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Primary ideals

Rodney Coleman, Laurent Zwald

December 4, 2020

Abstract

We introduce primary ideals and prove the Lasker-Noether theorem, namely that in a noetherian ring any ideal can be written as a minimal finite intersection of primary ideals. We also introduce minimal prime ideals and symbolic powers of prime ideals, which are closely related to primary ideals.

First notions

All rings are assumed to be commutative with identity.

Definition An ideal Q in a ring R is primary if

- 1. $Q \neq R$;
- 2. $xy \in Q \implies x \in Q$ or $y^n \in Q$, for some $n > 0$.

Clearly prime ideals are primary, so the notion of primary ideal extends that of prime ideal.

The next result is fundamental.

Proposition 1 *The ideal Q is primary in the ring R if and only if R/Q is nontrivial and every zero-divisor in R/Q is nilpotent.*

PROOF \implies) If Q is primary, then Q is properly contained in R , so R/Q is nontrivial. If $z + Q$ is a zero-divisor, then there exists $w \notin Q$ such that

$$zw + Q = Q \implies zw \in Q.$$

As $w \notin Q$, there exists $n > 0$ such that $z^n \in Q$ and so $(z + Q)^n = Q$.

\Leftarrow) Suppose that R/Q is nontrivial and that every zero-divisor in R/Q is nilpotent. As R/Q is nontrivial $Q \neq R$. Let $x, y \in R$ be such that $xy \in Q$. If $x \notin Q$, then

$$(x + Q)(y + Q) = xy + Q = Q,$$

so $y + Q$ is a zero-divisor and there exists $n > 0$ such that

$$y^n + Q = (y + Q)^n = Q \implies y^n \in Q.$$

Therefore Q is primary. □

Example The primary ideals in \mathbf{Z} are (0) and (p^i) , where p is a prime number and $i > 0$: It is easy to see that such ideals are primary. Suppose that Q is a proper ideal in \mathbf{Z} , which is not of this form; then $Q = (m)$, and there is a prime number p and an element $q > 1$ in \mathbf{Z} such that $m = p^i q$, with $i \geq 1$ and $p \nmid q$. However, $p + (m)$ is a zero-divisor in $\mathbf{Z}/(m)$, which is not nilpotent, because $p^n \in (m)$ implies that $p^i q | p^n$, which is not possible. So in this case Q is not primary. Thus the primary ideals in \mathbf{Z} are as stated.

Notation For an ideal Q in a ring R , we will write $r(Q)$ for the radical of Q , i.e., $r(Q)$ is the set of elements $x \in R$ with a positive power in Q . We recall that $r(Q)$ is the intersection of all prime ideals containing Q .

PRIMARYprop1

Proposition 2 *Let Q be a primary ideal in the ring R . Then the radical $r(Q)$ of Q is the smallest prime ideal containing Q .*

PROOF Let Q be a primary ideal in the ring R and suppose that $x, y \in R$ and $xy \in r(Q)$. There exists $n > 0$ such that $(xy)^n \in Q$, i.e., $x^n y^n \in Q$. As Q is primary, either $x^n \in Q$ or $(y^n)^m \in Q$, for some $m > 0$. By definition of the radical, either $x \in r(Q)$ or $y \in r(Q)$, so $r(Q)$ is prime.

Suppose now that P' is a prime ideal containing Q . If $x \in r(Q)$, then $x^n \in Q$, for some $n > 0$. As $Q \subset P'$ and P' is prime, we have $x \in P'$. Hence $r(Q) \subset P'$. \square

Corollary 1 *If P is a prime ideal, then $r(P) = P$.*

Definition If Q is a primary ideal and $r(Q) = P$, then we say that Q is P -primary. It is noticeable that P is a prime ideal.

Example If p is a prime number and $i > 0$, then the radical of the primary ideal (p^i) is (p) , so (p^i) is (p) -primary.

We have seen that powers of prime ideals in \mathbf{Z} are primary. We may generalize this to UFDs.

