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Minimal ideals and some applications

Rodney Coleman, Laurent Zwald

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Abstract

In these notes we introduce minimal prime ideals and some of their applications. We prove Krull's principal ideal and height theorems and introduce and study the notion of a system of parameters of a local ring. In addition, we give a detailed proof of the formula for the dimension of a polynomial ring over a noetherian ring.

Let R be a commutative ring and I a proper ideal in R . A prime ideal P is said to be a minimal prime ideal over I if it is minimal (with respect to inclusion) among all prime ideals containing I . A prime ideal is said to be minimal if it is minimal over the zero ideal (0) . Minimal prime ideals are those of height 0.

If I is a prime ideal, then I is the only minimal prime ideal over I . Thus, in an integral domain the only minimal prime ideal is (0) . It should also be noticed that, if I is an ideal contained in a prime ideal P and $\text{ht}(I) = \text{ht}(P)$, then P is a minimal prime ideal over I .

We now consider the existence of minimal prime ideals.

Theorem 1 *If I is a proper ideal in a ring R , then there exists a prime ideal P in R which is minimal over I .*

PROOF Let S be the set of prime ideals containing I . Since any proper ideal is contained in a maximal ideal, S is not empty. We order S by reverse inclusion, i.e., we write $P_a \leq P_b$ if $P_b \subset P_a$. If C is a chain in S , then any $P_a \in C$ is majored by the intersection P of all elements in the chain. P is clearly an ideal containing I . We claim that P is prime.

Suppose that $xy \in P$ and $x \notin P$, $y \notin P$. Then there exists $P_a, P_b \in C$ such that $x \notin P_a$, $y \notin P_b$. Without loss of generality, let us assume that $P_a \subset P_b$. Then $x, y \notin P_a$. Now, $xy \in P$ implies that $xy \in P_a$; as P_a is prime, we have $x \in P_a$ or $y \in P_a$, a contradiction. Hence P is prime and so the chain C has a maximum.

From Zorn's lemma, S has a maximal element Q , which is a minimal prime ideal over I . \square

If the ring R is noetherian, we can say a little more.

Theorem 2 *If I is a proper ideal in a noetherian ring R , then R has only a finite number of minimal prime ideals over I .*

PROOF First we show that every ideal containing I contains a finite product of prime ideals each containing I . Suppose that this is not the case, and let S be the set of ideals containing I which do not contain a finite product of prime ideals each containing I . By hypothesis, S is not empty. Let C be a chain in S . As R is noetherian, C has a maximum. From Zorn's lemma, S has a maximal element M . As R has a prime ideal containing I , $M \neq R$. Also, M is not prime.

There exist $a, b \in R$ such that $ab \in M$ and $a \notin M, b \notin M$. Setting $A = (M, a)$ and $B = (M, b)$, we obtain $AB \subset M$, with A and B strictly included in M . Since M is maximal, A and B both contain a finite product of prime ideals each containing I , hence so does M , which contradicts the fact that $M \in S$. It follows that every ideal containing I contains a finite product of prime ideals each containing I .

We now apply this result to the ideal I : there exist prime ideals P_1, \dots, P_n , each containing I , whose product is contained in I . We claim that any minimal prime P over I is among the P_i . Indeed, $P_1 \cdots P_n \subset I \subset P$. We deduce that $P_i \subset P$, for some i . However, P is minimal, so $P_i = P$ and it follows that there is only a finite number of minimal prime ideals over I . \square

The following result is interesting, and perhaps unexpected.

Proposition 1 *If P is a minimal prime ideal (over (0)), then every element $x \in P$ is a zero divisor.*

PROOF First we notice that $R_P P$ is the unique prime ideal in R_P and so the nilradical $N(R_P) = R_P P$. If $x \in P$, then $\frac{x}{1} \in R_P P$ and there exists $n \in \mathbf{N}^*$ such that $(\frac{x}{1})^n = 0$. Hence there exists $s \in R \setminus P$ such that $sx^n = 0$. If m is the smallest n for which this applies, then $sx^{m-1} \neq 0$ and so x is a zero divisor. \square

We aim now to show that every maximal ideal in an artinian ring is minimal (over (0)). We need a preliminary result.

Lemma 1 *Let R be a commutative ring, $I \subset A$ ideals in R , with A prime. Then A contains a minimal prime ideal over I .*

PROOF Let S be the set of prime ideals containing I and included in A . We order the elements in S by reverse inclusion, i.e., we write $P_a \leq P_b$ if $P_b \subset P_a$. Let C be a chain in S . If P is the intersection of all elements in C , then P is a maximum of all elements C . P is clearly an ideal. We claim that P is prime.

