

SUPPLEMENTARY MATERIAL FOR : ESTIMATION OF EXTREMES FOR  
HEAVY-TAILED AND LIGHT-TAILED DISTRIBUTIONS IN THE PRESENCE OF  
RANDOM CENSORING

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This document provides the proofs of all the Lemmas and Propositions of the manuscript *Estimation of extremes for heavy-tailed and light-tailed distributions in the presence of random censoring*. For ease of reference, a number of assumptions, definitions and notations are recalled here and there in this document. Propositions 2 and 3 are the cornerstones for proving the theorems stated in the main manuscript.

In all this document,  $cst$  will denote a positive generic constant which exact value does not need to be explicitied.

## 1. Assumptions

We recall here some of the assumptions under which our asymptotic results are proved.

### 1.1. Assumptions on the model

The main assumption is

**Assumption (A1)** : there exist  $\tau_X \in [0, 1]$ ,  $\tau_C \in [0, 1]$ ,  $\theta_X > 0$ ,  $\theta_C > 0$  such that

$$F_X \in A_1(\tau_X, \theta_X) \quad \text{and} \quad F_C \in A_1(\tau_C, \theta_C).$$

This means that there exists positive functions  $H_X$  and  $H_C$  such that

$$\bar{F}_X(x) = 1 - F_X(x) = \exp(-K_{\tau_X}^-(\log(H_X(x)))) \quad \text{and} \quad \bar{F}_C(x) = 1 - F_C(x) = \exp(-K_{\tau_C}^-(\log(H_C(x))))$$

and, for some slowly varying functions  $\bar{l}_X$  and  $\bar{l}_C$  at infinity,

$$H_X^-(x) = x^{\theta_X} \bar{l}_X(x) \quad \text{and} \quad H_C^-(x) = x^{\theta_C} \bar{l}_C(x).$$

It is clear that under this condition we also have  $H_X(x) = x^{1/\theta_X} l_X(x)$  and  $H_C(x) = x^{1/\theta_C} l_C(x)$  where both  $l_X$  and  $l_C$  are slowly varying functions at infinity.

Remind that  $(\tau_X, \tau_C) = (0, 0)$  and  $(\tau_X, \tau_C) = (1, 1)$  have not been considered in this paper.

The second important assumption is about the second order tail properties of  $F_X$  and  $F_C$  :

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**Assumption (A2)** : there exist some negative constants  $\rho_X$  and  $\rho_C$ , and some rate functions  $b_X$  and  $b_C$  having constant sign at  $+\infty$  and satisfying  $|b_X| \in RV_{\rho_X}$  and  $|b_C| \in RV_{\rho_C}$ , such that, as  $t \rightarrow +\infty$ ,

$$\frac{l_X(tx)/l_X(t) - 1}{b_X(t)} \rightarrow K_{\rho_X}(x), \quad \text{and} \quad \frac{l_C(tx)/l_C(t) - 1}{b_C(t)} \rightarrow K_{\rho_C}(x), \quad \forall x > 0. \quad (1)$$

where  $K_\rho(x) = (x^\rho - 1)/\rho$  if  $\rho < 0$ , and  $K_0(x) = \log(x)$ .

Consider the functions :

$$H_Z^-(x) = x^{\theta_Z} \tilde{l}(x) \quad \text{and} \quad H_X \circ H_Z^-(x) = x^a l(x) \quad \text{with} \quad a := \frac{\theta_Z}{\theta_X}, \quad (2)$$

where both  $\tilde{l}$  and  $l$  are slowly varying. The crucial parameter  $a = \theta_Z/\theta_X$  is equal to 1 in "low censoring" situations (in particular when  $\tau_X < \tau_C$ ).

Our important technical Lemma 1 ensures that functions  $H_Z^-$  and  $H_X \circ H_Z^-$  also satisfy a second order condition SR2. For technical reasons though, we need to consider the following stronger conditions on  $\tilde{l}$  and  $l$ , respectively noted  $R_{\tilde{l}}(\tilde{b}, \tilde{\rho})$  and  $R_l(b, \rho)$ , and defined by :

**Assumption  $R_\ell(B, \rho)$**  : for some constant  $\rho \leq 0$  and a rate function  $B$  satisfying  $\lim_{x \rightarrow +\infty} B(x) = 0$ , such that for all  $\epsilon > 0$ , we have

$$\sup_{\lambda \geq 1} \left| \frac{\ell(\lambda x)/\ell(x) - 1}{B(x)K_\rho(\lambda)} - 1 \right| \leq \epsilon, \quad \text{for } x \text{ sufficiently large.}$$

## 1.2. Assumptions on the sample fraction $(k_n)$

The basic assumption on  $(k_n)$  is

$$H_1 : k \rightarrow +\infty, \quad \frac{k}{n} \rightarrow 0, \quad \frac{\log k}{\log n} \rightarrow 0, \quad \text{as } n \rightarrow +\infty.$$

Introducing the important notation

$$L_{nk} = \log(n/k),$$

let  $v_n$  be the factor which contributes to the rates of convergence of our estimators, it depends on the censoring strength in the tail :

$$v_n := \begin{cases} 1 & \text{if } 0 < \tau_X < \tau_C \leq 1 \quad \text{or} \quad 0 < \tau_X = \tau_C < 1 \quad \text{or} \quad 0 = \tau_X < \tau_C \leq 1, \\ L_{nk}^{\frac{1}{2}(\frac{\tau_C}{\tau_X} - 1)} & \text{if } 0 < \tau_C < \tau_X \leq 1, \\ L_{nk}^{-1/2} (\log L_{nk})^{\frac{1}{2}(\frac{1}{\tau_X} - 1)} & \text{if } 0 = \tau_C < \tau_X \leq 1. \end{cases}$$

We also consider the following conditions

$$H_2 : 0 < \tau_X < \tau_C \leq 1 \quad \text{and} \quad \begin{cases} (i) \sqrt{k} L_{nk}^{\tau_X/\tau_C - 1} \rightarrow 0 \text{ if } \frac{1}{\tau_C} - \frac{1}{\tau_X} \geq -1 \\ (ii) \sqrt{k} L_{nk}^{-\tau_X} \rightarrow 0 \text{ if } \frac{1}{\tau_C} - \frac{1}{\tau_X} < -1 \end{cases}$$

$$H_3 : 0 < \tau_C < \tau_X \leq 1 \quad \text{and} \quad \begin{cases} (i) \sqrt{k} v_n \rightarrow +\infty \\ (ii) \sqrt{k} v_n L_{nk}^{\tau_C/\tau_X - 1} \rightarrow 0 \text{ if } \frac{1}{\tau_X} - \frac{1}{\tau_C} \geq -1 \\ (iii) \sqrt{k} v_n L_{nk}^{-\tau_C} \rightarrow 0 \text{ if } \frac{1}{\tau_X} - \frac{1}{\tau_C} < -1 \end{cases}$$

$$H_4 : 0 < \tau_X = \tau_C < 1 \quad \text{and} \quad \sqrt{k} L_{nk}^{-\tau_X} \rightarrow 0$$

$$H_5 : 0 = \tau_X < \tau_C \leq 1 \quad \text{and} \quad \exists \delta > 0, \sqrt{k} L_{nk}^{\tilde{\rho} + \delta} \rightarrow 0$$

$$H_6 : 0 = \tau_C < \tau_X \leq 1 \quad \text{and} \quad \begin{cases} (i) \sqrt{k} v_n \rightarrow +\infty \\ (ii) \sqrt{k} v_n (\log L_{nk})^{-1} \rightarrow 0 \end{cases}$$

(in assumption  $H_5$  above,  $\tilde{\rho}$  denotes the second order parameter associated to the slowly varying function  $\tilde{l}$ , which is negative in this case ; see formula (2) as well as Lemma 1)

## 2. Second Order expansions

**Proposition 1.** *Under Assumption (A1), the distribution function of  $Z = \min(X, C)$  satisfies condition  $A_1(\tau_Z, \theta_Z)$ , where*

$$\tau_Z = \min(\tau_X, \tau_C) \quad \text{and} \quad \theta_Z = \begin{cases} \theta_X & \text{if } 0 \leq \tau_X < \tau_C \leq 1 \\ \theta_C & \text{if } 0 \leq \tau_C < \tau_X \leq 1 \\ (\theta_X^{-1/\tau_Z} + \theta_C^{-1/\tau_Z})^{-\tau_Z} & \text{if } 0 < \tau_X = \tau_C \leq 1 \\ \min(\theta_X, \theta_C) & \text{if } \tau_X = \tau_C = 0 \end{cases}$$

Therefore, there exists  $x_* > 0$  such that for any  $x \geq x_*$ , we have

$$\mathbb{P}(Z > x) = \exp(-K_{\tau_Z}^-(\log(H_Z(x)))),$$

where  $H_Z^- \in RV_{\theta_Z}$ . Consequently, if  $E$  denotes a standard exponential distribution, we have

$$Z = H_Z^-(\exp K_{\tau_Z}(E)).$$

*Proof of Proposition 1*

Let us first consider the case  $0 < \tau_X < \tau_C \leq 1$  (note that the case  $\tau_X = \tau_C = 0$  is already treated in Worms and Worms (2019)). We are going to prove that  $\Lambda_Z(x) = K_{\tau_Z}^-(\log(H_Z(x)))$ , where  $H_Z(x) = x^{1/\theta_Z} l_Z(x)$ , with, in this case,  $\tau_Z = \tau_X$ ,  $\theta_Z = \theta_X$  and  $l_Z$  is a slowly varying function to be determined.

Recall that  $H_X(x) = x^{1/\theta_X} l_X(x)$ ,  $H_C(x) = x^{1/\theta_C} l_C(x)$ ,

$$K_{\tau_X}(t) = \frac{t^{\tau_X} - 1}{\tau_X} \quad \text{and} \quad K_{\tau_X}^-(y) = (1 + \tau_X y)^{1/\tau_X}.$$

We thus have by Assumption (A1)

$$\begin{aligned} \Lambda_X(x) &= (1 + \tau_X \log H_X(x))^{1/\tau_X} = \left(1 + \frac{\tau_X}{\theta_X} \log x + \tau_X \log l_X(x)\right)^{1/\tau_X} \\ &= \left(\frac{\tau_X}{\theta_X}\right)^{1/\tau_X} (\log x)^{1/\tau_X} (1 + \epsilon_X(x))^{1/\tau_X} \end{aligned} \quad (3)$$

and similarly

$$\Lambda_C(x) = \left(\frac{\tau_C}{\theta_C}\right)^{1/\tau_C} (\log x)^{1/\tau_C} (1 + \epsilon_C(x))^{1/\tau_C}$$

where we set  $\epsilon_V(x) := \frac{\theta_V/\tau_V + \theta_V \log l_V(x)}{\log x}$ , for  $V$  being  $X$  or  $C$ .

By independence of  $X$  and  $C$  (independent censoring), the cumulative hazard function of  $Z$  is the sum of those of  $X$  and  $C$ . Hence

$$\begin{aligned} K_{\tau_X}(\Lambda_Z(x)) &= K_{\tau_X}(\Lambda_X(x) + \Lambda_C(x)) \\ &= K_{\tau_X}(\Lambda_X(x)(1 + \lambda(x))), \end{aligned}$$

where

$$\begin{aligned} \lambda(x) &= \frac{\Lambda_C(x)}{\Lambda_X(x)} \\ &= \left(\frac{\tau_C}{\theta_C}\right)^{1/\tau_C} \left(\frac{\tau_X}{\theta_X}\right)^{-1/\tau_X} (\log x)^{1/\tau_C - 1/\tau_X} (1 + \epsilon_C(x))^{1/\tau_C} (1 + \epsilon_X(x))^{-1/\tau_X}. \end{aligned}$$

Since both  $\epsilon_X$  and  $\epsilon_C$  tend to 0, it is also the case for the function  $\lambda$ . Consequently, we define

$$\begin{aligned} H_Z(x) &:= \exp(K_{\tau_X}(\Lambda_Z(x))) \\ &= \exp\left\{\tau_X^{-1} [(\Lambda_X(x))^{\tau_X} ((1 + \lambda(x))^{\tau_X} - 1) + ((\Lambda_X(x))^{\tau_X} - 1)]\right\}. \end{aligned}$$

But  $\exp\left(\frac{(\Lambda_X(x))^{\tau_X} - 1}{\tau_X}\right) = K_{\tau_X}(\Lambda_X(x)) = \log H_X(x)$ , hence

$$H_Z(x) = H_X(x) \exp\left\{\tau_X^{-1} (\Lambda_X(x))^{\tau_X} [(1 + \lambda(x))^{\tau_X} - 1]\right\}.$$

Using (3) and a Taylor expansion of order 2 of  $(1 + \lambda(x))^{\tau_X}$ , we thus obtain

$$H_Z(x) = x^{1/\theta_X} l_Z(x) \quad \text{with} \quad l_Z(x) = l_X(x) \cdot \phi(x)$$

where, introducing the constant  $B(X, C) = \left(\frac{\tau_C}{\theta_C}\right)^{1/\tau_C} \left(\frac{\tau_X}{\theta_X}\right)^{1-1/\tau_X}$ ,

$$\begin{aligned}\phi(x) &= \exp\left(\theta_X^{-1}(\log x)(1 + \epsilon_X(x))(\tau_X \lambda(x) + \frac{\tau_X(\tau_X-1)}{2}\lambda^2(x) + o(\lambda^2(x)))\right) \\ &= \exp\left(B(X, C)(\log x)^r (1 + \epsilon_C(x))^{1/\tau_C} (1 + \epsilon_X(x))^{1-1/\tau_X} \left(1 + \frac{\tau_X-1}{2}\lambda(x) + o(\lambda(x))\right)\right),\end{aligned}$$

with  $r = 1 - \left|\frac{1}{\tau_C} - \frac{1}{\tau_X}\right| < 1$ .

