations of $\{1, \cdots, n+1\}$, generated by the n elementary transpositions $\sigma_i = (i, i+1)$ for $1 \leq i \leq n$. FC elements in $W(A_n)$ (AKA 321-*avoiding permutations* of the symmetric group), forming the subset denoted by $W^c(A_n)$, were used in the famous work of V. Jones [9] before they were officially defined and studied, for example in the works of Graham and Stembrige [6,11]. Nowadays the study of full-commutativity has taken its own place, forming a nice theory relating the Coxeter groups theory to many others: not starting by diagram algebras and not ending by algebraic combinatorics.

Poincaré polynomials. For a given Coxeter system (W, S) and a subset H of W we define:

$$H(q) = \sum_{0 \leq i} \#\{w \in H; l(w) = i\}\ q^i = \sum_{w \in H} q^{l(w)}. \tag{1}$$

Among the various conventions in the literature, we choose, following Bourbaki [5], Humphreys [8], Bjorner and Brenti [4], to call it the Poincaré series of H (or Poincaré polynomial of H, if H is finite) rather than another meaningful name: "the Coxeter length generating fonction" of H. Actually we focus on the series of polynomials $a_n = W^c(A_n)(q)$. Let

$$f(x, q) = \sum_{0 \leq n} a_n x^n.$$

In the literature many authors, seeking brevity, refer to $f(x, q)$ as the length generating fonction of $W^c(A_n)$, when it is really the generating fonction of the Poincaré polynomial or the generating fonction of the length generating fonction, not to mention the fact that sometimes even a three-variable series $f(x, y, q)$ is called a length generating fonction. So we choose to call a_n the Poincaré polynomial of $W^c(A_n)$ for the sake of distinguishing it from generating fonctions.

A generating fonction of the Poincaré polynomial for FC elements in type A was studied in [2] and was extended to the affine type $\tilde{A}$ in [7], while another expression for $\tilde{A}$ and formulas for FC elements in other affine cases appear in [3]. In this work we do not use generating fonctions, not even as a technical tool. In a forthcoming work we compute directly the Poincaré polynomial of $W^c(\tilde{A}_n)$, in which $W^c(A_n)$ forms the subset of elements of affine length 0, which is the most difficult case, while

for affine lengths equal to and greater than 1, the computation is remarkably easier, using the normal form established in [1], where the notion of affine length is defined.

More generally, our method of computation of the Poincaré series, starting from the normal forms of FC elements in the four infinite families of affine Coxeter groups, namely $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ and $\tilde{D}$, established by the first author, reduces this computation to elements of affine length 0, that is, to the Poincaré polynomial for FC elements in the three infinite families of finite Coxeter groups A, B and D. Among those three, the case A_n is the generic case. From these facts comes the importance of the Poincaré polynomial for FC elements in type A which is the focus of this work.

Catalan level. The starting point of this work is a partition of $W^c(A_n)$, the cardinal of which is the famous Catalan number $C_{n+1} = \frac{1}{n+2}\binom{2(n+1)}{n+1}$. In a future paper we explain many other partitions to re-count $W^c(A_n)$ and to count many distinguished subsets of $W^c(A_n)$, that give many interesting partitions of the Catalan number among which Narayana numbers and the Catalan triangle, the latter coming from the very partition that we use in this work (see Remark 2.3); we go down from the polynomials to the numbers related to the Catalan number by specializing q to 1. In this work we compute explicitly $\sum_{w\in W^c(A_n)} q^{l(w)}$ rather than $\sum_{0\leq i} \#\{w \in W^c(A_n); l(w) = i\}q^i$, in which specializing q to 1 gives a new partition of the Catalan number with the "Coxeter" color all over it, so we have an explicit "pretty sophisticated" a_n, which is the next-to-last step to answer the question: let r be a positive integer, how many elements do we have in $W^c(A_n)$ which are of length equal to r? That is: explaining the obvious equality of definition (1), supposing that H is $W^c(A_n)$.

The paper is organized as follows.

In section 2, we recall a normal form for FC elements in $W(A_n)$, following Stembridge, and we partition the set $W^c(A_n)$ into $\{1\}$ and the subsets A_n^j, $1 \leq j \leq n$, of FC elements having a normal form with rightmost element the j-th generator. We write a recurrence relation for the Poincaré polynomials $a_n^j = A_n^j(q)$ and obtain the quite intriguing fact that a_n^j is a linear combination of a_{n-1}, ..., a_{n-j} over $\mathbb{Z}[q]$, with coefficients depending on j, not on n, namely (6):

$$a_n^j = \sum_{k=1}^{j} q^{j-k+1} B_j^k(q) a_{n-k} \quad (1 \leq j \leq n).$$

where the family of polynomials B_j^k in $\mathbb{Z}[q]$ is uniquely determined (Proposition 2.4). This leads us to the main recurrence relation (7) for the Poincaré polynomial:

$$q^n a_{n-1} = q + q^2 + \cdots + q^n - \sum_{k=2}^{n} q^{n-k+1} B_n^k a_{n-k} \qquad (n \geq 1).$$

We observe that the value at 1 of the polynomial B_j^k is plus or minus a binomial coefficient and that, by specializing q to 1, we obtain a recurrence relation for Catalan numbers very similar to the one attributed by Stanley to Ming Antu in [10, B1].

In section 3 we proceed to the computation of the polynomials B_j^k. It is in fact more convenient to compute the polynomials b_j^k defined by $b_j^k = B_j^{j-k+1}$. The family b_j^k for fixed j and variable k can be viewed as a family of polynomials intermediate between $b_j^{j-1} = 1 - q^2 - \cdots - q^j$ and $b_j^1 = (1-q^2)\cdots(1-q^j)$ and indeed we are led to define polynomials $\Pi(a,b) = \prod_{a \leq i \leq b}(1-q^i)$ for $1 \leq a \leq b$ and $\Sigma\Pi(a,b)[e_1, \cdots, e_u]$, the sum of all possible products obtained from $\Pi(a,b)$ by removing e_i consecutive terms $(1-q^j)\cdots(1-q^{j+e_i-1})$ and replacing them by $(-q^j)$. Using these as basic bricks, we obtain in Theorem 3.3 a formula for b_j^k. We do not know if this family of polynomials has been used in other contexts.

We start section 4 with a general expression for a sequence satisfying a recurrence relation of the same shape as (7) (Proposition 4.1). As a direct application, we get an expression for the Poincaré polynomial a_n in Theorem 4.2. Our goal is achieved, yet we find of interest to record a property of the sequence a_n that we noticed on the way, this is Proposition 4.3 that says that, up to a shift, the sequence (a_n) is the first column of the inverse matrix of (b_j^k). This leads to a slightly different formula for a_n, to finish we write both formulas extensively.

We thank Mathieu Florence and Luc Lapointe for their challenging comments.

