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A Fixed-Parameter Algorithm for Scheduling Unit Dependent Tasks on Parallel Machines with Time Windows

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Abstract

This paper proves that the existence of a feasible schedule for a set of dependent tasks of unit execution times with release dates and deadlines on a limited number of processors is a fixed-parameter tractable problem. The parameter considered is the pathwidth of the interval graph associated with the time windows of the tasks. A fixed-parameter algorithm based on a dynamic programming approach is developed and proved to solve this decision problem.

Fixed-parameter algorithms for the two classical problems $P|prec, p_i = 1, r_i|C_{\max}$ and $P|prec, p_i = 1, r_i|L_{\max}$ are then derived using a binary search. They are, as far as we know, the first fixed-parameter algorithms for these scheduling problems.

Keywords: parallel identical machines, makespan, maximum lateness, parameterized complexity

1. Introduction

This paper tackles a basic scheduling problem defined as follows: we consider a set T of n tasks of unit duration. Precedence relations are expressed by an (acyclic) precedence graph noted $G = (T, A)$. If s_i^σ denotes the starting time of a task $i \in T$ according to a feasible schedule σ , each arc $e = (i, j) \in A$ expresses the precedence constraint $s_i^\sigma + 1 \leq s_j^\sigma$. Moreover, each task must be executed during its time window, i.e. for each task $i \in T$, $r_i \leq s_i^\sigma < d_i$ where r_i is the release time of i and d_i its deadline. We assume without loss of generality that $\min_{i \in T} r_i = 0$. Lastly, m identical machines are available,

10 so that at most m tasks can be scheduled at each time instant. The ques-
 11 tion is the existence of a feasible schedule. This problem is referred to as
 12 $P|prec, p_i = 1, r_i, d_i|*$ using standard notations [11].

13 Let $C(\sigma) = \max_{i \in T}(s_i^\sigma + 1)$ be the makespan of a schedule σ and $L(\sigma) =$
 14 $\max_{i \in T}(\max(0, s_i^\sigma + 1 - d_i))$ its maximum lateness. Let us consider the op-
 15 timization problem $P|prec, p_i = 1, r_i|C_{\max}$. For each upper bound $C \geq 0$
 16 of the makespan, the deadline of each task $i \in T$ can be set to $d_i = C - \ell_i$
 17 where ℓ_i is the maximum number of arcs of a path starting from i . We ob-
 18 serve that any solution of the corresponding instance of $P|prec, p_i = 1, r_i, d_i|*$
 19 is a feasible solution of $P|prec, p_i = 1, r_i|C_{\max}$. We conclude that any algo-
 20 rithm solving the decision problem $P|prec, p_i = 1, r_i, d_i|*$ can be combined
 21 with binary search to solve $P|prec, p_i = 1, r_i|C_{\max}$. A similar remark can
 22 be made for the optimisation problem $P|prec, p_i = 1, r_i|L_{\max}$. Indeed, for
 23 each instance of this problem and each upper bound $\lambda \geq 0$ of the maximum
 24 lateness, we can set the deadline of each task $i \in T$ to $d'_i = d_i + \lambda$ and solve
 25 the associated decision problem $P|prec, p_i = 1, r_i, d'_i|*$. So, each algorithm
 26 solving the decision problem $P|prec, p_i = 1, r_i, d_i|*$ can thus be extended for
 27 $P|prec, p_i = 1, r_i|C_{\max}$ or $P|prec, p_i = 1, r_i|L_{\max}$ by adding a binary search
 28 on the upper bound of respectively the makespan or the maximum lateness.

29 The optimisation problem $P|prec, p_i = 1|C_{\max}$ has been widely studied. It
 30 was proved to be NP-hard initially for opposing forests by Garey et al. [10].
 31 The recent survey of Prot and Bellenguez-Morineau [15] presents several
 32 complexity results depending on the structure of the precedence graph and
 33 the number of processors.

34 The development of fixed-parameter algorithms for NP-complete prob-
 35 lems is a way to get polynomial-time algorithms when some parameters are
 36 fixed [6, 8]. More formally, a fixed-parameter algorithm solves any instance
 37 of the problem of size n in time $f(k) \times \text{poly}(n)$, where f is allowed to be a
 38 computable superpolynomial function and k the associated parameter.

39 Several authors developed exact algorithms for $P|prec, p_i = 1|C_{\max}$. Dolev
 40 and Warmuth [7] developed a dynamic programming algorithm of time com-
 41 plexity $\mathcal{O}(n^{h(G)(m-1)+1})$ to solve this problem. The parameter $h(G)$ repre-
 42 sents the number of nodes of the longest path of the precedence graph G .
 43 Few years later, Möhring [14] developed another dynamic programming al-
 44 gorithm of time complexity $\mathcal{O}(n^{w(G)})$. The parameter $w(G)$ is the width of
 45 the precedence graph G defined as the size of its largest antichain. None of
 46 these algorithms are fixed-parameter ones.