Now, using the expression of  $l_X$  and  $l_C$  in Lemma 1 (i), we can perform a Taylor expansion of  $\epsilon_X$ ,  $\epsilon_C$  and  $\lambda$  to obtain that (after some careful computations)

$$\phi(x) = \exp\{B(X, C)(\log x)^r(1 + R(x))\}, \quad (4)$$

where

$$R(x) = \begin{cases} \frac{cst}{\log x}(1 + o(1)) & \text{if } r \leq 0 \\ \frac{cst}{(\log x)^{1-r}} + \frac{cst}{\log x}(1 + o(1)) & \text{if } r > 0. \end{cases}$$

It is then easy to check that  $\phi$  is slowly varying at infinity and so does  $l_Z$ . We thus have proved that  $\Lambda_Z(x) = K_{\tau_Z}^-(\log(H_Z(x)))$  with, in this case,  $\tau_Z = \tau_X$ ,  $\theta_Z = \theta_X$  and  $H_Z(x) = x^{1/\theta_Z}l_Z(x)$  where  $l_Z = l_X\phi$  is slowly varying. In addition, one can remark that

$$\phi(x) \rightarrow \begin{cases} 1 & \text{if } r < 0, \\ \exp(B(X, C)) & \text{if } r = 0, \\ \infty & \text{if } 0 < r < 1. \end{cases}$$

Concerning the other cases ( $0 < \tau_C < \tau_X \leq 1$ ,  $0 = \tau_X < \tau_C \leq 1$ ,  $0 = \tau_C < \tau_X \leq 1$  and  $0 < \tau_X = \tau_C \leq 1$ ), they are proved similarly : the functions  $\lambda(x)$  and  $\phi(x)$  slightly differ from the case above, but the proof is substantially the same. Of particular interest is the proof for the case  $0 < \tau_X = \tau_C \leq 1$  which starts by setting  $\tau_Z = \tau_X = \tau_C$  and writing

$$\begin{aligned}K_{\tau_Z}(\Lambda_Z(x)) &= \frac{1}{\tau_Z} \{(\Lambda_X(x) + \Lambda_C(x))^{\tau_Z} - 1\} \\ &= \frac{1}{\tau_Z} \left\{ \left[ \left( (\tau_Z/\theta_X)^{1/\tau_Z} + (\tau_Z/\theta_C)^{1/\tau_Z} \right) (\log x)^{1/\tau_Z} (1 + o(1)) \right]^{\tau_Z} - 1 \right\} \\ &= \left( \theta_X^{-1/\tau_Z} + \theta_C^{-1/\tau_Z} \right)^{\tau_Z} (\log x)(1 + o(1)) - \tau_Z^{-1}\end{aligned}$$

which identifies  $\theta_Z$  as  $\theta_Z = \left( \theta_X^{-1/\tau_Z} + \theta_C^{-1/\tau_Z} \right)^{-\tau_Z}$ , since  $H_Z(x) = \exp(K_{\tau_Z}(\Lambda_Z(x)))$  is then of the order  $x^{1/\theta_Z}$ .

Note that in this case, with a more refined expansion of  $K_{\tau_Z}(\Lambda_Z(x))$  (under the assumptions of Lemma 1 below), it can be proved that  $H_Z(x) = x^{1/\theta_Z} e^{-1/\tau_Z} \phi(x)$  with  $\lim_{x \rightarrow \infty} \phi(x) = \exp((c_X + c_C)\tau_Z/\theta_Z) \in ]0, +\infty[$ .  $\square$

Lemma 1 stated below provides details about the second order properties of the functions  $H_Z^-$  and  $H_X \circ H_Z^-$  (and therefore, on the behavior of the variables  $Z_i$  and  $\Lambda_X(Z_i)$ ). These properties not only depend on the position of the parameters  $\tau_X$  and  $\tau_C$  with respect to each other, but on their proximity through the parameter  $r$  defined by

$$r := 1 - \left| \frac{1}{\tau_C} - \frac{1}{\tau_X} \right| \in [-\infty, 1]$$

(if either  $\tau_X = 0$  or  $\tau_C = 0$ , indeed consider that  $r = -\infty$ ). This parameter  $r$  appears in the function  $\phi(x)$  which is introduced in the previous proof of Proposition 1.

**Lemma 1.** *Let conditions (A<sub>1</sub>) and (A<sub>2</sub>) hold.*

(i) *For different slowly varying functions generically noted  $v$ , we have*

$$\begin{aligned}l_X(x) &= c_X(1 - x^{\rho_X}v(x)) & \text{and} & & l_C(x) &= c_C(1 - x^{\rho_C}v(x)) \\ \bar{l}_X(x) &= c_X^{-\theta_X}(1 - x^{\theta_X\rho_X}v(x)) & \text{and} & & \bar{l}_C(x) &= c_C^{-\theta_C}(1 - x^{\theta_C\rho_C}v(x)).\end{aligned}$$

(ii) *The slowly varying functions  $\tilde{l}$  and  $l$  associated to  $H_Z^-$  and  $H_X \circ H_Z^-$  satisfy a second order condition SR2 : as  $t \rightarrow +\infty$ ,*

$$\frac{\tilde{l}(tx) - 1}{\tilde{l}(t)} \longrightarrow K_{\tilde{\rho}}(x) \quad \text{and} \quad \frac{l(tx) - 1}{l(t)} \longrightarrow K_{\rho}(x)$$

where

$$\tilde{\rho} = \rho = \begin{cases} \max(\theta_X \rho_X, -1) & \text{if } 0 = \tau_X < \tau_C < 1 \\ \max(\theta_C \rho_C, -1) & \text{if } 0 = \tau_C < \tau_X < 1 \\ 0 & \text{in the other cases,} \end{cases}$$

and  $|\tilde{b}| \in RV_{\tilde{\rho}}$  and  $|b| \in RV_{\rho}$ . When  $\rho = 0$ , both  $b(t)$  and  $\tilde{b}(t)$  are (as  $t \rightarrow +\infty$ ) of the order  $O((\log t)^{r-1})$  when  $r \neq 0$ , and of the order  $O((\log t)^{-2})$  when  $r = 0$ .

(iii) The slowly varying function  $l_Z$  associated to  $H_Z$  satisfies

$$\lim_{x \rightarrow +\infty} l_Z(x) = c_Z \begin{cases} \in ]0, +\infty[ & \text{if } \tau_X = \tau_C \text{ or } r \leq 0, \\ = +\infty & \text{if } \tau_X \neq \tau_C \text{ and } r \in ]0, 1[ \end{cases}$$

where in particular  $c_Z = c_X$  if  $\tau_X < \tau_C$  and  $r < 0$ , and  $c_Z = c_C$  if  $\tau_C < \tau_X$  and  $r < 0$ . Moreover we have (with the convention  $(+\infty)^{-\theta} = 0$  when  $\theta > 0$ )

$$\tilde{l}(t) \rightarrow \tilde{c} := c_Z^{-\theta z} \quad \text{and} \quad l(t) \rightarrow c := c_X \tilde{c}^{1/\theta x}, \quad \text{as } t \rightarrow +\infty.$$

When  $\tau_X = \tau_C$  or  $r \leq 0$ , both  $c$  and  $\tilde{c}$  are positive. When  $\tau_X \neq \tau_C$  and  $r \in ]0, 1[$ , both  $\tilde{c}$  and  $c$  are zero and the following relation holds for some  $\nu > 0$ , as  $x \rightarrow \infty$

$$\frac{\log l(\exp x)}{x} = -\nu x^{r-1}(1 + o(1)) \rightarrow 0 \quad \text{and} \quad \frac{\log \tilde{l}(\exp x)}{x} = -\theta_X \nu x^{r-1}(1 + o(1)) \rightarrow 0 \quad (5)$$

*Proof of Lemma 1*

*Part (i)*

Let us prove the statements for  $l_X$  and  $\bar{l}_X$ , the slowly varying functions involved in  $H_X$  and  $H_X^-$ . The expansion for  $l_X$  is a direct consequence of the assumption (A2) (which states that  $\rho_X < 0$ ) and Lemma 3 in Hua and Joe (2011) (a corollary of Theorem B.2.2 in de Haan and Ferreira (2006), thereafter referred to as Hua & Joe's Lemma). Assumptions (A1) and (A2) also imply that, for every  $x$

$$\frac{\frac{H_X(tx)}{H_X(t)} - x^{1/\theta_X}}{b_X(t)} \xrightarrow{t \rightarrow \infty} x^{1/\theta_X} \frac{x^{\rho_X} - 1}{\rho_X}.$$

It is a rather standard technique in extreme value theory that Vervaat's Lemma implies, for every  $y$ ,

$$\frac{\frac{H_X^-(ty)}{H_X^-(t)} - y^{\theta_X}}{-\theta_X^2 b_X(H_X^-(t))} \xrightarrow{t \rightarrow \infty} y^{\theta_X} \frac{y^{\theta_X \rho_X} - 1}{\rho_X} \quad \text{and thus} \quad \frac{\frac{\bar{l}_X(ty)}{\bar{l}_X(t)} - 1}{\bar{b}_X(t)} \xrightarrow{t \rightarrow \infty} \frac{y^{\theta_X \rho_X} - 1}{\theta_X \rho_X}$$

for some slowly varying  $\bar{b}_X$ . Therefore, the slowly varying function  $\bar{l}_X$  associated to  $H_X^-$  satisfies the second order condition with index  $\theta_X \rho_X < 0$  and, for the same reason as above (Hua & Joe's Lemma), we have the expansion  $\bar{l}_X(y) = \bar{c}_X(1 - y^{\theta_X \rho_X} v(y))$  for some slowly varying function  $v(y)$  and constant  $\bar{c}_X$ . Finally, the fact that the limit  $\bar{c}_X$  of  $\bar{l}_X$  at infinity is  $c_X^{-\theta_X}$  can be justified using the following lines : if  $y = H_X(x) = x^{1/\theta_X} l_X(x)$ , then at infinity  $y^{\theta_X} \bar{l}_X(y) = H_X^-(y) \sim x = y^{\theta_X} (l_X(x))^{-\theta_X}$ , and thus, since  $\lim_{\infty} l_X = c_X$ , we have  $\lim_{\infty} \bar{l}_X = c_X^{-\theta_X}$ .

*Part (ii)*

We only provide details in the case  $0 < \tau_X < \tau_C \leq 1$ , the other cases are similar. The proof of Proposition 1 yields that

$$H_Z(x) = x^{1/\theta_Z} l_Z(x) = x^{1/\theta_X} l_Z(x)$$

where  $l_Z(x) = l_X(x) \cdot \phi(x)$ . Starting from (4), one can prove that

$$\frac{\phi(tx)}{\phi(t)} - 1 = \begin{cases} cst(\log t)^{r-1}(1 + o(1))(\log x) & \text{if } r \neq 0, \\ cst(\log t)^{-2}(1 + o(1))(\log x) & \text{if } r = 0, \end{cases}$$

i.e.  $\phi$  satisfies the SR2 condition with a  $\rho$ -coefficient equal to 0. Moreover, since

$$\frac{l_Z(tx)}{l_Z(t)} - 1 = \frac{l_X(tx)}{l_X(t)} \left( \frac{\phi(tx)}{\phi(t)} - 1 \right) + \frac{l_X(tx)}{l_X(t)} - 1,$$

it is clear that  $l_Z$  satisfies the SR2 condition with  $\rho_Z = 0$  and

$$b_Z(t) = \begin{cases} O((\log t)^{r-1}) & \text{if } r \neq 0, \\ O((\log t)^{-2}) & \text{if } r = 0. \end{cases}$$

The result for  $\tilde{l}$  (associated to  $H_Z^-$ ) thus follows using Vervaat's Lemma :  $\tilde{l}$  satisfies the SR2 condition with  $\tilde{\rho} = 0$  and

$$\tilde{b}(t) = \begin{cases} O((\log t)^{r-1}) & \text{if } r \neq 0, \\ O((\log t)^{-2}) & \text{if } r = 0. \end{cases}$$

Now, recall that

$$H_X \circ H_Z^-(x) = (x^{1/\theta_Z} \tilde{l}(x))^{1/\theta_X} l_X(H_Z^-(x)) = x^a l(x),$$

with  $a = \frac{\theta_Z}{\theta_X}$  ( $= 1$  in the present case  $0 < \tau_X < \tau_C \leq 1$ ) and  $l(x) = l_X(H_Z^-(x))(\tilde{l}(x))^{1/\theta_X}$ . Consequently,

$$\frac{l(tx)}{l(t)} - 1 = \frac{l_X(H_Z^-(tx))}{l_X(H_Z^-(t))} \left( \frac{\tilde{l}(tx)}{\tilde{l}(t)} - 1 \right) + \frac{l_X(H_Z^-(tx))}{l_X(H_Z^-(t))} - 1,$$

and clearly  $l$  also satisfies an SR2 condition with  $\rho = \max(0, \rho_X) = 0$  and  $b$  of the same order as  $\tilde{b}$ .

