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# The Plasticine Ball Argument.

Hermann Weyl, the Homogeneity of Space and Mach's Principle\*

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Die einfache Tatsache, daß ich eine Plastelinkugel in meiner Hand zu einer beliebigen Mißgestalt zerdrücken kann, die ganz anders aussieht als eine Kugel, scheint den Riemannschen Standpunkt ad absurdum zu führen.

The simple fact that I can squeeze a ball of plasticine with my hands into any irregular shape totally différent from a sphere would seem to reduce Riemann's view to an absurdity<sup>1</sup>.

H. Weyl, *Space-Time-Matter*, first edition, p. 90.

**Keywords:** Hermann Weyl, Mach's principle, history of general relativity, philosophy of space, gravitational ether.

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\*This article has been translated from French to English by Pascale Pelletier, in collaboration with the author who thanks her for her patient and precise work.

<sup>1</sup>As in the following, we use Bose's translation, revised if necessary.



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# 1 Introduction

## 1.1 The Plasticine Ball Argument

Hermann Weyl's Work is difficult to classify as physics, mathematics, philosophy or history of sciences. Perhaps because of his wide audience, perhaps also because of his aesthetic preferences, Weyl likes to use analogies and metaphors in order to provide insights about the most difficult and abstract problems of the 20<sup>th</sup> century science.

One of these analogies attracted my attention. This gives its name to what I call *the Plasticine Ball Argument* (I will abbreviate this as the *pba* in the following). Weyl uses this argument to think about the relationships between the metric and the material content of space-time. One can find an entire family of Weyl's texts that develop this argument<sup>2</sup>. Weyl wants to legitimate, at the same time, the rise of differential geometry in the domain of physics, the adoption of a dynamical metric and the refusal of flat and fixed spaces (such as Newton's or Minkowski's). For these three reasons he poses the principle of total determination of the metric by matter. This is a radical version of what Einstein was soon to call "Mach's principle". The first version of the *pba* takes the form of an Eleatic aporia. Weyl shows that a too radical principle of determination of the metric by matter could lead to the impossibility of thinking about any kind of motion, or at least any kind of deformation. Slightly rephrasing Weyl: with such a principle, we would no longer understand how it is possible to squeeze a ball of plasticine in order to change its form.

Weyl's argument that leads to this aporia, and the way he answers it have some striking formal similarities with Einstein's famous "hole argument"<sup>3</sup>. Nevertheless, the two arguments differ by their functions as well as by the manner in which the cosmic matter is distributed in the respective thought experiments. Einstein considers a hole, that is a place empty of matter, which is surrounded by a cosmos that is not necessarily empty. On the contrary, Weyl considers a ball of plasticine, which is surrounded by a cosmos that is not necessarily full of matter.

Since Einstein gave it a name, "Mach's principle" has never ceased to be the focus of an abundant literature. It is difficult to determine how much this principle is fulfilled in general relativity, and to evaluate its contribution to the philosophy of space-time. This difficulty is due not only to the intrinsic mathematical and conceptual complexity of general relativity but also to the usually vague characterization of Mach's principle itself - at least Mach's and Einstein's formulations -. Barbour and Pfister enumerated more than twenty meanings to the expression "Mach's principle"<sup>4</sup>. The present article does not tend to review this delicate question<sup>5</sup> or to add another meaning again. Rather, I aim at using a precise corpus of Weyl's texts in order to evaluate his contributions in the debate on Mach's principle in the first half of the twentieth century, and show how his own position is connected to a web of philosophical issues on space. Just like Einstein, Weyl at first totally adhered to Mach's principle and then progressively retreated from that position. His intellectual

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<sup>2</sup>See section 1.3.

<sup>3</sup>The hole argument was conceived by Einstein at the time of the Grossman's-Einstein's "Entwurf" theory ([EG13]). Within the rich literature on the hole argument, one can refer in particular to [Nor99], [IS06], [Sta93] and [Nor87].

<sup>4</sup>[BP95, p. 530]

<sup>5</sup>The reader will find a good synthesis on Mach's principle up until the 1990s in [BP95], [Rei73], [Tor83, section 6.2], [Nor93, p. 808-sq.].

pathway is not however only a redundant repetition of Einstein's. Indeed, in Weyl's specific case, the principle of determination of the metric by matter is considered in the context of a resolutely idealistic philosophy.

## 1.2 The relativistic context of the argument.

The *pba* appeared in relativistic context. Indeed, during the years 1916-1923, general relativity<sup>6</sup> prompted Weyl to develop an important thought on space. An issue was imperative at this moment: how could the foundations of Einstein's new theory be exhaustively enumerated and precisely characterised? The author of the theory himself met difficulties in dealing with this issue. Through vague and changing characterizations, he spoke about a principle of covariance, a principle of relativity, a principle of equivalence, and "Mach's principle"<sup>7</sup>. Already in the first edition of *Raum-Zeit-Materie*, Weyl gives his own position: the central idea of general relativity cannot be reduced to the mere formal property of generalized covariance<sup>8</sup>, but it rather consists in a kind of relationship between matter and the metric, expressing in an indissociable way inertial and gravitational phenomena<sup>9</sup>. Afterwards, Weyl remained faithful to this position<sup>10</sup>. Therefore, among all the different principles expressed by Einstein, Weyl underlined what Einstein called "Mach's principle" even if Weyl did not use this expression in his first texts<sup>11</sup>.

Nevertheless, Weyl's epistemological investigation is not directly characterized as a simple search for the principles of general relativity. This is rather considered as a wide-ranging philosophical investigation on *space*, only partly directed by Einstein's theory. Let us develop an analogy. When he conceived his critical philosophy, E. Kant first accepted Euclidean geometry and the bases of Newtonian physics as apodictic sciences; only afterwards did he ask what made these sciences possible. Analogically, Weyl first accepted that Riemann's mathematical developments<sup>12</sup> and Einstein's general relativity let us enter a new stage of the understanding of the foundations of the notion of space; only afterwards did he develop a conceptual and epistemological theory in order to legitimate these new truths. However neither Kant nor Weyl considered that their respective epistemologies *were derived from* or *were based on* the scientific theories they had to account for. This would have

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<sup>6</sup>[Ein16].

<sup>7</sup>See [Nor93] for a good overview of the foundational debates on general relativity, during the first 80 years of existence of this theory.

<sup>8</sup>Among the bibliographical references given by Weyl for his chapter IV, we find Kretschmann's article "Über den physikalischen Sinn der Relativitätspostulate". Therefore, Weyl had probably been influenced by Kretschmann's famous argument, according to which the general covariance principle had no physical meaning by itself, since every physical theory can be expressed in a covariant form by a tensorial reinterpretation. See also [Nor93].

<sup>9</sup>[Wey18b, p. 181], [Wey19, §26, p. 192], [Wey10, p. 226]:

A new physical factor appears only when it is assumed that the metrical structure of the world is not given *a priori*, but that the above quadratic form is related to matter by generally invariant laws. Only this fact justifies us in assigning the name "general theory of relativity" to our reasoning; we are not simply giving it to a theory which has merely borrowed the mathematical form of relativity.

<sup>10</sup>For example, in 1924, see [Wey24, p. 197].

<sup>11</sup>The expression appeared first in [Ein18]. Cf. [BP95, p. 10]. Weyl does not explicitly refer to Mach within paragraph 12 of *Raum-Zeit-Materie*. Mach appears however in the bibliographical references of chapter IV, in [Wey21, p. 291, bibliographical note 2]. Weyl in [Wey24, p. 198] acknowledged that Mach was the father of the principle of determination of inertia by cosmic matter.

<sup>12</sup>In particular the famous text: [Rie19].

been a vicious circle, since the sought-after epistemological justifications are supposed to hold *a priori*. That is why Weyl, as well as Kant, thought that the respective scientific theories had been simple *opportunities* to reveal certain *a priori* epistemological elements. Weyl is peculiarly lucid and subtle when he thinks about the relationships between theory of knowledge, as aiming at *a priori* claims, and the factual development of positive science<sup>13</sup>. Thus Weyl asks: what did Einstein’s theory teach us (or confirm) about the nature of space and the way one can know it scientifically? How can we epistemologically justify that the “correct” notion of space is the one that was finally used by Einstein, after having been announced by Riemann? In order to answer this question, one must deal with two issues that give the global structure of Weyl’s thought on space in the period 1917-1923:

1. The first issue -according to the logical order- consists in justifying that the space-time metric is “of the Pythagorean type”. This means that it has the same properties, in the infinitesimal realm, as the (pseudo-)Euclidean metric. This is the technical meaning of what Weyl calls “the problem of space”<sup>14</sup>,
2. The second issue consists in justifying the claim that the space-time metric, away from the infinitesimal realm, is a metric the curvature of which is everywhere intrinsically indeterminate. More precisely, the determination of the finite metrical relations is only possible *a posteriori*, when geometry is articulated with physics. The metric is determined by the manner matter and forces are spatially distributed.

### 1.3 Marking out of the corpus. Why is the argument so recurring and polymorphic?

This article will address the second of the issues mentioned above. As early as the first edition of *Raum-Zeit-Materie*, Weyl wants to justify epistemologically the variable and dynamic character of Einstein’s metric and its link with matter. It is in this context that he elaborates his *pba*<sup>15</sup>. It is repeated with notable changes in the third edition of the same work<sup>16</sup>; then, with more changes, in the fourth edition<sup>17</sup>. The argument disappears from the fifth edition, but regularly reappears later in others of Weyl’s texts, in always changing forms: in “Massenträgheit und Kosmos”<sup>18</sup>, in both (German and English) editions of *Philosophie der Mathematik und Naturwissenschaft*<sup>19</sup> and in *Mind and Nature*<sup>20</sup>. The *pba* does not appear in *Mathematische Analyse des Raumproblems*, but many typical questions can be found there which give rise to the formulation of this argument in

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<sup>13</sup>See Weyl’s texts quoted in [Mic06].

<sup>14</sup>See the bibliography of [Ber15b] for the list of Weyl’s works on the *problem of space* - in its technical meaning - and a historical discussion. Secondary reading on this subject is abundant, see [CK01], [Sch04], [Lau58], [Ber15a], [Ber18], [Wey15, vol. 2].

<sup>15</sup>[Wey18b, p. 90].

<sup>16</sup>[Wey19, p. 90].

<sup>17</sup>[Wey21, p. 90].

<sup>18</sup>[Wey24, p. 198].

<sup>19</sup>[Wey49, pp. 86-87;105].

<sup>20</sup>[Wey34, p. 129].

the first texts<sup>21</sup>. Both in *Raum-Zeit-Materie* and in *Philosophy of Mathematics and Natural Science* the chapter containing the *pba* is amongst the most modified one in the different editions<sup>22</sup>. In the literature on Weyl we can find references to or analyses of some of these texts<sup>23</sup>, but, to my knowledge, no systematic studies of the entire collection of the occurrences of the *pba*.

Considering this inventory, with the *pba*, we are in front of a thematic which entertained Weyl's thinking about space during many years, and which develops in ever changing forms in a whole range of texts, as a musical variation of the same theme. *Why is this thought experiment so present, and why is it so polymorphic in Weyl's texts?*

We already have partly answered this question. Weyl relies on this thought experiment to have an imaginative and conceptual support on which he can base his thought, in order to address the second fundamental epistemological problem enunciated above<sup>24</sup>. Weyl first wants to see how adopting a dynamic metric with variable curvature can be justified, while not abandoning the thesis of space as a homogeneous form of appearances. Consequently the shifts of this argument, from work to work and from edition to edition, partly reflect the evolution of Weyl's thinking on this key issue.

But there is a second reason for this variability of the text. Indeed, to elaborate his thought experiment, Weyl is led to express precisely the way in which the metric properties of space-time are *determined* or at least *correlated with* the spatiotemporal distribution of matter. Therefore, even if it was *not* the determining of the metric by matter which *was the problem* for Weyl, when he began to elaborate his *pba*, some technical problems that he met led him to position himself more and more subtly on Mach's principle. But clarifying this principle is not just a *technical* problem which would replace the anterior *philosophical* problem which generated the argument. The critique of Mach's principle also has a philosophical dimension, which led Weyl to modify his position on the ontology of physics. In the version of *Raum, Zeit, Materie* of the *pba*, Weyl supported a form of *materialism*, inasmuch as all physical phenomena –including gravity which provides its foundation to spatiotemporal geometry– were to be reducible to the relationships between material elements<sup>25</sup>. In later versions of the argument, the critique of Mach's principle led Weyl to become anti-materialistic and argue in favour of a dynamical ether, partly autonomous from fields of matter, thus following an intellectual path close to Einstein's<sup>26</sup>.

These two reasons for the variations of the *pba* provide the two major objectives of the present article. Firstly we are going to use these variations as a means to underscore the decisive stages of the evolution of Weyl's philosophy of space at that time. Then by studying the technical problems across which Weyl stumbled in the first version of the argument, and by showing how the later versions brought an answer, we will be able to explain which role Weyl played towards clarifying

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<sup>21</sup>[Wey23, p. 44-45]. We find there the problem of the tension between the homogeneity of space, as a form of appearance, and the heterogeneity of the metric ; and the solution consisting in moving the metric simultaneously with matter.

<sup>22</sup>I will specify in this article the most important changes in *Raum-Zeit-Materie* that I personally noticed. Concerning *Philosophy of Mathematics and Natural Science*, I received the information from Carlos Lobo.

<sup>23</sup>Namely: [CK01, p. 266-267],[Cof79, p. 290],[Giopu],[Gio13, p. 130],[Sch17].

<sup>24</sup>See the end of section 1.2

<sup>25</sup>The notion of matter which is present in the first four editions of *Raum-Zeit-Materie* do not form a discrete set of particles, but a *field*. Weyl was then taking up the programme of Gustave Mie which consisted in bringing out the notion of matter from the notion of field.

<sup>26</sup>See for example [Ein20].

Mach's principle<sup>27</sup>.

Then, in the course of the article, we will pursue these two themes of thought simultaneously, going through and commenting on the different versions of the *pba*, following the order in which they were published. The natural evolution of Weyl's thought, from text to text, will progressively lead us from the problem of *the non-homogeneity of the metric* to the problem of *the validity of Mach's principle* and the existence of an ether, the transition being very gradual.

## 2 Preliminary: the homogeneity of space, and the exclusion of the metric from space.

To understand the reasons why Weyl first formulated his *pba*, we must keep in mind a problem which is at the core of all of Weyl's philosophy of space. That is the problematic tension between the homogeneity of space and the non-homogeneity of the metric. I have dealt with the theme before<sup>28</sup>, so I will only mention here the elements that are useful in order to understand the *pba*.

### 2.1 The issue of the non-homogeneity of the relativistic metric

Geometricians from the beginning of the twentieth century were the inheritors of two crucial developments of the nineteenth century: on the one hand differential geometry, and all the associated analytical tools in connection with it allowing the study of spaces with randomly variable curvature and, on the other hand, the discovery of the founding unifying function of the notion of group towards geometry (Helmholtz, Klein and Lie in particular). But these two legacies are not easily reconciled, inasmuch as Riemannian manifolds and the other related infinitesimal geometries generally have a trivial isometry group.

This is why Weyl just as Klein, Poincaré or Cartan, considered the tension between the notion of homogeneous space and differential geometry, as a -even for some *the-* central epistemological question on space raised by the nineteenth century<sup>29</sup>. If space is defined by the possibility of defining a group of displacements, must the rich infinite universe of Riemannian manifolds be drastically limited so as to keep only a few homogeneous geometries? Instead cannot the notions of homogeneity and group be transformed in order to become compatible with the perspective of differential geometry?

Weyl's specificity within this group of authors is due to the precise signification that he gives to the homogeneity requisite and his reasons for putting it forwards. In the Erlangen tradition which is not necessarily the one followed by Weyl, the homogeneity of the "spaces" considered is justified from within mathematical practice, by the unified treatment of a vast part of the geometry practices of the nineteenth century it allows (in particular: projective, affine, Euclidean, spherical, Lobatchevskian geometries). In contrast, for Weyl, even though the homogeneity of space is soon used to legitimate

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<sup>27</sup>We can already find in the literature works that refer to the role of Weyl in the history of Mach's principle. See in particular [BK16, "Weyl's Critique of Einstein's Machian Ideas"], [CK01, p. 264], [Cof79, p.290 sq.].