47 Mnich and van Bevern [13] surveyed main results on parameterized com-

48 plexity for scheduling problems and identified 15 open problems. They point
49 out that the fixed-parameter tractability of $P3|prec, p_i = 1|C_{\max}$ param-
50 eterized by the width $w(G)$ of the precedence graph G is an open chal-
51 lenging question. The width $w(G)$ leads to several negative results. Du et
52 al. [9] proved that $P2|chains|C_{\max}$ is strongly NP-hard for unbounded width.
53 Günther et al. [12] proved that $P2|chains, w(G) \leq 3|C_{\max}$ is weakly NP-hard.
54 Bodlaender and Fellows [4] proved that $P|prec, p_i = 1|C_{\max}$ is W[2]-hard pa-
55 rameterized by the width and the number of machines. More recently, van
56 Bevern et al. [2] proved that $P2|prec, p_j \in \{1, 2\}|C_{\max}$ is W[2]-hard parame-
57 terized by the width $w(G)$, and that this result is an argument to look at the
58 parameterized complexity of $P3|prec, p_i = 1|C_{\max}$ using $w(G)$ as parameter.
59 The existence of a fixed-parameter algorithm for the optimisation problem
60 $P|prec, p_i = 1|C_{\max}$ is a challenging question, but as we described above,
61 most of the authors focused their study on the width $w(G)$ as a parameter.

62 The parameter considered in this paper is the pathwidth $pw(\mathcal{I})$ of an
63 instance \mathcal{I} of the decision problem $P|prec, p_i = 1, r_i, d_i|\star$. Let us consider the
64 strictly increasing sequences u_α , $\alpha \in \mathbb{N}$ corresponding to all the time instants
65 of the beginnings and endings of the intervals $[r_i, d_i]$, $i \in T$. By definition of
66 release dates, $u_0 = 0$. Moreover, the last element u_{k^*} of the sequence satisfies
67 $k^* \leq 2n - 1$. Then, for any $\alpha \in \{1, \dots, k^*\}$, we set $X_\alpha = \{i \in T, (u_{\alpha-1}, u_\alpha) \cap$
68 $(r_i, d_i) \neq \emptyset\}$. The pathwidth is defined as $pw(\mathcal{I}) = \max_{\alpha \in \{1, \dots, k^*\}} (|X_\alpha| - 1)$.

69 The pathwidth $pw(\mathcal{I})$ is a simple measure of the parallelism of the in-
70 stance \mathcal{I} . One can observe that $pw(\mathcal{I})$ is the pathwidth of the interval graph
71 associated with the set of intervals $\{(r_i, d_i), i \in T\}$ [3]. As far as we know,
72 this measure was not yet considered as a parameter for a scheduling problem.

73 We prove in this paper that the decision problem $P|prec, p_i = 1, r_i, d_i|\star$
74 can be solved in time $\mathcal{O}(n^2 2^{4 \times pw(\mathcal{I})})$ using a dynamic programming approach
75 based on a multistage auxiliary graph which longest paths model all the
76 feasible schedules. By coupling our algorithm with a binary search based
77 on a lower bound of the respective objectives, we get that the two opti-
78 misation problems $P|prec, p_i = 1, r_i|C_{\max}$ and $P|prec, p_i = 1, r_i|L_{\max}$ are
79 fixed-parameter tractable parameterized by the pathwidth.

80 Baptiste and Timkovsky [1] developed a similar approach to get a polyno-
81 mial time algorithm of time complexity $\mathcal{O}(n^9)$ for $P2|prec, p_i = 1, r_i|\sum C_j$.
82 They also conjectured the existence of ideal schedules, i.e. that minimize both
83 the makespan and the total completion time for the previous instance. More
84 recently, Coffman et al. [5] proved that each optimal solution of $P2|prec, p_i =$
85 $1, r_i|\sum C_j$ minimizes the makespan of the same instance and they improved

86 the algorithm of Baptiste and Timkowsky to get a complexity in time $\mathcal{O}(n^3)$.

87 This paper is organised as follows: Section 2 presents additional notations
 88 and an example. Section 3 is devoted to the description of properties of
 89 feasible schedules. These properties will be considered in Section 4 to build
 90 the multistage auxiliary graph \mathcal{G} whose longest paths represent all the feasible
 91 solutions (if any). Section 5 details the complexity of the construction of \mathcal{G} .
 92 Section 6 is our conclusion.