This phenomenon  $\rho = \tilde{\rho} = 0$  holds in other cases (not detailed here), except when either  $\tau_X$  or  $\tau_C$  is equal to zero. For instance, when  $0 = \tau_X < \tau_C \leq 1$ , in the proof of Proposition 1, the function  $\epsilon(x) = \Lambda_C(x)/\Lambda_X(x)$  is of the order of  $(\log x)^{1/\tau_C} x^{-1/\theta_X}$ , and it implies that  $l_Z(x) = l_X(x)\phi(x)$  where, this time, the function  $\phi$  satisfies the SR2 condition with a  $\rho$ -coefficient equal to  $-1/\theta_X < 0$ . As a consequence, the coefficient  $\rho_Z$  associated to the SR2 condition for  $l_Z$  is now  $\max\{\rho_X, -1/\theta_X\}$ , which yields  $\tilde{\rho} = \max\{\theta_X \rho_X, -1\} < 0$  as announced.

*Part (iii)*

Again in the case  $0 < \tau_X < \tau_C \leq 1$ , the limiting result for  $l_Z$  is an easy consequence of the fact that  $l_Z = l_X \phi$ , with  $l_X$  tending to  $c_X$  and the limit of  $\phi$  stated in the proof of Proposition 1. In the case  $0 < \tau_X = \tau_C < 1$ , as said at the end of the proof of Proposition 1, we have  $\lim_{x \rightarrow +\infty} l_Z(x) \in ]0, +\infty[$ . The other cases are similar to the first one, details are omitted.

Concerning now the function  $\tilde{l}$ , the same argument as in Part (i) of the proof yields that  $\tilde{l}$  tends to  $\tilde{c} = c_Z^{-\theta_Z}$ , with the convention  $(+\infty)^{-\theta_X} = 0$ . Concerning the limit  $c = c_X \tilde{c}^{1/\theta_X}$  of  $l$ , it is a consequence of the relation  $l(x) = l_X(H_Z^-(x))(\tilde{l}(x))^{1/\theta_X}$  (in the case  $0 < \tau_X < \tau_C \leq 1$ ).

In the case where  $c_Z = 0$  (*i.e.* when  $r \in ]0, 1[$  and  $\tau_X \neq \tau_C$ ), both  $\tilde{c}$  and  $c$  are equal to 0 and we use the fact that

$$\tilde{l}(x) \sim (l_Z(H_Z^-(x)))^{-\theta_Z} \text{ and } l(x) = l_X(H_Z^-(x))(\tilde{l}(x))^{1/\theta_X}$$

to deduce the statement (5), after some calculations. These rates will prove useful in two occasions later in the proofs.  $\square$

Let us now turn our attention to the second order expansion for the function  $p(\cdot)$  defined by

$$p(x) = \mathbb{P}(\delta = 1 | Z = x).$$

The following lemma provides useful expansions of functions  $p(\cdot)$  and  $r(\cdot)$

$$r(t) = p \circ H_Z^-(\exp(K_{\tau_Z}(-\log t))),$$

which are crucial to derive the properties of the random proportion  $\hat{p}_k$  (and therefore the statements of Proposition 2).

**Lemma 2.** *Let us define the following constants (for  $\tau_X$  and  $\tau_C$  positive only)*

$$A_X = \theta_X(\tau_X^{-1} - 1)(\tau_X^{-1} + \log c_X), \quad A_C = \theta_C(\tau_C^{-1} - 1)(\tau_C^{-1} + \log c_C)$$

and

$$A = A_C - A_X \quad \text{and} \quad B = \frac{\theta_X}{\theta_C} \left( \frac{\tau_X}{\theta_X} \right)^{1-1/\tau_X} \left( \frac{\tau_C}{\theta_C} \right)^{1/\tau_C-1}.$$

Let assumptions  $(A_1)$  and  $(A_2)$  hold.

(i) We have, as  $x \rightarrow +\infty$ ,

$$p(x) \rightarrow p := \begin{cases} 1 & \text{if } 0 \leq \tau_X < \tau_C \leq 1, \\ 0 & \text{if } 0 \leq \tau_C < \tau_X \leq 1, \\ \frac{\theta_X^{1/\tau_X}}{(\theta_X^{1/\tau_X} + \theta_C^{1/\tau_X})} = a^{1/\tau_X} & \text{if } 0 < \tau_X = \tau_C < 1, \end{cases}$$

and, more precisely,

$$p(x) - p = \begin{cases} D(\log x)^{r-1} [1 + g(r)(\log x)^{\max(-1, r-1)}(1 + o(1))] & \text{if } 0 < \tau_X \neq \tau_C \leq 1, \\ D x^{-1/\theta_X} (\log x)^{\tau_C^{-1}-1} [1 + A_C(\log x)^{-1}(1 + o(1))] & \text{if } 0 = \tau_X < \tau_C \leq 1, \\ D x^{-1/\theta_C} (\log x)^{\tau_X^{-1}-1} [1 + A_X(\log x)^{-1}(1 + o(1))] & \text{if } 0 = \tau_C < \tau_X \leq 1, \\ D(\log x)^{-1}(1 + O(1/\log x)) & \text{if } 0 < \tau_C = \tau_X < 1, \end{cases}$$

where

$$D = \begin{cases} -B & \text{if } 0 < \tau_X < \tau_C \leq 1, \\ B^{-1} & \text{if } 0 < \tau_C < \tau_X \leq 1, \\ -(\tau_C/\theta_C)^{\tau_C^{-1}-1}(\theta_X/\theta_C c_X) & \text{if } 0 = \tau_X < \tau_C \leq 1, \\ (\tau_X/\theta_X)^{\tau_X^{-1}-1}(\theta_C/\theta_X c_C) & \text{if } 0 = \tau_C < \tau_X \leq 1, \\ -AB(1+B)^{-2} & \text{if } 0 < \tau_C = \tau_X < 1, \end{cases}$$

and

$$g(r) = \begin{cases} A\mathbb{1}_{r < 0} + (A-B)\mathbb{1}_{r=0} + (-B)\mathbb{1}_{r \in ]0,1[} & \text{if } 0 < \tau_X < \tau_C \leq 1, \\ (-A)\mathbb{1}_{r < 0} + (-A-B^{-1})\mathbb{1}_{r=0} + (-B^{-1})\mathbb{1}_{r \in ]0,1[} & \text{if } 0 < \tau_C < \tau_X \leq 1. \end{cases}$$

(ii) When  $\tau_Z > 0$  and  $\tau_X \neq \tau_C$ , as  $t \downarrow 0$  we have

$$r(t) - p = D(\theta_Z/\tau_Z)^{r-1}(-\log t)^{-\tau_Z(1-r)} \left(1 + O\left((-\log t)^{-\tau_Z \min\{1, 1-r\}}\right)\right),$$

in particular, when  $0 < \tau_C < \tau_X \leq 1$ ,

$$r(t) = a^{1/\tau_X} (\tau_X/\tau_C)^{\tau_X^{-1}-1} (-\log t)^{\frac{\tau_C}{\tau_X}-1} \left(1 + O\left((-\log t)^{\max\{-\tau_C, \tau_C/\tau_X-1\}}\right)\right).$$

When  $\tau_Z > 0$  and  $\tau_X = \tau_C$ , we have

$$r(t) - p = -AB \left[ (1+B)^2 (\theta_Z/\tau_Z) \right]^{-1} (-\log t)^{-\tau_Z} \left(1 + O\left((-\log t)^{-\tau_Z}\right)\right).$$

When  $\tau_Z = 0$ , if  $\tau_+ = \max(\tau_X, \tau_C)$  we have

$$r(t) - p = cst (-\log t)^{-1} (\log \log(1/t))^{\frac{1}{\tau_+}-1} \left(1 + O\left((\log \log(1/t))^{-1}\right)\right).$$

with the constant being equal to  $\tau_X^{\frac{1}{\tau_X}-1} a^{1/\tau_X}$  when  $0 = \tau_C < \tau_X \leq 1$ .

*Proof of Lemma 2 :*

This proof is even more technical than the previous ones. As in the main part of the paper, a complete proof of all the cases would be too lengthy, we only provide here a sketch of the proof, focusing on some subcases.

*Part (i)*

We start by introducing  $f_X$  and  $f_C$  the respective probability density functions of  $X$  and  $C$ . By Assumption (A1), the cumulative hazard function  $\Lambda_X$  of  $X$  is defined by  $\Lambda_X(x) = -\log \bar{F}_X(x) = K_{\tau_X}^-(\log H_X(x))$ , and its derivative is  $f_X(x)/\bar{F}_X(x)$ , which is therefore equal to  $(K_{\tau_X}^-)'(\log H_X(x)) \times H_X'(x)/H_X(x)$ . The following thus comes easily

$$p(x) = \frac{\bar{F}_C(x)f_X(x)}{\bar{F}_C(x)f_X(x) + \bar{F}_X(x)f_C(x)} = \left(1 + \frac{(K_{\tau_C}^-)'(\log H_C(x)) H_C'(x)/H_C(x)}{(K_{\tau_X}^-)'(\log H_X(x)) H_X'(x)/H_X(x)}\right)^{-1}.$$

Consider first the case where neither  $\tau_X$  nor  $\tau_C$  is zero. Since  $H_X(x) = x^{1/\theta_X} l_X(x)$  and  $(K_{\tau}^-)'(u) = (\tau u + 1)^{1/\tau-1}$ , we have

$$(K_{\tau_X}^-)'(\log H_X(x)) = \left(\frac{\tau_X}{\theta_X} \log x + \tau_X \log l_X(x) + 1\right)^{1/\tau_X-1} \quad \text{and} \quad \frac{H_X'(x)}{H_X(x)} = \frac{1}{\theta_X x} \left(1 + \theta_X \frac{x l_X'(x)}{l_X(x)}\right),$$

where the slow variation of  $l_X$  ensures that  $x l'_X(x)/l_X(x) \rightarrow 1$  as  $x \rightarrow \infty$ . With the corresponding formula for the  $C$  version, the following comes

$$\frac{1}{p(x)} = 1 + \frac{\theta_X}{\theta_C} \left( \frac{\tau_X}{\theta_X} \right)^{1-1/\tau_X} \left( \frac{\tau_C}{\theta_C} \right)^{1/\tau_C-1} (\log x)^{1/\tau_C-1/\tau_X} (1 + o(1)) = 1 + B (\log x)^{1/\tau_C-1/\tau_X} (1 + o(1)).$$

The convergence to 1 of  $p(x)$  is thus proved when  $0 < \tau_X < \tau_C \leq 1$ , and it can be proved similarly that it is also valid when  $\tau_X = 0$ . It is easy to see that when  $0 < \tau_C < \tau_X \leq 1$ , we have instead  $1/p(x) \rightarrow \infty$ , and thus  $p(x) \rightarrow 0$ , as  $x \rightarrow \infty$  (also valid when  $\tau_C = 0$ ). When  $0 < \tau_X = \tau_C < 1$ , we have  $1/p(x) = 1 + B(1 + o(1)) \rightarrow 1 + B$  which is equal in this case to  $1 + (\theta_X/\theta_C)^{1/\tau_X}$ , and this provides the announced limit for  $p(x)$ , belonging to  $]0, 1[$ .