<sup>28</sup>[Ber13], [Ber10], [Wey15, vol. 2, 2<sup>e</sup> introduction].

<sup>29</sup>See the general introduction of the current volume, [Poi02], [Cho09], [Car25].

the application of the group theory techniques<sup>30</sup>, homogeneity is not originally defined *by* the notion of group nor justified within a mere mathematical discourse. Instead homogeneity is given as an essential property of space, *in the name of a philosophical tradition*. For Weyl, *because* space is ideal, being only a “form of appearances”, it is necessarily homogeneous<sup>31</sup>. The homogeneity of space, which follows from its ideality, is defined this way:

Space [...] is a form of appearances (*Form der Erscheinungen*). Precisely the same content, identically the same thing, still remaining what it is, can equally well be at some place in space other than that at which it is actually. The new portion of space  $S'$  then occupied by it is equal to that portion  $S$  which it actually occupied.  $S$  and  $S'$  are said to be congruent. [...]<sup>32</sup>

Thus space is defined by Weyl as something the proprieties of which are, *by definition*, independent from matter (i.e. independent from sensory properties, physical properties and forces induced by matter) which fills it. Space does not yet belong to the domain of physics. It is a form the properties of which can entirely be characterised *a priori*, precisely because this form and its intrinsic characteristics are given to us prior to any matter which later fills it. In particular, the (topological, projective, affine, conformal, metric) properties intrinsic to space, if any, must be characterizable by a mathematical theory which precedes the study of the forces and the way in which matter occupies space-time. Therefore the phrase “physical space” has no more meaning for Weyl than for Poincaré, for instance.

Finally in the context of general relativity, the problem of the tension between the homogeneity of space and the non-homogeneity of the Riemannian metrics eventually takes the form: *how can a non-homogeneous spatiotemporal metric be accepted in physics, when space as such is by nature homogeneous?*<sup>33</sup>

However, this question does not only address the theory of general relativity, but also any physical

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<sup>30</sup>For the shift from the notion of homogeneity to the notion of congruence then to the notion of *group* of congruences, see [Wey10, pp. 5-6;11-15] or [Wey23, pp. 44-49]. For the use of the theory of groups to found the notion of metric in a context of differential geometry, See [Wey10, §18], [Eck11] or the texts in relation to the problem of space, in its technical meaning (cf. Note 14).

<sup>31</sup>[Wey10, p. 11]:

Space is a form of appearances (*Form der Erscheinungen*), and, by being so, is necessarily homogeneous. It would appear from this that out of the rich abundance of possible geometries included in Riemann’s conception, only the three special cases mentioned come into consideration from the outset, and that all the others must be rejected without further examination as being of no account: *parturiunt montes, nascetur ridiculus mus!* Riemann held a different opinion, as is evidenced by the concluding remarks of his essay [...] Only now that Einstein has removed the scales from our eyes by the magic light of his theory of gravitation do we see what these words actually mean.

<sup>32</sup>[Wey10, p. 11], [Wey19, p. 10]. See also [Wey15, p. 1], in which the opposition between form and matter becomes of a more psychological nature, inasmuch as “matter” refers to the sensory content of perception, and it is connected to Kant.

<sup>33</sup>In the specific case of general relativity, we have [Wey15, p. 44]:

According to Einstein, the metric structure of the universe is not homogeneous. How is this possible, given that space and time are forms of appearances?

theory based on differential geometry, i.e. on a variable curvature metric. This is why Weyl just like Poincaré addresses this problem to *Riemann*.

## 2.2 Solution: the metric is excluded from the intrinsic properties of space

In the first edition of *Raum-Zeit-Materie*<sup>34</sup>, Weyl provides part of solution to the problem expressed in the previous section. In the context of an infinitesimal geometry –in Riemann’s manner– the homogeneity of space must be expressed by the fact that a portion of matter can be moved from a region  $S$  of the spatial manifold towards any other region  $S'$ , while keeping all its properties invariant. Matter must be represented by a field, in which *material qualities*<sup>35</sup>, like mass or electric charge, are distributed:

To simplify this examination of the underlying principles we assume that the material content can be described fully by scalar phase quantities (*skalaren Zustandsgrößen*) such as mass-density, density of charge, and so forth. We fix our attention on a definite moment of time<sup>36</sup>.

Thus the simplification offered by Weyl is twofold: 1) reducing matter to a few scalar properties –he then keeps only one of them– and 2) eliminating the time factor. The text that follows actually considers two different distributions of matter (which I will express as  $\rho^{\text{before}}$  and  $\rho^{\text{after}}$ ), but each of them is considered as static, at its point of equilibrium; we do not consider the transitional stage. This eventually led Weyl to represent matter at first with a simple scalar function which depends only on position:

$$\rho^{\text{before}} : f(x_1x_2x_3);$$

the coordinates  $x_1x_2x_3$  vary so that they take all the values corresponding to the region  $S$  of space where matter initially is. The change of position to which we want to subject matter is expressed in our system of coordinates by a transformation:

$$x'_i = \phi_i(x_1x_2x_3).$$

Thus the region  $S'$  towards which we move the body is represented, always in the same coordinate system, by the set of  $x'_i$  (corresponding to  $x_i$  of  $S$ ). The movement of matter is technically expressed by the fact that the field  $\rho$  is *pulled forward*<sup>37</sup> on space by the transformation  $\phi$ . Weyl writes this

<sup>34</sup>See [Wey18b, p. 88-90] and the corresponding parts in the three editions of *Raum-Zeit-Materie* that follow.

<sup>35</sup>In *Raum-Zeit-Materie*, Weyl does not refer to *material qualities*. He only refers to “the material” (*das Materiale*). However in other texts of the same time, such as [Wey23, introduction], Weyl calls the material content “qualitative” and describes the homogeneity of space by the fact of being able to move these qualities towards any point.

<sup>36</sup>[Wey18b, p. 88]. The fact that Weyl choses as an exemple the density of electric charge (*Elektrizitätsdichte*) is meaningful. Perhaps he has already in mind his own theory (to be published in 1918) in which the metric field is the carrier of the gravitational and *electromagnetic* interactions, simultaneously. So, if something like a principle of metric determination by matter is to be considered in such a conceptual framework, it cannot take the form “mass determines the metric” any longer but, instead, “mass and electric charge determine the metric”.

<sup>37</sup>The “pull-forward” (“pull-back”) terminology is posterior to Weyl. See [IS06, p. 1243]. Besides Weyl does not mention here the fact that a region  $S$  cannot be moved towards *any* region  $S'$ . Instead, as shown by the process used, one must take a region  $S'$  *diffeomorphic* to the first one. For instance a simply connected region cannot be transformed into an annular region. Weyl is explicit about it further on in the text [Wey10, p. 98].

in coordinates:

$$\rho^{\text{after}} : f(x'_1 x'_2 x'_3).$$

Here we must understand that the letter  $f$  represents the same mathematical function as above. In other terms, the distribution of mater after the displacement must be expressed (still in the same coordinate system) by the function  $(f \circ \phi)$ .

By this process, we have moved matter, keeping all its intrinsic qualities unchanged. But, Weyl goes on, in order to assert that space *is* homogeneous, the *metric properties* of the material body that was moved must have been kept. These metric properties before the moving, were given by a metric field defined on  $S$ <sup>38</sup>:

$$\sum_{(i,k=1)}^3 g_{ik}^{\text{before}}(x_1 x_2 x_3) dx_i dx_k,$$

Since our space is supposed to be Riemannian, there is no reason to think that the values  $g_{ik}^{\text{before}}(x'_1 x'_2 x'_3)$  of the metric at the point  $S'$  are initially the same as in  $S$ . This is why, Weyl concludes, if the metric was fixed *a priori* once and for all, then space homogeneity could not be preserved, since we would have:

$$g_{ik}^{\text{after}}(x'_1 x'_2 x'_3) = g_{ik}^{\text{before}}(x'_1 x'_2 x'_3) \neq g_{ik}^{\text{before}}(x_1 x_2 x_3).$$

Any body moved in space would generally be metrically deformed.

However, Weyl continues, space homogeneity can be preserved if we say that metric is *dynamic* and determined by matter. Indeed according to this supposition, after moving our body, metric will change so as to conform with the moving of matter. Once the equilibrium between matter and the metric has been reestablished, the body shall have recovered its metric properties. So we will have:

$$g_{ik}^{\text{after}}(x'_1 x'_2 x'_3) = g_{ik}^{\text{before}}(x_1 x_2 x_3) \quad (\neq g_{ik}^{\text{before}}(x'_1 x'_2 x'_3)).$$

To justify this equality, Weyl plays with the twofold active/passive interpretation of  $\phi$ , as per a process also at work in Einstein's hole argument<sup>39</sup>. Once the body has been moved (transformation  $\phi$  actively interpreted as a pulling forward of  $\rho$  on the manifold), Weyl changes the coordinates. The

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<sup>38</sup>Weyl insists on the fact that, in order to determine the visual shape of a portion of matter, one must not only know the metric coefficients for the portion  $S$  of space-time where the matter is, but also for all the space-time points through which the light rays which, emitted from  $S$ , will reach the observer. The body of the latter is represented by a point-eye set on a point outside  $S$ . The necessity to take into account the metric on the intermediate trajectory is clear as soon as we think of phenomena such as light rays deflection by gravity or, in an anachronistic way, the gravitational lenses phenomena. Weyl will come back to this necessity to take into account the intermediate metric field to differentiate the rotation of the stellar compass from the rotation of the stars themselves in [Wey24, p. 198, left hand column].

For Weyl, having the visual observer intervene in order to define the shape of a material object is an important step, in view of his attachment to the Husserlian phenomenology during the years which we are considering; Weyl, here, uses significantly the term "experiences of consciousness" (*Bewußtseinserlebnissen*) [Wey18b, p. 89]. Concerning this point-eye idea or Ego-center (*Ich-Zentrum*), and its phenomenological function, see in particular analyses in [Ryc10, p. 286], [MR05], [Ber13, 241-sq.]. See also Kerszberg's article in the present volume.

However concerning the specific issue with which we are dealing, we do not need to discuss it further. The "pulling forward" of the metric in the space-time region which separates the observer from the element of matter that is observed is technically expressed in the same way as the pulling forward of the region  $S$  itself. The point-eye representing the observer is also pulled forward.

<sup>39</sup>See [Nor93, pp. 801-sq.], [Nor87, pp. 164-165], [Nor99, Appendix: "Active and Passive Covariance"].

point  $P'$  previously had the coordinates:  $(x'_1 x'_2 x'_3)$ . It will now have the new coordinates:  $(x_1 x_2 x_3)$ , that  $P$  had in the first coordinate system before the displacement. Thus Weyl now uses  $\phi^{-1}$  as a passive transformation. The intrinsic properties of matter, after the displacement, and in the new coordinate system will be expressed again by the function  $f = f \circ \phi \circ \phi^{-1}$ . So, if the metric functions  $g_{ik}$  are perfectly determined by the function  $f$  which represents matter, the conclusion shall be that metric will be moved exactly in the same way as matter. More precisely: it will have taken exactly the same values, in the new coordinate system, as it had before matter was moved, in the first system of coordinates. Thus the displaced body has kept its metric properties and the space homogeneity is preserved!

Weyl's argument can be transcribed in a more modern mathematical language which avoids the coordinate systems just as Stachel and Iftime did for Einstein's hole argument<sup>40</sup>. Such a rewriting may hide some of the problems met by Weyl and Einstein, but it can also clarify some aspects of the problem. To outline the problem briefly: it is supposed that matter is represented by a function  $\rho^{\text{before}} : M \rightarrow \mathbb{R}$  which associates its density to any point of the manifold  $M$ . A moving in the manifold is simply a diffeomorphism  $\phi : M \rightarrow M$ . Moving matter by means of  $\phi$  amounts to producing a new distribution  $\rho^{\text{after}}$  obtained by "pulling forward" the preceding distribution. So we have:  $\rho^{\text{after}} = \phi^* \rho^{\text{before}} = \rho^{\text{before}} \circ \phi$ . Let us suppose, in Weyl's manner, a law of total determination of the metric by matter. If the metric  $g^{\text{before}}$  is associated with the distribution of matter  $\rho^{\text{before}}$ , and if our law is generally covariant, then, to the distribution of matter  $\phi^* \rho^{\text{before}}$  we must necessarily associate the metric  $g^{\text{after}} = \phi^* g^{\text{before}}$ ; in which the pulling forward  $\phi^* g$  of a metric by a diffeomorphism is defined by:

$$\phi^* g(\phi^* \vec{x}, \phi^* \vec{y}) = g(\vec{x}, \vec{y});$$

and the pulling forward of a vector  $\vec{x}$  at  $P$  ( $\vec{x} \in T_P(M)$ ) is in turn defined by  $\phi^* \vec{x} = D\phi|_P(\vec{x})$  (it is a vector of  $T_{\phi(P)}(M)$ ). Thus, in this rewriting, the metric invariance that Weyl is aiming for is directly encoded in the fact that the law of the determination of the metric by matter is generally covariant.

Weyl in view of the technical solution that we have reported, concludes:

[...] Space in itself is nothing more than a three-dimensional manifold devoid of all form; it acquires a definite form only through the advent of the material content filling it and determining its metric relations.[...] the metrical groundform will alter in the course of time just as the disposition of matter in the world changes. <<sup>41</sup> We recover the possibility of displacing a body without altering its metric relations by making the body carry along with it the "metrical field" which it has produced<sup>42</sup> > [...] We shall illustrate in greater detail [...] that any two portions of space which can be transformed into one another by a continuous deformation, must be recognised as being congruent in the sense we have adopted, and that the same material content can fill one portion of space just as well as the other<sup>43</sup>.

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<sup>40</sup>[IS06].

<sup>41</sup>Added in the fourth edition.

<sup>42</sup>On this issue, also see the analogy of flexible sheet metal in [Wey15, p. 44].

<sup>43</sup>[Wey18b, p. 88], [Wey21, pp. 87-88].

The beginning of the text shows that the solution suggested by Weyl consists in excluding the metric from the intrinsic properties of space. It is rather part of the *content* of space, in the same manner as matter<sup>44</sup>. So when he refers to Riemann's-Einstein's dynamic metric, Weyl is careful not to call it "space" (*Raum*). At least, space, when it is endowed with Einstein's metric, has already ceased to be space *in itself*, space with only its intrinsic properties, but is already space *as being informed by matter*. Therefore, its onto-epistemological status has changed:

These metric relations are not the outcome of space being a form of phenomena (*Form der Erscheinungen*), but of the physical behaviour of measuring rods and light rays as determined by the gravitational field<sup>45</sup>.

The context of this quotation clearly shows that Weyl, here, does not aim at making the notion of space as a form of appearances obsolete, but at taking out the metric determinations. Thus, space(-time) the homogeneity of which Weyl can keep on ascertaining, and which can still continue to act as "form of appearances", is eventually reduced to the naked spatiotemporal manifold, i.e. deprived of any metric. That the spatiotemporal manifold, with regard to its only topological and differential properties, is homogeneous, is of course correct from a mathematical perspective<sup>46</sup>. However, this solution, in spite of often appearing in the literature of that time, and in spite of the fact that it effectively captures an important aspect of general relativity, is insufficient, from both mathematical and epistemological perspectives. We will not develop here what is the nature of these difficulties<sup>47</sup>. Let us just mention here that Weyl will qualify this thesis when he further develops his philosophy of space and his "infinitesimal geometry" or "contact geometry" (*Nahegeometrie*). He will then specify that the *infinitesimal* metric properties are indeed part of the essence of space, and can be *a priori* characterised, in contrast with the *fortuitous variations of metric relations* in a finite space-time region which alone has the status of an *a posteriori* determined physical field.

