Supplementary material for : Estimation of extremes for heavy-tailed and light-tailed distributions in the presence of random censoring

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ARTICLE HISTORY
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This document provides the proofs of all the Lemmas and Propositions of the manuscript *Estimation of extremes for heavy-tailed and light-tailed distributions in the presence of random censoring.* For ease of reference, a number of assumptions, definitions and notations are recalled here and there in this document. Propositions 2 and 3 are the cornerstones for proving the theorems stated in the main manuscript.

In all this document, *cst* will denote a positive generic constant which exact value does not need to be explicited.

1. Assumptions

We recall here some of the assumptions under which our asymptotic results are proved.

1.1. Assumptions on the model

The main assumption is

Assumption (A1) : there exist $\tau_X \in [0,1]$, $\tau_C \in [0,1]$, $\theta_X > 0$, $\theta_C > 0$ such that

$$F_X \in A_1(\tau_X, \theta_X) \quad \text{and} \quad F_C \in A_1(\tau_C, \theta_C).$$

This means that there exists positive functions H_X and H_C such that

$$\bar{F}_X(x) = 1 - F_X(x) = \exp(-K^-_{\tau_X}(\log(H_X(x)))) \quad \text{and} \quad \bar{F}_C(x) = 1 - F_C(x) = \exp(-K^-_{\tau_C}(\log(H_C(x))))$$

and, for some slowly varying functions $\bar{l}_X$ and $\bar{l}_C$ at infinity,

$$H^-_X(x) = x^{\theta_X}\bar{l}_X(x) \quad \text{and} \quad H^-_C(x) = x^{\theta_C}\bar{l}_C(x).$$

It is clear that under this condition we also have $H_X(x) = x^{1/\theta_X} l_X(x)$ and $H_C(x) = x^{1/\theta_C} l_C(x)$ where both l_X and l_C are slowly varying functions at infinity.

Remind that $(\tau_X, \tau_C) = (0,0)$ and $(\tau_X, \tau_C) = (1,1)$ have not been considered in this paper.

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The second important assumption is about the second order tail properties of F_X and F_C :

Assumption (A2) : there exist some negative constants ρ_X and ρ_C, and some rate functions b_X and b_C having constant sign at $+\infty$ and satisfying $|b_X| \in RV_{\rho_X}$ and $|b_C| \in RV_{\rho_C}$, such that, as $t \to +\infty$,

$$\frac{l_X(tx)/l_X(t) - 1}{b_X(t)} \longrightarrow K_{\rho_X}(x), \quad \text{and} \quad \frac{l_C(tx)/l_C(t) - 1}{b_C(t)} \longrightarrow K_{\rho_C}(x), \forall x > 0. \tag{1}$$

where $K_\rho(x) = (x^\rho - 1)/\rho$ if $\rho < 0$, and $K_0(x) = \log(x)$.

Consider the functions :

$$H_Z^-(x) = x^{\theta_Z}\tilde{l}(x) \quad \text{and} \quad H_X \circ H_Z^-(x) = x^a l(x) \quad \text{with} \quad a := \frac{\theta_Z}{\theta_X}, \tag{2}$$

where both $\tilde{l}$ and l are slowly varying. The crucial parameter $a = \theta_Z/\theta_X$ is equal to 1 in "low censoring" situations (in particular when $\tau_X < \tau_C$).

Our important technical Lemma 1 ensures that functions H_Z^- and $H_X \circ H_Z^-$ also satisfy a second order condition SR2. For technical reasons though, we need to consider the following stronger conditions on $\tilde{l}$ and l, respectively noted $R_{\tilde{l}}(\tilde{b}, \tilde{\rho})$ and $R_l(b, \rho)$, and defined by :

Assumption $R_\ell(B, \rho)$: for some constant $\rho \leqslant 0$ and a rate function B satisfying $\lim_{x\to+\infty} B(x) = 0$, such that for all $\epsilon > 0$, we have

$$\sup_{\lambda \geqslant 1} \left| \frac{\ell(\lambda x)/\ell(x) - 1}{B(x)K_\rho(\lambda)} - 1 \right| \leqslant \epsilon, \text{ for } x \text{ sufficiently large .}$$

1.2. Assumptions on the sample fraction (k_n)

The basic assumption on (k_n) is

$H_1 : k \to +\infty$, $\frac{k}{n} \to 0$, $\frac{\log k}{\log n} \to 0$, as $n \to +\infty$.

Introducing the important notation

$$L_{nk} = \log(n/k),$$

let v_n be the factor which contributes to the rates of convergence of our estimators, it depends on the censoring strength in the tail :

$$v_n := \begin{cases} 1 & \text{if } 0 < \tau_X < \tau_C \leqslant 1 \text{ or } 0 < \tau_X = \tau_C < 1 \text{ or } 0 = \tau_X < \tau_C \leqslant 1, \\ L_{nk}^{\frac{1}{2}(\frac{\tau_C}{\tau_X} - 1)} & \text{if } 0 < \tau_C < \tau_X \leqslant 1, \\ L_{nk}^{-1/2} (\log L_{nk})^{\frac{1}{2}(\frac{1}{\tau_X} - 1)} & \text{if } 0 = \tau_C < \tau_X \leqslant 1. \end{cases}$$

We also consider the following conditions

$$H_2 : 0 < \tau_X < \tau_C \leqslant 1 \text{ and } \begin{cases} (i)\sqrt{k}L_{nk}^{\tau_X/\tau_C - 1} \to 0 \text{ if } \frac{1}{\tau_C} - \frac{1}{\tau_X} \geqslant -1 \\ (ii)\sqrt{k}L_{nk}^{-\tau_X} \to 0 \text{ if } \frac{1}{\tau_C} - \frac{1}{\tau_X} < -1 \end{cases}$$

$$H_3 : 0 < \tau_C < \tau_X \leqslant 1 \text{ and } \begin{cases} (i)\sqrt{k}v_n \to +\infty \\ (ii)\sqrt{k}v_n L_{nk}^{\tau_C/\tau_X - 1} \to 0 \text{ if } \frac{1}{\tau_X} - \frac{1}{\tau_C} \geqslant -1 \\ (iii)\sqrt{k}v_n L_{nk}^{-\tau_C} \to 0 \text{ if } \frac{1}{\tau_X} - \frac{1}{\tau_C} < -1 \end{cases}$$

$$H_4 : 0 < \tau_X = \tau_C < 1 \text{ and } \sqrt{k}L_{nk}^{-\tau_X} \to 0$$

$$H_5 : 0 = \tau_X < \tau_C \leqslant 1 \text{ and } \exists \delta > 0, \sqrt{k}L_{nk}^{\tilde{\rho}+\delta} \to 0$$

$$H_6 : 0 = \tau_C < \tau_X \leqslant 1 \text{ and } \begin{cases} (i)\sqrt{k}v_n \to +\infty \\ (ii)\sqrt{k}v_n(\log L_{nk})^{-1} \to 0 \end{cases}$$

(in assumption H_5 above, $\tilde{\rho}$ denotes the second order parameter associated to the slowly varying function $\tilde{l}$, which is negative in this case ; see formula (2) as well as Lemma 1)

2. Second Order expansions

Proposition 1. *Under Assumption (A1), the distribution function of* $Z = \min(X, C)$ *satisfies condition* $A_1(\tau_Z, \theta_Z)$*, where*

$$\tau_Z = \min(\tau_X, \tau_C) \quad \textit{and} \quad \theta_Z = \begin{cases} \theta_X & \textit{if } 0 \leqslant \tau_X < \tau_C \leqslant 1 \\ \theta_C & \textit{if } 0 \leqslant \tau_C < \tau_X \leqslant 1 \\ (\theta_X^{-1/\tau_Z} + \theta_C^{-1/\tau_Z})^{-\tau_Z} & \textit{if } 0 < \tau_X = \tau_C \leqslant 1 \\ \min(\theta_X, \theta_C) & \textit{if } \tau_X = \tau_C = 0 \end{cases}$$

Therefore, there exists $x_* > 0$ *such that for any* $x \geqslant x_*$*, we have*

$$\mathbb{P}(Z > x) = \exp(-K^-_{\tau_Z}(\log(H_Z(x)))),$$

where $H_Z^- \in RV_{\theta_Z}$*. Consequently, if* E *denotes a standard exponential distribution, we have*

$$Z = H_Z^-(\exp K_{\tau_Z}(E)).$$

Proof of Proposition 1

Let us first consider the case $0 < \tau_X < \tau_C \leqslant 1$ (note that the case $\tau_X = \tau_C = 0$ is already treated in [30]). We are going to prove that $\Lambda_Z(x) = K^-_{\tau_Z}(\log(H_Z(x)))$, where $H_Z(x) = x^{1/\theta_Z} l_Z(x)$, with, in this case, $\tau_Z = \tau_X$, $\theta_Z = \theta_X$ and l_Z is a slowly varying function to be determined.

Recall that $H_X(x) = x^{1/\theta_X} l_X(x)$, $H_C(x) = x^{1/\theta_C} l_C(x)$,

$$K_{\tau_X}(t) = \frac{t^{\tau_X} - 1}{\tau_X} \quad \text{and} \quad K^-_{\tau_X}(y) = (1 + \tau_X y)^{1/\tau_X}.$$

We thus have by Assumption (A1)

$$\begin{aligned} \Lambda_X(x) = (1 + \tau_X \log H_X(x))^{1/\tau_X} &= \left(1 + \frac{\tau_X}{\theta_X} \log x + \tau_X \log l_X(x)\right)^{1/\tau_X} \\ &= \left(\frac{\tau_X}{\theta_X}\right)^{1/\tau_X} (\log x)^{1/\tau_X} (1 + \epsilon_X(x))^{1/\tau_X} \end{aligned} \tag{3}$$

and similarly

$$\Lambda_C(x) = \left(\frac{\tau_C}{\theta_C}\right)^{1/\tau_C} (\log x)^{1/\tau_C} (1 + \epsilon_C(x))^{1/\tau_C}$$

where we set $\epsilon_V(x) := \frac{\theta_V/\tau_V + \theta_V \log l_V(x)}{\log x}$, for V being X or C.

By independence of X and C (independent censoring), the cumulative hazard function of Z is the sum of those of X and C. Hence

$$\begin{aligned} K_{\tau_X}(\Lambda_Z(x)) &= K_{\tau_X}(\Lambda_X(x) + \Lambda_C(x)) \\ &= K_{\tau_X}(\Lambda_X(x)(1 + \lambda(x))), \end{aligned}$$

where

$$\begin{aligned}\lambda(x) &= \frac{\Lambda_C(x)}{\Lambda_X(x)} \\ &= \left(\tfrac{\tau_C}{\theta_C}\right)^{1/\tau_C}\left(\tfrac{\tau_X}{\theta_X}\right)^{-1/\tau_X}(\log x)^{1/\tau_C-1/\tau_X}\,(1+\epsilon_C(x))^{1/\tau_C}(1+\epsilon_X(x))^{-1/\tau_X}.\end{aligned}$$

Since both ϵ_X and ϵ_C tend to 0, it is also the case for the function λ. Consequently, we define

$$\begin{aligned}H_Z(x) &:= \exp(K_{\tau_X}(\Lambda_Z(x))) \\ &= \exp\left\{\tau_X^{-1}\left[(\Lambda_X(x))^{\tau_X}\left((1+\lambda(x))^{\tau_X}-1\right)+\left((\Lambda_X(x))^{\tau_X}-1\right)\right]\right\}.\end{aligned}$$

But $\exp\left(\frac{(\Lambda_X(x))^{\tau_X}-1}{\tau_X}\right) = K_{\tau_X}(\Lambda_X(x)) = \log H_X(x)$, hence

$$H_Z(x) = H_X(x)\exp\left\{\tau_X^{-1}(\Lambda_X(x))^{\tau_X}\left[(1+\lambda(x))^{\tau_X}-1\right]\right\}.$$

Using (3) and a Taylor expansion of order 2 of $(1+\lambda(x))^{\tau_X}$, we thus obtain

$$H_Z(x) = x^{1/\theta_X} l_Z(x) \quad \text{with} \quad l_Z(x) = l_X(x).\phi(x)$$

where, introducing the constant $B(X,C) = \left(\frac{\tau_C}{\theta_C}\right)^{1/\tau_C}\left(\frac{\tau_X}{\theta_X}\right)^{1-1/\tau_X}$,

$$\begin{aligned}\phi(x) &= \exp\left(\theta_X^{-1}(\log x)(1+\epsilon_X(x))(\tau_X\lambda(x)+\tfrac{\tau_X(\tau_X-1)}{2}\lambda^2(x)+o(\lambda^2(x)))\right) \\ &= \exp\left(B(X,C)\,(\log x)^r\,(1+\epsilon_C(x))^{1/\tau_C}(1+\epsilon_X(x))^{1-1/\tau_X}\left(1+\tfrac{\tau_X-1}{2}\lambda(x)+o(\lambda(x))\right)\right),\end{aligned}$$

with $r = 1-\left|\frac{1}{\tau_C}-\frac{1}{\tau_X}\right| < 1$.

