

SUPPLEMENTARY INFORMATION

Fourier transform of Green Tensor for electromagnetic scattering : detailed computations

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PRESENTATION

In this supplementary information, we explain the calculation in detail, and generalize some of the results of the main text. We also present a set of results on the distributions associated to $\frac{1}{x^n}$ and $\frac{\text{sgn}(x)}{x^n}$, and their Fourier Transform in the general notation.

I. DEFINITION AND FIRST PROPERTIES

A. Distributions associated to $\frac{1}{x^n}$ and $\frac{\text{sgn}(x)}{x^n}$

We provide a generalisation of these later functions by defining

$$T_{\frac{1}{x^n}} := \frac{(-1)^{n-1}}{(n-1)!} (\ln(|x|))^{(n)} \quad (\text{I.1})$$

$$T_{\frac{\text{sgn}(x)}{x^n}} := \frac{(-1)^{n-1}}{(n-1)!} (\text{sgn}(x) \ln(|x|))^{(n)} \quad (\text{I.2})$$

Remark I.1 1. One has the recurrence property:

$$T_{\frac{1}{x^n}} = \frac{-1}{n-1} T'_{\frac{1}{x^{n-1}}} \quad (\text{I.3})$$

$$T_{\frac{\text{sgn}(x)}{x^n}} = \frac{-1}{n-1} T'_{\frac{\text{sgn}(x)}{x^{n-1}}}. \quad (\text{I.4})$$

2. $T_{\frac{1}{x^n}}$ and $T_{\frac{\text{sgn}(x)}{x^n}}$ are tempered distributions, as they stem from the multiple differentiation (in the sense of distribution) of, respectively, $x \mapsto x \ln(|x|) - x$ and $x \mapsto |x| \ln(|x|) - |x|$, which are continuous and slowly increasing functions, – and therefore tempered distributions.

Proposition I.2 1. $x^n T_{\frac{1}{x^n}} = 1$.

2. $x^n T_{\frac{\text{sgn}(x)}{x^n}} = \text{sgn}(x)$.

3. $x T_{\frac{1}{x^{m+1}}} = T_{\frac{1}{x^m}}$.

4. $x^n T_{\frac{1}{x^{m+n}}} = T_{\frac{1}{x^m}}$.

5. $x^n T_{\frac{\text{sgn}(x)}{x^{n+1}}} = T_{\frac{\text{sgn}(x)}{x}} - 2 \left(\sum_{k=1}^n \frac{1}{k} \right) \delta_0$, where δ_0 is the Dirac distribution at 0.

6. $x T_{\frac{\text{sgn}(x)}{x^{m+1}}} = T_{\frac{\text{sgn}(x)}{x^m}} - \frac{2}{m!} (-1)^{m-1} \delta_0^{(m-1)}$, with the convention $\delta_0^{(0)} := \delta_0$.

7. $x^n T_{\frac{\text{sgn}(x)}{x^{m+n}}} = T_{\frac{\text{sgn}(x)}{x^m}} - (-1)^{m-1} P(m, n) \delta_0^{(m-1)}$

$$P(m, n) = \frac{2}{(m-1)!} \sum_{k=0}^{n-1} \frac{1}{m+k} = \frac{2(H_{m+n-1} - H_{m-1})}{(m-1)!}.$$

where H_n is the n -th Harmonic number:

$$H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}.$$

Let us remark that

$$\begin{aligned}
\int_0^{+\infty} \ln(x)(\varphi(x) + x\varphi'(x)) dx &= \int_0^{+\infty} \ln(x)\varphi(x) dx + \int_0^{+\infty} (x\ln(x) - x)\varphi'(x) dx \\
&\quad + \int_0^{+\infty} x\varphi'(x) dx \\
&= \int_0^{+\infty} \ln(x)\varphi(x) dx - \int_0^{+\infty} \ln(x)\varphi(x) dx \\
&\quad + \int_0^{+\infty} x\varphi'(x) dx \\
&= - \int_0^{+\infty} \varphi(x) dx
\end{aligned} \tag{I.5}$$

and

$$\begin{aligned}
\int_{-\infty}^0 \ln(-x)(\varphi(x) + x\varphi'(x)) dx &= \int_{-\infty}^0 \ln(-x)\varphi(x) dx \\
&\quad + \int_{-\infty}^0 (x\ln(-x) - x)\varphi'(x) dx + \int_{-\infty}^0 x\varphi'(x) dx \\
&= \int_{-\infty}^0 \ln(-x)\varphi(x) dx - \int_{-\infty}^0 \ln(-x)\varphi(x) dx \\
&\quad + \int_{-\infty}^0 x\varphi'(x) dx \\
&= - \int_{-\infty}^0 \varphi(x) dx.
\end{aligned} \tag{I.6}$$

1. By induction :

$$\begin{aligned}
\langle xT_{\frac{1}{x}}, \varphi \rangle &= \langle T_{\frac{1}{x}}, x\varphi \rangle \\
&= \langle (\ln(|x|))', x\varphi \rangle \\
&= -\langle \ln(|x|), (x\varphi)' \rangle \\
&= - \int_{-\infty}^0 \ln(-x)(\varphi(x) + x\varphi'(x)) dx - \int_0^{+\infty} \ln(x)(\varphi(x) + x\varphi'(x)) dx \\
&= \int_{-\infty}^0 \varphi(x) dx + \int_0^{+\infty} \varphi(x) dx \tag{with (I.5) and (I.6)} \\
&= \int_{-\infty}^{+\infty} \varphi(x) dx = \langle 1, \varphi \rangle.
\end{aligned}$$

Then (with $T_{\frac{1}{x^n}} = \frac{-1}{n-1} T'_{\frac{1}{x^{n-1}}}$),

$$\begin{aligned}
\langle x^n T_{\frac{1}{x^n}}, \varphi \rangle &= \langle T_{\frac{1}{x^n}}, x^n \varphi \rangle \\
&= -\langle \frac{-1}{n-1} T'_{\frac{1}{x^{n-1}}}, x^n \varphi \rangle \\
&= \frac{1}{n-1} \langle T_{\frac{1}{x^{n-1}}}, nx^{n-1} \varphi + x^n \varphi' \rangle \\
&= \frac{n}{n-1} \langle x^{n-1} T_{\frac{1}{x^{n-1}}}, \varphi \rangle + \frac{1}{n-1} \langle x^{n-1} T_{\frac{1}{x^{n-1}}}, x\varphi' \rangle \\
&= \frac{n}{n-1} \int_{-\infty}^{+\infty} \varphi(x) dx + \frac{1}{n-1} \int_{-\infty}^{+\infty} x\varphi'(x) dx \\
&= \frac{n}{n-1} \int_{-\infty}^{+\infty} \varphi(x) dx - \frac{1}{n-1} \int_{-\infty}^{+\infty} \varphi(x) dx \\
&= \int_{-\infty}^{+\infty} \varphi(x) dx = \langle 1, \varphi \rangle.
\end{aligned}$$

2. By induction:

If $n = 1$,

$$\begin{aligned}
 \langle xT_{\frac{\text{sgn}(x)}{x}}, \varphi \rangle &= -\langle \text{sgn}(x) \ln(|x|), (x\varphi)' \rangle \\
 &= \int_{-\infty}^0 \ln(-x)(\varphi(x) + x\varphi'(x)) dx - \int_0^{+\infty} \ln(x)(\varphi(x) + x\varphi'(x)) dx \\
 &= -\int_{-\infty}^0 \varphi(x) dx + \int_0^{+\infty} \varphi(x) dx && \text{(with (I.5) and (I.6))} \\
 &= \int_{-\infty}^{+\infty} \text{sgn}(x)\varphi(x) dx = \langle \text{sgn}(x), \varphi \rangle.
 \end{aligned}$$

Let us assume the property is true at rank $n - 1$, we then have (with $T_{\frac{\text{sgn}(x)}{x^n}} = \frac{-1}{n-1} T'_{\frac{\text{sgn}(x)}{x^{n-1}}}$),

$$\begin{aligned}
 \langle x^n T_{\frac{\text{sgn}(x)}{x^n}}, \varphi \rangle &= \langle T_{\frac{\text{sgn}(x)}{x^n}}, x^n \varphi \rangle \\
 &= \langle \frac{-1}{n-1} T'_{\frac{\text{sgn}(x)}{x^{n-1}}}, x^n \varphi \rangle \\
 &= \frac{1}{n-1} \langle T_{\frac{\text{sgn}(x)}{x^{n-1}}}, nx^{n-1} \varphi + x^n \varphi' \rangle \\
 &= \frac{n}{n-1} \langle \text{sgn}(x), \varphi \rangle + \frac{1}{n-1} \langle \text{sgn}(x), x\varphi' \rangle
 \end{aligned}$$

but,

$$\begin{aligned}
 \langle \text{sgn}(x), x\varphi' \rangle &= -\int_{-\infty}^0 x\varphi'(x) dx + \int_0^{\infty} x\varphi'(x) dx \\
 &= \int_{-\infty}^0 \varphi(x) dx - \int_0^{\infty} \varphi(x) dx \\
 &= -\langle \text{sgn}(x), \varphi \rangle
 \end{aligned}$$

then,

$$\langle x^n T_{\frac{\text{sgn}(x)}{x^n}}, \varphi \rangle = \frac{n}{n-1} \langle \text{sgn}(x), \varphi \rangle - \frac{1}{n-1} \langle \text{sgn}(x), \varphi \rangle = \langle \text{sgn}(x), \varphi \rangle.$$

3. By induction :

$$\begin{aligned}
 \langle xT_{\frac{1}{x^2}}, \varphi \rangle &= \langle T_{\frac{1}{x^2}}, x\varphi \rangle \\
 &= -\langle T'_{\frac{1}{x}}, x\varphi \rangle \\
 &= \langle T_{\frac{1}{x}}, \varphi \rangle + \langle T_{\frac{1}{x}}, x\varphi' \rangle \\
 &= \langle T_{\frac{1}{x}}, \varphi \rangle + \underbrace{\langle 1, \varphi' \rangle}_{= \int \varphi' = 0} \\
 &= \langle T_{\frac{1}{x}}, \varphi \rangle.
 \end{aligned}$$

Then

$$\begin{aligned}
 \langle xT_{\frac{1}{x^{m+1}}}, \varphi \rangle &= \langle T_{\frac{1}{x^{m+1}}}, x\varphi \rangle \\
 &= \frac{1}{m} \left(\langle T_{\frac{1}{x^m}}, \varphi \rangle + \langle xT_{\frac{1}{x^m}}, \varphi' \rangle \right) \\
 &= \frac{1}{m} \left(\langle T_{\frac{1}{x^m}}, \varphi \rangle + \langle T_{\frac{1}{x^{m-1}}}, \varphi' \rangle \right) \\
 &= \frac{1}{m} \left(\langle T_{\frac{1}{x^m}}, \varphi \rangle + (m-1) \langle T_{\frac{1}{x^m}}, \varphi \rangle \right) \\
 &= \langle T_{\frac{1}{x^m}}, \varphi \rangle.
 \end{aligned}$$