The second *a priori* notion of space which we have just characterised, which is metric and infinitesimal, does not replace the global and topological notion which we have characterised above (the naked manifold). Instead, both notions are at work in the foundational discourse of general relativity, or of any physical theory based on an infinitesimal geometry. In Weyl's work, the concomitance of these two *a priori* notions of space is expressed as follows. Weyl retains the idea that the naked spatiotemporal manifold plays the role of a (globally) homogeneous "space", used as an individuation principle, but he adds the idea that the infinitesimal metric structures, identical everywhere, are part of the "essence of space" (*Wesen des Raumes*). By integrating both ideas, the space of general relativity will appear not as form-less, but as multi-form, its changing form being illustrated by Weyl with the image of a snail shell which is built at the same time as the matter that fills it, and adapts to it<sup>48</sup>. It is that type of image that must be kept in mind as a basis for thinking

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<sup>44</sup>A question will remain whether matter could even totally emerge from matter itself. It is the question raised by Mach's principle below.

<sup>45</sup>[Wey18b, p. 91], [Wey19, p. 91], [Wey10, p. 102].

<sup>46</sup>Any  $n$ -dimensional manifold that is (arcwise) connected and of class  $\mathcal{C}^p$  for  $p = 0, 1, \dots, \infty$  is not only homogeneous, but even maximally isotropic. Given any two points  $P, P'$  of  $M$ , and given  $\{v_1, \dots, v_{n-1}\}, \{v'_1, \dots, v'_{n-1}\}$  two families of linearly independent vectors respectively taken in  $T_P(M)$  and in  $T_{P'}(M)$ . Then there is a diffeomorphism of class  $\mathcal{C}^p$  which sends  $P$  to  $P'$ , and sends the infinitesimal straight line  $\langle v_1 \rangle$  to  $\langle v'_1 \rangle$ , sends the infinitesimal plane  $\langle v_1, v_2 \rangle$  to  $\langle v'_1, v'_2 \rangle$ ,  $\dots$ , and finally sends the infinitesimal hyperplane  $\langle v_1, \dots, v_{n-1} \rangle$  to  $\langle v'_1, \dots, v'_{n-1} \rangle$ .

<sup>47</sup>On that question, see [Ber13, Chap. III, 3.].

<sup>48</sup>[Wey15, p. 44].

about the *pba*. Even though an infinitesimal structure can still be characterised *a priori*, as being part of the essence of space, the exact form of the metric in a finite region remains intrinsically indeterminate, waiting to be completely in-formed (shaped) by matter.

### 3 The *pba* in *Raum-Zeit-Materie*

#### 3.1 The *pba* as an Eleatic aporia

We are now technically armed to enter the *pba*. We shall start from the first text in the series, from a chronological perspective, namely the first edition of *Raum-Zeit-Materie*. The other versions shall then be considered in contrast with the first one.

In our section 2.2, we have shown how Weyl, to preserve the homogeneity of space, was led to presuppose that “matter determines the metric”. We can identify the idea later known as one of the many forms of “Mach’s principle”<sup>49</sup>. However it is introduced by Weyl in his own manner and without any reference to Mach. The totally Machian idea that *inertia* is determined by the masses of the cosmos is not clarified. The principle of equivalence and the generalized principle of relativity which will be closely related to Mach’s principle in Einstein’s thought, are absent too<sup>50</sup>. Weyl is more directly interested in the link between the metric and matter.

Weyl chooses Riemann instead of Einstein as a symbolic figure of the idea of a metric determined by matter. This attribution is justified by a small passage of Riemann’s Habilitation text which is enough, according to Weyl, to make him a prophet of general relativity<sup>51</sup>. Whatever the relevance of this attribution to Riemann, in any case it is significant. It shows that the conceptual framework to which Weyl belongs is indeed broad. The point is not to look for the bases of an individual physical theory (Einstein’s), but to work on a philosophical issue which more generally addresses all physical theories of fields based on an infinitesimal dynamic metric.

To meet both the current conventions and the specificity of Weyl’s point of view, let us attribute the idea to the triplet Riemann-Mach-Einstein ( $\mathcal{RME}$ ):

**$\mathcal{RME}$  Principle :** the values of the metric relations are perfectly determined by the distribution of matter and of its intrinsic qualities  $\rho$  (charge, mass $\dots$ )

The strategy of §12<sup>52</sup> of *Raum-Zeit-Materie* to preserve the homogeneity of space is problematic because it is too simplistic. In addition to the philosophical problems evoked at the end of the previous section, concerning the foundations of the infinitesimal metric structures, Weyl stumbles on another difficulty linked to his exaggeratedly strong interpretation of this principle. It will lead him to an aporia. While the intrinsic properties of matter are automatically kept in the course of time, and are expressed by scalar fields  $\rho$ , and if metric is perfectly determined by matter in the

<sup>49</sup>See our note 11.

<sup>50</sup>This point partly follows from the *Raum-Zeit-Materie* structure. The text which we have explained is taken from §12, therefore from chapter II, while general relativity is mentioned only in chapter IV. See what we have said p. 5 about Weyl’s standpoint towards the generalised covariance principle.

<sup>51</sup>See our comments in [Ber18, p. 3].

<sup>52</sup>I refer to the paragraphs numbering in the fourth edition.

sense described above, then it is the very *possibility* of some sort of transformation of matter which seems to be hampered, so that Weyl is led to state:

Riemann's point of view seems to be reduced to the absurd by the simple fact that I can shape a plasticine ball in my hands, and give it any irregular shape, totally different from the initial spherical shape<sup>53</sup>.

The plasticine ball represents any portion of matter, and the kneading represents any physical force able to move elements of matter in relation to others, in order to produce a deformation. Thus, what has become difficult to consider here is the very possibility that portions of matter may modify their reciprocal distances in the course of time. That is why I think that the difficulty on which Weyl stumbles can be reformulated as an aporia of the Zeno's paradoxes type. It is the very ability to understand the possibility for any change which seems compromised in such a framework.

**Reformulation of the *pba* as an aporia of the Eleatic type**

Given a physical theory based on the following assumptions:

1. Space is a Riemannian manifold, the metrical relations of which are not *a priori* fixed.
2. Matter which occupies this space is intrinsically characterised by one (or several) scalar field(s)  $\rho$  defined on the manifold.
3. Matter completely determines the metric. It means that a system of coordinates being fixed, if  $\rho(x)$  is determined for any  $x$ , then the  $g_{\mu\nu}(x)$  must also be determined for any  $x$ . In other words, there are  $n^2$  functions  $F_{\mu\nu}$  such as  $g_{\mu\nu} = F_{\mu\nu}(\rho)$ , where we must understand that  $g_{\mu\nu}(x)$  is not necessarily only dependent of  $\rho(x)$  but of the data of the entire field  $\rho$ .

Then, the value of the metric field associated with any point cannot evolve in time, a point of the manifold being identified by the element of matter that fills it. All the distances between the elements of matter are therefore invariant. Thus, as in Zeno's paradoxes, we come to the conclusion that any change in the universe is impossible.

To paraphrase Weyl: *we can no longer even understand how it is possible to knead a plasticine ball to change its shape.*

Thus, we have achieved a physical theory in which any motion and therefore, doubtless, any change can no more be conceived. It is not exactly a logical contradiction, but close to it. For what is a physical theory if not a theory of change? In this regard Aristotle's reaction to the Eleatic arguments which aimed at showing the impossibility for any motion is famous. For the Stagirite, a philosophy that negates what our senses teach us with the strongest evidence –i.e. the existence of motion– does not even deserve to be called “physics”<sup>54</sup>. So faced with such an argument, an

<sup>53</sup>[Wey18b, p. 90], [Wey19, p. 90], [Wey21, p. 90].

<sup>54</sup>Aristotle, *Physics*, book I, Chap. 2.

argumentative refutation seems pointless. Like Diogenes the Cynic, we can dismiss such a standpoint with the simple gesture which shows in an immediate intuition the possibility of motion, for instance walking<sup>55</sup>. It is somehow Weyl's starting point. He shows the absurdity to which his own standpoint was leading with the simple gesture consisting in kneading a plasticine ball. Luckily Weyl does not stop there but tries to identify where exactly the error lies which led to this unsustainable situation. A paradox meaning a manifestly wrong proposition but deduced from a plausible argument can only be useful for knowledge if it is analysed so that the flaw can be isolated and deconstructed. Showing that it is false is not enough, as Aristotle himself eventually admitted<sup>56</sup>.

How, starting from Weyl's text, have we elaborated our aporia? In order to interpret the assumption 2. above, we had to make two relevant choices concerning the text:

2 bis) we have established that the function  $\rho$  operating in the hypothesis 2) was a *scalar* in the formal sense, i.e. a variable represented by a number that is independent from the location of this element in space, and from the choice of the coordinate system.

2 ter) We have assumed (this is not explicit in the text) that the functions  $\rho$  were constants.

In spite of the formal analogies between Weyl's *pba* and Einstein's hole argument, we can notice that they follow clearly different intellectual paths. For Einstein the problem was to reach a physical theory in which the metric coefficients are perfectly determined by matter. *The* problem met by Einstein consisted in the fact that, whatever the equation of type:  $G^{\mu\nu} = T^{\mu\nu}$ , chosen as the fundamental law<sup>57</sup>, a total determining of the metric coefficients in a coordinate system where the factors  $T^{\mu\nu}$  are known is impossible. Einstein only shows it in the case of the existence of regions absolutely empty of matter (holes). In these empty regions, the factors  $T^{\mu\nu}$  are absolutely cancelled out and therefore do no more vary during the application of a diffeomorphism on the manifold, while  $g_{\mu\nu}$  continues to covary according to its tensorial nature.

In Weyl's text, the problem *is not* reaching a theory in which metric is totally determined by matter. On the contrary this is an *accepted assumption*, posited to try to solve Weyl's own issue (cf. section 2). It only becomes an issue because of its unexpected consequences, leading to negating the very possibility of any movement.

### **Covariance problem in the formulation of the *pba***

Before trying to find a solution to this aporia, let us notice that the way in which the problem is set down is open to doubt, because it seems incoherent at the level of the properties of covariance. Let us show that this nevertheless does not invalidate the problem set down by Weyl.

Weyl supposes that a specific scalar field  $\rho$ , representing matter, would totally determine the metric field  $g_{\mu\nu}$ . But this hypothesis seems absurd, since a scalar field and a metric field (that is a field of tensors with two covariant indices) do not have the same covariance properties. Starting from two such fields, we can find, at least in some cases, a change of coordinates (in modern language

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<sup>55</sup>Diogenes Laercius, *Lives, Doctrines and Sentences of Famous Philosophers*, VI, Chap. 2 [Diogenes].

<sup>56</sup>After the passage evoked previously Aristotle eventually admitted that even if the Eleatic opinion of the immobility and unity of the world is obviously false, dedicating efforts to refute Parmenides and Melissus may be physically instructive. It is of course also the case for Zeno.

<sup>57</sup> $T^{\mu\nu}$  is the energy-momentum tensor and  $G^{\mu\nu}$  a tensor only dependant of the metric field and its derivatives, which would still need to be determined. See [Nor87, p. 162-sq.].

a diffeomorphism) which leaves the field  $\rho$  invariant, while modifying the values of  $g_{\mu\nu}$ . By contrast, the Einstein equations  $G^{\mu\nu} = T^{\mu\nu}$  seem to be free from this covariance problem, both sides of those equations being of the same tensorial nature. Therefore, with Einstein, one needed to consider the very specific case of a “hole”, in order to find an application which modifies  $g$  without modifying  $T$ .

In general relativity, the distribution of matter is represented by the energy-momentum tensor  $T^{\mu\nu}$ . Yet, in very simple situations, this tensor is reduced to one or two scalars. Indeed if we consider the approximation of a *perfect fluid*<sup>58</sup> then, in a system of coordinates that is comoving with the fluid, the tensor  $T^{\mu\nu}$  is reduced to two scalars:  $\rho$  the density of matter at rest, and  $p$  the hydrostatic pressure:

$$T^{\mu\nu} : \begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{bmatrix}$$

if we choose the signature  $+ - - -$  for the metric and take  $c = 1$ .

This tensor is absolutely invariant for a *purely spatial* transformation of the coordinates, i.e. as long as we remain in a system of co-mobile coordinates. In this simplified case, the tensor  $T^{\mu\nu}$  is indeed eventually reduced to two scalars. But, the fully covariant definition remains:

$$T^{\mu\nu} = (\rho + p) u^\alpha u^\beta - p g^{\alpha\beta}.$$

This equation is coherent at the covariance level, but requires to show explicitly the metric<sup>59</sup>. Weyl specifies that generally the most natural form for the energy-momentum tensor is the mixed tensorial density  $\mathcal{T}_\nu^{\mu 60}$  which in that case has the value:

$$\mathcal{T}_\nu^\mu = \sqrt{-\det(g)} [(\rho + p) u_\alpha u^\beta - p \delta_\alpha^\beta]$$

and that includes the metric, even in the absence of pressure.

Schwarzschild’s pioneering works<sup>61</sup> use that type of simple matter characterisation. The simplification made by Weyl, when he says that matter is reduced to a few scalars, is therefore not at all incongruous. We can actually use Schwarzschild’s solution to interpret the *pba*.

Let us consider a general-relativistic space-time  $(M, g_{\mu\nu}, T_{\mu\nu})$ , which fulfills the Einstein equations without the cosmological term. Matter is supposed to be concentrated in a region  $S$ , the remaining of space being empty. This matter is admittedly an incompressible perfect fluid at rest which, in an adequate system of coordinates (co-mobile with the fluid), admits a spherical symmetry. Finally, we suppose that, in that same system of coordinates, the metric admits also a spherical symmetry and tends towards the Minkowski flat metric at infinity. Then, Schwarzschild’s metric (interior and exterior) is required<sup>62</sup>. The form of our “sphere” of matter is therefore well determined. Let

<sup>58</sup>Weyl uses himself this approximation in [Wey10, p. 205].

<sup>59</sup>[Wey10, p. 205; 262-263].

<sup>60</sup>[Wey10, p. 229].

<sup>61</sup>[Sch16a; Sch16b].

<sup>62</sup>Here, we leave aside the issues arising at the boundary, when joining the two solutions.

us now suppose that the elements of matter have been moved so that matter, after equilibrium is reestablished, can still be characterised as a perfect fluid at rest with the same mass and the same uniform density. Finally, let us suppose that the spherical symmetry is resumed. According to general relativity, the metric must resume the form imposed by Schwarzschild in the system of coordinates adapted to the new location of the “ball” at rest. It is a simple and precise interpretation of Weyl’s thought experiment, which can also be used as a very simple model for the moving of a “spherical” body with a uniform density in a flat cosmic environment.

### 3.2 Is the variability of matter properties sufficient to get out of the aporia?

Weyl’s strategy to get out of the aporia in *Raum-Zeit-Materie* consists in abandoning 2 ter). He keeps the idea that matter keeps intrinsic pre-metric properties  $\rho$ , which in turn completely determine metric (hypothesis 3). But these properties do not need to be constant. Here are the reasons why a plasticine can however be deformed:

So that the form that was squeezed may seem spherical to an observer from any perspective, we should need, among other things, a deformation of the internal atomic structure of the plasticine different from the one I can actually produce with my hand<sup>63</sup>.

Thus the type of physical change that we can induce on a plasticine ball, by kneading it, would be of another nature than a simple pulling forward of its intrinsic properties on the spatial manifold. In kneading the plasticine ball, we will fundamentally change its intrinsic properties, which in turn allows the modification of its *metric properties*. So the variability of the magnitudes  $\rho$  allows restoring the possibility of motion.

Weyl’s assertion teaches us that the properties  $\rho$  in his general formulation of the  $\mathcal{RME}$  principle are not physical magnitudes which would be fundamental constants of matter, absolutely invariant like the charge or mass density of the electron for instance. Rather, they are magnitudes capable to take different contingent values in the course of time for the same element of matter; as is the case for energy–momentum density which appears in the Einstein equations.

However, to get out of this aporia, we have been compelled to adopt a physical theory of a very specific type, in which no motion is possible without modifying the intrinsic properties (the  $\rho$ ) of matter. Something like “pure motion” has become impossible. We can nevertheless still imagine that the form of the ball can be modified without the intrinsic qualities  $\rho$  of its matter being affected. It is indeed possible if we modify the properties  $\rho$  of matter *outside the S region corresponding to the ball*. let us remember that the expression of the  $\mathcal{RME}$  principle is *non-local*. The problem then takes a cosmological turn. This is why in the 4<sup>th</sup> edition of *Raum-Zeit-Materie*, Weyl adds a phrase to the precedent sentence:

[to restore the spherical form, we would need to consider] a deformation of the internal atomic structure of the plasticine, *or a rearrangement of all the cosmos masses*

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<sup>63</sup>[Wey18b, p. 90].