2. Main recurrence relation and its general solution

2.1. **Fully commutative elements of type A.** Consider the A-type Coxeter group $W(A_n)$ (n being a positive integer; we let $W(A_0) = 1$) that has the following presentation by generators $\{\sigma_1, \cdots, \sigma_n\}$ and relations:

- $\sigma_i^2 = 1$ for $1 \leq i \leq n$;
- braid relations:
  - $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ where $1 \leq i \leq n-1$,
  - commutation relations:

$$\sigma_i\sigma_j = \sigma_j\sigma_i \text{ where } 1 \leq i, j \leq n \text{ and } |i-j| \geq 2.$$

Any element w in $W(A_n)$ can be expressed as a product of generators, say $w = \sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_s}$ with $s \geq 0$ (for $s = 0$ we have an empty product, equal to 1). Among such expressions of w, the ones for which s is minimal are called *reduced* and the corresponding (minimal) value of s is called the (Coxeter) length of w and denoted by $l(w)$.

Definition 2.1. *Elements w in $W(A_n)$ for which one can pass from any reduced expression to any other one only by applying commutation relations are called* fully commutative elements. *We denote by $W^c(A_n)$ the set of fully commutative elements in $W(A_n)$.*

Our aim in this paper is the computation of the length polynomial of fully commutative elements in $W(A_n)$, namely, denoting by q the indeterminate:

$$a_n = a_n(q) = \sum_{w \in W^c(A_n)} q^{l(w)} \quad (n \geq 0).$$

The basis of our enumeration of $W^c(A_n)$ is the following normal form of fully commutative elements, for which we refer to Stembridge. We let:

$$\lceil i, j \rceil = \sigma_i\sigma_{i+1}\ldots\sigma_j \text{ for } 1 \leq i \leq j \leq n\ .$$

Theorem 2.2. [12, Corollary 5.8] *Let n be a positive integer, then $W^c(A_n)$ is the set of elements of the form:*

$$\lceil i_1, j_1 \rceil \lceil i_2, j_2 \rceil \ldots \lceil i_p, j_p \rceil, \text{ with } 0 \leq p \leq n \text{ and } \begin{cases} n \geq j_1 > \cdots > j_p \geq 1, \\ n \geq i_1 > \cdots > i_p \geq 1, \\ j_t \geq i_t \text{ for } 1 \leq t \leq p. \end{cases} \tag{2}$$

It is well-known that the number of FC elements in $W^c(A_n)$ is the Catalan number $C_{n+1} = \frac{1}{n+2}\binom{2(n+1)}{n+1}$. We refer the reader to [12] for comments on this fact, and to the list of 214 appearances of the Catalan number established by Stanley in [10]: appearance number 107 indeed counts sequences of integers satisfying (2).

2.2. **Groundwork: two recurrence relations.** For $1 \leq j \leq n$, we define A_n^j as the set of fully commutative elements in $W(A_n)$ given by a normal form (2) that ends with σ_j on the right, in other terms such that $j_p = j$. We are looking for the length polynomial of these elements:

$$a_n^j = a_n^j(q) = \sum_{w \in A_n^j} q^{l(w)} \text{ for } 1 \leq j \leq n.$$

We make the convention that $a_n^j = 0$ for $j > n$ and $a_n^0 = 0$, for a reason that will appear soon. The length polynomial of elements in $W^c(A_n)$ is

$$a_n = 1 + \sum_{j=1}^{n} a_n^j \quad (n \geq 1), \quad a_0 = 1.$$

Remark 2.3. *We have already counted $a_n^j(1)$ which is the number of FC elements ending with σ_j on the right, that is:*

$$a_n^j(1) = \frac{j}{n+1}\binom{2n-j+1}{n}.$$

And this is but the famous Catalan's triangle, of which we give a full description in a forthcoming work. In other words, the relation above is a q-version of the Catalan triangle.

We partition A_n^j as follows. Let $w \in A_n^j$ for $1 \leq j \leq n$, given by its normal form (2). Then either σ_1 does not appear in w, i.e. $i_p > 1$, hence w corresponds, upon shifting the generators ($\sigma_i \mapsto \sigma_{i-1}$ for $i \geq 2$), to a unique $w' \in A_{n-1}^{j-1}$ with the same length; or σ_1 appears in w, i.e. $i_p = 1$, hence the rightmost bloc in w is $\sigma_1 \cdots \sigma_j$, of length j, and it is preceded on the left by either 1 or an element that, upon the same shift as before, belongs to $W^c(A_{n-1})$ with a normal form ending with some σ_s with s at least equal to j. We obtain:

$$a_n^j = a_{n-1}^{j-1} + q^j(1 + \sum_{s=j}^{n-1} a_{n-1}^s) \quad (1 \leq j \leq n). \tag{3}$$

We note that this relation does not hold for $j > n$ or $j = 0$. For $j = 1$ we get

$$a_n^1 = qa_{n-1}.$$

For $n \geq j \geq 2$ we can write as well

$$a_n^{j-1} = a_{n-1}^{j-2} + q^{j-1}(1 + \sum_{s=j-1}^{n-1} a_{n-1}^s).$$

Substracting q times this last equation to the previous one, we obtain

$$a_n^j - qa_n^{j-1} = a_{n-1}^{j-1} - qa_{n-1}^{j-2} - q^j a_{n-1}^{j-1} \quad (2 \leq j \leq n). \tag{4}$$

Using this relation for $j = 2$ gives:

$$a_n^2 = qa_n^1 + (1-q^2)a_{n-1}^1 = q^2 a_{n-1} + q(1-q^2)a_{n-2} \quad (n \geq 2).$$

We claim the following:

Proposition 2.4. *The initial conditions*

$$B_j^1(q) = 1 \text{ for } j \geq 1, \qquad B_j^j(q) = (1-q^2)\cdots(1-q^j) \text{ for } j \geq 2$$

and the recurrence relation:

$$B_j^k(q) = B_{j-1}^k(q) + (1-q^j)B_{j-1}^{k-1}(q) - B_{j-2}^{k-1}(q) \quad (2 \leq k \leq j-1) \tag{5}$$

define a unique family of polynomials $(B_j^k)_{1\leq k\leq j}$, in the variable q, with integer coefficients. Those polynomials satisfy $B_j^k(0) = 1$. We have the following equality:

$$a_n^j = \sum_{k=1}^{j} q^{j-k+1} B_j^k(q) a_{n-k} \quad (1 \leq j \leq n). \tag{6}$$

Proof. Existence and unicity of the family $(B_j^k)_{1\leq k\leq j}$ are an immediate consequence of the recurrence relation (5): thinking of (j,k) on a grid with j on the x axis and k on the y axis, we see that the knowledge of B_*^{k-1}, corresponding to the line $y = k-1$, and of B_k^k, the leftmost point on the line $y = k$, implies the knowledge of B_*^k, i.e. the line $y = k$. Since the initial conditions give us the bottom line $k = 1$ and the diagonal $j = k$, we are done. The value at 0 is easy.