93 2. Additional notations and example

94 For any task $i \in T$, the set of immediate successors and predecessors of i is
 95 respectively denoted by $\Gamma^+(i)$ and $\Gamma^-(i)$. For any couple of tasks $(i, j) \in T^2$,
 96 we denote by $i \rightarrow j$ if there is a path in G from i to j . The descendants of
 97 a task $i \in T$, denoted by $\Gamma^{+\star}(i)$ is the set of tasks $j \neq i$ such that $i \rightarrow j$.
 98 Similarly, $\Gamma^{-\star}(i)$ denotes the set of ancestors of i , i.e. the set of tasks $j \neq i$
 99 such that $j \rightarrow i$.

100 We assume without loss of generality that release dates and deadlines
 101 are consistent with respect to precedence constraints, i.e. if $(i, j) \in A$
 102 then $r_i + 1 \leq r_j$ and $d_i + 1 \leq d_j$. For any task $i \in T$, $f(i)$ is the small-
 103 est value $\alpha \in \{1, \dots, k^\star\}$ such that $i \in X_\alpha$.

104 Figure 1 presents an instance \mathcal{I} of our scheduling problem for two ma-
 105 chines. We get $k^\star = 5$ and $u_\alpha = \alpha$ for $\alpha \in \{0, \dots, 5\}$. The corresponding
 106 sets are $X_1 = \{1\}$, $X_2 = \{1, 2, 4\}$, $X_3 = \{2, 3, 4, 5, 6\}$, $X_4 = \{3, 5, 6, 7\}$ and
 107 $X_5 = \{7\}$. The pathwidth is $pw(\mathcal{I}) = |X_3| - 1 = 4$. We also have $f(1) = 1$,
 $f(2) = f(4) = 2$, $f(3) = f(5) = f(6) = 3$ and $f(7) = 4$.

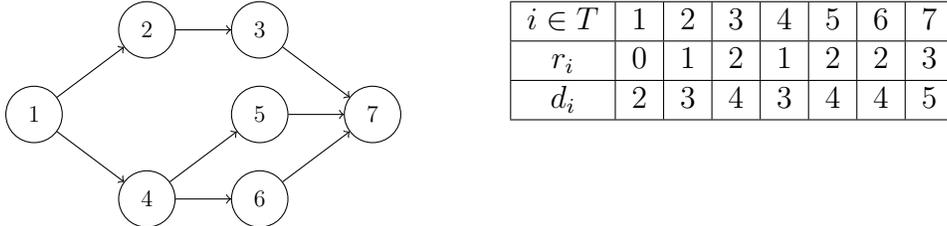


Figure 1: An instance \mathcal{I} of $P|prec, p_i = 1, r_i, d_i| \star$ for $m = 2$ machines.

108

109 **3. Basic properties of feasible solutions**

110 Let us suppose that σ is a feasible solution of our scheduling problem. For
 111 any value $\alpha \in \{1, \dots, k^*\}$, we define the slice $S_\alpha = \{i \in T, u_{\alpha-1} \leq s_i^\sigma < u_\alpha\}$.

112 Consider the feasible schedule σ for our example given in Figure 2. We
 113 get the slices $S_1 = \{1\}$, $S_2 = \{2, 4\}$, $S_3 = \{3, 5\}$, $S_4 = \{6\}$ and $S_5 = \{7\}$.



Figure 2: A feasible schedule σ associated with the example given in Figure 1.

114 **Definition 1.** A set of tasks $V \subseteq T$ is said to be consistent if for any couple
 115 $(i, j) \in V^2$ with $i \neq j$ such that $i \rightarrow j$, any task k of a path in G from i to j
 116 belongs to V .

117 **Definition 2.** A set $V \subseteq T$ is said to be strictly consistent if for any $x \in V$,
 118 $\Gamma^+(x) \subseteq V$.

119 We observe that any strictly consistent set V is consistent, but the reverse
 120 is not true. The consequence is that Definition 2 is more restrictive than
 121 Definition 1.

122 **Lemma 1.** For any value $\alpha \in \{1, \dots, k^*\}$, S_α is consistent.

123 *Proof.* Suppose, for the sake of contradiction, that S_α is not consistent. Then,
 124 there exist a couple $(i, j) \in S_\alpha^2$ and a task $k \in S_\beta$ with $\beta \neq \alpha$ and $i \rightarrow k \rightarrow j$.
 125 If $\beta > \alpha$, then the precedence $k \rightarrow j$ cannot be fulfilled since $s_k^\sigma \geq u_\alpha > s_j^\sigma$.
 126 Similarly, if $\beta < \alpha$, the precedence $i \rightarrow k$ cannot be fulfilled as $s_k^\sigma < u_{\alpha-1} \leq$
 127 s_i^σ , thus the lemma is proved. \square

128 **Lemma 2.** For any value $\alpha \in \{1, \dots, k^*\}$, $\cup_{\beta=\alpha}^{k^*} S_\beta$ is strictly consistent.