Of course, to derive the more precise expansions for  $p(x) - p$  stated in Part (i), second order properties are required. Suppose neither  $\tau_X$  nor  $\tau_C$  is zero, and assumption (A2) holds. Part (i) of Lemma 1 ensures that

$$(K_{\tau_X}^-)'(\log H_X(x)) = \left( \frac{\tau_X}{\theta_X} \right)^{1/\tau_X-1} (\log x)^{1/\tau_X-1} \left( 1 + (\tau_X^{-1} - 1) \frac{\theta_X (\log c_X + \tau_X^{-1})}{\log x} + o(1/\log x) \right)$$

and

$$\frac{H'_X(x)}{H_X(x)} = \frac{1}{\theta_X x} (1 - \theta_X \rho_X x^{\rho_X} \tilde{v}_X(x)),$$

where  $v_X$  is a slowly varying function. Similar formulas are valid for  $X$  instead of  $C$ . Therefore, after some efforts, we obtain

$$\frac{1}{p(x)} = 1 + B (\log x)^{1/\tau_C-1/\tau_X} (1 + A (\log x)^{-1} (1 + o(1))),$$

where  $A$  and  $B$  are the constants described in the statement of Lemma 2. This is the moment where knowing the position of  $\tau_X$  with respect of  $\tau_C$  is needed, and it can be easily checked (but it is a bit tedious) that the different expansions of  $p(x) - p$  stated in Part (ii) of Lemma 2 are valid when neither  $\tau_X$  nor  $\tau_C$  is zero. When either  $\tau_X$  or  $\tau_C$  is zero, the proof is very similar, with the expression of either  $(K_{\tau_X}^-)'(\log H_X(x))$  or  $(K_{\tau_C}^-)'(\log H_C(x))$  varying from what is detailed above.

*Part (ii)*

A complete description of all the cases would be too lengthy, let us focus on the case  $0 < \tau_C < \tau_X \leq 1$ . For  $t \in ]0, \infty[$ , if  $s$  denotes the quantity  $s = K_{\tau_Z}(-\log t)$ , we have

$$r(t) - p = p \circ H_Z^-(\exp s) - p = D (\log H_Z^-(\exp s))^{r-1} (1 + R_1(t)),$$

where  $R_1(t) = g(r) \{ \log H_Z^-(\exp K_{\tau_Z}(-\log t)) \}^{\max(-1, r-1)} (1 + o(1))$ . But

$$\log H_Z^-(\exp s) = \log \left( (e^s)^{\theta_Z} \tilde{l}(e^s) \right) = (\theta_Z s) \left( 1 + \theta_Z^{-1} (\log \tilde{l}(e^s))/s \right),$$

and thus

$$r(t) - p = (D \theta_Z^{r-1}) (K_{\tau_Z}(-\log t))^{r-1} (1 + R_1(t))(1 + R_2(t)),$$

where  $R_2(t) = (\log \tilde{l}(\exp K_{\tau_Z}(-\log t)))/(\theta_Z K_{\tau_Z}(-\log t))$ . When  $\tau_Z > 0$ , we have  $K_{\tau_Z}(-\log t) = \frac{1}{\tau_Z} (-\log t)^{\tau_Z} (1 - (-\log t)^{-\tau_Z})$ , and therefore we obtain

$$r(t) - p = D (\theta_Z/\tau_Z)^{r-1} (-\log t)^{-\tau_Z(1-r)} (1 + R_1(t))(1 + R_2(t))(1 + R_3(t)),$$

where  $R_3(t) = -(-\log t)^{-\tau_Z}$ . When  $0 < \tau_C < \tau_X \leq 1$ , it can be checked that the multiplying constant is indeed equal to  $a^{1/\tau_X} (\tau_X/\tau_C)^{\tau_X^{-1}-1}$ , and a careful study of the 3 multiplicative remainder terms leads to the stated big  $O$ , by relying on relation (5) in Part (iii) of Lemma 1. The other cases are similar, details are omitted.  $\square$



### 3. Proofs of the other propositions

**Proposition 2.** *Under assumptions (A1) and (A2), with  $(\tau_X, \tau_C) \in [0, 1]^2 \setminus \{(0, 0), (1, 1)\}$ , as well as  $R_l(b, \rho)$  and  $R_l(\tilde{b}, \tilde{\rho})$  and if  $(k_n)$  satisfies  $H_1$  and one of the conditions  $H_2, \dots, H_6$ , we have, as  $n \rightarrow \infty$ ,*

$$\begin{aligned} \text{if } 0 \leq \tau_X < \tau_C \leq 1, \quad & \sqrt{k}v_n A_{2,n} = \sqrt{k}v_n(\hat{p}_k - a) = \sqrt{k}(\hat{p}_k - 1) \xrightarrow{\mathbb{P}} 0, \\ \text{if } 0 < \tau_C < \tau_X \leq 1, \quad & \sqrt{k}v_n A_{2,n} = \sqrt{k}v_n \left( \left( \frac{a\tau_X}{\tau_C} \right)^{1-\frac{1}{\tau_X}} L_{nk}^{1-\tau_C/\tau_X} \hat{p}_k - a \right) \xrightarrow{d} N \left( 0, a^{2-1/\tau_X} \left( \frac{\tau_X}{\tau_C} \right)^{1-1/\tau_X} \right), \\ \text{if } 0 < \tau_X = \tau_C < 1, \quad & \sqrt{k}v_n A_{2,n} = \sqrt{k}(a^{1-1/\tau_X} \hat{p}_k - a) \xrightarrow{d} N \left( 0, a^{2-1/\tau_X} (1 - a^{1/\tau_X}) \right), \\ \text{if } 0 = \tau_C < \tau_X < 1, \quad & \sqrt{k}v_n A_{2,n} = \sqrt{k}v_n \left( (a\tau_X)^{1-\frac{1}{\tau_X}} L_{nk} (\log L_{nk})^{1-\frac{1}{\tau_X}} \hat{p}_k - a \right) \xrightarrow{d} N \left( 0, a^{2-1/\tau_X} \tau_X^{1-1/\tau_X} \right). \end{aligned}$$

*Proof of Proposition 2*

The function  $p(\cdot)$  being defined in the previous subsection, and proceeding as in Einmahl et al. (2008), we carry on the proof by considering now that  $\delta_i$  is related to  $Z_i$  by

$$\delta_i = \mathbb{I}_{U_i \leq p(Z_i)},$$

where  $(U_i)_{i \leq n}$  denotes an independent sequence of standard uniform variables, independent of the sequence  $(Z_i)_{i \leq n}$ . We denote by  $U_{[1,n]}, \dots, U_{[n,n]}$  the (unordered) values of the uniform sample pertaining to the order statistics  $Z_{1,n} \leq \dots \leq Z_{n,n}$  of the observed sample  $Z_1, \dots, Z_n$ .

Recall that  $Z_i = H_Z^-(\exp(K_{\tau_Z}(E_i)))$ , where  $E_1, \dots, E_n$  are independent standard exponential random variables (Proposition 1). We introduce, for every  $1 \leq i \leq n$ , the standard uniform random variables  $V_i = 1 - \exp(-E_i)$  such that

$$Z_i = H_Z^-(\exp(K_{\tau_Z}(-\log(1 - V_i)))) = r(1 - V_i)$$

where the function  $r(\cdot)$  was defined before the statement of Lemma 2, which provides valuable information about it. Let us provide a detailed proof of Proposition 2 in the case  $0 < \tau_C < \tau_X \leq 1$  (the non-Weibull-tail strong censoring case); all the other cases are treated similarly. We start by writing

$$\begin{aligned} \sqrt{k}v_n A_{2,n} &= \sqrt{k}v_n \left( \left( \frac{a\tau_X}{\tau_C} \right)^{1-1/\tau_X} (L_{nk})^{1-\tau_C/\tau_X} \hat{p}_k - a \right) \\ &= \sqrt{k}v_n \left( \frac{a\tau_X}{\tau_C} \right)^{1-1/\tau_X} (L_{nk})^{1-\tau_C/\tau_X} \frac{1}{k} \sum_{j=1}^k \left( \mathbb{I}_{U_{[n-j+1,n]} \leq r(1-V_{n-j+1,n})} - \mathbb{I}_{U_{[n-j+1,n]} \leq r(j/n)} \right) \\ &\quad + \sqrt{k}v_n \frac{1}{k} \sum_{j=1}^k \left( \left( \frac{a\tau_X}{\tau_C} \right)^{1-1/\tau_X} (L_{nk})^{1-\tau_C/\tau_X} \mathbb{I}_{U_{[n-j+1,n]} \leq r(j/n)} - a \right) \\ &=: T_{1,k} + T_{2,k}. \end{aligned}$$

We will prove below that the term  $T_{1,k}$  above converges to 0 in probability. Let us, first, treat the term  $T_{2,k}$ . We write

$$\begin{aligned} T_{2,k} &= \frac{1}{\sqrt{k}}v_n \left( \frac{a\tau_X}{\tau_C} \right)^{1-1/\tau_X} (L_{nk})^{1-\tau_C/\tau_X} \sum_{j=1}^k \left( \mathbb{I}_{U_{n-j+1,n} \leq r(j/n)} - r(j/n) \right) \\ &\quad + \frac{1}{\sqrt{k}}v_n \sum_{j=1}^k \left( \left( \frac{a\tau_X}{\tau_C} \right)^{1-1/\tau_X} (L_{nk})^{1-\tau_C/\tau_X} r(j/n) - a \right) \\ &=: T'_{2,k} + T''_{2,k}, \end{aligned}$$

Let us prove that  $T'_{2,k} \xrightarrow{d} N(0, D)$  where  $D = a^{2-1/\tau_X} \left( \frac{\tau_X}{\tau_C} \right)^{1-1/\tau_X}$ , while  $T''_{2,k} \xrightarrow{\mathbb{P}} 0$ .

We deduce from Lemma 2 that

$$r(t) = a^{1/\tau_X} \left( \frac{\tau_X}{\tau_C} \right)^{1/\tau_X - 1} (-\log t)^{\tau_C/\tau_X - 1} (1 + o(1)) \rightarrow 0.$$

Hence,

$$\begin{aligned} \mathbb{V}(T'_{2,k}) &= v_n^2 \left( \frac{a\tau_X}{\tau_C} \right)^{2-2/\tau_X} (L_{nk})^{2-2\tau_C/\tau_X} \frac{1}{k} \sum_{j=1}^k r(j/n)(1 - r(j/n)) \\ &= v_n^2 D (L_{nk})^{1-\tau_C/\tau_X} (1 + o(1)) \frac{1}{k} \sum_{j=1}^k \left( \frac{L_{nj}}{L_{nk}} \right)^{\tau_C/\tau_X - 1}, \end{aligned}$$

denoting  $L_{nj} = \log(n/j)$ . We have  $\frac{1}{k} \sum_{j=1}^k \left( \frac{L_{nj}}{L_{nk}} \right)^{\tau_C/\tau_X - 1}$  converges to 1, because  $\frac{L_{nj}}{L_{nk}}$  converges uniformly to 1. Consequently,

$$\mathbb{V}(T'_{2,k}) = Dv_n^2 (L_{nk})^{1-\tau_C/\tau_X} (1 + o(1)) \rightarrow D.$$

We conclude, for this term, using Lyapunov's Theorem (details are omitted).

Concerning  $T''_{2,k}$ , we see that  $\left( \frac{a\tau_X}{\tau_C} \right)^{1-1/\tau_X} (L_{nk})^{1-\tau_C/\tau_X} r(j/n) = a + o(1)$ . Hence, we need a second order development for  $r(j/n)$ . According to Lemma 2 (part (ii)), we have

$$\left( \frac{a\tau_X}{\tau_C} \right)^{1-\frac{1}{\tau_X}} L_{nk}^{1-\tau_C/\tau_X} r(j/n) - a = a \left( \left( \frac{L_{nj}}{L_{nk}} \right)^{\frac{\tau_C}{\tau_X} - 1} - 1 \right) + O(1) L_{nk}^{-\alpha} \left( \frac{L_{nj}}{L_{nk}} \right)^{-\alpha}.$$

where  $\alpha = \max\{-\tau_C, \tau_C/\tau_X - 1\}$ . Hence,

$$T''_{2,k} = a\sqrt{k}v_n \left( \frac{\tau_C}{\tau_X} - 1 \right) L_{nk}^{-1} (1 + o(1)) \frac{1}{k} \sum_{j=1}^k \log(k/j) + O(1)\sqrt{k}v_n L_{nk}^{-\alpha} (1 + o(1)) \frac{1}{k} \sum_{j=1}^k \left( \frac{L_{nj}}{L_{nk}} \right)^{-\alpha}.$$

But  $\frac{1}{k} \sum_{j=1}^k \log(k/j)$  and  $\frac{1}{k} \sum_{j=1}^k \left( \frac{L_{nj}}{L_{nk}} \right)^{-\alpha}$  both tend to 1. Hence, according to assumption  $H_3$  ((ii) or (iii)), depending on the closeness of  $\tau_X$  w.r.t.  $\tau_C$ ,  $T''_{2,k}$  indeed tends to 0. This concludes the proof for  $T_{2,k}$ .