Now, using the expression of l_X and l_C in Lemma 1 (i), we can perform a Taylor expansion of ϵ_X, ϵ_C and λ to obtain that (after some careful computations)

$$\phi(x) = \exp\left\{B(X,C)(\log x)^r(1+R(x))\right\}, \tag{4}$$

where

$$R(x) = \begin{cases} \frac{cst}{\log x}(1+o(1)) & \text{if} \quad r \leqslant 0 \\ \frac{cst}{(\log x)^{1-r}} + \frac{cst}{\log x}(1+o(1)) & \text{if} \quad r > 0. \end{cases}$$

It is then easy to check that ϕ is slowly varying at infinity and so does l_Z. We thus have proved that $\Lambda_Z(x) = K^-_{\tau_Z}(\log(H_Z(x)))$ with, in this case, $\tau_Z = \tau_X$, $\theta_Z = \theta_X$ and $H_Z(x) = x^{1/\theta_Z} l_Z(x)$ where $l_Z = l_X\phi$ is slowly varying. In addition, one can remark that

$$\phi(x) \to \begin{cases} 1 & \text{if} \quad r < 0, \\ \exp\left(B(X,C)\right) & \text{if} \quad r = 0, \\ \infty & \text{if} \quad 0 < r < 1. \end{cases}$$

Concerning the other cases ($0 < \tau_C < \tau_X \leqslant 1$, $0 = \tau_X < \tau_C \leqslant 1$, $0 = \tau_C < \tau_X \leqslant 1$ and $0 < \tau_X = \tau_C \leqslant 1$), they are proved similarly : the functions $\lambda(x)$ and $\phi(x)$ slightly differ from the case above, but the proof is substantially the same. Of particular interest is the proof for the case $0 < \tau_X = \tau_C \leqslant 1$ which starts by setting $\tau_Z = \tau_X = \tau_C$ and writing

$$\begin{aligned}K_{\tau_Z}(\Lambda_Z(x)) &= \frac{1}{\tau_Z}\left\{(\Lambda_X(x)+\Lambda_C(x))^{\tau_Z}-1\right\} \\ &= \frac{1}{\tau_Z}\left\{\left[\left((\tau_Z/\theta_X)^{1/\tau_Z}+(\tau_Z/\theta_C)^{1/\tau_Z}\right)(\log x)^{1/\tau_Z}(1+o(1))\right]^{\tau_Z}-1\right\} \\ &= \left(\theta_X^{-1/\tau_Z}+\theta_C^{-1/\tau_Z}\right)^{\tau_Z}(\log x)(1+o(1))-\tau_Z^{-1}\end{aligned}$$

which identifies θ_Z as $\theta_Z = \left(\theta_X^{-1/\tau_Z}+\theta_C^{-1/\tau_Z}\right)^{-\tau_Z}$, since $H_Z(x) = \exp(K_{\tau_Z}(\Lambda_Z(x)))$ is then

of the order x^{1/θ_Z}. Note that in this case, with a more refined expansion of $K_{\tau_Z}(\Lambda_Z(x))$ (under the assumptions of Lemma 1 below), it can be proved that $H_Z(x) = x^{1/\theta_Z} e^{-1/\tau_Z} \phi(x)$ with $\lim_{x\to\infty} \phi(x) = \exp((c_X + c_C)\tau_Z/\theta_Z) \in]0, +\infty[$. □

Lemma 1 stated below provides details about the second order properties of the functions H_Z^- and $H_X \circ H_Z^-$ (and therefore, on the behavior of the variables Z_i and $\Lambda_X(Z_i)$). These properties not only depend on the position of the parameters τ_X and τ_C with respect to each other, but on their proximity through the parameter r defined by

$$r := 1 - \left| \frac{1}{\tau_C} - \frac{1}{\tau_X} \right| \in [-\infty, 1]$$

(if either $\tau_X = 0$ or $\tau_C = 0$, indeed consider that $r = -\infty$). This parameter r appears in the function $\phi(x)$ which is introduced in the previous proof of Proposition 1.

Lemma 1. *Let conditions (A_1) and (A_2) hold.*

(i) For different slowly varying functions generically noted v, we have

$$l_X(x) = c_X(1 - x^{\rho_X} v(x)) \quad \text{and} \quad l_C(x) = c_C(1 - x^{\rho_C} v(x))$$
$$\bar{l}_X(x) = c_X^{-\theta_X}(1 - x^{\theta_X \rho_X} v(x)) \quad \text{and} \quad \bar{l}_C(x) = c_C^{-\theta_C}(1 - x^{\theta_C \rho_C} v(x)).$$

(ii) The slowly varying functions $\tilde{l}$ and l associated to H_Z^- and $H_X \circ H_Z^-$ satisfy a second order condition SR2 : as $t \to +\infty$,

$$\frac{\frac{\tilde{l}(tx)}{\tilde{l}(t)} - 1}{\tilde{b}(t)} \longrightarrow K_{\tilde{\rho}}(x) \quad \text{and} \quad \frac{\frac{l(tx)}{l(t)} - 1}{b(t)} \longrightarrow K_\rho(x)$$

where

$$\tilde{\rho} = \rho = \begin{cases} \max(\theta_X \rho_X, -1) & \text{if } 0 = \tau_X < \tau_C < 1 \\ \max(\theta_C \rho_C, -1) & \text{if } 0 = \tau_C < \tau_X < 1 \\ 0 & \text{in the other cases} , \end{cases}$$

and $|\tilde{b}| \in RV_{\tilde{\rho}}$ and $|b| \in RV_\rho$. When $\rho = 0$, both $b(t)$ and $\tilde{b}(t)$ are (as $t \to +\infty$) of the order $O((\log t)^{r-1})$ when $r \neq 0$, and of the order $O((\log t)^{-2})$ when $r = 0$.

(iii) The slowly varying function l_Z associated to H_Z satisfies

$$\lim_{x\to+\infty} l_Z(x) = c_Z \begin{cases} \in]0, +\infty[& \text{if } \tau_X = \tau_C \text{ or } r \leqslant 0, \\ = +\infty & \text{if } \tau_X \neq \tau_C \text{ and } r \in]0, 1[\end{cases}$$

where in particular $c_Z = c_X$ if $\tau_X < \tau_C$ and $r < 0$, and $c_Z = c_C$ if $\tau_C < \tau_X$ and $r < 0$. Moreover we have (with the convention $(+\infty)^{-\theta} = 0$ when $\theta > 0$)

$$\tilde{l}(t) \to \tilde{c} := c_Z^{-\theta_Z} \quad \text{and} \quad l(t) \to c := c_X \tilde{c}^{1/\theta_X}, \quad \text{as } t \to +\infty.$$

When $\tau_X = \tau_C$ or $r \leqslant 0$, both c and $\tilde{c}$ are positive. When $\tau_X \neq \tau_C$ and $r \in]0, 1[$, both $\tilde{c}$ and c are zero and the following relation holds for some $\nu > 0$, as $x \to \infty$

$$\frac{\log l(\exp x)}{x} = -\nu . x^{r-1}(1 + o(1)) \longrightarrow 0 \quad \text{and} \quad \frac{\log \tilde{l}(\exp x)}{x} = -\theta_X \nu . x^{r-1}(1 + o(1)) \longrightarrow 0 \quad (5)$$

Proof of Lemma 1

Part (i)

Let us prove the statements for l_X and $\bar{l}_X$, the slowly varying functions involved in H_X and H_X^-. The expansion for l_X is a direct consequence of the assumption (A2) (which states that $\rho_X < 0$) and Lemma 3 in [21] (a corollary of Theorem B.2.2 in [19], thereafter referred to as Hua & Joe's

Lemma). Assumptions (A1) and (A2) also imply that, for every x

$$\frac{\frac{H_X(tx)}{H_X(t)} - x^{1/\theta_X}}{b_X(t)} \xrightarrow{t\to\infty} x^{1/\theta_X} \frac{x^{\rho_X} - 1}{\rho_X}.$$

It is a rather standard technique in extreme value theory that Vervaat's Lemma implies, for every y,

$$\frac{\frac{H_X^-(ty)}{H_X^-(t)} - y^{\theta_X}}{-\theta_X^2 b_X(H_X^-(t))} \xrightarrow{t\to\infty} y^{\theta_X} \frac{y^{\theta_X\rho_X} - 1}{\rho_X} \quad \text{and thus} \quad \frac{\frac{\bar{l}_X(ty)}{\bar{l}_X(t)} - 1}{\bar{b}_X(t)} \xrightarrow{t\to\infty} \frac{y^{\theta_X\rho_X} - 1}{\theta_X\rho_X}$$

for some slowly varying $\bar{b}_X$. Therefore, the slowly varying function $\bar{l}_X$ associated to H_X^- satisfies the second order condition with index $\theta_X\rho_X < 0$ and, for the same reason as above (Hua & Joe's Lemma), we have the expansion $\bar{l}_X(y) = \bar{c}_X(1 - y^{\theta_X\rho_X} v(y))$ for some slowly varying function $v(y)$ and constant $\bar{c}_X$. Finally, the fact that the limit $\bar{c}_X$ of $\bar{l}_X$ at infinity is $c_X^{-\theta_X}$ can be justified using the following lines : if $y = H_X(x) = x^{1/\theta_X} l_X(x)$, then at infinity $y^{\theta_X}\bar{l}_X(y) = H_X^-(y) \sim x = y^{\theta_X}(l_X(x))^{-\theta_X}$, and thus, since $\lim_\infty l_X = c_X$, we have $\lim_\infty \bar{l}_X = c_X^{-\theta_X}$.

Part (ii)

We only provide details in the case $0 < \tau_X < \tau_C \leqslant 1$, the other cases are similar. The proof of Proposition 1 yields that

$$H_Z(x) = x^{1/\theta_Z} l_Z(x) = x^{1/\theta_X} l_Z(x)$$

where $l_Z(x) = l_X(x).\phi(x)$. Starting from (4), one can prove that

$$\frac{\phi(tx)}{\phi(t)} - 1 = \begin{cases} cst(\log t)^{r-1}(1 + o(1))(\log x) & \text{if} \quad r \neq 0, \\ cst(\log t)^{-2}(1 + o(1))(\log x) & \text{if} \quad r = 0, \end{cases}$$

i.e. ϕ satisfies the SR2 condition with a ρ-coefficient equal to 0. Moreover, since

$$\frac{l_Z(tx)}{l_Z(t)} - 1 = \frac{l_X(tx)}{l_X(t)}\left(\frac{\phi(tx)}{\phi(t)} - 1\right) + \frac{l_X(tx)}{l_X(t)} - 1,$$

it is clear that l_Z satisfies the SR2 condition with $\rho_Z = 0$ and

$$b_Z(t) = \begin{cases} O((\log t)^{r-1}) & \text{if } r \neq 0, \\ O((\log t)^{-2}) & \text{if } r = 0. \end{cases}$$

The result for $\tilde{l}$ (associated to H_Z^-) thus follows using Vervaat's Lemma : $\tilde{l}$ satisfies the SR2 condition with $\tilde{\rho} = 0$ and

$$\tilde{b}(t) = \begin{cases} O((\log t)^{r-1}) & \text{if } r \neq 0, \\ O((\log t)^{-2}) & \text{if } r = 0. \end{cases}$$

Now, recall that

$$H_X \circ H_Z^-(x) = (x^{1/\theta_Z}\tilde{l}(x))^{1/\theta_X} l_X(H_Z^-(x)) = x^a l(x),$$

with $a = \frac{\theta_Z}{\theta_X}$ ($= 1$ in the present case $0 < \tau_X < \tau_C \leqslant 1$) and $l(x) = l_X(H_Z^-(x))(\tilde{l}(x))^{1/\theta_X}$. Consequently,

$$\frac{l(tx)}{l(t)} - 1 = \frac{l_X(H_Z^-(tx))}{l_X(H_Z^-(t))}\left(\frac{\tilde{l}(tx)}{\tilde{l}(t)} - 1\right) + \frac{l_X(H_Z^-(tx))}{l_X(H_Z^-(t))} - 1,$$

and clearly l also satisfies an SR2 condition with $\rho = \max(0, \rho_X) = 0$ and b of the same order as $\tilde{b}$.