Suppose that $xy \in P$, but $x \notin P$ and $y \notin P$. Now, there exist prime ideals P_a, P_b such that $x \notin P_a, y \notin P_b$. Without loss of generality, let us assume that $P_a \subset P_b$, which implies that $x, y \notin P_a$. As $xy \in P_a$, we have a contradiction. It follows that P is prime and so C has a maximum. From Zorn's lemma, there is a maximal element Q of S , which is a minimal prime ideal over I . \square

Remark From the lemma, when considering the height of an ideal I , we only need to take into account the minimal prime ideals over I .

Theorem 3 *If R is an artinian ring and M a maximal ideal in R , then M is minimal.*

PROOF From Lemma 1, there exists a minimal prime ideal P contained in M . However, every prime ideal in an artinian ring is maximal ([1] Theorem 8), so P is maximal and we have $P = M$. \square

Krull's height theorem

We first consider a particular case of Krull's height theorem, namely Krull's principal ideal theorem: in a noetherian ring, if P is a minimal prime ideal over a principal ideal (a) , then $\text{ht}(P) \leq 1$. We begin with a preliminary result.

Lemma 2 *If (R, M) is a local noetherian domain and M a minimal prime ideal over some principal ideal (a) , then (0) and M are the only prime ideals in R .*

PROOF Let P be any prime ideal in R other than M . Then P is a proper subset of M . We aim to show that $P = (0)$. Since R is a domain, it is sufficient to show that $P^n = (0)$, for some $n \in \mathbf{N}^*$. We recall that $P^{(n)} = R \cap R_P P^n$, the n th symbolic power of P , is a P -primary ideal containing P^n ([2] Corollary 7), so it is sufficient to show that $P^{(n)} = (0)$, for some positive n . We also recall that $\bigcap_{n \geq 1} P^{(n)}$ is the kernel of the standard mapping $f : R \rightarrow R_P$ ([2] Proposition 11). We claim that the descending chain of ideals $(P^{(n)})_{n \geq 1}$ stabilizes after a certain n .

We set $\bar{R} = R/(a)$ and $\bar{M} = M/(a)$. Then \bar{M} is the only prime ideal in \bar{R} , because M is a minimal prime ideal over (a) . Hence \bar{R} is a noetherian ring of dimension 0. This implies that \bar{R} is an artinian ring ([1] Theorem 12). Hence the descending chain of ideals $(P^{(n)} + (a))_{n \geq 1}$ must stabilize. For sufficiently large n we have

$$P^{(n)} + (a) = P^{(n+1)} + (a).$$

Therefore, if $x \in P^{(n)}$, we may write $x = y + za$, where $y \in P^{(n+1)}$ and $z \in R$. As $P^{(n+1)} \subset P^{(n)}$, we have $x - y \in P^{(n)}$, from which we deduce that $z \in P^{(n)} : a$ ([2] Definition after Theorem 1). Next we observe that $a \notin P$: if $a \in P$, then $(a) \subset P$; however, M is minimal over (a) , which implies that $M \subset P$ and so $M = P$, a contradiction, thus $a \notin P$. As $a \notin P$, $P^{(n)} : a = P^{(n)}$ ([2] Proposition 5). It follows that $z \in P^{(n)}$ and so

$$P^{(n)} \subset P^{(n+1)} + P^{(n)}a.$$

Since

$$P^{(n+1)} + P^{(n)}a \subset P^{(n)},$$

we have

$$P^{(n)} = P^{(n+1)} + P^{(n)}a.$$

We set $N = P^{(n)}/P^{(n+1)}$. Then N is an ideal in the the noetherian ring $R/P^{(n+1)}$ so is finitely generated. Also,

$$N = P^{(n)}/P^{(n+1)} = (P^{(n+1)} + P^{(n)}a)/P^{(n+1)} = P^{(n)}a/P^{(n+1)} = aN.$$

Given that the Jacobson radical $J(R) = M$ and $(a) \subset M$, we may apply Nakayama's lemma version 2 ([3] Theorem 2) to obtain that $N = 0$, i.e., $P^{(n)} = P^{(n+1)}$, for n sufficiently large, as claimed. It follows that the kernel of the standard mapping $f : R \rightarrow R_P$ is equal to $\bigcap_{i=1}^n P^{(i)}$. As $P^{(n)}$ is included in each $P^{(i)}$, we have $P^{(n)} \subset \ker f$, which implies that $P^{(n)} = (0)$. \square

We are now in a position to prove Krull's principal ideal theorem.

Theorem 4 *Let R be a noetherian ring and $a \in R$.*

- **1.** *If P is a minimal prime ideal over (a) , then $ht(P) \leq 1$;*
- **2.** *If $a \in R^*$ is not a zerodivisor and P a minimal prime ideal over (a) , then $ht(P) = 1$.*

PROOF **1.** Suppose that P_2 is a minimal prime ideal over (a) and $P_0 \subset P_1 \subsetneq P_2$ a chain of prime ideals. We must show that $P_1 = P_0$.