(Here I emphasize the added part of the sentence).

I will however add that the use of cosmology, introduced by Weyl in the 4<sup>th</sup> edition, is dubious. Indeed let us suppose that we kneaded our plasticine without changing its internal properties  $\rho$ . If its form has changed, Weyl's text suggests, it is because we have modified the properties of matter outside the region filled with the ball. However, we could modify the field of matter only in the immediate environment of the ball. The need to place ourselves at a cosmological level does not seem very relevant. Besides, if the immediate environment of the ball is empty or nearly empty, it is represented by  $\rho=0$ , and we cannot really see how the change of shape of the ball could be ascribed to it. Finally, last difficulty, let us recall that the *pba* appeared in the context of a philosophical challenge specific to Weyl: saving the homogeneity of space understood as the possibility of moving a material content without modifying its nature. When the material content that is moved is *finite*, such a move can mean: keeping the properties of the ball, including the metric properties, while changing its metric relations with the other bodies of the cosmos. This has a clear *physical* sense. But if the matter that is to be moved is the cosmic matter as a whole, and that all the cosmic metric relations are kept by such a "motion", it then seems that motion can only have a purely ideal meaning<sup>64</sup>.

To summarise, Weyl only gets out of his aporia by allowing a variability of the magnitudes  $\rho$ , or inside or in the cosmic environment of the plasticine ball. However, the text leaves in abeyance several fundamental problems, which will come up again in later texts. We are going to discuss the two main ones.

### 3.3 Why is Weyl so focused on finding a *physical* interpretation for diffeomorphisms?

A reader familiar with modern literature on the covariance principle and Einstein's hole argument may look back on Weyl's text suspiciously. let us recall that, indeed, Einstein, after elaborating his hole argument, had been temporarily led to reject all the generally covariant formulations of gravitation. It is generally accepted that Einstein had made a conceptual error while elaborating his argument: he had wrongly thought that a system of coordinates had, *per se*, a physical meaning. In fact it is *per se* only a mathematical artefact, as long as it is not linked with physical entities (matter and metric fields). So, two ordered pairs  $(g_{\mu\nu}(x), T^{\mu\nu}(x))$  and  $(g_{\mu\nu}(x'), T^{\mu\nu}(x'))$  representing a field of matter and a metric field, related by a simple active diffeomorphism (using a modern language), would in fact only be two mathematical representations of the same physical situation<sup>65</sup>.

A reader who is aware of these developments may be surprised at the apparent naivety with which Weyl tries at all costs to give a *physical* significance to the operation of pulling forward the matter and the metric field on the manifold. Why does he not conclude, with a spirit close to Einstein's, that the spatiotemporal manifold has lost all objectivity and, therefore, the operation consisting in pulling forward simultaneously the metric and matter on the manifold *has no physical*

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<sup>64</sup>Giving some sense to such a moving of matter would require setting the problem in a really dynamic framework, without only considering the initial state and the final equilibrium state. In *Massenträgheit und Kosmos*, Weyl will be able to give sense to such a global movement with his "Boats-<Lake Analogy". See further section 4.7.

<sup>65</sup>[IS06, p. 1243], [Nor99, p. 804-805], [Nor93], [Nor87, p. 170-171;177], [Sta89].

*significance but is only the expression of a mathematical latitude in representing the same physical situation?*

Bluntly accepting it would lead to a form of ideality of spatial (or spatiotemporal) manifold. It would not be a physical reality, but a mere mathematical artefact used to label spatial points. Only matter and metric would have a physical reality, but not the manifold. That form of ideality had been considered by certain forms of neo-Kantism of the time<sup>66</sup>. Why does Weyl, in spite of his idealism, stop before reaching that position? This question can only give rise to speculation. He was perhaps looking for a form of idealism closer to the original Kantian form, which makes space a “form of our intuition” and not a mere analytical mathematical artefact. It seems that Weyl construes the idea of space as a form of intuition involving the possibility of a “real” motion (not only a mathematical transformation) taking a body from one point to another “real” point, without modifying its properties.

It is clear that, when Weyl speaks of moving elements of matter from a region  $S$  towards a region  $S'$ , he has in mind much more than a mere transformation of the mathematical representation of a physical situation otherwise left unchanged. In fact, in order to give form to his thought experiment, Weyl takes as his model the idea of an electrically charged body in equilibrium with an electrical field; moreover, he considers that the moving of matter that is considered generates a temporary physical perturbation.

We are tempted to ask Weyl: how, in a physical theory based on a dynamical geometry, could we *physically* identify, in the course of time, a point of the manifold? We can *a priori* see only two solutions:

- Either we consider a simplified framework in which metric is static, and the system of coordinates is chosen relatively to that metric. This solution can indeed allow for physically identifying points, for instance to give sense to the moving of a test-particle in a static field. But we must assume that the displacement of the body (the test-particle) does not perturb the fundamental metric. Such a solution is therefore inapplicable in the *pba* case.
- Or we attach the coordinates to elements of matter taken as physical markers of the position. It is therefore a system of coordinates which is co-mobile towards a specific material background. In that case, moving the plasticine ball means: leading its constitutive points to coinciding with new elements from the “material background”. In that case *we only fall into the pba aporia if we suppose that the material background has a negligible influence on the determination of the metric, compared with the ball that has been moved*. It is in this cosmological fault of the *pba* aporia that Weyl rushes when he adds the sentence element quoted above.

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<sup>66</sup>In particular, Cassirer, in the line of the Marbourgh school, criticises Kant’s philosophy of space, in so far as it is too strongly connected to perception data, leading to an exclusive focus on Euclidean geometry. See [Cas10, chapitre III, particularly p 106] and [Cas23, chap. V]. According to Cassirer, the *a priori* notion of space which must be incorporated to science needs rather to be based on the driving forces of mathematical analysis and numerical symbols.

### 3.4 On the impossibility of a primitive separation of matter and metric

Another difficulty emerges as early as the first version of the *pba*: the impossibility to radically differentiate, in the formulation of the  $\mathcal{RME}$  principle, matter from the metric.

Several features of Weyl’s text seem to confirm that Weyl –just as Einstein in his first formulation of Mach’s principle– was initially driven by the idea of a total emergence of the metric from matter (ontological anteriority of matter). This provides reasons for researching a manner to characterise matter totally independently from any metric consideration, the metric being supposed to then emerge from this originally non-metric matter. This is suggested by the fact that Weyl formulates the  $\mathcal{RME}$  principle in a very radical manner, according to which matter is to *completely* determine metric. This idea is still reinforced when Weyl concludes, at the end of the passage on the *pba*, that space-time, prior to any considering of matter and forces, is absolutely deprived of any form. However it is on this formless space that the field of matter is originally defined. Moreover by making the qualities of the matter,  $\rho$ , mere *scalars*, Weyl seems to be trying to suppress any dependence of matter towards a prior metric structure.

Yet, this supposed total anteriority of matter on the metric is at the core of the problems met by Weyl in his aporia. We have seen that, even assuming that the quantities  $\rho$  characterising matter are variable, we can only get out of the aporia by adopting a specific type of physical theory in which a “pure” motion, leaving matter properties untouched, has become impossible. Besides that, we spontaneously wonder what kind of variability of the properties  $\rho$  is concerned. We are led to think that the different possible values of the  $\rho$ s for the same elements of matter refer to different manners, for matter, to *distribute* spatially. This is clear if we have in mind the nature of the energy-momentum tensor in general relativity, which is a specification of the general idea of matter  $\rho$  which is at work in the Weyl’s  $\mathcal{RME}$  principle. It is also clear in Weyl’s text, since he refers to (matter or charge) *density* as the prototype of what must be understood by the functions  $\rho$  describing matter. However the idea of density clearly has a *metric* meaning, and not only a topological one. If we did not presuppose a metric, it seems that we could not provide meaning to the simple “rest-mass density” scalar. The properties  $\rho$  seem to be modes of matter spatialisation, presupposing a metric. This questions the very possibility to distinguish, within matter properties, between pre-metric qualities, supposed to be primitive, and metric properties, supposed to be derived.

It seems therefore that we are forced to abandon the ontological anteriority of matter upon metric in order to get out of the *pba* aporia. Weyl does not reach that conclusion in *Raum-Zeit-Materie*. In the first edition he informs the reader, in a footnote, that things will get clearer as regards the *pba*, in chapter IV in which general relativity will be developed<sup>67</sup>. However, Weyl does not explicitly come back to it, and, moreover, his footnote is deleted from the third edition onwards. It is only in *Massenträgheit und Kosmos* that Weyl will develop all the consequences of his aporia.

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<sup>67</sup>[Wey18b, p. 88]: “Genauerer hierüber in Kap. IV.”.

## 4 The *pba* in *Massenträgheit und Kosmos*

### 4.1 A new issue

In the *Massenträgheit und Kosmos* article published in March 1924, we find numerous elements from *Raum-Zeit-Materie* which we have discussed. These elements are numerous enough for us to consider that it is indeed the same argument, the “plasticine ball” one, which comes back in an altered form. These common elements are: the metaphor of the plasticine ball itself<sup>68</sup>, the issue of the relations between the metric and matter, as well as many technical notions and considerations.

However the issue which motivates the argument is now different. The idea of space as the form of appearances, the homogeneity<sup>69</sup> requisite that goes with it and the tension triggered by this homogeneity towards adopting variable curvature metrics do not appear anymore. Instead, the *pba* is motivated by a questioning on the legitimacy of the principle of determination of inertia and metric by matter. What was only part of the argumentation in *Raum-Zeit-Materie*, vaguely expressed, has now become the very core of the questioning. The problematic is now closer to Einstein’s, focus on the legitimacy of “Mach’s principle”, the reference to Mach being newly introduced by Weyl p. 197.

We will begin this section by discussing the dialogue form of this text, and the function of the characters. Then we shall proceed with analysing the *pba* in the first part of the dialogue. Finally we will address the cosmological aspect of the debate which corresponds to the second part of the dialogue.

### 4.2 The dialogue form of *Massenträgheit und Kosmos*. Who are Paulus and Petrus?

In the dialogue, both characters, Paul <Paulus> and Peter <Petrus>, meet to resume a discussion, that was interrupted in 1915, on the foundations of general relativity. We are led to think that Paul impersonates Weyl as the one who leads the dialogue and takes it to its conclusion. Moreover, the intellectual stages through which Paul tells us he went remind us of the ones Weyl actually experienced. Paul introduces himself as somebody who initially strongly believed in Mach’s principle, in its most radical aspect, similar to the *RME* principle in §12 of *Raum-Zeit-Materie*, before retracting. Now, he no longer believes in the validity of Mach’s principle, and the dialogue unfolds as Paul explains to Peter the reasons for his change of mind. Paul says that if the belief in Mach’s principle forms the “stone base on which the relativity Church lies”<sup>70</sup>, then he has become an apostate, a heretic. He has changed from Saul to Paul<sup>71</sup>.

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<sup>68</sup>[Wey24, p. 198]: (*Plastelinmasse*).

<sup>69</sup>The notion of homogeneity appears punctually in *Massenträgheit und Kosmos*; however it is not the homogeneity as an *a priori* property required from space “per se”, but, instead, the (local or global) homogeneity of some configurations of matter, considered as particular cases, or the metric homogeneity of some specific solutions to the Einstein equations, particularly the de Sitter’s one. Homogeneity has become the exception rather than the *a priori* rule.

<sup>70</sup>[Wey24, p. 197]. There is an implicit reference to the famous biblical sentence “And I tell you that you are Peter (Céphas=Rock), and on this rock I will build my church”.

<sup>71</sup>Saul of Tarsus was the Jewish name of the man later known as the Apostle Paul or Saint Paul in the *New Testament*. It is said that he was initially a Pharisee, violent towards Christians, before converting and joining Jesus Christ. He changed his name from Saul to Paul to mark this conversion. There is a German phrase “change from

It is difficult to precisely follow the manner in which Weyl wants to use the episode of Paul’s conversion to illustrate, apparently, his own intellectual journey. The limit of the image used by Weyl comes from the fact that the intellectual journey of Paul in *Massenträgheit und Kosmos*, is in fact a double *conversion then reconversion* movement. By detaching himself from the old “religion”, Newton’s and the belief in absolute space, Paul has temporarily joined the new Einstein’s-Mach’s Church. Then, he has discovered the error of this point of view and renounced Mach’s principle, thus becoming an apostate, this time from the perspective of the new religion.

As we will develop below, this reconversion is not however a mere return to the former belief (Newton’s). Paul will believe again that space cannot be reduced to matter, that it does not simply emerge from it. Nevertheless, this space with an autonomous existence will no longer be an absolute space with fixed properties, as for Newton, but a dynamical “ether”, interacting with matter, without any ontological hierarchy between them. Paul does not consider this reconversion as just leaving the general relativity “Church”, but, instead as a deviation from the orthodox interpretation of the theory –this is 1923–, based on Mach’s principle. From this point of view the word “heresy” is rather well-suited.

We can note that Einstein followed an intellectual path very similar to Weyl’s, at first subscribing without any restriction to Mach’s principle, before retracting<sup>72</sup>. So he could also very well be the person represented by Paul in the dialogue<sup>73</sup>.

### 4.3 Typology of the principles of determination of the metric by matter, rejection of Mach’s principle

Weyl and his relativist contemporaries – starting with Einstein – had met many difficulties in applying Mach’s principle in general relativity. This is highlighted by the technical difficulties met by Weyl while developing his first versions of the *pba*. Weyl seems to have become gradually aware that the encountered technical problems were not contingent – only due to an oversimplified conceptual framework – but on the contrary fell within the range of the actual difficulties inherent to Mach’s principle itself. This leads Weyl, in *Massenträgheit und Kosmos*, to refine his thought about the adequate expression and the relevance of the principle which links matter, inertia and metric. The problem is all the more important since Weyl remains convinced that such a principle

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Saul to Paul” (*sich vom Saulus zu Paulus wandeln*), used to describe a radical change of personality or behaviour.

<sup>72</sup>About Einstein’s abandonment of Mach’s principle, see letter of 02.02 1954 to F. Pirani, the extract of which is reported and translated into English in [Ren07, p. 61]. In [Nor87, p. 180-sq.], it explains that this abandonment by Einstein of Mach’s principle takes the form of a shift from an overt anti-realism towards space (then identified with the naked manifold) to a realism towards space (then identified with the metric field, called “ether”). Other references about Einstein’s position concerning Mach’s principle in: [BP95, P. 10; 67-90], [Nor93, p. 808-sq.], [Tor83, section 6.2]. Before abandoning Mach’s principle, Einstein gave it very variable forms. According to Norton and [Tor83, p. 201], Einstein’s change of mind about Mach’s principle began in the years 1918-1919.

<sup>73</sup>The question whether Weyl had real persons in mind behind his characters is minor. What is important is to underscore the fact that Paul’s intellectual evolution is close to Einstein’s and Weyl’s. Paul and Peter, in the dialogue, say that they first met in the United States in 1915. Einstein and Weyl met as early as 1913 at the E.T.H. of Zurich. Weyl arrived at the institute when Einstein was there, working with his friend Grossman at elaborating general relativity. In the dialogue, Paul tells Peter that the latter should well know the axial symmetry solutions of the theory of general relativity, since he raised the problem of their existence. Weyl (with Lense and Thirring) is amongst the first scientists who published such solutions (see [Wey10, §32; and bibliography note 22 of chapter IV]).

(and not the covariance one) must be at the core of our understanding of general relativity<sup>74</sup>.

While §12 of *Raum-Zeit-Materie* only contained a radical and vaguely expressed version of Mach's principle ("RME principle" above), the dialogue from 1924 includes a series of more or less important variations of Mach's principle and of related principles. The function of this plurality is to successively isolate the difficulties that make Mach's principle inapplicable in general relativity, or more widely in any field theory that adopts a dynamic metric. Weyl starts from the following principle which he classically attributes to Mach:

(*M*) the inertia of a body comes to existence (*Zustande kommt*) due to the interactions of all the universe masses.