This phenomenon $\rho = \tilde{\rho} = 0$ holds in other cases (not detailed here), except when either τ_X or τ_C is equal to zero. For instance, when $0 = \tau_X < \tau_C \leqslant 1$, in the proof of Proposition 1, the function $\epsilon(x) = \Lambda_C(x)/\Lambda_X(x)$ is of the order of $(\log x)^{1/\tau_C} x^{-1/\theta_X}$, and it implies that $l_Z(x) = l_X(x)\phi(x)$ where, this time, the function ϕ satisfies the SR2 condition with a ρ-coefficient equal to $-1/\theta_X < 0$. As a consequence, the coefficient ρ_Z associated to the SR2 condition for l_Z is now $\max\{\rho_X, -1/\theta_X\}$, which yields $\tilde{\rho} = \max\{\theta_X \rho_X, -1\} < 0$ as announced.

Part (iii)

Again in the case $0 < \tau_X < \tau_C \leqslant 1$, the limiting result for l_Z is an easy consequence of the fact that $l_Z = l_X \phi$, with l_X tending to c_X and the limit of ϕ stated in the proof of Proposition 1. In the case $0 < \tau_X = \tau_C < 1$, as said at the end of the proof of Proposition 1, we have $\lim_{x\to+\infty} l_Z(x) \in]0, +\infty[$. The other cases are similar to the first one, details are omitted.

Concerning now the function $\tilde{l}$, the same argument as in Part (i) of the proof yields that $\tilde{l}$ tends to $\tilde{c} = c_Z^{-\theta_Z}$, with the convention $(+\infty)^{-\theta_X} = 0$. Concerning the limit $c = c_X \tilde{c}^{1/\theta_X}$ of l, it is a consequence of the relation $l(x) = l_X(H_Z^-(x))(\tilde{l}(x))^{1/\theta_X}$ (in the case $0 < \tau_X < \tau_C \leqslant 1$).

In the case where $c_Z = 0$ (*i.e.* when $r \in]0, 1[$ and $\tau_X \neq \tau_C$), both $\tilde{c}$ and c are equal to 0 and we use the fact that

$$\tilde{l}(x) \sim (l_Z(H_Z^-(x)))^{-\theta_Z} \text{ and } l(x) = l_X(H_Z^-(x))(\tilde{l}(x))^{1/\theta_X}$$

to deduce the statement (5), after some calculations. These rates will prove useful in two occasions later in the proofs. □

Let us now turn our attention to the second order expansion for the function $p(\cdot)$ defined by

$$p(x) = \mathbb{P}(\delta = 1 | Z = x).$$

The following lemma provides useful expansions of functions $p(\cdot)$ and $r(\cdot)$

$$r(t) = p \circ H_Z^-(\exp(K_{\tau_Z}(-\log t))),$$

which are crucial to derive the properties of the random proportion $\hat{p}_k$ (and therefore the statements of Proposition 2).

Lemma 2. *Let us define the following constants (for τ_X and τ_C positive only)*

$$A_X = \theta_X(\tau_X^{-1} - 1)(\tau_X^{-1} + \log c_X) \ , \ A_C = \theta_C(\tau_C^{-1} - 1)(\tau_C^{-1} + \log c_C)$$

and

$$A = A_C - A_X \quad \textit{and} \quad B = \frac{\theta_X}{\theta_C} \left(\frac{\tau_X}{\theta_X}\right)^{1-1/\tau_X} \left(\frac{\tau_C}{\theta_C}\right)^{1/\tau_C - 1}.$$

Let assumptions (A_1) and (A_2) hold.

(i) We have, as $x \to +\infty$,

$$p(x) \to p := \begin{cases} 1 & \textit{if } 0 \leqslant \tau_X < \tau_C \leqslant 1, \\ 0 & \textit{if } 0 \leqslant \tau_C < \tau_X \leqslant 1, \\ \dfrac{\theta_X^{1/\tau_X}}{(\theta_X^{1/\tau_X} + \theta_C^{1/\tau_X})} = a^{1/\tau_X} & \textit{if } 0 < \tau_X = \tau_C < 1, \end{cases}$$

and, more precisely,

$$p(x)-p=\begin{cases} D\,(\log x)^{r-1}\left[1+g(r)(\log x)^{\max(-1,r-1)}(1+o(1))\right] & \text{if } 0<\tau_X\neq\tau_C\leqslant 1,\\ D\,x^{-1/\theta_X}(\log x)^{\tau_C^{-1}-1}\left[1+A_C(\log x)^{-1}(1+o(1))\right] & \text{if } 0=\tau_X<\tau_C\leqslant 1,\\ D\,x^{-1/\theta_C}(\log x)^{\tau_X^{-1}-1}\left[1+A_X(\log x)^{-1}(1+o(1))\right] & \text{if } 0=\tau_C<\tau_X\leqslant 1,\\ D\,(\log x)^{-1}(1+O(1/\log x)) & \text{if } 0<\tau_C=\tau_X<1,\end{cases}$$

where

$$D=\begin{cases} -B & \text{if } 0<\tau_X<\tau_C\leqslant 1,\\ B^{-1} & \text{if } 0<\tau_C<\tau_X\leqslant 1,\\ -(\tau_C/\theta_C)^{\tau_C^{-1}-1}(\theta_X/\theta_C c_X) & \text{if } 0=\tau_X<\tau_C\leqslant 1,\\ (\tau_X/\theta_X)^{\tau_X^{-1}-1}(\theta_C/\theta_X c_C) & \text{if } 0=\tau_C<\tau_X\leqslant 1,\\ -AB(1+B)^{-2} & \text{if } 0<\tau_C=\tau_X<1,\end{cases}$$

and

$$g(r)=\begin{cases} A\mathbb{I}_{r<0}+(A-B)\mathbb{I}_{r=0}+(-B)\mathbb{I}_{r\in]0,1[} & \text{if } 0<\tau_X<\tau_C\leqslant 1,\\ (-A)\mathbb{I}_{r<0}+(-A-B^{-1})\mathbb{I}_{r=0}+(-B^{-1})\mathbb{I}_{r\in]0,1[} & \text{if } 0<\tau_C<\tau_X\leqslant 1.\end{cases}$$

(ii) When $\tau_Z>0$ and $\tau_X\neq\tau_C$, as $t\downarrow 0$ we have

$$r(t)-p=D(\theta_Z/\tau_Z)^{r-1}(-\log t)^{-\tau_Z(1-r)}\left(1+O\left((-\log t)^{-\tau_Z\min\{1,1-r\}}\right)\right),$$

in particular, when $0<\tau_C<\tau_X\leqslant 1$,

$$r(t)=a^{1/\tau_X}(\tau_X/\tau_C)^{\tau_X^{-1}-1}(-\log t)^{\frac{\tau_C}{\tau_X}-1}\left(1+O\left((-\log t)^{\max\{-\tau_C,\tau_C/\tau_X-1\}}\right)\right).$$

When $\tau_Z>0$ and $\tau_X=\tau_C$, we have

$$r(t)-p=-AB\left[(1+B)^2(\theta_Z/\tau_Z)\right]^{-1}(-\log t)^{-\tau_Z}\left(1+O\left((-\log t)^{-\tau_Z}\right)\right).$$

When $\tau_Z=0$, if $\tau_+=\max(\tau_X,\tau_C)$ we have

$$r(t)-p=cst(-\log t)^{-1}(\log\log(1/t))^{\frac{1}{\tau_+}-1}\left(1+O\left((\log\log(1/t))^{-1}\right)\right).$$

with the constant being equal to $\tau_X^{\frac{1}{\tau_X}-1}a^{1/\tau_X}$ when $0=\tau_C<\tau_X\leqslant 1$.

Proof of Lemma 2 :

This proof is even more technical than the previous ones. As in the main part of the paper, a complete proof of all the cases would be too lengthy, we only provide here a sketch of the proof, focusing on some subcases.

Part (i)

We start by introducing f_X and f_C the respective probability density functions of X and C. By Assumption (A1), the cumulative hazard function Λ_X of X is defined by $\Lambda_X(x)=-\log\bar{F}_X(x)=K^-_{\tau_X}(\log H_X(x))$, and its derivative is $f_X(x)/\bar{F}_X(x)$, which is therefore equal to $(K^-_{\tau_X})'(\log H_X(x))\times H'_X(x)/H_X(x)$. The following thus comes easily

$$p(x)=\frac{\bar{F}_C(x)f_X(x)}{\bar{F}_C(x)f_X(x)+\bar{F}_X(x)f_C(x)}=\left(1+\frac{(K^-_{\tau_C})'(\log H_C(x))}{(K^-_{\tau_X})'(\log H_X(x))}\frac{H'_C(x)/H_C(x)}{H'_X(x)/H_X(x)}\right)^{-1}.$$

Consider first the case where neither τ_X nor τ_C is zero. Since $H_X(x)=x^{1/\theta_X}l_X(x)$ and $(K^-_\tau)'(u)=(\tau u+1)^{1/\tau-1}$, we have

$$(K^-_{\tau_X})'(\log H_X(x))=\left(\frac{\tau_X}{\theta_X}\log x+\tau_X\log l_X(x)+1\right)^{1/\tau_X-1}\quad\text{and}\quad\frac{H'_X(x)}{H_X(x)}=\frac{1}{\theta_X x}\left(1+\theta_X\frac{xl'_X(x)}{l_X(x)}\right),$$

where the slow variation of l_X ensures that $xl'_X(x)/l_X(x)\to 1$ as $x\to\infty$. With the corresponding

formula for the C version, the following comes

$$\frac{1}{p(x)} = 1+\frac{\theta_X}{\theta_C}\left(\frac{\tau_X}{\theta_X}\right)^{1-1/\tau_X}\left(\frac{\tau_C}{\theta_C}\right)^{1/\tau_C-1}(\log x)^{1/\tau_C-1/\tau_X}\,(1+o(1)) = 1+B\,(\log x)^{1/\tau_C-1/\tau_X}\,(1+o(1)).$$

The convergence to 1 of $p(x)$ is thus proved when $0 < \tau_X < \tau_C \leqslant 1$, and it can be proved similarly that it is also valid when $\tau_X = 0$. It is easy to see that when $0 < \tau_C < \tau_X \leqslant 1$, we have instead $1/p(x) \to \infty$, and thus $p(x) \to 0$, as $x \to \infty$ (also valid when $\tau_C = 0$). When $0 < \tau_X = \tau_C < 1$, we have $1/p(x) = 1 + B(1 + o(1)) \to 1 + B$ which is equal in this case to $1 + (\theta_X/\theta_C)^{1/\tau_X}$, and this provides the announced limit for $p(x)$, belonging to $]0, 1[$.

Of course, to derive the more precise expansions for $p(x) - p$ stated in Part (i), second order properties are required. Suppose neither τ_X nor τ_C is zero, and assumption (A2) holds. Part (i) of Lemma 1 ensures that

$$(K^-_{\tau_X})'(\log H_X(x)) = \left(\frac{\tau_X}{\theta_X}\right)^{1/\tau_X-1}(\log x)^{1/\tau_X-1}\left(1+(\tau_X^{-1}-1)\frac{\theta_X(\log c_X+\tau_X^{-1})}{\log x}+o(1/\log x)\right)$$

and

$$\frac{H'_X(x)}{H_X(x)} = \frac{1}{\theta_X x}\left(1-\theta_X\rho_X x^{\rho_X}\tilde{v}_X(x)\right),$$

where v_X is a slowly varying function. Similar formulas are valid for X instead of C. Therefore, after some efforts, we obtain

$$\frac{1}{p(x)} = 1 + B(\log x)^{1/\tau_C-1/\tau_X}\left(1+A(\log x)^{-1}(1+o(1))\right),$$

where A and B are the constants described in the statement of Lemma 2. This is the moment where knowing the position of τ_X with respect of τ_C is needed, and it can be easily checked (but it is a bit tedious) that the different expansions of $p(x) - p$ stated in Part (ii) of Lemma 2 are valid when neither τ_X nor τ_C is zero. When either τ_X or τ_C is zero, the proof is very similar, with the expression of either $(K^-_{\tau_X})'(\log H_X(x))$ or $(K^-_{\tau_C})'(\log H_C(x))$ varying from what is detailed above.