129 *Proof.* Suppose, for the sake of contradiction, that there exists $\alpha \in \{1, \dots, k^*\}$
 130 such that $\cup_{\beta=\alpha}^{k^*} S_\beta$ is not strictly consistent. Let then $i \in \cup_{\beta=\alpha}^{k^*} S_\beta$ with
 131 $\Gamma^+(i) \not\subseteq \cup_{\beta=\alpha}^{k^*} S_\beta$ and $j \in \Gamma^+(i) - \cup_{\beta=\alpha}^{k^*} S_\beta$. Thus, $j \in S_\beta$ with $\beta < \alpha$,
 132 and then $s_j^\sigma < u_{\alpha-1} \leq s_i^\sigma$. The precedence $i \rightarrow j$ is not fulfilled by σ ,
 133 a contradiction. \square

134 **Lemma 3.** For any $\alpha \in \{1, \dots, k^*\}$, let $Z_\alpha = \{i \in T, f(i) \geq \alpha\}$. If σ
135 is a feasible schedule, then any subset $V_\alpha = \cup_{\beta=\alpha}^{k^*} S_\beta$ can be partitioned such as
136 $V_\alpha = Z_\alpha \cup p(V)$ with $p(V) \subseteq X_\alpha$.

137 *Proof.* Any element $i \in Z_\alpha$ satisfies $f(i) \geq \alpha$ and thus $r_i \geq u_{\alpha-1}$. Since σ
138 is feasible, $s_i^\sigma \geq r_i$, and thus $i \in V_\alpha$. We conclude that $Z_\alpha \subseteq V_\alpha$.

139 Now, let $i \in p(V_\alpha) = V_\alpha - Z_\alpha$, then $f(i) < \alpha$, thus $r_i < u_{\alpha-1}$. Then,
140 there exists $\beta < \alpha$ with $i \in X_\beta$. Since $i \in V_\alpha$, there exists $\beta' \in \{\alpha, \dots, k^*\}$
141 with $i \in S_{\beta'}$, and then $s_i^\sigma \leq u_{\beta'} \leq d_i$. Since $\beta' \geq \alpha$, $u_{\beta'} \geq u_\alpha$. As $r_i < u_{\alpha-1}$
142 and $d_i \geq u_\alpha$, $(r_i, d_i) \cap (u_{\alpha-1}, u_\alpha) \neq \emptyset$ and thus $i \in X_\alpha$, which concludes
143 the proof. \square

144 For the example given by Figure 1, we get $Z_5 = \emptyset$, $Z_4 = \{7\}$, $Z_3 =$
145 $\{3, 5, 6, 7\}$, $Z_2 = \{2, 3, 4, 5, 6, 7\}$ and $Z_1 = T$. For the schedule σ from
146 Figure 2, $V_1 = T = Z_1$, $V_2 = \{2, 3, 4, 5, 6, 7\} = Z_2$, $V_3 = \{3, 5, 6, 7\} = Z_3$
147 $V_4 = \{6, 7\} = \{6\} \cup Z_4$ and $V_5 = \{7\} = \{7\} \cup Z_5$. We deduce that
148 $p(V_1) = p(V_2) = p(V_3) = \emptyset$, $p(V_4) = \{6\}$ and $p(V_5) = \{7\}$.

149 4. Representation of feasible solutions and description of the algo- 150 rithm

151 Let \mathcal{I} be an instance of $P|prec, p_i = 1, r_i, d_i|*$. The idea of the algorithm
152 is to build a multistage auxiliary graph $\mathcal{G} = (\mathcal{T}, \mathcal{A})$ whose longest paths
153 describe all the feasible solutions (if any) of \mathcal{I} .

154 Elements of \mathcal{T} are associated with subsets of T . The set \mathcal{T} is partitioned
155 into k^* sets $\mathcal{T}_1, \dots, \mathcal{T}_{k^*}$ such that, for any value $\alpha \in \{1, \dots, k^*\}$, elements
156 of \mathcal{T}_α are strictly consistent subsets of $X_\alpha - Z_\alpha$. Then, every element $v \in \mathcal{T}_\alpha$
157 verifies $v \cup Z_\alpha \subseteq X_\alpha \cup Z_\alpha \subseteq \bigcup_{\beta=\alpha}^{k^*} X_\beta$. Moreover, $p(v \cup Z_\alpha) = v$.

158 Now, let us suppose that v and v' are two vertices respectively of \mathcal{T}_α and
159 $\mathcal{T}_{\alpha+1}$ for $\alpha \in \{1, \dots, k^* - 1\}$. Then, there is an arc (v, v') in \mathcal{A} if:

- 160 1. $v' \cup Z_{\alpha+1} \subseteq v \cup Z_\alpha$ and $S = v \cup Z_\alpha - (v' \cup Z_{\alpha+1})$ is in X_α and is consistent;
- 161 2. There exists a schedule of tasks from S on m machines of makespan
162 $C_\alpha = u_\alpha - u_{\alpha-1}$ meeting the precedence constraints induced by the sub-
163 graph $G' = (S, A)$ of G .