We set $\bar{R} = R/P_0$. Then \bar{R} is a noetherian domain. Setting $\bar{P}_1 = P_1/P_0$ and $\bar{P}_2 = P_2/P_0$, we obtain a chain of prime ideals in \bar{R} : $(0) \subset \bar{P}_1 \subsetneq \bar{P}_2$, and \bar{P}_2 is a minimal prime ideal over $(\bar{a}) = (a + P_0)$.

We now localize with respect to \bar{P}_2 , to obtain a local noetherian domain $\bar{R}_{\bar{P}_2}$, with maximal ideal $\bar{R}_{\bar{P}_2}\bar{P}_2$. In addition, $\bar{R}_{\bar{P}_2}\bar{P}_2$ is a minimal prime ideal over $(\frac{\bar{a}}{1})$. From Lemma 2, we know that the only prime ideals in $\bar{R}_{\bar{P}_2}$ are (0) and $\bar{R}_{\bar{P}_2}\bar{P}_2$, so $\bar{R}_{\bar{P}_2}\bar{P}_1 = (0)$ or $\bar{R}_{\bar{P}_2}\bar{P}_1 = \bar{R}_{\bar{P}_2}\bar{P}_2$. Only

the first alternative is possible, which implies that $\bar{P}_1 = (0)$, which in turn implies that $P_1 = P_0$, as required.

2. Suppose that a is not a zerodivisor and that a minimal prime P over (a) has height 0. If P' is a prime ideal and $(0) \subset P' \subset P$, then $P' = P$, because $\text{ht}(P) = 0$. This implies that P is a minimal prime (over (0)). From Proposition 1, the elements of a minimal prime ideal are nilpotent. As $a \in P$, a is nilpotent, hence a zerodivisor, which is a contradiction. It follows that $\text{ht}(P) = 1$. \square

There is a corollary to the above result. We need a definition. If R is a commutative ring and $P_0 \subsetneq P_2$ are prime ideals, then a prime ideal P_1 such that $P_0 \subsetneq P_1 \subsetneq P_2$ is said to be intermediate between P_0 and P_2 .

Corollary 1 *Let R be a noetherian ring and $P_0 \subsetneq P_2$ prime ideals in R . If there exists an intermediate prime ideal between P_0 and P_2 , then, for each $a \in P_2$, there exists an intermediate prime ideal between P_0 and P_2 containing a .*

PROOF Suppose that $a \in P_2$ and there is no intermediate prime ideal containing a . We set $\bar{R} = R/P_0$ and $\bar{P}_2 = P_2/P_0$. Then \bar{R} is a domain and \bar{P}_2 is minimal over $(\bar{a}) = (a + P_0)$. From Theorem 4, we have $\text{ht}(\bar{P}_2) \leq 1$, so there is no prime ideal \bar{P}' such that $(0) \subsetneq \bar{P}' \subsetneq \bar{P}_2$, because (0) is a prime ideal. It follows that there is no intermediate prime ideal between P_0 and P_2 , a contradiction. Hence the result. \square

We now generalize Theorem 4. The following result is referred to as Krull's height theorem.

Theorem 5 *Let R be a noetherian ring. If P is a minimal prime ideal over an ideal $I = (x_1, \dots, x_n)$, then $\text{ht}(P) \leq n$.*

PROOF By induction on n . For $n = 1$, we have Krull's principal ideal theorem. Suppose now that $n \geq 2$ and that the result is true for $k < n$. We must show that the result is true for n . Let us assume that this is not the case. Then there is a chain (C) of distinct prime ideals

$$P_0 \subset P_1 \subset \dots \subset P_n \subset P_{n+1} = P.$$

Let us suppose that one of the x_i belongs to P_1 . Without loss of generality, let this element be x_1 . Then

$$(x_1, \dots, x_n) \subset P_1 + (x_2, \dots, x_n) \subset P.$$

If $Q \subset P$ is a prime ideal containing $P_1 + (x_2, \dots, x_n)$, then Q contains (x_1, \dots, x_n) , so $P = Q$, because P is minimal. It follows that P is a minimal prime ideal over $P_1 + (x_2, \dots, x_n)$, which implies that P/P_1 is a minimal prime ideal over $(\bar{x}_2, \dots, \bar{x}_n)$ in R/P_1 , where $\bar{x}_i = x_i + P_1$. As the chain of distinct prime ideals

$$(0) \subset P_2/P_1 \subset \dots \subset P_{n+1}/P_1$$

has length n , we have a contradiction to the induction hypothesis. Thus it is sufficient to show that the chain (C) can be modified in such a way that $x_1 \in P_1$. This we will now do.