Part (ii)

A complete description of all the cases would be too lengthy, let us focus on the case $0 < \tau_C < \tau_X \leqslant 1$. For $t \in]0, \infty[$, if s denotes the quantity $s = K_{\tau_Z}(-\log t)$, we have

$$r(t) - p = p \circ H_Z^-(\exp s) - p = D\left(\log H_Z^-(\exp s)\right)^{r-1}(1 + R_1(t)),$$

where $R_1(t) = g(r)\left\{\log H_Z^-(\exp K_{\tau_Z}(-\log t))\right\}^{\max(-1,r-1)}(1 + o(1))$. But

$$\log H_Z^-(\exp s) = \log\left((e^s)^{\theta_Z}\tilde{l}(e^s)\right) = (\theta_Z s)\left(1+\theta_Z^{-1}(\log\tilde{l}(e^s))/s\right),$$

and thus

$$r(t) - p = \left(D\theta_Z^{r-1}\right)(K_{\tau_Z}(-\log t))^{r-1}(1 + R_1(t))(1 + R_2(t)),$$

where $R_2(t) = (\log\tilde{l}(\exp K_{\tau_Z}(-\log t)))/(\theta_Z K_{\tau_Z}(-\log t))$. When $\tau_Z > 0$, we have $K_{\tau_Z}(-\log t) = \frac{1}{\tau_Z}(-\log t)^{\tau_Z}(1 - (-\log t)^{-\tau_Z})$, and therefore we obtain

$$r(t) - p = D\,(\theta_Z/\tau_Z)^{r-1}(-\log t)^{-\tau_Z(1-r)}(1 + R_1(t))(1 + R_2(t))(1 + R_3(t)),$$

where $R_3(t) = -(-\log t)^{-\tau_Z}$. When $0 < \tau_C < \tau_X \leqslant 1$, it can be checked that the mutliplying constant is indeed equal to $a^{1/\tau_X}(\tau_X/\tau_C)^{\tau_X^{-1}-1}$, and a careful study of the 3 multiplicative remainder terms leads to the stated big O, by relying on relation (5) in Part (iii) of Lemma 1. The other cases are similar, details are omitted. □

3. Proofs of the other propositions

Proposition 2. *Under assumptions (A1) and (A2), with* $(\tau_X, \tau_C) \in [0,1]^2 \backslash \{(0,0), (1,1)\}$*, as well as* $R_l(b,\rho)$ *and* $R_{\tilde{l}}(\tilde{b},\tilde{\rho})$ *and if* (k_n) *satisfies* H_1 *and one of the conditions* $H_2, \ldots, H_6$*, we have, as* $n \to \infty$*,*

$$
\begin{aligned}
&\text{if} \quad 0 \leqslant \tau_X < \tau_C \leqslant 1, \quad \sqrt{k} v_n A_{2,n} = \sqrt{k} v_n (\hat{p}_k - a) = \sqrt{k}(\hat{p}_k - 1) \xrightarrow{\mathbb{P}} 0, \\
&\text{if} \quad 0 < \tau_C < \tau_X \leqslant 1, \quad \sqrt{k} v_n A_{2,n} = \sqrt{k} v_n \left(\left(\tfrac{a\tau_X}{\tau_Z} \right)^{1-\frac{1}{\tau_X}} L_{nk}^{1-\tau_Z/\tau_X} \hat{p}_k - a \right) \xrightarrow{d} N\left(0, a^{2-1/\tau_X} \left(\tfrac{\tau_X}{\tau_C} \right)^{1-1/\tau_X} \right), \\
&\text{if} \quad 0 < \tau_X = \tau_C < 1, \quad \sqrt{k} v_n A_{2,n} = \sqrt{k}(a^{1-1/\tau_X} \hat{p}_k - a) \xrightarrow{d} N\left(0, a^{2-1/\tau_X}(1 - a^{1/\tau_X})\right), \\
&\text{if} \quad 0 = \tau_C < \tau_X < 1, \quad \sqrt{k} v_n A_{2,n} = \sqrt{k} v_n \left((a\tau_X)^{1-\frac{1}{\tau_X}} L_{nk} (\log L_{nk})^{1-\frac{1}{\tau_X}} \hat{p}_k - a \right) \xrightarrow{d} N\left(0, a^{2-1/\tau_X} \tau_X^{1-1/\tau_X}\right).
\end{aligned}
$$

Proof of Proposition 2

The function $p(\cdot)$ being defined in the previous subsection, and proceeding as in [13], we carry on the proof by considering now that δ_i is related to Z_i by

$$\delta_i = \mathbb{I}_{U_i \leqslant p(Z_i)},$$

where $(U_i)_{i \leqslant n}$ denotes an independent sequence of standard uniform variables, independent of the sequence $(Z_i)_{i \leqslant n}$. We denote by $U_{[1,n]}, \ldots, U_{[n,n]}$ the (unordered) values of the uniform sample pertaining to the order statistics $Z_{1,n} \leqslant \ldots \leqslant Z_{n,n}$ of the observed sample $Z_1, \ldots, Z_n$.

Recall that $Z_i = H_Z^- (\exp(K_{\tau_Z}(E_i)))$, where $E_1, \ldots, E_n$ are independent standard exponential random variables (Proposition 1). We introduce, for every $1 \leqslant i \leqslant n$, the standard uniform random variables $V_i = 1 - \exp(-E_i)$ such that

$$Z_i = H_Z^- (\exp(K_{\tau_Z}(-\log(1 - V_i)))) = r(1 - V_i)$$

where the function $r(\cdot)$ was defined before the statement of Lemma 2, which provides valuable information about it. Let us provide a detailed proof of Proposition 2 in the case $0 < \tau_C < \tau_X \leqslant 1$ (the non-Weibull-tail strong censoring case) ; all the other cases are treated similarly. We start by writing

$$
\begin{aligned}
\sqrt{k} v_n A_{2,n} &= \sqrt{k} v_n \left(\left(\tfrac{a\tau_X}{\tau_C} \right)^{1-1/\tau_X} (L_{nk})^{1-\tau_C/\tau_X} \hat{p}_k - a \right) \\
&= \sqrt{k} v_n \left(\frac{a\tau_X}{\tau_C} \right)^{1-1/\tau_X} (L_{nk})^{1-\tau_C/\tau_X} \frac{1}{k} \sum_{j=1}^{k} \left(\mathbb{I}_{U_{[n-j+1,n]} \leqslant r(1 - V_{n-j+1,n})} - \mathbb{I}_{U_{[n-j+1,n]} \leqslant r(j/n)} \right) \\
&\quad + \sqrt{k} v_n \frac{1}{k} \sum_{j=1}^{k} \left(\left(\frac{a\tau_X}{\tau_C} \right)^{1-1/\tau_X} (L_{nk})^{1-\tau_C/\tau_X} \mathbb{I}_{U_{[n-j+1,n]} \leqslant r(j/n)} - a \right) \\
&=: T_{1,k} + T_{2,k}.
\end{aligned}
$$

We will prove below that the term $T_{1,k}$ above converges to 0 in probability. Let us, first, treat the term $T_{2,k}$. We write

$$
\begin{aligned}
T_{2,k} &= \tfrac{1}{\sqrt{k}} v_n \left(\tfrac{a\tau_X}{\tau_C} \right)^{1-1/\tau_X} (L_{nk})^{1-\tau_C/\tau_X} \textstyle\sum_{j=1}^{k} \left(\mathbb{I}_{U_{n-j+1,n} \leqslant r(j/n)} - r(j/n) \right) \\
&\quad + \tfrac{1}{\sqrt{k}} v_n \textstyle\sum_{j=1}^{k} \left(\left(\tfrac{a\tau_X}{\tau_C} \right)^{1-1/\tau_X} (L_{nk})^{1-\tau_C/\tau_X} r(j/n) - a \right) \\
&=: T'_{2,k} + T''_{2,k},
\end{aligned}
$$

Let us prove that $T'_{2,k} \xrightarrow{d} N(0, D)$ where $D = a^{2-1/\tau_X} \left(\tfrac{\tau_X}{\tau_C} \right)^{1-1/\tau_X}$, while $T''_{2,k} \xrightarrow{\mathbb{P}} 0$.

We deduce from Lemma 2 that

$$r(t) = a^{1/\tau_X} \left(\frac{\tau_X}{\tau_C}\right)^{1/\tau_X - 1} (-\log t)^{\tau_C/\tau_X - 1} (1 + o(1)) \to 0.$$

Hence,

$$\begin{aligned}
\mathbb{V}(T'_{2,k}) &= v_n^2 \left(\tfrac{a\tau_X}{\tau_C}\right)^{2-2/\tau_X} (L_{nk})^{2-2\tau_C/\tau_X} \tfrac{1}{k}\textstyle\sum_{j=1}^k r(j/n)(1-r(j/n)) \\
&= v_n^2 D\, (L_{nk})^{1-\tau_C/\tau_X} (1+o(1)) \tfrac{1}{k}\textstyle\sum_{j=1}^k \left(\tfrac{L_{nj}}{L_{nk}}\right)^{\tau_C/\tau_X - 1},
\end{aligned}$$

denoting $L_{nj} = \log(n/j)$. We have $\frac{1}{k}\sum_{j=1}^k \left(\frac{L_{nj}}{L_{nk}}\right)^{\tau_C/\tau_X - 1}$ converges to 1, because $\frac{L_{nj}}{L_{nk}}$ converges uniformly to 1. Consequently,

$$\mathbb{V}(T'_{2,k}) = D v_n^2 \, (L_{nk})^{1-\tau_C/\tau_X} (1 + o(1)) \to D.$$

We conclude, for this term, using Lyapunov's Theorem (details are omitted).

Concerning $T''_{2,k}$, we see that $\left(\frac{a\tau_X}{\tau_C}\right)^{1-1/\tau_X} (L_{nk})^{1-\tau_C/\tau_X} r(j/n) = a + o(1)$. Hence, we need a second order development for $r(j/n)$. According to Lemma 2 (part (ii)), we have

$$\left(\frac{a\tau_X}{\tau_C}\right)^{1-\frac{1}{\tau_X}} L_{nk}^{1-\tau_C/\tau_X} r(j/n) - a = a\left(\left(\frac{L_{nj}}{L_{nk}}\right)^{\frac{\tau_C}{\tau_X}-1} - 1\right) + O(1) L_{nk}^{-\alpha} \left(\frac{L_{nj}}{L_{nk}}\right)^{-\alpha}.$$

where $\alpha = \max\{-\tau_C, \tau_C/\tau_X - 1\}$. Hence,

$$T''_{2,k} = a\sqrt{k} v_n \left(\frac{\tau_C}{\tau_X} - 1\right) L_{nk}^{-1}(1+o(1)) \frac{1}{k}\sum_{j=1}^k \log(k/j) + O(1)\sqrt{k} v_n L_{nk}^{-\alpha}(1+o(1)) \frac{1}{k}\sum_{j=1}^k \left(\frac{L_{nj}}{L_{nk}}\right)^{-\alpha}.$$

But $\frac{1}{k}\sum_{j=1}^k \log(k/j)$ and $\frac{1}{k}\sum_{j=1}^k \left(\frac{L_{nj}}{L_{nk}}\right)^{-\alpha}$ both tend to 1 . Hence, according to assumption H_3 ((ii) or (iii), depending on the closeness of τ_X w.r.t. τ_C), $T''_{2,k}$ indeed tends to 0. This concludes the proof for $T_{2,k}$.