4. By induction : from item 3., it is true for $n = 1$ (and for all m).

Then

$$\begin{aligned}
 \langle x^{n+1} T_{\frac{1}{x^{m+n+1}}}, \varphi \rangle &= \langle x^n T_{\frac{1}{x^{(m+1)+n}}}, x\varphi \rangle \\
 &= \langle T_{\frac{1}{x^{m+1}}}, x\varphi \rangle \\
 &= \langle x T_{\frac{1}{x^{m+1}}}, \varphi \rangle \\
 &= \langle T_{\frac{1}{x^m}}, \varphi \rangle.
 \end{aligned}
 \tag{from item 3.}$$

5.

$$\begin{aligned}
 \langle x^{n+1} T_{\frac{\text{sgn}(x)}{x^{n+1+1}}}, \varphi \rangle &= \langle T_{\frac{\text{sgn}(x)}{x^{(n+1)+1}}}, x^{n+1} \varphi \rangle \\
 &= \frac{-1}{n+1} \langle T'_{\frac{\text{sgn}(x)}{x^{n+1}}}, x^{n+1} \varphi \rangle \\
 &= \frac{1}{n+1} \langle T_{\frac{\text{sgn}(x)}{x^{n+1}}}, x^{n+1} \varphi' + (n+1)x^n \varphi \rangle \\
 &= \frac{1}{n+1} \langle x^{n+1} T_{\frac{\text{sgn}(x)}{x^{n+1}}}, \varphi' \rangle + \langle T_{\frac{\text{sgn}(x)}{x^{n+1}}}, x^n \varphi \rangle \\
 &= \frac{1}{n+1} \langle \text{sgn}(x), \varphi' \rangle + \langle x^n T_{\frac{\text{sgn}(x)}{x^{n+1}}}, \varphi \rangle \\
 &= \frac{-2}{n+1} \varphi(0) + \langle T_{\frac{\text{sgn}(x)}{x}} - 2 \left(\sum_{k=1}^n \frac{1}{k} \right) \delta_0, \varphi \rangle \\
 &= \langle T_{\frac{\text{sgn}(x)}{x}} - 2 \left(\sum_{k=1}^{n+1} \frac{1}{k} \right) \delta_0, \varphi \rangle.
 \end{aligned}$$

6. By induction. $m = 1$,

$$\begin{aligned}
 \langle x T_{\frac{\text{sgn}(x)}{x^2}}, \varphi \rangle &= \langle T_{\frac{\text{sgn}(x)}{x^2}}, x\varphi \rangle \\
 &= -\langle T'_{\frac{\text{sgn}(x)}{x}}, x\varphi \rangle \\
 &= \langle T_{\frac{\text{sgn}(x)}{x}}, \varphi \rangle + \langle T_{\frac{\text{sgn}(x)}{x}}, x\varphi' \rangle
 \end{aligned}$$

With

$$\langle T_{\frac{\text{sgn}(x)}{x}}, x\varphi' \rangle = \langle x T_{\frac{\text{sgn}(x)}{x}}, \varphi' \rangle = \langle \text{sgn}(x), \varphi' \rangle = -2\varphi(0),$$

the property is well verified.

We can write :

$$\begin{aligned}
 \langle x T_{\frac{\text{sgn}(x)}{x^{m+1}}}, \varphi \rangle &= \langle T_{\frac{\text{sgn}(x)}{x^{m+1}}}, x\varphi \rangle \\
 &= \frac{1}{m} \left(\langle T_{\frac{\text{sgn}(x)}{x^m}}, \varphi \rangle + \langle x T_{\frac{\text{sgn}(x)}{x^m}}, \varphi' \rangle \right)
 \end{aligned}$$

We then write the latter expression m times, replacing φ by its first derivatives $\varphi^{(q)}$ with $q \in [0..(m-1)]$. A linear combination of these m equations gives:

$$\langle x T_{\frac{\text{sgn}(x)}{x^{m+1}}}, \varphi \rangle = \langle T_{\frac{\text{sgn}(x)}{x^m}}, \varphi \rangle + \frac{1}{m!} \langle x T_{\frac{\text{sgn}(x)}{x}}, \varphi^{(m)} \rangle
 \tag{I.7}$$

Then, one gets

$$\begin{aligned}
\frac{1}{m!} \langle x T_{\frac{\text{sgn}(x)}{x}}, \varphi^{(m)} \rangle &= \frac{1}{m!} \langle \text{sgn}(x), \varphi^{(m)} \rangle && \text{(item 2.)} \\
&= -\frac{2}{m!} \varphi^{(m-1)}(0) \\
&= -\frac{2}{m!} (-1)^{(m-1)} \langle \delta_0^{(m-1)}, \varphi \rangle,
\end{aligned}$$

what permits to obtain the result.

7. By induction on n . For $n = 1$ (and for all m), we recover the previous formula, if we define $P(m, 1) := \frac{2}{m!}$. Now, let us assume that, for all m , and at some rank n ,

$$\langle x^n T_{\frac{\text{sgn}(x)}{x^{m+n}}}, \varphi \rangle = \langle T_{\frac{\text{sgn}(x)}{x^m}}, \varphi \rangle - P(m, n) \varphi^{(m-1)}(0). \quad (\text{I.8})$$

We then have:

$$\begin{aligned}
\langle x^{n+1} T_{\frac{\text{sgn}(x)}{x^{m+n+1}}}, \varphi \rangle &= \langle T_{\frac{\text{sgn}(x)}{x^{m+1+n}}}, x^{n+1} \varphi \rangle \\
&= -\frac{1}{m+n} \langle T'_{\frac{\text{sgn}(x)}{x^{m+n}}}, x^{n+1} \varphi \rangle \\
&= +\frac{1}{m+n} \langle T_{\frac{\text{sgn}(x)}{x^{m+n}}}, (x^{n+1} \varphi)' \rangle \\
&= \frac{n+1}{m+n} \langle x^n T_{\frac{\text{sgn}(x)}{x^{m+n}}}, \varphi \rangle + \frac{1}{m+n} \langle T_{\frac{\text{sgn}(x)}{x^{m+n}}}, x^{n+1} \varphi' \rangle
\end{aligned}$$

But,

$$\begin{aligned}
\langle T_{\frac{\text{sgn}(x)}{x^{m+n}}}, x^{n+1} \varphi' \rangle &= \langle x^n T_{\frac{\text{sgn}(x)}{x^{m+n}}}, x \varphi' \rangle \\
&= \langle T_{\frac{\text{sgn}(x)}{x^m}}, x \varphi' \rangle - P(m, n) (x \varphi')^{(m-1)}(0)
\end{aligned}$$

and

$$\begin{aligned}
\langle T_{\frac{\text{sgn}(x)}{x^m}}, x \varphi' \rangle &= \langle x T_{\frac{\text{sgn}(x)}{x^m}}, \varphi' \rangle \\
&= \langle T_{\frac{\text{sgn}(x)}{x^{m-1}}}, \varphi' \rangle - \frac{2}{(m-1)!} \varphi^{(m-1)}(0),
\end{aligned}$$

where the last equality has been demonstrated previously, see Eq. (I.7).

Gathering, and using the induction hypothesis, we obtain:

$$\begin{aligned}
(m+n) \langle x^{n+1} T_{\frac{\text{sgn}(x)}{x^{m+n+1}}}, \varphi \rangle &= (n+1) \langle T_{\frac{\text{sgn}(x)}{x^m}}, \varphi \rangle - P(m, n) (n+1) \varphi^{(m-1)}(0) \\
&\quad + (m-1) \langle T_{\frac{\text{sgn}(x)}{x^m}}, \varphi \rangle - \frac{2}{(m-1)!} \varphi^{(m-1)}(0) \\
&\quad - P(m, n) (x \varphi')^{(m-1)}(0)
\end{aligned}$$

The value of the $(m-1)^{\text{th}}$ derivative of $x \mapsto (x \varphi')$ at $x = 0$ can be evaluated easily to $(m-1) \varphi^{(m-1)}(0)$. We finally obtain:

$$\begin{aligned}
(m+n) \langle x^{n+1} T_{\frac{\text{sgn}(x)}{x^{m+n+1}}}, \varphi \rangle &= (m+n) \langle T_{\frac{\text{sgn}(x)}{x^m}}, \varphi \rangle - P(m, n) (n+m) \varphi^{(m-1)}(0) \\
&\quad - \frac{2}{(m-1)!} \varphi^{(m-1)}(0)
\end{aligned}$$

Let us assume that $P(m, n)$ obeys to:

$$P(m, n+1) = P(m, n) + \frac{2}{(m+n)(m-1)!} \quad (\text{I.9})$$

if a set of numbers $P(m, n)$, that verifies Eq.(I.9), as well as the condition $P(m, 1) = \frac{2}{m!}$ exists, then, we have proved the equation Eq. (I.8) at rank $n+1$.

It is actually trivial to find an explicit expression for $P(m, n)$, that matches those requirements :

$$P(m, n) = \frac{2}{(m-1)!} \sum_{p=0}^{n-1} \frac{1}{m+p}, \quad (\text{I.10})$$

Note that we assume $P(m, 0) = 0$.

Using the definition of the Harmonic numbers[18], we can write

$$P(m, n) = \frac{2(H_{m+n-1} - H_{m-1})}{(m-1)!}.$$

B. Theorem for change of variable

Theorem I.3 *Let $\varphi : U \rightarrow D$ a \mathcal{C}^1 diffeomorphism. Then, for all measurable (resp. integrable) function $f : D \rightarrow \mathbb{R}_+$ (resp. \mathbb{R}),*

$$\int_{D=\varphi(U)} f(x) dx = \int_U f(\varphi(u)) |\det(\varphi'(u))| du.$$

II. FOURIER TRANSFORMS: GENERAL NOTATIONS

A. “Universal” notation for the Fourier Transform

The general definitions for the Fourier transform are given in the main text. Let us recall them here, for the sake of clarity. With $f, g : \mathbb{R}^n \rightarrow \mathbb{R}$:

$$\mathcal{F}^{a,b}(f)(k) := \left(\frac{|b|}{(2\pi)^{1-a}} \right)^{n/2} \int_{\mathbb{R}^n} f(t) e^{ib\langle k, t \rangle} dt \quad (\text{II.11})$$

$$(\mathcal{F}^{a,b})^{-1}(g)(t) := \left(\frac{|b|}{(2\pi)^{1+a}} \right)^{n/2} \int_{\mathbb{R}^n} g(k) e^{-ib\langle t, k \rangle} dk \quad (\text{II.12})$$

To jump from a convention to another, we have the simple following corresponding formula :

$$\mathcal{F}^{a,b}(f)(k) = \left(\frac{|b/b'|}{(2\pi)^{a'-a}} \right)^{n/2} \mathcal{F}^{a',b'}(f)\left(\frac{b}{b'}k\right). \quad (\text{II.13})$$

B. Fourier Transform of a tempered distribution

Let us recall that if $T \in \mathcal{S}'(\mathbb{R}^n)$ is a tempered distribution, we define de Fourier transform of $\mathcal{F}^{a,b}T$ by

$$\langle \mathcal{F}^{a,b}T, \varphi \rangle := \langle T, \mathcal{F}^{a,b}\varphi \rangle \quad (\text{II.14})$$

In order to give the formula to jump from a convention to another, we have to define the following operator : for $a \in \mathbb{R}$, we set $m_a : f \mapsto (x \mapsto f(ax), x \in \mathbb{R}^n)$.

