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Tropical Effective Primary and Dual Nullstellensätze*

Dima Grigoriev · Vladimir V. Podolskii

Abstract Tropical algebra is an emerging field with a number of applications in various areas of mathematics. In many of these applications appeal to tropical polynomials allows to study properties of mathematical objects such as algebraic varieties and algebraic curves from the computational point of view. This makes it important to study both mathematical and computational aspects of tropical polynomials.

In this paper we prove a tropical Nullstellensatz and moreover we show an effective formulation of this theorem. Nullstellensatz is a natural step in building algebraic theory of tropical polynomials and its effective version is relevant for computational aspects of this field.

On our way we establish a simple formulation of min-plus and tropical linear dualities. We also observe a close connection between tropical and min-plus polynomial systems.

Keywords Nullstellensatz · Min-plus algebra · Tropical algebra

Mathematics Subject Classification (2000) 14T05 · 68W30 · 52C99

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1 Introduction

A *min-plus* or a *tropical semiring* is defined by a set \mathbb{K} , which can be \mathbb{R} , $\mathbb{R}_\infty = \mathbb{R} \cup \{+\infty\}$, \mathbb{Q} or $\mathbb{Q}_\infty = \mathbb{Q} \cup \{+\infty\}$ endowed with two operations, *tropical addition* \oplus and *tropical multiplication* \odot , defined in the following way:

$$x \oplus y = \min\{x, y\}, \quad x \odot y = x + y.$$

Tropical polynomials are a natural analog of classical polynomials. In classical terms a tropical polynomial is an expression of the form $f(\vec{x}) = \min_i M_i(\vec{x})$, where each $M_i(\vec{x})$ is a linear polynomial (a tropical monomial) in variables $\vec{x} = (x_1, \dots, x_n)$, and all the coefficients of all M_i 's are nonnegative integers except for a free coefficient which can be any element of \mathbb{K} (free coefficient corresponds to a coefficient of the tropical monomial and other coefficients correspond to the degrees of variables in the tropical monomial).

The degree of a tropical monomial M is the sum of its coefficients (except the free coefficient) and the degree of a tropical polynomial f denoted by $\deg(f)$ is the maximal degree of its monomials. A point $\vec{a} \in \mathbb{K}^n$ is a root of the polynomial f if the minimum $\min_i \{M_i(\vec{a})\}$ is either attained on at least two different monomials M_i , or is infinite. We defer more detailed definitions on the basics of min-plus algebra to Preliminaries.

Tropical polynomials have appeared in various areas of mathematics and found many applications (see, for example, [18, 25, 32, 26, 29, 17]; one of the earliest papers in tropical mathematics is [35]). An early forerunner source of the tropical approach was the Newton's method for solving algebraic equations in Newton-Puiseux series [32]. An important advantage of tropical algebra is that it makes some properties of classical mathematical objects computationally accessible [34, 18, 25, 32]: on one hand tropical analogs reflect certain properties of classical objects and on the other hand tropical objects have much more simple and discrete structure and thus are more accessible to algorithms. One of the main goals of min-plus mathematics is to build a theory of tropical polynomials which would help to work with them and would possibly lead to new results in related areas. Computational reasons, on the other hand, make it important to keep the theory maximally computationally efficient.

The case studied best so far is the one of tropical linear polynomials and systems of tropical linear polynomials. For them an analog of a large part of the classical theory of linear polynomials was established. This includes studies of tropical analogs of the rank of a matrix and the independence of vectors [8, 20, 1], an analog of the determinant of a matrix and its properties [29], an analog of Gauss triangular form [12]. Also the solvability problem for tropical linear systems was studied from the complexity point of view. Interestingly,

this problem turns out to be polynomially equivalent to the mean payoff games problem [13] which received considerable attention in computational complexity theory.

For tropical polynomials of arbitrary degree less is known. In [30] the radical of a tropical ideal was explicitly described. In [29,31] a tropical version of the Bezout theorem was proved for tropical polynomial systems for the case when the number of polynomials in the system is equal to the number of variables. In [7] the Bezout bound was extended to systems with an arbitrary number of polynomials. In [34] it was shown that the solvability problem for tropical polynomial systems is NP-complete.

Along with tropical polynomials there were also studied min-plus polynomials. A min-plus polynomial is a pair of tropical polynomials $(f(\vec{x}), g(\vec{x}))$. A point $\vec{a} \in \mathbb{K}^n$ is a root of the polynomial $(f(\vec{x}), g(\vec{x}))$ if $f(\vec{a}) = g(\vec{a})$. We call an equation $f(\vec{x}) = g(\vec{x})$ a *min-plus polynomial equation*.

Min-plus polynomials were studied mainly for their connections to dynamic programming (see [6,21]). As in the case of tropical polynomials here the best studied case is the one of linear min-plus polynomials [6]. Also in [13] the connection between min-plus and tropical linear polynomials was established.

As for the min-plus polynomials of arbitrary degree much less is known. We are only aware of the result on the computational complexity of the solvability problem of a system of min-plus polynomials: the paper [15] shows that this problem is NP-complete.

Our results. A next natural step in developing of the theory of tropical polynomials would be an analog of the classical Hilbert's Nullstellensatz, which for the classical polynomials constitutes one of the cornerstones of algebraic geometry. Concerning the tropical Nullstellensatz, the problem was already addressed in the paper [11]. That paper came up with a general idea to approach this theorem in the tropical case through the dual formulation (a naive tropical analog of Nullstellensatz trivially fails, see below). Moreover, in [11] there was formulated a conjecture (which we restate below as Conjecture 1) capturing the formulation of the tropical dual Nullstellensatz and this conjecture was proven for the case of polynomials in 1 variable. Previously in [33] a tropical dual Nullstellensatz was established for a pair of polynomials in 1 variable. This result relied on the classical resultant and on the Kapranov's theorem [9,33].

More specifically, in [11] there was considered the Macaulay matrix of a system of tropical polynomials $F = \{f_1, \dots, f_k\}$. This matrix can be easily constructed from F : we just consider all the polynomials of the form $f_i + m_j$ (in the classical notation) of degree at most N , where N is a parameter and m_j is a tropical monomial. We put the coefficients of these polynomials in the rows of the matrix, while the columns of the matrix correspond to monomials. Empty entries of the matrix we fill with ∞ . The resulting matrix we denote by M_N . In [11] it was conjectured that the system of polynomials F has a root iff the tropical linear system with the matrix M_N has a solution, and moreover

N can be bounded by some function on n , k and the degrees of polynomials in F (this refers to effectiveness).

In this paper we prove this conjecture establishing an effective version of the tropical dual Nullstellensatz. Surprisingly, it turns out that the cases of tropical semiring with and without ∞ differ dramatically. More specifically, in the case of tropical semirings $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{Q}$ we show that F has a root iff the tropical linear system with the matrix M_N has a solution, where $N = (n+2)kd$, d is the maximal degree of polynomials in F , k is the number of polynomials in F and n is the number of variables. For the case of tropical semirings $\mathbb{K} = \mathbb{R}_\infty$ or $\mathbb{K} = \mathbb{Q}_\infty$ we show a similar result, but with $N = (C_1 d)^{\min(n,k)+C_2}$ for some constants C_1 and C_2 . Thus for the case without ∞ the bound on N is polynomial in n, k, d and for the case with ∞ the bound on N is still polynomial in d , but is exponential in n and k . We give examples showing that our bounds on N are qualitatively optimal, that is the difference of the values of N in these cases is not an artifact of the proof, but is unavoidable. However, quantitatively there is a gap between upper and lower bounds, see Section 3 for details.

We note that our result extend from \mathbb{R} and \mathbb{Q} to any linearly ordered vector space over rational numbers. On the other hand, our result does not extend to the case of \mathbb{Z} and \mathbb{Z}_∞ . In our proof it is important that the semiring is closed under division by integers. This property corresponds to the fact that each tropical polynomial with at least two monomials has a solution. Thus this property exactly corresponds to algebraic closure of the field in the classical Nullstellensatz.

The effective version of the tropical dual Nullstellensatz does not give direct general computational consequences since even for the case of tropical semirings $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{Q}$ the resulting Macaulay matrix is of exponential size in the size of the system of polynomials F . This is quite natural since the solvability problem for tropical polynomial systems is known to be NP-complete [34]. However, the upper bound on the degree is important for computational reasons in settings in which some specific small systems of polynomials are considered. Such settings analysing specific tropical objects of small size are quite common in this field (see, for example, [29, 28]).

Regarding the substantial gap between the required degree in the finite and infinite cases we observe that there is a somewhat similar situation for the classical Nullstellensatz. Indeed, we show that in case of the semiring \mathbb{R} the bound in a tropical effective Nullstellensatz is roughly equal to the sum of the degrees of the polynomials, while in case of the semiring \mathbb{R}_∞ the bound is roughly equal to the product of the degrees (Theorems 1 and 4). Recall that for systems of classical polynomials over an algebraically closed field the bound in the effective Nullstellensatz is roughly equal to the sum of the degrees of polynomials in the homogeneous (projective) case [23, 24] while the bound is roughly equal to the product of the degrees for arbitrary polynomials (affine case) [5, 10, 22].

As a consequence of the tropical dual Nullstellensatz we obtain its infinite version. Namely, a system of tropical polynomials has a root iff the infinite

tropical linear system with the infinite Macaulay matrix M (that is, with no bound on the degree) has a solution. Note that the latter system is well defined since each row of M contains just a finite number of finite entries. This infinite version was conjectured in [11], where it was also observed that a similar infinite dual version of the classical Nullstellensatz holds.

Next we show a primary version of the tropical Nullstellensatz. We view Nullstellensatz as a duality¹ result for systems of polynomials: if there is *no* root to the system of polynomials then some positive property holds (something *does* exist). In the classical case this positive property is the containment of 1 in the ideal generated by polynomials (over an algebraically closed field). A naive analog does not hold for the tropical case. Indeed, for example, the system of tropical polynomials $\{\min(x, 0), \min(x, 1)\}$ has no roots but the tropical ideal generated by this system does not contain a constant polynomial and more generally, any polynomial in this tropical ideal has a root. Basically, the point is that in the tropical semiring there is no subtraction, so in any algebraic combination of tropical polynomials no monomials cancel out. To overcome this difficulty we introduce the notion of a *nonsingular* tropical algebraic combination of tropical polynomials (see the definition in Preliminaries; here we only note that the nonsingularity property is simple and straightforward to check). For the tropical primary Nullstellensatz we show that there is no root to the tropical polynomial system F iff there is a nonsingular tropical algebraic combination of polynomials in F of degree at most N . We show this result for both cases of tropical semiring with and without ∞ and the value of N in both cases corresponds the value of N in the tropical dual Nullstellensatz.

To establish the primary Nullstellensatz we need a duality for tropical linear systems. We show this duality result as a sidestep. However, we note that the duality for tropical linear systems is heavily based on already known results [2] and should be considered more as an observation.

We also prove analogs of all mentioned results for the case of min-plus polynomials. As another sidestep of our analysis we study the connection between tropical and min-plus systems of polynomials. We argue that these two settings are very closely connected and that this connection can be used to establish new results in tropical algebra. The observation is that some results (like linear duality) are easier to obtain for min-plus polynomials and then translate to tropical polynomials, and some other results (like the dual Nullstellensatz), on the other hand, are easier to obtain for tropical polynomials and then translate to min-plus polynomials. In our opinion it is fruitful for further development of the theory to consider both settings simultaneously.

¹ To avoid a confusion we note that we use the word ‘dual’ in two different meanings. First, we use it in the term “dual Nullstellensatz” as opposed to the standard version of Nullstellensatz. This means that the dual Nullstellensatz is obtained from the standard Nullstellensatz by the (linear) duality (see [11] and below in this paper). Second, we use the word ‘dual’ in term “duality result” to denote the general type of results. Since the standard Nullstellensatz is a duality result itself, applying the linear duality to it results in a non-duality result. Thus, the dual Nullstellensatz is not a duality result in a proper sense, but rather the word “dual” is used in contrast to the customary Nullstellensatz which we name “primary”.

Our techniques We use the general approach of the paper [11] to Nullstellensatz through the dual formulation.

To establish the dual Nullstellensatz we use methods of discrete geometry dealing with integer polyhedra. First we obtain dual Nullstellensatz for the case without ∞ . The case with ∞ requires much more additional technical work.

To obtain the primary Nullstellensatz we apply the duality results for tropical linear polynomials. We note that these results rely on the completely different combinatorial techniques, namely on the connection to mean payoff games [2].

Other works on tropical Nullstellensatz After the paper was already submitted the authors learned that the primary min-plus Nullstellensatz in a different language was independently proved in [3]. Their proof is completely different and has a more algebraic flavor. Also, the proof is given in [3] for the case of \mathbb{R}_∞ only and there is no effective version of the theorem in [3].

In [19] there was established Nullstellensatz for the tropical semiring augmented with additional elements (called ghosts). This result is in the line with other results [32] trying to capture tropical mathematics by the means of the classical ones. However, the tropical semiring augmented with ghosts constitutes (logically) a completely different model compared to the usual tropical semiring. Thus our results are incomparable with the ones of [19].

We also note that [30] (which has Nullstellensatz in the title) takes completely different view on Nullstellensatz. We consider Nullstellensatz as a result on the solvability of a system of polynomials, and [30] views Nullstellensatz as a result on the structure of the radical of a tropical ideal. As it can be easily seen, for example, from our results during the translation from the classical world to the tropical one, the connection between the solvability and the ideal changes drastically (cf. the example $F = \{\min(x, 0), \min(x, 1)\}$ above). Thus, our results are incomparable with the results of [30] as well.

In [27] a version of Nullstellensatz was shown for a related structure of amoebas. However, [27] proposes a view on Nullstellensatz different from the one suggested in the present paper. An analogous to [27] result in tropical setting was obtained in [4].

The rest of the paper is organized as follows. In Section 2 we introduce the main definitions. In Section 3 we state our results. In Section 4 we prove the tropical and min-plus dual Nullstellensätze. In Section 5 we establish a connection between the sets of roots of tropical and of min-plus polynomial systems. As an illustration we deduce the min-plus dual Nullstellensatz with slightly worse parameter from the tropical dual Nullstellensatz. In Section 6 we show the tropical and min-plus primary Nullstellensätze. In Section 7 we show the min-plus and tropical linear dualities. Sections 5 and 7 depend only on Sections 2 and 3, and can be read independently of other sections.

2 Preliminaries

Tropical and min-plus polynomials. A *min-plus* or a *tropical semiring* is defined by a set \mathbb{K} , which can be \mathbb{R} , $\mathbb{R}_\infty = \mathbb{R} \cup \{+\infty\}$, \mathbb{Q} or $\mathbb{Q}_\infty = \mathbb{Q} \cup \{+\infty\}$ endowed with two operations, *tropical addition* \oplus and *tropical multiplication* \odot , defined in the following way:

$$x \oplus y = \min\{x, y\}, \quad x \odot y = x + y.$$

Below we mainly consider $\mathbb{K} = \mathbb{R}$ and $\mathbb{K} = \mathbb{R}_\infty$. The proofs however literally translate to the cases of \mathbb{Q} and \mathbb{Q}_∞ and moreover to any linearly ordered vector space over rational numbers (see Remark 1 below).

A tropical (or min-plus) monomial in variables $\vec{x} = (x_1, \dots, x_n)$ is defined as

$$m(\vec{x}) = c \odot x_1^{\odot i_1} \odot \dots \odot x_n^{\odot i_n}, \quad (1)$$

where c is an element of the semiring \mathbb{K} and i_1, \dots, i_n are nonnegative integers. In the usual notation the monomial is the linear function

$$m(\vec{x}) = c + i_1 x_1 + \dots + i_n x_n.$$

For $\vec{x} = (x_1, \dots, x_n)$ and $I = (i_1, \dots, i_n)$ we introduce the notation

$$\vec{x}^I = x_1^{\odot i_1} \odot \dots \odot x_n^{\odot i_n} = i_1 x_1 + \dots + i_n x_n.$$

The degree of the monomial m is defined as the sum $i_1 + \dots + i_n$. We denote this sum by $|I|$.

A *tropical polynomial* is the tropical sum of tropical monomials

$$f(\vec{x}) = \bigoplus_i m_i(\vec{x})$$

with pairwise distinct exponent vectors $I = (i_1, \dots, i_n)$, or in the usual notation $f(\vec{x}) = \min_i m_i(\vec{x})$. The *degree* of the tropical polynomial f denoted by $\deg(f)$ is the maximal degree of its monomials. A point $\vec{a} \in \mathbb{K}^n$ is a *root* of the polynomial f if the minimum $\min_i \{m_i(\vec{a})\}$ is either attained on at least two different monomials m_i or is infinite. We also say in this case that f is *satisfied* by \vec{a} .

Geometrically, over the semiring \mathbb{R} a tropical polynomial $f(\vec{x})$ is a concave piece-wise linear function over \mathbb{R}^n and the roots of f are non-smoothness points of this function.

We say that \vec{a} is a root for the system of tropical polynomials $F = \{f_1, \dots, f_k\}$ in variables \vec{x} if \vec{a} is a root to each polynomial $f_i \in F$.

A min-plus polynomial is a pair of tropical polynomials

$$(f(\vec{x}), g(\vec{x})).$$

The degree of a min-plus polynomial is the maximum of the degrees of f and g . A point $\vec{a} \in \mathbb{K}^n$ is a root of this polynomial if the following equality holds: $f(\vec{a}) = g(\vec{a})$. We call an equation $f(\vec{x}) = g(\vec{x})$ a *min-plus polynomial equation*.

Talking about tropical and min-plus polynomials we sometimes omit the adjectives ‘‘tropical’’ and ‘‘min-plus’’ when it does not cause a confusion.