164 Figure 3 presents the multistage auxiliary graph $\mathcal{G} = (\mathcal{T}, \mathcal{A})$ associated with
165 the instance given in Figure 1.

166 We prove in Lemma 4 that any feasible schedule σ is associated with
167 a path of the multistage auxiliary graph \mathcal{G} from a vertex of \mathcal{T}_1 to a vertex

168 of \mathcal{T}_{k^*} . Indeed, let S_1, \dots, S_{k^*} be the slices of σ and the corresponding se-
 169 quence $V_\alpha = \cup_{\beta=\alpha}^{k^*} S_\beta$, for each $\alpha \in \{1, \dots, k^*\}$. Then, σ is associated with
 170 the path $p(V_1), p(V_2), \dots, p(V_{k^*})$ of \mathcal{G} .

171 Let us consider for example the schedule σ given in Figure 2. Recall that
 172 $p(V_1) = p(V_2) = p(V_3) = \emptyset$, $p(V_4) = \{6\}$ and $p(V_5) = \{7\}$. The schedule σ
 173 is thus associated with the path $\emptyset \rightarrow \emptyset \rightarrow \emptyset \rightarrow \{6\} \rightarrow \{7\}$.

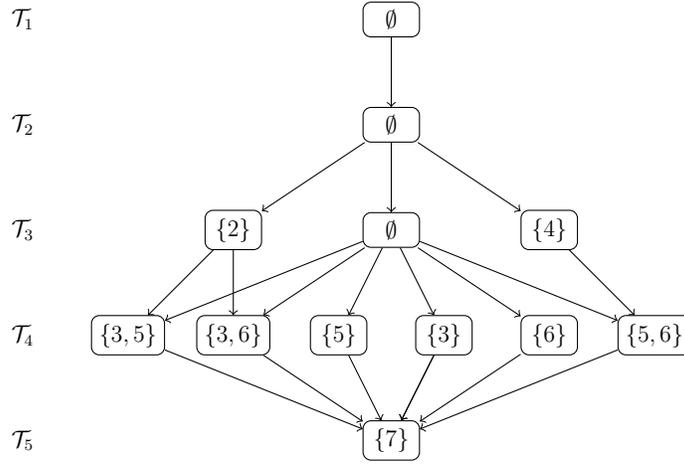


Figure 3: The multistage auxiliary graph $\mathcal{G} = (\mathcal{T}, \mathcal{A})$ associated with the example given in Figure 1. Vertices without successor are removed.

174 **Lemma 4.** *Each feasible solution σ of the instance \mathcal{I} is associated with a path*
 175 *of the multistage auxiliary graph \mathcal{G} from a vertex of \mathcal{T}_1 to a vertex of \mathcal{T}_{k^*} .*

176 *Proof.* Let S_1, \dots, S_{k^*} be the slices of σ and $V_\alpha = \cup_{\beta=\alpha}^{k^*} S_\beta$ for each $\alpha \in$
 177 $\{1, \dots, k^*\}$. For any value $\alpha \in \{1, \dots, k^*\}$, $V_\alpha \subseteq \cup_{\beta=\alpha}^{k^*} X_\beta$. V_α is strictly
 178 consistent by Lemma 2 and $Z_\alpha \subseteq V_\alpha$ by Lemma 3. Thus, $p(V_\alpha) \in \mathcal{T}_\alpha$.

179 Now, for any value $\alpha \in \{1, \dots, k^* - 1\}$, $V_{\alpha+1} \subseteq V_\alpha$, and the set $S_\alpha =$
 180 $V_\alpha - V_{\alpha+1} \subseteq X_\alpha$ is consistent by Lemma 1. Lastly, as σ is a feasible solution,
 181 S_α can be scheduled according to the resource constraints and the precedence
 182 expressed by the sub-graph (S_α, \mathcal{A}) . Thus the arc $(p(V_\alpha), p(V_{\alpha+1})) \in \mathcal{A}$,
 183 which concludes the proof. \square

184 Let us consider now a path $p(V_1), \dots, p(V_{k^*})$ of \mathcal{G} . We prove in Lemma 5
 185 that the sequence of slices $S_{k^*} = p(V_{k^*}) \cup Z_{k^*}$ and for each $\alpha \in \{1, \dots, k^* - 1\}$,
 186 $S_\alpha = p(V_\alpha) \cup Z_\alpha - (p(V_{\alpha+1}) \cup Z_{\alpha+1})$ defines a feasible schedule.