Weyl specifies that Mach's principle is a particular case of an absolutely general principle, which he calls "causality principle":

(*C*) all the [physical] events are causally univocally determined by matter, that is by charge, mass and the state of motion of matter constitutive elements.

Considering the way Weyl uses this principle, the name is rather ill-chosen. For it is not a question of opposing causality to causeless phenomena. Instead the debate is about knowing whether we can relate the ultimate causes of physical phenomena to pure relations between elements of matter, or whether we are led to adopt an immaterial physical entity like Newton's absolute space or ether such as it appears at the end of the nineteenth century, deprived of any material consistency. The principle (*C*) opts for the first alternative, and should therefore be called a *materialism*.

So, Weyl turns Mach's principle into a restricted version of a principle which concerns the ontology of physics (the word "ontology" is however not used by Weyl). The point is to postulate that all physical events -starting with inertia- would come to existence (*Zustande Kommen*) from masses and their interaction. It is therefore a reductionist principle which, if it turned to be true, would lead to a materialistic ontology for physics, in which matter (and its "interactions") would be the only primitive entity. The formulation (*C*) develops the idea, subjacent to (*M*), of an ontological reduction in the terms of a *univocal causal determination*. Paul will then develop some arguments which lead to abandon (*M*) and a *fortiori* (*C*).

At first we note that the principle (*M*), contrary to the one at work in *Raum-Zeit-Materie*, does not directly refer to the metric notion, but only to the notions of inertia and mass. In this it is close to Mach's original thought. However, Weyl demonstrates that, here, the metric is an inevitable element of thought. In fact, let us suppose that we attribute to cosmic masses the causal origin of an inertial phenomenon, for instance the flattening of the Earth at the poles. The simple presence (in the sense of the determination of the *positions*) of cosmic masses could not be sufficient to be used as the causal origin to the phenomenon. We would have to say that it is the *motion* of the Earth relatively to the big cosmic masses, which is the cause of its flattening<sup>75</sup>. Yes, but, Weyl goes on, general relativity teaches us that:

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<sup>74</sup>[Wey24, p. 187, right hand column]. See our note 9 above.

<sup>75</sup>Thus, since at the time one consensually believed in the static nature of the cosmos, Newton's bucket experiment was explained by Mach by the fact that the bucket is in motion (rotating) relatively to the referential defined by the whole of cosmic masses, supposed to be static relative to each other.

(A) [Independently from the metric field], the concept of the relative motion of several bodies separate from one another is as untenable as the absolute motion of a single one.

To understand this radical affirmation, we must clarify the meaning of the word “separate” (*getrennter*). Separate bodies are bodies that are located in topologically disjoint regions. What can allow us to assert that two such distant bodies are in motion relative to each other? Observation will not suffice. While we commonly say that we can see fixed stars turn around us, in reality what we see turning is the stellar “compass” (*Sternenkompas*), that is the beam of all the light rays which reach our eyes from the stars. But we cannot make any inference from the rotation of the stellar compass (relatively to us) to the rotation of the stars themselves, without making hypotheses relative to a metric field which occupies the intermediate region between the stars and us, and which determines light trajectories as geodesic. Mach did not consider the necessity to take the metric into account, because in his time everybody believed in the static nature of the cosmos and in the rigid Euclidean body which could ideally extend all over the cosmos. Thus, Mach’s criticism of Newton’s absolute space mainly targeted its ontological independence towards matter, but the Euclidean nature of the metric was hardly questioned<sup>76</sup>.

With general relativity, remarks Weyl, we become aware of the contingency of the hypothesis of the indefinitely extended rigid Euclidean body. Just as inertial phenomena, metric phenomena are physical and can vary contingently. Thus, to give sense to (*M*), we need to break free from any metric hypothesis in defining matter. Just as in §12 of *Raum-Zeit-Materie*, we are led to characterise matter only through its intrinsic features (charge, mass) and its distribution (in a purely topological sense) within space-time. It is in that context that we must understand Weyl’s assertion (A). The notion of motion –even *relative* motion– between two separate bodies, loses all its meaning if we overlook the metric as medium. That is what the *fibred plasticine ball argument* (I will abbreviate this as the *fpba* in the following) will show.

Before going into this argument in detail, let us remark that its conclusion is strikingly akin to Weyl’s discourse in the introduction to *Mathematische Analyse des Raumproblems*. Indeed, Weyl has just shown to us that one cannot inquire into the relationships between matter and inertia without involving a third intermediate element: the metric. Similarly, in the *Mathematische Analyse des Raumproblems*, Weyl had exposed the impossibility to correctly conceive the problem of space only on the basis of the “space/content (=matter)” duality favoured by Kant. Instead, one must consider the triplet “space/matter/metric”<sup>77</sup>. So, the metric must be introduced as a third irreducible element, between matter and space, in both the infinitesimal sphere which is addressed in *Mathematische Analyse des Raumproblems* and the finite sphere which is addressed in the texts about Mach’s principle.

#### 4.4 The Fibred Plasticine Ball Argument

The fibred plasticine ball argument (the *fpba*) is a variant of the *pba* which, to our knowledge, appears in *Massenträgheit und Kosmos*:

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<sup>76</sup>However, there is no consensus on Mach’s real purpose. See [BP95, 9-65;90-sq.].

<sup>77</sup>[Wey15, p. 1].

[On can] think about the four-dimensional universe as a plasticine mass penetrated by fibres, world-lines of particles of matter, which cannot converge into a single one, but which can otherwise spread arbitrarily [...]

In *Raum-Zeit-Materie*, the *pba* concerned space rather than space-time. The idea of motion was there reduced to a transformation from an initial static equilibrium state into a final state of the same type. In contrast, in the present text, the fibred plasticine represents a four-dimensional Lorentzian manifold endowed with a still undetermined metric. The fibres in the plasticine represent the world-lines of the material points, which are one-dimensional submanifolds of space-time<sup>78</sup>. At the end of the text, Weyl specifies in a literary therefore imprecise way that this family of lines must constitute what is nowadays called a foliation (at least local, and generally not unique) of space-time of the type  $\mathbb{R} \times \mathbb{R}^3$  (with one-dimensional sheets). Weyl's formulation is not precise enough to indicate whether he has in mind only a *local* foliation (which exists in all Lorentzian manifolds) or, in a more restrictive manner, a *global* foliation. In that case, the hypothesis would not be trivially verified in general relativity, but would require an additional cosmological hypothesis, similar to what is nowadays known as "Weyl's principle"<sup>79</sup>. In other words, it should be assumed that space-time can accept a globally defined (not necessarily unequivocal) temporal orientation.

Local or global, Weyl's hypothesis is in any case sufficient for him to develop his argument:

[One can] continuously deform the plasticine so that, not only one fibr, but all the fibres become straight and vertical. If the vertical axis represents the axis of times, it is construed as follows: each body remains in its place in space.

The technical background is quite clear. Let us take any system of coordinates compatible with our foliation. It will transform each line in our family into a "vertical line" of  $\mathbb{R}^4$ , that is a set of the form:

$$\{(t, a, b, c) | t \in \mathbb{R}\}$$

in which  $(a, b, c) \in \mathbb{R}^3$  is a fixed triplet, the coordinate  $t$  representing time, the three others representing space. Thus, by choosing coordinates co-mobile with matter particles, we have simultaneously "put at rest" all matter, this notion of rest being understood only in a topological (pre-metric) sense.

So, the *fpba* shows us that, as long as we do not provide ourselves with a metric, the very difference between rest and motion is only illusory since a motion defined only in a topological way, on the spatiotemporal manifold, can be destroyed by a mere change of coordinates. This justifies the affirmation (A) above. We can instantiate this idea on the concrete example of a matter reduced to a perfect fluid without pressure ("dust"), represented by the twofold covariant tensor  $T^{\mu\nu} = \rho u^\mu u^\nu$ . In this particular case<sup>80</sup>, the tensor  $T^{\mu\nu}$  has no metric feature, since it is definable as a simple tensor on the naked manifold. It shows therefore that differentiating between matter at rest and matter in motion is impossible only on the basis of this tensor. Whatever the initial value of this tensor,

<sup>78</sup>When a metric is attributed to space-time, we shall expect the world-lines to be timelike.

<sup>79</sup>[Ker86].

<sup>80</sup>This case contrasts with the tensorial density  $\mathcal{T}^{\mu\nu}$  which always depends on the metric (or at least its determinant), and with tensors of more complex matter (with pressure, etc.) which we shall consider later.

one can ensure that the four-velocity field is identically  $(1, 0, 0, 0)$  (matter “at rest”) by shifting to co-mobile coordinates. Then, the value of the tensor  $T^{\mu\nu}$  is everywhere

$$T^{\mu\nu} : \begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The *fpba* shows us that Mach’s principle, in some radical form, is not only contradicted by experience, it is even *absurd*; moreover, in contrast with *Raum-Zeit-Materie*, Weyl explicitly says so<sup>81</sup>. The fibred plasticine argument forbids us to differentiate motion and rest, when we originally refuse to provide ourselves with the metric, in conformity with the full interpretation of  $(M)$ . We can therefore attribute to matter only static properties such as charge or mass<sup>82</sup>. The belief that the metric could be univocally derived from such static properties is what Weyl (Paul) qualifies as absurd. This qualification confirms backwards our interpretation of the § 12 of *Raum-Zeit-Materie* as developing an aporia of the Eleatic type<sup>83</sup>.

In summary: the *fpba* shows that, at a simple topological level, motion cannot be differentiated from rest. But Mach’s principle, in its strongest versions, requires that the metric be entirely determined by the field of matter, the latter being described at a purely topological level, as the simple data of world-lines of matter and/or scalar fields. Such a principle can only lead to immobility (this time, immobility in its full sense, that is *metrical*). It is an absurd requirement for a physical theory. It seems to us that the too radical forms of Mach’s principle which are denounced here encompass in particular the *RMÉ* principle in §12 of *Raum-Zeit-Materie*. It is doubtless partly the numerous difficulties met by Weyl with the first versions of the *pba* which led him to reach this conclusion.

## 4.5 The existence of the inertial-gravitational ether

Weyl (Paul) then unfolds the philosophical consequences of the radical impossibility to give sense to Mach’s principle. A metric must be originally given, at the same ontological level as matter itself. It is on this metric field that the inertial motion shall be based (which is identified to gravitational or perhaps gravitational-electromagnetic motion<sup>84</sup>). Only then shall we be able to physically differentiate rest and movement. To designate this field, Weyl either uses Einstein’s word “ether”, or speaks of a “guide field” (*Führungsfeld*).

Since without ether, motion cannot have any significance, Weyl challenges the materialistic

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<sup>81</sup>[Wey24, p. 198, 2<sup>nd</sup> column]: (*Da dies offenbar ist absurd*). See also earlier in the text [Wey24, p. 197, right hand column], in which Weyl (Paul) says that he understands *a priori* that the principle  $(M)$  is inapplicable. I insist here on the phrase “*a priori*”.

<sup>82</sup>See the formulation of the principle  $(C)$  above, in which the properties considered in order to characterise matter were motion, charge and mass. If, according to the *fpba*, motion disappeared, there would only be mass and charge left, construed as simple scalars, as in §12 of *Raum-Zeit-Materie*.

<sup>83</sup>Section 3.1.

<sup>84</sup>The reader is referred to the literature on the evolution of Weyl’s belief in his theory of unification of gravitation with electromagnetism (1918). See [Afr09].

interpretation (in the sense of the principle (C) above) of general relativity, which de-substantiates space and alleges that space can emerge from pure relations between material elements:

The theory of relativity, well construed, does not attempt to eradicate absolute motion in favour of relative movement, but it destroys the concept of kinematic motion and replaces it with dynamic motion.<sup>85</sup>

General relativity has thus not replaced absolute motion -i.e. relative to space-, with a motion only relative to matter. Instead, it has kept absolute motion but has fundamentally changed its nature. We have shifted from kinematic motion (Newton) to dynamic motion (Einstein)<sup>86</sup>. In fact, the notion of space on which this absolute motion is based, ether, has no longer the same status as Newton's absolute space had. It does not have a rigid structure fixed *a priori* from all eternity. Instead, the ether has dynamic metric properties interacting with the field of matter. In this dynamic coupling of matter and ether, none of the two partners can entirely be deduced from the other one. Matter and ether are *co-original*. Finally, we must accept to abandon Mach's principle (M)<sup>87</sup>, and adopt a weaker principle for the relationships between matter and the metric, which subsumes Einstein's equivalence principle:

(G) the guide (ether) is a field of physical state (like the electromagnetic field) which interacts with matter. Gravitation belongs to the guide and not the force; and it is only in that way that we have an in-depth understanding of the equality between the inertial mass and the weight mass.<sup>88</sup>

## 4.6 Cosmological consequences of the *pba*

### 4.6.1 Preliminary to the discussion: the Einstein equations and the Cauchy problem

In the second part of *Massenträgheit und Kosmos* the protagonists of the dialogue give a cosmological dimension to the problem posed<sup>89</sup>, coming still closer to the problem as it was formulated by Mach and Einstein. In fact, even though he does not see any fault in the *a priori* argument which demonstrates the absurdity of Mach's principle, Peter cannot be convinced because:

[it seems that] Einstein has already done what you refute [, realise Mach's principle], - in the work in which he has generalized his original gravity laws, by [introducing the] "cosmological term". In view of this fact, any proof of its impossibility is therefore invalid.

The fact that general relativity is on the right track to express Mach's principle will be supported further down in the text, by Thirring's work. The latter demonstrated that, according to general relativity, a massive and hollow sphere, rotating about itself, exerts, on a mass inside it, an effect comparable to the centrifugal force<sup>90</sup>. Thus, general relativity seems to confirm Mach's answer to

<sup>85</sup>[Wey24, p. 199, right hand column].

<sup>86</sup>[Wey24, p. 199].

<sup>87</sup>See our note 72 about the similar rejection found in Einstein's thought .

<sup>88</sup>[Wey24, p. 199, left hand column].

<sup>89</sup>This late arrival of cosmology in the debate had already been noticed in *Raum-Zeit-Materie*. See our note 41.

<sup>90</sup>[Wey24, p. 199-200].

Newton’s bucket experiment<sup>91</sup>. Paul then accepts to re-examine Mach’s principle not on the basis of an *a priori* thought but in criticizing more precisely the way in which Einstein and his immediate successors tried to realise it, within general relativity, and more specifically in the 1917 cosmological article.

The idea of a determination of the metric -therefore of inertia- by masses is expressed in general relativity by the Einstein equations. They are partial differential equations which pose the proportionality between Einstein’s tensor

$$G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}.R (+\Lambda.g^{\mu\nu}),$$

which has a metric significance, and the energy-momentum tensor  $T^{\mu\nu}$  which represents matter. If knowing  $T^{\mu\nu}$  is enough to univocally determine the metric, then, according to Einstein, we shall be allowed to assert that Mach’s principle is indeed theoretically verified<sup>92</sup>. These are *second order* equations in regard with the metric, Einstein’s tensor expressing a form of spatiotemporal curvature of the metric. More specifically  $G^{\mu\nu}$  is a determined function of the metric and its first and second derivatives. Therefore, an infinite number of non-isometric  $g^{\mu\nu}$  may correspond to a single Einstein tensor  $G^{\mu\nu}$ . In particular, there are an infinite number of “Einsteinian metrics” which are the solutions of the Einstein vacuum equations  $G^{\mu\nu} = 0$ .

Because of the nature of these equations, we can approach the issue of the determination of the metric by matter in the form of a “Cauchy problem”. Typically we start from a spacelike hypersurface, on which, besides the matter  $T_{\mu\nu}$ , we set as initial conditions the metric and its first derivatives<sup>93</sup>. The Einstein equations being of the second order we then may expect to develop one single solution (up to an isometry) for the couple  $(g_{\mu\nu}, T_{\mu\nu})$  at least on a neighbourhood of the initial hypersurface<sup>94</sup>. Thus construed as providing a solution to the Cauchy problem, the Einstein equations do not directly express a radical genesis of the metric but only indicate how a metric

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<sup>91</sup>It is a classic in relativistic literature. See [BP95, “bucket experiment” p. 531].