Proposition 3 *If R is a UFD and $p \in R$ a prime element, then all powers of the principal ideal (p) are primary ideals.*

PROOF Let us consider $(p)^n$, with $n > 0$. If $ab \in (p)^n$, then $p^n | ab$. Since R is a UFD, if $a = cp^k$, with $k < n$, then $b = dp^l$, with $l \geq n - k \neq 0$. There exists s such that $ls \geq n$, so $b^s = d^s p^{ls} \in (p)^n$. \square

PRIMARYcor1

Corollary 2 *If F is a field and $\lambda \in F$, then $F[X](-\lambda + X)^n$ is a primary ideal in $F[X]$, for any $n > 0$.*

Remark We might be tempted to think that powers of primary ideals are always prime or that primary ideals are always powers of prime ideals. Both of these statements are false. We will give a counter-example to each of these assertions in an appendix.

In Proposition ^{PRIMARYprop1}2 we saw that the radical of a primary ideal is a prime ideal. We have a partial converse to this statement.

PRIMARYprop2

Proposition 4 *If Q is a proper ideal in the ring R and $r(Q)$ is maximal, then Q is primary.*

PROOF Suppose that $r(Q) = M$, with M maximal. As $r(Q)$ is the intersection of all prime ideals containing Q , if P is such an ideal, then $M \subset P$. As M is maximal, we have $M = P$. Thus M is the unique prime ideal in R containing Q . It follows that M/Q is the unique prime ideal in R/Q . As every nonunit is contained in a maximal ideal, every nonunit in R/Q is contained in M/Q . However, all elements in M have powers in Q , so every nonunit in R/Q is nilpotent. As zero-divisors are nonunits, every zero-divisor in R/Q is nilpotent. It follows from Proposition 1 that Q is primary. □

Although powers of prime ideals are not always primary, this is the case for the subclass of maximal ideals.

PRIMARYcor1

Corollary 3 *If M is a maximal ideal in the ring R , then any positive power of M is M -primary.*

PROOF Let M be a maximal ideal in R and $n > 0$. We claim that $r(M^n) = M$: If $x \in r(M^n)$, then there exists $m > 0$ such that $x^m \in M^n \subset M$, so $x \in r(M) = M$, because M is prime. Hence $r(M^n) \subset M$. Suppose now that $x \in M$; then $x^n \in M^n$, which implies that $x \in r(M^n)$, so $M \subset r(M^n)$. This proves the claim. As $r(M^n) = M$, $r(M^n)$ is maximal, so, from Proposition 1, M^n is primary. Also, because $r(M^n) = M$, M^n is M -primary. □

Intersections of primary ideals

A finite intersection of prime ideals is not necessarily a prime ideal. However, a finite intersection of primary ideals is primary, if we impose that all the ideals are P -primary for a given prime P . To establish this, we need a preliminary result.

PRIMARYlem1

Lemma 1 *If Q_1, \dots, Q_n are ideals in a ring R and $Q = \cap_{i=1}^n Q_i$, then*

$$r(Q) = \cap_{i=1}^n r(Q_i).$$

PROOF If $x \in r(Q)$, then $x^m \in Q$, for some $m > 0$, and so $x^m \in Q_i$, for all i . Thus, $x \in \cap_{i=1}^n r(Q_i)$ and it follows that $r(Q) \subset \cap_{i=1}^n r(Q_i)$.

Now suppose that $x \in \cap_{i=1}^n r(Q_i)$. For all i , there exists $m_i > 0$ such that $x^{m_i} \in Q_i$. Setting $m = \max\{m_i\}$, we obtain

$$\forall i, x^m \in Q_i \implies x^m \in Q \implies x \in r(Q).$$

Thus $\cap_{i=1}^n r(Q_i) \subset r(Q)$ and so $r(Q) = \cap_{i=1}^n r(Q_i)$. □

PRIMARYthm1

Theorem 1 *Let P be a prime ideal in the ring R and Q_1, \dots, Q_n P -primary ideals. Then $Q = \cap_{i=1}^n Q_i$ is also P -primary.*

PROOF From Lemma 1, □

$$r(Q) = \cap_{i=1}^n r(Q_i) = \cap_{i=1}^n P = P,$$

so it remains to show that Q is primary. Suppose that $xy \in Q$. Then $xy \in Q_i$, for all i . If $x \notin Q_j$ for some j , then $y^m \in Q_j$, for some $m > 0$, because Q_j is primary. This implies that $y \in r(Q_j) = P$. Since $r(Q) = P$, there exists $n > 0$ such that $y^n \in Q$, so Q is primary. □

Ideals $Q : x$

Definition If Q is a proper ideal in a ring R and $x \in R$, then we set

$$Q : x = \{y \in R : xy \in Q\}.$$

There is no difficulty in seeing that $Q : x$ is an ideal in R and that $Q \subset Q : x$.