We now prove (6) by induction on j. The cases $j = 1$ and $j = 2$ have been established above. We assume that (6) holds for any t with $1 \leq t < j \leq n$ and we use (4) to prove it for j, $3 \leq j \leq n$. Indeed:

$$\begin{aligned}
a_n^j &= qa_n^{j-1} + (1-q^j)a_{n-1}^{j-1} - qa_{n-1}^{j-2} \\
&= \sum_{k=1}^{j-1} q^{j-k+1} B_{j-1}^k(q) a_{n-k} + (1-q^j)(\sum_{k=1}^{j-1} q^{j-k} B_{j-1}^k(q) a_{n-1-k}) \\
&\qquad - \sum_{k=1}^{j-2} q^{j-k} B_{j-2}^k(q) a_{n-1-k} \\
&= \sum_{k=1}^{j-1} q^{j-k+1} B_{j-1}^k(q) a_{n-k} + (1-q^j)(\sum_{k=2}^{j} q^{j-k+1} B_{j-1}^{k-1}(q) a_{n-k}) \\
&\qquad - \sum_{k=2}^{j-1} q^{j-k+1} B_{j-2}^{k-1}(q) a_{n-k} \\
&= q^j B_{j-1}^1(q) a_{n-1} + \sum_{k=2}^{j-1} q^{j-k+1}(B_{j-1}^k(q) + (1-q^j)B_{j-1}^{k-1}(q) - B_{j-2}^{k-1}(q)) a_{n-k} \\
&\qquad + q(1-q^j) B_{j-1}^{j-1}(q) a_{n-j}
\end{aligned}$$

so (6) holds for j. □

In what follows we shorten $B_j^k(q)$ to B_j^k.

2.3. **Recurrence relation for the Poincaré polynomial.** Now (6) leads to:

$$a_n = 1 + \sum_{j=1}^{n}\sum_{k=1}^{j} q^{j-k+1} B_j^k a_{n-k} = 1 + \sum_{k=1}^{n}\left[\sum_{j=k}^{n} q^{j-k+1} B_j^k\right] a_{n-k}.$$

This is a recurrence relation that might allow to compute a_n from $a_{n-1}, \dots, a_0$. But we can do better. From the above we have, for $n \geq 1$:

$$a_n^n = \sum_{k=1}^{n} q^{n-k+1} B_n^k a_{n-k}.$$

We can also compute directly a_n^n: if $j_p = n$, then $p = 1$ and $1 \leq i_p \leq n$ so that

$$a_n^n = q + q^2 + \cdots + q^n = q\frac{1-q^n}{1-q}.$$

We get the recurrence relation:

$$q^n a_{n-1} = q + q^2 + \cdots + q^n - \sum_{k=2}^{n} q^{n-k+1} B_n^k a_{n-k} \qquad (n \geq 1). \tag{7}$$

We postpone to the last section the study of this recurrence relation. In the next section we compute the polynomials B_j^k, actually the related polynomials b_j^k.

2.4. **Value at 1 and Catalan number.** The family of values $B_j^k(1)$ is uniquely defined by Proposition 2.4 at $q = 1$, with recurrence relation

$$B_j^k(1) = B_{j-1}^k(1) - B_{j-2}^{k-1}(1) \quad (2 \leq k \leq j-1).$$

We check that $B_j^k(1) = (-1)^{k-1}\binom{j-k}{k-1}$ and write (7) at $q = 1$, replacing the various $a_k(1)$ by Catalan numbers. We get:

$$C_n = n - \sum_{k=2}^{n} (-1)^{k-1}\binom{n-k}{k-1} C_{n-k+1} \qquad (n \geq 1).$$

Hence (7) can be seen as a q-analog of the relation above and the polynomials B_j^k can be viewed as q-analogs of the coefficients $(-1)^{k-1}\binom{j-k}{k-1}$.

3. A family of polynomials

3.1. **First step.** Our recurrence relation (5) for the polynomials B_j^k translates into

$$b_j^k = b_{j-1}^{k-1} + (1-q^j)\, b_{j-1}^k - b_{j-2}^{k-1} \quad (2 \leq k \leq j-1) \tag{8}$$

for the polynomials $b_j^k = B_j^{j-k+1}$ and the initial conditions in Proposition 2.4 become:

$$b_j^j = 1 \text{ for } j \geq 1, \qquad b_j^1 = (1-q^2)\cdots(1-q^j) \text{ for } j \geq 2.$$

From (8) we have $b_j^{j-1} = b_{j-1}^{j-2} - q^j$, giving by iteration:

$$b_j^{j-1} = 1 - q^2 - \cdots - q^j = 1 - \psi(j-1) \tag{9}$$

where we define $\psi(k) = q^2 + \cdots + q^{k+1}$ for $k \geq 1$.

Remembering that $b_j^1 = (1-q^2)\cdots(1-q^j)$, and with the help of some computations for small j and k, we are led to consider the family b_j^k for fixed j and variable k as a family of polynomials intermediate between $1 - q^2 - \cdots - q^j$ and $(1-q^2)\cdots(1-q^j)$. They can be expressed in terms of the polynomials $\Sigma\Pi(a,b)[l_1, l_2, \cdots, l_u]$ that we define in the next subsection.

3.2. **The polynomials Π and $\Sigma\Pi$.** We define $\Pi(u,v) = 1$ for $u > v$ and

$$\Pi(u,v) = (1-q^u)(1-q^{u+1})\cdots(1-q^v) = \prod_{t=u}^{v}(1-q^t) \text{ for } 1 \leq u \leq v.$$

Starting with a product $\Pi(u,v)$, we make the following transformation attached to $s \geq 2$ and i, $u \leq i \leq v - s + 1$: we replace the subproduct $(1-q^i)\cdots(1-q^{i+s-1})$ by the monomial $-q^i$. We denote the resulting polynomial by $\Pi(u,v)[i(s)]$: s stands for the number of consecutive terms $(1-q^x)$ suppressed, which we will refer to as the *length of the gap*, and i means that the term of lowest degree suppressed in this gap is $(1-q^i)$, replaced by $-q^i$. For instance:

$$\Pi(4,10)[5(3)] = (1-q^4)(-q^5)(1-q^8)(1-q^9)(1-q^{10}).$$

Now let $s_1, \cdots, s_p$ be integers at least equal to 2 and let $i_1, \cdots, i_p$ satisfy

$$u \leq i_1,\ i_t + s_t \leq i_{t+1} \text{ for } 1 \leq t \leq p-1,\ i_p + s_p - 1 \leq v.$$

The notation

$$\Pi(u,v)[i_1(s_1), i_2(s_2), \cdots, i_p(s_p)]$$

is almost self-explanatory: we cut a gap of length s_t at i_t and replace it by $-q^{i_t}$. The conditions specify that we can have two gaps next to each other but not overlapping. For instance:

$$\Pi(4,10)[5(3), 8(2)] = (1-q^4)(-q^5)(-q^8)(1-q^{10}),$$
$$\Pi(4,10)[5(2), 8(3)] = (1-q^4)(-q^5)(1-q^7)(-q^8).$$

We write, for $a \leq b$ and $l \geq 2$:

$$\Sigma\Pi(a,b)[l] = \sum_{i=a}^{b-l+1} \Pi(a,b)[i(l)]$$

and observe that it is the maximal meaningful sum of terms $\Pi(a,b)[i(l)]$. Following this observation we notice that, in a term with gaps at $i_1, \ldots, i_u$, we may accept that the gaps neighbour each other, but not overlap each other. So if we are to write the

maximal meaningful sum of such terms, the relevant information is the sequence of the lengths of the gaps, say $(l_1, l_2, \cdots, l_u)$.