<sup>92</sup>It is a way to express Mach’s principle which is most often used by Einstein. See [Ein18, p. 241-242], [BP95, 67–sq.]. We will discuss later the apparent circularity of the process, the metric seeming necessary in order to interpret the tensor  $T_{\mu\nu}$ .

<sup>93</sup>Typically, along the lines of Choquet-Bruhat and Geroch, the first derivatives are not given but, instead, a second order tensor giving the *external* curvature of the initial hypersurface within the manifold which is to be generated.

<sup>94</sup>The difficult problem of the existence and unicity (up to a diffeomorphism) of the solutions to the Einstein equations, with several types of initial conditions, and regularity hypotheses, has been subject since the 1950s to major progress thanks to Choquet’s work. See [CB52; CB69]. She has shown the existence and unicity (up to a diffeomorphism) of a local solution to the Einstein equations, within the neighbourhood of a spacelike hypersurface on which the Cauchy boundary conditions were given. The existence and unicity results are valid for the Einstein equations without sources but also with sources like perfect fluids or electromagnetic fields. See [CBG69, p. 331].

To obtain global results, we must add hypotheses such as the global hyperbolic character of the manifold. See [CBG69]. The problem becomes more complicated, sometimes with no solution or no unicity, if the correct regularity conditions are not posed, if the initial conditions are ill-defined or if hypotheses similar to global hyperbolicity are not available (which has everything to do with Weyl’s principle in cosmology).

In the 1920s, we do not know whether results, at least partial, similar to the ones obtained by Choquet, were already available. The most ancient reference given by Choquet is: [Dar27]. Einstein’s and Weyl’s convictions on the possibility to correctly pose the Cauchy problem for the Einstein equations, could be based on the similarity of these equations with the Laplace and Poisson equations, for which the results of existence and unicity were well known, and on some successful attempts to the univocal determining of a metric in a few specific cases (in the first place those considered by Schwarzschild). For a presentation by Weyl of the theorem of existence and unicity of the solution to a system of partial differential equations, see [Wey15, Appendix 3, 2<sup>nd</sup> part].

*initially given* must *evolve*, in view of the (metric) distribution of matter on the initial hypersurface. Rather than expecting a metric to entirely emerge from a purely non-metric field of matter, we simply expect the (metric) distribution of matter at a specific time to determine the metric (and the distribution of matter) for any ulterior time.

The works discussed by Weyl in *Massenträgheit und Kosmos*, whether they are Einstein's, Thirring's or Schwarzschild's, do not pose the Cauchy problem in general relativity in all its generality but are limited to the static case. It means that they assume a distribution of matter  $T_{\mu\nu}$  and a metric  $g_{\mu\nu}$  that are invariant in relation to the time coordinate. In this simplified framework, the Einstein equations correlate a form of *spatial* curvature of the metric to the properties of matter (like its density  $\rho$ ) and possibly its pressure  $p$ ) as do the Poisson equations for classic static gravitation. Let us suppose that we know the values of these properties of matter, and we are trying to determine the metric on a domain  $\mathcal{D}$  of the spatial manifold. The Einstein equations, as well as Poisson's, enable us to univocally determine a solution only if we give ourselves the "boundary conditions", i.e. the values of  $g_{\mu\nu}$  on the spatial boundary of the domain  $\partial\mathcal{D}$ . Taking an infinite domain does not make a difference since we shall always need to know the values at spatial infinity to univocally solve the equations. The Einstein equations, in this case, lead us to a Dirichlet problem instead of a Cauchy problem.

#### 4.6.2 Return to the idea that $T_{\mu\nu}$ makes sense only in an already metrical context

Einstein tried to solve his equations without having to impose any boundary conditions for the metric. Having such a goal, was he under the range of Weyl's *a priori* argument which doomed some formulations of Mach's principle to being absurd? Einstein asserts that, if his project was realised, then the metric would be *entirely* determined by the tensor  $T_{\mu\nu}$ . It seems that we must suppose the tensor  $T_{\mu\nu}$  to be deprived of metric properties, before the solving of the field equations. Can this idea be given any sense?

It is clear that the tensor  $T_{\mu\nu}$ , once the equations have been solved and a metric has been determined, takes a clearly metric signification, since it encompasses in particular the definitely metrical notions of density and momentum. In contrast, before the correlation with the metric, the tensor  $T_{\mu\nu}$  admits no definite interpretation. It may be one of the reasons why Weyl prefers mixed tensorial density  $\mathcal{T}_\nu^\mu$ , which explicitly involves the determinant of the metric. It is also why, in the Cauchy problem, the tensor  $T_{\mu\nu}$  is immediately correlated to the metric on the initial hypersurface. This leads Ehlers<sup>95</sup> to assert that  $T_{\mu\nu}$  can in no case correctly describe the state of the field of matter, until it has been correlated to a metric.

However, following Einstein, we can try to see whether the values of the  $T_{\mu\nu}$  may be determined before a determined metric has been given. In the simplest cases, as in the hypothesis of a matter

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<sup>95</sup>In [BP95, p. 93], Ehlers notes that the energy-momentum tensor, until it is coupled with a metric, does not properly describe the field of matter, and to this day, no physical theory can describe the field of matter before a metric is given. Thus he agrees that Mach's principle, if it stipulated that "matter *in itself* [i.e. prior to any metric consideration] determines the metric" would be neither true nor false but even pure nonsense. He mentions that Einstein eventually admitted it in his letter to Pirani of 02.02.1954: "the  $T_{\mu\nu}$  which must represent 'matter', always presupposes  $g_{\mu\nu}$ ", quoted from Einstein's letter to Pirani of 02.02.1954 in [Tor83, p. 202]. Einstein then suggests to avoid from now on speaking of Mach's principle in regard to general relativity.

reduced to a perfect fluid without pressure, we saw that the tensor  $T_{\mu\nu}$  could be defined independently from the metric<sup>96</sup>. It is doubtless these simple cases that Einstein had in mind when he formulated Mach's principle by the request for a determining of the metric field by the tensor  $T_{\mu\nu}$ .

Nevertheless, in the most general cases (when the matter-energy comprises a pressure factor, or an electromagnetic field, etc.), the metric appears on both sides of the Einstein equations. It is actively involved in the general form given to the tensor  $T_{\mu\nu}$ . In that case, the metric seems both *determined* (by the integration of  $G_{\mu\nu}$ ) and *determining* (as an ingredient to give sense to  $T_{\mu\nu}$  or to  $\mathcal{T}_\nu^\mu = \sqrt{-\det(g)}T_\nu^\mu$ ). Therefore in the most general Cauchy problem, we cannot firstly calculate the tensor  $T_{\mu\nu}$  (outside the initial hypersurface), and then determine  $g_{\mu\nu}$ . Instead both fields are simultaneously co-calculated, except in a few cases which were studied later<sup>97</sup>.

So the initial data of the Cauchy problem for general relativity usually include information on the metric and its first derivatives. The  $T_{\mu\nu}$  alone is insufficient<sup>98</sup>. We can illustrate it with a simple case. In the numerical space of coordinates, we note  $S$  the set  $x_1^2 + x_2^2 + x_3^2 = 1$ . We then consider the following initial distribution of matter:

$$T^{\mu\nu}(x_1x_2x_3) = \rho\delta_0^\mu\delta_0^\nu \text{ if } (x_1, x_2, x_3) \in S, T^{\mu\nu}(x_1x_2x_3) = 0 \text{ otherwise,}$$

( $\delta$  is the Kronecker symbol,  $\rho$  a fixed positive constant). This represents a cosmos with a differentiated region, where lies a perfect fluid without pressure with a constant density (a cloud of dust), the rest being void. Even so, can we say that our tensor has univocally determined a distribution of matter? According to the metric that is correlated to that  $T^{\mu\nu}$ , the matter that is described may

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<sup>96</sup>See p. 26 above.

<sup>97</sup>[Sta69] showed that, when we limit ourselves to a field of matter with restricted properties, then we can find dynamic variables describing sources, independently from any metric data. In these particular cases, the tensor  $T_{\mu\nu}$  has the properties of a simple tensorial field that can be defined on the naked manifold. Stachel illustrates it with three cases:

- a scalar field without a mass,
- an incoherent matter ([dust]),
- an incoherent radiation.

We can then calculate the  $T_{\mu\nu}$  outside the initial Cauchy surface (by solving the conservation equations), before solving the Einstein equations to have  $g_{\mu\nu}$ . Imposing from the start the conservation equations, enables then to obtain the conditions to the integration of the Einstein equations, whatever the metric ultimately retained. In that sense, we can calculate the dynamic of the sources before knowing the space-time geometry. In these particular cases, the metric only appears on one side of the Einstein equations, contrary to the general case. However, this does not invalidate the fact that a given field  $T_{\mu\nu}$ , even of one of these very simple types, shall take totally different physical significations according to the specific metric to which it is correlated. Moreover, Stachel shows that this early calculation of  $T_{\mu\nu}$  on the whole manifold does not generate any extra restriction on  $g_{\mu\nu}$ , which still fully depends on the initial conditions that can be freely chosen.

<sup>98</sup>Afriat and Caccese in [Afr, p. 16-17] argue that we can sometimes attain a notion of matter without using any metric. After having considered various types of metric tensor, they conclude:

Generally, then, the reliance of matter on the metric seems to depend on the kind of matter; in particular on how rich, structured and complicated it is. The simplest matter –absent matter– can do without the metric; the more frills it acquires, the more it will need the metric.

Of course, these affirmations do not raise any problem if we replace everywhere the word “matter” by “tensor  $T_{\mu\nu}$ ”. However what is precisely debatable is the possibility that the tensor  $T_{\mu\nu}$  alone, before being coupled with a metric, represents a well determined state of matter. So, for example, even if the tensor  $T_{\mu\nu} = \rho.u_\mu u_\nu$  (“dust”) does not depend on  $g_{\mu\nu}$ , it will represent a very different state of the matter, depending on the metric to which it is correlated on a considered hypersurface.

be a small sphere which can attain a stable state, or, if it is given a large enough radius, a black hole<sup>99</sup>, or again a body without a spherical symmetry, if the metric adopted does not itself have this symmetry.

The circularity exposed does not however immediately invalidates Einstein’s idea. Here we only reproach him with the unfortunate slogan “the  $T_{\mu\nu}$  determines the  $g_{\mu\nu}$ ” which can work to describe field equations only in the elementary cases where the  $T_{\mu\nu}$  does not explicitly contain the metric. Even here, we do not know all the properties of matter from the beginning, but only some global properties, independent of the precise determination of the metric, such as the *scalar fields*  $\rho$  and  $p$ <sup>100</sup>. So, from the perspective of differentiating between what is a starting data, and what is deduced, the usual separation of the Einstein equations in two terms,  $G_{\mu\nu}$  and  $T_{\mu\nu}$  is misleading. Both terms of the Einstein equations need dismantling. The starting data -i.e. before solving the field equations- are actually limited to some general hypotheses on the nature of the matter involved, and to the data of the scalar fields ( $\rho, p \dots$ ) which describe some of its properties, independently of the precise metrical distribution. The  $g_{\mu\nu}$  (and its derivatives), which is involved in both the term  $G^{\mu\nu}$  and the term  $T^{\mu\nu}$ , is then only obtained by integrating the equations. This is confirmed by the study of the texts of Einstein’s contemporaries<sup>101</sup>. Finally, in general, the only necessary metric data, to solve the Einstein equations, are the *initial* or *boundary* conditions. Einstein is therefore right to be concerned with it.

### 4.6.3 Does Mach’s principle require to eliminate boundary conditions?

To solve the Einstein equations as a Cauchy problem, we cannot do in general without the metric data on the Cauchy hypersurface representing the initial time. Let us call it “the initial metric”. It correlates to the tensor  $T_{\mu\nu}$  in order to give it a full sense. Is it sufficient to invalidate the fact that these equations can realise Mach’s principle? Several specialists of Mach’s principle of the second

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<sup>99</sup>Let us remember that in fact the Schwarzschild radius of a massive body is proportional to its mass, while the geometrical radius of the object grows much more slowly based on the mass (if geometry was Euclidean, this radius would of course grow as  $\sqrt[3]{m}$ ). Therefore, the initial density being fixed, a ball of matter will become a black hole as soon as its radius is large enough. The hypothesis of a constant density then loses its coherence.

<sup>100</sup>This is in particular the case when we solve the Einstein equations in the peculiar case of a stationary solution, with specific symmetries, as in the calculus made by Schwarzschild for his “interior metric”. In this kind of simple situation, the application of “Mach’s principle” takes a form that is reminiscent of Weyl’s formulation of the  $\mathcal{RM}\mathcal{E}$  principle in *Raum-Zeit-Materie*, that is:  $g = F(\rho \dots)$ .

<sup>101</sup>Of course, for any relativistic calculation of the metric on a void region, the problem does not arise since the energy-momentum tensor is simply null, therefore, independent *per se* from any metric. This particularly includes the approximate derivation, by Einstein[Ein15], of the metric which surrounds a point mass, the exact solution suggested by Schwarzschild [Sch16a], or again the metric interior to Thirring’s hollow sphere ([Thi18; Thi21]) or the metric near a Lense-Thirring rotating massive body ([LT18]). However in the case of the calculation by Schwarzschild of the metric *interior* to a spheric mass of perfect fluid, incompressible and at rest [Sch16b], the problem arises since  $T_{\mu\nu}$  should appear under the general form:

$$T^{\mu\nu} = (\rho + p) u^\alpha u^\beta - p g^{\alpha\beta}$$

which explicitly depends on the metric. In Schwarzschild’s text, it is however clear that the  $T^{\mu\nu}$  is not a starting data of the problem. In a significant manner, Schwarzschild starts with the *mixed* tensor  $T_1^1 = T_2^2 = T_3^3 = -p$  and  $T_4^4 = \rho_0$  ( $\rho_0$  is a constant, since the fluid is incompressible,  $p$  depends on the radial coordinate as per a function which will be determined by the stability hypothesis). The presence of symmetry hypotheses indeed enables Schwarzschild to specify the *general* form of the metric, before calculating. But it is clear that the metric (therefore the  $T_{\mu\nu}$ ) is only perfectly determined after the field equations have been solved.

half of the twentieth century, of whom J.W. Wheeler<sup>102</sup> is an illustrious representative, argued that the existence and unicity of a solution to the Einstein equations, for given Cauchy data, is enough to express Mach's principle, in spite of the inevitably metric status of these initial data. It is perhaps possible to support the fact that the initial metric does not represent an autonomous entity, in the manner of Newton's absolute space, but would be reducible to the set of all the metric relations *in concreto* between the elements which constitute the field of matter.

In any case, this position was not Einstein's. He thought that no one could legitimately pretend that Mach's principle is actually realised in relativistic cosmology, as long as we need to use initial conditions or boundary conditions on the metric, to solve the equations. Einstein's programme, in the years 1917-1918, consists in looking for all possible subterfuges to avoid boundary conditions. Weyl shows in the last part of *Massenträgheit und Kosmos* that this programme is in principle unrealisable. All cosmological subterfuges imagined by Einstein and others<sup>103</sup> to get rid of the boundary conditions are doomed to failure. These conditions, whether they concern the spatial infinite, or the past (infinitely distant or not) are inevitable<sup>104</sup>.

Weyl ends his argumentation by specifying that a choice has to be made between all the possible universes without matter, which are the solutions to the Einstein equations with  $T_{\mu\nu} = 0$ . This choice is a real physical hypothesis, since this solution will represent the ether in its "normal state", when it is not disturbed by any matter. Concerning this choice, Weyl prefers de Sitter's hypothesis<sup>105</sup> for:

- The spectral galaxy redshift suggests an expansion of the universe<sup>106</sup>.
- De Sitter's universe has good properties (infinitely distant past and future are topologically disjoint) in order to prevent any time loop<sup>107</sup>.

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<sup>102</sup>See in particular [BP95, p. 188-sq.].

<sup>103</sup>About Mach's principle construed as a selection principle of some cosmological models which force boundary conditions, see [Rei73, p. 531-534] and [BP95, p. 39;77;79-83;95;97;148;190-195;228;238-239;443]. About the idea of realising Mach's ideas within the limits of initial relational data between elements of matter, in a context wider than general relativity, see [BP95, pp. 107;111-112;204-207;218-222;443-444].