As $x_1 \in P$ and $P_{n-1} \subset P_n \subset P$, from Corollary 2 there exists an intermediate prime ideal P'_n such that $x_1 \in P'_n$ and $P_{n-1} \subset P'_n \subset P$. Now $P_{n-2} \subset P_{n-1} \subset P'_n$ and $x_1 \in P'_n$, so,

using Corollary 2 again, we obtain an intermediate prime ideal P'_{n-1} such that $x_1 \in P'_{n-1}$ and $P_{n-2} \subset P'_{n-1} \subset P'_n$. Repeating the process, we finally obtain a prime ideal P'_1 with $x_1 \in P'_1$ and

$$P_0 \subset P'_1 \subset \cdots \subset P'_n \subset P_{n+1} = P,$$

a chain whose length is $n + 1$. This completes the proof. \square

Remark Let R be a noetherian ring and $I = (x_1, \dots, x_n)$ a proper ideal in R . Then $\text{ht}(I) \leq n$: from Theorem 1 there is a prime ideal P minimal over I , hence $\text{ht}(I) \leq \text{ht}(P) \leq n$. In particular, if (R, M) is a local noetherian ring and (x_1, \dots, x_n) a minimal generating set of M , then $\text{ht}(M) \leq n$, hence $\dim(R) \leq n$. It follows that the dimension of a local noetherian ring is finite.

There is a question which naturally arises. Suppose that we have a prime ideal P , with $\text{ht}(P) = n$. Does there exist an ideal I with n generators such that P is minimal over I ? In fact, this is the case as we will now see and we can say a little more:

Theorem 6 *If R is a noetherian ring, I an ideal in R such that $\text{ht}(I) = n$, with $n \geq 1$, and $1 \leq k \leq n$, then there exist $x_1, \dots, x_k \in I$ such that $\text{ht}((x_1, \dots, x_k)) = k$.*

PROOF First suppose that $n = 1$. Let Q_1, \dots, Q_k be the minimal prime ideals in R . (From Theorem 2 there is a finite number of such ideals.) If $I = \cup_{i=1}^k Q_i$ then, by the Prime Avoidance Lemma ([1] Theorem 9), $I \subset Q_i$, for some i . However, this is not possible, because $\text{ht}(Q_i) = 0$. Hence there exists $x \in I \setminus \cup_{i=1}^k Q_i$. Then $(x) \subset I$ and $\text{ht}((x)) \leq \text{ht}(I) = 1$. If $\text{ht}((x)) = 0$, then $(x) = Q_j$, for some j , a contradiction, so $\text{ht}((x)) = 1$.

Now suppose that $n > 1$. As in the case $n = 1$, we may find x_1 such that $(x_1) \subset I$ and $\text{ht}((x_1)) \geq 1$. Using the remark after Theorem 5 we see that $\text{ht}((x_1)) = 1$. We claim that there exists $x_2 \in I$ such that $\text{ht}((x_1, x_2)) = 2$. Let Q'_1, \dots, Q'_l be the minimal prime ideals in R over (x_1) . If $I \subset \cup_{i=1}^l Q'_i$, then, by Prime Avoidance Lemma, $I \subset Q'_i$, for some i , which is not possible, because $\text{ht}(Q'_i) = 1$ and $\text{ht}(I) > 1$. Hence there exists $x_2 \in I \setminus \cup_{i=1}^l Q'_i$. Then $(x_1, x_2) \subset I$. We claim that $\text{ht}(x_1, x_2) = 2$.

Let P be a minimal prime ideal over (x_1, x_2) . By Krull's height theorem (Theorem 5), we have $\text{ht}(P) \leq 2$. Next we notice that $(x_1) \subset P$, therefore there exists a minimal prime ideal Q'_i over (x_1) such that $(x_1) \subset Q'_i \subset P$. However, by construction, $x_2 \in P$ and $x_2 \notin Q'_i$, which implies that $\text{ht}(P) > \text{ht}(Q'_i) = 1$, i.e., $\text{ht}(P) \geq 2$. Since for any minimal prime ideal P over (x_1, x_2) we have $\text{ht}(P) \geq 2$, we must have $\text{ht}((x_1, x_2)) \geq 2$. Using the remark after Theorem 5 we obtain $\text{ht}((x_1, x_2)) = 2$.

If $n = 2$, then we are finished. If $n > 2$ and $k = 3$, then we may find $x_3 \in I \setminus \cup_{i=1}^m Q''_i$, where the Q''_i are the minimal prime ideals over (x_1, x_2) . By an analogous argument to that previously used, we obtain that (x_1, x_2, x_3) has height 3. If $n > 3$, continuing in the same way up to k , we find $x_1, \dots, x_k \in I$, such that $\text{ht}(x_1, \dots, x_k)$ has height k . \square

Remark It should be noticed that we have used the remark after Lemma 1.