Linear polynomials. An important special case of tropical and min-plus polynomials are linear polynomials which are just tropical and min-plus polynomials of degree 1. If there is no monomial of degree 0 (constant monomial) in a linear polynomial, we say that the linear polynomial is *homogeneous*.

It is convenient to express a homogeneous linear polynomial f in the form

$$\min_{1 \leq j \leq n} \{a_j + x_j\}.$$

In particular, if some variable x_j is not presented in the polynomial for notational convenience we still write it in this expression and just set the corresponding coefficient a_j to ∞ (even if we consider the semiring \mathbb{R}).

A tropical homogeneous linear system

$$\min_{1 \leq j \leq n} \{a_{ij} + x_j\}, \quad 1 \leq i \leq m, \quad (2)$$

then can be naturally associated with its matrix $A \in \mathbb{R}_\infty^{m \times n}$. We will also use a matrix notation $A \odot \vec{x}$ for such systems. Thus, we consider tropical linear systems $A \odot \vec{x}$ with the matrices $A \in \mathbb{R}_\infty^{m \times n}$ over both semirings \mathbb{R} and \mathbb{R}_∞ . We assume however, that there are no rows and columns in A consisting entirely of ∞ . One can delete each infinite row since any vector is its root. One can also delete each infinite column since this has no effect on the solvability of the system. It is convenient to call the roots of tropical linear systems *solutions*.

We note that it is also common to consider tropical linear systems $A \odot \vec{x}$ over \mathbb{R} with matrices $A \in \mathbb{R}^{m \times n}$ only. Some of our sidestep results (Corollaries 3 and 4) address this setting.

Analogously min-plus homogeneous linear systems

$$\left(\min_{1 \leq j \leq n} \{a_{ij} + x_j\}, \min_{1 \leq j \leq n} \{b_{ij} + x_j\} \right), \quad 1 \leq i \leq m,$$

can be associated with a pair of matrices A and B corresponding to the left-hand side and the right-hand side of an equation. We will also write a min-plus homogeneous linear system of equations in the matrix form as $A \odot \vec{x} = B \odot \vec{x}$. It will be also convenient to consider min-plus linear systems of (componentwise) inequalities $A \odot \vec{x} \leq B \odot \vec{x}$. It is not hard to see that their expressive power is the same as of equations.

Lemma 1 *For any min-plus system of linear homogeneous equations there is an equivalent system of min-plus linear inequalities and visa versa.*

Proof Indeed, each min-plus linear equation $L_1(\vec{x}) = L_2(\vec{x})$ is equivalent to the pair of min-plus inequalities $L_1(\vec{x}) \geq L_2(\vec{x})$ and $L_1(\vec{x}) \leq L_2(\vec{x})$. On the other hand min-plus linear inequality $L_1(\vec{x}) \leq L_2(\vec{x})$ is equivalent to the min-plus equation $L_1(\vec{x}) = \min(L_1(\vec{x}), L_2(\vec{x}))$. It is not hard to see that the last equation can be transformed to the form of min-plus linear equation.

There is one more important convention we make concerning the case of a tropical semiring with infinity. For two matrices $A, B \in \mathbb{R}_\infty^{m \times n}$ we say that the system $A \odot \vec{x} < B \odot \vec{x}$ has a solution if there is $\vec{x} \in \mathbb{R}_\infty^n$ such that for each row of the system if one of sides is finite, then the strict inequality holds, but also the case where both sides are equal to ∞ is allowed (informally, we can say that $\infty < \infty$).

We also consider non-homogeneous tropical linear systems

$$\min_{1 \leq j \leq n} \{a_{ij} + x_j\} \cup \{a_i\}, \quad 1 \leq i \leq m. \quad (3)$$

This system can be naturally associated to the matrix $A \in \mathbb{K}^{m \times (n+1)}$ and written in the matrix form as $A \odot (\vec{x}, 0)$. Analogously, we can consider a non-homogeneous min-plus linear systems $A \odot (\vec{x}, 0) = B \odot (\vec{x}, 0)$. We note that over \mathbb{R}^n the tropical system $A \odot (\vec{x}, 0)$ is solvable iff the homogeneous system $A \odot \vec{x}'$ is solvable, where $\vec{x}' = (\vec{x}, x_{n+1})$. Indeed, having a solution \vec{x}' for the latter system we can add the same number to all coordinates of \vec{x}' to make $x_{n+1} = 0$ and thus obtain a solution of the former system. The same is true for the min-plus case. But this is not true over \mathbb{R}_∞ : a homogeneous system always has a solution (just let $\vec{x} = (\infty, \dots, \infty)$), but a non-homogeneous system does not always have a solution. However, over \mathbb{R}_∞ we have that $A \odot (\vec{x}, 0)$ has a solution iff there is a solution to $A \odot \vec{x}'$ with $x_{n+1} \neq \infty$.

3 Statement of the results

3.1 Tropical and Min-plus Nullstellensätze

Definition 1 For a given system of tropical polynomials $F = \{f_1, \dots, f_k\}$ in n variables we introduce its infinite Macaulay matrix M . The columns of M correspond to nonnegative integer vectors $I \in \mathbb{Z}_+^n$ and the rows of M correspond to the pairs (j, J) , where $1 \leq j \leq k$ and $J \in \mathbb{Z}_+^n$. For a given I and (j, J) we let the entry $m_{(j, J), I}$ be equal to the coefficient of the monomial \vec{x}^I in the polynomial $\vec{x}^J \odot f_j$ (if there is no such monomial in the polynomial we assume that the entry is equal to ∞). By M_N we denote the finite submatrix of the matrix M consisting of the columns I such that $|I| = i_1 + \dots + i_n \leq N$ and the rows that have all their finite entries in these columns. The tropical linear system $M_N \odot \vec{y}$ will be of interest to us. Over \mathbb{R}_∞ we consider the non-homogeneous system $M_N \odot (0, \vec{y})$. The column corresponding to the constant monomial is a non-homogeneous column.

For a system of min-plus polynomials $F = \{(f_1, g_1), \dots, (f_k, g_k)\}$ we analogously introduce the pair of matrices Ml and Mr corresponding to the left-hand sides and the right-hand sides of polynomials respectively. In the same way we introduce matrices Ml_N , Mr_N and the corresponding min-plus linear system $Ml_N \odot \vec{y} = Mr_N \odot \vec{y}$. Analogously, for the case of \mathbb{R}_∞ we consider the non-homogeneous system $Ml_N \odot (0, \vec{y}) = Mr_N \odot (0, \vec{y})$.

In [11] there were conjectured three forms of a tropical dual Nullstellensatz. We state the strongest of them, namely an effective Nullstellensatz conjecture.

Conjecture 1 ([11]) There is a function N of n and of $\deg(f_i)$ for $1 \leq i \leq k$ such that a system of polynomials $F = \{f_1, \dots, f_k\}$ has a common tropical root iff the tropical linear system corresponding to the matrix M_N has a solution.

Note that the classical analog of this statement is precisely the effective Nullstellensatz in the dual form (see [11] for the detailed discussion).

In [11] the conjecture was proven for the case of $n = 1$. In this paper we prove the general case of the conjecture.

Theorem 1 (Tropical Dual Nullstellensatz) *Consider a system of tropical polynomials $F = \{f_1, \dots, f_k\}$ in n variables. Denote by d_i the degree of the polynomial f_i and let $d = \max_i d_i$.*

(i) *Over the semiring \mathbb{R} the system F has a root iff the Macaulay tropical linear system $M_N \odot \vec{y}$ for*

$$N = (n + 2)(d_1 + \dots + d_k)$$

has a solution.

(ii) *Over the semiring \mathbb{R}_∞ the system F has a root iff the Macaulay tropical non-homogeneous linear system $M_N \odot (0, \vec{y})$ for*

$$N = \text{poly}(n, k, d) (4d)^{\min(n, k)}$$

has a solution.

Note that one direction of the theorem is simple. Indeed, if the system of polynomials has a root $\vec{a} \in \mathbb{R}^n$ then it is not hard to see that there is a solution to $M_N \odot \vec{y}$. Indeed, note that the coordinates y_I of \vec{y} correspond to monomials \vec{x}^I , so let $y_I = \vec{a}^I$. Since each row of M_N corresponds to the polynomial of the form $\vec{x}^J \odot f_j(\vec{x})$ and \vec{a} is a root of any such polynomial, we have that \vec{y} satisfies all the rows of M_N . Thus, the essence of the theorem is to prove the other direction. The same argument works over \mathbb{R}_∞ .

We note that we can also consider an infinite Macaulay tropical linear system $M \odot \vec{y}$. It is well defined since each row of M has only finite number of finite entries. As a corollary of the previous theorem we deduce an infinite version of the tropical dual Nullstellensatz.

Corollary 1 *A system of tropical polynomials $F = \{f_1, \dots, f_k\}$ of n variables has a root iff the infinite Macaulay tropical linear system with the matrix M has a solution. The result holds for both \mathbb{R} and \mathbb{R}_∞ semirings.*

The proof in the simple direction is the same as for Theorem 1 and the hard part of the corollary follows trivially from Theorem 1.

We show a dual Nullstellensatz for the min-plus case.

Theorem 2 (Min-Plus Dual Nullstellensatz) Consider a system of min-plus polynomials $F = \{(f_1, g_1), \dots, (f_k, g_k)\}$ in n variables. Let d_i be the degree of the polynomial (f_i, g_i) and let $d = \max_i d_i$.

(i) Over the semiring \mathbb{R} the system F has a root iff the Macaulay min-plus linear system $Ml_N \odot \vec{y} = Mr_N \odot \vec{y}$ for

$$N = (n + 2)(d_1 + \dots + d_k)$$

has a solution.

(ii) Over the semiring \mathbb{R}_∞ the system F has a root iff the non-homogeneous Macaulay min-plus linear system $Ml_N \odot (0, \vec{y}) = Mr_N \odot (0, \vec{y})$ for

$$N = \text{poly}(n, k, d) (4d)^{\min(n, k)}$$

has a solution.

As in the tropical case an infinite version of the min-plus dual Nullstellensatz follows.

Corollary 2 Consider the system of min-plus polynomials $F = \{f_1=g_1, \dots, f_k=g_k\}$ of n variables. The system F has a root iff the infinite Macaulay min-plus linear system with the pair of matrices (Ml, Mr) has a solution. The result holds for both \mathbb{R} and \mathbb{R}_∞ semirings.

We provide examples showing that our bounds on N are qualitatively tight. Namely for the semiring \mathbb{R} , for any $d > 1$ and any n we construct a family F of $(n + 1)$ tropical (or min-plus) polynomials in n variables and of degree d such that F has no root, but the Macaulay tropical (or min-plus) linear system for $N = (d - 1)(n - 1)$ has a solution. For the semiring \mathbb{R}_∞ , for any $d > 1$ and any n we construct a system F of $n + 1$ tropical (or min-plus) polynomials in $n + 1$ variables and of degree d such that F has no root, but the Macaulay tropical (or min-plus) linear system for $N = d^{n-1} - 1$ has a solution.

We note that quantitatively there is a room for an improvement between our lower and upper bounds on N . The gap is more noticeable in the case of the semiring \mathbb{R} . Assuming for the sake of simplicity that $n \approx k$ our upper bound gives $N \sim dn^2$ and our lower bound gives $N \sim dn$. Thus we can formulate the following open problem.

Open Problem Close the gap between upper and lower bounds on N in the tropical Nullstellensatz.

Next we establish the Nullstellensatz in a more standard primary form.

We start with a more intuitive min-plus Nullstellensatz. For this we need to introduce the notion of an algebraic combination of min-plus polynomials.

Definition 2 An algebraic combination of a system of min-plus polynomials $F = \{(f_1, g_1), \dots, (f_k, g_k)\}$ is a min-plus polynomial (f, g) that can be represented in the form

$$(f, g) = \left(\bigoplus_{j=1}^K h_j^{(l)}, \bigoplus_{j=1}^K h_j^{(r)} \right),$$

where for each j we have either $(h_j^{(l)}, h_j^{(r)}) = (m_j \odot f_{i_j}, m_j \odot g_{i_j})$ or $(h_j^{(l)}, h_j^{(r)}) = (m_j \odot g_{i_j}, m_j \odot f_{i_j})$ for some monomial m_j and some i_j .

Theorem 3 (Min-Plus Primary Nullstellensatz) *Consider a system of min-plus polynomials $F = \{(f_1, g_1), \dots, (f_k, g_k)\}$ in n variables. Denote by d_i the degree of the polynomial (f_i, g_i) and let $d = \max_i d_i$.*

(i) *Over the semiring \mathbb{R} the system F has no root iff we can construct an algebraic min-plus combination (f, g) of F with degree at most*

$$N = (n + 2)(d_1 + \dots + d_k)$$

such that for each monomial $m = x_1^{\odot j_1} \odot \dots \odot x_n^{\odot j_n}$ its coefficient in f is greater than its coefficient in g .

(ii) *Over the semiring \mathbb{R}_∞ the system F has no root iff we can construct an algebraic combination (f, g) of F with degree at most*

$$N = \text{poly}(n, k, d) (4d)^{\min(n, k)}$$

such that for each monomial $m = x_1^{\odot j_1} \odot \dots \odot x_n^{\odot j_n}$ its coefficient in f is greater than its coefficient in g and with the additional property that the constant term in g is finite.

For the tropical case we will need the following definition.

Definition 3 For a system of tropical polynomials $F = \{f_1, \dots, f_k\}$ and tropical monomials m_1, \dots, m_K the algebraic combination

$$g = \bigoplus_{j=1}^K g_j,$$

where

$$g_j = m_j \odot f_{i_j},$$

is called *nonsingular* if the following two properties hold:

- for each monomial m of g there is a (unique) $1 \leq l(m) \leq K$ such that the coefficient of m in the polynomial $g_{l(m)}$ is less than the coefficients of m in all other polynomials g_j for $j \neq l(m)$;
- for different m and m' we have $l(m) \neq l(m')$.

We illustrate this notion on a small example. Consider a tropical polynomial system $F = \{f_1, f_2\}$, where $f_1 = 0 \odot x \oplus 0$ and $f_2 = 0 \odot x \oplus 1$. We already observed in the introduction that this system does not have a root. Now we note that this system has a nonsingular algebraic combination. Indeed, consider $g_1 = \frac{1}{2} \odot f_1$, $g_2 = 0 \odot f_2$ and

$$g = g_1 \oplus g_2 = \frac{1}{2} \odot (0 \odot x \oplus 0) \oplus 0 \odot (0 \odot x \oplus 1) = \left(\frac{1}{2} \odot x \oplus \frac{1}{2} \right) \oplus (0 \odot x \oplus 1).$$

Note that for the constant monomial its coefficient is minimal exclusively in g_1 , and for monomial x its coefficient is minimal only in g_2 .

Now we can formulate the tropical Nullstellensatz in a primary form, that shows that the above example is not a coincidence.

Theorem 4 (Tropical Primary Nullstellensatz) *Consider a system of tropical polynomials $F = \{f_1, \dots, f_k\}$ in n variables. Denote by d_i the degree of the polynomial f_i and let $d = \max_i d_i$.*

(i) *The system F has no root over \mathbb{R} iff there is a nonsingular algebraic combination g of F with the degree at most*

$$N = (n + 2)(d_1 + \dots + d_k).$$

(ii) *The system F has no root over \mathbb{R}_∞ iff there is a nonsingular algebraic combination g of F with the degree at most*

$$N = \text{poly}(n, k, d) (4d)^{\min(n, k)}$$

and with a finite constant monomial.

For the proofs of the last two theorems we use the following min-plus and tropical linear duality.

3.2 Linear Duality

We prove the following result on the min-plus linear duality.

Lemma 2 *Let $A, B \in \mathbb{R}_\infty^{m \times n}$ be two matrices.*

- (i) *For any subset $S \subseteq \{1, \dots, n\}$ exactly one of the following two statements is true.*
1. *There is a solution to $A \odot \vec{x} \leq B \odot \vec{x}$ with finite coordinates x_i for $i \in S$.*
 2. *There is a solution to the transposed system $B^T \odot \vec{y} < A^T \odot \vec{y}$ such that for some $i \in S$ the i -th coordinate of the vector $B^T \odot \vec{y}$ is finite.*
- (ii) *For any subset $S \subseteq \{1, \dots, n\}$ exactly one of the following two statements is true.*
1. *There is a solution to $A \odot \vec{x} \leq B \odot \vec{x}$ such that for some $i \in S$ the coordinate x_i is finite.*
 2. *There is a solution to $B^T \odot \vec{y} < A^T \odot \vec{y}$ such that the i -th coordinate of the vector $B^T \odot \vec{y}$ is finite for all $i \in S$.*

The proof of this lemma is based on the connection of min-plus linear systems with mean payoff games established in [2]. Though the proof is rather simple as soon as one invokes this connection, we are not aware of the claim and the proof of this result in the literature.

As a simple corollary of this lemma we show the following clean formulation of the min-plus linear duality.

Corollary 3 (i) *For two matrices $A, B \in \mathbb{R}^{m \times n}$ exactly one of the following two statements is true.*

1. *There is a solution to $A \odot \vec{x} \leq B \odot \vec{x}$ over \mathbb{R} .*
2. *There is a solution to $B^T \odot \vec{y} < A^T \odot \vec{y}$ over \mathbb{R} .*

- (ii) For two matrices $A, B \in \mathbb{R}_{\infty}^{m \times n}$ exactly one of the following two statements is true.
1. There is a solution $\vec{x} \neq (\infty, \dots, \infty)$ to $A \odot \vec{x} \leq B \odot \vec{x}$.
 2. There is a finite solution to $B^T \odot \vec{y} < A^T \odot \vec{y}$.
- (iii) For two matrices $A, B \in \mathbb{R}_{\infty}^{m \times n}$ exactly one of the following two statements is true.
1. There is a finite solution to $A \odot \vec{x} \leq B \odot \vec{x}$.
 2. There is a solution $\vec{y} \neq (\infty, \dots, \infty)$ to $B^T \odot \vec{y} < A^T \odot \vec{y}$.