It remains to prove that  $T_{1,k}$  above converges to 0 in probability. Following the same lines as in the proof of Lemma 2 (Subsection C.3) in Worms and Worms (2019), it turns out that this amounts to proving that, for some positive sequence  $s_n = k^{-\delta}/n$  ( $\delta > 0$ ) and some constant  $c > 0$ ,

$$\sqrt{k}v_n S_{n,k} \xrightarrow{n \rightarrow \infty} 0 \quad \text{where} \quad S_{n,k} := \sup \left\{ |r(s) - r(t)| ; \frac{1}{n} \leq t \leq \frac{k}{n}, |s - t| \leq c\sqrt{k}/n, s \geq s_n \right\}. \quad (6)$$

In the case considered here,  $0 < \tau_C < \tau_X \leq 1$ ,  $r(t) = cst(-\log t)^{\tau_C/\tau_X - 1} v(-\log t)$ , where  $v$  is a slowly varying function such that  $v(-\log t)$  tends to 1 when  $t \rightarrow 0$ . Let  $h(t) = (-\log t)^{\tau_C/\tau_X - 1}$ . Applying the mean value theorem, we obtain

$$\begin{aligned} |r(t) - r(s)| &\leq cst|t - s| \sup_{u \in [s,t]} \left| h'(u)v(-\log u) \left( 1 + \frac{(-\log u)v'(-\log u)}{v(-\log u)} \right) \right| \\ &\leq cst|t - s| \sup_{u \in [s,t]} |h'(u)|, \end{aligned}$$

since  $\frac{tv'(t)}{v(t)}$  tends to 1, as  $t$  tends to infinity. This entails that

$$S_{n,k} \leq cst k^{1/2+\delta} L_{nk}^{\tau_C/\tau_X - 2}.$$

Recall that in this case  $v_n = L_{nk}^{\frac{1}{2}(\tau_C/\tau_X - 1)}$ . Hence

$$\sqrt{k}v_n S_{n,k} \leq cst \left( \sqrt{k} L_{nk}^{\alpha + \delta'} \right)^{2(1+\delta)},$$

for some  $\delta' > 0$  and  $\alpha = \frac{3}{4}(\tau_C/\tau_X - 1) - \frac{1}{2}$ . We easily prove that, if we choose  $0 < \delta' < \frac{1}{2}$ ,  $\sqrt{k} L_{nk}^{\alpha + \delta'} \rightarrow 0$ , under assumption  $H_3$  (ii) or  $H_3$  (iii).  $\square$

Let us now turn our attention to the proof of Proposition 3. This proof often implies the random functions  $\mu_{q,\tau_Z}(t)$  (defined for  $q \in \mathbb{N}^*$ ) and  $\sigma_{1,\tau_Z}^2(t)$  which are defined by and satisfy (see Lemma 2 of Gardes et al. (2011))

$$\mu_{q,\tau_Z}(t) := \mathbb{E}(\theta_{n,q}(t)) = \int_0^\infty (K_{\tau_Z}(x+t) - K_{\tau_Z}(t))^q e^{-x} dx = (q!) t^{q(\tau_Z-1)} (1 + o(1)) \quad (\text{as } t \rightarrow +\infty) \quad (7)$$

and  $\sigma_{1,\tau_Z}^2(t) := \mu_{2,\tau_Z}(t) - \mu_{1,\tau_Z}^2(t)$ . In addition, if for a sample of standard exponential variables  $(F_i)_{i \leq k_n}$  we define

$$\theta_{n,1}(t) = \frac{1}{k} \sum_{j=1}^k (K_{\tau_Z}(F_j + t) - K_{\tau_Z}(t)),$$

then we recall that Lemma 5 of Gardes et al. (2011) establishes that

$$\sqrt{k}A_{1,n} \xrightarrow{d} N(0,1) \quad \text{where} \quad A_{1,n} := \frac{\theta_{n,1}(E_{n-k}) - \mu_{1,\tau_Z}(E_{n-k})}{\sigma_{1,\tau_Z}(E_{n-k})}. \quad (8)$$

**Proposition 3.** *Under the conditions of Proposition 2, for all  $1 \leq i \leq 5$ ,  $\sqrt{k}v_n T_{i,n} \xrightarrow{\mathbb{P}} 0$ , as  $n$  tends to infinity.*

*Proof of Proposition 3*

The proofs for the terms  $T_{1,n}, \dots, T_{5,n}$  are respectively detailed in parts (1),  $\dots$ , (5) below.

(1) Recall that  $T_{1,n} = R_{n,\tilde{l}}/D_{k,\tau_X}$ , where

$$R_{n,\tilde{l}} = \frac{1}{k} \sum_{j=1}^k \log \left( \frac{\tilde{l}(\exp(K_{\tau_Z}(E_{n-i+1,n})))}{\tilde{l}(\exp(K_{\tau_Z}(E_{n-k,n})))} \right).$$

According to assumption  $R_{\tilde{l}}(\tilde{b}, \tilde{\rho})$ , we have  $\log \left( \frac{\tilde{l}(tx)}{\tilde{l}(t)} \right) \sim \tilde{b}(t)K_{\tilde{\rho}}(x)$ , uniformly for  $x \geq 1$ , as  $t \rightarrow +\infty$ .

The Renyi representation yields that  $E_{n-i+1,n} - E_{n-k} \stackrel{d}{=} F_{k-i+1,k}$ , where  $F_1, \dots, F_k$  are  $k$  independent standard exponential random variables. Consequently, taking  $t = \exp(K_{\tau_Z}(E_{n-k,n})) \rightarrow +\infty$  and  $x = \exp(K_{\tau_Z}(E_{n-i+1,n}) - K_{\tau_Z}(E_{n-k,n})) \geq 1$ , we obtain

$$R_{n,\tilde{l}} \stackrel{d}{=} \tilde{b}(\exp(K_{\tau_Z}(E_{n-k,n}))) (1 + o_{\mathbb{P}}(1)) \frac{1}{k} \sum_{j=1}^k K_{\tilde{\rho}}(\exp(K_{\tau_Z}(F_j + E_{n-k,n}) - K_{\tau_Z}(E_{n-k,n}))).$$

But on one hand,  $\sqrt{k}v_n \tilde{b}(\exp(K_{\tau_Z}(E_{n-k,n})))$  tends to 0, under conditions  $H_2$ - $H_6$ . On the other hand, since  $\frac{\mu_{1,\tau_Z}(E_{n-k})}{\sigma_{1,\tau_Z}(E_{n-k})}$  tends to 1 (thanks to (7)), Corollary 1 (a corollary of Proposition 2 and the result for  $T_{2,n}$  proved in the next bullet) yields that  $\sigma_{1,\tau_Z}(E_{n-k})/D_{k,\tau_X} \xrightarrow{\mathbb{P}} 1/a$ . Consequently, in order to have negligibility of  $\sqrt{k}v_n T_{1,n}$ , it thus remains to prove that

$$\frac{\frac{1}{k} \sum_{j=1}^k K_{\tilde{\rho}}(\exp(K_{\tau_Z}(F_j + E_{n-k,n}) - K_{\tau_Z}(E_{n-k,n})))}{\sigma_{1,\tau_Z}(E_{n-k})}$$

is bounded in probability.

In the cases where  $\tilde{\rho}$  is equal to 0, we readily have

$$\frac{1}{k} \sum_{j=1}^k K_{\tilde{\rho}}(\exp(K_{\tau_Z}(F_j + E_{n-k,n}) - K_{\tau_Z}(E_{n-k,n}))) = \frac{1}{k} \sum_{j=1}^k (K_{\tau_Z}(F_j + E_{n-k,n}) - K_{\tau_Z}(E_{n-k,n})) = \theta_{n,1}(E_{n-k}),$$

and  $\frac{\theta_{n,1}(E_{n-k})}{\sigma_{1,\tau_Z}(E_{n-k})} \xrightarrow{\mathbb{P}} 1$  (see (8)). In the cases where  $\tilde{\rho} < 0$ , we use the fact that  $|K_{\tilde{\rho}}(e^u) - u| \leq |\tilde{\rho}| \frac{u^2}{2}$ , and we easily prove (following the lines of the proof of (8)) that

$$\frac{\frac{1}{k} \sum_{j=1}^k (K_{\tau_Z}(F_j + E_{n-k,n}) - K_{\tau_Z}(E_{n-k,n}))^2}{\sigma_{1,\tau_Z}(E_{n-k})} \xrightarrow{\mathbb{P}} 0.$$

This concludes the proof for  $T_{1,n}$ .

(2) Recall that  $T_{2,n} = -\theta_X \frac{R_{1,n}}{D_{k,\tau_X}}$ , where  $R_{1,n}$  is defined in Lemma 3 and we have (also in Lemma 3)

$$D_{k,\tau_X} = \hat{\Lambda}_k^{\tau_X-1} \hat{\rho}_k + R_{1,n}.$$

It suffices to prove that  $\sqrt{k}v_n \frac{R_{1,n}}{\hat{\Lambda}_k^{\tau_X-1} \hat{\rho}_k} \xrightarrow{\mathbb{P}} 0$ . Let us consider the case where  $\tau_X \neq 0$  and  $\tau_C \neq 0$ , and introduce the notations

$$\Lambda_j := \Lambda_X(Z_{n-j+1,n}) \quad \text{and} \quad \hat{\Lambda}_j := \hat{\Lambda}_{nX}(Z_{n-j+1,n}).$$

In this case (except when  $\tau_X = 1$ , since in that case  $R_{1,n} = 0$ ),

$$R_{1,n} = \frac{\tau_X - 1}{2} \hat{\Lambda}_k^{\tau_X} \frac{1}{k} \sum_{j=1}^k \left( \frac{\hat{\Delta}_{j,k}}{\hat{\Lambda}_k} \right)^2 (1 + T_{j,k})^{\tau_X-2},$$

with  $\hat{\Delta}_{j,k} = \hat{\Lambda}_j - \hat{\Lambda}_k$  and  $T_{j,k} \in ]0, \frac{\hat{\Delta}_{j,k}}{\hat{\Lambda}_k}[$ . Since  $\tau_X - 2 < 0$ , we are led to prove that

$$\sqrt{kv_n} \frac{\hat{\Lambda}_k}{\hat{p}_k} \frac{1}{k} \sum_{j=1}^k \left( \frac{\hat{\Delta}_{j,k}}{\hat{\Lambda}_k} \right)^2 \xrightarrow{\mathbb{P}} 0,$$

and, introducing

$$\xi_{j,k} := \frac{\hat{\Lambda}_j}{\hat{\Lambda}_k} \frac{\Lambda_k}{\Lambda_j} - 1 \quad \text{and} \quad d_{j,k} := \frac{\Lambda_j}{\Lambda_k} - 1,$$

we have  $(\hat{\Delta}_{j,k}/\hat{\Lambda}_k)^2 = (\frac{\Lambda_j}{\hat{\Lambda}_k} \xi_{j,k} + d_{j,k})^2 \leq 2((\Lambda_j/\Lambda_k)^2 \xi_{j,k}^2 + d_{j,k}^2)$ . We thus need to prove that

$$\sqrt{kv_n} \frac{\hat{\Lambda}_k}{\hat{p}_k} \frac{1}{k} \sum_{j=1}^k \left( \frac{\Lambda_j}{\Lambda_k} \right)^2 \xi_{j,k}^2 \xrightarrow{\mathbb{P}} 0 \quad \text{and} \quad \sqrt{kv_n} \frac{\hat{\Lambda}_k}{\hat{p}_k} \frac{1}{k} \sum_{j=1}^k d_{j,k}^2 \xrightarrow{\mathbb{P}} 0. \quad (9)$$

Let  $E_1, \dots, E_n$  be i.i.d. standard exponential random variables. We have (see Lemma 4 (i))

$$\frac{\Lambda_j}{\Lambda_k} - 1 \stackrel{d}{=} (1 + x_{j,k})^{1/\tau_X} - 1,$$

where

$$\begin{aligned} x_{j,k} &= \frac{\tau_X a K_{\tau_Z}(E_{n-j+1,n}) + \tau_X \log l(\exp(K_{\tau_Z}(E_{n-j+1,n}))) + 1}{\tau_X a K_{\tau_Z}(E_{n-k,n}) + \tau_X \log l(\exp(K_{\tau_Z}(E_{n-k,n}))) + 1} - 1 \\ &= (1 + o_{\mathbb{P}}(1))(A_{j,k} + B_{j,k}), \end{aligned}$$

with

$$A_{j,k} = 1 - \frac{K_{\tau_Z}(E_{n-j+1,n})}{K_{\tau_Z}(E_{n-k,n})} \quad \text{and} \quad B_{j,k} = \frac{1}{a K_{\tau_Z}(E_{n-j+1,n})} \log \left( \frac{l(\exp(K_{\tau_Z}(E_{n-j+1,n})))}{l(\exp(K_{\tau_Z}(E_{n-j+1,n})))} \right).$$

Hence,  $d_{j,k} = \tau_X^{-1}(A_{j,k} + B_{j,k})(1 + o_{\mathbb{P}}(1))$ . Moreover, the Renyi representation yields that  $E_{n-i+1,n} - E_{n-k,n} \stackrel{d}{=} F_{k-i+1,k}$ , where  $F_1, \dots, F_k$  are  $k$  independent standard exponential random variables. Consequently,

$$\begin{aligned} A_{j,k} &= 1 - \frac{E_{n-j+1,n}^{\tau_Z} - 1}{E_{n-k,n}^{\tau_Z} - 1} \\ &\stackrel{d}{=} -\tau_Z \frac{F_{k-j+1,k}}{E_{n-k,n}} (1 + o_{\mathbb{P}}(1)). \end{aligned}$$