This operator can be extended to the distributions : for all $a \neq 0$,

$$\langle m_a T, \varphi \rangle := \frac{1}{|a|^n} \langle T, m_{\frac{1}{a}} \varphi \rangle \quad (\text{II.15})$$

From (II.13) and (II.15) we have

$$\mathcal{F}_n^{a,b} T = \left(\frac{|b'/b|}{(2\pi)^{d'-a}} \right)^{n/2} \mathcal{F}_n^{a',b'} (m_{\frac{b'}{b}} T). \quad (\text{II.16})$$

(please notice that $|b/b'|$ in (II.13) “becoming” $|b'/b|$ in (II.16) is not a typo!)

Then, we can prove

Proposition II.1

$$m_\ell \mathcal{F}^{a,b}(T) = \frac{1}{|\ell|} \mathcal{F}^{a,b}(m_{\frac{1}{\ell}} T) \quad (\text{II.17})$$

$$\begin{aligned} \langle m_\ell \mathcal{F}^{a,b}(T), \varphi \rangle &= \frac{1}{|\ell|} \langle \mathcal{F}^{a,b}(T), m_{\frac{1}{\ell}} \varphi \rangle \\ &= \frac{1}{|\ell|} \langle T, \mathcal{F}^{a,b}(m_{\frac{1}{\ell}} \varphi) \rangle \\ &\quad \mathcal{F}^{a,b}(m_{\frac{1}{\ell}} \varphi)(x) = c_{a,b} \int_{\mathbb{R}} \varphi\left(\frac{y}{\ell}\right) e^{ibxy} dy \\ \text{with} \quad &= c_{a,b} |\ell| \int_{\mathbb{R}} \varphi(t) e^{ib\ell x t} dt \\ &= |\ell| \mathcal{F}^{a,b}(\varphi)(\ell x) = |\ell| m_\ell \mathcal{F}^{a,b}(\varphi)(x) \\ &= \langle T, m_\ell \mathcal{F}^{a,b}(\varphi) \rangle \\ &= \frac{1}{|\ell|} \langle m_{\frac{1}{|\ell|}} T, \mathcal{F}^{a,b}(\varphi) \rangle = \frac{1}{|\ell|} \langle \mathcal{F}^{a,b}(m_{\frac{1}{\ell}} T), \varphi \rangle. \end{aligned}$$

Using a straightforward multidimensional version of (II.17) and (II.16), we obtain

$$\mathcal{F}^{a,b} T = \left(\frac{|b/b'|}{(2\pi)^{d'-a}} \right)^{n/2} m_{\frac{b}{b'}} \mathcal{F}^{a',b'}(T). \quad (\text{II.18})$$

C. Elementary Formulas

In this section we denote by T distributions of $\mathcal{S}'(\mathbb{R})$

Notation II.2 In the sequel we will give general results using the $\mathcal{F}^{a,b}$ notation, but also the simpler notation

$$\hat{u} := \mathcal{F}^{1,-1}(u)$$

where u is a function or a distribution, to obtain more concise results.

1. Double Fourier Transform

Lemma II.3 Let f a function in $L^1 \cap L^2$.

$$\mathcal{F}^{a,b}(\mathcal{F}^{a,b} f)(t) = (2\pi)^{an} f_\sigma(t)$$

(where $f_\sigma(t) := f(-t)$).

We will just use the fact that

$$(\mathcal{F})^{-1} \mathcal{F}(f) = f$$

with

$$\mathcal{F}(f)(k) := \mathcal{F}^{0,-2\pi}(f)(k) = \int_{\mathbb{R}^n} f(t) e^{-2\pi i \langle k, t \rangle} dt.$$

and

$$(\mathcal{F})^{-1}(g)(t) = \int_{\mathbb{R}^n} g(x) e^{2i\pi \langle t, x \rangle} dx.$$

Since

$$\mathcal{F}(f)(x) := \mathcal{F}^{0,-2\pi}(f)(x) = \left(\frac{|\frac{2\pi}{b}|}{(2\pi)^a} \right)^{n/2} \mathcal{F}^{a,b}(f)\left(-\frac{2\pi}{b}x\right),$$

$$\begin{aligned} (\mathcal{F})^{-1}\mathcal{F}(f) &= \int_{\mathbb{R}^n} e^{2i\pi \langle t, x \rangle} \mathcal{F}(f)(x) dx \\ &= \left(\frac{|\frac{2\pi}{b}|}{(2\pi)^a} \right)^{n/2} \int_{\mathbb{R}^n} e^{2i\pi \langle t, x \rangle} \mathcal{F}^{a,b}(f)\left(-\frac{2\pi}{b}x\right) dx \\ &= \left(\frac{|\frac{2\pi}{b}|}{(2\pi)^a} \right)^{n/2} \left(\frac{|b|}{2\pi} \right)^n \int_{\mathbb{R}^n} e^{-ib \langle u, t \rangle} \mathcal{F}^{a,b}(f)(u) du \\ &= \left(\frac{|\frac{2\pi}{b}|}{(2\pi)^a} \right)^{n/2} \left(\frac{|b|}{2\pi} \right)^n \left(\frac{(2\pi)^{1-a}}{|b|} \right)^{n/2} \mathcal{F}^{a,b}(\mathcal{F}^{a,b}(f))(-t) \end{aligned}$$

and the result is then straightforward.

We shall now give the FT of the convolution and the product of two distributions in the general notation. Note that these formula can be found in many textbooks, see e.g. [1], but are not usually derived in the general notation.

2. FT of a convolution

Let us recall, see e.g. [1], the definitions of convolution between S and T , two generalized functions of \mathbb{R}^n :

$$\langle S \star T, \varphi \rangle = \left\langle [\xi \mapsto S(\xi)] \otimes [\eta \mapsto T(\eta)], [(\xi, \eta) \mapsto \varphi(\xi + \eta)] \right\rangle$$

where \otimes is the tensorial product defined by:

$$\langle f \otimes g, [(x, y) \mapsto \varphi(x, y)] \rangle = \int_{\mathbb{R}^n \times \mathbb{R}^n} f(x) g(y) \varphi(x, y) dx dy$$

Therefore, the FT of the convolution writes:

$$\begin{aligned} \mathcal{F}^{a,b}(S \star T) &= \left\langle S \star T, [t \mapsto c_{a,b}^n e^{ib \langle k, t \rangle}] \right\rangle \\ &= c_{a,b}^n \left\langle [\xi \mapsto S(\xi)] \otimes [\eta \mapsto S(\eta)], [(\xi, \eta) \mapsto e^{ib \langle (k, \xi) + (k, \eta) \rangle}] \right\rangle \\ &= c_{a,b}^n \left\langle [\xi \mapsto S(\xi)], [\xi \mapsto e^{ib \langle k, \xi \rangle}] \right\rangle \cdot \left\langle [\eta \mapsto T(\eta)], [\eta \mapsto e^{ib \langle k, \eta \rangle}] \right\rangle \\ &= c_{a,b}^n \mathcal{F}^{a,b}(S) \cdot \mathcal{F}^{a,b}(T), \end{aligned}$$

Where we have used the notation

$$c_{a,b}^n = \left(\frac{|b|}{(2\pi)^{1-a}} \right)^{n/2}. \quad (\text{II.19})$$

3. FT of a product

Starting from

$$\begin{aligned}
\langle \mathcal{F}^{a,b}(S) \star \mathcal{F}^{a,b}(T), \varphi \rangle &= \int_{\mathbb{R}^n \times \mathbb{R}^n} \mathcal{F}^{a,b}(S_\xi) \mathcal{F}^{a,b}(T_\eta) \varphi(\xi + \eta) d\xi d\eta \\
&= c_{a,b}^{2n} \int_{\mathbb{R}^n \times \mathbb{R}^n} \int_{\mathbb{R}^n} S_t e^{ib\langle \xi, t \rangle} dt \int_{\mathbb{R}^n} T_{t'} e^{ib\langle \eta, t' \rangle} dt' \varphi(\xi + \eta) d\xi d\eta \\
&= c_{a,b}^{2n} \int_{\mathbb{R}^{4n}} S_t T_{t'} e^{ib\langle (\xi + \eta), (t+t')/2 \rangle} e^{ib\langle (\xi - \eta), (t-t')/2 \rangle} \varphi(\xi + \eta) dt dt' d\xi d\eta \\
&= c_{a,b}^{2n} \int_{\mathbb{R}^{4n}} S_{u^+ + u^-} T_{u^+ - u^-} e^{ib\langle v^+, u^+ \rangle} e^{ib\langle v^-, u^- \rangle} \varphi(v^+) dv^+ dv^- du^+ du^-,
\end{aligned}$$

where we have set $u^\pm = \frac{t \pm t'}{2}$, $v^\pm = \xi \pm \eta$. Using now

$$\int_{\mathbb{R}^n} e^{ib\langle v^-, u^- \rangle} dv^- = \frac{(2\pi)^n}{|b|^n} \delta_{u^-},$$

we obtain

$$\begin{aligned}
\langle \mathcal{F}^{a,b}(S) \star \mathcal{F}^{a,b}(T), \varphi \rangle &= c_{a,b}^{2n} \int_{\mathbb{R}^{2n}} S_{u^+} T_{u^+} e^{ib\langle v^+, u^+ \rangle} \varphi(v^+) dv^+ du^+ \\
&= \frac{(2\pi)^n}{|b|^n} c_{a,b}^n \langle \mathcal{F}^{a,b}(ST), \varphi \rangle
\end{aligned}$$

We eventually obtain:

$$\mathcal{F}^{a,b}(S.T) = \left(\frac{|b|}{(2\pi)^{1+a}} \right)^{n/2} \mathcal{F}^{a,b}(S) \star \mathcal{F}^{a,b}(T)$$