Corollary 2 *If P is a prime ideal in a noetherian ring R such that $\text{ht}(P) = n \geq 1$, then there exist $x_1, \dots, x_n \in P$ such that P is a minimal prime ideal over (x_1, \dots, x_n) .*

PROOF From Theorem 6, we may find $x_1, \dots, x_n \in P$ such that $\text{ht}((x_1, \dots, x_n)) = n$. As $\text{ht}((x_1, \dots, x_n)) = \text{ht}(P)$, P is a minimal prime ideal over (x_1, \dots, x_n) . \square

Systems of parameters

We investigate further properties of local rings. If R is a local ring with maximal ideal M , then we usually write (R, M) for R .

Proposition 2 *Let (R, M) be a local ring and $x_1, \dots, x_n \in M$. The following conditions are equivalent:*

- **1.** M is the only prime ideal containing (x_1, \dots, x_n) ;
- **2.** M is a minimal prime ideal over (x_1, \dots, x_n) ;
- **3.** $r((x_1, \dots, x_n)) = M$;
- **4.** (x_1, \dots, x_n) is M -primary.

PROOF **1.** \Rightarrow **2.** From Lemma 1, there exists a minimal prime ideal $M' \subset M$ over (x_1, \dots, x_n) . Since M is the only prime ideal containing (x_1, \dots, x_n) , we have $M' = M$.

2. \Rightarrow **1.** Let P be a prime ideal containing (x_1, \dots, x_n) . P must lie in a maximal ideal, hence $P \subset M$. As M is a minimal prime ideal over (x_1, \dots, x_n) , we have $M = P$.

1. \Rightarrow **3.** $r((x_1, \dots, x_n))$ is the intersection of all prime ideals containing (x_1, \dots, x_n) . As M is the only prime ideal containing (x_1, \dots, x_n) , we have $r((x_1, \dots, x_n)) = M$.

3. \Rightarrow **4.** From [2] Proposition 4, if the radical of a proper ideal is maximal, then the ideal is primary. Given that $r((x_1, \dots, x_n)) = M$, (x_1, \dots, x_n) is primary. As $r((x_1, \dots, x_n)) = M$, (x_1, \dots, x_n) is M -primary.

4. \Rightarrow **1.** Let P be a prime ideal containing (x_1, \dots, x_n) . Then the intersection of all prime ideals containing (x_1, \dots, x_n) is included in P , i.e., $r((x_1, \dots, x_n)) \subset P$. Since (x_1, \dots, x_n) is M -primary, we have $r((x_1, \dots, x_n)) = M$, so $M \subset P$. Given that M is a maximal ideal, we must have $P = M$. \square

There may be sets $\{x_1, \dots, x_n\}$ of different cardinals satisfying the conditions of Proposition 2. If (R, M) is noetherian, then we may characterize the smallest possible cardinal.

Proposition 3 *Let (R, M) be a noetherian local ring. Then $\dim(R)$ is the smallest integer n for which there exist $x_1, \dots, x_n \in M$ satisfying the equivalent conditions of Proposition 2.*

PROOF By [1] Lemma 2, we have $\dim(R) = \text{ht}(M)$. We note this common value m . If x_1, \dots, x_n are elements of R satisfying the conditions of Proposition 2, then $m \leq n$, by Krull's height theorem (Theorem 5). However, from Corollary 2 there exist elements $x_1, \dots, x_m \in M$ such that M is a minimal prime ideal over (x_1, \dots, x_m) , hence the result. \square

Definition Let (R, M) be a local ring and $n = \dim(R)$. A set $\{x_1, \dots, x_n\}$ is a system of parameters for M , if any one of the conditions of Proposition 2 is satisfied. From Corollary 2, if R is noetherian, then M has a system of parameters.

Let (R, M) be a noetherian local ring. Given a collection of elements $x_1, \dots, x_r \in M$, we aim to find conditions under which the collection may be extended to a system of parameters for M . We set $\bar{R} = R/(x_1, \dots, x_r)$. Then \bar{R} is a noetherian local ring, with maximal ideal \bar{M} , where \bar{M} is the image of M under the standard mapping of R onto \bar{R} .

Lemma 3 $\dim(\bar{R}) \geq \dim(R) - r$.

PROOF Let $s = \dim(\bar{R})$ and $y_1, \dots, y_s \in R$ be such that $\bar{y}_1, \dots, \bar{y}_s$ form a system of parameters for \bar{M} . Then \bar{M} is the only prime ideal in \bar{R} containing $(\bar{y}_1, \dots, \bar{y}_s)$. It follows that M is the only prime ideal in R containing $(x_1, \dots, x_r, y_1, \dots, y_s)$. From Proposition 3, $\dim(R) \leq r + s$, which implies that $\dim(\bar{R}) \geq \dim(R) - r$. \square

Theorem 7 *If (R, M) is a noetherian local ring and $x_1, \dots, x_r \in M$, then the following statements are equivalent:*

- **1.** x_1, \dots, x_r can be extended to a system of parameters for M ;
- **2.** $\dim(\bar{R}) = \dim(R) - r$.