187 Let us consider for example the path of the graph \mathcal{G} given by Figure 3 from
188 $p(V_1)$ from $p(V_5)$ such that $p(V_1) = p(V_2) = \emptyset$, $p(V_3) = \{2\}$, $p(V_4) = \{3, 6\}$
189 and $p(V_5) = \{7\}$. Corresponding slices are $S_5 = \{7\}$, $S_4 = \{3, 6, 7\} -$
190 $\{7\} = \{3, 6\}$, $S_3 = \{2, 3, 5, 6, 7\} - \{3, 6, 7\} = \{2, 5\}$, $S_2 = \{2, 3, 4, 5, 6, 7\} -$
191 $\{2, 3, 5, 6, 7\} = \{4\}$, and $S_1 = T - \{2, 3, 4, 5, 6, 7\} = \{1\}$. We get the sched-
192 ule σ defined as $s_1^\sigma = 0$, $s_4^\sigma = 1$, $s_2^\sigma = s_5^\sigma = 2$, $s_3^\sigma = s_6^\sigma = 3$ and $s_7^\sigma = 4$.

193 **Lemma 5.** *Each path of the multistage auxiliary graph \mathcal{G} from a vertex of \mathcal{T}_1*
194 *to a vertex of \mathcal{T}_{k^*} is associated with a feasible solution.*

195 *Proof.* Let $p(V_1), \dots, p(V_{k^*})$ be a path of \mathcal{G} with $p(V_\alpha) \in \mathcal{T}_\alpha$. Let us de-
196 fine the sequences $V_\alpha = p(V_\alpha) \cup Z_\alpha$ for $\alpha \in \{1, \dots, k^*\}$, $S_{k^*} = V_{k^*}$ and
197 for each $\alpha \in \{1, \dots, k^* - 1\}$, $S_\alpha = V_\alpha - V_{\alpha+1}$.

198 By definition of \mathcal{G} , each set S_α is consistent and can be scheduled fol-
199 lowing resource and precedence constraints of the sub-graph (S_α, A) within
200 a makespan $C_\alpha = u_\alpha - u_{\alpha-1}$. We prove that the schedule obtained by merging
201 the schedules of the successive slices S_1, \dots, S_{k^*} is feasible.

202 By definition, $Z_1 \subseteq V_1$ and $Z_1 = \{i \in T, f(i) \geq 1\} = T$, thus $V_1 =$
203 T . Moreover, for each arc $(p(V_\alpha), p(V_{\alpha+1}))$ of \mathcal{G} with $\alpha \in \{1, \dots, k^* - 1\}$,
204 $p(V_{\alpha+1}) \cup Z_{\alpha+1} \subseteq p(V_\alpha) \cup Z_\alpha$, thus $V_{\alpha+1} \subseteq V_\alpha$ and then $V_{k^*} \subseteq V_{k^*-1} \subseteq \dots \subseteq$
205 V_1 . Then, each task $i \in T$ belongs to exactly one slice S_α and S_1, \dots, S_{k^*}
206 is a partition of T .

207 By definition, $S_\alpha \subseteq X_\alpha$, thus for any task of S_α , release dates and
208 deadlines are fulfilled. Resource constraints are fulfilled by definition of
209 each set S_α , for each $\alpha \in \{1, \dots, k^*\}$.

210 Lastly let us consider two tasks $(i, j) \in T^2$ with $(i, j) \in A$. Let suppose
211 that $i \in S_\alpha$ and $j \in S_\beta$. If $\alpha = \beta$, the constraint is fulfilled since there
212 exists a feasible schedule of S_α for the sub-graph (S_α, A) . If $\alpha < \beta$, the
213 set S_α is scheduled before S_β , the constraint is then fulfilled. Lastly, if
214 $\alpha > \beta$, then V_α is not strictly consistent since $i \in V_\alpha$ and $j \notin V_\alpha$, which
215 is a contradiction. Thus, the lemma is proved. \square

216 The next theorem is a simple outcome of Lemmas 4 and 5.

217 **Theorem 1.** *There exists a feasible schedule of makespan C if and only if*
218 *there exists a path in the multistage auxiliary graph \mathcal{G} from a vertex of \mathcal{T}_1 to*
219 *a vertex of \mathcal{T}_{k^*} .*

220 Our algorithm simply builds the multistage auxiliary graph \mathcal{G} starting
221 from \mathcal{T}_{k^*} to \mathcal{T}_1 . Vertices without successor from \mathcal{T}_α with $\alpha \neq k^*$ are removed.
222 In the next section, we investigate the time complexity for computing \mathcal{G} .