4. Formula on distributions

Proposition II.4

$$\mathcal{F}^{a,b}(T') = -ibx \mathcal{F}^{a,b}(T) \tag{II.20}$$

$$\mathcal{F}^{a,b}(e^{ivx}T) = \tau_{-\frac{v}{b}} \mathcal{F}^{a,b}(T) \tag{II.21}$$

$$\mathcal{F}^{a,b}(T_\sigma) = \mathcal{F}^{a,b}(T)_\sigma \tag{II.22}$$

A direct calculation gives:

$$\begin{aligned}
\langle \mathcal{F}^{a,b}(T'), \varphi \rangle &= \langle T', \langle \mathcal{F}^{a,b}(\varphi) \rangle \rangle \\
&= -\langle T, \mathcal{F}^{a,b}(\varphi)' \rangle \quad (\text{As } \mathcal{F}^{a,b}(\varphi)' = c_{a,b} \left(\int_{\mathbb{R}} \varphi(x) e^{ib\xi x} dx \right)' = c_{a,b} \int_{\mathbb{R}} ibx \varphi(x) e^{ib\xi x} dx = \mathcal{F}^{a,b}(x \mapsto ibx\varphi(x))) \\
&= -\langle T, \mathcal{F}^{a,b}(ibx\varphi) \rangle \\
&= -\langle \mathcal{F}^{a,b}(T), ibx\varphi \rangle = \langle -ibx \mathcal{F}^{a,b}(T), \varphi \rangle.
\end{aligned}$$

$$\begin{aligned}
\langle \mathcal{F}^{a,b}(e^{ivx}T), \varphi \rangle &= \langle e^{ivx}T, \mathcal{F}^{a,b}(\varphi) \rangle \\
&= \langle T, e^{ivx} \mathcal{F}^{a,b}(\varphi) \rangle \\
&\quad e^{ivx} \mathcal{F}^{a,b}(\varphi)(x) = c_{a,b} \int_{\mathbb{R}} \varphi(y) e^{ix(by + b\frac{v}{b})} dy \\
\text{with} \quad &= c_{a,b} \int_{\mathbb{R}} \varphi(t - \frac{v}{b}) e^{ibxt} dt \\
&= c_{a,b} \int_{\mathbb{R}} \tau_{\frac{v}{b}} \varphi(t) e^{-ibxt} dt = \mathcal{F}^{a,b}(\tau_{\frac{v}{b}} \varphi)(x) \\
&= \langle T, \mathcal{F}^{a,b}(\tau_{\frac{v}{b}} \varphi) \rangle \\
&= \langle \mathcal{F}^{a,b}(T), \tau_{\frac{v}{b}} \varphi \rangle = \langle \tau_{-\frac{v}{b}} \mathcal{F}^{a,b}(T), \varphi \rangle.
\end{aligned}$$

$$\begin{aligned}
\langle \widehat{T}_\sigma, \varphi \rangle &= \langle \widehat{T}, \varphi_\sigma \rangle \\
&= \langle T, \widehat{\varphi}_\sigma \rangle \\
&\quad \widehat{\varphi}_\sigma(x) = \int_{\mathbb{R}} \varphi_\sigma(y) e^{-ixy} dy \\
\text{avec} \quad &= \int_{\mathbb{R}} \varphi(-y) e^{-ixy} dy \\
&= \int_{\mathbb{R}} \varphi(t) e^{-i(-x)t} dt = \widehat{\varphi}(-x) = \widehat{\varphi}_\sigma(x) \\
&= \langle T, \widehat{\varphi}_\sigma \rangle \\
&= \langle T_\sigma, \widehat{\varphi} \rangle = \langle \widehat{T}_\sigma, \varphi \rangle.
\end{aligned}$$

Corollary II.5 *Let T be a distribution, then*

$$\mathcal{F}^{a,b}(\sin(kr)T) = \frac{1}{2i} \tau_{-k} \mathcal{F}^{a,b}(T) - \frac{1}{2i} \tau_k \mathcal{F}^{a,b}(T) \quad (\text{II.23})$$

$$\mathcal{F}^{a,b}(\cos(kr)T) = \frac{1}{2} \tau_{-k} \mathcal{F}^{a,b}(T) + \frac{1}{2} \tau_k \mathcal{F}^{a,b}(T) \quad (\text{II.24})$$

Let us recall that, with the $\mathcal{F}^{1,-1}$ notation, we would obtain:

$$\widehat{\sin(kr)T} = \frac{1}{2i} \tau_k \widehat{T} - \frac{1}{2i} \tau_{-k} \widehat{T} \quad (\text{II.25})$$

$$\widehat{\cos(kr)T} = \frac{1}{2} \tau_k \widehat{T} + \frac{1}{2} \tau_{-k} \widehat{T} \quad (\text{II.26})$$

D. Fourier Transform of a few distributions

In this section, we present the FT of some distributions, that have been used in the main text. Note that we go beyond the derivation presented in [2], as each formula is given here in the general $\mathcal{F}^{a,b}$ notation.

1. Conversion in general notations

Most of the calculations in the main text have been carried out in the simplified notation for FT. We give here the method to convert the FT in the general notation, in \mathbb{R}^3 .

Using Eq. (A.6) of the main text, we can write

$$\begin{aligned}
\langle \mathcal{F}^{a,b}(T), \varphi \rangle &= c_{a,b}^3 \langle m_{-b} \mathcal{F}^{1,-1}(T), \varphi \rangle \\
&= c_{a,b}^3 \langle \mathcal{F}^{1,-1}(T), m_{-\frac{1}{b}} \varphi \rangle \frac{1}{|b|^3}
\end{aligned}$$

This permits to obtain $\mathcal{F}^{a,b}(T)$ in most cases. Let us give an example :

conversion of $\mathcal{F}^{1,-1}(T) := k \mapsto P.V. \left(\frac{k_b}{k-k_b} \right)$

One writes, using a radial test function $x \mapsto \varphi(x)$:

$$\begin{aligned}
\langle \mathcal{F}^{a,b}(T), \varphi \rangle &= c_{a,b}^3 \frac{1}{|b|^3} \int \frac{k_b}{k-k_b} \varphi\left(\frac{-x}{b}\right) dx \\
&= c_{a,b}^3 \frac{k_b}{|b|^3} \int \frac{1}{|b||u|-k_b} \varphi(u) |b|^3 du \\
&= c_{a,b}^3 \frac{k_b}{|b|} \int \frac{\varphi(u)}{|u| - \frac{k_b}{|b|}} du
\end{aligned}$$

Therefore,

$$\mathcal{F}^{a,b}(T) = c_{a,b}^3 \frac{k_b}{|b|} \frac{1}{k - \frac{k_b}{|b|}}. \quad (\text{II.27})$$

Using the same method, we obtain:

| $\mathcal{F}^{1,-1}$ | $\mathcal{F}^{a,b}$ |
|--|--|
| $k \mapsto P.V.\left(\frac{k_b}{k-k_b}\right)$ | $k \mapsto c_{a,b}^3 \frac{k_b}{ b } \frac{1}{k-\frac{k_b}{ b }}$ |
| $k \mapsto k_b \delta_{S(0,k_b)}(k)$ | $k \mapsto c_{a,b}^3 \frac{k_b}{ b } \delta_{S(0,\frac{k_b}{ b })}(k)$ |
| $k \mapsto \frac{k_i k_j}{k^2}$ | $k \mapsto c_{a,b}^3 \frac{k_i k_j}{k^2}$ |
| $k \mapsto \frac{k_i k_j}{k^2} P.V.\left(\frac{1}{k-k_b}\right)$ | $k \mapsto c_{a,b}^3 \frac{k_i k_j}{k^2} \frac{1}{k-\frac{k_b}{ b }}$ |
| $k \mapsto \frac{k_i k_j}{k_b} \delta_{S(0,k_b)}(k)$ | $k \mapsto c_{a,b}^3 k_i k_j \frac{ b }{k_b} \delta_{S(0,\frac{k_b}{ b })}(k)$ |

2. Principal Value

Let us give the Fourier transform of Principal Value distribution, in general notations. As well known,

Proposition II.6

$$\widehat{\text{pv}}\left(\frac{1}{x}\right)(\xi) = -i\pi \operatorname{sgn}(\xi). \quad (\text{II.28})$$

Corollary II.7 We now extend Eq. (II.28) to the general notation :

$$\boxed{\mathcal{F}^{a,b}\left(\text{vp}\left(\frac{1}{x}\right)\right)(\xi) = i\sqrt{|b|} 2^{\frac{a-1}{2}} \pi^{\frac{a+1}{2}} \operatorname{sgn}(b\xi)} \quad (\text{II.29})$$

3. Distribution $T_{\frac{1}{|x|}}$

from [2]

Proposition II.8

$$\widehat{T_{\frac{1}{|x|}}}(\xi) = -2\gamma - 2\ln(|\xi|) \quad (\text{II.30})$$

Corollary II.9 We now extend Eq. (II.30) to the general notation :

$$\boxed{\mathcal{F}^{a,b}\left(T_{\frac{1}{|x|}}\right)(\xi) = -2\left(\frac{|b|}{(2\pi)^{1-a}}\right)^{1/2} (\gamma + \ln(|b\xi|))} \quad (\text{II.31})$$

4. Distribution $T_{\frac{1}{r^n}}$ and $T_{\frac{\operatorname{sgn}(r)}{r^n}}$

Corollary II.10 For every $n \geq 1$, see [2],

$$\widehat{T_{\frac{1}{r^n}}} = -i\pi \frac{(-ir)^{n-1}}{(n-1)!} \operatorname{sgn}(r) \quad (\text{II.32})$$

$$\widehat{T_{\frac{\operatorname{sgn}(r)}{r^n}}} = -2 \frac{(-ir)^{n-1}}{(n-1)!} (\gamma + \ln(|r|)) \quad (\text{II.33})$$

Corollary II.11 We now extend these last results to the general notation : For every $n \geq 1$,

$$\mathcal{F}^{a,b}(T_{\frac{1}{r^n}}) = i \left(\frac{|b|}{(2\pi)^{1-a}} \right)^{1/2} \pi \frac{(ibr)^{n-1}}{(n-1)!} \text{sgn}(br) \quad (\text{II.34})$$

$$\mathcal{F}^{a,b}(T_{\frac{\text{sgn}(r)}{r^n}}) = -2 \left(\frac{|b|}{(2\pi)^{1-a}} \right)^{1/2} \frac{(ibr)^{n-1}}{(n-1)!} (\gamma + \ln(|br|)) \quad (\text{II.35})$$

E. Some useful Lemmas

Lemma II.12 Let f be a continuous real valued function. Let us assume moreover f being differentiable except over a finite set and $f' \in L^1_{loc}$. Then (with T_g denoting the distribution associated to g),

$$T'_f = T_{f'} \quad (\text{II.36})$$

It is sufficient to prove it when the finite set is $\{a\}$ for some $a \in \mathbb{R}$. Let φ be a test function. Then (for every $\varepsilon > 0$)

$$\begin{aligned} \langle T'_f, \varphi \rangle &= -\langle T_f, \varphi' \rangle \\ &= -\int f \varphi' \\ &= -\left(\int_{-\infty}^{a-\varepsilon} f \varphi' + \int_{a-\varepsilon}^{a+\varepsilon} f \varphi' + \int_{a+\varepsilon}^{+\infty} f \varphi' \right) \\ &= \int_{-\infty}^{a-\varepsilon} f' \varphi - f(a-\varepsilon)\varphi(a-\varepsilon) - \int_{a-\varepsilon}^{a+\varepsilon} f \varphi' + \int_{a+\varepsilon}^{+\infty} f' \varphi + f(a+\varepsilon)\varphi(a+\varepsilon) \end{aligned}$$

It comes, with $\varepsilon \rightarrow 0$

$$\langle T'_f, \varphi \rangle = \int f' \varphi = \langle T_{f'}, \varphi \rangle.$$