<sup>104</sup>In [Wey24, p. 201], Weyl remarks that the hypothesis of a static universe, like in [Ein17], is equivalent to determining the state of the metric in the past:

The difficulty that arises from the spatial horizon is evidently resolved by [the choice of a] closed space ; but it remains nevertheless, since it is located everywhere in the universe continuum which can deform [...] in the same manner as a mollusc. The restriction to static conditions is indeed opaque and debatable.

Weyl then develop an analogy with electromagnetism, and asks how Coulomb's equations, in the static case, derive from Maxwell's equations. Then he concludes:

The formation of this field  $F$  inevitably results from the *variable* electromagnetic field laws, if we add the hypothesis that *space was deprived of a field at the beginning of the sequence*. If so, it is not because the field is fixed on the infinitely distant spatial horizon, but, instead the link comes from the world boundary of the past which goes back to an infinitely distant [time].

This argument is also developed in [Wey49, §23 C]. Besides the hypothesis of a static nature, on which Weyl insists, it seems to me that the homogeneity and symmetry hypotheses on which Einstein and cosmologists usually rely in their derivations, have also a metric significance.

<sup>105</sup>For an extensive development of the subject: [Ker89], [BM99].

<sup>106</sup>[Wey24, p. 202, right hand column].

<sup>107</sup>[Wey24, p. 202, left hand column].

- This universe solves the dark night sky paradox and avoids repeating the images of galaxies, eon after eon, as we are compelled to do in Einstein’s model<sup>108</sup>.

## 4.7 Weyl’s Boats-Lake Analogy

Let us conclude our discussion of *Massenträgheit und Kosmos* with a precise study of the passage in which Weyl develops the following analogy: matter is to the ether what boats are to the surface of a lake. This analogy is important for us inasmuch as it shows an evolution of Weyl’s position on some issues of the *pba* of *Raum-Zeit-Materie*:

Your objection being based on the principle of continuity I can doubtless weaken it in the best manner, intuitively, with an analogy in which I compare the ether with the surface of a lake, and matter with boats that plough it. The different possibilities that you have mentioned lie here in the fact that what can be materially realised in an infinity of different manners is the same form of the surface of the lake, the same qualitative state; the “material state” is in fact considered as determined only when it has been established in which point of the lake basin each particle of water is. Here, the arbitrary marking (for instance, by numbering) which helps differentiating the identical individual particles of water corresponds to the setting of a system of coordinates in the Ether, [that is], to the relations to a medium. If the water is at rest in the evening, when all the boats are in port, then the qualitative state is exactly the same as in the morning before the boats plough it: the surface of the lake is a “homogeneous” smooth plane. But the material state hidden behind it may have completely shifted. It is impossible (as it happened for the guide-field before Einstein) to recompose the actual position of all the particles of the water in the lake that were stirred by the boats, starting from a rest position fixed once and for all and from an elongation caused by the boats<sup>109</sup>.

This analogy is developed by Weyl/Paul in reply to an objection raised by Peter<sup>110</sup>. The latter wonders how it is possible to attribute a metric –the Minkowski one for instance<sup>111</sup>– to a portion of the universe void of matter. He then develops an argument against that type of possibility, which is a reminiscence of Einstein’s hole argument<sup>112</sup>.

There is only one way to describe the void (i.e. by the vanishing of the energy-momentum tensor), while there is an infinity of ways to position a given metric structure on a manifold by means of diffeomorphic pulling forward<sup>113</sup>. Let us suppose that the metric structure tends towards the structure we have chosen (for us: the Minkowski one), where matter tends to vanish. We are

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<sup>108</sup>[Wey24, p. 202, between both columns].

<sup>109</sup>[Wey24, p. 203].

<sup>110</sup>[Wey24, p.202, right hand column].

<sup>111</sup>In *Massenträgheit und Kosmos*, it is de Sitter’s. In *Philosophy of Mathematics and Natural Science*, Weyl will return to the same argument using Minkowski metric as the rest metric of the ether. This changes nothing to the argument that follows.

<sup>112</sup>Weyl explicitly refers to Einstein’s article from 1914.

<sup>113</sup>[Wey24, p. 202, right hand column]: “though [de Sitter’s metric] is, *per se*, qualitatively totally determined, there are however an infinity of possible ways for this state to be realised in the continuum of the world”.

embarrassed for we do not know which concrete realisation of this structure we must choose, within the infinite class obtained by the pulling forward on the manifold. So that it seems absurd to Peter that a determined metric may be associated to a void region of the universe.

Weyl/Paul answers by denying that the different manners to position the metric on the void region respond to situations that are *physically different*:

A difference [between two ways to position de Sitter's metric on a void region of the spatiotemporal manifold] would exist only if the four-dimensional world were a subsisting environment, in which, in some manner, traces of the material processes were discernible. And it is only then that one could acknowledge as distinct the possibilities of realisation that you have mentioned. But this subsisting environment will be completely rejected by the theory of relativity, probably with your applause<sup>114</sup>.

In this text, Weyl does not just assert that the naked spatiotemporal manifold, with neither matter nor metric, is deprived of a form. He goes as far as to say that it has no physical existence. The different realisations (obtained by diffeomorphic pulling forward) of a single metric does not represent different *physical situations*, but rather different *representations* of a single physical state. This position is quite different from the one he had in §12 of *Raum-Zeit-Materie* in which Weyl seemed to be desirous, at all cost, to give a *physical* significance to the system of coordinates. In reality, it is not a total reversal on Weyl's part. For, in the next lines of the text, we are told that the system of coordinates (or the manifold which it enables to describe) may acquire a physical significance if we connect it to the trajectories of some determined elements of matter.

For a better perception, let us develop the analogy to its end. Let us suppose that our starting point is a region of the universe,  $\mathcal{B}_0$ , void of matter, where the metric takes a form:

$$g_{\mu\nu}(t, x, y, z) = \eta_{\mu\nu} = \text{diag}(+1, -1, -1, -1)$$

which we consider as characteristic of the naked ether. This state is compared with the plane surface of a lake where water is at rest. The specific form  $g_{\mu\nu} = \eta_{\mu\nu}$  is dependent on the fact that we have chosen an adapted natural system of coordinates  $(t, x, y, z)$  (which Weyl calls, in his analogy, a specific "numbering of the molecules of water" which compose the surface of the lake). This system of coordinates is naturally related to a congruence of geodesic timelike world-lines, which we can imagine as being traversed by small free-falling label-particles of negligible mass. This congruence is given by the lines:

$$\mathcal{L}_{x,y,z} \{(t, x, y, z) \mid t \in \mathbb{R}\}$$

The coordinate  $t$  which parameterises each of these geodesic lines corresponds to the proper time measured along it. The simultaneity hypersurfaces are orthogonal to these geodesic lines, and are provided with an Euclidean distance. The spatial distance between two label-particles do not evolve with  $t$ . In other words, the value of the distance between  $(t, x, y, z, t)$  and  $(t, x', y', z', t)$  is:

$$-(x - x')^2 - (y - y')^2 - (z - z')^2,$$

independently from  $t$ . We can then graphically represent the congruence of the  $\mathcal{L}_{x,y,z}$  by vertical lines, and the simultaneity surfaces by horizontal lines (we delete two spatial dimensions in order

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<sup>114</sup>[Wey24, p. 203, left hand column].

to give a simple representation). It is the usual foliation of Minkowski space defined by an inertial reference frame. Here, the verticality of the lines has a stronger sense than in the fibred plasticine argument, for it designates a true *metric* invariance. Let us suppose that these metric properties are valid for a first interval of time when our geodesic lines remain in the region  $\mathcal{R}_0$ .

Let us now suppose that the boats come to disturb the surface of the water. The lines  $\mathcal{L}_{x,y,z}$  are extended (as geodesic lines) in a new region  $\mathcal{R}_1$  of space-time where a mass  $M$  curves the metric. Minkowski metric is no longer valid: we no longer have  $g_{\mu\nu}(t, x, y, z) = \eta_{\mu\nu}$  either in the system of coordinates defined by the congruence of the  $\mathcal{L}_{x,y,z}$ , or in any other one.

What happens now when the boats return to port, i.e. if the  $\mathcal{L}_{x,y,z}$  are again extended into a third region  $\mathcal{R}_3$  which is void like the first one? The surface of the water gradually becomes plane again. That is: the metric will converge again towards a Minkowski metric. However, Weyl insists, the position of each particle of water will not necessarily be the same, within this new plane surface of the water, as before the passing of the boats.

To what does Weyl analogically refer? The Minkowski metric, which will be restored in the new environment devoid of mass, will generally be “orientated”<sup>115</sup> differently. That is: it is not necessarily expressed by  $g_{\mu\nu}(t, x, y, z) = \eta_{\mu\nu}$  but more generally by  $g_{\mu\nu}(t, x, y, z) = \phi^* \eta_{\mu\nu}$  ( $\phi^*$ : diffeomorphic pulling forward) for a specific  $\phi$ ). Actually, the congruence of the geodesic lines  $\mathcal{L}_{xyz}$  which was defined in the region  $\mathcal{R}_1$ , then extended to  $\mathcal{R}_3$ , will no longer be adapted to the recovered Minkowskian structure. While these particles were at rest in relation to one another at the beginning of the process, these particles are no longer at (metric) rest, relative to one another after passing near the mass  $M$ . A new system of coordinates  $(t', x', y', z')$  should then be established, by choosing a new family of label-particles, so that the congruence  $\mathcal{L}_{x',y',z'}$  of their geodesic lines enables us to re-establish  $g_{\mu\nu}(t', x', y', z') = \eta_{\mu\nu}$  in  $\mathcal{R}_3$ .

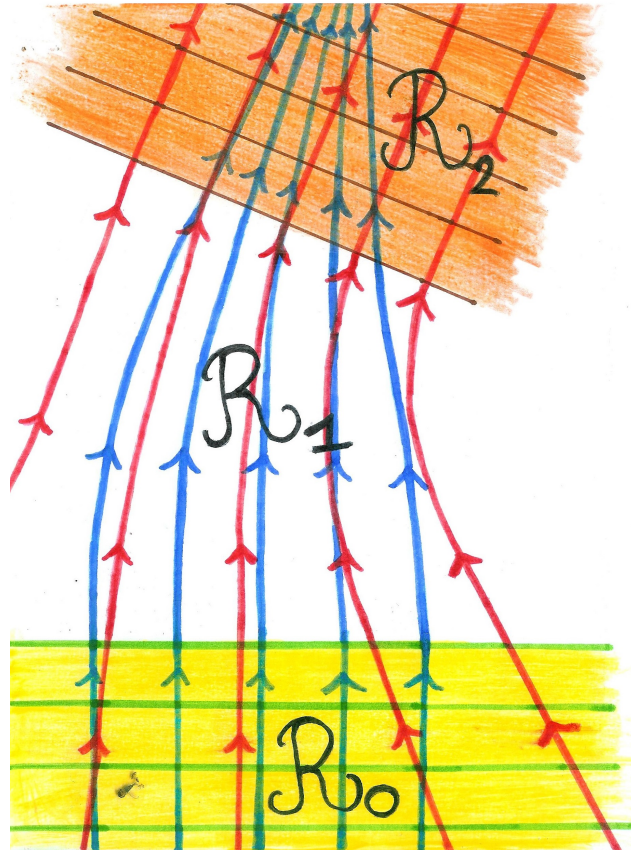
With this analogy, Weyl suggests a fine answer to the question: has the spatiotemporal manifold a physical significance? The changes of coordinates may be defined in a purely mathematical manner. Thus two fields  $g_{\mu\nu}$  and  $\phi^* g_{\mu\nu}$  can be seen as different representations of a single metric. But as soon as a system of coordinates is associated to a material reality -as the congruence of the geodesic lines of material label-points- then a diffeomorphism becomes physically significant. So, the change of orientation to which Minkowski metric was subjected in our example has a precise physical significance. To materialise the (flat) recovered Minkowskian structure, once we have penetrated into the region  $\mathcal{R}_3$ , the family of label-particles used to define our system of coordinates must be changed. Finally we can see that if we consider two flat regions topologically disjoint of space-time, it is meaningless to question whether the Minkowski metrics of the two regions appear with the same

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<sup>115</sup>Here we use Weyl’s terminology. He very often uses the word “orientation” to refer to the different manners in which a same geometrical object may be expressed in coordinates, that is the different manners in which it can unfold on the manifold. See for instance [Wey15, p. 44-45] or [Wey21, p. 126].

The orientation of a geometrical object, in that sense, may have a purely subjective status, resulting from an arbitrary choice, as when we consider the orientation of the Riemannian metric *at a singular point of the manifold*, or, on the contrary, have an objective invariant sense, as when we consider the variation of the orientation of the metric *throughout an open domain*.

In the same manner, here Weyl says that the absolute orientation of Minkowski metric in *one* region considered in isolation has no physical significance. But the relative change of the orientation of the metric in passing from a region to another one makes sense. Weyl generally illustrates this type of behaviour by referring to the discussions on the differentiation between right hand and left hand in Kant’s and Leibniz’s works.



**Weyl's analogy of the lake and boats.** In a region  $\mathcal{R}_0$  (here in yellow), void of matter, a congruence of geodesic lines  $\mathcal{L}_{x,y,z}$  have been selected (represented in blue) which materialise the Minkowskian structure of space-time in that region. That is: these geodesic lines, to which negligible label-particles are associated, remain at a constant spatial distance in the course of time. The simultaneity hypersurfaces (represented here by green horizontal lines) are provided with a constant Euclidean structure in the course of time.

Then, these geodesic lines enter the region  $\mathcal{R}_1$  (not coloured) where a mass responsible for a non-null curvature lies (we did not represent the world-lines of the elements of matter). The geodesic lines  $\mathcal{L}_{x,y,z}$  start to diverge from one another.

When these geodesic lines reach the region  $\mathcal{R}_2$  (in orange) void of matter again, they are no longer adapted to reveal the recovered Minkowskian structure. According to Weyl's analogy: "the position of the particles of water was disturbed by the passing of the boats". To reveal the recovered Minkowskian structure, we must change geodesic congruences, and take the  $\mathcal{L}_{x',y',z'}$  (represented in red).

So, even if the *absolute orientation* of the Minkowskian structure of a void region of the universe has no sense, by considering the intermediate region, we could give a sense to the *relative change of orientation* of the metric of a region in relation to the metric of another region.

orientation. This is a merely subjective matter which is based on a choice of coordinates. But as soon as the two regions are connected by an intermediate region, the geodesic lines can be extended from one of the regions towards the other one, and the orientation acquires a physical significance.

This enlightens more effectively the position, surprising at first sight, that Weyl had adopted in *Raum-Zeit-Materie* about the physical significance of a change of coordinates.

## 5 The *pba* in later texts

To our knowledge, Weyl revisited the *pba*, after 1924, in two texts only: his vast philosophical monograph *Philosophy of Mathematics and Natural Science* and in *Mind and Nature*.

### 5.1 In *Philosophy of Mathematics and Natural Science*

*Philosophy of Mathematics and Natural Science* was first published in German in 1927, then, in English in 1949. As early as 1927 we find paragraph 16, called “The Structure of Space and Time in their Physical Effectiveness”. It goes over the *fpba* and the analogy of the lake and boats again. It is the first paragraph of the chapter “Space and Time, the Transcendental External World”. The objective of the whole paragraph is to question the content and the origin of the space-time structure, namely its metric structure. Part of the content of the paragraph already appeared in Weyl’s previous literature, but its arrangement is new, and, as in the other chapters of *Philosophy of Mathematics and Natural Science*, we can find definite references to Leibniz.

The purpose of the passage in which we are particularly interested<sup>116</sup> is to solve the problem of the non-equivalence between the kinematic perspective and the dynamic perspective on the analysis of motion. From the kinematic perspective, any reference frame should be equivalent to describe motion. But from the dynamic point of view, it seems that there are privileged reference frames, in which solely physics laws can be expressed with simplicity. It was Newton’s point of view, which was not outdated by special relativity. Is it possible to go beyond this apparent limitation, on a dynamical level, and to enunciate the laws of physics independently from referring to a privileged point of view?