It remains to prove that $T_{1,k}$ above converges to 0 in probability. Following the same lines as in the proof of Lemma 2 (Subsection C.3) in [30], it turns out that this amounts to proving that, for some positive sequence $s_n = k^{-\delta}/n$ ($\delta > 0$) and some constant $c > 0$,

$$\sqrt{k} v_n S_{n,k} \overset{n\to\infty}{\longrightarrow} 0 \quad \text{where} \quad S_{n,k} := \sup\left\{ |r(s) - r(t)| \; ; \; \frac{1}{n} \leqslant t \leqslant \frac{k}{n} \; , \; |s-t| \leqslant c\sqrt{k}/n \; , \; s \geqslant s_n \right\}. \tag{6}$$

In the case considered here, $0 < \tau_C < \tau_X \leqslant 1$, $r(t) = cst(-\log t)^{\tau_C/\tau_X - 1} v(-\log t)$, where v is a slowly varying function such that $v(-\log t)$ tends to 1 when $t \to 0$. Let $h(t) = (-\log t)^{\tau_C/\tau_X - 1}$. Applying the mean value theorem, we obtain

$$\begin{aligned}
|r(t) - r(s)| &\leqslant cst|t-s| \sup_{u\in[s,t]} \left| h'(u) v(-\log u) \left(1 + \tfrac{(-\log u) v'(-\log u)}{v(-\log u)}\right)\right| \\
&\leqslant cst|t-s| \sup_{u\in[s,t]} |h'(u)|,
\end{aligned}$$

since $\frac{tv'(t)}{v(t)}$ tends to 1, as t tends to infinity. This entails that

$$S_{n,k} \leqslant cst \; k^{1/2+\delta} L_{nk}^{\tau_C/\tau_X - 2}.$$

Recall that in this case $v_n = L_{nk}^{\frac{1}{2}(\tau_C/\tau_X - 1)}$. Hence

$$\sqrt{k} v_n S_{n,k} \leqslant cst \left(\sqrt{k} L_{nk}^{\alpha+\delta'}\right)^{2(1+\delta)},$$

for some $\delta' > 0$ and $\alpha = \frac{3}{4}(\tau_C/\tau_X - 1) - \frac{1}{2}$. We easily prove that, if we choose $0 < \delta' < \frac{1}{2}$,

$\sqrt{k}L_{nk}^{\alpha+\delta'} \to 0$, under assumption $H_3(ii)$ or $H_3(iii)$. □

Let us now turn our attention to the proof of Proposition 3. This proof often implies the random functions $\mu_{q,\tau_Z}(t)$ (defined for $q \in \mathbb{N}^*$) and $\sigma^2_{1,\tau_Z}(t)$ which are defined by and satisfy (see Lemma 2 of [15])

$$\mu_{q,\tau_Z}(t) := \mathbb{E}(\theta_{n,q}(t)) = \int_0^\infty (K_{\tau_Z}(x+t) - K_{\tau_Z}(t))^q \, e^{-x} \, dx = (q!)\, t^{q(\tau_Z - 1)}(1+o(1)) \quad (\text{as } t \to +\infty) \tag{7}$$

and $\sigma^2_{1,\tau_Z}(t) := \mu_{2,\tau_Z}(t) - \mu^2_{1,\tau_Z}(t)$. In addition, if for a sample of standard exponential variables $(F_i)_{i \leqslant k_n}$ we define

$$\theta_{n,1}(t) = \frac{1}{k}\sum_{j=1}^{k} (K_{\tau_Z}(F_i + t) - K_{\tau_Z}(t)),$$

then we recall that Lemma 5 of [15] establishes that

$$\sqrt{k}A_{1,n} \xrightarrow{d} N(0,1) \quad \text{where} \quad A_{1,n} := \frac{\theta_{n,1}(E_{n-k}) - \mu_{1,\tau_Z}(E_{n-k})}{\sigma_{1,\tau_Z}(E_{n-k})}. \tag{8}$$

Proposition 3. *Under the conditions of Proposition 2, for all* $1 \leqslant i \leqslant 5$, $\sqrt{k}v_n T_{i,n} \xrightarrow{\mathbb{P}} 0$, *as* n *tends to infinity.*

Proof of Proposition 3

The proofs for the terms $T_{1,n}, \ldots, T_{5,n}$ are respectively detailed in parts $(1), \ldots, (5)$ below.

(1) Recall that $T_{1,n} = R_{n,\tilde{l}}/D_{k,\tau_X}$, where

$$R_{n,\tilde{l}} = \frac{1}{k}\sum_{j=1}^{k} \log\left(\frac{\tilde{l}(\exp(K_{\tau_Z}(E_{n-i+1,n})))}{\tilde{l}(\exp(K_{\tau_Z}(E_{n-k,n})))}\right).$$

According to assumption $R_{\tilde{l}}(\tilde{b}, \tilde{\rho})$, we have $\log\left(\frac{\tilde{l}(tx)}{\tilde{l}(t)}\right) \sim \tilde{b}(t)K_{\tilde{\rho}}(x)$, uniformly for $x \geqslant 1$, as $t \to +\infty$. The Renyi representation yields that $E_{n-i+1,n} - E_{n-k} \stackrel{d}{=} F_{k-i+1,k}$, where $F_1, \ldots, F_k$ are k independent standard exponential random variables. Consequently, taking $t = \exp(K_{\tau_Z}(E_{n-k,n})) \to +\infty$ and $x = \exp(K_{\tau_Z}(E_{n-i+1,n}) - K_{\tau_Z}(E_{n-k,n})) \geqslant 1$, we obtain

$$R_{n,\tilde{l}} \stackrel{d}{=} \tilde{b}(\exp(K_{\tau_Z}(E_{n-k,n})))(1 + o_{\mathbb{P}}(1))\frac{1}{k}\sum_{j=1}^{k} K_{\tilde{\rho}}(\exp(K_{\tau_Z}(F_i + E_{n-k,n}) - K_{\tau_Z}(E_{n-k,n}))).$$

But on one hand, $\sqrt{k}v_n\tilde{b}(\exp(K_{\tau_Z}(E_{n-k,n})))$ tends to 0, under conditions H_2-H_6. On the other hand, since $\frac{\mu_{1,\tau_Z}(E_{n-k})}{\sigma_{1,\tau_Z}(E_{n-k})}$ tends to 1 (thanks to (7)), Corollary 1 (a corollary of Proposition 2 and the result for $T_{2,n}$ proved in the next bullet) yields that $\sigma_{1,\tau_Z}(E_{n-k})/D_{k,\tau_X} \xrightarrow{\mathbb{P}} 1/a$. Consequently, in order to have negligibility of $\sqrt{k}v_nT_{1,n}$, it thus remains to prove that

$$\frac{\frac{1}{k}\sum_{j=1}^{k} K_{\tilde{\rho}}(\exp(K_{\tau_Z}(F_i + E_{n-k,n}) - K_{\tau_Z}(E_{n-k,n})))}{\sigma_{1,\tau_Z}(E_{n-k})}$$

is bounded in probability.

In the cases where $\tilde{\rho}$ is equal to 0, we readily have

$$\frac{1}{k}\sum_{j=1}^{k} K_{\tilde{\rho}}(\exp(K_{\tau_Z}(F_i+E_{n-k,n}) - K_{\tau_Z}(E_{n-k,n}))) = \frac{1}{k}\sum_{j=1}^{k}(K_{\tau_Z}(F_i+E_{n-k,n}) - K_{\tau_Z}(E_{n-k,n})) = \theta_{n,1}(E_{n-k}),$$

and $\frac{\theta_{n,1}(E_{n-k})}{\sigma_{1,\tau_Z}(E_{n-k})} \xrightarrow{\mathbb{P}} 1$ (see (8)). In the cases where $\tilde{\rho} < 0$, we use the fact that $|K_{\tilde{\rho}}(e^u) - u| \leqslant |\tilde{\rho}|\frac{u^2}{2}$, and we easily prove (following the lines of the proof of (8)) that

$$\frac{\frac{1}{k}\sum_{j=1}^{k}\left(K_{\tau_Z}(F_i + E_{n-k,n}) - K_{\tau_Z}(E_{n-k,n})\right)^2}{\sigma_{1,\tau_Z}(E_{n-k})} \xrightarrow{\mathbb{P}} 0.$$

This concludes the proof for $T_{1,n}$.

(2) Recall that $T_{2,n} = -\theta_X \frac{R_{1,n}}{D_{k,\tau_X}}$, where $R_{1,n}$ is defined in Lemma 3 and we have (also in Lemma 3)

$$D_{k,\tau_X} = \hat{\Lambda}_k^{\tau_X - 1}\hat{p}_k + R_{1,n}.$$

It suffices to prove that $\sqrt{k}v_n \frac{R_{1,n}}{\hat{\Lambda}_k^{\tau_X-1}\hat{p}_k} \xrightarrow{\mathbb{P}} 0$. Let us consider the case where $\tau_X \neq 0$ and $\tau_C \neq 0$, and introduce the notations

$$\Lambda_j := \Lambda_X(Z_{n-j+1,n}) \quad \text{and} \quad \hat{\Lambda}_j := \hat{\Lambda}_{nX}(Z_{n-j+1,n}).$$

In this case (except when $\tau_X = 1$, since in that case $R_{1,n} = 0$),

$$R_{1,n} = \frac{\tau_X - 1}{2}\hat{\Lambda}_k^{\tau_X}\frac{1}{k}\sum_{j=1}^{k}\left(\frac{\hat{\Delta}_{j,k}}{\hat{\Lambda}_k}\right)^2 (1 + T_{j,k})^{\tau_X - 2},$$

with $\hat{\Delta}_{j,k} = \hat{\Lambda}_j - \hat{\Lambda}_k$ and $T_{j,k} \in]0, \frac{\hat{\Delta}_{j,k}}{\hat{\Lambda}_k}[$. Since $\tau_X - 2 < 0$, we are led to prove that

$$\sqrt{k}v_n \frac{\hat{\Lambda}_k}{\hat{p}_k}\frac{1}{k}\sum_{j=1}^{k}\left(\frac{\hat{\Delta}_{j,k}}{\hat{\Lambda}_k}\right)^2 \xrightarrow{\mathbb{P}} 0,$$

and, introducing

$$\xi_{j,k} := \frac{\hat{\Lambda}_j}{\Lambda_j}\frac{\Lambda_k}{\hat{\Lambda}_k} - 1 \quad \text{and} \quad d_{j,k} := \frac{\Lambda_j}{\Lambda_k} - 1,$$

we have $(\hat{\Delta}_{j,k}/\hat{\Lambda}_k)^2 = (\frac{\Lambda_j}{\Lambda_k}\xi_{j,k} + d_{j,k})^2 \leqslant 2((\Lambda_j/\Lambda_k)^2\xi_{j,k}^2 + d_{j,k}^2)$. We thus need to prove that

$$\sqrt{k}v_n \frac{\hat{\Lambda}_k}{\hat{p}_k}\frac{1}{k}\sum_{j=1}^{k}\left(\frac{\Lambda_j}{\Lambda_k}\right)^2\xi_{j,k}^2 \xrightarrow{\mathbb{P}} 0 \quad \text{and} \quad \sqrt{k}v_n \frac{\hat{\Lambda}_k}{\hat{p}_k}\frac{1}{k}\sum_{j=1}^{k} d_{j,k}^2 \xrightarrow{\mathbb{P}} 0. \tag{9}$$

Let $E_1, \ldots E_n$ be i.i.d. standard exponential random variables. We have (see Lemma 4 (i))

$$\frac{\Lambda_j}{\Lambda_k} - 1 \stackrel{d}{=} (1 + x_{j,k})^{1/\tau_X} - 1,$$

where

$$\begin{aligned} x_{j,k} &= \frac{\tau_X a K_{\tau_Z}(E_{n-j+1,n}) + \tau_X \log l(\exp(K_{\tau_Z}(E_{n-j+1,n})))+1}{\tau_X a K_{\tau_Z}(E_{n-k,n}) + \tau_X \log l(\exp(K_{\tau_Z}(E_{n-k,n})))+1} - 1 \\ &= (1 + o_{\mathbb{P}}(1))(A_{j,k} + B_{j,k}), \end{aligned}$$

with

$$A_{j,k} = 1 - \frac{K_{\tau_Z}(E_{n-j+1,n})}{K_{\tau_Z}(E_{n-k,n})} \text{ and } B_{j,k} = \frac{1}{aK_{\tau_Z}(E_{n-j+1,n})}\log\left(\frac{l(\exp(K_{\tau_Z}(E_{n-j+1,n})))}{l(\exp(K_{\tau_Z}(E_{n-j+1,n})))}\right).$$

Hence, $d_{j,k} = \tau_X^{-1}(A_{j,k} + B_{j,k})(1 + o_{\mathbb{P}}(1))$. Moreover, the Renyi representation yields that $E_{n-i+1,n} - E_{n-k,n} \stackrel{d}{=} F_{k-i+1,k}$, where $F_1, \ldots, F_k$ are k independent standard exponential

random variables. Consequently,

$$\begin{aligned} A_{j,k} &= 1 - \frac{E^{\tau_Z}_{n-j+1,n}-1}{E^{\tau_Z}_{n-k,n}-1} \\ &\stackrel{d}{=} -\tau_Z \frac{F_{k-j+1,k}}{E_{n-k,n}}(1+o_{\mathbb{P}}(1)). \end{aligned}$$