Definition 3.1. *Let l_1, l_2, ..., l_u be integers at least equal to 2. We write*

$$\Sigma\Pi(a,b)[l_1, l_2, \cdots, l_u] = \sum_{(i_1,i_2,\ldots,i_u)\in I} \Pi(a,b)[i_1(l_1), i_2(l_2), \cdots, i_u(l_u)] \qquad \textit{where}$$

$$I = \{(i_1, i_2, \cdots, i_u) \ / \ a \le i_1, i_t + l_t \le i_{t+1} \textit{ for } 1 \le t < u, i_u + l_u - 1 \le b\}.$$

We make the usual convention that the value is 0 if I is empty, which happens if and only if $a + l_1 + \cdots + l_u - 1 > b$.

Remark 3.2. *If $a + l_1 + \cdots + l_u - 1 = b$ i.e. $l_1 + \cdots + l_u = b - a + 1$, the set I has only one element and*

$$\Sigma\Pi(a,b)[l_1, l_2, \cdots, l_u] = (-1)^u q^{ua+(u-1)l_1+\cdots+l_{u-1}}.$$

3.3. **The polynomials b_j^k.** We are ready to write a formula for b_j^k.

Theorem 3.3. *Let $\psi(k) = q^2 + \cdots + q^{k+1}$ for $k \ge 1$ and let $(b_j^k)_{1\le k\le j}$ be the unique family of polynomials in the variable q satisfying the initial conditions*

$$b_j^j = 1 \textit{ for } j \ge 1, \qquad b_j^1 = (1-q^2)\cdots(1-q^j) \textit{ for } j \ge 2$$

and recurrence relation (8):

$$b_j^k = b_{j-1}^{k-1} + (1-q^j)\ b_{j-1}^k - b_{j-2}^{k-1} \quad (2 \le k \le j-1).$$

Then b_j^k is given, for $1 \le k \le j-1$, by the following formula:

$$b_j^k = \sum_{t=1}^{k} (1-\psi(t))\ b(j,k,t) \tag{10}$$

involving the polynomials $b(j,k,t)$ defined for $t = k$ by

$$b(j,k,k) = \Pi(k+2, j)$$

and for $1 \le t < k \le j-1$ by

$$b(j,k,t) = \sum_{u=1}^{\min\{j-k-1,k-t\}} \sum_{d_1+\cdots+d_u=k-t} \Sigma\Pi(t+2,j)[d_1+1, \cdots, d_u+1]$$

where the indices $d_1, \cdots, d_u$ are positive integers.

Before proceeding with the proof, we remark that the upper bound in u in the sum defining $b(j,k,t)$ may be increased whenever convenient. Indeed, for $u > k-t$, the sum over $(d_1, \cdots, d_u)$ such that $d_1 + \cdots + d_u = k-t$ is equal to 0 since the summation set is empty, whereas for $u > j-k-1$, the sum of the lengths of the gaps, equal to $u+k-t$, is greater than the available length $j-t-1$, so that the corresponding term $\Sigma\Pi$ is equal to 0 by convention.

Proof. The formula holds for $k = 1$ and $k = j - 1$ (recall (9)). The family (b_j^k) is unique by Proposition 2.4, and following the proof of this Proposition, it is enough to prove that, assuming b_{j-1}^{k-1}, b_{j-1}^k and b_{j-2}^{k-1} are given by (10), then relation (8) implies that b_j^k is also given by (10). We proceed, taking k such that $2 \leq k \leq j-2$.

We see b_{j-1}^{k-1}, b_{j-1}^k and b_{j-2}^{k-1} as sums of k or $k-1$ terms and write accordingly b_j^k as a sum of k terms, namely

$$b_j^k = \sum_{t=1}^{k} (1 - \psi(t))\beta(j, k, t)$$

where we let

$$\beta(j, k, t) = b(j-1, k-1, t) + (1 - q^j)b(j-1, k, t) - b(j-2, k-1, t)$$

and agree on $b(x, y, z) = 0$ if $z > y$.

We first note that

$$\beta(j, k, k) = (1 - q^j)\Pi(k+2, j-1) = \Pi(k+2, j) = b(j, k, k).$$

We look next at the terms in $t = k - 1$.

$$\begin{aligned}\beta(j, k, k-1) &= \Pi(k+1, j-1) - \Pi(k+1, j-2) \\ &\quad + (1 - q^j) \sum_{u=1}^{\min\{j-k-2,1\}} \sum_{d_1+\cdots+d_u=1} \Sigma\Pi(k+1, j-1)[d_1+1, \cdots, d_u+1].\end{aligned}$$

If $k = j - 2$ the middle term is 0 and we have

$$\Pi(j-1, j-1) - \Pi(j-1, j-2) = -q^{j-1} = \Sigma\Pi(j-1, j)[2] = b(j, j-2, j-3).$$

If $k < j - 2$ we claim that

$$\Pi(k+1, j-1) + (1 - q^j)\Sigma\Pi(k+1, j-1)[2] - \Pi(k+1, j-2) = \Sigma\Pi(k+1, j)[2].$$

Indeed: $\Sigma\Pi(k+1, j)[2] = \sum_{i=k+1}^{j-1} \Pi(k+1, j)[i(2)]$. In this sum, the terms for $i < j-1$ end with a $(1 - q^j)$ and their sum is equal to $(1 - q^j)\Sigma\Pi(k+1, j-1)[2]$. The last term, for $i = j-1$, is equal to $-q^{j-1}\Pi(k+1, j-2)$ which is the sum of the first and third terms above, q.e.d. We get

$$\beta(j, k, k-1) = b(j, k, k-1).$$

For $t < k-1$ we don't have such an equality, nonetheless we compute $\beta(j,k,t)$. We use the remark preceeding the proof to simplify the upper bound in u.

$$\begin{aligned}\beta(j,k,t) = &\sum_{u=1}^{k-t} \sum_{d_1+\cdots+d_u=k-1-t} \Sigma\Pi(t+2,j-1)[d_1+1,\cdots,d_u+1] \\ &+(1-q^j)\sum_{u=1}^{k-t} \sum_{d_1+\cdots+d_u=k-t} \Sigma\Pi(t+2,j-1)[d_1+1,\cdots,d_u+1] \\ &-\sum_{u=1}^{k-t} \sum_{d_1+\cdots+d_u=k-1-t} \Sigma\Pi(t+2,j-2)[d_1+1,\cdots,d_u+1].\end{aligned} \tag{11}$$