Since the corollary follows from Lemma 2 almost immediately, we present the proof here.

Proof For matrices $A, B \in \mathbb{R}^{m \times n}$ with finite entries we can use Lemma 2(i) with $S = \{1, \dots, n\}$. Then the first statement in Lemma 2(i) coincides with the first statement in Corollary 3(i). The second statement in Lemma 2(i) is equivalent to the second statement in Corollary 3(i). Indeed, if there is a finite solution to the system $B^T \odot \vec{y} < A^T \odot \vec{y}$ then clearly for this \vec{y} all the coordinates of the vector $B^T \odot \vec{y}$ are finite. In the reverse direction, if there is a solution \vec{y} such that $B^T \odot \vec{y}$ has a finite coordinate, then the vector \vec{y} itself has a finite coordinate. Since all the entries in A and B are finite, then all the coordinates of $A^T \odot \vec{y}$ and $B^T \odot \vec{y}$ are finite. If in the vector \vec{y} there are infinite coordinates we can replace them by large enough finite numbers in such a way that the vectors $A^T \odot \vec{y}$ and $B^T \odot \vec{y}$ do not change. The resulting vector is a finite solution of the system $B^T \odot \vec{y} < A^T \odot \vec{y}$.

For the proof of Corollary 3(ii) let $S = \{1, \dots, n\}$ and apply Lemma 2(ii). Then the first statement in Lemma 2(ii) is equivalent to the first statement in Corollary 3(ii). To see that the equivalence holds also for the second statements of Lemma 2(ii) and Corollary 3(ii) note that if for some \vec{y} all the coordinates of $B^T \odot \vec{y}$ are finite, then we can assume that all the coordinates of \vec{y} are also finite. Indeed, if there are infinite coordinates in \vec{y} we can just set them to constants large enough without changing the value of the vector $B^T \odot \vec{y}$.

The last part of the corollary can be shown analogously by letting $S = \{1, \dots, n\}$ and applying Lemma 2(i).

We show a similar result for the tropical duality.

Lemma 3 Let $A \in \mathbb{R}_{\infty}^{m \times n}$ be a matrix.

- (i) For any subset $S \subseteq \{1, \dots, n\}$ exactly one of the following two statements is true.
1. There is a solution to $A \odot \vec{x}$ with finite coordinates x_i for $i \in S$.
 2. There is \vec{z} such that in each row of $A^T \odot \vec{z}$ the minimum is attained only once or is equal to ∞ , for each two rows with the finite minimum the (unique) minimums are in different columns and for some $i \in S$ the i -th coordinate of $A^T \odot \vec{z}$ is finite.
- (ii) For any subset $S \subseteq \{1, \dots, n\}$ exactly one of the following two statements is true.

1. There is a solution to $A \odot \vec{x}$ such that for some $i \in S$ the coordinate x_i is finite.
2. There is \vec{z} such that in each row of $A^T \odot \vec{z}$ the minimum is attained only once or is equal to ∞ , for each two rows with the finite minimum the minimums are in different columns and the i -th coordinate of $A^T \odot \vec{z}$ is finite for all $i \in S$.

This result can be proven either through a reduction to min-plus linear systems, or through the analysis of [12]. We give a proof through the reduction to min-plus linear systems in Section 7.

Just like in the case of min-plus linear systems we can show the following corollary.

Corollary 4 (i) For a matrix $A \in \mathbb{R}^{m \times n}$ exactly one of the following two statements is true.

1. There is a solution to $A \odot \vec{x}$ over \mathbb{R} .
2. There is $\vec{z} \in \mathbb{R}^m$ such that in each row of $A^T \odot \vec{z}$ the minimum is attained only once and for each two rows their minimums are in different columns.

(ii) For a matrix $A \in \mathbb{R}_{\infty}^{m \times n}$ exactly one of the following two statements is true.

1. There is a solution $\vec{x} \neq (\infty, \dots, \infty)$ to $A \odot \vec{x}$.
2. There is a finite \vec{z} such that in each row of $A^T \odot \vec{z}$ the minimum is attained only once and for each two rows their minimums are in different columns.

(iii) For a matrix $A \in \mathbb{R}_{\infty}^{m \times n}$ exactly one of the following two statements is true.

1. There is a finite solution to $A \odot \vec{x}$.
2. There is $\vec{z} \neq (\infty, \dots, \infty)$ such that in each row of $A^T \odot \vec{z}$ the minimum is attained only once or is equal to ∞ and for each two rows with the finite minimum their minimums are in different columns.

The proof of this corollary is completely analogous to the proof of Corollary 3.

3.3 Tropical vs. Min-plus

We also establish the connection between tropical and min-plus polynomial systems.

Lemma 4 Over both \mathbb{R} and \mathbb{R}_{∞} given a system of tropical polynomials we can construct a system of min-plus polynomials over the same set of variables, with the same set of roots and the same degree. The same is true for the semirings \mathbb{Q} and \mathbb{Q}_{∞} . In this case the construction is effective and can be computed in polynomial time.

In the opposite direction we do not have such a simple connection, but we can still prove the following lemma.

Lemma 5 *Over both \mathbb{R} and \mathbb{R}_∞ for any system of min-plus polynomials F in n variables there is a system of tropical polynomials T in $2n$ variables and an injective linear transformation $H: \mathbb{R}_\infty^n \rightarrow \mathbb{R}_\infty^{2n}$ such that the image of the set of roots of F coincides with the set of roots of T . The same is true for the semirings \mathbb{Q} and \mathbb{Q}_∞ . In this case the construction is effective and can be computed in polynomial time.*

The proof of this lemma follows the lines of the proof of the analogous statement for the case of linear polynomials in [13].

Remark 1 We note that all of our results translate from \mathbb{R} and \mathbb{Q} to arbitrary linearly ordered vector space over the field of rationals. Indeed, note that all theorems can be formulated in first-order logic, where each atomic formula is a linear equation or inequality with integer coefficients. It is known that for such formulas a quantifier elimination is possible (implicit in [36]) and thus by the standard logical argument (the transfer principle) any statement without free variables is either true in all such vector spaces, or false in all such vector spaces.

Here we briefly explain why our results can be formulated in the first-order logic of linear equations and inequalities with integer coefficients. We show this on the example of Theorem 1. We formulate the theorem in the first-order logic for each n and k and each tuple (d_1, \dots, d_k) separately. Coefficients of the polynomials and coordinates of roots and solutions are variables in the first-order formula. The fact that a polynomial has some point as a root means that there are coordinates of this root such that the minimum is attained on two monomials, that is there is a pair of monomials that has equal values and that are less or equal to the values of all other monomials. This can be stated as a first-order formula consisting of an OR over all pairs of monomials of ANDs of linear equations and inequalities stating that this particular pair of monomials has minimal values. Note that the value of the monomial in a certain point is a linear combination of coordinates of a point with integer coefficients (coefficients are the exponents of variables in the monomial). Thus each inequality or equation between the values of monomials is a linear inequality or equation as required.

The fact that a polynomial has a root can be obtained now by appending to the previous formula existential quantifiers over coordinates of the root. Analogously, we can express existence of a root to a system of polynomials. A similar formula expresses existence of a solution to Macaulay matrix of fixed size. Overall, Theorem 1 (for fixed n, k, d_1, \dots, d_k) can now be expressed as a formula starting with universal quantifiers over the coefficients of polynomials followed by an equivalence of two statements: the existence of a root to the system of polynomials and the existence of a solution to Macaulay system.

In a similar way we can formulate statements about algebraic combinations (now coefficients in the combinations are variables). So in the same way we can formulate all other theorems within the described first-order logic.

4 Tropical and Min-plus Dual Nullstellensätze

Throughout the whole section we assume that we are given a system of tropical polynomials $F = \{f_1, \dots, f_k\}$ in n variables $\vec{x} = (x_1, \dots, x_n)$.

The proofs of Theorems 1 and 2 are analogous. We present the proof of Theorem 1 (which is more intuitive) and specify what should be changed to obtain the proof of Theorem 2.

This section is organized as follows. In Subsection 4.1 we introduce a required notation and show preliminary results. In Subsections 4.2 and 4.3 we give the proofs of Theorems 1(i) and 2(i). In Subsection 4.4 we provide some examples illustrating the certain difficulties behind the proof. In Subsection 4.5 we prove Theorems 1(ii) and 2(ii). Finally, in Subsection 4.6 we show that the upper bounds in Theorems 1 and 2 are tight.

4.1 Preliminary definitions and results

Geometrical interpretation of tropical polynomials. All functions $\varphi: \mathbb{Z}^n \rightarrow \mathbb{R}$ we consider in this section are partial, that is they are defined on some subset $\text{Dom}(\varphi) \subseteq \mathbb{Z}^n$.

Definition 4 For two functions $\varphi, \psi: \mathbb{Z}^n \rightarrow \mathbb{R}$ let $D = \text{Dom} \varphi \cap \text{Dom} \psi$ and consider $t \in \mathbb{R}$ (if there is one) such that

1. for all $\vec{x} \in D$ we have $\varphi(\vec{x}) + t \leq \psi(\vec{x})$;
2. there is $\vec{x} \in D$ such that $\varphi(\vec{x}) + t = \psi(\vec{x})$.

It is easy to see that t is unique provided it exists. We denote the set of points satisfying property 2 by $\text{Sing}(\varphi, \psi)$ and call them *singularity points* for the pair (φ, ψ) . If such t does not exist we let $\text{Sing}(\varphi, \psi) = \emptyset$. We say that φ is *singular* to ψ iff $|\text{Sing}(\varphi, \psi)| \neq 1$.

Assume that in the case that D is non-empty the function $\psi(x) - \varphi(x)$ attains its minimum on D (this will always be the case for the functions we will consider, but this is not necessarily the case if D is infinite). Then geometrically, φ is singular to ψ if either the domains of φ and ψ do not intersect, or if we can adjust the graph of φ in \mathbb{R}^{n+1} space along the $(n+1)$ -th coordinate in such a way that this graph lies below the graph of ψ and has with it at least two common points. For a function φ we denote by $G(\varphi)$ the graph of the function in \mathbb{R}^{n+1} .

Note that the notion of singularity is asymmetric. It might be that φ is singular to ψ , but ψ is not singular to φ . For example, consider $\varphi, \psi: \mathbb{Z} \rightarrow \mathbb{R}$ given by $\varphi(x) = 0$ for all x , $\psi(0) = 1$ and $\psi(x) = 0$ for all $x \neq 0$.

The next lemma follows directly from the definition.

Lemma 6 *We have $\vec{x} \in \text{Sing}(\varphi, \psi)$ iff \vec{x} minimizes the function $\psi - \varphi$ (on its domain). Or, equivalently, iff for all \vec{y} we have*

$$\varphi(\vec{x}) - \varphi(\vec{y}) \geq \psi(\vec{x}) - \psi(\vec{y}).$$

In this paper we consider rows of the matrix M_N , solutions to $M_N \odot \vec{y}$, coefficients of f_i 's. All of them constitute vectors \vec{a} whose coordinates are labeled by $I \in D$ for some $D \subseteq \mathbb{Z}_+^n$, that is by vectors with integer non-negative coordinates. With a vector \vec{a} we associate a function $\varphi_{\vec{a}}: \mathbb{Z}^n \rightarrow \mathbb{R}$ letting $\varphi_{\vec{a}}(I) = a_I$ for $I \in D$ and $a_I \neq \infty$ and leaving $\varphi_{\vec{a}}(I)$ undefined otherwise. When this vector is the vector of the coefficients of a polynomial f we shortly denote the resulting function by φ_f . When a polynomial f is one of the polynomials $f_i \in F$ we simplify the notation further to φ_i . Note that due to the definition of M_N if \vec{r} is a row of M_N labeled by (i, J) then $\varphi_{\vec{r}}(I) = \varphi_i(I - J)$.

In what follows we reserve Greek letters for the functions representing the coefficients of polynomials and entries of Macaulay matrix to distinguish them from the functions f_i 's.

The motivation for our notion of singularity is that it captures the solvability of tropical polynomials.

Lemma 7 *A vector $\vec{a} = \{a_I\}_{|I| \leq N}$ over \mathbb{R} or \mathbb{R}_∞ is a solution to a tropical linear polynomial $\min_I \{y_I + r_I\}$ corresponding to the vector $\vec{r} = \{r_I\}_{|I| \leq N}$ iff the function $-\varphi_{\vec{a}}$ is singular to $\varphi_{\vec{r}}$.*

Proof Consider an arbitrary vector \vec{r} over \mathbb{R} or \mathbb{R}_∞ and the corresponding tropical linear polynomial. The vector \vec{a} is a root of this linear polynomial if the minimum in $\{\varphi_{\vec{a}}(I) + \varphi_{\vec{r}}(I)\}_I$ is attained at least twice or is equal to ∞ . This minimum is ∞ iff $\text{Dom}(\varphi_{\vec{a}}) \cap \text{Dom}(\varphi_{\vec{r}}) = \emptyset$ and thus $\text{Sing}(-\varphi_{\vec{a}}, \varphi_{\vec{r}}) = \emptyset$. If the minimum is finite let t be the minimal number such that $\varphi_{\vec{a}}(I) + \varphi_{\vec{r}}(I) + t \geq 0$ for all I . Then $\varphi_{\vec{a}}(I) + \varphi_{\vec{r}}(I) + t = 0$ equals zero for at least two different I 's. This means that $-\varphi_{\vec{a}}(I) - t \leq \varphi_{\vec{r}}(I)$ and the equality holds for at least two points. Thus, the function $-\varphi_{\vec{a}}$ is singular to $\varphi_{\vec{r}}$.

The proof in the opposite direction follows the same lines.

In particular, a vector \vec{y} is a solution to $M_N \odot \vec{y}$ iff $-\varphi_{\vec{y}}$ is singular to all $\varphi_{\vec{r}}$, where \vec{r} is a row of M_N .

The difference between the semirings \mathbb{R} and \mathbb{R}_∞ is that over \mathbb{R} we have $\text{Dom} \varphi_{\vec{y}} = \{I \mid |I| \leq N\}$.

Now let \vec{r} be a vector of coefficients of a tropical polynomial f , that is r_I is the coefficient of the monomial \vec{x}^I in f . Then a root of the polynomial f is a vector $\vec{x} = (x_1, \dots, x_n)$ and \vec{y} described above in this case is given by

$$y_I = \vec{x}^I = \sum_j i_j x_j = \langle \vec{x}, I \rangle,$$

that is by the (classical) inner product of vectors \vec{x} and I . Thus in this case $\varphi_{\vec{y}}(I) = \langle \vec{x}, I \rangle$ is a partial linear function (note, that some coordinates x_j of \vec{x} might be infinite and then $y_I = \infty$ once $i_j \neq 0$ for at least one infinite coordinate x_j), whose graph is a part of a hyperplane in $(n+1)$ -dimensional space. Note that here we assume that $0 \cdot \infty = 0$. We introduce the notation $\chi_{\vec{x}} = -\varphi_{\vec{y}}$ and we say that $\chi_{\vec{x}}$ is a *partial hyperplane*. Over \mathbb{R} the function

$\chi_{\vec{x}}$ corresponds to an ordinary hyperplane. Thus, from Lemma 7 we get the following result.

Lemma 8 *A vector $\vec{x} \in \mathbb{R}_{\infty}^n$ is a root to f iff the partial hyperplane $\chi_{\vec{x}}$ is singular to the function φ_f .*

In particular, the system of polynomials F has a root over \mathbb{R}_{∞} iff there is a partial hyperplane singular to φ_i for all $i = 1, \dots, k$.

As a result we have that, if there is a partial hyperplane singular to all φ_i for all $i = 1, \dots, k$, then it clearly provides a solution to M_N . This repeats the proof of the simple direction of the tropical dual Nullstellensatz theorem. What we need to show for the opposite direction is that if there is some function singular to all translations of all φ_i 's within some simplex $|I| \leq N$, then there is also a singular partial hyperplane.

For the proof of Theorem 1 it is convenient to use the language of polytopes. We summarize it in the next definition.

Definition 5 To switch to polytope notation for a polynomial $f \in F$ we consider the graph $G(\varphi_f) = \{(I, \varphi_f(I)) \mid I \in \mathbb{Z}^n, \varphi_f(I) \neq \infty\}$ of the function φ_f and along with each point $(I, \varphi_f(I))$ we consider all points (I, t) above it, that is such that $t > \varphi_f(I)$. We take the convex hull in \mathbb{R}^{n+1} of all these points and call the resulting polytope $P(f)$ the *(extended) Newton polytope* of f . For the given system F of polynomials f_1, \dots, f_k we denote the resulting convex polytopes by P_1, \dots, P_k . We note that this construction is quite standard [18, 29, 32]. By the *bottom* of $P(f)$ we denote the set of points $\vec{x} = (x_1, \dots, x_n, x_{n+1}) \in P(f)$ such that there are no points of $P(f)$ below them, that is for any $\epsilon > 0$ we have that $(x_1, \dots, x_n, x_{n+1} - \epsilon) \notin P(f)$. Note that the bottom of $P(f)$ can be considered as a graph of a partial function on \mathbb{R}^n . We denote the restriction of this function to \mathbb{Z}^n by $\beta_P: \mathbb{Z}^n \rightarrow \mathbb{R}$. Note that this is an extension of φ_f . For the case of polytopes P_i we shorten this notation to β_i . It is not hard to see geometrically that a partial hyperplane is singular to φ_f iff it is singular to β_P . This is not necessarily true for an arbitrary function $\varphi_{\vec{a}}$ instead of a hyperplane.