Concerning  $B_{j,k}$ , we use the second order condition  $R_l(b, \rho)$  for  $l$  to write

$$B_{j,k} = \frac{b(\exp(K_{\tau_Z}(E_{n-k,n})))}{a K_{\tau_Z}(E_{n-k,n})} K_{\rho}(\exp(K_{\tau_Z}(E_{n-j+1,n}) - K_{\tau_Z}(E_{n-k,n}))) (1 + o_{\mathbb{P}}(1)).$$

Since  $(A_{j,k} + B_{j,k})^2 \leq 2(A_{j,k}^2 + B_{j,k}^2)$ , we only have to prove that  $\sqrt{kv_n} \frac{\hat{\Lambda}_k}{\hat{p}_k} \frac{1}{k} \sum_{j=1}^k A_{j,k}^2 \xrightarrow{\mathbb{P}} 0$  and  $\sqrt{kv_n} \frac{\hat{\Lambda}_k}{\hat{p}_k} \frac{1}{k} \sum_{j=1}^k B_{j,k}^2 \xrightarrow{\mathbb{P}} 0$ . Moreover  $\Lambda_k \stackrel{d}{=} \left( \frac{a\tau_X}{\tau_Z} \right)^{1/\tau_X} (E_{n-k,n})^{\tau_Z/\tau_X} (1 + o_{\mathbb{P}}(1))$ , where  $\frac{E_{n-k,n}}{L_{nk}} \xrightarrow{\mathbb{P}} 1$  and  $\frac{\hat{\Lambda}_k}{\Lambda_k} \xrightarrow{\mathbb{P}} 1$ . Hence

$$\sqrt{kv_n} \frac{\hat{\Lambda}_k}{\hat{p}_k} \frac{1}{k} \sum_{j=1}^k A_{j,k}^2 \stackrel{d}{=} cste(1 + o_{\mathbb{P}}(1)) \sqrt{kv_n} \frac{L_{nk}^{\tau_Z/\tau_X - 2}}{\hat{p}_k} \frac{1}{k} \sum_{j=1}^k F_j^2.$$

But  $\left( \frac{a\tau_X}{\tau_Z} \right)^{1 - \frac{1}{\tau_X}} (L_{nk})^{1 - \tau_Z/\tau_X} \hat{p}_k \xrightarrow{\mathbb{P}} a$ , according to Proposition 2. Consequently,  $\sqrt{kv_n} \frac{\hat{\Lambda}_k}{\hat{p}_k} \frac{1}{k} \sum_{j=1}^k A_{j,k}^2 \stackrel{d}{=} O_{\mathbb{P}}(1) \sqrt{kv_n} L_{nk}^{-1}$ , which, using assumptions  $H_2, \dots, H_4$ , goes to 0 in probability.

Now, according to Lemma 5 in Gardes et al. (2011), we have

$$\frac{1}{\mu_{2,\tau_Z}(E_{n-k})} \frac{1}{k} \sum_{j=1}^k K_{\rho}^2(\exp(K_{\tau_Z}(E_{n-j+1,n}) - K_{\tau_Z}(E_{n-k,n}))) \xrightarrow{\mathbb{P}} cst.$$

Hence,

$$\begin{aligned} \sqrt{kv_n} \frac{\hat{\Lambda}_k}{\hat{p}_k} \frac{1}{k} \sum_{j=1}^k B_{j,k}^2 &\stackrel{d}{=} cst(1 + o_{\mathbb{P}}(1)) \sqrt{kv_n} \frac{L_{nk}^{\tau_Z/\tau_X}}{\hat{p}_k} \left( \frac{b(\exp(K_{\tau_Z}(E_{n-k,n})))}{a K_{\tau_Z}(E_{n-k,n})} \right)^2 \mu_{2,\tau_Z}(E_{n-k}) \\ &\stackrel{d}{=} cst(1 + o_{\mathbb{P}}(1)) \sqrt{kv_n} L_{nk}^{-1} b^2(\exp(K_{\tau_Z}(E_{n-k,n}))), \end{aligned}$$

since  $\mu_{2,\tau_Z}(E_{n-k}) \sim 2L_{nk}^{2(\tau_Z-1)}$ , according to Lemma 2 in Gardes et al. (2011). The second part of relation (9) is thus proved.

Let us now deal with the first part of relation (9). We have

$$\xi_{j,k} = \frac{\hat{\Lambda}_j \Lambda_k}{\Lambda_j \hat{\Lambda}_k} - 1 = \left( \frac{\Lambda_k}{\hat{\Lambda}_k} \right) \left( \Delta_j \frac{\Lambda_k}{\Lambda_j} - \Delta_{k+1} \right) \Lambda_k^{-1},$$

where  $\Delta_j := \hat{\Lambda}_j - \Lambda_j$  and  $\Delta_{k+1} := \hat{\Lambda}_k - \Lambda_k$ . Lemmas 6 and 7 in Worms and Worms (2019) ensure that  $|\Delta_j| = O_{\mathbb{P}}(1/\sqrt{j-1})$  for all  $j = 2, \dots, k+1$ ,  $|\Delta_1| = O_{\mathbb{P}}(1)$  and  $\frac{E_{n-k,n}}{L_{nk}} \xrightarrow{\mathbb{P}} 1$ . Since in addition both  $\frac{\Lambda_k}{\hat{\Lambda}_k}$  and  $\frac{\Lambda_k}{\hat{\Lambda}_j}$  tend to 1 in probability, and the latter is  $\leq 1$ , we thus obtain  $|\xi_{1,n}| \leq (1 + o_{\mathbb{P}}(1)) \left( O_{\mathbb{P}}(1) + O_{\mathbb{P}}(1/\sqrt{k}) \right) \Lambda_k^{-1}$  and

$$|\xi_{j,n}| \leq (1 + o_{\mathbb{P}}(1)) \left( O_{\mathbb{P}}(1/\sqrt{j-1}) + O_{\mathbb{P}}(1/\sqrt{k}) \right) \Lambda_k^{-1}, \text{ for } j = 2, \dots, k.$$

Therefore,

$$\sqrt{k} v_n \frac{\hat{\Lambda}_k}{\hat{p}_k} \frac{1}{k} \sum_{j=1}^k \left( \frac{\Lambda_j}{\Lambda_k} \right)^2 \xi_{j,k}^2 \leq (1 + o_{\mathbb{P}}(1)) \frac{v_n}{\sqrt{k}} (\Lambda_k \hat{p}_k)^{-1} \left( O_{\mathbb{P}}(1) + \sum_{j=2}^k O_{\mathbb{P}}(1/(j-1)) \right).$$

But  $\Lambda_k \stackrel{d}{=} cst(1 + o_{\mathbb{P}}(1)) L_{nk}^{\tau_Z/\tau_X}$  and, according to Proposition 2,  $L_{nk}^{1-\tau_Z/\tau_X} \hat{p}_k = cst(1 + o_{\mathbb{P}}(1))$ . Consequently

$$\sqrt{k} v_n \frac{\hat{\Lambda}_k}{\hat{p}_k} \frac{1}{k} \sum_{j=1}^k \left( \frac{\Lambda_j}{\Lambda_k} \right)^2 \xi_{j,k}^2 \leq O_{\mathbb{P}}(1) \sqrt{k} v_n L_{nk}^{1-2\tau_Z/\tau_X} \frac{\log k}{k},$$

due to  $\frac{1}{k} \sum_{j=1}^k \frac{1}{j} \sim \frac{\log k}{k}$ . If  $\tau_Z = \tau_X$  (thus  $v_n = 1$ ), then the right-hand side above becomes  $O_{\mathbb{P}}(1) \sqrt{k} L_{nk}^{-1} \frac{\log k}{k}$ , which tends to 0 in probability, under assumption  $H_2$  or  $H_4$ . If  $\tau_Z = \tau_C < \tau_X$  (thus  $v_n = L_{nk}^{(\tau_C/\tau_X-1)/2}$ ), let  $0 < \epsilon < \frac{1}{2}$  and write

$$\sqrt{k} v_n L_{nk}^{1-2\tau_Z/\tau_X} \frac{\log k}{k} = \sqrt{k} v_n L_{nk}^{1-2\tau_C/\tau_X} k^{\epsilon-1} o(1) = L_{nk}^{\frac{3}{2} \frac{\tau_C}{\tau_X} - \frac{1}{2}} k^{\epsilon-1/2} o(1) = (\sqrt{k} L_{nk}^{-b})^{2\epsilon-1} o(1),$$

where  $-b > \frac{3}{2} \frac{\tau_C}{\tau_X} - \frac{1}{2}$ . It remains to ensure that  $\sqrt{k} L_{nk}^{\frac{3}{2} \frac{\tau_C}{\tau_X} - \frac{1}{2}}$  tends to infinity : this is the case under assumption  $H_3(i)$ .

- (3) Recall that  $T_{3,n} = -\theta_X (\hat{\Lambda}_k^{\tau_X-1} - \Lambda_k^{\tau_X-1}) (D_{k,\tau_X})^{-1} \hat{p}_k$ . Since  $D_{k,\tau_X} = \hat{\Lambda}_k^{\tau_X-1} \hat{p}_k + R_{1,n}$ , according to Lemma 3 (stated in 4 below) and  $R_{1,n}/D_{k,\tau_X} = o_{\mathbb{P}}(1)$  (term  $T_{2,n}$  in Proposition 3), we obtain that

$$T_{3,n} = O_{\mathbb{P}}(1) \left( 1 - \left( \frac{\Lambda_k}{\hat{\Lambda}_k} \right)^{\tau_X-1} \right).$$

But  $|\Lambda_k - \hat{\Lambda}_k| = O_{\mathbb{P}}(k^{-1/2})$  (see Lemma 7 in Worms and Worms (2019)). Hence

$$|T_{3,n}| \leq O_{\mathbb{P}}(k^{-1/2}) \Lambda_k^{-1}.$$

But  $\Lambda_k = K_{\tau_X}^- (aK_{\tau_Z}(E_{n-k}) + \log l(\exp(K_{\tau_Z}(E_{n-k}))))$  (see statement (i) of Lemma 4). In the case where both  $\tau_X$  and  $\tau_C$  are not equal to 0 (the other cases are treated similarly), this yields that  $\Lambda_k = O_{\mathbb{P}}(1) L_{nk}^{\tau_Z/\tau_X}$ . Since  $v_n L_{nk}^{-\tau_Z/\tau_X} = o_{\mathbb{P}}(1)$ , this concludes the proof for  $T_{3,n}$ .

- (4) Recall that  $T_{4,n} = -\theta_X \frac{R_{2,n}}{D_{k,\tau_X}}$ , where  $R_{2,n}$  is defined in the statement of Lemma 5.

Let us consider the case where  $\tau_X > 0$  and  $\tau_C > 0$ . If  $\tau_X = 1$ , then  $R_{2,n} = 0$  and there is nothing to prove, so we suppose  $\tau_X \in ]0, 1[$ . We then have

$$R_{2,n} = \left( \frac{a\tau_X}{\tau_Z} \right)^{1-\frac{1}{\tau_X}} (E_{n-k,n})^{\tau_Z(1-\frac{1}{\tau_X})} \hat{p}_k \left( (1 - E_{n-k,n}^{-\tau_Z})^{1-\frac{1}{\tau_X}} \left( 1 + \frac{1+\tau_X \log l(\exp(K_{\tau_Z}(E_{n-k,n})))}{a\tau_X K_{\tau_Z}(E_{n-k,n})} \right)^{1-\frac{1}{\tau_X}} - 1 \right)$$

According to Lemma 3 (stated in Section 4 below) and the fact that  $\frac{\hat{\Lambda}_k}{\Lambda_k} \xrightarrow{\mathbb{P}} 1$ , since

$$\Lambda_k = \left( \frac{a\tau_X}{\tau_Z} \right)^{1/\tau_X} (E_{n-k,n})^{\tau_Z/\tau_X} (1 + o_{\mathbb{P}}(1)),$$

it remains to prove that  $\sqrt{k}v_n R_n$ , where

$$R_n := \left( (1 - E_{n-k,n}^{-\tau_Z})^{1-\frac{1}{\tau_X}} \left( 1 + \frac{1 + \tau_X \log l(\exp(K_{\tau_Z}(E_{n-k,n})))}{a\tau_X K_{\tau_Z}(E_{n-k,n})} \right)^{1-\frac{1}{\tau_X}} - 1 \right).$$

But  $l(x)$  tends to a constant  $c$  that can be 0, as  $x$  tends to  $+\infty$ . Hence,

$$R_n = bE_{n-k,n}^{-\tau_Z}(1 + o_{\mathbb{P}}(1)) \quad \text{if } c \neq 0 \quad \text{and} \quad R_n = cst \frac{\log l(\exp(K_{\tau_Z}(E_{n-k,n})))}{K_{\tau_Z}(E_{n-k,n})}(1 + o_{\mathbb{P}}(1)) \quad \text{if } c = 0,$$

where  $b = (1/\tau_X - 1)(1 - a^{-1}\tau_Z/\tau_X - \tau_Z/a \log c)$ . According to Lemma 1 (part (iii)), in the cases when  $c = 0$ , we have  $\frac{\log l(e^x)}{x} = cst \cdot x^{r-1}(1 + o(1))$  as  $x \rightarrow +\infty$ . Consequently,

$$R_n = cst \cdot L_{nk}^{\tau_Z(r-1)}(1 + o_{\mathbb{P}}(1)).$$

Hence,  $\sqrt{k}v_n R_n \xrightarrow{\mathbb{P}} 0$ , under assumption  $H_2$  or  $H_3$ . The cases when  $c \neq 0$  are treated similarly. This concludes the proof for  $T_{4,n}$  when  $\tau_X > 0$  and  $\tau_C > 0$ . The other cases ( $\tau_X = 0$  or  $\tau_C = 0$ ) can be treated similarly, details are omitted.