PRIMARYprop2a

Proposition 5 Let P be a prime ideal in the ring R , Q a P -primary ideal and $x \in R$. Then

- 1. $x \in Q \implies Q : x = R$;
- 2. $x \notin Q \implies Q : x$ is P -primary;
- 3. $x \notin P \implies Q : x = Q$.

PROOF 1. If $x \in Q$, then $x1 \in Q$, so $1 \in Q : x$, which implies that $Q : x = R$.

2. Suppose that $x \notin Q$. If $y \in Q : x$, then $xy \in Q$. As Q is primary and $x \notin Q$, there exists $k > 0$ such that $y^k \in Q$. Hence $y \in r(Q) = P$. Thus we have

$$Q \subset Q : x \subset P.$$

yielding

$$P = r(Q) \subset r(Q : x) \subset r(P) = P$$

hence

$$r(Q : x) = P.$$

It remains to show that $Q : x$ is primary. Since $x \notin Q$, we have $1 \notin Q : x$, so $Q : x \neq R$. Suppose that $ab \in Q : x$. If $b^k \notin Q : x$, for all $k > 0$, then $b \notin r(Q : x) = P$. However, $abx \in Q$ implies that $ax \in Q$ or $b^l \in Q$, for some $l > 0$. In the latter case, $b \in r(Q) = P$, a contradiction, so $ax \in Q$, which implies that $a \in Q : x$. Therefore $Q : x$ is primary.

3. Suppose that $x \notin P$. If $y \notin Q$ and $xy \in Q$, then $x^k \in Q$, for some $k > 0$, because Q is primary, so $x \in r(Q) = P$, a contradiction, hence $xy \notin Q$, which implies that $y \notin Q : x$. As $Q \subset Q : x$, we have $Q = Q : x$. \square

The following property is useful.

PRIMARYprop3

Proposition 6 If Q_1, \dots, Q_n are ideals in a ring R and $x \in R$, then

$$(\cap_{i=1}^n Q_i) : x = \cap_{i=1}^n (Q_i : x).$$

PROOF If $a \in \cap_{i=1}^n (Q_i : x)$, then $ax \in Q_i$, for all i . It follows that $ax \in \cap_{i=1}^n Q_i$ and so $a \in (\cap_{i=1}^n Q_i) : x$.

On the other hand, if $a \in (\cap_{i=1}^n Q_i) : x$, then $ax \in \cap_{i=1}^n Q_i$. Thus $ax \in Q_i$, for all i , and it follows that $a \in (\cap_{i=1}^n Q_i) : x$. \square

Primary decomposition

Definition A primary decomposition of an ideal I in a ring R is an expression of the form

$$I = \cap_{i=1}^n Q_i,$$

where the Q_i are primary ideals. The expression is said to be minimal if

- (i) the radicals $r(Q_1), \dots, r(Q_n)$ are distinct;
- (ii) $\cap_{j \neq i} Q_j \not\subset Q_i$, for all i .

If an ideal has a primary decomposition, then we say that it is decomposable.

Proposition 7 *A primary decomposition may be replaced by a minimal primary decomposition.*

PROOF Consider a primary decomposition

$$I = \cap_{i=1}^n Q_i.$$

If $r(Q_{i_1}) = \dots = r(Q_{i_k}) = P_1$, then, from Theorem PRIMARYthm1,

$$Q = \cap_{j=1}^k Q_{i_j}$$

is P_1 -primary, so we may replace the ideals Q_{i_1}, \dots, Q_{i_k} by Q . Continuing in the same way we can guarantee that the condition (i) holds.

If condition (ii) does not hold, then we may eliminate ideals until it does hold, without changing the overall intersection. □

In an arbitrary ring there may be ideals which do not have a primary decomposition. However, every ideal in a noetherian ring has a primary decomposition. We will now set about proving this. To do so we introduce the notion of an irreducible ideal.