Let $d=(d_1,\cdots,d_u)$ and, following Definition 3.1, write $I_s(d)$ for the summation set of $\Sigma\Pi(t+2,s)[d_1+1,\cdots,d_u+1]$. We have:

$$I_s(d) = \{(i_1,i_2,\cdots,i_u) \ / \ t+2 \le i_1, i_t+d_t+1 \le i_{t+1} \text{ for } 1 \le t < u, i_u+d_u \le s\}.$$

Hence $I_{s-1}(d)$ is contained in $I_s(d)$, with complement

$$X_s(d) = \{(i_1,i_2,\cdots,i_u) \ / \ t+2 \le i_1, i_t+d_t+1 \le i_{t+1} \text{ for } 1 \le t < u, i_u+d_u = s\}.$$

We can write

$$\begin{aligned}&\Sigma\Pi(t+2,s)[d_1+1,\cdots,d_u+1] = \\ &\qquad (1-q^s)\ \Sigma\Pi(t+2,s-1)[d_1+1,\cdots,d_u+1] \\ &\quad + \sum_{(i_1,i_2,\ldots,i_u)\in X_s(d)} \Pi(t+2,s)[i_1(d_1+1),\cdots,i_u(d_u+1)].\end{aligned} \tag{12}$$

We analyse the sum over $X_s(d)$ according to the value of d_u.

- If $d_u = 1$, we let $d^- = (d_1,\cdots,d_{u-1})$. We have:

$$\begin{aligned}&\sum_{(i_1,i_2,\ldots,i_u)\in X_s(d)} \Pi(t+2,s)[i_1(d_1+1),\cdots,i_u(d_u+1)] \\ &\quad = -\,q^{s-1} \sum_{(i_1,i_2,\ldots,i_{u-1})\in I_{s-2}(d^-)} \Pi(t+2,s-2)[i_1(d_1+1),\cdots,i_{u-1}(d_{u-1}+1)] \\ &\quad = -\,q^{s-1}\ \Sigma\Pi(t+2,s-2)[d_1+1,\cdots,d_{u-1}+1].\end{aligned}$$

  This equality holds even for $u=1$ if we consider this last expression as meaning $-q^{s-1}\ \Pi(t+2,s-2)$ if $u=1$.
- If $d_u > 1$, we let $d' = (d_1,\cdots,d_{u-1},d_u-1)$. The condition $(i_1,i_2,\ldots,i_u) \in X_s(d)$ is equivalent to the condition $(i_1,i_2,\ldots,i_u) \in X_{s-1}(d')$. We get

$$\begin{aligned}&\sum_{(i_1,i_2,\ldots,i_u)\in X_s(d)} \Pi(t+2,s)[i_1(d_1+1),\cdots,i_u(d_u+1)] \\ &\quad = \sum_{(i_1,i_2,\ldots,i_u)\in X_{s-1}(d')} \Pi(t+2,s-1)[i_1(d'_1+1),\cdots,i_u(d'_u+1)].\end{aligned}$$

We sum up (12) over $(d_1, \cdots, d_u)$, for $s = j$:

(13)
$$\begin{aligned}
(1-q^j) &\sum_{d_1+\cdots+d_u=k-t} \Sigma\Pi(t+2, j-1)[d_1+1, \cdots, d_u+1] \\
&= \sum_{d_1+\cdots+d_u=k-t} \Sigma\Pi(t+2, j)[d_1+1, \cdots, d_u+1] \\
&\quad - \sum_{\substack{d_1+\cdots+d_u=k-t \\ d_u=1}} \sum_{(i_1,i_2,\ldots,i_u)\in X_j(d)} \Pi(t+2, j)[i_1(d_1+1), \cdots, i_u(d_u+1)] \\
&\quad - \sum_{\substack{d_1+\cdots+d_u=k-t \\ d_u>1}} \sum_{(i_1,i_2,\ldots,i_u)\in X_j(d)} \Pi(t+2, j)[i_1(d_1+1), \cdots, i_u(d_u+1)] \\
&= \sum_{d_1+\cdots+d_u=k-t} \Sigma\Pi(t+2, j)[d_1+1, \cdots, d_u+1] \\
&\quad + q^{j-1} \sum_{d_1+\cdots+d_{u-1}=k-t-1} \Sigma\Pi(t+2, j-2)[d_1+1, \cdots, d_{u-1}+1] \\
&\quad - \sum_{d_1+\cdots+d_u=k-t-1} \sum_{(i_1,i_2,\ldots,i_u)\in X_{j-1}(d)} \Pi(t+2, j-1)[i_1(d_1+1), \cdots, i_u(d_u+1)].
\end{aligned}$$

In these equalities we have $u \leq k-t$, otherwise all sums are 0, and if $u = k-t$ the last line is 0. For $u \leq k-t$, the term in u in $\beta(j, k, t)$ is then equal to:

(14)
$$\begin{aligned}
&\sum_{d_1+\cdots+d_u=k-1-t} [\Sigma\Pi(t+2, j-1)[d_1+1, \cdots, d_u+1] \\
&\qquad\qquad -\Sigma\Pi(t+2, j-2)[d_1+1, \cdots, d_u+1]] \\
&+ \sum_{d_1+\cdots+d_u=k-t} \Sigma\Pi(t+2, j)[d_1+1, \cdots, d_u+1] \\
&+ q^{j-1} \sum_{d_1+\cdots+d_{u-1}=k-t-1} \Sigma\Pi(t+2, j-2)[d_1+1, \cdots, d_{u-1}+1] \\
-&\sum_{d_1+\cdots+d_u=k-t-1} \sum_{(i_1,i_2,\ldots,i_u)\in X_{j-1}(d)} \Pi(t+2, j-1)[i_1(d_1+1), \cdots, i_u(d_u+1)].
\end{aligned}$$

Using (12) again we see that the first, second and last line add up to

$$-q^{j-1} \sum_{d_1+\cdots+d_u=k-t-1} \Sigma\Pi(t+2, j-2)[d_1+1, \cdots, d_u+1]$$

We can now compute $\beta(j,k,t)$.