Lemma II.13 If $f \in \mathcal{C}(\mathbb{R}^n)$ (or in L^1 and radial), differentiable on $\mathbb{R}^n \setminus \{0\}$ with $\partial_{x_i} f \in L^1(\mathbb{R}^n) \cap \mathcal{C}(\mathbb{R}^n \setminus \{0\})$, then

$$D_{x_i}(T_f) = T_{\partial_{x_i} f}.$$

Let φ be a test function. Then (for every $\varepsilon > 0$)

$$\begin{aligned} \langle \partial_{x_i}(T_f), \varphi \rangle &= -\langle T_f, \partial_{x_i} \varphi \rangle \\ &= -\int_{\mathbb{R}^n} f \partial_{x_i} \varphi \\ &= -\int_{\mathbb{R}^{n-1}} \left(\int_{-\varepsilon}^{-\varepsilon} f \partial_{x_i} \varphi dx_i + \int_{-\varepsilon}^{+\varepsilon} f \partial_{x_i} \varphi dx_i + \int_{+\varepsilon}^{+\infty} f \partial_{x_i} \varphi dx_i \right) \prod_{j \neq i} dx_j \\ &= +\int_{\mathbb{R}^{n-1}} \left(\int_{-\varepsilon}^{-\varepsilon} \partial_{x_i} f \varphi dx_i - f(-\varepsilon)\varphi(-\varepsilon) - \int_{-\varepsilon}^{+\varepsilon} f \partial_{x_i} \varphi \right. \\ &\quad \left. + \int_{+\varepsilon}^{+\infty} \partial_{x_i} f \varphi dx_i + f(+\varepsilon)\varphi(+\varepsilon) \right) \prod_{j \neq i} dx_j \end{aligned}$$

(with the notation $g(\pm\varepsilon) := g(x_1, \dots, \pm\varepsilon, \dots, x_n)$, at the j th place, for $g = f, \varphi$). It comes, with $\varepsilon \rightarrow 0$

$$\langle D_{x_i} T_f, \varphi \rangle = \int \partial_{x_i} f \varphi = \langle T_{\partial_{x_i} f}, \varphi \rangle.$$

III. FOURIER TRANSFORM OF THE SINGULAR PART : SI

A. Theorem for change of variable

Theorem III.1 Let $\varphi : U \rightarrow D$ a \mathcal{C}^1 diffeomorphism. Then, for all measurable (resp. integrable) function $f : D \rightarrow \mathbb{R}_+$ (resp. \mathbb{R}),

$$\int_{D=\varphi(U)} f(x) dx = \int_U f(\varphi(u)) |\det(\varphi'(u))| du.$$

B. Spherical properties

Here, we prove a set of lemma necessary to obtain explicitly the FT of the singular part of the LS Equation.

In the multidimensional case, we have

$$\int_{R\mathbb{S}^{n-1}} f(x) d\sigma(x) = \int_{\varphi_1=0}^{\pi} \cdots \int_{\varphi_{n-2}=0}^{\pi} \int_{\varphi_{n-1}=0}^{2\pi} f(x(\varphi)) J(n, R, \varphi) d\varphi_{n-1} \cdots d\varphi_1 \quad (\text{III.37})$$

Where

$$\begin{aligned} x_1 &= R \cos(\varphi_1), \\ x_2 &= R \sin(\varphi_1) \cos(\varphi_2), \\ x_3 &= R \sin(\varphi_1) \sin(\varphi_2) \cos(\varphi_3), \\ &\vdots \\ x_{n-1} &= R \sin(\varphi_1) \cdots \sin(\varphi_{n-2}) \cos(\varphi_{n-1}), \\ x_n &= R \sin(\varphi_1) \cdots \sin(\varphi_{n-2}) \sin(\varphi_{n-1}), \end{aligned}$$

and

$$J(n, R, \varphi) = R^{n-1} (\sin(\varphi_1))^{n-2} \cdots (\sin(\varphi_{n-3}))^2 \sin(\varphi_{n-2}). \quad (\text{III.38})$$

Note that $\varphi_1, \varphi_2, \dots, \varphi_{n-2} \in [0, \pi]$ and $\varphi_{n-1} \in [0, 2\pi]$.

Lemma III.2 *Let f be in \mathbb{S}^{n-1} and A a rotation. Then*

$$\int_{\mathbb{S}^{n-1}} f(A\theta) d\theta = \int_{\mathbb{S}^{n-1}} f(\theta) d\theta.$$

We define a function $\tilde{f}: B(0, 1) \rightarrow \mathbb{R}$ by

$$\tilde{f}(y) := c_n f\left(\frac{y}{\|y\|}\right), \quad \forall y \in B(0, 1)$$

with $c_n := \frac{1}{\int_0^1 r^{n-1} dr}$.

We have, then

$$\int_{\mathbb{S}^{n-1}} f(\theta) d\theta = \int_{B(0,1)} \tilde{f}(x) dx,$$

since, using spherical coordinates

$$\begin{aligned} \int_{B(0,1)} \tilde{f} &= \int_{r=0}^1 \int_{\varphi_1=0}^{\pi} \cdots \int_{\varphi_{n-2}=0}^{\pi} \int_{\varphi_{n-1}=0}^{2\pi} \tilde{f}(rx(\varphi)) J(n, r, \varphi) d\varphi_{n-1} \cdots d\varphi_1 dr \\ &= c_n \int_{r=0}^1 \int_{\varphi_1=0}^{\pi} \cdots \int_{\varphi_{n-2}=0}^{\pi} \int_{\varphi_{n-1}=0}^{2\pi} f(x(\varphi)) r^{n-1} J(n, 1, \varphi) d\varphi_{n-1} \cdots d\varphi_1 dr \\ &= c_n \int_{r=0}^1 r^{n-1} dr \int_{\varphi_1=0}^{\pi} \cdots \int_{\varphi_{n-2}=0}^{\pi} \int_{\varphi_{n-1}=0}^{2\pi} f(x(\varphi)) J(n, 1, \varphi) d\varphi_{n-1} \cdots d\varphi_1 \\ &= \int_{\varphi_1=0}^{\pi} \cdots \int_{\varphi_{n-2}=0}^{\pi} \int_{\varphi_{n-1}=0}^{2\pi} f(x(\varphi)) J(n, 1, \varphi) d\varphi_{n-1} \cdots d\varphi_1 = \int_{\mathbb{S}^{n-1}} f(\theta) d\theta. \end{aligned}$$

If A is a rotation, $\|Ay\| = \|y\|$, and it follows that

$$\begin{aligned} \int_{\mathbb{S}^{n-1}} f(A\theta) d\theta &= \int_{B(0,1)} \tilde{f}(Ax) dx \\ &= \int_{B(0,1)} \tilde{f}(x) dx && \text{(using Theorem III.1)} \\ &= \int_{\mathbb{S}^{n-1}} f(\theta) d\theta \end{aligned}$$

Lemma III.3

$$\int_{\mathbb{S}^{n-1}} f(\theta_i) g(\theta_1, \dots, \theta_i, \dots, \theta_n) d\theta = \int_{\mathbb{S}^{n-1}} f(\theta_1) g(\theta_i, \dots, -\theta_1, \dots, \theta_n) d\theta.$$

Let A be the rotation defined by $Ae_i = e_1, Ae_1 = -e_i, Ae_j = e_j$ elsewhere ; then (since $'Ae_1 = e_i$ and $'Ae_i = -e_1$)

$$\begin{aligned} \int_{\mathbb{S}^{n-1}} f(\theta_i) g(\theta_1, \dots, \theta_i, \dots, \theta_n) d\theta &= \int_{\mathbb{S}^{n-1}} f(e_i \cdot \theta) g(e_1 \cdot \theta, \dots, e_i \cdot \theta, \dots, e_n \cdot \theta) d\theta \\ &= \int_{\mathbb{S}^{n-1}} f(Ae_1 \cdot \theta) g(A'Ae_1 \cdot \theta, \dots, A'Ae_i \cdot \theta, \dots, A'Ae_n \cdot \theta) d\theta \\ &= \int_{\mathbb{S}^{n-1}} f(e_1 \cdot 'A\theta) g('Ae_1 \cdot 'A\theta, \dots, 'Ae_i \cdot 'A\theta, \dots, 'Ae_n \cdot 'A\theta) d\theta \\ &= \int_{\mathbb{S}^{n-1}} f(e_1 \cdot \theta) g('Ae_1 \cdot \theta, \dots, 'Ae_i \cdot \theta, \dots, 'Ae_n \cdot \theta) d\theta \\ &= \int_{\mathbb{S}^{n-1}} f(\theta_1) g(\theta_i, \dots, -\theta_1, \dots, \theta_n) d\theta \end{aligned}$$

Lemma III.4 Let $\alpha_1, \dots, \alpha_n$ be positive integers.

$$\exists j \neq i, \quad \alpha_j \text{ is odd} \quad \implies \quad \int_{\mathbb{S}^{n-1}} f(\theta_i) \prod_{j=1}^n \theta_j^{\alpha_j} d\theta = 0.$$

Introducing $\tau := (1 \ i)$, the permutation between i^{th} and 1^{st} elements, defined by $\tau(1) = i, \tau(i) = 1$ and $\tau(j) = j$ if $j \neq 1$ and i , we have by Lemma III.3, with the slight notation abuse $\tau(\alpha_j) := \alpha_{\tau(j)}$,

$$\int_{\mathbb{S}^{n-1}} f(\theta_i) \prod_{j=1}^n \theta_j^{\alpha_j} d\theta = (-1)^{\alpha_i} \int_{\mathbb{S}^{n-1}} f(\theta_1) \prod_{j=1}^n \theta_j^{\tau(\alpha_j)} d\theta.$$

Now, let us set

$$k := \inf\{j \neq i / \tau(\alpha_j) \text{ is odd}\}.$$

If the hypothesis ($\exists j \neq i, \alpha_j$ is odd) is true, then $k \geq 2$. Let us denote $\beta_j := \tau(\alpha_j)$.

We have

$$\theta_j = \cos(\varphi_1) \sin(\varphi_2) \cdots \sin(\varphi_{j-1}) \cos(\varphi_j), \quad j \leq n-1 \quad (\text{III.39})$$

$$\theta_n = \cos(\varphi_1) \sin(\varphi_2) \cdots \sin(\varphi_{n-2}) \sin(\varphi_{n-1}), \quad (\text{III.40})$$

and (recalling that $J(n, 1, \varphi) = (\sin(\varphi_1))^{n-2} \cdots (\sin(\varphi_{n-3}))^2 \sin(\varphi_{n-2})$, using the Fubini Theorem, one of the factors is,

- If $k = n$,

$$\int_0^{2\pi} (\sin(\varphi_{n-1}))^{\beta_n} d\varphi_{n-1} = 0$$

since β_n is supposed to be odd;

- if $k = n-1$,

$$\int_0^{2\pi} (\cos(\varphi_{n-1}))^{\beta_{n-1}} (\sin(\varphi_{n-1}))^{\beta_n} d\varphi_{n-1} = 0$$

- if $k \leq n-2$

$$\begin{aligned} &\int_0^\pi (\cos(\varphi_k))^{\beta_k} \prod_{j=k+1}^n (\sin(\varphi_k))^{\beta_j} (\sin(\varphi_k))^{n-1-k} d\varphi_k \\ &= \int_0^\pi (\cos(\varphi_k))^{\beta_k} (\sin(\varphi_k))^{\sum_{j=k+1}^n \beta_j + n-1-k} d\varphi_k \\ &= \int_0^\pi (\cos(\varphi_k))^{\beta_k} (\sin(\varphi_k))^\gamma d\varphi_k = 0, \end{aligned}$$

indeed, since β_k is supposed to be odd, we have,

$$\begin{aligned} \int_0^\pi (\cos(\varphi_k))^{\beta_k} (\sin(\varphi_k))^\gamma d\varphi_k &= \int_{-\pi/2}^{\pi/2} (\cos(\varphi + \frac{\pi}{2}))^{\beta_k} (\sin(\varphi + \frac{\pi}{2}))^\gamma d\varphi_k \\ &= (-1)^{\beta_k} \int_{-\pi/2}^{\pi/2} \underbrace{(\sin(\varphi))^{\beta_k} (\cos(\varphi))^\gamma}_{\text{Odd function since } \beta_k \text{ is odd}} d\varphi_k = 0. \end{aligned}$$