PROOF **1.** \Rightarrow **2.** Suppose that we can extend the set x_1, \dots, x_r to a system of parameters $x_1, \dots, x_r, y_1, \dots, y_s$ for M . Then $r + s = \dim(R)$. We claim that \bar{M} is the unique prime ideal in \bar{R} containing $(\bar{y}_1, \dots, \bar{y}_s)$. If this is not the case, then there is a prime ideal \bar{P} strictly included in \bar{M} containing $(\bar{y}_1, \dots, \bar{y}_s)$. If ϕ is the standard mapping from R onto \bar{R} and $P = \phi^{-1}(\bar{P})$, then P is a prime ideal strictly included in M and $(x_1, \dots, x_r, y_1, \dots, y_s) \subset P$, contradicting condition **1.** of Proposition 2. Therefore our claim is correct.

Using Proposition 2 again, we obtain that \bar{M} is a minimal prime ideal over $(\bar{y}_1, \dots, \bar{y}_s)$, so, from Krull's height theorem (Theorem 5), $\dim(\bar{R}) = \text{ht}(\bar{M}) \leq s$. However, from Lemma 3, $\dim(\bar{R}) \geq \dim(R) - r = s$, hence $\dim(\bar{R}) = s$, as required.

2. \Rightarrow **1.** Let $s = \dim(\bar{R}) = \dim(R) - r$ and $y_1, \dots, y_s \in M$ such that $\bar{y}_1, \dots, \bar{y}_s$ form a system of parameters for \bar{M} . Then \bar{M} is the only prime ideal in \bar{R} containing $(\bar{y}_1, \dots, \bar{y}_s)$, which implies that M is the only prime ideal in R containing $(x_1, \dots, x_r, y_1, \dots, y_s)$. In addition, $r + s = \dim(R)$. Therefore $x_1, \dots, x_r, y_1, \dots, y_s$ is a required extension. \square

This theorem has two useful corollaries.

Corollary 3 *Let (R, M) be a noetherian local ring and $x_1, \dots, x_r \in M$. If $\text{ht}((x_1, \dots, x_r)) = r$, then x_1, \dots, x_r may be extended to a system of parameters for M .*

PROOF From Lemma 3 we have $\dim(\bar{R}) \geq \dim(R) - r$. However, from PMI Lemma 1, we know that $\dim(\bar{R}) + \text{ht}((x_1, \dots, x_r)) \leq \dim(R)$, which implies that $\dim(\bar{R}) = \dim(R) - r$. From Theorem 7 we deduce that x_1, \dots, x_r may be extended to a system of parameters for M . \square

Corollary 4 *Let (R, M) be a noetherian local ring and $x \in M$, with $\text{ht}((x)) = 1$. Then*

$$\dim(R/(x)) = \dim(R) - 1.$$

In particular, if x is a nonzero element in M , which is not a zero divisor, then $\text{ht}((x)) = 1$ and so the equality applies.

PROOF From Corollary 3 we may extend x to a system of parameters x, y_1, \dots, y_{n-1} for M , where $\dim(R) = n$. We deduce from Theorem 7 that

$$\dim(R/(x)) = \dim(R) - 1,$$

as required.

Now suppose that x is a nonzero element in M , which is not a zero divisor. From Theorem 4, if P a minimal prime ideal over (x) , then $\text{ht}(P) = 1$. It follows that $\text{ht}((x)) = 1$. \square

Polynomial rings over noetherian rings

From [1] Theorem 2 we know that if R is a commutative ring, then

$$\dim(R) + 1 \leq \dim(R[X]) \leq 2 \dim(R) + 1.$$

If R is noetherian, then we can be more precise, namely show that $\dim(R[X]) = \dim(R) + 1$. In the following we aim to prove this. We begin with a preliminary result. We recall a definition: if (A, M) and (B, N) are local rings, then a homomorphism $f : A \rightarrow B$ is a local homomorphism if $f(M) \subset N$.

Let A and B be commutative rings and $f : A \rightarrow B$ a surjective ring homomorphism. We suppose that M is a maximal ideal in A whose image M' under f is not equal to B . Then M' is a maximal ideal in B , hence a prime ideal, and f induces a surjective local ring homomorphism f_1 from A_M onto $B_{M'}$:

For $\frac{r}{s} \in A_M$, we set $f_1(\frac{r}{s}) = \frac{f(r)}{f(s)}$. We need to show that $f(s) \notin M'$ and that f_1 is well-defined. To see that $f(s) \notin M'$ it is sufficient to show that $M = f^{-1}(M')$. Clearly, $M \subset f^{-1}(M')$. If $M \neq f^{-1}(M')$, then $f^{-1}(M') = A$, but in this case all the elements of A are mapped onto a proper subset of B , which contradicts the surjectivity.