Definition A proper ideal I in a ring R is irreducible if there is no pair of ideals $\{J_1, J_2\}$, both distinct from I , such that $I = J_1 \cap J_2$. Alternatively, if $I = J_1 \cap J_2$, then $J_1 = I$ or $J_2 = I$.

PRIMARYlem2

Lemma 2 *An irreducible ideal in a noetherian ring is primary.*

PROOF Let R be a noetherian ring and Q an irreducible ideal in R . By definition, we have $Q \neq R$, so R/Q is nontrivial. Let \bar{x} be a zero-divisor in R/Q . Then there exists $\bar{y} \neq \bar{0}$ in R/Q such that $\bar{x}\bar{y} = \bar{0}$. We consider the chain of ideals in R/Q :

$$\text{Ann}(\bar{x}) \subset \text{Ann}(\bar{x}^2) \subset \text{Ann}(\bar{x}^3) \subset \dots,$$

where $\text{Ann}(\bar{a})$ is the annihilator of the element \bar{a} , i.e., $\text{Ann}(\bar{a}) = \{\bar{u} \in R/Q : \bar{u}\bar{a} = \bar{0}\}$. As R/Q is noetherian, there exists $n > 0$ such that $\text{Ann}(\bar{x}^n) = \text{Ann}(\bar{x}^{n+1})$. We claim that $(\bar{y}) \cap (\bar{x}^n) = (\bar{0})$. Indeed, suppose that $\lambda\bar{y} = \mu\bar{x}^n$, for some $\lambda, \mu \in R/Q$. Then

$$\bar{0} = \lambda\bar{y}\bar{x} = \mu\bar{x}^{n+1},$$

hence $\mu \in \text{Ann}(\bar{x}^{n+1}) = \text{Ann}(\bar{x}^n)$. Thus $\mu\bar{x}^n = \bar{0}$ and so $(\bar{y}) \cap (\bar{x}^n) = (\bar{0})$, as claimed.

Since Q is irreducible in R , the ideal $(\bar{0})$ is irreducible in R/Q . As $(\bar{y}) \neq (\bar{0})$, we must have $(\bar{x}^n) = (\bar{0})$. Hence \bar{x}^n is nilpotent and it follows from Proposition PRIMARYprop1a that Q is primary. □

We are now in a position to show that any ideal in a noetherian has a primary decomposition. From Lemma PRIMARYlem2 it is sufficient to show that we may express an ideal as an intersection of irreducible ideals.

Theorem 2 *If I is an ideal in a noetherian ring, then I has a primary decomposition.*

PROOF As we have remarked above, it is sufficient to show that there are irreducible ideals Q_1, \dots, Q_n whose intersection is I . Let S be the set of ideals which are not finite intersections of irreducible ideals. Suppose that S is nonempty. Let C be a chain of ideals in S . If C does not have a maximum, then we can extract from C an infinite chain of distinct ideals. However, this is not possible, because R is noetherian. Therefore every chain has a maximum and so by Zorn's lemma, S has a maximal element M . As M is not irreducible, there exist ideals J and K , such that $M = J \cap K$, with $M \subsetneq J$ and $M \subsetneq K$. Since M is maximal, both J and K do not belong to S , i.e., J and K are both finite intersections of irreducible ideals. However, this implies that $M = I \cap J$ is such an intersection, a contradiction. It follows that S is empty, which finishes the proof. \square

We might be tempted to think that primary decompositions are unique, or at least that minimal primary decompositions are unique. This is not the case; however, we do have certain uniqueness properties.

Theorem 3 *Let I be a decomposable ideal in a ring R and*

$$I = \bigcap_{i=1}^n Q_i$$

a minimal primary decomposition. We set $P_i = r(Q_i)$, for $i = 1, \dots, n$. Then the set $\{P_1, \dots, P_n\}$ is composed of the prime ideals P in R such that $P = r(I : x)$, for some $x \in R$.