Remark 2 We note that in the paper [11] the conjecture on the tropical dual Nullstellensatz was considered not for the original Macaulay matrix, but for the Macaulay matrix in which we already switch to the convex hulls of the graphs of polynomials in F , that is instead of values of functions φ_i rows of Macaulay matrix contained graphs of functions β_{P_i} . Our proofs work for both settings, but we consider the original Macaulay matrix being more natural.

Remark 3 For the min-plus case we analogously associate to each polynomial $(f_i, g_i) \in F$ the function φ_i . Let $\varphi_i(I)$ be the minimum of the coefficients of the monomial \vec{x}^I in f_i and g_i . Additionally we introduce colors to the points of the graphs $G(\varphi_i)$. If $\varphi_i(I)$ is equal to the coefficient of \vec{x}^I in f_i , then we color $(I, \varphi_i(I))$ in black, and if $\varphi_i(I)$ is equal to the coefficient of \vec{x}^I in g_i , then we color $(I, \varphi_i(I))$ in white. Note that we allow the same point I be labeled by both colors simultaneously.

In the notion of singularity of some function φ to ψ we now consider ψ to be colored function. For φ to be singular to ψ we require now that there are either no singular points or at least one black singular point and at least one white singular point. Note, however, that we are satisfied if there is only one singular point, but labeled with both colors. The analog of Lemma 7 is proven in the same way. Note that now we consider min-plus polynomials and to have a solution we need to have minimum to be attained on monomials in both parts of the polynomial. This directly corresponds to having singular points of both colors. The analog of Lemma 8 is proven in the same way.

Newton polytopes P_i are introduced in the same way as before (in the construction of the polytope we ignore the colors). But note that now some points of P_i are labeled by colors. In particular, all vertices of polytopes are labeled.

Convex polytopes. A convex polytope P in n -dimensional space can be specified by a set of (classical) linear functions $E_1(\vec{x}), \dots, E_l(\vec{x}), L_1(\vec{x}), \dots, L_k(\vec{x})$, where $\vec{x} \in \mathbb{R}^n$: P is the set of points $\vec{x} \in \mathbb{R}^n$ such that $E_i(\vec{x}) = 0$ for all $i = 1, \dots, l$ and $L_i(\vec{x}) \geq 0$ for all $i = 1, \dots, k$. We assume that none of $L_i(\vec{x})$ evaluates to 0 on the set $\{\vec{x} | \forall i E_i(\vec{x}) = 0\}$. Any face of a polytope can be specified by a nonempty set $S \subseteq \{1, \dots, k\}$. The face corresponding to S is the set of points $\vec{x} \in P$ such that $L_i(\vec{x}) = 0$ for all $i \in S$.

The boundary of the polytope is the union of all its faces. The interior $\overset{\circ}{P}$ of the polytope P is the set of its non-boundary points.

For polytopes P_1 and P_2 we denote by $P_1 + P_2$ the Minkowski sum of these polytopes. For natural k we use the notation $kP = P + \dots + P$, where there are k summands on the right-hand side. For an n -dimensional vector $\vec{\alpha}$ we denote by $P + \vec{\alpha}$ the translation of P by the vector $\vec{\alpha}$. That is,

$$P + \vec{\alpha} = \{\vec{x} + \vec{\alpha} | \vec{x} \in P\}.$$

By the homothety with a center $\vec{x} \in \mathbb{R}^n$ and a coefficient $\lambda > 0$ we denote the following bijective transformation $h_{\vec{x}}^\lambda$ of the space \mathbb{R}^n : the point $\vec{y} \in \mathbb{R}^n$ is sent to the point $h_{\vec{x}}^\lambda(\vec{y}) = \vec{x} + \lambda(\vec{y} - \vec{x})$. Note that kP is an image of P under the homothety $h_{\vec{0}}^k$. It is well known that translations and homotheties with the composition operation form a group called dilation group. In particular, an arbitrary composition of translations and homotheties results in a homothety or a translation. Below we will use this fact without mentioning.

Definition 6 Consider a polytope $P \subseteq \mathbb{R}^n$, a set of points $Q \subseteq \mathbb{R}^n$ and a point \vec{x} on the boundary of P . We say that Q *touches* P at \vec{x} iff

1. $Q \subseteq P$;
2. $\vec{x} \in Q$;
3. if Q contains a point \vec{y} on the boundary of P , then \vec{y} lies in a face of P containing \vec{x} .

Below we collect some facts we will need about the structure of convex polytopes. Though they are simple and intuitive we give the proofs of them for the sake of completeness.

Lemma 9 *Let P be a convex polytope and let $\vec{x}, \vec{y}, \vec{z}$ be three distinct points in P lying on the same line in the specified order. Then if \vec{y} belongs to some face of P then \vec{x} also belongs to the same face of P .*

Proof Suppose on the contrary that \vec{y} belongs to some face and \vec{x} does not. Then there is some inequality L among linear inequalities defining P such that $L(\vec{x}) > 0$ and $L(\vec{y}) = 0$. The restriction of values of L to the line containing \vec{x}, \vec{y} and \vec{z} is a linear function and hence, clearly, $L(\vec{z}) < 0$. Therefore, \vec{z} is not in P and thus, we have a contradiction.

Corollary 5 *Let P be a convex polytope and let $\vec{x}, \vec{y}, \vec{z}, \vec{t}$ be four distinct points in it lying on the same line in the specified order. Then \vec{y} belongs to some face of P iff \vec{z} belongs to the same face.*

Proof Just apply Lemma 9 to the points $\vec{y}, \vec{z}, \vec{t}$ and to the points $\vec{z}, \vec{y}, \vec{x}$.

Lemma 10 *Let P be a convex polytope and let \vec{x} be a point in P . Let P' be an image of P under $h_{\vec{x}}^\lambda$ for $\lambda > 1$. If P contains a point on some face of P' then this face contains \vec{x} .*

Proof Let \vec{y} be a point of P . Then the point

$$\vec{z} = h_{\vec{x}}^\lambda(\vec{y}) = \vec{x} - \lambda(\vec{y} - \vec{x})$$

lies in P' and $\vec{x}, \vec{y}, \vec{z}$ lie on the same line in the specified order. Thus, by Lemma 9 if \vec{y} is on some face of P' then \vec{x} is also on this face.

4.2 The enveloping polytope

The key idea of the proof of Theorem 1(i) is to consider a large extended Newton polytope P_0 “enveloping” all polytopes P_1, \dots, P_k corresponding to tropical polynomials f_1, \dots, f_k . The main property of P_0 we will ensure in this subsection is that for each point \vec{x} on its bottom and for any i there is a translation $P_i + \vec{\alpha}$ of the polytope P_i such that $P_i + \vec{\alpha}$ touches P_0 at \vec{x} (Lemma 12). Using this property in the next subsection we will show that for a solution \vec{a} of the tropical linear system $M_N \odot \vec{y}$ there is a singular point in $\text{Sing}(\vec{a}, \beta_{P_0})$ such that the facet of P_0 containing the point $(\vec{a}, \beta_{P_0}(\vec{a}))$ gives a root to the system F (Lemma 13).

It turns out that for P_0 we can just take the Minkowski sum of P_1, \dots, P_k multiplied by a large enough number. We just let

$$P_0 = (n+2) \cdot (P_1 + \dots + P_k). \quad (4)$$

To ensure the desired property of P_0 we need the following general fact on convex polytopes.

Lemma 11 *Let P be an n -dimensional convex polytope and let $P' = (n+2)P$. Then for each point $\vec{x} \in P'$ there is a translation $P + \vec{\alpha}$ of P and the homothety $h_{\vec{y}}^{n+2}$ mapping $P + \vec{\alpha}$ to P' with the following properties:*

1. the center \vec{y} of the homothety lies in \mathring{P}' ;
2. \vec{x} is a vertex of $P + \vec{\alpha}$.

It is easy to see that the first property is equivalent to the fact that $P + \vec{\alpha} \subseteq \mathring{P}'$, but the current form of the lemma will be more convenient for us.

The main tools in the proof of this lemma are Caratheodory's Theorem, the notion of the center of mass and homothety transformations.

Proof We first give a proof sketch and then proceed to the detailed formal proof. Since \vec{x} is in P' it lies in some simplex S' generated by $n + 1$ vertices of P' . For S' we consider each of its vertices and make a homothety with the center in it and the coefficient $(n + 1)/(n + 2)$. The resulting $(n + 1)$ simplices cover all S' (even with overlap). So \vec{x} lies in one of them, say in the one determined by the vertex v'_1 . Then we can consider the translation $S + \vec{\alpha}$ of the simplex S which is $(n + 2)$ times smaller than S' such that its vertex corresponding to v'_1 is mapped into \vec{x} . Then $S + \vec{\alpha}$ lies in S' . Now we can consider P' and note that $P + \vec{\alpha}$ is in P' .

Now we give a formal proof following the outline above. Since \vec{x} is a point in the convex polytope P' it lies in the convex hull of its vertices. By Caratheodory's Theorem there are $n + 1$ vertices v'_1, \dots, v'_{n+1} of P' such that $\vec{x} \in \text{Conv}\{v'_1, \dots, v'_{n+1}\}$. We denote the latter simplex by S' and denote the corresponding vertices of P by v_1, \dots, v_{n+1} .

Let w_1, \dots, w_{n+1} be the barycentric coordinates of \vec{x} with respect to v'_1, \dots, v'_{n+1} , that is $w_i \geq 0$ for all i , $\sum_i w_i = 1$ and

$$\vec{x} = \sum_i w_i v'_i.$$

Without loss of generality let w_1 be the largest among w_i . Then $nw_1 \geq w_2 + \dots + w_{n+1}$. Let

$$v' = \frac{1}{\sum_{i=2}^{n+1} w_i} \sum_{i=2}^{n+1} w_i v'_i.$$

Then $v' \in \text{Conv}\{v'_2, \dots, v'_{n+1}\}$, there is a relation

$$\vec{x} = w_1 v'_1 + \left(\sum_{i=2}^{n+1} w_i \right) v'$$

and thus the points v'_1 , \vec{x} and v' are on the same line. Moreover, $|\vec{x} - v'_1| \leq n|v' - \vec{x}| < (n + 1)|v' - \vec{x}|$ (observe that $|v' - \vec{x}|$ is nonzero since w_1 is nonzero being the largest weight). Consider a point \vec{y} on the same line between the points \vec{x} and v' and such that $(n + 1)|\vec{x} - \vec{y}| = |v'_1 - \vec{x}|$.

Now consider the polytope P and consider its translation $P + \alpha$ by which v_1 is mapped to \vec{x} . The homothety $h_{\vec{y}}^{n+2}$ sends \vec{x} to v'_1 and since $(n + 2)P$ and P' are equal, the image of $P + \alpha$ under this homothety is P' .

It is only left to note that the points \vec{x} , \vec{y} , v' lie on the same line in the specified order and all lie in P' . Thus by Lemma 9 since $\vec{x} \in \mathring{P}'$ we have $\vec{y} \in \mathring{P}'$.

Remark 4 We note that Lemma 11 does not hold for $P' = (n+1)P$. The example is very simple, just let P to be a standard simplex, that is a convex hull of points $\{0, \vec{e}_1, \dots, \vec{e}_n\}$. Then $P' = (n+1)P$ is a convex hull of points $\{\vec{0}, (n+1)\vec{e}_1, \dots, (n+1)\vec{e}_n\}$. Let \vec{x} be the center of the polytope P' , that is $\vec{x} = \vec{e}_1 + \dots + \vec{e}_n$. Then for \vec{x} to be a vertex of $P + \vec{\alpha}$ we should have that either $\vec{\alpha} = \vec{e}_1 + \dots + \vec{e}_n$, or $\vec{\alpha} = \vec{e}_1 + \dots + \vec{e}_{i-1} + \vec{e}_{i+1} + \dots + \vec{e}_n$ for some i . In the first case $\vec{y} = (n+1)(\vec{e}_1 + \dots + \vec{e}_n)/n$ and in the second case $\vec{y} = (n+1)(\vec{e}_1 + \dots + \vec{e}_{i-1} + \vec{e}_{i+1} + \dots + \vec{e}_n)/n$. In both cases \vec{y} lies on the boundary of P' : in the first case it is in the convex hull of $\{\vec{e}_1, \dots, \vec{e}_n\}$ and in the second case it is in the convex hull of $\{\vec{0}, \vec{e}_1, \dots, \vec{e}_{i-1}, \vec{e}_{i+1}, \dots, \vec{e}_n\}$.

Lemma 12 *For any point \vec{x} on the bottom of P_0 and for any P_j there is $\vec{\alpha}$ such that $P_j + \vec{\alpha}$ touches P_0 at \vec{x} .*

Proof Let

$$P = P_1 + \dots + P_k.$$

Thus $P_0 = (n+2)P$.

First we show that there is a translation of P touching P_0 at \vec{x} .

If \vec{x} is a vertex of P_0 then just note that there is a translation $P + \vec{\alpha}'$ lying inside of P_0 and containing \vec{x} (since P_0 is a Minkowski sum and P is a summand, P_0 can be viewed as a union of translations of P). Since \vec{x} is a vertex of P_0 it is also a vertex of $P + \vec{\alpha}'$. The homothety $h_{\vec{x}}^{n+2}$ sends \vec{x} as a vertex of $P + \vec{\alpha}'$ into \vec{x} as a corresponding vertex of P_0 and thus sends $P + \vec{\alpha}'$ to P_0 . Then by Lemma 10 $P + \vec{\alpha}'$ touches P_0 at \vec{x} .

If \vec{x} is not a vertex of P_0 denote the minimal dimension face of P_0 containing \vec{x} by Q_0 . Clearly, \vec{x} is in the interior of Q_0 . Since $P_0 = (n+2)P$ we have that there is a face Q of P such that $Q_0 = (n+2)Q$. By Lemma 11 we can find a translation $Q + \vec{\alpha}'$ such that \vec{x} is a vertex of $Q + \vec{\alpha}'$ and $Q + \vec{\alpha}' \subseteq Q_0$. This lemma also gives us the homothety $h_{\vec{y}}^{n+2}$ whose center \vec{y} lies in the interior of Q_0 . Now let us consider $P + \vec{\alpha}'$ and consider its image under $h_{\vec{y}}^{n+2}$. The vertex \vec{x} goes under this homothety to the corresponding vertex of P_0 and thus $P + \vec{\alpha}'$ goes to P_0 . Note that by Lemma 10 we also get that $P + \vec{\alpha}'$ intersects P_0 only in the faces incident to \vec{y} and thus only in the faces incident to \vec{x} .

Now note that $P + \vec{\alpha}'$ is the translation of the Minkowski sum of $P_1 + \dots + P_k$, thus for each of P_j there is a translation $\vec{\alpha}$ such that $P_j + \vec{\alpha}$ is in $P + \vec{\alpha}'$ and contains the point \vec{x} . Since this point is a vertex of $P + \vec{\alpha}'$ we have that \vec{x} is a vertex of $P_j + \vec{\alpha}$. Note that $P_j + \vec{\alpha}$ lies inside of $P + \vec{\alpha}'$ and thus also can intersect the boundary of P_0 only in the faces containing \vec{x} .

Corollary 6 *For any point \vec{x} on the bottom of P_0 and for any f_j there is $\vec{\alpha}$ such that $G(\varphi_j) + \vec{\alpha}$ touches P_0 at \vec{x} .*

Proof Note that the set $G(\varphi_j) + \vec{\alpha}$ is a subset of $P_j + \vec{\alpha}$, but on the other hand contains all its vertices. Thus, $G(\varphi_j) + \vec{\alpha}$ touches P_0 at \vec{x} .

Remark 5 The analysis of this subsection can be translated to the min-plus case literally.

4.3 A facet of P_0 is singular

In this subsection we are going to finish the proof of Theorem 1(i).

For the sake of convenience we will throughout this subsection call an $(n+1)$ dimensional vector \vec{a} (or a point in \mathbb{R}^{n+1}) integer if its first n coordinates are integers.

We will not need the following observation in the proof, but it helps to clarify the intuition.

Proposition 1 *Consider the bottom β_{P_0} of P_0 , consider the vector $\{a_I\}_I$ corresponding to it, that is $a_I = \beta_{P_0}(I)$ if $\beta_{P_0}(I)$ is defined, and $a_I = \infty$ otherwise. Consider the tropical polynomial $g = \oplus_I (a_I \odot \vec{x}^I)$. Then for each f_j the polynomial g lies in the tropical ideal generated by f_j (a tropical ideal is just the set of all the polynomials that can be represented as tropical algebraic combinations of the generating polynomials).*

Proof It is easier to give a proof in geometric terms. For each integer point \vec{x} on the bottom of P_0 consider a translation $G(\varphi_j) + \vec{a}_{\vec{x}}$ touching P_0 at \vec{x} which exists by Corollary 6. This translation corresponds to the tropical multiplication of f_j by a monomial. Then it is easy to see that all the integer points on the bottom of P_0 lie in the union of $G(\varphi_j) + \vec{a}_{\vec{x}}$ over all \vec{x} and on the other hand, all other integer points of this union lie in P_0 . The union operation corresponds to the minimum operation (tropical addition) for polynomials.