223 **5. Complexity for the computation of the multistage auxiliary graph**

224 Lemmas 6 and 7 study the complexity of two specific tests considered by
 225 our previous algorithm.

226 **Lemma 6.** *Checking the consistency of a set $V \subseteq T$ can be done in time $\mathcal{O}(n^3)$
 227 and checking the strict consistency of a set $V \subseteq T$ can be done in time $\mathcal{O}(n^2)$
 228 (with $n = |T|$).*

229 *Proof.* For any task $x \in T$, the computation of the sets of descendants $\Gamma(x)^{+\star}$
 230 and of ancestors $\Gamma(x)^{-\star}$ can be done once in time $\mathcal{O}(n^2)$ using a depth-first
 231 search algorithm. Checking that V is consistent can be done by checking
 232 that, for any couple $(i, j) \in V^2$ with $i \neq j$, $\Gamma(i)^{+\star} \cap \Gamma(j)^{-\star} \subseteq V$. The overall
 233 complexity is then $\mathcal{O}(n^3)$.

Similarly, checking that for any task $x \in V$, $\Gamma^+(x) \subseteq V$ can be done in
 time $\mathcal{O}(|\Gamma^+(x)|)$. The overall complexity is bounded by

$$\sum_{x \in V} |\Gamma^+(x)| \leq \sum_{x \in T} |\Gamma^+(x)| \leq n^2,$$

234 the complexity of the algorithm is then $\mathcal{O}(n^2)$. □

235 **Lemma 7.** *The complexity of checking the existence of a feasible schedule
 236 of fixed duration C on m machines for a consistent set of tasks $V \subseteq T$ with
 237 the precedence relations modelled by $G' = (V, A)$ is $\mathcal{O}(|V|^3 \times 2^{2|V|})$.*

238 *Proof.* Let us denote by $\mathcal{P}(V)$ the set of consistent subsets of V . Let us
 239 consider the directed graph $H = (\mathcal{P}(V), E)$ which arcs are built as follows:
 240 an arc $(v, v') \in E$ if $v \subsetneq v'$, $|v' - v| \leq m$ and for any task $i \in v' - v$, $\Gamma_{G'}^{\star}(i) \subseteq v$
 241 where $\Gamma_{G'}^{\star}(i)$ is the set of the ancestors of i in G' .

242 Paths of H from \emptyset to V are associated to feasible schedules. Indeed, let
 243 us consider a path $v_0 = \emptyset, v_1, \dots, v_p = V$ of H . For any value $\beta \in \{1, \dots, p\}$,
 244 $Y_\beta = v_\beta - v_{\beta-1}$ are executed at time $\beta - 1$. Conversely any feasible schedule
 245 corresponds to a path of H . Thus, there exists a feasible schedule of makespan
 246 at most C if the value of a shortest path (considering each arc valued by 1)
 247 from \emptyset to V is inferior than or equal to C .

248 The number of vertices of H satisfies $|\mathcal{P}(V)| \leq 2^{|V|}$. By Lemma 6, check-
 249 ing that any element $v \in \mathcal{P}(V)$ is consistent can be done in time $\mathcal{O}(|V|^3 \times$
 250 $2^{|V|})$. Checking that $e = (v, v') \in E$ has a time complexity bounded by
 251 $\mathcal{O}(|V|^2)$ thus the construction of H can be done in time $\mathcal{O}(|V|^3 \times 2^{2|V|})$.

252 Now, any arc $e = (v, v') \in E$ verifies $v \subsetneq v'$ and thus $|v| < |v'|$. Then, H
 253 has no circuit. The shortest path can be computed from a topological sort
 254 in time complexity bounded by $\mathcal{O}(2^{2|V|})$, thus the lemma holds. \square

255 **Lemma 8.** *For any instance \mathcal{I} , the number of vertices $|\mathcal{T}|$ of the multi-*
 256 *stage auxiliary graph \mathcal{G} is in $\mathcal{O}(n2^{pw(\mathcal{I})})$, while the number of arcs $|\mathcal{A}|$ is in*
 257 *$\mathcal{O}(n2^{2pw(\mathcal{I})})$.*

258 *Proof.* By Lemma 3, for any value $\alpha \in \{1, \dots, k^*\}$, each element $v \in \mathcal{T}_\alpha$
 259 is included in X_α thus $|\mathcal{T}_\alpha| \leq 2^{|X_\alpha|} \leq 2^{pw(\mathcal{I})+1}$. Then, $|\mathcal{T}| = \sum_{\alpha=1}^{k^*} |\mathcal{T}_\alpha| \leq$
 260 $k^* \times 2^{pw(\mathcal{I})+1}$. Since $k^* \leq 2n$, we get $|\mathcal{T}| \leq 2n2^{pw(\mathcal{I})+1}$.