PROOF Let $x \in R$. Then, using Proposition [PRIMARYprop3](#) and Lemma [PRIMARYlem1](#), we obtain

$$r(I : x) = r(\bigcap_{i=1}^n Q_i : x) = r(\bigcap_{i=1}^n (Q_i : x)) = \bigcap_{i=1}^n r(Q_i : x).$$

Finally, from parts [1.](#) and [2.](#) of Proposition [PRIMARYprop2a](#), we obtain

$$r(I : x) = \bigcap_{i=1}^n r(Q_i : x) = \bigcap_{i, x \notin Q_i} P_i.$$

If the intersection of a finite set of ideals is a prime ideal, then the intersection is equal to one of the ideals; thus, if $r(I : x)$ is prime, then

$$r(I : x) \in \{P_i : x \notin Q_i\} \subset \{P_1, \dots, P_n\}.$$

Now we consider the converse. Let $i \in \{1, \dots, n\}$. Because the primary decomposition is minimal, for each i , there exists

$$x_i \in (\bigcap_{j \neq i} Q_j) \setminus Q_i.$$

If $y \in Q_i : x_i$, then $yx_i \in Q_i$, so

$$yx_i \in Q_i \cap (\bigcap_{j \neq i} Q_j) = I \implies y \in I : x_i.$$

Hence,

$$Q_i : x_i \subset I : x_i \subset Q_i : x_i.$$

(The latter inclusion follows from the fact that $I \subset Q_i$.) Therefore $Q_i : x_i = I : x_i$ and so, using part [2.](#) of Proposition [PRIMARYprop2a](#), we obtain

$$r(I : x_i) = r(Q_i : x_i) = P_i.$$

It follows that the P_i form the set of those ideals $r(I : x)$ which are prime. \square

Corollary 4 *In a minimal decomposition*

$$I = \bigcap_{i=1}^n Q_i$$

the prime ideals $P_i = r(Q_i)$ are uniquely determined.

Minimal prime ideals

In this paragraph I is supposed to be a decomposable ideal and $I = \bigcap_{i=1}^n Q_i$ a minimal primary decomposition. We denote $r(Q_i) = P_i$. We say that the prime ideals P_1, \dots, P_n belong to I . The minimal elements of the set $S = \{P_1, \dots, P_n\}$ with respect to inclusion are said to be isolated and the others embedded. At least one isolated prime ideal exists, because the set S is finite.

Let $V_I = \{P \in \text{Spec}(R) : I \subset P\}$. Then $S \subset V_I$, because $I \subset Q_i \subset r(Q_i) = P_i$, for $i = 1, \dots, n$. We will show that the minimal elements of the set S are the minimal elements of the set V_I . We need the following result.

PRIMARYprop4

Proposition 8 *Let I be a decomposable ideal in a ring R , $I = \bigcap_{i=1}^n Q_i$ a minimal primary decomposition, with S the set of prime ideals belonging to I . If $P \in V_I$, then P contains an isolated prime ideal P_j .*

PROOF Let $I = \bigcap_{i=1}^n Q_i$ be a minimal primary decomposition. We set $P_i = r(Q_i)$, for $i = 1, \dots, n$. Then

$$P = r(P) \supset r(I) = \bigcap_{i=1}^n r(Q_i) = \bigcap_{i=1}^n P_i,$$

where we have used Lemma I. However, if a prime ideal contains an intersection of ideals, then at least one of the ideals in the intersection is contained in the prime ideal, therefore $P_j \subset P$, for some j . The result now follows. \square

Corollary 5 *The isolated prime ideals are the minimal elements in V_I .*

PROOF Above we saw that $S \subset V_I$, so *a fortiori* if P_j is an isolated prime in S , then $P_j \in V_I$. Suppose now that $P \in V_I$, with $P \subset P_j$. From Proposition 8 there exists an isolated prime ideal $P_k \subset P$, so $P_k \subset P_j$. Since P_j is isolated, $P_k = P_j$, hence $P = P_j$ and it follows that P_j is minimal in V_I .

Suppose now that P is minimal in V_I . From Proposition 8, P contains an isolated prime ideal $P_j \in S$. As P is minimal, we have $P = P_j$. \square

Although the primary ideals in different minimal decompositions of an ideal are not necessarily the same, the primary ideals whose radicals are isolated are the same. We aim now to establish this.