Lemma 13 *Suppose the tropical linear system $M_N \odot \vec{y}$ has a solution \vec{a} .*

- (i) *For the case of the tropical semiring \mathbb{R} there is a face of P_0 such that some hyperplane containing it provides a root to the tropical system F .*
- (ii) *For the case of the tropical semiring \mathbb{R}_∞ if there is $\vec{x} \in \mathbb{Z}^n$ such that $x \in \text{Dom } \beta_{P_0} \cap \text{Dom } \varphi_{\vec{a}}$ then there is a face of P_0 such that some hyperplane containing it provides a root to the tropical system F .*

Proof Consider the functions $\varphi_{\vec{a}}$ and β_{P_0} . Since in both cases \mathbb{R} and \mathbb{R}_∞ there is $\vec{x} \in \text{Dom } \beta_{P_0} \cap \text{Dom } \varphi_{\vec{a}}$, we have that there is a singularity point in $\text{Sing}(\varphi_{\vec{a}}, \beta_{P_0})$. Further proof works for both cases.

For each point $\vec{x} \in \text{Sing}(\varphi_{\vec{a}}, \beta_{P_0})$ consider the lowest dimension of the faces of P_0 to which the point $(\vec{x}, \beta_{P_0}(\vec{x}))$ belongs and further on denote by \vec{x} the point in $\text{Sing}(\varphi_{\vec{a}}, \beta_{P_0})$ which maximizes this minimal dimension. In simple words, we look for a singularity point in the most general position w.r.t. the polytope P_0 . Let us denote the minimal dimension face of P_0 containing $(\vec{x}, \beta_{P_0}(\vec{x}))$ by Q_0 . Below we show that this face satisfies the formulation of the lemma.

Consider some polynomial f_j . By Corollary 6 there is a vector \vec{a} such that $G(\varphi_j) + \vec{a}$ touches P_0 at $(\vec{x}, \beta_{P_0}(\vec{x}))$. Denote by ψ the function with the graph $G(\varphi_j) + \vec{a}$ (note that due to the choice of N we have that ψ corresponds to one of the rows of M_N up to an additive constant). Then, in particular, we have that $\vec{x} \in \text{Sing}(\beta_{P_0}, \psi)$. Since we also have $\vec{x} \in \text{Sing}(\varphi_{\vec{a}}, \beta_{P_0})$ clearly we have $\vec{x} \in \text{Sing}(\varphi_{\vec{a}}, \psi)$ (indeed, since \vec{x} by Lemma 6 minimizes functions $\psi - \beta_{P_0}$

and $\beta_{P_0} - \varphi_{\vec{a}}$, it also minimizes their sum; note that all three functions are defined at the point \vec{x} , so all minimums exist). However, recall that \vec{a} is a solution to the system $M_N \odot \vec{y}$ and ψ corresponds to one of the rows of M_N . Thus, $|\text{Sing}(\varphi_{\vec{a}}, \psi)| \geq 2$. But any point minimizing $\psi - \varphi_{\vec{a}}$ should also minimize $\psi - \beta_{P_0}$ and $\beta_{P_0} - \varphi_{\vec{a}}$ (since \vec{x} does), hence any point in $\text{Sing}(\varphi_{\vec{a}}, \psi)$ should be also in both $\text{Sing}(\varphi_{\vec{a}}, \beta_{P_0})$ and $\text{Sing}(\beta_{P_0}, \psi)$. In particular, $|\text{Sing}(\beta_{P_0}, \psi)| \geq |\text{Sing}(\varphi_{\vec{a}}, \psi)|$ and thus there is another common point of $G(\varphi_j) + \vec{a}$ and the bottom of P_0 .

Since $G(\varphi_j) + \vec{a}$ touches P_0 at \vec{x} we have that any point in $|\text{Sing}(\beta_{P_0}, \psi)|$ lies in a face of P_0 incident to $(\vec{x}, \beta_{P_0}(\vec{x}))$. If it does not lie in the face Q_0 , then the minimal dimension face containing this point has a larger dimension than the dimension of Q_0 and we get the contradiction with the maximality property of $(\vec{x}, \beta_{P_0}(\vec{x}))$. Therefore, there are at least two common points of $G(\varphi_j) + \vec{a}$ and Q_0 . Hence any hyperplane H going through Q_0 and not intersecting the interior of P_0 is singular to the function corresponding to $G(\varphi_j) + \vec{a}$ and thus provides a root to f_j . Since the argument above works for all f_j and Q_0 does not depend on f_j , we get that H is singular to all f_1, \dots, f_k and thus, defines a root to the system F .

Remark 6 In the min-plus case the formulation of Lemma 13 remains the same. The proof also almost repeats the proof in the tropical case. We consider a solution \vec{a} to the min-plus Macaulay system, consider a singular point of $\varphi_{\vec{a}}$ and β_{P_0} with exactly the same maximization property as before. We consider the translation of arbitrary $G(\varphi_j)$ for arbitrary j as before. And here we have the only difference to the previous proof. The singularity of $\varphi_{\vec{a}}$ and ψ gives us that there is a white singular point and a black singular point (see Remark 3). Note that this might be the same point with two colors. Next, the same argument shows that the points of both colors are in Q_0 , which finishes the proof.

Thus Lemma 13(i) gives that over \mathbb{R} if a tropical linear system $M_N \odot \vec{y}$ has a solution, then the tropical polynomial system F has a root. This finishes the proof of Theorem 1(i). We will use Lemma 13(ii) in the proof of Theorem 1(ii) later. In the same way Theorem 2(i) follows from the min-plus analog of Lemma 13(i) as outlined in Remark 6. In Section 5 we give another proof for the min-plus dual Nullstellensatz with somewhat worse parameters deducing it directly from the tropical dual Nullstellensatz.

4.4 Examples

We provide several examples illustrating why the case of $n > 1$ in Theorem 1(i) is substantially harder than the case $n = 1$.

Stepped pyramid. In the case $n = 1$ it was actually shown in [11] that if we consider any solution $\{a_i\}_i$ to the infinite Macaulay system and we consider its coordinates a_i for the large enough i 's, then in some natural sense they already

form a hyperplane solution (that is a linear function in i). This hyperplane solution immediately gives a root to the polynomial system.

Now we show that this is not the case already for two variables $n = 2$.

To illustrate this consider a tropical polynomial f with the graph

$$G(f) = \{(0, 0, 0), (0, 1, 0), \\ (1, 0, 0), (1, 1, 0)\}.$$

Its convex hull is just a square.

Consider the tropical polynomial system consisting of one polynomial f . Consider the Macaulay linear system corresponding to $\{f\}$. For this system we will construct a solution $\varphi: \mathbb{Z}^2 \rightarrow \mathbb{R}$ such that even if we consider this solution $\varphi(x, y)$ only for large enough (in absolute values) variables (x, y) , the function φ is not linear no matter how large x and y are.

It is easier to describe the continuous version of the solution. The discrete solution is defined by the integer points of the continuous solution.

Let $S_k = \{(x, y) | 10(k-1) \leq |x|, |y| \leq 10k\}$ for $k = 1, 2, \dots$. For each odd k we let the solution $\varphi: \mathbb{R}^2 \rightarrow \mathbb{R}$ to be constant on S_k . For each even k we divide S_k into 4 regions by lines $y = x$ and $y = -x$. On the region with $x \geq |y|$ we let $\varphi(x, y) = x + C$, where C will be chosen later. Analogously for $x \leq -|y|$ we let $\varphi(x, y) = -x + C$, for $y \geq |x|$ let $\varphi(x, y) = y + C$ and for $y \leq -|x|$ let $\varphi(x, y) = -y + C$. We choose constants in these linear and constant functions in such a way that φ is continuous on the whole real plane.

It is not hard to see that the graph of φ is singular to all the translation of the convex hull of $G(f)$ and thus, is a solution to the Macaulay matrix. Indeed, consider an arbitrary translation of f by a vector (a, b) and denote the corresponding function by ψ . Its domain is $D = \{(a, b), (a, b + 1), (a + 1, b), (a + 1, b + 1)\}$. Consider a singular point (c, d) in $\text{Sing}(\varphi, \psi)$. Without loss of generality assume that $|c| \geq |d|$. Then it is easy to see that another point in D with the first coordinate c is also a singular point. This can be done by a direct computation with Lemma 6: the value of ψ is the same in both points and the value of φ in the second point is greater or equal than the value of φ in the first point.

Stripes. Now we provide an example demonstrating that solutions of the Macaulay system can behave wildly. Specifically, we describe almost everywhere “non-continuous” solution, that is the solution having arbitrary large gaps in the neighboring points. For this example also $n = 2$ variables suffice.

Consider a polynomial f with

$$G(f) = \{(0, 0, 0), (0, 1, -1), (0, 2, 0), \\ (1, 0, 0), (1, 1, -1), (1, 2, 0)\}.$$

The shape of the convex hull of this graph is a prism.

Consider a set of points described by the following function $\varphi: \mathbb{Z}^2 \rightarrow \mathbb{R}$:

$$\varphi(x, y) = \begin{cases} y, & \text{if } \lfloor x/2 \rfloor \text{ is even;} \\ -y, & \text{if } \lfloor x/2 \rfloor \text{ is odd.} \end{cases}$$

It is not hard to see that the graph of φ is singular to the convex hull of all the translation of $G(f)$. Indeed, consider a translation of φ by a vector (a, b) and denote the corresponding function by ψ . Its domain is the set $D = \{(a, b), (a, b + 1), (a, b + 2), (a + 1, b), (a + 1, b + 1), (a + 1, b + 2)\}$. Consider a singularity point $\vec{\alpha}$ in $\text{Sing}(\varphi, \psi)$. Without loss of generality assume that its x -coordinate is a . Then it turns out that one of the neighboring points in D with x -coordinate a is also a singular point. This can be seen by a direct computation with Lemma 6.

Thus, the vector $\{\varphi(I)\}_{|I| \leq N}$ is a solution to the Macaulay system corresponding to f . On the other hand, note that the gaps between the values of φ in the neighboring points in \mathbb{Z}^2 can be arbitrarily large.

4.5 Tropical and Min-plus Dual Nullstellensätze over \mathbb{R}_∞

In this subsection we prove the following more precise version of Theorem 1(ii).

Theorem 5 *Over the semiring \mathbb{R}_∞ a system of tropical polynomials $F = \{f_1, \dots, f_k\}$ of degree at most d and in n variables has a root iff the Macaulay tropical non-homogeneous linear system with the matrix M_N for*

$$N = 2(n + 2)^2 k (4d)^{\min(n, k) + 2}$$

has a solution.

Proof Suppose we have a system of tropical polynomials F and consider the corresponding Macaulay tropical non-homogeneous linear system $M_N \odot (0, \vec{y})$. We have already shown one direction: if F has a root then $M_N \odot (0, \vec{y})$ also has a solution.

Suppose in the opposite direction that we have a solution \vec{a} to the non-homogeneous system with the matrix M_N . This means that there is a solution \vec{a} with the finite coordinate corresponding to constant monomial. If for the enveloping polytope P_0 constructed in Subsection 4.2 there is $\vec{x} \in \mathbb{Z}^n$ such that $\vec{x} \in \text{Dom } \beta_{P_0} \cap \text{Dom } \varphi_{\vec{a}}$ then we can directly apply Lemma 13(ii). But initially we know only that $\vec{0} \in \text{Dom } \varphi_{\vec{a}}$ and it can be that $\vec{0} \notin \text{Dom } \beta_{P_0}$ (and there can be no translation $P_0 + \vec{\alpha}$ of P_0 within \mathbb{Z}_+^n such that $\vec{0} \in \text{Dom } \beta_{P_0 + \vec{\alpha}}$). Below we describe how we deal with this problem.

Consider the column of M_N corresponding to the constant monomial. If it has no finite entry, the non-homogeneous Macaulay system has the infinite solution. At the same time the system of polynomials also has the infinite solution. Indeed, note that no polynomial in the system in this case has a finite constant term. So, this case is simple and further we can assume that the column of M_N corresponding to the constant monomial has a finite entry. This means that there is a polynomial in F with a finite constant term. For simplicity of notation assume that it is f_1 .

Now based on the system of polynomials F we construct a system of polynomials F' such that

1. each polynomial in F' is a (tropical) algebraic combination of polynomials in F ;
2. each polynomial in F' has a finite constant term;
3. F' has a root iff F also has a root;
4. the number K of polynomials in F' is at most $(n+2)k$ and the maximal degree d' of polynomials in F' is at most $2(4d)^{\min(n,k)+2}$.

Claim If a family of polynomials F' with the properties described above exists then Theorem 5 follows.

Proof (Proof of the claim) We only need to show the opposite direction of Theorem 5. Consider the solution \vec{a} to the system $M_N \odot (0, \vec{y})$. Consider the non-homogeneous Macaulay matrix M'_N corresponding to F' . Since all the polynomials in F' are tropical algebraic combinations of the polynomials in F , the rows of M'_N are tropical linear combinations of the rows of M_N . Hence \vec{a} is a solution to $M'_N \odot (0, \vec{y})$. Consider the extended Newton polytopes P'_1, P'_2, \dots, P'_K for the polynomials in the system F' and consider the enveloping polytope P'_0 (see Subsection 4.2). Note that for each function $f \in F'$ we have $\vec{0} \in \text{Dom } \varphi_f$. Thus the same is true for the corresponding polytopes and for the enveloping polytope P'_0 as well. Note that we have $N = 2(n+2)^2 k (4d)^{\min(n,k)+2} = (n+2)Kd'$. Therefore Lemma 13(ii) is applicable and we obtain a root $\vec{b} = (-b_1, \dots, -b_n) \in \mathbb{R}^n$ for F' . This in turn implies that F has a root.

Thus to finish the proof of Theorem 5 it is left to construct the system of polynomials F' and ensure its properties.

The idea for the construction of F' is to incorporate f_1 into all polynomials f_i in F in order to insure that their constant terms are finite. That is, each polynomial in F' will be a (tropical) sum of f_1 and an algebraic combination of other polynomials in F . In particular, an extended Newton polytope of a polynomial in F' is a convex hull of a union of P_1 with some translations of other polytopes among P_2, \dots, P_k . However, we would like to avoid new roots for the system F' . If some face of P_1 is a face for all polytopes of the polynomials in F' it gives a new root to F' . So we have to construct new polytopes P'_i in such a way that as much as possible of the vertices of P_1 lie in the interior of P'_i and thus to reduce the number of faces of P_1 in the polytope P'_i . The necessity to avoid new roots in F' adds more technical complications.

We now proceed to the construction of F' .

Below we will need the following value:

$$\Delta = \max_{1 \leq i_1, i_2 \leq k} \max_{\vec{x} \in \text{Dom } \varphi_{i_1}, \vec{y} \in \text{Dom } \varphi_{i_2}} |\varphi_{i_1}(\vec{x}) - \varphi_{i_2}(\vec{y})|. \quad (5)$$

Informally, it measures the maximal joint variation of φ -functions for the system F .

We also can assume that $\min_I \varphi_i(I) = 0$ for all i since adding (in the classical sense) a constant to each coefficient of a polynomial does not change singularity.

To construct F' we first for all $i = 2, \dots, k$ and $j = 1, \dots, n$ define polynomials of the following form:

$$g_{ij} = (-C) \odot x_j^\alpha \odot f_i.$$

Here the parameters C and α can be fixed in the following way:

$$C = 3\Delta(4d)^{2\min(n,k)+2}, \quad \alpha = (4d)^{\min(n,k)+2}.$$

Next for all $i > 1$ we define

$$f'_i = f_1 \oplus g_{i1} \oplus g_{i2} \oplus \dots \oplus g_{in}. \quad (6)$$

Also for each $i = 2, \dots, k$ and $j = 1, \dots, n$ we introduce a polynomial

$$f'_{ij} = f_1 \oplus g_{i1} \oplus g_{i2} \oplus \dots \oplus g_{i,j-1} \oplus (-1) \odot g_{ij} \oplus g_{i,j+1} \oplus \dots \oplus g_{in},$$

that is the difference between f'_i and f'_{ij} is that in the latter the coefficient of $x_j^\alpha \odot f_i$ is $-C - 1$ instead of $-C$. We let

$$F' = \{f_1\} \cup \{f'_i \mid i = 2, \dots, k\} \cup \{f'_{ij} \mid i = 2, \dots, k, j = 1, \dots, n\}.$$

Overall, F' consists of $K = (n+1)(k-1)+1 \leq (n+2)k$ polynomials of degree at most $\alpha + d \leq 2(4d)^{\min(n,k)+2}$. For a function f'_i we denote its extended Newton polytope by P'_i and the function $\varphi_{f'_i}$ by φ'_i . Analogously, for a function f'_{ij} we denote its extended Newton polytope by P'_{ij} and the function $\varphi_{f'_{ij}}$ by φ'_{ij} .

The tropical summands of the sum (6) will be called below the *components* of the polynomial f'_i . We specifically distinguish f_1 -component. All other components are called f_i -components. When we need to distinguish them, the component g_{ij} will be called the j -th component of f'_i .

All properties of F' are clear from the construction except the property 3. Moreover, one direction of the property 3 is simple: since F' consists of algebraic combinations of polynomials of F , any root of F is also a root of F' . Thus it is left to show the following lemma.

Lemma 14 *If there is a root to the system F' then there is a root to the system F .*

The proof of this lemma has a geometric intuition, but it is not easy to see the intuition behind the technical details. So, before proceeding with the proof we would like to explain this intuition in the case of $n = 2$ and $k = 3$. After that we provide a formal proof for the general case.

Informal proof for $n = 2$ and $k = 3$. Informally it is convenient to think of constants C and α as of very large parameters to be fixed later. In the formal proof of Lemma 14 we will show that the values of C and α specified above suffice.

It is instructive to look at the extended Newton polytope P'_2 corresponding to the function f'_2 . It can be obtained in two steps: we first take a union of the graph of φ_1 and of two copies of the graph of φ_2 translated far away along each of the axes x_1 and x_2 and far below along the vertical axis and then take an extended Newton polytope of the result. The idea behind the construction of f'_2 is that all the points of the polytope P_1 (corresponding to φ_1) except possibly the points on x_1 -axis and x_2 -axis are in the interior of the polytope P'_2 .