$$\begin{aligned}\beta(j,k,t) = &\sum_{u=1}^{k-t}\Bigg[-q^{j-1}\sum_{d_1+\cdots+d_u=k-t-1}\Sigma\Pi(t+2,j-2)[d_1+1,\cdots,d_u+1]\\ &\qquad+\sum_{d_1+\cdots+d_u=k-t}\Sigma\Pi(t+2,j)[d_1+1,\cdots,d_u+1]\\ &\qquad+q^{j-1}\sum_{d_1+\cdots+d_{u-1}=k-t-1}\Sigma\Pi(t+2,j-2)[d_1+1,\cdots,d_{u-1}+1]\Bigg]\\ &= b(j,k,t)\\ &\quad+q^{j-1}\Bigg[-\sum_{u=1}^{k-t-1}\sum_{d_1+\cdots+d_u=k-t-1}\Sigma\Pi(t+2,j-2)[d_1+1,\cdots,d_u+1]\\ &\qquad\qquad+\sum_{u=0}^{k-t-1}\sum_{d_1+\cdots+d_u=k-t-1}\Sigma\Pi(t+2,j-2)[d_1+1,\cdots,d_u+1]\Bigg].\end{aligned}$$

The two last lines cancel one another, except for "$u=0$", for which we use our convention below (12): those terms come from terms with $u=1$ and $d_u=d_1=k-t=1$ which only happens if $t=k-1$, a case treated separately before. Here we have $t\le k-2$, giving finally:

$$\beta(j,k,t) = b(j,k,t).$$

We obtain:

$$\begin{aligned}b_j^k &= \sum_{t=1}^{k}(1-\psi(t))\beta(j,k,t)\\ &= (1-\psi(k))b(j,k,k)+(1-\psi(k-1))b(j,k,k-1)\\ &\qquad+\sum_{t=1}^{k-2}(1-\psi(t))b(j,k,t)\end{aligned}$$

which is exactly (10). □

Remark 3.4. *We have actually proved that the family of polynomials $b(j,k,t)$ defined in the Theorem for $1\le t\le k\le j-1$ and extended by $b(x,y,z)=0$ if $x>y$ and $z>y$, satisfies the following recurrence relation, for $2\le k\le j-2$:*

$$b(j,k,t) = b(j-1,k-1,t)+(1-q^j)\ b(j-1,k,t)-b(j-2,k-1,t).$$

4. Conclusion and further questions

4.1. **A general formula.** We established the following in the course of our computations. We do not know if it is a known formula, in any case we couldn't find a reference for it. We point out that in the Proposition below, we make no assumption on the double sequence b_n^i, in other words, in this subsection and the next, the notation b_n^i does not refer to the polynomials that we have computed in the previous section, nor does ψ refer to the function used before.

Proposition 4.1. *Let A be a commutative ring. Let $n \mapsto \psi(n)$, $n \geq 1$, be a function from $\mathbb{N}$ to A and let $(b_n^i)_{n\geq 2, 1\leq i\leq n-1}$ be a double sequence of elements in A. Let $(u_n)_{n\geq 1}$ be the sequence of elements in A defined by $u_1 = \psi(1)$ and the following recurrence relation:*

$$u_n = \psi(n) - \sum_{i=1}^{n-1} b_n^i \, u_i \qquad (n \geq 2). \tag{15}$$

Then we have, for $n \geq 1$:

$$u_n = \psi(n) + \sum_{i=1}^{n-1} \psi(i) \sum_{s=0}^{n-1-i} (-1)^{s+1} \sum_{n=v_0>\cdots>v_s>v_{s+1}=i} \prod_{u=0}^{s} b_{v_u}^{v_{u+1}}. \tag{16}$$

Proof. This holds for $n = 1$. We assume it holds for any $m < n$ and prove it for n.

$$\begin{aligned} u_n &= \psi(n) - \sum_{i=1}^{n-1} b_n^i \, u_i \\ &= \psi(n) - \sum_{i=1}^{n-1} b_n^i \left(\psi(i) + \sum_{j=1}^{i-1} \psi(j) \sum_{s=0}^{i-1-j} (-1)^{s+1} \sum_{i=v_0>\cdots>v_s>v_{s+1}=j} \prod_{u=0}^{s} b_{v_u}^{v_{u+1}} \right). \end{aligned}$$

We exchange the sums in i and j; a term $\psi(j)$ appears in the i-th term if and only if $i \geq j$, whence:

$$\begin{aligned} u_n &= \psi(n) - \sum_{j=1}^{n-1} \psi(j) \left(b_n^j + \sum_{i=j+1}^{n-1} b_n^i \sum_{s=0}^{i-1-j} (-1)^{s+1} \sum_{i=v_0>\cdots>v_s>v_{s+1}=j} \prod_{u=0}^{s} b_{v_u}^{v_{u+1}} \right) \\ &= \psi(n) + \sum_{j=1}^{n-1} \psi(j) \left(-b_n^j + \sum_{i=j+1}^{n-1} \sum_{s=0}^{i-1-j} (-1)^{s+2} \sum_{i=v_0>\cdots>v_s>v_{s+1}=j} b_n^i \prod_{u=0}^{s} b_{v_u}^{v_{u+1}} \right). \end{aligned}$$

We change the index s into $s-1$ with $1 \leq s \leq i-j$, then shift the indices in the last sum to let in $v_0 = n$:

$$\begin{aligned} u_n &= \psi(n) + \sum_{j=1}^{n-1} \psi(j) \left(-b_n^j + \sum_{i=j+1}^{n-1} \sum_{s=1}^{i-j} (-1)^{s+1} \sum_{n=v_0>i=v_1>\cdots>v_s>v_{s+1}=j} \prod_{u=0}^{s} b_{v_u}^{v_{u+1}} \right) \\ &= \psi(n) + \sum_{j=1}^{n-1} \psi(j) \left(-b_n^j + \sum_{s=1}^{n-1-j} (-1)^{s+1} \sum_{i=j+s}^{n-1} \sum_{n=v_0>i=v_1>\cdots>v_s>v_{s+1}=j} \prod_{u=0}^{s} b_{v_u}^{v_{u+1}} \right) \end{aligned}$$

where the coefficient of $\psi(j)$ is exactly

$$\sum_{s=0}^{n-1-j} (-1)^{s+1} \sum_{n=v_0>\cdots>v_s>v_{s+1}=j} \prod_{u=0}^{s} b_{v_u}^{v_{u+1}}$$

as expected. □

4.2. **Matrix interpretation.** Recurrence relation (15) can be viewed in matrix form as:

$$PU = \Psi \qquad \text{i.e. } U = P^{-1}\Psi$$

with

$$U = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \\ \vdots \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 0 & \cdots & \cdots & \cdots \\ b_2^1 & 1 & 0 & \cdots & \cdots \\ \vdots & \ddots & \ddots & \ddots & \cdots \\ b_n^1 & \cdots & b_n^{n-1} & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}, \quad \Psi = \begin{pmatrix} \psi(1) \\ \psi(2) \\ \vdots \\ \psi(n) \\ \vdots \end{pmatrix}.$$

We also write

$$P^{-1} = \begin{pmatrix} 1 & 0 & \cdots & \cdots & \cdots \\ c_2^1 & 1 & 0 & \cdots & \cdots \\ \vdots & \ddots & \ddots & \ddots & \cdots \\ c_n^1 & \cdots & c_n^{n-1} & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

and obtain

$$u_n = \psi(n) + \sum_{i=1}^{n-1} c_n^i \psi(i) = \sum_{i=1}^{n} c_n^i \psi(i).$$