Lemma 3 *Let I be a decomposable ideal in the ring R and Q_j an ideal in a minimal decomposition of I such that $r(Q_j)$ is an isolated prime ideal. Then Q_j is composed of the elements $a \in R$ for which there exists $b \notin r(Q_j)$ with $ab \in I$.*

PROOF Let $I = \bigcap_{i=1}^n Q_i$ be a minimal decomposition, with $r(Q_j)$ isolated. We claim that $Q_i \not\subset r(Q_j)$, for $i \neq j$: Suppose that $Q_i \subset r(Q_j)$ and let $x \in r(Q_i)$; then $x^n \in Q_i$, for some $n > 0$, which implies that there exists $m > 0$ such that $x^{nm} \in Q_j$, for some $m > 0$, because $Q_i \subset r(Q_j)$, hence $x \in r(Q_j)$. It follows that $r(Q_i) \subset r(Q_j)$, which is impossible, because $r(Q_j)$ is isolated. Thus $Q_i \not\subset r(Q_j)$, as claimed.

Now let $a \in Q_j$. From what we have just seen, for $i \neq j$, there exists $b_i \in Q_i \setminus r(Q_j)$. We set $b = \prod_{i \neq j} b_i$. As $r(Q_j)$ is a prime ideal, we have $b \notin r(Q_j)$. However, $ab \in Q_i$, for $i \neq j$, because

$b_i \in Q_i$. Also, given that $a \in Q_j$, we also have $ab \in Q_j$ and so $ab \in \bigcap_{i=1}^n Q_i = I$. Therefore, a is an element in R for which there exists $b \notin r(Q_j)$ with $ab \in I$.

We now consider the converse. Suppose that $a \in R$ and $b \notin r(Q_j)$ are such that $ab \in I$. Since $I \subset Q_j$, we have $ab \in Q_j$. As Q_j is primary and $b^n \notin Q_j$, for any $n > 0$, we must have $a \in Q_j$. This concludes the proof. \square

Corollary 6 *If I is a decomposable ideal in a ring R , then the primary ideals in a minimal decomposition of I corresponding to isolated prime ideals are uniquely determined.*

Symbolic powers of prime ideals

Let P be a prime ideal in a commutative ring R . For $n \in \mathbf{N}^*$, we call $P^{(n)} = R_P P^n \cap R$ the n th symbolic power of P . It is not difficult to see that

$$P^{(n)} = \{r \in R : sr \in P^n \text{ for some } s \notin P\}.$$

PRIMARYprop5

Proposition 9 *Let $f : A \rightarrow B$ be a ring homomorphism and $Q \subset B$ a primary ideal. Then $P = f^{-1}(Q)$ is a primary ideal.*

PROOF It is not difficult to show that P is an ideal. Since $1 \notin P$, because $f(1) = 1 \notin Q$, P is a proper ideal and it follows that $A/P \neq 0$. f induces a mapping $F : A/P \rightarrow B/Q$ defined by $F(a + P) = f(a) + Q$, which is a well-defined ring homomorphism. If $F(a + P) = 0$, then $f(a) \in Q$, which implies that $a \in P$, so $a + P = 0$, hence F is injective.

We now use Proposition 1. Let $a + P$ be a zero-divisor in A/P . Then $F(a + P)$ is a zero-divisor in B/Q . From Proposition 1, $F(a + P)$ is nilpotent in B/Q , i.e., $F(a + P)^n = 0$, for some $n \in \mathbf{N}^*$. As F is injective, $(a + P)^n = 0$, i.e., $a + P$ is nilpotent. It follows that P is primary. \square

Corollary 7 *The n th symbolic power $P^{(n)}$ of P is P -primary.*

PROOF First we show that $P^{(n)}$ is primary. Since $R_P P$ is maximal in R_P , from Corollary 3, $R_P P^n = (R_P P)^n$ is $R_P P$ -primary. The standard mapping $f : R \rightarrow R_P$ is a ring homomorphism and so, from Proposition 9, $f^{-1}(R_P P^n) = R_P P^n \cap R$ is primary, i.e., $P^{(n)}$ is primary.