Concerning $B_{j,k}$, we use the second order condition $R_l(b,\rho)$ for l to write

$$B_{j,k} = \frac{b(\exp(K_{\tau_Z}(E_{n-k,n})))}{aK_{\tau_Z}(E_{n-k,n}))} K_\rho\left(\exp(K_{\tau_Z}(E_{n-j+1,n}) - K_{\tau_Z}(E_{n-k,n}))\right)(1+o_{\mathbb{P}}(1)).$$

Since $(A_{j,k}+B_{j,k})^2 \leqslant 2(A^2_{j,k}+B^2_{j,k})$, we only have to prove that $\sqrt{k}v_n \frac{\hat{\Lambda}_k}{\hat{p}_k}\frac{1}{k}\sum_{j=1}^k A^2_{j,k} \xrightarrow{\mathbb{P}} 0$ and $\sqrt{k}v_n \frac{\hat{\Lambda}_k}{\hat{p}_k}\frac{1}{k}\sum_{j=1}^k B^2_{j,k} \xrightarrow{\mathbb{P}} 0$. Moreover $\Lambda_k \stackrel{d}{=} \left(\frac{a\tau_X}{\tau_Z}\right)^{1/\tau_X}(E_{n-k,n})^{\tau_Z/\tau_X}(1+o_{\mathbb{P}}(1))$, where $\frac{E_{n-k,n}}{L_{nk}} \xrightarrow{\mathbb{P}} 1$ and $\frac{\hat{\Lambda}_k}{\Lambda_k} \xrightarrow{\mathbb{P}} 1$. Hence

$$\sqrt{k}v_n \frac{\hat{\Lambda}_k}{\hat{p}_k}\frac{1}{k}\sum_{j=1}^k A^2_{j,k} \stackrel{d}{=} cste(1+o_{\mathbb{P}}(1))\sqrt{k}v_n \frac{L^{\tau_Z/\tau_X-2}_{nk}}{\hat{p}_k}\frac{1}{k}\sum_{j=1}^k F_j^2.$$

But $\left(\frac{a\tau_X}{\tau_Z}\right)^{1-\frac{1}{\tau_X}}(L_{nk})^{1-\tau_Z/\tau_X}\hat{p}_k \xrightarrow{\mathbb{P}} a$, according to Proposition 2. Consequently, $\sqrt{k}v_n \frac{\hat{\Lambda}_k}{\hat{p}_k}\frac{1}{k}\sum_{j=1}^k A^2_{j,k} \stackrel{d}{=} O_{\mathbb{P}}(1)\sqrt{k}v_n L^{-1}_{nk}$, which, using assumptions $H_2,\dots H_4$, goes to 0 in probability.

Now, according to Lemma 5 in [15], we have

$$\frac{1}{\mu_{2,\tau_Z}(E_{n-k})}\frac{1}{k}\sum_{j=1}^k K^2_\rho\left(\exp(K_{\tau_Z}(E_{n-j+1,n}) - K_{\tau_Z}(E_{n-k,n}))\right) \xrightarrow{\mathbb{P}} cst.$$

Hence,

$$\begin{aligned} \sqrt{k}v_n \frac{\hat{\Lambda}_k}{\hat{p}_k}\frac{1}{k}\sum_{j=1}^k B^2_{j,k} &\stackrel{d}{=} cst(1+o_{\mathbb{P}}(1))\sqrt{k}v_n \frac{L^{\tau_Z/\tau_X}_{nk}}{\hat{p}_k}\left(\frac{b(\exp(K_{\tau_Z}(E_{n-k,n})))}{aK_{\tau_Z}(E_{n-k,n}))}\right)^2 \mu_{2,\tau_Z}(E_{n-k}) \\ &\stackrel{d}{=} cst(1+o_{\mathbb{P}}(1))\sqrt{k}v_n L^{-1}_{nk} b^2(\exp(K_{\tau_Z}(E_{n-k,n})))), \end{aligned}$$

since $\mu_{2,\tau_Z}(E_{n-k}) \sim 2L^{2(\tau_Z-1)}_{nk}$, according to Lemma 2 in [15]. The second part of relation (9) is thus proved.

Let us now deal with the first part of relation (9). We have

$$\xi_{j,k} = \frac{\hat{\Lambda}_j}{\Lambda_j}\frac{\Lambda_k}{\hat{\Lambda}_k} - 1 = \left(\frac{\Lambda_k}{\hat{\Lambda}_k}\right)\left(\Delta_j \frac{\Lambda_k}{\Lambda_j} - \Delta_{k+1}\right)\Lambda_k^{-1},$$

where $\Delta_j := \hat{\Lambda}_j - \Lambda_j$ and $\Delta_{k+1} := \hat{\Lambda}_k - \Lambda_k$. Lemmas 6 and 7 in [30] ensure that $|\Delta_j| = O_{\mathbb{P}}(1/\sqrt{j-1})$ for all $j = 2,\dots,k+1$, $|\Delta_1| = O_{\mathbb{P}}(1)$ and $\frac{E_{n-k,n}}{L_{nk}} \xrightarrow{\mathbb{P}} 1$. Since in addition both $\frac{\Lambda_k}{\hat{\Lambda}_k}$ and $\frac{\Lambda_k}{\Lambda_j}$ tend to 1 in probability, and the latter is $\leqslant 1$, we thus obtain $|\xi_{1,n}| \leqslant (1+o_{\mathbb{P}}(1))\left(O_{\mathbb{P}}(1)+O_{\mathbb{P}}(1/\sqrt{k})\right)\Lambda_k^{-1}$ and

$$|\xi_{j,n}| \leqslant (1+o_{\mathbb{P}}(1))\left(O_{\mathbb{P}}(1/\sqrt{j-1})+O_{\mathbb{P}}(1/\sqrt{k})\right)\Lambda_k^{-1}, \text{ for } j=2,\dots,k.$$

Therefore,

$$\sqrt{k}v_n \frac{\hat{\Lambda}_k}{\hat{p}_k}\frac{1}{k}\sum_{j=1}^k \left(\frac{\Lambda_j}{\Lambda_k}\right)^2 \xi^2_{j,k} \leqslant (1+o_{\mathbb{P}}(1))\frac{v_n}{\sqrt{k}}(\Lambda_k \hat{p}_k)^{-1}\left(O_{\mathbb{P}}(1) + \sum_{j=2}^k O_{\mathbb{P}}(1/(j-1))\right).$$

But $\Lambda_k \stackrel{d}{=} cst(1+o_{\mathbb{P}}(1))L^{\tau_Z/\tau_X}_{nk}$ and, according to Proposition 2, $L^{1-\tau_Z/\tau_X}_{nk}\hat{p}_k = cst(1+o_{\mathbb{P}}(1))$.

Consequently

$$\sqrt{k}v_n\frac{\hat{\Lambda}_k}{\hat{p}_k}\frac{1}{k}\sum_{j=1}^{k}\left(\frac{\Lambda_j}{\Lambda_k}\right)^2\xi_{j,k}^2 \leqslant O_{\mathbb{P}}(1)\sqrt{k}v_nL_{nk}^{1-2\tau_Z/\tau_X}\frac{\log k}{k},$$

due to $\frac{1}{k}\sum_{j=1}^{k}\frac{1}{j}\sim\frac{\log k}{k}$. If $\tau_Z=\tau_X$ (thus $v_n=1$), then the right-hand side above becomes $O_{\mathbb{P}}(1)\sqrt{k}L_{nk}^{-1}\frac{\log k}{k}$, which tends to 0 in probability, under assumption H_2 or H_4. If $\tau_Z=\tau_C<\tau_X$ (thus $v_n=L_{nk}^{(\tau_C/\tau_X-1)/2}$), let $0<\epsilon<\frac{1}{2}$ and write

$$\sqrt{k}v_nL_{nk}^{1-2\tau_Z/\tau_X}\frac{\log k}{k}=\sqrt{k}v_nL_{nk}^{1-2\tau_C/\tau_X}k^{\epsilon-1}o(1)=L_{nk}^{\frac{3}{2}\frac{\tau_C}{\tau_X}-\frac{1}{2}}k^{\epsilon-1/2}o(1)=(\sqrt{k}L_{nk}^{-b})^{2\epsilon-1}o(1),$$

where $-b>\frac{3}{2}\frac{\tau_C}{\tau_X}-\frac{1}{2}$ It remains to ensure that $\sqrt{k}L_{nk}^{\frac{3}{2}\frac{\tau_C}{\tau_X}-\frac{1}{2}}$ tends to infinity : this is the case under assumption $H_3(i)$.

(3) Recall that $T_{3,n}=-\theta_X(\hat{\Lambda}_k^{\tau_X-1}-\Lambda_k^{\tau_X-1})(D_{k,\tau_X})^{-1}\hat{p}_k$. Since $D_{k,\tau_X}=\hat{\Lambda}_k^{\tau_X-1}\hat{p}_k+R_{1,n}$, according to Lemma 3 (stated in 4 below) and $R_{1,n}/D_{k,\tau_X}=o_{\mathbb{P}}(1)$ (term $T_{2,n}$ in Proposition 3), we obtain that

$$T_{3,n}=O_{\mathbb{P}}(1)\left(1-\left(\frac{\Lambda_k}{\hat{\Lambda}_k}\right)^{\tau_X-1}\right).$$

But $|\Lambda_k-\hat{\Lambda}_k|=O_{\mathbb{P}}(k^{-1/2})$ (see Lemma 7 in [30]). Hence

$$|T_{3,n}|\leqslant O_{\mathbb{P}}(k^{-1/2})\Lambda_k^{-1}.$$

But $\Lambda_k=K_{\tau_X}^{-}\left(aK_{\tau_Z}(E_{n-k})+\log l(\exp(K_{\tau_Z}(E_{n-k})))\right)$ (see statement (i) of Lemma 4). In the case where both τ_X and τ_C are not equal to 0 (the other cases are treated similarly), this yields that $\Lambda_k=O_{\mathbb{P}}(1)L_{nk}^{\tau_Z/\tau_X}$. Since $v_nL_{nk}^{-\tau_Z/\tau_X}=o_{\mathbb{P}}(1)$, this concludes the proof for $T_{3,n}$.

(4) Recall that $T_{4,n}=-\theta_X\frac{R_{2,n}}{D_{k,\tau_X}}$, where $R_{2,n}$ is defined in the statement of Lemma 5.
Let us consider the case where $\tau_X>0$ and $\tau_C>0$. If $\tau_X=1$, then $R_{2,n}=0$ and there is nothing to prove, so we suppose $\tau_X\in]0,1[$. We then have

$$R_{2,n}=\left(\frac{a\tau_X}{\tau_Z}\right)^{1-\frac{1}{\tau_X}}(E_{n-k,n})^{\tau_Z(1-\frac{1}{\tau_X})}\hat{p}_k\left((1-E_{n-k,n}^{-\tau_Z})^{1-\frac{1}{\tau_X}}\left(1+\frac{1+\tau_X\log l(\exp(K_{\tau_Z}(E_{n-k,n})))}{a\tau_XK_{\tau_Z}(E_{n-k,n})}\right)^{1-\frac{1}{\tau_X}}-1\right)$$

According to Lemma 3 (stated in Section 4 below) and the fact that $\frac{\hat{\Lambda}_k}{\Lambda_k}\xrightarrow{\mathbb{P}}1$, since

$$\Lambda_k=\left(\frac{a\tau_X}{\tau_Z}\right)^{1/\tau_X}(E_{n-k,n})^{\tau_Z/\tau_X}(1+o_{\mathbb{P}}(1)),$$

it remains to prove that $\sqrt{k}v_nR_n$, where

$$R_n:=\left((1-E_{n-k,n}^{-\tau_Z})^{1-\frac{1}{\tau_X}}\left(1+\frac{1+\tau_X\log l(\exp(K_{\tau_Z}(E_{n-k,n})))}{a\tau_XK_{\tau_Z}(E_{n-k,n})}\right)^{1-\frac{1}{\tau_X}}-1\right).$$

But $l(x)$ tends to a constant c that can be 0, as x tends to $+\infty$. Hence,

$$R_n=bE_{n-k,n}^{-\tau_Z}(1+o_{\mathbb{P}}(1))\ \text{ if }\ c\neq 0\ \text{ and }\ R_n=cst\frac{\log l(\exp(K_{\tau_Z}(E_{n-k,n})))}{K_{\tau_Z}(E_{n-k,n})}(1+o_{\mathbb{P}}(1))\ \text{ if }\ c=0,$$

where $b=(1/\tau_X-1)(1-a^{-1}\tau_Z/\tau_X-\tau_Z/a\log c)$. According to Lemma 1 (part (iii)), in the cases when $c=0$, we have $\frac{\log l(e^x)}{x}=cst.x^{r-1}(1+o(1))$ as $x\to+\infty$. Consequently,

$$R_n=cst.L_{nk}^{\tau_Z(r-1)}(1+o_{\mathbb{P}}(1)).$$

Hence, $\sqrt{k}v_nR_n \xrightarrow{\mathbb{P}} 0$, under assumption H_2 or H_3. The cases when $c \neq 0$ are treated similarly. This concludes the proof for $T_{4,n}$ when $\tau_X > 0$ and $\tau_C > 0$. The other cases ($\tau_X = 0$ or $\tau_C = 0$) can be treated similarly, details are omitted.