The k-th column Q_k of P^{-1} satisfies $PQ_k = L_k$, where L_k is the k-th column of the identity matrix, so by the general formula (16) we have

$$(Q_k)_n = (L_k)_n + \sum_{i=1}^{n-1} (L_k)_i \sum_{s=0}^{n-1-i} (-1)^{s+1} \sum_{n=v_0>\cdots>v_s>v_{s+1}=i} \prod_{u=0}^{s} b_{v_u}^{v_{u+1}}$$

Indeed for $n < k$ this formula yields 0, for $n = k$ it yields 1, and for $n > k$ it yields

$$c_n^k = \sum_{s=0}^{n-1-k} (-1)^{s+1} \sum_{n=v_0>\cdots>v_s>v_{s+1}=k} \prod_{u=0}^{s} b_{v_u}^{v_{u+1}}.$$

In other words, formula (16) amounts to the calculation of the inverse of a triangular matrix, which is most likely known,

4.3. **The Poincaré polynomial for $W^c(A_n)$.** We now apply Proposition 4.1 to our sequence $(a_n)_{n\geq 0}$. The dictionary between (7), multiplied by q for convenience, and (15) is:

$$u_n = q^{n+1}a_{n-1}, \quad \psi(n) = q^2 + \cdots + q^{n+1}, \quad b_n^i = B_n^{n-i+1}.$$

We get immediately formula (17) in the following Theorem:

Theorem 4.2. *For $1 \leq k \leq j$ we let $(b_j^k)_{1\leq k\leq j}$ be the unique family of polynomials described in Theorem 3.3. The Poincaré polynomial $a_n(q)$ is given, for $n \geq 1$, by the following formula:*

$$\begin{aligned} q^{n+2}a_n =& q^2 + \cdots + q^{n+2} \\ &+ \sum_{i=1}^{n} (q^2 + \cdots + q^{i+1}) \sum_{s=0}^{n-i}(-1)^{s+1} \sum_{n+1=v_0>\cdots>v_s>v_{s+1}=i} \prod_{u=0}^{s} b_{v_u}^{v_{u+1}}. \end{aligned} \tag{17}$$

4.4. **A slightly shorter formula.** We use the notation in subsection 4.2 and consider the first column $(c_i^1)_{i\geq 1}$ of the inverse matrix of $P = (b_i^j)$ (notation extended to $b_i^j = 0$ for $j > i$). In the case of the Poincaré polynomial, there is actually a shortcut in the computation:

Proposition 4.3.

$$c_{n+2}^1 = -(1-q^2)u_n = -(1-q^2)q^{n+1}a_{n-1} \quad \textit{for } n \geq 1.$$

As in subsection 4.2, we get a matrix form of this identity as follows:

$$SQ_1 = SP^{-1}L_1 = -(1-q^2)U = -(1-q^2)P^{-1}\Psi$$

where S is the matrix of the double shift:

$$S = \begin{pmatrix} 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ \vdots & & \ddots & \ddots & \ddots \\ \vdots & & & \ddots & \ddots \end{pmatrix}, \; SU = \begin{pmatrix} u_1' \\ u_2' \\ \vdots \\ u_n' \\ \vdots \end{pmatrix}, u_i' = u_{i+2}.$$

This amounts to $PSP^{-1}L_1 = -(1-q^2)\Psi$, that is, the first column of PSP^{-1} is $-(1-q^2)\Psi$. This formulation is equivalent to the Proposition and gives relations involving the b_j^k. We write the generic one:

$$\sum_{k=1}^{n} b_n^k c_{k+2}^1 = -(1-q^2)\psi(n) \tag{18}$$

i.e.

$$(1-q^2)\psi(n) = -\sum_{k=1}^{n} b_n^k \sum_{s=0}^{k}(-1)^{s+1} \sum_{k+2=v_0>\cdots>v_s>v_{s+1}=1} \prod_{u=0}^{s} b_{v_u}^{v_{u+1}}.$$

We proceed to the proof of the Proposition.

Proof. Since the sequence $(-(1-q^2)u_n)_{n\geq 1}$ is uniquely determined by relation

$$-(1-q^2)u_n = -(1-q^2)\psi(n) - \sum_{i=1}^{n-1} b_n^i \ (-(1-q^2)u_i) \qquad (n\geq 2) \tag{19}$$

as in (15), together with the first term $-(1-q^2)u_1 = -(1-q^2)q^2$, it is enough to prove that the sequence $(c_{n+2}^1)_{n\geq 1}$ satisfies the same conditions, as announced in (18).

We check the first term. Indeed:

$$\begin{aligned} c_3^1 &= \sum_{s=0}^{1}(-1)^{s+1} \sum_{3=v_0>\cdots>v_s>v_{s+1}=1} \prod_{u=0}^{s} b_{v_u}^{v_{u+1}} \\ &= -b_3^1 + b_3^2 b_2^1 \\ &= -(1-q^2)(1-q^3) + (1-q^2-q^3)(1-q^2) \\ &= -q^2(1-q^2). \end{aligned}$$

Now we must show (19) for $(c_{n+2}^1)_{n\geq 1}$, namely:

$$c_{n+2}^1 = -(1-q^2)\psi(n) - \sum_{i=1}^{n-1} b_n^i \ c_{i+2}^1 \qquad (n\geq 2), \tag{20}$$

whereas $(c_{n+2}^1)_{n\geq 1}$ satisfies $b_{n+2}^1 + \sum_{i=2}^{n+1} b_{n+2}^i c_i^1 + c_{n+2}^1 = 0$ i.e.

$$c_{n+2}^1 = -b_{n+2}^1 - \sum_{i=2}^{n+1} b_{n+2}^i c_i^1. \tag{21}$$

A main difference between the two relations is that the coefficient of c_i^1 in (20) is b_n^{i-2}, instead of b_{n+2}^i in (21). We thus use recurrence relation (8) on b_j^k to drop the indices, and we use it with an extended range of values: it actually holds for $k=1$ provided we set $b_j^0 = 0$ for $j\geq 0$. We replace in (21) b_{n+2}^i by

$$b_{n+2}^i = (1-q^{n+2})\ b_{n+1}^i + b_{n+1}^{i-1} - b_n^{i-1} \quad (1\leq i\leq n+1)$$

and gather first all terms coming from the first one above, the one with a factor $(1-q^{n+2})$. We get the product of $(1-q^{n+2})$ and

$$-b^1_{n+1} - \sum_{i=2}^{n+1} b^i_{n+1} c^1_i = 0,$$

since the term for $i = n+1$ in the sum is $-c^1_{n+1}$, while the rest is equal to c^1_{n+1}. Hence we can replace in (21) b^i_{n+2} by $b^{i-1}_{n+1} - b^{i-1}_n$, getting:

$$c^1_{n+2} = -\sum_{i=2}^{n+1} (b^{i-1}_{n+1} - b^{i-1}_n) c^1_i. \tag{22}$$

We use once again (8) in the following form:

$$b^{i-1}_{n+1} - b^{i-1}_n = -q^{n+1}\ b^{i-1}_n + b^{i-2}_n - b^{i-2}_{n-1} \quad (1 \le i-1 \le n)$$

and replace this in the previous expression:

$$c^1_{n+2} = -\sum_{i=2}^{n+1} (-q^{n+1}\ b^{i-1}_n + b^{i-2}_n - b^{i-2}_{n-1}) c^1_i = \Phi - \sum_{i=3}^{n+1} b^{i-2}_n c^1_i \tag{23}$$

(recall $b^0_n = 0$), letting $\Phi = -\sum_{i=2}^{n+1} (-q^{n+1}\ b^{i-1}_n - b^{i-2}_{n-1}) c^1_i$. We compute Φ.