Note that we have set $\gamma = \sum_{j=k+1}^n \beta_j + n - 1 - k$

C. Detailed computation of the integrals for \mathbb{G}_0

First, we can manage to obtain $\int_{\mathbb{S}^{n-1}} \log |\theta_1| \theta_2^2 d\theta$ from $\int_{\mathbb{S}^{n-1}} \log |\theta_1| \theta_1^2 d\theta$ and $\int_{\mathbb{S}^{n-1}} \log |\theta_1| d\theta$.
With the notation $J(n, \varphi) := J(n, 1, \varphi)$ (cf. (III.38))

$$\begin{aligned} \int g(\varphi_1) \cos^2(\varphi_2) J(n, \varphi) d\varphi &= \int g(\varphi_1) J(n, \varphi) d\varphi - \int g(\varphi_1) \sin^2(\varphi_2) J(n, \varphi) d\varphi \\ &= \int g(\varphi_1) J(n, \varphi) d\varphi - \frac{n-2}{n-1} \int g(\varphi_1) J(n, \varphi) d\varphi \\ &= \frac{1}{n-1} \int g(\varphi_1) J(n, \varphi) d\varphi \end{aligned}$$

Using Fubini and the properties of Wallis integrals : $W_n = \int_0^{\pi/2} \sin^n(x) dx$ that obey the well-known relation $(n-1)W_{n-2} = nW_n$, for all $n \geq 2$. (here $n \geq 4$),

$$\begin{aligned} \int_0^\pi \sin^2(\varphi_2) \sin(\varphi_2)^{n-3} d\varphi_2 &= 2W_{n-1} \\ &= \frac{n-2}{n-1} 2W_{n-3} \\ &= \frac{n-2}{n-1} \int_0^\pi \sin(\varphi_2)^{n-3} d\varphi_2 \end{aligned}$$

So

$$\begin{aligned} \int_{\mathbb{S}^{n-1}} \log |\theta_1| \theta_2^2 d\theta &= \int \log |\cos(\varphi_1)| \sin^2(\varphi_1) \cos^2(\varphi_2) J(n, \varphi) d\varphi \\ &= \frac{1}{n-1} \int \log |\cos(\varphi_1)| \sin^2(\varphi_1) J(n, \varphi) d\varphi \\ &= \frac{1}{n-1} \left(\int_{\mathbb{S}^{n-1}} \log |\theta_1| d\theta - \int_{\mathbb{S}^{n-1}} \log |\theta_1| \theta_1^2 d\theta \right) \end{aligned}$$

To compute these integrals, we use [23, p. 442], that links the integral of $\theta \mapsto K(x \cdot \theta)$ on the \mathbb{S}^{n-1} sphere to the following integral on the [0 1] interval:

$$\begin{aligned} \int_0^1 s^\alpha (1-s^2)^\beta \log(s) ds &= \frac{\partial}{\partial \alpha} \int_0^1 s^\alpha (1-s^2)^\beta ds \\ &= \frac{\partial}{2\partial \alpha} \int_0^1 u^{(\alpha-1)/2} (1-u)^\beta du \\ &= \frac{\partial}{2\partial \alpha} B\left(\frac{\alpha+1}{2}, \beta+1\right) \\ &= \frac{1}{4} B\left(\frac{\alpha+1}{2}, \beta+1\right) \left(\psi\left(\frac{\alpha+1}{2}\right) - \psi\left(\frac{\alpha+1}{2} + \beta+1\right) \right) \end{aligned}$$

where $\psi(x) := \frac{\Gamma'(x)}{\Gamma(x)}$ is the digamma function, and B is defined by $B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$, and verifies $\frac{\partial}{\partial x} B(x, y) = B(x, y)(\psi(x) - \psi(x+y))$

Then, using the relations $\Gamma(z+1) = z\Gamma(z)$, $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, $\psi(z+1) = \psi(z) + \frac{1}{z}$, $\psi(\frac{1}{2}) = -2\log(2) - \gamma$, we obtain:

$$\int_{\mathbb{S}^{n-1}} \log |\theta_1| d\theta = -\frac{\pi^{n/2}}{\Gamma(n/2)} (\psi(n/2) + \gamma + 2\log(2)) \quad (\text{III.41})$$

and

$$\int_{\mathbb{S}^{n-1}} \log |\theta_1| \theta_1^2 d\theta = -\frac{\pi^{n/2}}{n\Gamma(n/2)} (\psi(n/2) + \frac{2}{n} + \gamma + 2\log(2) - 2). \quad (\text{III.42})$$

We also obtain

$$\begin{aligned} \int_{\mathbb{S}^{n-1}} \log |\theta_1| \theta_1^2 d\theta &= \frac{1}{n-1} (\int_{\mathbb{S}^{n-1}} \log |\theta_1| d\theta - \int_{\mathbb{S}^{n-1}} \log |\theta_1| \theta_1^2 d\theta) \\ &= -\frac{\pi^{n/2}}{n\Gamma(n/2)} (\psi(n/2) + \frac{2}{n} + \gamma + 2\log(2)). \end{aligned} \quad (\text{III.43})$$

IV. FOURIER TRANSFORM OF THE REGULAR PART : SI

A. Detailed computation of the FT for G_{1I}

1. Imaginary part of \hat{v}_1

First, the FT of $r \mapsto -i \sin(k_b r) T_{\frac{1}{r^3}} + ik_b \cos(k_b r) T_{\frac{1}{r^2}} + ik_b^2 \sin(k_b r) T_{\frac{1}{r}}$ is the function $k \mapsto V_{1I}(k)$, given by (cf. Corollaries II.5 and II.10) :

$$\begin{aligned} V_{1I}(k) &:= \frac{-1}{2} (\tau_{k_b} \widehat{T}_{\frac{1}{r^3}} - \tau_{-k_b} \widehat{T}_{\frac{1}{r^3}}) + i \frac{k_b}{2} (\tau_{k_b} \widehat{T}_{\frac{1}{r^2}} + \tau_{-k_b} \widehat{T}_{\frac{1}{r^2}}) + \frac{k_b^2}{2} (\tau_{k_b} \widehat{T}_{\frac{1}{r}} - \tau_{-k_b} \widehat{T}_{\frac{1}{r}}) \\ &= \frac{-1}{2} \left(\frac{i\pi}{2} (k-k_b)^2 \text{sgn}(k-k_b) - \frac{i\pi}{2} (k+k_b)^2 \text{sgn}(k+k_b) \right) + i \frac{k_b}{2} \left(-\pi(k-k_b) \text{sgn}(k-k_b) - \pi(k+k_b) \text{sgn}(k+k_b) \right) \\ &\quad + \frac{k_b^2}{2} \left(-i\pi \text{sgn}(k-k_b) + i\pi \text{sgn}(k+k_b) \right) \\ &= \frac{i\pi}{4} (k^2 + k_b^2) (\text{sgn}(k+k_b) - \text{sgn}(k-k_b)). \end{aligned}$$

2. Real part of \hat{v}_1

The FT of $r \mapsto (1 - \cos(k_b r)) T_{\frac{1}{|r|^3}} - k_b \sin(k_b r) T_{\frac{\text{sgn}(r)}{r^2}} + k_b^2 \cos(k_b r) T_{\frac{1}{|r|}}$ is the function $k \mapsto V_{1R}(k)$ given by (cf. Corollaries II.5 and II.10) :

$$\begin{aligned} V_{1R}(k) &:= \widehat{T}_{\frac{1}{|r|^3}} - \frac{1}{2} (\tau_{k_b} \widehat{T}_{\frac{1}{|r|^3}} + \tau_{-k_b} \widehat{T}_{\frac{1}{|r|^3}}) - \frac{k_b}{2i} (\tau_{k_b} \widehat{T}_{\frac{\text{sgn}(r)}{r^2}} - \tau_{-k_b} \widehat{T}_{\frac{\text{sgn}(r)}{r^2}}) + \frac{k_b^2}{2} (\tau_{k_b} \widehat{T}_{\frac{1}{|r|}} + \tau_{-k_b} \widehat{T}_{\frac{1}{|r|}}) \\ &= -\frac{1}{2} \left((k-k_b)^2 (\gamma + \ln(|k-k_b|)) + (k+k_b)^2 (\gamma + \ln(|k+k_b|)) \right) + k^2 (\gamma + \ln(|k|)) \\ &\quad - k_b \left((k-k_b) (\gamma + \ln(|k-k_b|)) - (k+k_b) (\gamma + \ln(|k+k_b|)) \right) - k_b^2 \left((\gamma + \ln(|k-k_b|)) + (\gamma + \ln(|k+k_b|)) \right) \\ &= -k_b^2 \gamma + k^2 \ln(|k|) - \frac{1}{2} (k^2 + k_b^2) (\ln(|k-k_b|) + \ln(|k+k_b|)) \end{aligned}$$

3. Computation of \hat{v}_3 : real part

We give below the detail of the computation of $-\frac{\omega_2}{2} \langle k T_{\frac{1}{k-k_b}} + k T_{\frac{1}{k+k_b}}, \varphi^o(k) \rangle$.

As shown in the main text,

$$-\frac{\omega_2}{2} \langle kT_{\frac{1}{k-k_b}} + kT_{\frac{1}{k+k_b}}, \varphi^o(k) \rangle = -\frac{\omega_2}{2} \lim_{\varepsilon \rightarrow 0^+} \left(\int_{-\infty}^{k_b-\varepsilon} \frac{k}{k-k_b} \varphi^o(k) dk + \int_{k_b+\varepsilon}^{+\infty} \frac{k}{k-k_b} \varphi^o(k) dk + \int_{-\infty}^{-k_b-\varepsilon} \frac{k}{k+k_b} \varphi^o(k) dk + \int_{-k_b+\varepsilon}^{+\infty} \frac{k}{k+k_b} \varphi^o(k) dk \right)$$

But

$$\begin{aligned} & \int_{-\infty}^{k_b-\varepsilon} \frac{k}{k-k_b} \varphi^o(k) dk + \int_{k_b+\varepsilon}^{+\infty} \frac{k}{k-k_b} \varphi^o(k) dk + \int_{-\infty}^{-k_b-\varepsilon} \frac{k}{k+k_b} \varphi^o(k) dk + \int_{-k_b+\varepsilon}^{+\infty} \frac{k}{k+k_b} \varphi^o(k) dk = \\ & \int_{-\infty}^0 \frac{k}{k-k_b} \varphi^o(k) dk + \int_0^{k_b-\varepsilon} \frac{k}{k-k_b} \varphi^o(k) dk + \int_{k_b+\varepsilon}^{+\infty} \frac{k}{k-k_b} \varphi^o(k) dk \\ & + \int_{-\infty}^{-k_b-\varepsilon} \frac{k}{k+k_b} \varphi^o(k) dk + \int_{-k_b+\varepsilon}^0 \frac{k}{k+k_b} \varphi^o(k) dk + \int_0^{+\infty} \frac{k}{k+k_b} \varphi^o(k) dk \\ & = 2 \times \left(\int_0^{k_b-\varepsilon} \frac{k}{k-k_b} \varphi^o(k) dk + \int_{k_b+\varepsilon}^{+\infty} \frac{k}{k-k_b} \varphi^o(k) dk + \int_0^{+\infty} \frac{k}{k+k_b} \varphi^o(k) dk \right) \\ & = \frac{2}{\omega_2} \left(\int_{B(0, k_b-\varepsilon)} \frac{1}{|x|(|x|-k_b)} \varphi^o(|x|) dx + \int_{\mathbb{R}^3 \setminus B(0, k_b+\varepsilon)} \frac{1}{|x|(|x|-k_b)} \varphi^o(|x|) dx + \int_{\mathbb{R}^3} \frac{1}{|x|(|x|+k_b)} \varphi^o(|x|) dx \right). \end{aligned} \quad (\text{IV.44})$$