(5) Recall that $T_{5,n} = \theta_Z(1 + o_{\mathbb{P}}(1))R_{3,n}$, and that, in the case $\tau_X \neq 0$ and $\tau_C \neq 0$,

$$R_{3,n} = \left(\frac{a\tau_X}{\tau_Z}\right)^{1-1/\tau_X} \hat{p}_k \left(\frac{(E_{n-k})^{\tau_Z(1-1/\tau_X)}}{\mu_{1,\tau_Z}(E_{n-k})} - (L_{nk})^{1-\tau_Z/\tau_X}\right).$$

But, according to Proposition 2, $R_{3,n} = a(1 + o_{\mathbb{P}}(1))R_n$, where

$$R_n := \frac{L_{nk}^{\frac{\tau_Z}{\tau_X}-1}(E_{n-k})^{\tau_Z(1-\frac{1}{\tau_X})}}{\mu_{1,\tau_Z}(E_{n-k})} - 1 = R_n^{(1)} + R_n^{(2)} + R_n^{(3)},$$

and

$$\begin{aligned}
R_n^{(1)} &:= \frac{L_{nk}^{\frac{\tau_Z}{\tau_X}-1}}{\mu_{1,\tau_Z}(E_{n-k})}\left((E_{n-k})^{\tau_Z(1-\frac{1}{\tau_X})} - L_{nk}^{\tau_Z(1-\frac{1}{\tau_X})}\right),\\
R_n^{(2)} &:= L_{nk}^{\tau_Z-1}\left(\frac{1}{\mu_{1,\tau_Z}(E_{n-k})} - \frac{1}{\mu_{1,\tau_Z}(L_{nk})}\right)\\
R_n^{(3)} &:= \frac{L_{nk}^{\tau_Z-1}}{\mu_{1,\tau_Z}(L_{nk})} - 1.
\end{aligned}$$

Let us prove that $\sqrt{k}v_nR_n^{(i)}$ tend to 0 , for $i = 1, 2, 3$.

Concerning $R_n^{(1)}$, we use Lemma 4 of [15] to prove that $\sqrt{k}$ times the large brackets in the definition of $R_n^{(1)}$ is $O_{\mathbb{P}}(1)L_{nk}^{\tau_Z(1-\frac{1}{\tau_X})-1}$. Moreover, $\frac{L_{nk}^{\tau_Z-1}}{\mu_{1,\tau_Z}(E_{n-k})}$ tends to 1, in probability, according to (7). Consequently, $\sqrt{k}v_nR_n^{(1)} = O_{\mathbb{P}}(1)v_nL_{nk}^{-1}$, which tends to 0.

Concerning $R_n^{(2)}$, we also use Lemma 4 of [15] to prove that $\sqrt{k}$ times the large brackets in the definition of $R_n^{(2)}$ is $O_{\mathbb{P}}(1)\frac{\mu'_{1,\tau_Z}(L_{nk}(1+o_{\mathbb{P}}(1)))}{\mu^2_{1,\tau_Z}(L_{nk}(1+o_{\mathbb{P}}(1)))}$. Since $\frac{L_{nk}^{\tau_Z-1}}{\mu_{1,\tau_Z}(L_{nk})}$ tends to 1, we obtain that

$$\sqrt{k}v_nR_n^{(2)} = O_{\mathbb{P}}(1)v_n\frac{\mu'_{1,\tau_Z}(L_{nk}(1+o_{\mathbb{P}}(1)))}{\mu_{1,\tau_Z}(L_{nk}(1+o_{\mathbb{P}}(1)))}\frac{\mu_{1,\tau_Z}(L_{nk})}{\mu_{1,\tau_Z}(L_{nk}(1+o_{\mathbb{P}}(1)))},$$

which tends to 0, according to Lemma 2 (iii) of [15].

Concerning $R_n^{(3)}$, recall that, if $\tau \neq 0$, $\mu_{1,\tau}(t) = \int_0^{+\infty}(K_\tau(x+t) - K_\tau(t))e^{-x}\,dx$ and $t^{\tau-1} = K'_\tau(t)$. This entails that

$$\begin{aligned}
\tfrac{\mu_{1,\tau}(t)}{t^{\tau-1}} &= \int_0^{+\infty} x\tfrac{K_\tau(x+t)-K_\tau(t)}{xK'_\tau(t)}e^{-x}\,dx - \int_0^{+\infty} xe^{-x}\,dx\\
&= \int_0^{+\infty} \tfrac{x}{2}\tfrac{K''_\tau(t+\alpha)}{K'_\tau(t)}xe^{-x}\,dx\ (\alpha \in]0, x[)\\
&= \int_0^{+\infty} \tfrac{\tau-1}{2}\tfrac{x^2}{t}(1+\eta\tfrac{x}{t})^{\tau-2}e^{-x}\,dx\ (\eta \in]0, 1[)
\end{aligned}$$

Hence $R_n^{(3)} = \frac{1-\tau_Z}{2}L_{nk}^{-1}(1 + o_{\mathbb{P}}(1))$ and $\sqrt{k}v_nR_n^{(3)} = O_{\mathbb{P}}(1)v_nL_{nk}^{-1}$, which tends to 0 under assumptions H_2, H_3, H_4. □

The following corollary is a consequence of Proposition 2 and Proposition 3 (term $T_{2,n}$), and is also related to Lemma 3 stated a few lines below. It was used above in the proof of the term $T_{1,n}$ of Proposition 3.

Corollary 1. *Under the conditions of Proposition 2, we have* $\dfrac{D_{k,\tau_X}}{\mu_{1,\tau_Z}(E_{n-k})} \xrightarrow{\mathbb{P}} a$, *as n tends to infinity.*

Its proof is particularly short : according to Lemma 3, and since $\mu_{1,\tau_Z}(t) \sim t^{\tau_Z-1}$ as $t \to \infty$ (see

relation (7)), we have indeed

$$\frac{D_{k,\tau_X}}{\mu_{1,\tau_Z}(E_{n-k})} = L_{nk}^{1-\tau_Z}\Lambda_k^{\tau_X-1}\hat{p}_k(1+o(1)) \stackrel{d}{=} (A_{2,n}+a)(1+o(1)) \xrightarrow{\mathbb{P}} a.$$

The next proposition is the version of Proposition 3 adapted to the setting of Theorem 2. Its proof is very similar to the proof of Proposition 3, and is omitted.

Proposition 4. *Under the conditions of Proposition 2, for all* $1 \leqslant i \leqslant 5$, $\sqrt{k}v_nTT_{i,n} \xrightarrow{\mathbb{P}} 0$, *as* n *tends to infinity.*

4. Other technical Lemmas

Lemma 3. *The denominator of the estimator* $\hat{\theta}_{X,\tau_X}$ *satisfies the relation*

$$D_{k,\tau_X} = \frac{1}{k}\sum_{j=1}^{k} K_{\tau_X}(\hat{\Lambda}_{nX}(Z_{n-j+1,n})) - K_{\tau_X}(\hat{\Lambda}_{nX}(Z_{n-k,n})) = \hat{\Lambda}_k^{\tau_X-1}\hat{p}_k + R_{1,n},$$

where

$$R_{1,n} = \begin{cases} \frac{\tau_X-1}{2}\hat{\Lambda}_k^{\tau_X}\frac{1}{k}\sum_{j=1}^{k}\left(\frac{\hat{\Delta}_{j,k}}{\hat{\Lambda}_k}\right)^2(1+T_{j,k})^{\tau_X-2}, & \text{if } 0<\tau_X<1, \\ \frac{1}{k}\sum_{j=1}^{k}\left(\log\left(1+\frac{\hat{\Delta}_{j,k}}{\hat{\Lambda}_k}\right)-\frac{\hat{\Delta}_{j,k}}{\hat{\Lambda}_k}\right) & \text{if } \tau_X=0, \\ 0 & \text{if } \tau_X=1 \end{cases}$$

with, for each $j=1,\dots,k$, $\hat{\Delta}_{j,k} := \hat{\Lambda}_{nX}(Z_{n-j+1,n}) - \hat{\Lambda}_{nX}(Z_{n-k,n})$ *and the random variable* $T_{j,k}$ *lies between* 0 *and* $\frac{\hat{\Delta}_{j,k}}{\hat{\Lambda}_k}$.

Proof of Lemma 3
It is straightforward via Taylor's formula and the definition of function K_{τ_Z} (the negligibility of $R_{1,n}$ is another story, it is dealt with in the proof of Proposition 3, part (2)). □

For the following lemma, recall that (E_i) denote the i.i.d. standard exponential variable (E_i) satisfying $Z_i = H_Z^-(\exp(K_{\tau_Z}(E_i))$, and that $l(\cdot)$ denotes the slowly varying function which properties are described in Lemma 1 and which is such that $H_X \circ H_Z^-(x) = x^a l(x)$. Note that in part (ii) of this lemma, the results also hold when one replaces $E_{n-k,n}$ by L_{nk}, or replaces $Z_{n-k,n}$ and $E_{n-k,n}$ by $Z_{n-j+1,n}$ and $E_{n-j+1,n}$ (this will occasionally prove useful).

Lemma 4. (i) *For every* $i=1,\dots,n$, *and whether* $\tau_Z>0$ *or is equal to* 0, *we have*

$$\Lambda_X(Z_i) = K_{\tau_X}^-\big(aK_{\tau_Z}(E_i) + \log l(\exp K_{\tau_Z}(E_i))\big).$$

(ii) *When* $\tau_Z>0$, *we have*

$$\Lambda_X(Z_{n-k,n}) = \left(a\frac{\tau_X}{\tau_Z}\right)^{1/\tau_X} E_{n-k,n}^{\tau_Z/\tau_X}(1+o_{\mathbb{P}}(1)) = \left(a\frac{\tau_X}{\tau_Z}\right)^{1/\tau_X} E_{n-k,n}^{\tau_Z/\tau_X}\left(1+\beta E_{n-k,n}^{-\alpha}(1+o_{\mathbb{P}}(1))\right) \tag{10}$$

for some constant β *and exponent* $\alpha = \begin{cases} \tau_Z & \text{when either } \tau_X=\tau_C, \text{ or } \tau_X\neq\tau_C \text{ and } r\leqslant 0, \\ \tau_Z(1-r) & \text{when } \tau_X\neq\tau_C \text{ and } r\in]0,1[. \end{cases}$

When $0=\tau_X<\tau_C$, *we have* $\Lambda_X(Z_{n-k,n}) = E_{n-k,n}l(E_{n-k,n}) = E_{n-k,n}(1+o_{\mathbb{P}}(1))$.
When $0=\tau_C<\tau_X$, *we have*

$$\Lambda_X(Z_{n-k,n}) = (a\tau_X)^{1/\tau_X}(\log E_{n-k,n})^{1/\tau_X}\left(1+\beta(\log E_{n-k,n})^{-1}(1+o_{\mathbb{P}}(1))\right).$$

Note that the constant β is negative in the case $\tau_X \neq \tau_C$ and $r \in]0,1[$.

Proof of Lemma 4

The first statement (i) holds because on one hand, since $\bar{F}_X \in A_1(\tau_X, \theta_X)$, we have $\Lambda_X(x) = K^-_{\tau_X}(\log H_X(x))$, and on the other hand, $Z_i = H^-_Z(\exp(K_{\tau_Z}(E_i)))$ where $H_X \circ H^-_Z(x) = x^a l(x)$.