We now show that $P^{(n)}$ is P -primary. Suppose that $x \in r(P^{(n)})$. For some $s \in \mathbf{N}^*$, $x^s \in P^{(n)} = R_P P^n \cap R \subset R_P P^n$. Hence $x^s \in R_P P$ and so $x \in R_P P$, because $R_P P$ is a prime ideal. As $x \in R$, we have $x \in R_P P \cap R = P$ and so $r(P^{(n)}) \subset P$. Now suppose that $x \in P$. Then $x \in R_P P = r(R_P P^n)$. Hence there exists $s \in \mathbf{N}^*$ such that $x^s \in R_P P^n$. However, $x^s \in R$, so $x^s \in R_P P^n \cap R = P^{(n)}$, hence $P \subset r(P^{(n)})$. It follows that $P^{(n)}$ is P -primary. \square

We may go a little further.

Proposition 10 *$P^{(n)}$ is the smallest P -primary ideal containing P^n .*

PROOF We will use the expression for $P^{(n)}$ mentioned above, namely

$$P^{(n)} = \{r \in R : sr \in P^n \text{ for some } s \notin P\}.$$

First we notice that $1 \notin P$ implies that $P^n \subset P^{(n)}$. Let Q be another P -primary ideal such that $P^n \subset Q$ and suppose that $r \in P^{(n)}$. We aim to show that $r \in Q$. As $r \in P^{(n)}$, there exists $s \notin P$ such that $sr \in P^n \subset Q$. Then $r \in Q$ or a power of s lies in Q . In the latter case $s \in r(Q) = P$, which is a contradiction, so $r \in Q$, as required. Therefore $P^{(n)} \subset Q$. \square

Let R be a noetherian ring, P a prime ideal in R and $f : R \rightarrow R_P$ the standard mapping. We may use symbolic powers to characterize the kernel of f .

Proposition 11 *We have*

$$\text{Ker}(f) = \bigcap_{n \geq 1} P^{(n)}.$$

PROOF Let $M = R_P P$, the unique maximal ideal in R_P . Thus the Jacobson radical of R_P is M and we may apply Theorem 5 of 'Nakayama's lemma and applications' to obtain $\bigcap_{n \geq 1} M^n = (0)$. Then

$$\text{Ker}(f) = f^{-1}((0)) = f^{-1}(\bigcap_{n \geq 1} M^n) = \bigcap_{n \geq 1} f^{-1}(M^n) = \bigcap_{n \geq 1} M^n \cap R = \bigcap_{n \geq 1} P^{(n)},$$

as required. \square

APPENDIX

Primary ideals are not necessarily powers of prime ideals

Let $R = F[X, Y]$, where F is a field. We set

$$Q = \langle X, Y^2 \rangle = RX + RY^2.$$

We aim to show that Q is a primary ideal which is not a power of a prime ideal. First, we will show that all zero-divisors in R/Q , which is not trivial, are nilpotent and hence that Q is a primary ideal (Proposition 1).

We define a mapping ϕ from R into $F[Y]/\langle Y^2 \rangle$ by

$$\phi(f(X, Y)) = f(0, Y) + \langle Y^2 \rangle.$$

ϕ is clearly a surjective ring homomorphism and

$$\ker \phi = \{f(X, Y) \in R : f(0, Y) \in \langle Y^2 \rangle\}.$$

Certainly, $\phi(X)$ and $\phi(Y^2)$ belong to $\langle Y^2 \rangle$ so $Q \subset \ker \phi$. Suppose now that $f(X, Y) \in \ker \phi$. There exist $f_1 \in F[Y]$ and $f_2 \in F[X, Y]$ such that

$$f(X, Y) = f_1(Y) + Xf_2(X, Y).$$

As $f_1(Y) = f(0, Y) \in \langle Y^2 \rangle \subset Q$, we have $\ker \phi \subset Q$. Thus $\ker \phi = Q$ and so $R/Q \simeq F[Y]/\langle Y^2 \rangle$.

Now $\langle Y^2 \rangle = F[Y]Y^2$, which is a primary ideal by Corollary 3, so all zero-divisors of $F[Y]/\langle Y^2 \rangle$ are nilpotent. It follows that all zero-divisors of R/Q are nilpotent, which implies that Q is primary, as required.