Using the relations

$$\frac{1}{|x|(|x|+k_b)} = \frac{1}{k_b} \left(\frac{1}{|x|} - \frac{1}{|x|+k_b} \right), \quad \frac{1}{|x|(|x|-k_b)} = \frac{1}{k_b} \left(\frac{1}{|x|-k_b} - \frac{1}{|x|} \right),$$

and denoting: $B_{ext} := B(0, k_b - \varepsilon) \cup B^c(0, k_b + \varepsilon)$, $B_{surf} := \mathbb{R}^3 \setminus \{B(0, k_b - \varepsilon) \cup B^c(0, k_b + \varepsilon)\}$, Eq. (IV.44) finally reads:

$$\frac{2}{k_b \omega_2} \left(\int_{B_{ext}} \frac{1}{|x|-k_b} \varphi^o(|x|) dx - \int_{\mathbb{R}^3} \frac{1}{|x|+k_b} \varphi^o(|x|) dx + \int_{B_{surf}} \frac{1}{|x|} \varphi^o(|x|) dx \right)$$

Which eventually brings:

$$-\frac{\omega_2}{2} \left\langle kT_{\frac{1}{k-k_b}} + kT_{\frac{1}{k+k_b}}, \varphi^o(k) \right\rangle = \frac{1}{k_b} \left(-\left\langle \text{vp} \left(\frac{1}{|x|-k_b} \right), \varphi(x) \right\rangle + \left\langle \frac{1}{|x|+k_b}, \varphi(x) \right\rangle \right)$$

B. Detailed computation of the FT for $G_{1Q}(l)\underline{Q}$

1. Imaginary part of \hat{v}_1

The FT of $r \mapsto 3i \sin(k_b r) T_{\frac{1}{r^3}} - 3ik_b \cos(k_b r) T_{\frac{1}{r^4}} - ik_b^2 \sin(k_b r) T_{\frac{1}{r^3}}$ is the function $k \mapsto U_{1I}(k)$ given by (cf. Corollaries II.5 and II.10) :

$$\begin{aligned} U_{1I}(k) &= \frac{3}{2} \left(\tau_{k_b} \widehat{T}_{\frac{1}{r^3}} - \tau_{-k_b} \widehat{T}_{\frac{1}{r^3}} \right) - k_b i \frac{3}{2} \left(\tau_{k_b} \widehat{T}_{\frac{1}{r^4}} + \tau_{-k_b} \widehat{T}_{\frac{1}{r^4}} \right) - \frac{k_b^2}{2} \left(\tau_{k_b} \widehat{T}_{\frac{1}{r^3}} - \tau_{-k_b} \widehat{T}_{\frac{1}{r^3}} \right) \\ &= \frac{3}{2} \left(\frac{-i\pi}{24} (k-k_b)^4 \text{sgn}(k-k_b) + \frac{i\pi}{24} (k+k_b)^4 \text{sgn}(k+k_b) \right) - ik_b \frac{3}{2} \left(\frac{\pi}{6} (k-k_b)^3 \text{sgn}(k-k_b) + \frac{\pi}{6} (k+k_b)^3 \text{sgn}(k+k_b) \right) \\ &\quad - \frac{k_b^2}{4} \left(i\pi (k-k_b)^2 \text{sgn}(k-k_b) - i\pi (k+k_b)^2 \text{sgn}(k+k_b) \right) \\ &= i\pi \left[-\left(\frac{1}{16} (k-k_b)^4 + \frac{k_b}{4} (k-k_b)^3 + \frac{k_b^2}{4} (k-k_b)^2 \right) \text{sgn}(k-k_b) + \left(\frac{1}{16} (k+k_b)^4 - \frac{k_b}{4} (k+k_b)^3 + \frac{k_b^2}{4} (k+k_b)^2 \right) \text{sgn}(k+k_b) \right] \\ &= \frac{i\pi}{16} (k+k_b)^2 (k-k_b)^2 (\text{sgn}(k+k_b) - \text{sgn}(k-k_b)). \end{aligned}$$

2. Real part of \hat{v}_1

The FT of $r \mapsto 3(\cos(k_b r) - 1)T_{\frac{1}{|r|^5}} + 3k_b \sin(k_b r)T_{\frac{\text{sgn}(r)}{r^4}} - k_b^2 \cos(k_b r)T_{\frac{1}{|r|^3}}$ is the function $k \mapsto U_{1R}(k)$ given by (cf. Corollaries II.5 and II.10) :

$$\begin{aligned} U_{1R}(k) &:= \frac{3}{2} \left(\tau_{k_b} \widehat{T}_{\frac{1}{|r|^5}} + \tau_{-k_b} \widehat{T}_{\frac{1}{|r|^5}} \right) - 3 \widehat{T}_{\frac{1}{|r|^5}} - k_b^2 \frac{3}{2} \left(\tau_{k_b} \widehat{T}_{\frac{\text{sgn}(r)}{r^4}} - \tau_{-k_b} \widehat{T}_{\frac{\text{sgn}(r)}{r^4}} \right) - \frac{k_b^2}{2} \left(\tau_{k_b} \widehat{T}_{\frac{1}{|r|^3}} + \tau_{-k_b} \widehat{T}_{\frac{1}{|r|^3}} \right) \\ &= -\frac{1}{8} \left((k - k_b)^4 (\gamma + \ln(|k - k_b|)) + (k + k_b)^4 (\gamma + \ln(|k + k_b|)) \right) + \frac{1}{4} k^4 (\gamma + \ln(|k|)) \\ &\quad - \frac{k_b}{2} \left((k - k_b)^3 (\gamma + \ln(|k - k_b|)) - (k + k_b)^3 (\gamma + \ln(|k + k_b|)) \right) \\ &\quad - \frac{k_b^2}{2} \left((k - k_b)^2 (\gamma + \ln(|k - k_b|)) + (k + k_b)^2 (\gamma + \ln(|k + k_b|)) \right) \end{aligned}$$

Using again the factorisation $\frac{1}{4}(k - k_b)^4 + k_b(k - k_b)^3 + k_b^2(k - k_b)^2 = \frac{1}{4}(k + k_b)^2(k - k_b)^2$ and $\frac{1}{4}(k + k_b)^4 - k_b(k + k_b)^3 + k_b^2(k + k_b)^2 = \frac{1}{4}(k + k_b)^2(k - k_b)^2$, a nice and straightforward simplification follows.

To sum up, we already obtained :

$$\Re(\hat{v}_1) = \frac{k^4}{4} (\gamma + \ln(|k|)) - \frac{1}{8} (k + k_b)^2 (k - k_b)^2 (2\gamma + \ln(|k^2 - k_b^2|)) \quad (\text{IV.45})$$

$$\Im(\hat{v}_1) = \frac{\pi}{16} (k + k_b)^2 (k - k_b)^2 (\text{sgn}(k + k_b) - \text{sgn}(k - k_b)). \quad (\text{IV.46})$$

3. Computation of the derivatives

We need to compute the derivative of \hat{v}_1 expressions (in the sense of the distributions) ; using the Lemma II.12, we obtain, for the real part :

$$D_k \Re(\hat{v}_1) = k k_b^2 \left(\gamma + \frac{1}{4} \right) + k^3 \ln(|k|) - \frac{1}{2} k (k + k_b) (k - k_b) \ln(|k^2 - k_b^2|) \quad (\text{IV.47})$$

For the imaginary part, we obtain:

$$D_k \Im(\hat{v}_1) = \frac{\pi}{4} k (k + k_b) (k - k_b) (\text{sgn}(k + k_b) - \text{sgn}(k - k_b)) \quad (\text{IV.48})$$

4. Computation of $\mathcal{F}(T)$

To compute \hat{v}_3 , using the Corollary IV.5 in the main text, we need to evaluate the scalar product between $k D_k \hat{v}_1$ and a test function $k \mapsto \varphi^o(k)$. For the real part, we get:

$$\begin{aligned} \frac{\omega_2}{2} \left\langle k \left(k k_b^2 \left(\gamma + \frac{1}{4} \right) + k^3 \ln(|k|) \right), \varphi^o(k) \right\rangle &= \frac{\omega_2}{2} \int_{\mathbb{R}} \left(k^2 k_b^2 \left(\gamma + \frac{1}{4} \right) + k^4 \ln(|k|) \right) \varphi^o(k) dk \\ &= \omega_2 \int_0^{+\infty} \left(k^2 k_b^2 \left(\gamma + \frac{1}{4} \right) + k^4 \ln(k) \right) \varphi^o(k) dk \\ &= \int_{\mathbb{R}^3} k_b^2 \left(\gamma + \frac{1}{4} \right) + |x|^2 \ln(|x|) \varphi^o(|x|) dx \\ &= \left\langle k_b^2 \left(\gamma + \frac{1}{4} \right) + l^2 \ln(l), \varphi \right\rangle \end{aligned}$$

and, omiting the prefactor $-1/2$ in Eq. (IV.51) :

$$\begin{aligned}
\frac{\omega_2}{2} \left\langle k(k^2 - k_b^2) (\ln(|k + k_b|) + \ln(|k - k_b|)), \varphi^o(k) \right\rangle &= \frac{\omega_2}{2} \int_{\mathbb{R}} (k^2(k^2 - k_b^2) (\ln(|k + k_b|) + \ln(|k - k_b|))) \varphi^o(k) dk \\
&= \omega_2 \int_0^{+\infty} (k^2(k^2 - k_b^2) (\ln(|k + k_b|) + \ln(|k - k_b|))) \varphi^o(k) dk \\
&= \int_{\mathbb{R}^3} (|x|^2 - k_b^2) (\ln(|x| + k_b) + \ln(|x| - k_b)) \varphi^o(|x|) dx \\
&= \left\langle (l^2 - k_b^2) (\ln(|l + k_b|) + \ln(|l - k_b|)), \varphi \right\rangle
\end{aligned}$$