We will explain the presence of the polynomials $f'_{21}, f'_{22}, f'_{31}, f'_{32}$ in F' once we actually need them.

Assume that there is a root $\vec{b} = (-b_1, -b_2)$ to the system F' . Note that since each polynomial in F' has a finite constant monomial, the value of all the polynomials in F' is finite on any input. So we can assume that $\vec{b} \in \mathbb{R}^2$. If it is not the case, just substitute infinite coordinates of \vec{b} by large enough finite numbers. Recall, that the root corresponds to the plane $\chi_{\vec{b}}(\vec{x}) = b_1x_1 + b_2x_2$ (in 3-dimensional space) singular to φ_f for all the polynomials $f \in F'$. The first attempt would be to deduce that this hyperplane is also singular to functions $\varphi_1, \varphi_2, \varphi_3$, corresponding to polynomials f_1, f_2, f_3 . We already know that it is singular to φ_1 since $f_1 \in F'$. To show that it is singular to φ_2 and φ_3 we look closer at polynomials f'_2 and f'_3 . Without loss of generality let us consider f'_2 .

We know that the hyperplane $\chi_{\vec{b}}$ has at least two singular points with φ'_2 . Note that singularity points can be chosen among the vertices of P'_2 . However, if two singular points belong to different components of f'_2 it does not give us anything about the singularity of $\chi_{\vec{b}}$ to any φ_i . Thus, we would like to show that there are two singular points in one of the components of f'_2 . Suppose that there is at most one singular point in each component of f'_2 . We note that if there is at least one singular point in f_1 -component, then there are two singular points there, since the hyperplane $\chi_{\vec{b}}$ is singular to φ_1 . The case when the hyperplane $\chi_{\vec{b}}$ has only one singular point in one of f_2 -components is precisely the case, where we need polynomials f'_{21}, f'_{22} . Indeed, it is not hard to see that in this case one of these polynomials has only one singular point overall, and thus the hyperplane is not singular to φ'_{21} or φ'_{22} .

Thus, we have that each of the polynomials f'_2 and f'_3 has at least two singular points in the same component. If these are f_2 -component and f_3 -component respectively then we are done: clearly, the hyperplane is singular to both φ_2 and φ_3 . Thus, it is left to consider the case when one of the polynomials (or both) has two singular points in f_1 -component.

Here we encounter a serious obstacle. For example, assume that $\text{Dom } \varphi_2$ and $\text{Dom } \varphi_3$ do not intersect $\{(t, 0) \mid t \in \mathbb{R}\}$ — the set of points on x_1 -axis. Then the hyperplane having two singular points with φ_1 on the x_1 -axis and decreasing dramatically along the x_2 -axis provides a root to F' , but not necessarily to F .

Thus, it is not always true that a root of F' constitutes a root to F . However, in the example described above we can replace b_2 by $b'_2 = -\infty$ to obtain a root \vec{b}' for F .

It turns out that this trick with some additional work is enough to finish the proof. Indeed, suppose that the singular points of the hyperplane $\chi_{\vec{b}}$ and, say, φ'_2 are in f_1 -component. Then it is not hard to see that all these singular points lie either on x_1 -axis, or on x_2 -axis. Indeed, for any point $\vec{a} \in \text{Dom } \varphi'_2$ with both positive coordinates the point $(\vec{a}, \varphi'_2(\vec{a})) \in \mathbb{R}^3$ lies in P_1 and thus is in the interior of P'_2 by the construction of φ'_2 . Thus, \vec{a} is not a singular point. If on the other hand, there is a singular point \vec{a} with a positive x_1 -coordinate and another singular point \vec{a}' with a positive x_2 -coordinate, then the point $((\vec{a} + \vec{a}')/2, (\varphi_1(\vec{a}) + \varphi_1(\vec{a}'))/2)$ lie in P_1 due to its convexity and thus lie in the interior of P'_2 , which contradicts to the singularity of \vec{a} and \vec{a}' . Thus this case is also impossible and all singular points lie on one of the axes.

Without loss of generality assume that all the singular points for f'_2 lie on the x_1 -axis. Since there are at least two singular points on this axis in the f_1 -component we have that b_1 is not too large and not too small, or more formally, its absolute value is bounded by Δ (and thus, does not depend on C and α). Since we are allowed to fix C as large as we want, this in particular means that $\text{Dom}(\varphi_2)$ does not intersect the x_1 -axis. Otherwise, the singular point of the hyperplane $\chi_{\vec{b}}$ with φ'_2 would be in the 1-component and not in the f_1 -component. Thus, to obtain a root of the system $\{f_1, f_2\}$ we can just let $b'_2 = -\infty$ to obtain a new potential root $\vec{b}' = (-b_1, -b'_2)$.

We would like to stress here that at this point we have shown the theorem for the case $k = 2$. However we need one more observation for the case $k = 3$.

Consider the other polynomial f'_3 . If the domain of φ_3 also does not intersect the x_1 -axis, then \vec{b}' is indeed a root of f'_3 .

Thus we can assume that there is a point \vec{y} in $\text{Dom}(\varphi_3)$ on the x_1 -axis. Then just like in the case of φ'_2 the singular points of φ'_3 are not in the f_1 -component. Thus, they are in some f_3 -components and thus, $\chi_{\vec{b}}$ is singular to φ_3 itself. But we have set $b'_2 = -\infty$ and the singularity might not be translated to \vec{b}' . This happens if there is only one singular point in $\text{Sing}(\chi_{\vec{b}}, \varphi_3)$ on the x_1 -axis. So there is another singular point \vec{z} in $\text{Sing}(\chi_{\vec{b}}, \varphi_3)$ not on the x_1 -axis.

Consider both points $\vec{y}, \vec{z} \in \mathbb{Z}^2$ on the two-dimensional grid. To get from \vec{y} to \vec{z} in this grid we have to make several (at most d) steps along x_1 -axis in the positive or the negative direction and at least one step in the positive direction along the x_2 -axis. During this path the value of φ_3 and thus of $\chi_{\vec{b}}$, cannot decrease by more than Δ . Indeed, since \vec{z} is a singular point the difference of the values of $\chi_{\vec{b}}$ is bounded from below by the difference of the value of φ_3 at the same points. Since the absolute value of b_1 is also bounded, from this we can deduce that b_2 is bounded from below by some value depending only on F .

Now choosing C large enough we can get a contradiction with the assumption that the singular points of $\varphi_{f'_2}$ are in the f_1 -component: both b_1 and b_2

are not too small and if we place f_2 -components low enough the singular point will be in one of these components.

This proof (with some additional technical tricks) can be extended to the general case.

Next we proceed to the formal proof of Lemma 14.

Proof of Lemma 14. The plan is to consider a root of F' and replace some of its coordinates by the infinity. Below we describe how to choose the appropriate set of these coordinates. The construction is rather straightforward: we only keep the coordinates which we have a reason to keep and the others replace by the infinity.

Consider a root $\vec{b} = (-b_1, \dots, -b_n) \in \mathbb{R}^n$ of F' . We can assume that the root is in \mathbb{R}^n by the same argument as in the informal proof. As discussed in Section 4.1 this means that the hyperplane $\chi_{\vec{b}}(\vec{x}) = \sum_i b_i x_i$ is singular to all φ_f for $f \in F'$.

Note that for each polynomial f'_i there are two singularity points in the same component. Indeed, if this is not the case consider a j -component with one singularity point and consider the polynomial f'_{ij} . It has only one singularity point which is a contradiction (the same arguments works for the f_1 -component: we should consider the polynomial f_1 in this case).

Below for a set $T \subseteq \mathbb{R}^n$ and for a set $S \subseteq \{1, \dots, n\}$ we denote by $T|_S$ the set of points $\vec{x} \in T$ such that $x_j = 0$ for all $j \notin S$.

We define a sequence of sets of the coordinates in the following iterative way. First consider the set $\text{Sing}(\chi_{\vec{b}}, \varphi_1)$ of the singularity points for $\chi_{\vec{b}}$ and φ_1 . We let $j \in S_0$ iff there is $\vec{x} \in \text{Sing}(\chi_{\vec{b}}, \varphi_1)$ such that $x_j \neq 0$. Suppose we have defined S_l by recursion on $l \geq 0$. If there is a polynomial $f_i \in F$ such that $|\text{Sing}(\chi_{\vec{b}}, \varphi_i)| > |\text{Sing}(\chi_{\vec{b}}, \varphi_i)|_{S_l}$ and $\text{Dom}(\varphi_i)|_{S_l} \neq \emptyset$ then we define S_{l+1} letting $S_l \subseteq S_{l+1}$ and $j \in S_{l+1} \setminus S_l$ iff there is $\vec{x} \in \text{Sing}(\chi_{\vec{b}}, \varphi_i)$ such that $x_j \neq 0$. If there is no such f_i the process stops.

This procedure results in a sequence S_0, S_1, \dots, S_r and in the corresponding sequence of polynomials g_1, g_2, \dots, g_r , where for each l we have $g_l \in F$. For the sake of convenience denote $g_0 = f_1$. Note that $r \leq k$, since each polynomial from F can appear in the sequence at most once. Also $r \leq n$, since each S_l is a subset of $\{1, \dots, n\}$ and each next set is larger than the previous one. Thus $r \leq \min(n, k)$.

We can pose the following bounds on the coordinates of \vec{b} in S_r .

Claim For all $l = 0, \dots, r$ if there is $j \in S_l$ such that $b_j \leq -2\Delta(4d)^l$ then there is $j' \in S_l$ such that $b_{j'} \geq |b_j|/(4d)^{l+1}$.

Informally, if there is a very small b_j , then there is rather large $b_{j'}$.

Proof We argue by induction on l .

For the case of S_0 consider the coordinate j with $b_j \leq -2\Delta$ and consider $\vec{x} \in \text{Sing}(\chi_{\vec{b}}, \varphi_1)$ such that $x_j \neq 0$ (there is such an \vec{x} by the definition of S_0). Consider $\chi_{\vec{b}}(\vec{x}) - \chi_{\vec{b}}(\vec{0}) = \sum_p x_p b_p$. Since $\vec{x} \in \text{Sing}(\chi_{\vec{b}}, \varphi_1)$ by Lemma 6

$\chi_{\vec{b}}(\vec{x}) - \chi_{\vec{b}}(\vec{0}) \geq \varphi_1(\vec{x}) - \varphi_1(\vec{0}) \geq -\Delta$. Note that $x_j > 0$ (and thus $x_j \geq 1$) and for all p we have $x_p \geq 0$, so

$$\sum_{p \neq j} x_p b_p \geq -x_j b_j - \Delta \geq -b_j - \Delta \geq -b_j/2.$$

On the other hand, note that

$$\sum_{p \neq j} x_p b_p \leq \max_{p \neq j} b_p \sum_{p \neq j} x_p \leq d \max_{p \neq j} b_p.$$

Thus, there is j' such that $b_{j'} = \max_{p \neq j} b_p \geq -b_j/2d$.

For the induction step consider $j \in S_l$ such that $b_j \leq -2\Delta(4d)^l$. If $j \in S_{l-1}$, we are done by the induction hypothesis. Suppose $j \notin S_{l-1}$. Consider the polynomial g_l . By its definition we have the following

1. There is $\vec{y} \in \text{Dom}(\varphi_{g_l})|_{S_{l-1}}$. In particular, $y_j = 0$.
2. There is a singular point $\vec{x} \in \text{Sing}(\chi_{\vec{b}}, \varphi_{g_l})$ such that $x_j \neq 0$.

Consider $\chi_{\vec{b}}(\vec{x}) - \chi_{\vec{b}}(\vec{y}) = \sum_p (x_p - y_p) b_p$. Due to the singularity of \vec{x} by Lemma 6 we have $\chi_{\vec{b}}(\vec{x}) - \chi_{\vec{b}}(\vec{y}) \geq \varphi_{g_l}(\vec{x}) - \varphi_{g_l}(\vec{y}) \geq -\Delta$. Just like in the base of induction we have

$$\sum_{p \neq j} (x_p - y_p) b_p \geq -(x_j - y_j) b_j - \Delta \geq -b_j - \Delta \geq -b_j/2.$$

Let us partition the leftmost sum into two parts

$$\sum_{p \neq j, p \in S_{l-1}} (x_p - y_p) b_p + \sum_{p \neq j, p \notin S_{l-1}} (x_p - y_p) b_p \geq -b_j/2.$$

Note that in the second sum $(x_p - y_p)$ is nonnegative, because $y_p = 0$. Since

$$\sum_{p \neq j} |x_p - y_p| \leq 2d$$

there is either $p \notin S_{l-1}$ such that $b_p \geq -b_j/4d$, or $p \in S_{l-1}$ such that $|b_p| \geq -b_j/4d$. In the first case we are done immediately and in the second case we are done by the induction hypothesis.

To obtain the new root we fix all the coordinates of the root not in S_r to ∞ , that is we let $b'_j = -\infty$ if $j \notin S_r$ and $b'_j = b_j$ if $j \in S_r$. We denote the result by $\vec{b}' = (-b'_1, \dots, -b'_n) \in \mathbb{R}_\infty^n$.

We claim that \vec{b}' is a root of F .

Indeed, suppose on the contrary that there is a polynomial $f_i \in F$ such that there is only one $\vec{z} \in \text{Sing}(\chi_{\vec{b}'}, \varphi_i)$. Clearly, $\vec{z} \in \mathbb{Z}^n|_{S_r}$. Moreover, no other point can be a singular point of the original hyperplane $\chi_{\vec{b}}$ with φ_i . Indeed, other singular points can be only outside of $\mathbb{Z}^n|_{S_r}$ and if there is at least one, then following our construction we would have added some more coordinates to S_r . Thus, there is only one singular point in $\text{Sing}(\chi_{\vec{b}'}, \varphi_i)$ and as a result

singular points of $\chi_{\vec{b}}$ with $\varphi_{f'_i}$ are in the f_1 -component. Let $\vec{y} \in \text{Sing}(\chi_{\vec{b}}, \varphi_{f'_i})$ be one of these singular points. Note that by the definition of S_0 we have $\vec{y} \in \mathbb{Z}^n|_{S_0} \subseteq \mathbb{Z}^n|_{S_r}$. We are going to get a contradiction.

Let $Min = \min_{j \in S_r} b_j$ and $Max = \max_{j \in S_r} b_j$. Consider j and j' such that $b_j = Min$ and $b_{j'} = Max$. Consider the j' -component of f'_i and let \vec{x} be the translation of \vec{z} in this component, that is $\vec{x} = \vec{z} + \alpha \cdot e_{j'}$. Since $\vec{z} \in \mathbb{Z}^n|_{S_r}$ and $j' \in S_r$ we have $\vec{x} \in \mathbb{Z}^n|_{S_r}$.

Our goal is to show that $\chi_{\vec{b}}(\vec{x}) - \chi_{\vec{b}}(\vec{y}) > \varphi_{f'_i}(\vec{x}) - \varphi_{f'_i}(\vec{y})$ which will contradict Lemma 6 since $\vec{y} \in \text{Sing}(\chi_{\vec{b}}, \varphi_{f'_i})$.

Note that

$$\varphi_{f'_i}(\vec{x}) - \varphi_{f'_i}(\vec{y}) = \varphi_{f'_i}(\vec{z} + \alpha \cdot e_{j'}) - \varphi_1(\vec{y}) = (\varphi_i(\vec{z}) - C) - \varphi_1(\vec{y}) \leq -C + \Delta,$$

where the second equality follows from the definition of f'_i (6) and the last inequality follows from the definition of Δ (5). Thus it is enough to show that $\chi_{\vec{b}}(\vec{x}) - \chi_{\vec{b}}(\vec{y}) > -C + \Delta$.

Note now that

$$x_{j'} - y_{j'} = x_{j'} - z_{j'} + z_{j'} - y_{j'} \geq \alpha - 2d$$

and

$$\sum_{p \neq j'} |x_p - y_p| = \sum_{p \neq j'} |z_p - y_p| \leq 2d.$$

Consider the sum

$$\begin{aligned} \chi_{\vec{b}}(\vec{x}) - \chi_{\vec{b}}(\vec{y}) &= \\ & \sum_{p \neq j', x_p - y_p > 0} (x_p - y_p) b_p + \sum_{p \neq j', x_p - y_p \leq 0} (x_p - y_p) b_p + (x_{j'} - y_{j'}) b_{j'} \geq \\ Min \cdot \sum_{p \neq j', x_p - y_p > 0} (x_p - y_p) + Max \cdot \sum_{p \neq j', x_p - y_p \leq 0} (x_p - y_p) + Max \cdot (x_{j'} - y_{j'}). \end{aligned}$$

If $Max \leq 0$, then by the claim above we have $Min \geq -2\Delta(4d)^r$ and thus $Max \geq -2\Delta(4d)^r$. We have

$$\begin{aligned} \chi_{\vec{b}}(\vec{x}) - \chi_{\vec{b}}(\vec{y}) &\geq \\ & -2d2\Delta(4d)^r + 0 - 2\Delta(4d)^r(\alpha - 2d) = -\alpha 2\Delta(4d)^r. \end{aligned}$$

If $Max \geq 0$ we have

$$\begin{aligned} \chi_{\vec{b}}(\vec{x}) - \chi_{\vec{b}}(\vec{y}) &\geq \\ Min \cdot \sum_{p \neq j', x_p - y_p > 0} (x_p - y_p) - Max \cdot 2d + Max \cdot (\alpha - 2d) &= \\ Min \cdot \sum_{p \neq j', x_p - y_p > 0} (x_p - y_p) + Max \cdot (\alpha - 4d). \end{aligned}$$

If $Min \geq -2\Delta(4d)^r$ this sum is greater than $-2\Delta 2d(4d)^r$.