261 Now, the number of arcs of \mathcal{G} verifies $|\mathcal{A}| \leq \sum_{\alpha=1}^{k^*-1} (|\mathcal{T}_\alpha| \times |\mathcal{T}_{\alpha+1}|) \leq$
 262 $k^*(2^{pw(\mathcal{I})+1})^2 \leq 2n \times 2^{2pw(\mathcal{I})+2}$ which concludes the lemma. \square

263 We are now ready to prove our main theorem:

264 **Theorem 2.** *Computing the graph \mathcal{G} is in time $\mathcal{O}(n^4 2^{4pw(\mathcal{I})})$ for any in-*
 265 *stance \mathcal{I} .*

266 *Proof.* Let us consider all the steps for the construction of the multistage
 267 auxiliary graph \mathcal{G} for an instance \mathcal{I} of the scheduling problem with $n = |\mathcal{T}|$.
 268 The computation of sets X_α and Z_α for $\alpha \in \{1, \dots, k^*\}$ can be done in
 269 time complexity $\mathcal{O}(n \log n)$. For the definition of sets \mathcal{T}_α , $\alpha = 1, \dots, k^*$, we
 270 observe that for each subset $v \subseteq X_\alpha$, we have to test if $v \cup Z_\alpha$ is strictly
 271 consistent, which requires $\mathcal{O}(n^2)$ instructions by Lemma 6. By Lemma 8,
 272 the construction of \mathcal{T} is thus in time $\mathcal{O}(n^3 \times 2^{pw(\mathcal{I})})$.

273 Now, for any couple $(v, v') \in \mathcal{T}_\alpha \times \mathcal{T}_{\alpha+1}$, the consistency of $S = (v \cup$
 274 $Z_\alpha) - (v' \cup Z_{\alpha+1})$ is checked in time $\mathcal{O}(n^3)$ by Lemma 6. The existence
 275 of a schedule of S on m machines within a makespan $C_\alpha = u_\alpha - u_{\alpha-1}$ is
 276 in time $\mathcal{O}(|S|^3 \times 2^{2|S|})$ following Lemma 7. Now, since $S \subseteq X_\alpha$, $|S| \leq$
 277 $pw(\mathcal{I}) + 1$ and thus checking the existence of an arc (v, v') in \mathcal{A} is in time
 278 $\mathcal{O}(n^3 + pw(\mathcal{I})^3 \times 2^{2(pw(\mathcal{I}))})$. Following Lemma 8, the overall complexity for
 279 the determination of \mathcal{A} is then $\mathcal{O}(n \times 2^{2pw(\mathcal{I})} \times (n^3 + pw(\mathcal{I})^3 \times 2^{2pw(\mathcal{I})}))$ and
 280 thus $\mathcal{O}(n^4 \times 2^{4pw(\mathcal{I})})$. The theorem follows. \square

281 **Corollary 1.** *$P|prec, p_i = 1, r_i, d_i|_{\star}$ is fixed-parameter tractable by the path-*
 282 *width $pw(\mathcal{I})$.*

283 *Proof.* By Theorem 1, an instance \mathcal{I} has a solution if and only if there exists
 284 a path in \mathcal{G} from a vertex of \mathcal{T}_1 to one from \mathcal{T}_{k^*} . By removing vertices
 285 without successor, it is equivalent to test that the remaining set $|\mathcal{T}_1| > 0$. By
 286 Theorem 2, building \mathcal{G} is in time $\mathcal{O}(n^4 2^{4pw(\mathcal{I})})$, the corollary. \square

287 Corollary 2 is a simple outcome of Theorem 2.

288 **Corollary 2.** *Let \mathcal{I} be an instance of $P|prec, p_i = 1, r_i, d_i|*$ such that $pw(\mathcal{I})$*
289 *belongs to $\mathcal{O}(\log_2 n)$. The existence of a feasible schedule can be solved in*
290 *time $\mathcal{O}(n^8)$.*

291 By augmenting our algorithm by a binary search on an upper bound of the
292 objectives, we obtain in the next corollary the fixed-parameter tractability
293 of two optimization problems:

294 **Corollary 3.** *Scheduling problems $P|prec, p_i = 1, r_i|C_{\max}$ and $P|prec, p_i =$*
295 *$1, r_i|L_{\max}$ are fixed-parameter tractable by the pathwidth $pw(\mathcal{I})$.*

296 6. Conclusion and perspectives

297 For any instance \mathcal{I} of the decision problem $P|prec, p_i = 1, r_i, d_i|*$, we in-
298 troduced the pathwidth $pw(\mathcal{I})$ as a simple measure of the parallelism. We
299 proved that the previous decision problem is fixed-parameter tractable by
300 the parameter $pw(\mathcal{I})$. This result can be easily extended to $P|prec, p_i =$
301 $1, r_i|C_{\max}$ and $P|prec, p_i = 1, r_i|L_{\max}$ using a binary search. As far as
302 we know, it is the first fixed-parameter algorithm for these scheduling prob-
303 lems. Extensions of this result should be considered to solve more general
304 scheduling problems with precedence and resource constraints.

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