Lemma 4.4. *For any k, $0 \le k \le n-1$, we define*

$$\Phi(k) = \sum_{u=k+2}^{n+1} ((q^{n+1} + \cdots + q^{n-k+1}) b^{u-k-1}_{n-k} + b^{u-k-2}_{n-k-1})\ c^1_{u-k}.$$

Then for any k, $0 \le k \le n-1$, we have $\Phi = \Phi(k)$. Consequently

$$\Phi = \Phi(n-1) = -(1-q^2)(q^{n+1} + \cdots + q^2) = -(1-q^2)\psi(n).$$

Proof. The statement for $k = 0$ is just the definition of Φ. The conclusion comes with $c^1_2 = -b^1_2 = -(1-q^2)$. It remains to take some k, $0 \le k \le n-2$, such that $\Phi = \Phi(k)$ and prove that $\Phi = \Phi(k+1)$.

Indeed, the term with $u = n+1$ in $\Phi(k)$ is, using (22):

$$(q^{n+1} + \cdots + q^{n-k+1} + 1) c^1_{n-k+1} = -(q^{n+1} + \cdots + q^{n-k+1} + 1) \sum_{s=2}^{n-k} (b^{s-1}_{n-k} - b^{s-1}_{n-k-1}) c^1_s$$

so that

$$\begin{aligned}\Phi(k) &= \sum_{u=k+2}^{n} \left((q^{n+1}+\cdots+q^{n-k+1})b_{n-k}^{u-k-1} + b_{n-k-1}^{u-k-2}\right)\, c_{u-k}^{1} \\ &\quad - (q^{n+1}+\cdots+q^{n-k+1}+1)\sum_{s=2}^{n-k}(b_{n-k}^{s-1}-b_{n-k-1}^{s-1})c_s^1 \\ &= \sum_{u=k+2}^{n} \left(-b_{n-k}^{u-k-1} + b_{n-k-1}^{u-k-2} + (q^{n+1}+\cdots+q^{n-k+1}+1)b_{n-k-1}^{u-k-1}\right)\, c_{u-k}^{1}\end{aligned}$$

Relation (8) now does the trick, since

$$-b_{n-k}^{u-k-1} = -b_{n-k-1}^{u-k-2} - (1-q^{n-k})\, b_{n-k-1}^{u-k-1} + b_{n-k-2}^{u-k-2}$$

giving

$$\Phi(k) = \sum_{u=k+2}^{n} \left((q^{n+1}+\cdots+q^{n-k})b_{n-k-1}^{u-k-1} + b_{n-k-2}^{u-k-2}\right)\, c_{u-k}^{1}$$

which is exactly $\Phi(k+1)$. Lemma 4.4 is proved. □

We obtain (20) by replacing Φ by its value $-(1-q^2)\psi(n)$ in (23). □

Proposition 4.3 may shorten computer implementations since it decreases indices by 2. The method of proof can be iterated but we don't pursue this here.

4.5. **Extensive formulas.** To conclude this work, we write again the formula for the Poincaré polynomial in an extensive form, plugging in the value of the b_j^k.

Corollary 4.5. *The Poincaré polynomial $a_n(q)$ is given, for $n \geq 1$, by the following formula:*

$$\begin{aligned}a_n &= \frac{1}{q^{n+2}}\,(q^2+\cdots+q^{n+2}) \\ &\quad + \frac{1}{q^{n+2}}\sum_{i=1}^{n}(q^2+\cdots+q^{i+1})\sum_{s=0}^{n-i}(-1)^{s+1}\sum_{n+1=v_0>\cdots>v_s>v_{s+1}=i} \\ &\quad \prod_{u=0}^{s}\Bigg[(1-\psi(v_{u+1}))\,\Pi(v_{u+1}+2,v_u) \\ &+ \sum_{t=1}^{v_{u+1}-1}(1-\psi(t))\sum_{r=1}^{\min\{v_u-v_{u+1}-1,v_{u+1}-t\}}\sum_{d_1+\cdots+d_r=v_{u+1}-t}\Sigma\Pi(t+2,v_u)[d_1+1,\cdots,d_r+1]\Bigg]\end{aligned}$$

where $\psi(k) = q^2+\cdots+q^{k+1}$ for $k \geq 1$, $\Pi(a,b) = \prod_{a\leq i\leq b}(1-q^i)$ for $1 \leq a \leq b$ and $\Sigma\Pi(a,b)[e_1,\cdots,e_u]$ is the sum of all possible products obtained from $\Pi(a,b)$ by removing e_i consecutive terms $(1-q^j)\cdots(1-q^{j+e_i-1})$ and replacing them by $(-q^j)$, for $1 \leq i \leq u$.

For the sake of completeness we also write the formula that arises from Proposition 4.3, since $c^1_{n+3} = -q^{n+2}(1-q^2)a_n$.

Corollary 4.6. *The Poincaré polynomial $a_n(q)$ is given, for $n \geq 1$, by the following formula:*

$$\begin{aligned} a_n &= -\frac{1}{q^{n+2}(1-q^2)} \sum_{s=0}^{n+1} (-1)^{s+1} \sum_{n+3=v_0>\cdots>v_s>v_{s+1}=1} \prod_{u=0}^{s} b_{v_u}^{v_{u+1}} \\ &= -\frac{1}{q^{n+2}(1-q^2)} \sum_{s=0}^{n+1} (-1)^{s+1} \sum_{n+3=v_0>\cdots>v_s>v_{s+1}=1} \prod_{u=0}^{s} \\ &\quad \Bigg[(1-\psi(v_{u+1}))\ \Pi(v_{u+1}+2, v_u) \\ &+ \sum_{t=1}^{v_{u+1}-1} (1-\psi(t)) \sum_{r=1}^{\min\{v_u - v_{u+1}-1, v_{u+1}-t\}} \sum_{d_1+\cdots+d_r=v_{u+1}-t} \Sigma\Pi(t+2, v_u)[d_1+1, \cdots, d_r+1] \Bigg] \end{aligned}$$

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