If $Min \leq -2\Delta(4d)^r$ then by Claim 4.5 $Max \geq -Min/(4d)^{r+1}$ and we have

$$\begin{aligned} \chi_{\vec{b}}(\vec{x}) - \chi_{\vec{b}}(\vec{y}) &\geq \\ 2dMin - (\alpha - 4d)Min/(4d)^{r+1} &\geq 0. \end{aligned}$$

In all these cases $\chi_{\vec{b}}(\vec{x}) - \chi_{\vec{b}}(\vec{y})$ is greater than $-C + \Delta$ and we have a contradiction with the singularity of \vec{y} .

This finishes the proof of Lemma 14, Theorem 5 and thus Theorem 1(ii).

Remark 7 For the min-plus case the construction of the system F' and its properties are translated with the obvious changes only (a min-plus polynomial is a pair of tropical polynomials; there is a pair of matrices instead of one Macaulay matrix). The proof of Claim 4.5 also translates with the obvious changes only. We do not attempt to translate informal proof since this does not add to the intuition. The formal proof is given in terms of φ_i and φ'_i functions, so it also translates easily. The only difference is that instead of the property “to have only one singular point” we use the property “to have singular points of only one color” (see Remark 3). The rest of the proof remains the same. In Section 5 we give another proof for the min-plus dual Nullstellensatz with somewhat worse parameters deducing it directly from the tropical dual Nullstellensatz.

4.6 Lower Bounds

In this subsection we provide examples showing that our bounds on N in Theorem 1 are not far from being optimal. At the same time we provide the similar lower bounds for Theorem 2. We will translate these lower bounds to Theorems 3 and 4 in Section 6.

Lower bound for Theorem 1(i) First we show a lower bound for the case of \mathbb{R} . Namely for any $d \geq 2$ we provide a family F of $n+1$ polynomials in n variables and of degree at most d such that F has no root, but the corresponding Macaulay system $M_{(d-1)(n-1)} \odot \vec{y}$ has a solution.

The construction is an adaptation of the known example for a lower bound for classical Nullstellensatz due to Lazard, Mora and Philippon (unpublished, see [5, 16]).

Consider the following set F of tropical polynomials

$$\begin{aligned} f_1 &= 0 \oplus 0 \odot x_1, \\ f_{i+1} &= 0 \odot x_i^{\odot d} \oplus 0 \odot x_{i+1}, \quad 1 \leq i \leq n-1 \\ f_{n+1} &= 0 \oplus 1 \odot x_n. \end{aligned}$$

It is not hard to see that this system has no roots. Indeed, if there is a root, then from f_1 we get that $x_1 = 0$, then from f_2 we get that $x_2 = 0$ etc., finally from f_n we conclude that $x_n = 0$. However from f_{n+1} we have that $x_n = -1$ which is a contradiction.

Thus, it remains to show that the Macaulay tropical system with the matrix $M_{(d-1)(n-1)}$ corresponding to the system F has a solution.

Recall that the columns of $M_{(d-1)(n-1)}$ correspond to the monomials. We associate an undirected graph G to the matrix $M_{(d-1)(n-1)}$ in a natural way. The vertices of G are the monomials in the variables x_1, \dots, x_n of degree at most $(d-1)(n-1)$ (or, which is the same, the columns of $M_{(d-1)(n-1)}$). We connect two monomials by an edge if they occur in the same polynomial of the form $\vec{x}^I \odot f_i$ for $i \in \{1, \dots, n\}$. Or, to state it the other way, we connect two monomials if there is a row of $M_{(d-1)(n-1)}$ not corresponding to a polynomial f_{n+1} and such that the entries in the columns corresponding to these monomials are both finite in this row.

We assign the weight $w(m)$ to a tropical monomial m in the following way. First, $w(x_i) = d^{i-1}$ for all $i = 1, \dots, n$. Second, for all the monomials m_1 and m_2 we let $w(m_1 \odot m_2) = w(m_1) + w(m_2)$. That is, if $m = x_1^{\odot a_1} \odot \dots \odot x_n^{\odot a_n}$ then $w(m) = a_1 + a_2 d + \dots + a_n d^{n-1}$.

It turns out that the following lemma holds.

Lemma 15 *If for two monomials m_1 and m_2 we have $w(m_1) \geq kd^{n-1}$ and $w(m_2) < kd^{n-1}$ for some integer k , then m_1 and m_2 are not connected in G .*

Proof Note that if two monomials are connected by an edge corresponding to one of the polynomials f_2, \dots, f_n , then their weights coincide. If they are connected by an edge corresponding to f_1 , then their weights differ by 1.

Note that for an arbitrary k any monomial of the weight $kd^{n-1} - 1$ has the degree at least $(k-1) + (d-1)(n-1)$. Indeed, consider such a monomial $m = x_1^{\odot a_1} \odot \dots \odot x_n^{\odot a_n}$ of the minimal degree. If there is $i = 1, \dots, n-1$ such that $a_i \geq d$, then we can replace a_i by $a_i - d$ and a_{i+1} by $a_{i+1} + 1$ and obtain another monomial of the same weight but with a smaller degree. Thus (a_1, \dots, a_{n-1}) corresponds to d -ary representation of the residue of $kd^{n-1} - 1$ modulo d^{n-1} . So for all $i = 1, \dots, n-1$ we have $a_i = d-1$ and thus $a_n = k-1$.

Due to the restriction on the degree in the graph G there is only one monomial of the weight $d^{n-1} - 1$ and no monomials of the weight $kd^{n-1} - 1$ for $k > 1$. Moreover, the unique monomial of the weight $d^{n-1} - 1$ has the maximal degree $(d-1)(n-1)$ and thus, is not connected to a monomial of a higher weight by an edge.

From all this the lemma follows. Indeed, if monomials m_1 and m_2 are connected, then on the path between them there is an edge connecting monomials of the weights $kd^{n-1} - 1$ and kd^{n-1} . However, as we have shown, this is impossible.

Now we are ready to provide a solution to the Macaulay system $M_{(d-1)(n-1)} \odot \vec{y}$. For a monomial of weight $kd^{n-1} + s$, where $s < d^{n-1}$, set the corresponding variable of \vec{y} to k . Note that due to Lemma 15 if two monomials are connected, then the values of the corresponding variables are the same. Thus, the constructed \vec{y} satisfies the equations of $M_{(d-1)(n-1)} \odot \vec{y}$ corresponding to the polynomials f_1, \dots, f_n . The rows corresponding to f_{n+1} are satisfied since the weights of monomials differ by precisely d^{n-1} .

Remark 8 For the case of min-plus polynomials (see Theorem 2(i)) a straightforward adaptation works. Indeed, since in each polynomial there are only two monomials, the only way to satisfy them is to make their values equal. Thus, it is enough to consider a system of min-plus polynomials F

$$\begin{aligned} &(0, 0 \odot x_1), \\ &(0 \odot x_i^{\odot d}, 0 \odot x_{i+1}), \quad 1 \leq i \leq n-1, \\ &(0, 1 \odot x_n). \end{aligned}$$

Lower bound for Theorem 1(ii) Now we show a lower bound for the case of \mathbb{R}_∞ .

Consider the following system F of tropical polynomials in the variables x_1, \dots, x_n, y .

$$\begin{aligned} f_1 &= 0 \odot x_1 \odot y \oplus 0, \\ f_{i+1} &= 0 \odot x_i^{\odot d} \oplus 0 \odot x_{i+1}, \text{ for } i = 1, \dots, n-1, \\ f_{n+1} &= 0 \odot x_{n-1}^{\odot d} \oplus 1 \odot x_n. \end{aligned}$$

This system clearly has no roots. Indeed, assuming the contrary we can consecutively show that all the coordinates of a root should be finite and then the polynomials f_n and f_{n+1} give a contradiction.

Now consider the Macaulay non-homogeneous system $M_{d^{n-1}-1} \odot (0, \vec{z})$. We are going to construct a solution for it. For a tropical monomial $x_1^{a_1} \dots x_n^{a_n} y^b$ let its weight be

$$a_1 + da_2 + d^2a_3 + \dots + d^{n-1}a_n$$

Note that the degree in y is not counted. Consider all the monomials whose y -degree coincides with their weight and let the corresponding coordinates of \vec{z} be equal to 0. For all other monomials let the corresponding coordinates of \vec{z} be equal to ∞ . We show that indeed, this provides a solution to $M_{d^{n-1}-1} \odot (0, \vec{z})$. Consider the graph on the coordinates of $(0, \vec{z})$ in which two coordinates are connected if the corresponding monomials appear in the same row of the Macaulay matrix $M_{d^{n-1}-1}$. It is not hard to see that all the monomials on which $(0, \vec{z})$ is finite constitute a connected component of the graph, containing the coordinate corresponding to the constant monomial. Moreover, due to the constraint on the size of the matrix, no monomials in this component contain x_n variable. Thus, all the rows of the Macaulay matrix are satisfied.

Remark 9 For the min-plus case (see Theorem 2(ii)) note that the same observation as in Remark 8 works. Just consider the system of min-plus polynomials

$$\begin{aligned} &(0 \odot x_1 \odot y, 0), \\ &(0 \odot x_i^{\odot d}, 0 \odot x_{i+1}), \text{ for } i = 1, \dots, n-1, \\ &(0 \odot x_{n-1}^{\odot d}, 1 \odot x_n). \end{aligned}$$

5 Tropical polynomials vs. Min-plus polynomials

In this section we show that there is a tight connection between the sets of roots of systems of min-plus polynomials and of tropical polynomials. We will later use this connection to obtain the min-plus dual Nullstellensatz.

The connection is given by Lemmas 4 and 5. We proceed to the proofs of these lemmas.

Proof (Proof of Lemma 4)

Let T be some tropical polynomial system over \mathbb{R} . For each polynomial $f \in T$ we construct a min-plus polynomial system over the same set of variables which is equivalent to f .

For this let

$$f = \min\{l_1, l_2, \dots, l_m\}, \quad (7)$$

where l_i 's are tropical monomials.

It is easy to see that the minimum in (7) is attained at least twice iff for all $i = 1, \dots, m$ it is true that

$$\begin{aligned} \min\{l_1, \dots, l_{i-1}, l_i, l_{i+1}, \dots, l_m\} = \\ \min\{l_1, \dots, l_{i-1}, l_{i+1}, \dots, l_m\}. \end{aligned}$$

These equations are min-plus polynomial equations and thus, we have that any tropical polynomial is equivalent to a system of min-plus polynomials. To get a min-plus system equivalent to the tropical system we just unite the min-plus systems for all the polynomials of T .

Exactly the same analysis works for the case of \mathbb{R}_∞ and also for the cases of \mathbb{Q} and \mathbb{Q}_∞ . Clearly, the construction is effective.

Proof (Proof of Lemma 5)

In the opposite direction we do not have such a direct connection, but the connection we show below still preserves many properties.

We first for a given min-plus polynomial system F construct a corresponding tropical polynomial system T and then prove a relation between F and T .

Let us denote the variables of F by (x_1, \dots, x_n) . The tropical polynomial system T for each variable x_i of F will have two variables x_i and x'_i , thus, the set of the variables of T will be $(x_1, \dots, x_n, x'_1, \dots, x'_n)$.

The system T consists of the following polynomials.

1. For each $i = 1, \dots, n$ we add to T a polynomial

$$x_i \oplus x'_i.$$

2. Let $(\min_j m_j(\vec{x}), \min_p l_p(\vec{x}))$ be an arbitrary polynomial of F . For each p we add to T a tropical polynomial

$$\min(m_1(\vec{x}), m_1(\vec{x}'), \dots, m_k(\vec{x}), m_k(\vec{x}'), l_p(\vec{x})). \quad (8)$$

For each j we add to T a tropical polynomial

$$\min(l_1(\vec{x}), l_1(\vec{x}'), \dots, l_k(\vec{x}), l_k(\vec{x}'), m_j(\vec{x})). \quad (9)$$

We denote the monomials m_1, \dots, m_k in these polynomials by m -monomials. We denote the monomials l_1, \dots, l_k by l -monomials.

This completes the construction of T . Clearly, the construction is effective. Note that the maximal degree of polynomials in T is equal to the maximal degree of polynomials in F . Now we are ready to show how F and T are related.

Let $H(\vec{a}) = (\vec{a}, \vec{a})$ for all \vec{a} . Clearly, H is an injective linear transformation.

Note that the polynomial of T of the first type $x_i \oplus x'_i$ is satisfied iff $x_i = x'_i$. Thus, all the roots of T lie in the image of H .

If there is a root \vec{a} to F then it is easy to see that its image (\vec{a}, \vec{a}) under H satisfies all the polynomials of the form (8) and (9) in T . Indeed, since $\min_j m_j(\vec{a}) = \min_p l_p(\vec{a})$, then there is j such that $m_j(\vec{a}) = \min_p l_p(\vec{a})$. Then the minimum in the corresponding tropical polynomials (8) will be attained in the monomials $m_j(\vec{x})$ and $m_j(\vec{x}')$. The symmetric argument works for the tropical polynomials (9).

If there is a root of T , we already noted that it has the form (\vec{a}, \vec{a}) . Then it is not hard to see that for each min-plus polynomial of F we have $\min_j m_j(\vec{a}) = \min_p l_p(\vec{a})$. Indeed, since the corresponding tropical polynomials (8) are satisfied, we have that $\min_j m_j(\vec{a}) \leq \min_p l_p(\vec{a})$. On the other hand, the tropical polynomials (9) guarantee that

$$\min_j m_j(\vec{a}) \geq \min_p l_p(\vec{a}).$$

The proof works over the both semirings \mathbb{R} and \mathbb{R}_∞ as well as over \mathbb{Q} and \mathbb{Q}_∞ .

The sets of the roots of tropical (or min-plus) systems of the polynomials are called *tropical (respectively, min-plus) prevarieties*. In particular, it follows that the classes of tropical prevarieties and min-plus prevarieties are topologically equivalent.

To illustrate the possible applications of this connection we deduce another proof of the min-plus dual Nullstellensatz with somewhat worse values of N directly from the tropical dual Nullstellensatz.

We present the proof for the semiring \mathbb{R} . Exactly the same proof works also for \mathbb{R}_∞ .

As usually one direction is simple, that is if a system F has a root \vec{a} , then the Macaulay min-plus linear system $Ml_N \odot \vec{y} = Mr_N \odot \vec{y}$ for any N also has a solution: just let each coordinate y_I of \vec{y} to be equal to \vec{a}^I .

For the opposite direction, suppose the system $Ml_N \odot \vec{y} = Mr_N \odot \vec{y}$ for $N = (2n+2)(d_1 + \dots + d_k)$ has a solution \vec{a} . For the min-plus polynomial system F consider the corresponding tropical polynomial system T from Lemma 5. Let us denote by M'_N its Macaulay matrix. We will show that the tropical linear system $M'_N \odot \vec{z}$ has a solution. From this by Theorem 1 it follows immediately

that T has a root (note that the number of variables is $2n$, hence the change in the value of N), and by Lemma 5 we have that F has a root.

Thus, it is left to construct a solution to the tropical system $M'_N \odot \vec{z}$ based on the solution to the min-plus system $Ml_N \odot \vec{y} = Mr_N \odot \vec{y}$. The construction is straightforward: for each monomial $m(\vec{x}, \vec{x}')$ we partition it into two parts $m_1(\vec{x}) \odot m_2(\vec{x}')$ containing variables \vec{x} and \vec{x}' respectively, and we let the variable of \vec{z} corresponding to $m(\vec{x}, \vec{x}')$ be equal to the variable of \vec{y} corresponding to the monomial $m_1(\vec{x}) \odot m_2(\vec{x}')$.

Now we have to check that all the rows of the system $M'_N \odot \vec{z}$ are satisfied. This is obviously true for the rows corresponding to the polynomials $\vec{x}^I \odot \vec{x}'^{I'} \odot (x_i \oplus x'_i)$, since we clearly assign the same value to the variables corresponding to both monomials. For the polynomials of the form (8) we consider the corresponding equation in $Ml_N \odot \vec{y} = Mr_N \odot \vec{y}$. Since the minimum in them is attained in m -monomials in the corresponding row of $M'_N \odot \vec{z}$ the minimum will be attained in two corresponding m -monomials. The same works for the polynomials of the form (9).

6 Tropical and Min-plus Primary Nullstellensätze

Now we will deduce primary forms of the tropical and min-plus Nullstellensätze. We start with the min-plus primary Nullstellensatz.

Proof (Proof of Theorem 3) We will use the min-plus linear duality for the proof of this theorem. We start with the case of the semiring \mathbb{R} .

By Theorem 2(i) the system of polynomials F has no roots over \mathbb{R} iff the corresponding Macaulay linear system

$$Ml_N \odot \vec{y} = Mr_N \odot \vec{y} \quad (10)$$

has no finite solution. This system is equivalent to the system of min-plus inequalities

$$\begin{pmatrix} Ml_N \\ Mr_N \end{pmatrix} \odot \vec{y} \leq \begin{pmatrix} Mr_N \\ Ml_N \end{pmatrix} \odot \vec{y}.$$

Indeed, we just substituted each equation $\vec{a} \odot \vec{y} = \vec{b} \odot \vec{y}$ in (10) by a pair of inequalities $\vec{a} \odot \vec{y} \leq \vec{b} \odot \vec{y}$ and $\vec{a} \odot \vec{y} \geq \vec{b} \odot \vec{y}$.