If $\frac{r}{s} = \frac{r'}{s'}$, then there exists $t \notin M$ such that $t(r's - rs') = 0$, which implies that $f(t)(f(r')f(s) - f(r)f(s')) = 0$. As $f(t) \notin M'$, we have $f_1(\frac{r}{s}) = f_1(\frac{r'}{s'})$ and so f_1 is well-defined.

It is easy to check that f_1 is a surjective ring homomorphism. As $f_1(A_M M) \subset B_{M'} M'$, f_1 is a local ring homomorphism.

Theorem 8 *Let R be a noetherian ring and M a maximal ideal in $R[X]$. If $P = R \cap M$, then P is a prime ideal in R and $\text{ht}(M) = \text{ht}(P) + 1$.*

PROOF Let

$$P_0 \subset P_1 \subset \cdots \subset P_s = P$$

form a chain of distinct prime ideals in R . Then the ideals

$$R[X]P_0 \subset R[X]P_1 \subset \cdots \subset R[X]P_s$$

form a chain of distinct prime ideals in $R[X]$. (If $R[X]P_i = R[X]P_j$, then $P_i[X] = P_j[X]$, and so $P_i = P_j$.) Moreover, $R[X]P_s$ is strictly included in M , because $R[X]/R[X]P$ is not a field. Therefore $\text{ht}(M) \geq \text{ht}(P) + 1$. We will show that $\text{ht}(M) \leq \text{ht}(P) + 1$ by induction on $n = \text{ht}(P)$, which will imply that $\text{ht}(M) = \text{ht}(P) + 1$, as required.

If $n = 0$, then P is a minimal prime ideal in R . Let Q be a prime ideal in $R[X]$ contained in M . Then $Q \subset M$ implies that $R \cap Q \subset R \cap M = P$. Since $\text{ht}(P) = 0$, we have $P = R \cap Q = R \cap M$. From [1] Lemma 4, if Q is properly contained in M , then $Q = R[X]P$. Hence no chain of prime ideals contained in M can be longer than 1 and it follows that $\text{ht}(M) \leq 1 = \text{ht}(P) + 1$, so the result is true for $n = 0$.

Now suppose that $n \geq 1$ and that the result is true up to $n - 1$. From Theorem 6 there is an element $x \in P$ such that $\text{ht}((x)) = 1$. Let $B = R/(x)$ and ψ be the canonical mapping from R onto B . We define a mapping ϕ from $R[X]$ onto $B[X]$ by

$$\phi(a_0 + a_1X + \cdots + a_nX^n) = \psi(a_0) + \psi(a_1)X + \cdots + \psi(a_n)X^n.$$

The mapping ϕ is clearly a surjective homomorphism. Let M be a maximal ideal in $R[X]$ and $M' = \phi(M)$. We claim that $M' \neq B[X]$. Suppose that this is not the case. Then there exists

$$f = a_0 + a_1X + \cdots + a_mX^m \in M,$$

with $\phi(f) = 1 \in B[X]$. As $\phi(1) = 1$, we have

$$\psi(1 - a_0) = \psi(a_1) = \cdots = \psi(a_m) = 0 \implies 1 - a_0, a_1, \dots, a_m \in (x) \subset P \subset M.$$

As $1 \notin M$, we must have $a_0 \notin M$. However, $a_1X, \dots, a_mX^m \in M$, because $R[X]P \subset M$, and $f \in M$ implies that $a_0 \in M$, a contradiction. It follows that $\phi(f) \neq 1$ and hence $M' \neq B[X]$, as claimed. Thus M' is a maximal ideal in $B[X]$ and so ϕ induces a surjective local ring homomorphism ϕ_1 from $R[X]_M$ onto $B[X]_{M'}$.

We claim that the kernel of ϕ_1 is $R[X]_Mx$. First we notice that $\phi_1(\frac{r}{s}) = 0$ if and only if there exists $\beta \in B[X] \setminus M'$ such that $\beta\phi(r) = 0$. If $u \in R[X]_Mx$, then $u = \frac{r}{s}x$; as $\phi(rx) = \phi(r)\phi(x)$ and $\phi(x) = 0$, we have $u \in \ker(\phi_1)$. Hence $\ker R[X]_Mx \subset \ker(\phi_1)$. Now suppose that $v \in R[X]_M$ and $v \in \ker(\phi_1)$. Then $v = \frac{r}{s}$, with $s \notin M$ and there exists $\beta \in B[X] \setminus M'$ such that $\beta\phi(r) = 0$. As ϕ is surjective, there exists $\alpha \in R[X]$ such that $\phi(\alpha) = \beta$ and, given that $\beta \notin M'$, we have $\alpha \notin M$. Since ϕ is ring homomorphism, we may write $\phi(\alpha r) = 0$, i.e., $\alpha r \in \ker(\phi)$, which implies that x divides αr . It follows that $\frac{\alpha r}{s} \in R[X]_Mx$. However, $\alpha \notin M$ implies that $\frac{1}{\alpha} \in R[X]_M$ and so $\frac{1}{\alpha} \frac{\alpha r}{s} \in R[X]_Mx$, i.e., $\frac{r}{s} \in R[X]_Mx$. This completes the proof that $\ker(\phi_1) = R[X]_Mx$, as claimed.