Mach’s ideas are described here as an attempt to reach the largest generality from the point of view of the dynamical reference frames, without having to assume the existence of anything excepted matter. Weyl names Huygens as a predecessor in this regard, and of course Einstein as a successor on this path, at least for some time. It seems however that the issue of general relativity is partly back projected on Huygens and Mach by Weyl. Weyl then refutes as absurd Huygens’s and Mach’s path. This refutation closely follows the one in *Massenträgheit und Kosmos*. Weyl repeats the *fpba* to justify the fact that the naked spatiotemporal manifold, without metric, cannot be used to support a difference between the motion and the rest of two separate elements of matter. Ultimately, it is on the *inertial structure* (itself included in the metric structure) that the existence of a (local) dynamically privileged reference frame is based. According to Weyl, a reasonable solution to our

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<sup>116</sup>[Wey49, p. 104-107].

problem could not be found as long as we had not understood that the inertial structure itself was a physical field, “a real thing which not only exerts effects on matter, but undergoes effects from it”. Here, as in the very first version of the *pba*, Weyl suggests that Riemann is a precursor of that idea, which Einstein would have only developed –though in a decisive manner– nearly 70 years later, by introducing the principle of equivalence between inertia and gravitation.

Weyl does not dwell on the autonomous physical reality granted to the “ether” as much as in the previous text. This derives however from the *fpba*: this metric field, the existence of which was intuited by Riemann, and which Einstein’s theory rightly identified with the inertial gravitational field, could in no way be a pure emergence of the field of matter, as it is defined on the naked spatiotemporal manifold. It must be granted a particularly autonomous existence.

In the 1949 version of our paragraph, Weyl then takes up the analogy of the lake and boats. The content of the analogy is not different from the version of *Massenträgheit und Kosmos*. However, the function given to the argument is much more important. In the 1924 version, it seemed that this analogy only had a technical function, insuring that the ether had the potential to recover its “rest shape”<sup>117</sup>, away from any matter, without falling into an aporia due to the indetermination of the “orientation” of the metric of the ether. In *Philosophy of Mathematics and Natural Science*, Weyl sees in the analogy of the lake and boats an illustration of the essential difference between the Newtonian and the relativistic points of view, concerning the relations between gravity and inertia. In the Newtonian physics, there was a rigid structure, fixed once for all, inertia. When bodies gravitationally interacted, they left the tracks of the inertial structure, and returned to them as soon as they were far enough so that gravitation could be ignored again. In contrast, in general relativity there is no structure of ether at rest, fixed once for all from the start. Instead, one knows how to *qualitatively* characterise which metric structure the ether must adopt in the absence of any matter. But the new “orientation” taken by this structure will depend on the dynamical history followed by the ether in its relation to matter, in the intermediate region connecting the two regions void of any gravitating matter.

## 5.2 In *Mind and Nature*

The *pba* appears again in *Mind and Nature* in 1934<sup>118</sup>. The context is different again. The general purpose of this article is to show that the subjects, by their body actions and passions, and by their conscious minds, are inevitable constitutive elements of physical science. This is how he concludes the end of chapter IV:

I dare hope that we will have made the following point intelligible: how and up to what point the structure of our scientific knowledge is conditioned by the circumstance that the world, which is the purpose of all our scientific research, is not something that exists *per se*, but only exists and occurs from the encounter between the subject and the object.

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<sup>117</sup>In *Philosophy of Mathematics and Natural Science*, contrary to *Massenträgheit und Kosmos*, Weyl uses Minkowski’s metric (and not de Sitter’s) for the ether, when he develops the analogy. The cosmological preference for de Sitter’s metric will nevertheless be reasserted (and justified in the same manner) by Weyl a few pages further.

<sup>118</sup>It is therefore posterior to the German edition of *Philosophy of Mathematics and Natural Science*, but not to the English edition.

To reach that conclusion, Weyl, in the different sections of the article, gradually moves up in the hierarchy of the knowledge related to the world. He moves from the perception data to the primitive physical concepts (Locke, Descartes) in which the sole sensory qualities are questioned with regard to their objectivity. Finally, he comes to the questioning of the objectivity of space and time. Now the subject has no direct relation with the physical properties of the object. Instead he is necessarily led to reach objectivity through symbolic representation.

In chapter IV, Weyl decides to illustrate that with the particular case of the relativity of space and time. The problem which he reaches, after a detour through the correspondence between Leibniz and Clarke, is the same one as in *Philosophy of Mathematics and Natural Science*, namely the “antinomy between kinematics and dynamics”<sup>119</sup>.

To explain how general relativity solved this problem, every system of coordinates must firstly be deprived of any objectivity:

We are led to see the concept of coordinates in an essentially more fundamental manner. Coordinates are no longer measured, but are nothing more than arbitrary numberings of the universe [i.e. of space-time]; they are only symbols used to label and differentiate the universe points from one another<sup>120</sup>.

Thus Weyl enunciates more neatly than in the previous texts the absence of physical objectivity of the naked spatiotemporal manifold (without the metric or any other structure), or systems of coordinates that represent it. But this assertion will be qualified further in the text in a passage in which the spatiotemporal manifold is illustrated by a *pba*:

The sole relations [that can be expressed on the naked manifold] which have an objective signification are the ones that are preserved by any deformation of the plasticine. The intersection of two world-lines is, for instance, of that kind.

We see that Weyl uses the plasticine to encode the naked manifold, as in the two previous occurrences (*fpba* versions), even though he is less explicit here concerning the fibration as such. The pure subjectivity of coordinates, mentioned above, is now qualified. The topological invariants keep a form of objectivity.

Thus, assuming that space-time is numerically locally represented by an open set of  $\mathbb{R}^4$ , we can give as examples of non-objective properties of the naked manifold:

- *The individual identity of a point.*
- *The fact for a world-line of being straight or curved, vertical or not*
- Etc.

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<sup>119</sup>[Wey34, p. 125]. See above p. 38.

<sup>120</sup>[Wey34, p. 128].

However, two lines that intersect in a system of coordinates, will continue to do so in any other system. These are the only kinds of objective data actually encoded by the manifold. This argument was also put forward by Einstein as early as 1916<sup>121</sup>.

In a second stage, Weyl introduces the metric structure on the manifold. It is a field which has a physical significance, which is broken down in two component fields. On the one hand, we have the *inertial structure*, which determines the trajectories of the bodies not subjected to any influence other than gravity. On the other hand, we have the *causal structure*, which corresponds to the data of the light cones (one for every point) and which determines which events of the universe can be causally linked to which ones, and in which order. This breaking down of the metric did not appear in the texts in which the *pba* appeared previously. They however were long established by Weyl<sup>122</sup>. Weyl then shows that the inertial and causal structures respectively replace Newton's absolute space and time, to account for the gap between the total kinematic homogeneity of space-time, and its dynamical non-homogeneity. Physical realities were indeed responsible for this gap. However, they were not immutable and fixed entities, but dynamical fields interacting with matter. These developments clarify the original idea, present in the first edition of *Raum-Zeit-Materie*, stating that the variable coefficients of the metric are not properties inherent to space, but the correlate of a physical reality which determines the behaviour of rulers and clocks<sup>123</sup>.

The elements that are really new in this text<sup>124</sup> are present in the conclusion of Chapter IV. The subject, because of its singular place in the world, and because of its consciousness appears as a necessary mediation in order to root the knowledge of the physical world in something absolutely given. The singularity of the subject is expressed by the contingent form given to the "plasticine" (the system of coordinates). This singularity is then neutralised, to reach objectivity, through the principle of relativity. This neutralisation is somehow an impoverishment. The same subject, in the same conditions, and facing the same objective situation, will be led to feel the same conscious experiment of moving forms expanding in time and space. But this space and time experience goes much further than the sole objective spatiotemporal structure, which is indeed only a poor formal skeleton.

Finally, the objective inertial and causal structures are only measured by means of sending test bodies and light rays in free fall. But, even though minimally, this necessarily disturbs the structure to be measured. Therefore, the system of coordinates (the "plasticine" pattern) is not the only way the subject takes part in the determining of spatiotemporal forms. It is also an entity which can only know the metric structure by operating on it, and therefore by disturbing it. This consideration prepares, in Weyl's text, the evaluation of the position of the observing subject in quantum mechanics.

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<sup>121</sup>This is what the Einsteinian literature has called the "point-coincidence argument" since Stachel's suggestion, see [Nor99].

<sup>122</sup>See [Wey15, p.17-19] in which the inertial and causal structures respectively correspond to the projective and conformal structures.

<sup>123</sup>[Wey10, p. 125]. See above p. 13

<sup>124</sup>They are in great number *in comparison with the text discussed previously*.

## 6 Summary and Conclusion

We have shown that the *pba*, by its richness, its recurrence and its polymorphic character in Weyl's work, is a very valuable material. The text corpus which includes the different versions of the argument shows the long evolution of a complex and audacious thought on space, with several reversals. They are a consequence of Weyl's confrontation with physical reality, with his contemporaries' thought and with the difficulties inherent to his own philosophical standpoints.

The original problem which led to the emergence of the argument consists in justifying the adoption of a metric with variable coefficients. This problem, which has been at the core of geometry since the middle of the nineteenth century, and which Einstein's theory has made even more pressing, is addressed by Weyl under the very specific angle of his idealism: how could we adopt a non-homogeneous metric, while space, as a form of our intuition, is necessarily homogeneous?

According to Weyl, in *Raum-Zeit-Materie*, adopting a dynamical metric solves this tension. In fact, if we pose a principle of determination of the metric by matter, then a material body may be displaced in space while keeping its properties -*metric* properties included-. This comes to no longer considering the metric as a property inherent to space, but like a property emerging from its content.

This argument, however, leads Weyl into a series of technical and philosophical difficulties.

Firstly his solution, by excluding the metric of the essential properties of space, is too radical. The infinitesimal metric structure does not emerge contingently from matter, but is part of the essence of space, and should be justified *a priori*. On that issue, Weyl will rectify his standpoint when developing his infinitesimal geometry, the "problem of space" –in its technical sense– and the epistemological discourse which goes with it.

Secondly, the principle of determination of the metric by matter is difficult to formulate coherently. At each step, we may fall into an aporia. For if matter is characterised by a simple scalar field, and if it totally determines the metric, we may come to a theory in which every deformation, and thereby any change, have become impossible. It is the argument which is illustrated by the thought experiment of the plasticine ball, and that I have interpreted as an Eleatic aporia, negating the possibility of change. The modifications made in *Raum-Zeit-Materie*, one edition after the other, shows the difficulties met by Weyl in order to avoid this aporia. Is it possible to accept a theory in which any displacement generates a modification of the field of matter? Is it acceptable to hide behind a cosmological argument, the plasticine ball only recovering its shape after a cosmological rearrangement? How is it possible to give sense to the idea of a *density* of matter, and to the idea of a change of the field  $\rho$  of matter, while supposedly standing at a pre-metric level?

In *Massenträgheit und Kosmos*, Weyl's stance towards Mach's principle changes radically. This principle, which was accepted in *Raum-Zeit-Materie*, under Riemann's patronage, as an assumption used to solve a philosophical problem, becomes problematic itself and the subject of a critical investigation. It is as if the difficulties anticipated by Weyl, while developing his *pba*, had gradually gnawed at his belief in the pertinence of this principle. Perhaps Einstein's own disappointment towards this principle also played a part in Weyl's reversal.

In this new text, Weyl offers a truly spatiotemporal version of his argument, using a four-dimensional and fibred plasticine. If matter is characterised only by a simple congruence of trajectories, it is still possible for a simple diffeomorphism to rectify all the trajectories, that is put matter at rest. Then, the absurdity of claiming to define the motion of matter prior to any metric -in order to be able, then, to define the metric - becomes blatant. Therefore, Weyl provides another signification to Einstein's failure. If, in general relativity, Einstein could not realise Mach's principle, it is not a contingent fact, related to the lessons of experience. Rather, it was an inevitable fact since an *a priori* investigation on the signification of the most radical version of Mach's principle reveals its absurdity, or, at the very least, brings it back to an Eleatic negation of motion. This why a simple thought experiment, like the *pba*, is sufficient.

Weyl, in the second part of the dialogue, still engages in a criticism specifically targeting Einstein's attempt at realising Mach's principle in the context of relativistic cosmology. Mach's principle must be expressed through the Einstein equations which correlate Einstein metric tensor  $G^{\mu\nu}$  with the energy-momentum tensor  $T^{\mu\nu}$ . By slightly departing from Weyl's text we have been able to highlight that  $T^{\mu\nu}$  itself generally depends on the metric. As such, in opposition to Einstein's assertions, even if Mach's principle could be realised in general relativity, it could not be in the form "the  $T^{\mu\nu}$  is sufficient to determine the metric" (which entails a circularity). The only coherent starting point, for a total determination of the metric, should be the  $T_{\mu\nu}$  *modulo its correlation with a metric*. This retrospectively justifies the form taken by Mach's principle in Weyl's work, as early as the first edition of *Raum-Zeit-Materie*, as referring to a notion of matter determined by scalar fields.

However, once the problem is rightly posed, the Einstein equations being equations with second order partial derivatives, it cannot be solved without giving some initial (Cauchy or Dirichlet) conditions. This expresses a form of unsurpassable autonomy of the metric in relation to matter. Weyl shows that all cosmological subterfuge imagined by Einstein to break free from these conditions is illusory. Equilibrium or symmetry hypotheses always hide metric determinations. And how could it be otherwise since Mach's principle which Einstein is researching, when it is correctly formulated, falls within the range of Weyl's *a priori* demonstration: in no case could a metric univocally emerge from a purely topological notion of matter.

Weyl then draws the consequences for the ontology of physics. In order to re-establish the possibility of motion, hence the elaboration of physics, one must accept the existence, alongside matter, of a metric field, the *ether*, which is partly autonomous. This metric field does not have immutable properties, like Newton's absolute space, but is in dynamical interaction with matter. Due to its partial autonomy, when we move away from any matter, the ether will go back to its "rest state", which is specific to it. Several possibilities being open regarding the metric properties of this state, a choice must be made which shall be a true physical hypothesis of a cosmological nature, to be evaluated in connection with the observational data. Weyl, through his character Paul, then says that he is in favour of choosing the de Sitter metric (this is 1923), because it has good topological properties and seems compatible with the observations.

Weyl then returns to a question left pending since *Raum-Zeit-Materie*: can a spatiotemporal diffeomorphism which acts upon the metric have a *physical* meaning, or does it only express a simple mathematical license in expressing a single physical reality? Because of the modern point of view, in relation to covariance in general relativity and to the Einstein hole argument, we would tend to choose the second option. Weyl offers a more nuanced vision. His analogy of the surface

of the lake enables him to give a physical sense to a global diffeomorphism. Two regions of space-time, distant from all matter, will be isometric, but the metric can be expressed with a “different orientation”, following the disturbance generated by a mass filling the intermediate region.

In *Philosophy of Mathematics and Natural Science*, Weyl endorses the conclusions of *Massen-trägheit und Kosmos*; he reconstructs the history of the principle of determination of the metric by matter, making Huygens and Mach the initiators of the contradictory version (because it is too strong) of the principle, and Riemann and Einstein the moderators who gradually succeeded in expressing a coherent version of it. Finally in *Mind and Nature*, Weyl places the *pba* in the global context of the subject/object inter-relations in the construction of physical knowledge.

Ultimately, we can see how a single argument, the *pba*, was repeated in Weyl’s work with substantial modifications which not only reflect the evolution of the technical apparatus and the precision in the expression of Mach’s principle, but also the evolution of the philosophical problems which guided Weyl’s thought, and led him to re-use the same argument to very different purposes. The study of the corpus, in its evolving continuity, shows that Weyl’s reversals concerning his philosophical positions are not the consequences of an unstable nature or a deeply volatile temperament of the German mathematician. Instead, these changes result from the time needed by Weyl to clarify and solve the difficulties which appeared as early as the first edition of *Raum-Zeit-Materie*. Weyl’s philosophy is not a frozen system, constructed prior to science, but it is constructed with a time consuming reflection, on the more and more complex scientific theories of his time. Therefore, if Weyl’s philosophy may at times seem very unstable, in the light of the great lasting secular systems of the tradition, it is the result of an interaction with science which develops rapidly in that lively era of the beginning of the twentieth century.

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