Our next step is to show that Q is not a power of its radical. First we notice that

$$r(Q) = \langle X, Y \rangle = RX + RY :$$

Clearly X and Y lie in $r(Q)$, hence $\langle X, Y \rangle \subset r(Q)$. As no power of a constant polynomial lies in Q , we must have the equality stated. Then

$$(r(Q))^2 = (RX)^2 + (RY)^2 + (RX)(RY) = RX^2 + RY^2 + RXY,$$

so, for $n \geq 2$, we have

$$r(Q)^n \subset r(Q)^2 \subsetneq Q \subsetneq r(Q)$$

and it follows that Q is not a power of its radical.

We are now in a position to show that Q is not a power of a prime ideal. Suppose that $Q = P^n$, where P is a prime ideal. If $n = 1$, then Q is prime and so $r(Q) = Q$, thus Q is a power of its radical, which is impossible. Now suppose that $n \geq 2$. We claim that $r(Q) = P$. If $\alpha \in P$, then $\alpha^n \in P^n = Q \subset r(Q)$, so $P \subset r(Q)$. Now let β be an element of $r(Q)$. Then $\beta^2 \in r(Q)^2 \subsetneq Q = P^n \subset P$. As P is prime, we have $\beta \in P$ and so $r(Q) \subset P$. This establishes the claim. Therefore we may write

$$P^2 = r(Q)^2 \subsetneq Q = P^n \subset P.$$

If $n = 2$, then $r(Q)^2 = Q$, which is impossible. If $n > 2$, then

$$P^n \subset P^2 \subsetneq Q = P^n,$$

which is also not possible. It follows that Q is not a power of a prime ideal.

Powers of prime ideals are not necessarily primary ideals

Let $R = F[X, Y, Z]$, where F is a field and $I = \langle XY - Z^2 \rangle \subset R$. Also, we note $B = R/I$ and $P = \langle X + I, Z + I \rangle \subset B$. We aim to show that P is a prime ideal, with P^2 not primary.

To establish that P is a prime ideal, we show that B/P is an integral domain. We define a mapping ϕ from R into $F[Y]$ by

$$\phi(f(X, Y, Z)) = f(0, Y, 0).$$

Then ϕ is a surjective ring homomorphism and

$$\phi(XY - Z^2) = 0 \implies I \subset \ker \phi.$$

Thus ϕ induces a surjective ring homomorphism $\bar{\phi}$ from B onto $F[Y]$:

$$\bar{\phi}(f(X, Y, Z) + I) = f(0, Y, 0).$$

We claim that $\ker \bar{\phi} = P$. First, we notice that $X + I$ and $Z + I$ belong to $\ker \bar{\phi}$, so $P \subset \ker \bar{\phi}$. Showing that $\ker \bar{\phi} \subset P$ is more difficult. We may write

$$f(X, Y, Z) = f_1(Y) + Xf_2(X, Y) + Zf_3(X, Y, Z),$$

for some $f_1(Y) \in F[Y]$, $f_2(X, Y) \in F[X, Y]$ and $f_3(X, Y, Z) \in F[X, Y, Z]$. Then, if $f(X, Y, Z) + I \in \ker \bar{\phi}$,

$$f_1(Y) = f(0, Y, 0) = \bar{\phi}(f(X, Y, Z) + I) = 0,$$

so

$$f(X, Y, Z) \in \langle Y, Z \rangle \implies f(X, Y, Z) + I \in \langle X + I, Z + I \rangle = P \implies \ker \bar{\phi} \subset P,$$

and it follows that $\ker \bar{\phi} = P$. We now have $B/P \simeq F[Y]$, which implies that B/P is an integral domain and so P a prime ideal.

We now show that P^2 is not primary. First,

$$(X + I)(Y + I) = XY + I = XY - (XY - Z^2) + I = Z^2 + I = (Z + I)^2 \in P^2.$$

Also,

$$P^2 = \langle X^2 + I, XZ + I, Z^2 + I \rangle.$$

If P^2 is primary, then $X + I \in P^2$ or $Y^k + I = (Y + I)^k \in P^2$, for some $k > 0$, so that X or Y^k belong to the ideal $\langle X^2, XZ, Z^2, XY - Z^2 \rangle \subset R$. However, the elements of this ideal have the form

$$\alpha X^2 + \beta XY + \gamma Z^2 + \delta(XY - Z^2),$$

where $\alpha, \beta, \gamma, \delta \in R$. As X and Y^k do not have such a form, the ideal P^2 is not primary.