For the imaginary part, omitting the prefactor $\pi/4$ in Eq. (IV.52), we obtain:

$$\begin{aligned}
\frac{\omega_2}{2} \langle k^2(k^2 - k_b^2) (\text{sgn}(k + k_b) - \text{sgn}(k - k_b)), \varphi^o(k) \rangle &= \frac{\omega_2}{2} \int_{\mathbb{R}} (k^2(k^2 - k_b^2) (\text{sgn}(k + k_b) - \text{sgn}(k - k_b))) \varphi^o(k) dk \\
&= \omega_2 \int_0^{+\infty} (k^2(k^2 - k_b^2) (\text{sgn}(k + k_b) - \text{sgn}(k - k_b))) \varphi^o(k) dk \\
&= \omega_2 \left(\int_0^{+\infty} k^2(k^2 - k_b^2) \varphi^o(k) dk - \int_0^{k_b} -(k^2 - k_b^2) k^2 \varphi^o(k) dk \right. \\
&\quad \left. + \int_{k_b}^{\infty} -k^2(k^2 - k_b^2) \varphi^o(k) dk \right) \\
&= \omega_2 \int_0^{k_b} 2k^2(k^2 - k_b^2) \varphi^o(k) dk \\
&= 2 \int_{B(0, k_b)} (|x|^2 - k_b^2) \varphi^o(|x|) dx = 2 \left\langle \mathbb{1}_{B(0, k_b)}(l) (l^2 - k_b^2), \varphi \right\rangle
\end{aligned}$$

Gathering all the terms, we have

$$\frac{-2\gamma_3}{\omega_2} = k_b^2 \left(\gamma + \frac{1}{4} \right) + k^2 \ln(k) - \frac{1}{2} (k^2 - k_b^2) \ln(|k^2 - k_b^2|) + i \frac{\pi}{2} \mathbb{1}_{B(0, k_b)}(k) (k^2 - k_b^2) \quad (\text{IV.49})$$

5. Green formula

Definition IV.1 Let Ω be an open set of \mathbb{R}^n , and $k \in \mathbb{N}^* \cup \{\infty\}$. Ω is said to be a \mathcal{C}^k -open set if there exists a function $\rho \in \mathcal{C}^k(\mathbb{R}^n, \mathbb{R})$ such that

$$\begin{cases} \Omega = \{x \in \mathbb{R}^n : \rho(x) < 0\}, \\ \partial\Omega = \{x \in \mathbb{R}^n : \rho(x) = 0\}, \text{ grad}\rho(x) \neq 0, \quad \forall x \in \partial\Omega, \end{cases} \quad (\text{IV.50})$$

with $\partial\Omega := \overset{\circ}{\Omega} \setminus \overline{\Omega}$.

Remark IV.2 $\Omega := \{x \in \mathbb{R}^n, |x| < R\}$ (resp. $\Omega := \{x \in \mathbb{R}^n, |x| < R\}$) is a \mathcal{C}^∞ -open set with $\rho(x) := |x|^2 - R^2$.

Theorem IV.3 Let Ω be a \mathcal{C}^1 -open set (and ρ the associated function). There exists a positive measure $d\sigma$ on $\partial\Omega$ such that for all $\varphi \in \mathcal{C}_0^1(\mathbb{R}^n)$ and all $f_1, \dots, f_n \in \mathcal{C}^1(\mathbb{R}^n)$

$$\int_{\Omega} \text{div}F(x) \varphi(x) dx = - \int_{\Omega} F(x) \cdot \text{grad}\varphi(x) dx + \int_{\partial\Omega} \varphi(x) F(x) \cdot n(x) d\sigma \quad (\text{IV.51})$$

where $F(x) = (f_1(x), \dots, f_n(x))$, $X \cdot Y = \sum_{i=1}^n X_i Y_i$ and $n(x) := \frac{\text{grad}\rho(x)}{\|\text{grad}\rho(x)\|}$.

Corollary IV.4 For $\Omega = B(0, r)$ (resp. $\Omega = \overline{B(0, r)}^c$) \mathcal{C}^2 .

6. Partial Derivative of $\mathcal{F}(T)$: detailed computation

The first derivative of the function $k \mapsto (k^2 - k_b^2) \ln(|k^2 - k_b^2|)$ is

$$\partial_i f(k) = 2k_i (\ln(|k + k_b|) + \ln(|k - k_b|)) + 2k_i$$

, which is not L^1 anymore.

We will use the Green Formula (see Theorem IV.3) in order to compute $\partial_i \partial_j f(k)$ with $\Omega_{-\varepsilon} := B(0, k_b - \varepsilon)$ (resp. $\Omega_\varepsilon := B^c(0, k_b + \varepsilon)$).

$$\int_{\Omega_{-\varepsilon}} f(x) \partial_i \psi(x) dx = - \int_{\Omega_{-\varepsilon}} \partial_i f(x) \psi(x) dx + \int_{\partial\Omega_{-\varepsilon}} \psi(x) f(x) \cdot n_i^{-\varepsilon}(x) d\sigma$$

Using again this formula for $\psi := \partial_j \varphi$, we have

$$\begin{aligned} \int_{\Omega_{-\varepsilon}} f(x) \partial_i \partial_j \varphi(x) dx &= \int_{\Omega_{-\varepsilon}} f(x) \partial_i \psi(x) dx \\ &= - \int_{\Omega_{-\varepsilon}} \partial_i f(x) \psi(x) dx + \int_{\partial\Omega_{-\varepsilon}} \psi(x) f(x) n_i^{-\varepsilon}(x) d\sigma \\ &= \int_{\Omega_{-\varepsilon}} \partial_j \partial_i f(x) \varphi(x) dx - \int_{\partial\Omega_{-\varepsilon}} \varphi(x) \partial_i f(x) n_j^{-\varepsilon}(x) d\sigma + \int_{\partial\Omega_{-\varepsilon}} \psi(x) f(x) n_i^{-\varepsilon}(x) d\sigma \end{aligned} \quad (\text{IV.52})$$

By dominated convergence, $\lim_{\varepsilon \rightarrow 0} \int_{\partial\Omega_{\pm\varepsilon}} \psi(x) f(x) n_i^{\pm\varepsilon}(x) d\sigma = 0$, so

$$\begin{aligned} \int_{\mathbb{R}^3} f(x) \partial_i \partial_j \varphi(x) dx &= \lim_{\varepsilon \rightarrow 0} \int_{\Omega_{-\varepsilon} \cup \Omega_\varepsilon} \partial_j \partial_i f(x) \varphi(x) dx \\ &\quad - \int_{\partial\Omega_{-\varepsilon}} \varphi(x) \partial_i f(x) n_j^{-\varepsilon}(x) d\sigma \\ &\quad - \int_{\partial\Omega_\varepsilon} \varphi(x) \partial_i f(x) n_j^\varepsilon(x) d\sigma \end{aligned}$$

With $\Theta := (\cos(\theta) \sin(\varphi), \sin(\theta) \sin(\varphi), \cos(\varphi))$, in spherical coordinate, $n^{-\varepsilon} = \Theta$, $n^\varepsilon = -\Theta$ and with $\Theta(\pm\varepsilon) := (k_b \pm \varepsilon)\Theta$, (and with $J_{\varphi,\varepsilon} := (k_b - \varepsilon)^2 \sin(\varphi)$)

$$\begin{aligned} &\int_{\partial\Omega_{-\varepsilon}} \varphi(x) \partial_i f(x) n_j^{-\varepsilon}(x) d\sigma + \int_{\partial\Omega_\varepsilon} \varphi(x) \partial_i f(x) n_j^\varepsilon(x) d\sigma \\ &= \int_0^{2\pi} \int_0^\pi \varphi(\Theta(-\varepsilon)) (2(k_b - \varepsilon)\Theta_i (\ln(\varepsilon) + \ln(|2k_b - \varepsilon|)) + 2(k_b - \varepsilon)\Theta_i) \Theta_j J_{\varphi,\varepsilon} d\theta d\varphi \\ &\quad - \int_0^{2\pi} \int_0^\pi \varphi(\Theta(\varepsilon)) (2(k_b + \varepsilon)\Theta_i (\ln(\varepsilon) + \ln(|2k_b + \varepsilon|)) + 2(k_b + \varepsilon)\Theta_i) \Theta_j J_{\varphi,\varepsilon} d\theta d\varphi \\ &= \int_0^{2\pi} \int_0^\pi \Theta_i \Theta_j \ln(\varepsilon) 2 \left((k_b - \varepsilon)\varphi(\Theta(-\varepsilon)) - (k_b + \varepsilon)\varphi(\Theta(\varepsilon)) \right) J_{\varphi,\varepsilon} d\theta d\varphi + R(\varepsilon) \\ &= \int_0^{2\pi} \int_0^\pi \Theta_i \Theta_j \varepsilon \ln(\varepsilon) \left(\frac{(k_b - \varepsilon)\varphi(\Theta(-\varepsilon)) - (k_b + \varepsilon)\varphi(\Theta(\varepsilon))}{\varepsilon} \right) J_{\varphi,\varepsilon} d\theta d\varphi + R(\varepsilon) \end{aligned}$$

where the quantity $R(\varepsilon)$ is the part which converge to zero by a straightforward application of the dominated convergence theorem.

Noticing that

$$\frac{(k_b - \varepsilon)\varphi(\Theta(-\varepsilon)) - (k_b + \varepsilon)\varphi(\Theta(\varepsilon))}{\varepsilon} = k_b \frac{\varphi(\Theta(-\varepsilon)) - \varphi(\Theta(\varepsilon))}{\varepsilon} - \varphi(\Theta(-\varepsilon)) - \varphi(\Theta(\varepsilon))$$

and recalling that if

$$\partial_v f(a) := \lim_{t \rightarrow 0} \frac{f(a + tv) - f(a)}{t}$$

exists, then

$$\partial_v f(a) = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(a) \cdot v_i,$$

we can apply then again the dominated convergence theorem, since φ is smooth and

$$\begin{aligned} k_b \frac{\varphi(\Theta(-\varepsilon)) - \varphi(\Theta(\varepsilon))}{\varepsilon} &= k_b \frac{\varphi((k_b - \varepsilon)\Theta) - \varphi(k_b\Theta)}{\varepsilon} + k_b \frac{\varphi(k_b\Theta) - \varphi((k_b + \varepsilon)\Theta)}{\varepsilon} \\ &\xrightarrow{\varepsilon \rightarrow 0} k_b \partial_{-\Theta} \varphi(k_b\Theta) - k_b \partial_{\Theta} \varphi(k_b\Theta) \end{aligned}$$

(with $\|\partial_{\pm\Theta} \varphi(k_b\Theta)\| \leq \|\partial_1 \varphi\|_L + \|\partial_2 \varphi\|_L + \|\partial_3 \varphi\|_L$, with L a big enough compact set).

Finally

$$\int_{\mathbb{R}^3} f(x) \partial_i \partial_j \varphi(x) \, dx = \lim_{\varepsilon \rightarrow 0} \int_{\Omega_{-\varepsilon} \cup \Omega_{\varepsilon}} \partial_j \partial_i f(x) \varphi(x) \, dx$$

Eventually, we obtain:

$$D_{k_i} D_{k_j} (k^2 - k_b^2) \ln(|k^2 - k_b^2|) = 2(1 + \ln(|k^2 - k_b^2|)) \delta_{i,j} + 2 \frac{k_i k_j}{k} \left(\text{v.p.} \frac{1}{k - k_b} + \frac{1}{k + k_b} \right)$$

[1] L. Schwartz, *Mathematics for the Physical Sciences*, Hermann and Addison-Wesley Ed. (1966).

[2] M. Rahman, *Application of Fourier transform to generalized functions*, Wit Press (2011).