By Corollary 3 this system has no finite solution iff the dual system

$$(Mr_N^T \ Ml_N^T) \odot \begin{pmatrix} \vec{z}_1 \\ \vec{z}_2 \end{pmatrix} < (Ml_N^T \ Mr_N^T) \odot \begin{pmatrix} \vec{z}_1 \\ \vec{z}_2 \end{pmatrix} \quad (11)$$

has a solution $(\vec{y}, \vec{z}) \neq (\infty, \dots, \infty)$ (recall that we allow for both sides to be infinite in some rows).

This system can be interpreted back in terms of polynomials. That is we claim that the linear system (11) has a solution iff there is an algebraic combination (f, g) of the min-plus polynomials in F such that for each monomial its coefficient in f is smaller than its coefficient in g . Indeed, note that the

columns of the matrices in (11) correspond to the equations of F multiplied by some monomial \bar{x}^J and the rows correspond to some monomials \bar{x}^I . Both sides of (11) can be viewed as a tropical sum of the columns of the matrices. Each column of the matrix represents the coefficients of one of the polynomials in F multiplied by a monomial. A tropical sum of these columns thus, represents the coefficients of an algebraic combination of polynomials in F . Thus, a solution to the system (11) corresponds to an algebraic combination of polynomials in F , such that each coefficient of the algebraic combination on the left-hand side is smaller than the respective coefficient of the algebraic combination on the right-hand side. The fact that we allow both sides to be infinite in some row corresponds to the fact that some monomials might not occur in the sum. The fact that we allow infinite coordinates in the solution to (11) corresponds to the fact that we do not have to use all the columns in the tropical sum, that is we do not have to use all the polynomials of the form $(\bar{x}^I \odot f_j, \bar{x}^I \odot g_j)$ in the algebraic combination.

The proof of the second part of the theorem is almost the same. The only difference is that this time we should use non-homogeneous Macaulay system, which by an application of Lemma 2 results in a linear combination of the polynomials with a finite constant term.

Now we proceed to the tropical primary Nullstellensatz.

Proof (Proof of Theorem 4) By Theorem 1(i) the system of polynomials F has no roots over \mathbb{R} iff the corresponding Macaulay system

$$M_N \odot \vec{y}$$

has no finite solution.

By Corollary 4 this is equivalent to the fact that there is $\vec{z} \neq (\infty, \dots, \infty)$ in \mathbb{R}_∞^n such that in each row of

$$M_N^T \odot \vec{z}$$

the minimum is attained only once or is equal to ∞ and for each two rows the minimums are in different columns. Recall, that each column in M_N^T corresponds to a polynomial $\bar{x}^J \odot f_j$ and the rows correspond to the monomials \bar{x}^I in these polynomials. Thus, \vec{z} corresponds to an algebraic combination of the polynomials of F and the properties of \vec{z} described above are equivalent to the singularity of the corresponding algebraic combination.

The proof of the \mathbb{R}_∞ case is completely analogous.

The lower bounds on N in Theorems 3 and 4 can be proved along the same lines as the proofs above by considering polynomial systems constructed in Subsection 4.6. It was shown there that for these polynomial systems the Macaulay linear systems of certain sizes have solutions. Using the tropical and the min-plus linear duality and interpreting the results in terms of polynomials just like in the proofs above we can show that these polynomial systems provide lower bounds for the tropical and min-plus primary Nullstellensätze.

7 Linear duality in min-plus algebra

7.1 Min-plus linear duality

In this subsection we prove Lemma 2 on the duality for min-plus linear systems.

The proof of this lemma is based on the interpretation of min-plus linear systems as mean payoff games. Namely, given a min-plus linear system we construct a mean payoff game G such that a solution to the system corresponds to a winning strategy for one of the players. This connection between min-plus linear systems and mean payoff games was established in [2]. We present the details here for the sake of completeness.

Given two matrices $A, B \in \mathbb{R}_\infty^{m \times n}$ the corresponding mean payoff game G can be described as follows. Consider a directed bipartite graph whose vertices on the left side are r_1, \dots, r_m and the vertices on the right side are c_1, \dots, c_n . The left-side vertices correspond to the rows of the matrices A and B and the right-hand side vertices correspond to the columns of the matrices. From each vertex r_i there is an edge to a vertex c_j labeled by $-a_{ij}$. From each vertex c_j there is an edge to a vertex r_i labeled by b_{ij} . We denote the label of an edge (v, u) by $w(v, u)$. Thus $w(r_i, c_j) = -a_{ij}$ and $w(c_j, r_i) = b_{ij}$. There are two players which we call the row-player and the column-player and who in turns are moving a token over the vertices of the graph. In the beginning of the game the token is placed to some fixed vertex. On each turn one of the two players moves the token to some other node of the graph. Each turn of the game is organized as follows. If the token is currently in some node r_i then the column-player can move it to any node c_j (the column-player chooses a column). If, on the other hand, the token is in some node c_j then the row-player can move the token to any node r_i (the row-player chooses a row). The game is infinite and the process of the game can be described by a sequence of nodes v_0, v_1, v_2, \dots which the token visits. The column-player wins the game if

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \sum_{i=1}^t w(v_{i-1}, v_i) > 0. \quad (12)$$

If this limit is negative then the row-player wins. If the limit is zero we have a draw. If some entries of the matrices A, B are infinite we assume that there are no corresponding edges in the graph. Alternatively, we can assume that there are edges labeled by ∞ and the player following such an edge losses immediately.

The process of the game can be viewed in the following way. After each move of the column-player he receives from the row-player some amount $-a_{ij}$ and after each move of the row-player he receives from the column-player some amount b_{ij} . The goal of both players is to maximize their amount. If one of them can play in such a way that his amount grows to infinity as the game proceeds, then he wins. And if the amounts of the players always stay between some limits, then the result of the game is the draw.

Note that if all the entries of the matrices are finite the game has a complete bipartite graph and it is easy to see that this property implies that the winner of the game does not depend on the starting position. The situation is different in the case of matrices with entries from \mathbb{R}_∞ .

For the constructed game G the following property holds. It is implicit in [2].

Lemma 16 *There is a finite solution to $A \odot \vec{x} \leq B \odot \vec{x}$ iff the column-player has a non-losing strategy starting from any position.*

There is a solution $\vec{x} \neq (\infty, \dots, \infty)$ to $A \odot \vec{x} \leq B \odot \vec{x}$ iff the column-player has a non-losing strategy starting from some position.

There is a solution to $A \odot \vec{x} \leq B \odot \vec{x}$ with a finite coordinate x_i iff the column-player has a non-losing strategy starting from the position c_i .

There is a solution to $A \odot \vec{x} \leq B \odot \vec{x}$ such that the j -th coordinate of $A \odot \vec{x}$ is finite iff the column-player has a non-losing strategy starting from the position r_j .

Proof We always can add the same number to all the coordinates of the solution. In particular, we have that there is a solution $\vec{x} \neq (\infty, \dots, \infty)$ to $A \odot \vec{x} \leq B \odot \vec{x}$ iff there is a solution such that all $x_j \geq 0$ and $\min_j x_j = 0$.

We are going to show that the existence of such a solution is equivalent to the existence of non-losing strategy for the column-player. The proof is very intuitive, but to make the intuition clear we have to explain what does \vec{x} mean in terms of the game. To do this assume that the column-player has a non-losing strategy starting from some position. We know that if the player follows the strategy, then his amount does not decrease to $-\infty$. But it might become negative at some moments of the game. For an arbitrary vertex c_j let us denote by x'_j the minimal amount such that if the game starts in c_j and the column-player has x'_j in the beginning then he can keep his amount above zero playing optimally. If in some position c_j the column-player has no winning strategy we naturally set $x'_j = \infty$. It turns out that the vector \vec{x}' is a solution to the min-plus linear system.

Indeed, suppose that the column-player has a non-losing strategy and consider \vec{x}' corresponding to it. Assume that we are in position c_j . Then for each move of the row-player (j, i) there is a move of the column-player (i, k) such that the remaining amount of the column-player after these two moves is at least x'_k (so he does not go below his budget in the future). That is for each i and j there is a k such that $x'_j + b_{ij} - a_{ik} \geq x'_k$ or

$$x'_k + a_{ik} \leq x'_j + b_{ij}.$$

And this precisely means that $A \odot \vec{x}' \leq B \odot \vec{x}'$.

Now, suppose that there is a solution \vec{x} to the min-plus linear system. Let us give the column-player the amount x_j if the game starts in c_j . Then reversing the argument we have that for each i and j there is k such that $x_j + b_{ij} - a_{ik} \geq x_k$. And this means that for each position c_j and for each move (j, i) of the row-player there is a move (i, k) of the column-player, such

that the amount of the column-player does not go below x_k . Thus, we have that the column-player indeed does not go below the amounts \vec{x} and thus, does not lose the game if he makes moves in the described way.

This analysis shows all the statements of the lemma except for the last one. For this statement note that the column-player does not lose in the position r_j if he has a move to some position c_i such that first, he does not lose immediately, and second, he does not lose in position c_i . Thus a_{ij} is finite and x_i is finite and thus the i -th coordinate of $A \odot \vec{x}$ is finite. It is easy to reverse this argument.

The next observation seems to be a new step towards the min-plus linear duality.

Lemma 17 *There is a finite solution to $A \odot \vec{x} < B \odot \vec{x}$ iff the column-player has a winning strategy starting from any position.*

There is a solution $\vec{x} \neq (\infty, \dots, \infty)$ to $A \odot \vec{x} < B \odot \vec{x}$ iff the column-player has a winning strategy starting from some position.

There is a solution to $A \odot \vec{x} < B \odot \vec{x}$ with a finite coordinate x_i iff the column-player has a winning strategy starting from the position c_i .

There is a solution to $A \odot \vec{x} < B \odot \vec{x}$ such that the j -th coordinate of $A \odot \vec{x}$ is finite iff the column-player has a winning strategy starting from the position r_j .

Proof Suppose there is a solution \vec{x} to $A \odot \vec{x} < B \odot \vec{x}$. Then for a small enough positive ϵ there is a solution to $A \odot \vec{x} \leq (B - \epsilon) \odot \vec{x}$, where we subtract ϵ from each entry of B . Then by Lemma 16 there is a non-losing strategy for the column-player in the mean payoff game G' corresponding to the system $A \odot \vec{x} \leq (B - \epsilon) \odot \vec{x}$. Let the column-player apply the same strategy to the game G corresponding to $A \odot \vec{x} < B \odot \vec{x}$. Then compared to the game G' after k moves the column-player will have at least the value $k\epsilon$ added to his amount. Since the amount of the column-player is bounded from below in G' it will grow to infinity in G . Thus in the game G the column-player has a winning strategy.

For the opposite direction, assume that the column-player has a winning strategy. Then if we add a small enough ϵ to all payoffs of the row-player, the column-player will still have a winning strategy, which is in particular non-losing. Thus we have by Lemma 16 that there is a solution \vec{x} to $A \odot \vec{x} \leq (B - \epsilon) \odot \vec{x}$, where we subtract ϵ from each entry of B . Clearly, the very same \vec{x} is a solution to $A \odot \vec{x} < B \odot \vec{x}$ and we are done.

Now to get the Lemma 2 it is only left to use a duality of mean payoff games. For this note that for a game G either the column-player has a winning strategy starting from some position, or the row-player has a non-losing strategy starting from the same position. Also note that if along with the game G corresponding to the min-plus linear system $A \odot \vec{x} \leq B \odot \vec{x}$ we consider a game G' corresponding to the min-plus linear system $B^T \odot \vec{y} \leq A^T \odot \vec{y}$, then the games G and G' are the same except the roles of the players switched.

7.2 Tropical duality

Suppose we are given a tropical linear system $A \odot \vec{x}$ for $A \in \mathbb{R}^{m \times n}$ and we are interested whether it has a solution. First of all it is known that if the number of variables is greater than the number of equations, then there is always a solution [8]. So we can assume that $m \geq n$. Next note that if we add the same number to all the entries in some row of A then the set of solutions does not change. One simple obstacle for $A \odot \vec{x}$ to have a solution is that we can add some numbers to all the columns of A and possibly permute the rows and the columns in such a way that the minimums in the first n rows of the resulting matrix are unique in each row and are attained just in the entries $(1, 1), (2, 2), \dots, (n, n)$. It is easy to see that if this is the case then there is no solutions to $A \odot \vec{x}$ [8]. It turns out that this is the only obstacle. We give a proof below however we note that this is already implicit in [12, 20].

Proof (Proof of Lemma 3) Given a tropical product of a matrix by a vector $A \odot \vec{a}$, where $A \in \mathbb{R}_{\infty}^{m \times n}$ it is convenient to introduce a value $n_i(A \odot \vec{a})$ for all $i = 1, \dots, m$ which is equal to the number of the column in which the finite minimum in the row i is situated (if there is one). If there are several minimums, $n_i(A \odot \vec{a})$ corresponds to the first one. When the matrix and the vector are clear from the context we simply write n_i .

Denote by C_i for $i = 1, \dots, n$ the matrix in $\mathbb{R}^{m \times n}$ with 1 entries in the i -th column and 0 entries in other columns. Denote by R_i for $i = 1, \dots, n$ the matrix in $\mathbb{R}^{n \times m}$ with 1 entries in the i -th row and 0 entries in other rows. Note that $R_i = C_i^T$.

We will show the first part of the lemma. The proof of the second part is completely analogous.

Suppose we are given a matrix $A \in \mathbb{R}_{\infty}^{m \times n}$ and consider the tropical linear system $A \odot \vec{x}$. As shown in [13] (cf. Section 5) \vec{x} is a solution to this system iff for all small enough $\epsilon > 0$ \vec{x} is a solution to the following min-plus system:

$$\begin{pmatrix} A + \epsilon C_1 \\ A + \epsilon C_2 \\ \vdots \\ A + \epsilon C_n \end{pmatrix} \odot \vec{x} \leq \begin{pmatrix} A \\ A \\ \vdots \\ A \end{pmatrix} \odot \vec{x}.$$

By Lemma 2 this system has a solution \vec{x} with finite coordinates x_i for $i \in S$ if and only if the system

$$\begin{aligned} (A^T \ A^T \ \dots \ A^T) \odot \vec{y} < \\ (A^T + \epsilon R_1 \ A^T + \epsilon R_2 \ \dots \ A^T + \epsilon R_n) \odot \vec{y} \end{aligned} \quad (13)$$

has no solution \vec{y} such that for some $i \in S$ the i -th coordinate of

$$(A^T \ A^T \ \dots \ A^T) \odot \vec{y}$$

is finite.

On the right-hand side of (13) we have a block matrix with blocks $A^T + \epsilon R_i$.

It is left to show that the system (13) has a specified solution iff there is \vec{z} such that in each row of $A^T \odot \vec{z}$ the minimum is attained at least once or is equal to ∞ , in addition for each two rows with the finite minimums these minimums are in different columns and for some $i \in S$ the i -th coordinate of $A^T \odot \vec{z}$ is finite. Note that if \vec{y} is a solution to (13) then in each row i , where the minimum on the left-hand side is finite we have $m(i-1) < n_i \leq mi$, that is n_i is in the i -th block. Indeed, otherwise the minimum in this row in the left-hand side is greater or equal than the minimum in the right-hand side, since the i -th rows on the left-hand side and on the right-hand side differ only in the i -th block. Thus, if \vec{y} is a solution then for each row i with the finite minimum there exists a column $j_i = n_i \pmod{m}$ of the i -th block such that the minimum is attained in this column. Note that for $i_1 \neq i_2$ with the finite minimums in the rows of the system we have $j_{i_1} \neq j_{i_2}$. Otherwise, the rows i_1, i_2 and the columns $n_{i_1} = mi_1 + j_{i_1}, n_{i_2} = mi_2 + j_{i_1}$ would form a 2×2 subsystem

$$\begin{pmatrix} a_{i_1, j_{i_1}} & a_{i_1, j_{i_1}} \\ a_{i_2, j_{i_1}} & a_{i_2, j_{i_1}} \end{pmatrix} \odot \begin{pmatrix} y_{mi_1 + j_{i_1}} \\ y_{mi_2 + j_{i_1}} \end{pmatrix} < \begin{pmatrix} a_{i_1, j_{i_1}} + \epsilon & a_{i_1, j_{i_1}} \\ a_{i_2, j_{i_1}} & a_{i_2, j_{i_1}} + \epsilon \end{pmatrix} \odot \begin{pmatrix} y_{mi_1 + j_{i_1}} \\ y_{mi_2 + j_{i_1}} \end{pmatrix},$$

with the finite $y_{mi_1 + j_{i_1}}, y_{mi_2 + j_{i_1}}$, which is impossible. Thus, the columns j_i correspond to the different columns of the matrix A^T . Let us consider the tropical system

$$A^T \odot \vec{z}$$

and consider the following vector \vec{z} . For all i with the finite minimum in the row i let $z_{j_i} = y_{mi + j_i}$. Set all other coordinates of \vec{z} to ∞ . For this \vec{z} the minimum in each row is either infinite or is attained once and no two minimums are in the same column. Indeed, if a finite minimum is attained twice for some row, then for the same row of (13) we would have the equality. Note also that the i -th coordinate of the vector

$$(A^T \ A^T \ \dots \ A^T) \odot \vec{y}$$

is finite iff the i -th coordinate of $A^T \odot \vec{z}$ is finite.

In the opposite direction, suppose we have a vector \vec{z} such that in each row $A^T \odot \vec{z}$ the minimum is either infinite or is attained once and no two minimums are in the same column. Then for each row i with the finite minimum consider a column j_i in which this minimum is attained and let $y_{im + j_i} = z_{j_i}$. Set all other coordinates of \vec{y} to ∞ . Then for any small enough ϵ we will have a solution of (13) and the i -th coordinate of $A^T \odot \vec{z}$ is finite iff the i -th coordinate of

$$(A^T \ A^T \ \dots \ A^T) \odot \vec{y}$$

is finite.

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