(5) Recall that  $T_{5,n} = \theta_Z(1 + o_{\mathbb{P}}(1))R_{3,n}$ , and that, in the case  $\tau_X \neq 0$  and  $\tau_C \neq 0$ ,

$$R_{3,n} = \left( \frac{a\tau_X}{\tau_Z} \right)^{1-1/\tau_X} \hat{p}_k \left( \frac{(E_{n-k})^{\tau_Z(1-1/\tau_X)}}{\mu_{1,\tau_Z}(E_{n-k})} - (L_{nk})^{1-\tau_Z/\tau_X} \right).$$

But, according to Proposition 2,  $R_{3,n} = a(1 + o_{\mathbb{P}}(1))R_n$ , where

$$R_n := \frac{L_{nk}^{\frac{\tau_Z}{\tau_X}-1} (E_{n-k})^{\tau_Z(1-\frac{1}{\tau_X})}}{\mu_{1,\tau_Z}(E_{n-k})} - 1 = R_n^{(1)} + R_n^{(2)} + R_n^{(3)},$$

and

$$\begin{aligned} R_n^{(1)} &:= \frac{L_{nk}^{\frac{\tau_Z}{\tau_X}-1}}{\mu_{1,\tau_Z}(E_{n-k})} \left( (E_{n-k})^{\tau_Z(1-\frac{1}{\tau_X})} - L_{nk}^{\tau_Z(1-\frac{1}{\tau_X})} \right), \\ R_n^{(2)} &:= L_{nk}^{\tau_Z-1} \left( \frac{1}{\mu_{1,\tau_Z}(E_{n-k})} - \frac{1}{\mu_{1,\tau_Z}(L_{nk})} \right) \\ R_n^{(3)} &:= \frac{L_{nk}^{\tau_Z-1}}{\mu_{1,\tau_Z}(L_{nk})} - 1. \end{aligned}$$

Let us prove that  $\sqrt{k}v_n R_n^{(i)}$  tend to 0, for  $i = 1, 2, 3$ .

Concerning  $R_n^{(1)}$ , we use Lemma 4 of Gardes et al. (2011) to prove that  $\sqrt{k}$  times the large brackets in the definition of  $R_n^{(1)}$  is  $O_{\mathbb{P}}(1)L_{nk}^{\tau_Z(1-\frac{1}{\tau_X})-1}$ . Moreover,  $\frac{L_{nk}^{\tau_Z-1}}{\mu_{1,\tau_Z}(E_{n-k})}$  tends to 1, in probability, according to (7). Consequently,  $\sqrt{k}v_n R_n^{(1)} = O_{\mathbb{P}}(1)v_n L_{nk}^{-1}$ , which tends to 0.

Concerning  $R_n^{(2)}$ , we also use Lemma 4 of Gardes et al. (2011) to prove that  $\sqrt{k}$  times the large brackets in the definition of  $R_n^{(2)}$  is  $O_{\mathbb{P}}(1)\frac{\mu'_{1,\tau_Z}(L_{nk}(1+o_{\mathbb{P}}(1)))}{\mu_{1,\tau_Z}(L_{nk}(1+o_{\mathbb{P}}(1)))}$ . Since  $\frac{L_{nk}^{\tau_Z-1}}{\mu_{1,\tau_Z}(L_{nk})}$  tends to 1, we obtain that

$$\sqrt{k}v_n R_n^{(2)} = O_{\mathbb{P}}(1)v_n \frac{\mu'_{1,\tau_Z}(L_{nk}(1+o_{\mathbb{P}}(1)))}{\mu_{1,\tau_Z}(L_{nk}(1+o_{\mathbb{P}}(1)))} \frac{\mu_{1,\tau_Z}(L_{nk})}{\mu_{1,\tau_Z}(L_{nk}(1+o_{\mathbb{P}}(1)))},$$

which tends to 0, according to Lemma 2 (iii) of Gardes et al. (2011).

Concerning  $R_n^{(3)}$ , recall that, if  $\tau \neq 0$ ,  $\mu_{1,\tau}(t) = \int_0^{+\infty} (K_{\tau}(x+t) - K_{\tau}(t))e^{-x} dx$  and  $t^{\tau-1} = K'_{\tau}(t)$ . This entails that

$$\begin{aligned} \frac{\mu_{1,\tau}(t)}{t^{\tau-1}} &= \int_0^{+\infty} x \frac{K_{\tau}(x+t) - K_{\tau}(t)}{xK'_{\tau}(t)} e^{-x} dx - \int_0^{+\infty} x e^{-x} dx \\ &= \int_0^{+\infty} x \frac{K''_{\tau}(t+\alpha)}{K'_{\tau}(t)} x e^{-x} dx \quad (\alpha \in ]0, x[) \\ &= \int_0^{+\infty} \frac{\tau-1}{2} \frac{x^2}{t} (1 + \eta \frac{x}{t})^{\tau-2} e^{-x} dx \quad (\eta \in ]0, 1[) \end{aligned}$$

Hence  $R_n^{(3)} = \frac{1-\tau_Z}{2} L_{nk}^{-1}(1 + o_{\mathbb{P}}(1))$  and  $\sqrt{k}v_n R_n^{(3)} = O_{\mathbb{P}}(1)v_n L_{nk}^{-1}$ , which tends to 0 under assumptions  $H_2, H_3, H_4$ .  $\square$

The following corollary is a consequence of Proposition 2 and Proposition 3 (term  $T_{2,n}$ ), and is also related to Lemma 3 stated a few lines below. It was used above in the proof of the term  $T_{1,n}$  of Proposition 3.

**Corollary 1.** *Under the conditions of Proposition 2, we have  $\frac{D_{k,\tau_X}}{\mu_{1,\tau_Z}(E_{n-k})} \xrightarrow{\mathbb{P}} a$ , as  $n$  tends to infinity.*

Its proof is particularly short : according to Lemma 3, and since  $\mu_{1,\tau_Z}(t) \sim t^{\tau_Z-1}$  as  $t \rightarrow \infty$  (see relation (7)), we have indeed

$$\frac{D_{k,\tau_X}}{\mu_{1,\tau_Z}(E_{n-k})} = L_{nk}^{1-\tau_Z} \Lambda_k^{\tau_X-1} \hat{p}_k(1+o(1)) \stackrel{d}{=} (A_{2,n} + a)(1+o(1)) \xrightarrow{\mathbb{P}} a.$$

The next proposition is the version of Proposition 3 adapted to the setting of Theorem 2. Its proof is very similar to the proof of Proposition 3, and is omitted.

**Proposition 4.** *Under the conditions of Proposition 2, for all  $1 \leq i \leq 5$ ,  $\sqrt{k}v_n TT_{i,n} \xrightarrow{\mathbb{P}} 0$ , as  $n$  tends to infinity.*

#### 4. Other technical Lemmas

**Lemma 3.** *The denominator of the estimator  $\hat{\theta}_{X,\tau_X}$  satisfies the relation*

$$D_{k,\tau_X} = \frac{1}{k} \sum_{j=1}^k K_{\tau_X}(\hat{\Lambda}_{nX}(Z_{n-j+1,n})) - K_{\tau_X}(\hat{\Lambda}_{nX}(Z_{n-k,n})) = \hat{\Lambda}_k^{\tau_X-1} \hat{p}_k + R_{1,n},$$

where

$$R_{1,n} = \begin{cases} \frac{\tau_X-1}{2} \hat{\Lambda}_k^{\tau_X} \frac{1}{k} \sum_{j=1}^k \left( \frac{\hat{\Delta}_{j,k}}{\hat{\Lambda}_k} \right)^2 (1+T_{j,k})^{\tau_X-2}, & \text{if } 0 < \tau_X < 1, \\ \frac{1}{k} \sum_{j=1}^k \left( \log \left( 1 + \frac{\hat{\Delta}_{j,k}}{\hat{\Lambda}_k} \right) - \frac{\hat{\Delta}_{j,k}}{\hat{\Lambda}_k} \right) & \text{if } \tau_X = 0, \\ 0 & \text{if } \tau_X = 1 \end{cases}$$

with, for each  $j = 1, \dots, k$ ,  $\hat{\Delta}_{j,k} := \hat{\Lambda}_{nX}(Z_{n-j+1,n}) - \hat{\Lambda}_{nX}(Z_{n-k,n})$  and the random variable  $T_{j,k}$  lies between 0 and  $\frac{\hat{\Delta}_{j,k}}{\hat{\Lambda}_k}$ .

*Proof of Lemma 3*

It is straightforward via Taylor's formula and the definition of function  $K_{\tau_Z}$  (the negligibility of  $R_{1,n}$  is another story, it is dealt with in the proof of Proposition 3, part (2)).  $\square$

For the following lemma, recall that  $(E_i)$  denote the i.i.d. standard exponential variable  $(E_i)$  satisfying  $Z_i = H_{\bar{Z}}(\exp(K_{\tau_Z}(E_i)))$ , and that  $l(\cdot)$  denotes the slowly varying function whose properties are described in Lemma 1 and which is such that  $H_X \circ H_{\bar{Z}}(x) = x^\alpha l(x)$ . Note that in part (ii) of this lemma, the results also hold when one replaces  $E_{n-k,n}$  by  $L_{nk}$ , or replaces  $Z_{n-k,n}$  and  $E_{n-k,n}$  by  $Z_{n-j+1,n}$  and  $E_{n-j+1,n}$  (this will occasionally prove useful).

**Lemma 4.** (i) *For every  $i = 1, \dots, n$ , and whether  $\tau_Z > 0$  or is equal to 0, we have*

$$\Lambda_X(Z_i) = K_{\tau_X}^-(aK_{\tau_Z}(E_i) + \log l(\exp K_{\tau_Z}(E_i))).$$

(ii) *When  $\tau_Z > 0$ , we have*

$$\Lambda_X(Z_{n-k,n}) = \left( a \frac{\tau_X}{\tau_Z} \right)^{1/\tau_X} E_{n-k,n}^{\tau_Z/\tau_X} (1 + o_{\mathbb{P}}(1)) = \left( a \frac{\tau_X}{\tau_Z} \right)^{1/\tau_X} E_{n-k,n}^{\tau_Z/\tau_X} \left( 1 + \beta E_{n-k,n}^{-\alpha} (1 + o_{\mathbb{P}}(1)) \right) \quad (10)$$

for some constant  $\beta$  and exponent  $\alpha = \begin{cases} \tau_Z & \text{when either } \tau_X = \tau_C, \text{ or } \tau_X \neq \tau_C \text{ and } r \leq 0, \\ \tau_Z(1-r) & \text{when } \tau_X \neq \tau_C \text{ and } r \in ]0, 1[. \end{cases}$

When  $0 = \tau_X < \tau_C$ , we have  $\Lambda_X(Z_{n-k,n}) = E_{n-k,n} l(E_{n-k,n}) = E_{n-k,n} (1 + o_{\mathbb{P}}(1))$ .

When  $0 = \tau_C < \tau_X$ , we have

$$\Lambda_X(Z_{n-k,n}) = (a\tau_X)^{1/\tau_X} (\log E_{n-k,n})^{1/\tau_X} (1 + \beta (\log E_{n-k,n})^{-1} (1 + o_{\mathbb{P}}(1))).$$

Note that the constant  $\beta$  is negative in the case  $\tau_X \neq \tau_C$  and  $r \in ]0, 1[$ .

*Proof of Lemma 4*

The first statement (i) holds because on one hand, since  $\bar{F}_X \in A_1(\tau_X, \theta_X)$ , we have  $\Lambda_X(x) = K_{\tau_X}^-(\log H_X(x))$ , and on the other hand,  $Z_i = H_Z^-(\exp(K_{\tau_Z}(E_i)))$  where  $H_X \circ H_Z^-(x) = x^{\alpha}l(x)$ .