We now notice that $R[X]_M M$ is the unique maximal ideal in the local ring $R[X]_M$ and that its height in $R[X]_M$ is the same as that of M in $R[X]$. Thus, using [1] Lemma 2, Lemma 3 and the isomorphism which have just established, we obtain

$$\text{ht}(M) = \dim(R[X]_M) \leq \dim(R[X]_M/R[X]_Mx) + 1 = \dim(B[X]_{M'}) + 1 = \text{ht}(M') + 1.$$

We now return to the canonical mapping ψ from R onto $B = R/(x)$. Then ψ is a surjective homomorphism. Let $P' = \psi(P)$. We claim that $u \notin P$ implies that $\psi(u) \notin P'$: If $u + (x) \in P'$, then there exists $v \in P$ such that $u + (x) = v + (x)$, which implies that $u - v \in (x) \subset P$, because $x \in P$. It follows that $u \in P$, a contradiction, hence $\psi(u) \notin P'$. Proceeding as above for M and M' we may construct a ring homomorphism ψ_1 from R_P onto $B_{P'}$ whose kernel is R_Px , hence R_P/R_Px is isomorphic to $B_{P'}$. Since $B_{P'}P'$ is the unique maximal ideal in the local ring $B_{P'}$ and the height of P' in R is that of $B_{P'}P'$ in $B_{P'}$, we have $\text{ht}(P') = \dim(B_{P'})$. In the same way $\text{ht}(P) = \dim(R_P)$. Thus, using [1] Lemma 1,

$$\text{ht}(P') = \dim(B_{P'}) = \dim(R_P/R_Px) \leq \dim(R_P) - \text{ht}(R_Px).$$

However, $\text{ht}(R_Px) \geq 1$ implies that $-\text{ht}(R_Px) \leq -1$, thus

$$\text{ht}(P') \leq \dim(R_P) - 1 = \text{ht}(P) - 1,$$

where we have used [1] Lemma 2.

We recall that M' is a maximal ideal in $B[X]$ and P' a prime ideal in B . Our next step is to show that $B \cap M' = P'$. We recall that the mapping $\phi : R[X] \rightarrow B[X]$ has the form

$$\phi(a_0 + a_1X + \cdots + a_mX^m) = \psi(a_0) + \psi(a_1)X + \cdots + \psi(a_m)X^m.$$

Then

$$M' = \phi(M) = \{\psi(a_0) + \psi(a_1)X + \cdots + \psi(a_m)X^m : a_0 + a_1X + \cdots + a_mX^m \in M\}.$$

Then $B \cap M'$ is composed of the elements $\psi(a_0)$, where $a_0 + a_1X + \cdots + a_nX^n \in M$ and $\psi(x_1) = \cdots = \psi(x_n) = 0$. Hence $B \cap M' \subset \{\psi(a_0) : a_0 \in M \cap R\} = P'$. On the other hand, it is clear that $P' \subset B \cap M'$, therefore we have the equality $B \cap M' = P'$, as required.

We may now complete the proof. We have seen that $\text{ht}(P') \leq \text{ht}(P) - 1 = n - 1$, so, by the induction hypothesis, $\text{ht}(M') \leq \text{ht}(P') + 1$. Therefore

$$\text{ht}(M) \leq \text{ht}(M') + 1 \leq \text{ht}(P') + 2 = \text{ht}(P) + 1,$$

as asserted. □

Corollary 5 *If R is a noetherian ring, then*

$$\dim(R[X_1, \dots, X_n]) = \dim(R) + n.$$

PROOF It is sufficient to prove the result for $n = 1$, and for this we only need to show that $\dim(R[X]) \leq \dim(R) + 1$. Using Theorem 8, for any maximal ideal $M \subset R[X]$ we have

$$\text{ht}(M) = \text{ht}(R \cap M) + 1 \leq \sup_{P \in \text{Spec}(R)} \text{ht}(P) + 1 = \dim(R) + 1$$

and so $\dim(R[X]) \leq \dim(R) + 1$. However, from [1] Theorem 2 we know that $\dim(R) + 1 \leq \dim(R[X])$, therefore $\dim(R[X]) = \dim(R) + 1$. □

References

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