The second statement is essentially a consequence of the first one and of some of the second order results contained in Lemma 1. Suppose for the moment that $\tau_Z > 0$, *i.e.* $\tau_X > 0$ and $\tau_C > 0$. We thus have $K^-_{\tau_X}(x) = (\tau_X x + 1)^{1/\tau_X}$. Hence, noting temporarily $\phi(x) = \log l(\exp x)/x$, it is easy to see that (i) implies

$$\begin{aligned}\Lambda_X(Z_{n-k,n}) &= \{(a\tau_X K_{\tau_Z}(E_{n-k,n}) + \tau_X \log l(\exp(K_{\tau_Z}(E_{n-k,n}))) + 1\}^{1/\tau_X} \\ &= (a\tau_X)^{1/\tau_X} \left(K_{\tau_Z}(E_{n-k,n})\right)^{1/\tau_X} \left\{1 + (a\tau_X K_{\tau_Z}(E_{n-k,n}))^{-1} + a^{-1}\phi(K_{\tau_Z}(E_{n-k,n}))\right\}^{1/\tau_X}\end{aligned}$$

But $K_{\tau_Z}(E_{n-k,n}) = E^{\tau_Z}_{n-k,n}(1 - E^{-\tau_Z}_{n-k,n})/\tau_Z = E^{\tau_Z}_{n-k,n}(1 + o_{\mathbb{P}}(1))$, so

$$\Lambda_X(Z_{n-k,n}) = (a\tau_X/\tau_Z)^{1/\tau_X} E^{\tau_Z/\tau_X}_{n-k,n}\left(1 - \frac{1}{\tau_X}E^{-\tau_Z}_{n-k,n}(1 + o_{\mathbb{P}}(1))\right) \times B_n$$

where B_n denotes the quantity in curly brackets above. Thanks to part (iii) of Lemma 1, we have

$$B_n = 1 + \frac{\tau_Z}{a\tau_X}E^{-\tau_Z}_{n-k,n}(1 + o_{\mathbb{P}}(1)) + cst.E^{-\alpha}_{n-k,n}(1 + o_{\mathbb{P}}(1))$$

where either $\alpha = \tau_Z$ and $cst = (\log c)\tau_Z/a$ (when $\tau_X = \tau_C$ or $\tau_X \neq \tau_C$ and $r \leqslant 0$) or $\alpha = \tau_Z$ and $cst = -\nu a^{-1}\tau_Z^{1-r} < 0$ (when $\tau_X \neq \tau_C$ and $r \in]0,1[$). The proof is thus over when $\tau_Z > 0$.

The cases $\tau_X = 0$ and $\tau_C > 0$, or $\tau_C = 0$ and $\tau_X > 0$, can be proved similarly. When $0 = \tau_X < \tau_C$, we have $\tau_Z = 0$ and $a = 1$ so it immediately comes $\Lambda_X(Z_{n-k,n}) = E_{n-k,n}l(E_{n-k,n}) = E_{n-k,n}(1 + o_{\mathbb{P}}(1))$ (because $c = 1$ in that case, see Lemma 1). When $0 = \tau_C < \tau_X$, we have $\tau_Z = 0$ and thus

$$\Lambda_X(Z_{n-k,n}) = \{a\tau_X \log(E_{n-k,n}) + \tau_X \log l(E_{n-k,n}) + 1\}^{1/\tau_X}$$

The end of the proof is then very similar to the first case covered in details above.

The fact that relation (10) also holds when $E_{n-k,n}$ is replaced by L_{nk} is due to Lemma 4 in [15], which states that $\sqrt{k}(E_{n-k,n} - L_{nk})$ converges in distribution to a standard normal variable. □

Lemma 5. *Let $E_1, \ldots, E_n$ be i.i.d. standard exponential random variables.*

$$\Lambda_k^{\tau_X - 1}\hat{p}_k = \begin{cases} \left(\frac{a\tau_X}{\tau_Z}\right)^{1-1/\tau_X} E^{\tau_Z(1-1/\tau_X)}_{n-k,n}\hat{p}_k + R_{2,n}, & \text{if } \tau_X \neq 0 \text{ and } \tau_C \neq 0 \\ \frac{\hat{p}_k}{E_{n-k,n}} + R_{2,n}, & \text{if } 0 = \tau_X < \tau_C < 1 \\ (a\tau_X)^{1-1/\tau_X}\left(\log(E_{n-k,n})\right)^{1-1/\tau_X}\hat{p}_k + R_{2,n} & \text{if } 0 = \tau_C < \tau_X < 1, \end{cases}$$

where

$$R_{2,n} = \begin{cases} \left(\frac{a\tau_X}{\tau_Z}\right)^{1-\frac{1}{\tau_X}} E^{\tau_Z(1-\frac{1}{\tau_X})}_{n-k,n}\hat{p}_k\left((1 - E^{-\tau_Z}_{n-k,n})^{1-\frac{1}{\tau_X}}\left(1 + \frac{1+\tau_X \log l(\exp(K_{\tau_Z}(E_{n-k,n})))}{a\tau_X K_{\tau_Z}(E_{n-k,n})}\right)^{1-\frac{1}{\tau_X}} - 1\right), \\ \quad \text{if } 0 < \tau_X < 1 \text{ and } \tau_C \neq 0 \\ \frac{\hat{p}_k}{E_{n-k,n}}\left(\frac{1}{l(E_{n-k,n})} - 1\right), \text{ if } 0 = \tau_X < \tau_C < 1 \\ (a\tau_X)^{1-\frac{1}{\tau_X}}\left(\log(E_{n-k,n})\right)^{1-\frac{1}{\tau_X}}\hat{p}_k\left(\left(1 + \frac{1+\tau_X \log l(E_{n-k,n})}{a\tau_X \log(E_{n-k,n})}\right)^{1-\frac{1}{\tau_X}} - 1\right), \quad \text{if } 0 = \tau_C < \tau_X < 1 \\ 0, \quad \text{if } \tau_X = 1 \end{cases}$$

Proof of Lemma 5

Using part (i) of Lemma 4, we have

$$\Lambda_k = K^-_{\tau_X}\left(aK_{\tau_Z}(E_{n-k,n}) + \log l(\exp(K_{\tau_Z}(E_{n-k,n})))\right),$$

which yields, in the case $\tau_X \neq 0$ and $\tau_C \neq 0$,

$$\Lambda_k^{\tau_X - 1} = \left(\frac{a\tau_X}{\tau_Z}\right)^{1-\frac{1}{\tau_X}} E_{n-k,n}^{\tau_Z(1-\frac{1}{\tau_X})} (1 - E_{n-k,n}^{-\tau_Z})^{1-\frac{1}{\tau_X}} \left(1 + \frac{1 + \tau_X \log l(\exp(K_{\tau_Z}(E_{n-k,n})))}{a\tau_X K_{\tau_Z}(E_{n-k,n})}\right)^{1-\frac{1}{\tau_X}}.$$

The expression of $R_{2,n}$ follows in this case. The other cases are similar. □

Lemma 6. *Let $E_1, \ldots, E_n$ be i.i.d. standard exponential random variables.*

$$\Lambda_k^{-1}\hat{p}_k = \begin{cases} \left(\frac{a\tau_X}{\tau_Z}\right)^{-1/\tau_X} E_{n-k,n}^{-\tau_Z/\tau_X}\hat{p}_k + RR_{2,n}, & \text{if } \tau_X \neq 0 \text{ and } \tau_C \neq 0 \\ \frac{\hat{p}_k}{E_{n-k,n}} + RR_{2,n}, & \text{if } 0 = \tau_X < \tau_C < 1 \\ (a\tau_X)^{-1/\tau_X} \left(\log(E_{n-k,n})\right)^{-1/\tau_X} \hat{p}_k + RR_{2,n} & \text{if } 0 = \tau_C < \tau_X < 1, \end{cases}$$

where

$$RR_{2,n} = \begin{cases} \left(\frac{a\tau_X}{\tau_Z}\right)^{-\frac{1}{\tau_X}} E_{n-k,n}^{-\frac{\tau_Z}{\tau_X}}\hat{p}_k \left((1 - E_{n-k,n}^{-\tau_Z})^{-\frac{1}{\tau_X}} \left(1 + \frac{1+\tau_X \log l(\exp(K_{\tau_Z}(E_{n-k,n})))}{a\tau_X K_{\tau_Z}(E_{n-k,n})}\right)^{-\frac{1}{\tau_X}} - 1\right), \\ \quad \text{if } 0 < \tau_X < 1 \text{ and } \tau_C \neq 0 \\ \frac{\hat{p}_k}{E_{n-k,n}} \left(\frac{1}{l(E_{n-k,n})} - 1\right), \text{ if } 0 = \tau_X < \tau_C < 1 \\ (a\tau_X)^{-\frac{1}{\tau_X}} \left(\log(E_{n-k,n})\right)^{-\frac{1}{\tau_X}} \hat{p}_k \left(\left(1 + \frac{1+\tau_X \log l(E_{n-k,n})}{a\tau_X \log(E_{n-k,n})}\right)^{-\frac{1}{\tau_X}} - 1\right), \quad \text{if } 0 = \tau_C < \tau_X < 1 \\ 0, \quad \text{if } \tau_X = 1 \end{cases}$$

The proof of the previous lemma is very similar to the one of Lemma 5, it is therefore omitted. The following one is an easy consequence of Lemma 4.

Lemma 7. *Under the assumptions of Proposition 2, we have, as $n \to \infty$,*

$$\begin{array}{ll} \text{if } \tau_X \neq 0 \text{ and } \tau_C \neq 0, & \log(\Lambda_k) = \frac{\tau_Z}{\tau_X} \log L_{nk}(1 + o_{\mathbb{P}}(1)) \\ \text{if } \tau_X = 0, & \log(\Lambda_k) = a \log L_{nk}(1 + o_{\mathbb{P}}(1)) \\ \text{if } \tau_X \neq 0, \text{ and } \tau_C = 0 & \log(\Lambda_k) = \frac{1}{\tau_X} \log\log L_{nk}(1 + o_{\mathbb{P}}(1)) \end{array}$$

Finally, the next lemma is used inside the proof of Theorem 4.

Lemma 8. *Under the assumptions of Theorem 4, we have, as n tends to infinity,*

$$\int_{L_k}^{\Lambda_k} u^{\tau_X - 1} \log u \, du = \begin{cases} O_{\mathbb{P}}(\log L_{nk}) & \text{if } \tau_X \neq 0, \tau_C \neq 0 \text{ and } (\tau_X = \tau_C \text{ or } r \leqslant 0), \\ O_{\mathbb{P}}(L_{nk}^{r\tau_Z} \log L_{nk}) & \text{if } \tau_X \neq 0, \tau_C \neq 0, \tau_X \neq \tau_C \text{ and } r \in]0,1[, \\ O_{\mathbb{P}}(\log\log L_{nk}) & \text{if } \tau_X \neq 0 \text{ and } \tau_C = 0, \\ o_{\mathbb{P}}(\log L_{nk}) & \text{if } \tau_X = 0. \end{cases}$$

Proof of Lemma 8

We only treat the case where both τ_X and τ_C are positive. In this case, recall that $L_k = (a\tau_X/\tau_Z)^{1/\tau_X} (L_{nk})^{\tau_Z/\tau_X}$ and, according to Lemma 4, $\frac{\Lambda_k}{L_k} \xrightarrow{\mathbb{P}} 1$. We have (with $v = u/L_k$)

$$\begin{aligned} \textstyle\int_{L_k}^{\Lambda_k} u^{\tau_X - 1} \log u \; du &= L_k^{\tau_X} \textstyle\int_1^{\Lambda_k/L_k} v^{\tau_X - 1}(\log v + \log L_k) \; dv \\ &= \frac{L_k^{\tau_X}}{\tau_X} \log\left(\frac{\Lambda_k}{L_k}\right)\left(\frac{\Lambda_k}{L_k}\right)^{\tau_X} - \frac{L_k^{\tau_X}}{\tau_X^2}\left(\left(\frac{\Lambda_k}{L_k}\right)^{\tau_X} - 1\right) + \log L_k \frac{L_k^{\tau_X}}{\tau_X}\left(\left(\frac{\Lambda_k}{L_k}\right)^{\tau_X} - 1\right). \end{aligned}$$

An immediate consequence of Lemma 4 is that both $\log\left(\frac{\Lambda_k}{L_k}\right)$ and $\left(\frac{\Lambda_k}{L_k}\right)^{\tau_X} - 1$ are $O_{\mathbb{P}}\left((L_{nk})^{-\tau_Z}\right)$ if $\tau_X = \tau_C$ or $r \leqslant 0$, and are $O_{\mathbb{P}}\left((L_{nk})^{-\tau_Z(r-1)}\right)$ if $\tau_X \neq \tau_C$ and $r \in]0,1[$. The result follows easily. □

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