The second statement is essentially a consequence of the first one and of some of the second order results contained in Lemma 1. Suppose for the moment that  $\tau_Z > 0$ , *i.e.*  $\tau_X > 0$  and  $\tau_C > 0$ . We thus have  $K_{\tau_X}^-(x) = (\tau_X x + 1)^{1/\tau_X}$ . Hence, noting temporarily  $\phi(x) = \log l(\exp x)/x$ , it is easy to see that (i) implies

$$\begin{aligned} \Lambda_X(Z_{n-k,n}) &= \{(a\tau_X K_{\tau_Z}(E_{n-k,n}) + \tau_X \log l(\exp(K_{\tau_Z}(E_{n-k,n})))) + 1\}^{1/\tau_X} \\ &= (a\tau_X)^{1/\tau_X} (K_{\tau_Z}(E_{n-k,n}))^{1/\tau_X} \{1 + (a\tau_X K_{\tau_Z}(E_{n-k,n}))^{-1} + a^{-1}\phi(K_{\tau_Z}(E_{n-k,n}))\}^{1/\tau_X} \end{aligned}$$

But  $K_{\tau_Z}(E_{n-k,n}) = E_{n-k,n}^{\tau_Z}(1 - E_{n-k,n}^{-\tau_Z})/\tau_Z = E_{n-k,n}^{\tau_Z}(1 + o_{\mathbb{P}}(1))$ , so

$$\Lambda_X(Z_{n-k,n}) = (a\tau_X/\tau_Z)^{1/\tau_X} E_{n-k,n}^{\tau_Z/\tau_X} \left(1 - \frac{1}{\tau_X} E_{n-k,n}^{-\tau_Z}(1 + o_{\mathbb{P}}(1))\right) \times B_n$$

where  $B_n$  denotes the quantity in curly brackets above. Thanks to part (iii) of Lemma 1, we have

$$B_n = 1 + \frac{\tau_Z}{a\tau_X} E_{n-k,n}^{-\tau_Z}(1 + o_{\mathbb{P}}(1)) + cst.E_{n-k,n}^{-\alpha}(1 + o_{\mathbb{P}}(1))$$

where either  $\alpha = \tau_Z$  and  $cst = (\log c)\tau_Z/a$  (when  $\tau_X = \tau_C$  or  $\tau_X \neq \tau_C$  and  $r \leq 0$ ) or  $\alpha = \tau_Z$  and  $cst = -\nu a^{-1}\tau_Z^{1-r} < 0$  (when  $\tau_X \neq \tau_C$  and  $r \in ]0, 1[$ ). The proof is thus over when  $\tau_Z > 0$ .

The cases  $\tau_X = 0$  and  $\tau_C > 0$ , or  $\tau_C = 0$  and  $\tau_X > 0$ , can be proved similarly. When  $0 = \tau_X < \tau_C$ , we have  $\tau_Z = 0$  and  $a = 1$  so it immediately comes  $\Lambda_X(Z_{n-k,n}) = E_{n-k,n}l(E_{n-k,n}) = E_{n-k,n}(1 + o_{\mathbb{P}}(1))$  (because  $c = 1$  in that case, see Lemma 1). When  $0 = \tau_C < \tau_X$ , we have  $\tau_Z = 0$  and thus

$$\Lambda_X(Z_{n-k,n}) = \{a\tau_X \log(E_{n-k,n}) + \tau_X \log l(E_{n-k,n}) + 1\}^{1/\tau_X}$$

The end of the proof is then very similar to the first case covered in details above.

The fact that relation (10) also holds when  $E_{n-k,n}$  is replaced by  $L_{nk}$  is due to Lemma 4 in Gardes et al. (2011), which states that  $\sqrt{k}(E_{n-k,n} - L_{nk})$  converges in distribution to a standard normal variable.  $\square$

**Lemma 5.** *Let  $E_1, \dots, E_n$  be i.i.d. standard exponential random variables.*

$$\Lambda_k^{\tau_X-1} \hat{p}_k = \begin{cases} \left(\frac{a\tau_X}{\tau_Z}\right)^{1-1/\tau_X} E_{n-k,n}^{\tau_Z(1-1/\tau_X)} \hat{p}_k + R_{2,n}, & \text{if } \tau_X \neq 0 \text{ and } \tau_C \neq 0 \\ \frac{\hat{p}_k}{E_{n-k,n}} + R_{2,n}, & \text{if } 0 = \tau_X < \tau_C < 1 \\ (a\tau_X)^{1-1/\tau_X} (\log(E_{n-k,n}))^{1-1/\tau_X} \hat{p}_k + R_{2,n} & \text{if } 0 = \tau_C < \tau_X < 1, \end{cases}$$

where

$$R_{2,n} = \begin{cases} \left(\frac{a\tau_X}{\tau_Z}\right)^{1-\frac{1}{\tau_X}} E_{n-k,n}^{\tau_Z(1-\frac{1}{\tau_X})} \hat{p}_k \left( (1 - E_{n-k,n}^{-\tau_Z})^{1-\frac{1}{\tau_X}} \left(1 + \frac{1+\tau_X \log l(\exp(K_{\tau_Z}(E_{n-k,n})))}{a\tau_X K_{\tau_Z}(E_{n-k,n})}\right)^{1-\frac{1}{\tau_X}} - 1 \right), \\ \text{if } 0 < \tau_X < 1 \text{ and } \tau_C \neq 0 \\ \frac{\hat{p}_k}{E_{n-k,n}} \left( \frac{1}{l(E_{n-k,n})} - 1 \right), & \text{if } 0 = \tau_X < \tau_C < 1 \\ (a\tau_X)^{1-\frac{1}{\tau_X}} (\log(E_{n-k,n}))^{1-\frac{1}{\tau_X}} \hat{p}_k \left( \left(1 + \frac{1+\tau_X \log l(E_{n-k,n})}{a\tau_X \log(E_{n-k,n})}\right)^{1-\frac{1}{\tau_X}} - 1 \right), & \text{if } 0 = \tau_C < \tau_X < 1 \\ 0, & \text{if } \tau_X = 1 \end{cases}$$

*Proof of Lemma 5*

Using part (i) of Lemma 4, we have

$$\Lambda_k = K_{\tau_X}^-(aK_{\tau_Z}(E_{n-k,n}) + \log l(\exp(K_{\tau_Z}(E_{n-k,n})))) ,$$

which yields, in the case  $\tau_X \neq 0$  and  $\tau_C \neq 0$ ,

$$\Lambda_k^{\tau_X-1} = \left(\frac{a\tau_X}{\tau_Z}\right)^{1-\frac{1}{\tau_X}} E_{n-k,n}^{\tau_Z(1-\frac{1}{\tau_X})} (1 - E_{n-k,n}^{-\tau_Z})^{1-\frac{1}{\tau_X}} \left(1 + \frac{1 + \tau_X \log l(\exp(K_{\tau_Z}(E_{n-k,n})))}{a\tau_X K_{\tau_Z}(E_{n-k,n})}\right)^{1-\frac{1}{\tau_X}} .$$



The expression of  $R_{2,n}$  follows in this case. The other cases are similar.  $\square$

**Lemma 6.** *Let  $E_1, \dots, E_n$  be i.i.d. standard exponential random variables.*

$$\Lambda_k^{-1} \hat{p}_k = \begin{cases} \left( \frac{a\tau_X}{\tau_Z} \right)^{-1/\tau_X} E_{n-k,n}^{-\tau_Z/\tau_X} \hat{p}_k + RR_{2,n}, & \text{if } \tau_X \neq 0 \text{ and } \tau_C \neq 0 \\ \frac{\hat{p}_k}{E_{n-k,n}} + RR_{2,n}, & \text{if } 0 = \tau_X < \tau_C < 1 \\ (a\tau_X)^{-1/\tau_X} (\log(E_{n-k,n}))^{-1/\tau_X} \hat{p}_k + RR_{2,n} & \text{if } 0 = \tau_C < \tau_X < 1, \end{cases}$$

where

$$RR_{2,n} = \begin{cases} \left( \frac{a\tau_X}{\tau_Z} \right)^{-\frac{1}{\tau_X}} E_{n-k,n}^{-\frac{\tau_Z}{\tau_X}} \hat{p}_k \left( (1 - E_{n-k,n}^{-\tau_Z})^{-\frac{1}{\tau_X}} \left( 1 + \frac{1+\tau_X \log l(\exp(K_{\tau_Z}(E_{n-k,n})))}{a\tau_X K_{\tau_Z}(E_{n-k,n})} \right)^{-\frac{1}{\tau_X}} - 1 \right), \\ \text{if } 0 < \tau_X < 1 \text{ and } \tau_C \neq 0 \\ \frac{\hat{p}_k}{E_{n-k,n}} \left( \frac{1}{l(E_{n-k,n})} - 1 \right), & \text{if } 0 = \tau_X < \tau_C < 1 \\ (a\tau_X)^{-\frac{1}{\tau_X}} (\log(E_{n-k,n}))^{-\frac{1}{\tau_X}} \hat{p}_k \left( \left( 1 + \frac{1+\tau_X \log l(E_{n-k,n})}{a\tau_X \log(E_{n-k,n})} \right)^{-\frac{1}{\tau_X}} - 1 \right), & \text{if } 0 = \tau_C < \tau_X < 1 \\ 0, & \text{if } \tau_X = 1 \end{cases}$$

The proof of the previous lemma is very similar to the one of Lemma 5, it is therefore omitted. The following one is an easy consequence of Lemma 4.

**Lemma 7.** *Under the assumptions of Proposition 2, we have, as  $n \rightarrow \infty$ ,*

$$\begin{aligned} \text{if } \tau_X \neq 0 \text{ and } \tau_C \neq 0, & \quad \log(\Lambda_k) = \frac{\tau_Z}{\tau_X} \log L_{nk}(1 + o_{\mathbb{P}}(1)) \\ \text{if } \tau_X = 0, & \quad \log(\Lambda_k) = a \log L_{nk}(1 + o_{\mathbb{P}}(1)) \\ \text{if } \tau_X \neq 0, \text{ and } \tau_C = 0 & \quad \log(\Lambda_k) = \frac{1}{\tau_X} \log \log L_{nk}(1 + o_{\mathbb{P}}(1)) \end{aligned}$$

Finally, the next lemma is used inside the proof of Theorem 4.

**Lemma 8.** *Under the assumptions of Theorem 4, we have, as  $n$  tends to infinity,*

$$\int_{L_k}^{\Lambda_k} u^{\tau_X-1} \log u \, du = \begin{cases} O_{\mathbb{P}}(\log L_{nk}) & \text{if } \tau_X \neq 0, \tau_C \neq 0 \text{ and } (\tau_X = \tau_C \text{ or } r \leq 0), \\ O_{\mathbb{P}}(L_{nk}^{r\tau_Z} \log L_{nk}) & \text{if } \tau_X \neq 0, \tau_C \neq 0, \tau_X \neq \tau_C \text{ and } r \in ]0, 1[, \\ O_{\mathbb{P}}(\log \log L_{nk}) & \text{if } \tau_X \neq 0 \text{ and } \tau_C = 0, \\ o_{\mathbb{P}}(\log L_{nk}) & \text{if } \tau_X = 0. \end{cases}$$

*Proof of Lemma 8*

We only treat the case where both  $\tau_X$  and  $\tau_C$  are positive. In this case, recall that  $L_k = (a\tau_X/\tau_Z)^{1/\tau_X} (L_{nk})^{\tau_Z/\tau_X}$  and, according to Lemma 4,  $\frac{\Lambda_k}{L_k} \xrightarrow{\mathbb{P}} 1$ . We have (with  $v = u/L_k$ )

$$\begin{aligned} \int_{L_k}^{\Lambda_k} u^{\tau_X-1} \log u \, du &= L_k^{\tau_X} \int_1^{\Lambda_k/L_k} v^{\tau_X-1} (\log v + \log L_k) \, dv \\ &= \frac{L_k^{\tau_X}}{\tau_X} \log \left( \frac{\Lambda_k}{L_k} \right) \left( \frac{\Lambda_k}{L_k} \right)^{\tau_X} - \frac{L_k^{\tau_X}}{\tau_X^2} \left( \left( \frac{\Lambda_k}{L_k} \right)^{\tau_X} - 1 \right) + \log L_k \frac{L_k^{\tau_X}}{\tau_X} \left( \left( \frac{\Lambda_k}{L_k} \right)^{\tau_X} - 1 \right). \end{aligned}$$

An immediate consequence of Lemma 4 is that both  $\log \left( \frac{\Lambda_k}{L_k} \right)$  and  $\left( \frac{\Lambda_k}{L_k} \right)^{\tau_X} - 1$  are  $O_{\mathbb{P}}((L_{nk})^{-\tau_Z})$  if  $\tau_X = \tau_C$  or  $r \leq 0$ , and are  $O_{\mathbb{P}}((L_{nk})^{-\tau_Z(r-1)})$  if  $\tau_X \neq \tau_C$  and  $r \in ]0, 1[$ . The result follows easily.  $\square$

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