



**HAL**  
open science

# CLT FOR FLUCTUATIONS OF BETA-ENSEMBLES WITH GENERAL POTENTIAL

Florent Bekerman, Thomas Leblé, Sylvia Serfaty

► **To cite this version:**

Florent Bekerman, Thomas Leblé, Sylvia Serfaty. CLT FOR FLUCTUATIONS OF BETA-ENSEMBLES WITH GENERAL POTENTIAL. 2020. hal-03040294

**HAL Id: hal-03040294**

**<https://hal.science/hal-03040294>**

Preprint submitted on 4 Dec 2020

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# CLT FOR FLUCTUATIONS OF $\beta$ -ENSEMBLES WITH GENERAL POTENTIAL

FLORENT BEKERMAN, THOMAS LEBLÉ, AND SYLVIA SERFATY

ABSTRACT. We prove a central limit theorem for the linear statistics of one-dimensional log-gases, or  $\beta$ -ensembles. We use a method based on a change of variables which allows to treat fairly general situations, including multi-cut and, for the first time, critical cases, and generalizes the previously known results of Johansson, Borot-Guionnet and Shcherbina. In the one-cut regular case, our approach also allows to retrieve a rate of convergence as well as previously known expansions of the free energy to arbitrary order.

**keywords:**  $\beta$ -ensembles, Log Gas, Central Limit Theorem, Linear statistics.

**MSC classification:** 60F05, 60K35, 60B10, 60B20, 82B05, 60G15.

## 1. INTRODUCTION

Let  $\beta > 0$  be fixed. For  $N \geq 1$ , we are interested in the  $N$ -point canonical Gibbs measure<sup>1</sup> for a one-dimensional log-gas at the *inverse temperature*  $\beta$ , defined by

$$(1.1) \quad d\mathbb{P}_{N,\beta}^V(\vec{X}_N) = \frac{1}{Z_{N,\beta}^V} \exp\left(-\frac{\beta}{2} \mathcal{H}_N^V(\vec{X}_N)\right) d\vec{X}_N,$$

where  $\vec{X}_N = (x_1, \dots, x_N)$  is an  $N$ -tuple of points in  $\mathbb{R}$ , and  $\mathcal{H}_N^V(\vec{X}_N)$ , defined by

$$(1.2) \quad \mathcal{H}_N^V(\vec{X}_N) := \sum_{1 \leq i \neq j \leq N} -\log|x_i - x_j| + \sum_{i=1}^N NV(x_i),$$

is the energy of the system in the state  $\vec{X}_N$ , given by the sum of the pairwise repulsive logarithmic interaction between all particles plus the effect on each particle of an external field or confining potential  $NV$  whose intensity is proportional to  $N$ . We will use  $d\vec{X}_N$  to denote the Lebesgue measure on  $\mathbb{R}^N$ . The constant  $Z_{N,\beta}^V$  in the definition (1.1) is the normalizing constant, called the *partition function*, and is equal to

$$Z_{N,\beta}^V := \int_{\mathbb{R}^N} \exp\left(-\frac{\beta}{2} \mathcal{H}_N^V(\vec{X}_N)\right) d\vec{X}_N.$$

Such systems of particles with logarithmic repulsive interaction on the line have been extensively studied, in particular because of their connection with random matrix theory, see [For10] for a survey.

Under mild assumptions on  $V$ , it is known that the empirical measure of the particles converges almost surely to some deterministic probability measure on  $\mathbb{R}$  called the *equilibrium*

---

*Date:* Wednesday 28<sup>th</sup> June, 2017.

<sup>1</sup>We use  $\frac{\beta}{2}$  instead of  $\beta$  in order to match the existing literature. The first sum in (1.2), over indices  $i \neq j$ , is twice the physical one, but is more convenient for our analysis.

measure  $\mu_V$ , with no simple expression in terms of  $V$ . For any  $N \geq 1$ , let us define the *fluctuation measure*

$$(1.3) \quad \text{fluct}_N := \sum_{i=1}^N \delta_{x_i} - N\mu_V,$$

which is a random signed measure. For any test function  $\xi$  regular enough we define the *fluctuations of the linear statistics associated to  $\xi$*  as the random real variable

$$(1.4) \quad \text{Fluct}_N(\xi) := \int_{\mathbb{R}} \xi d\text{fluct}_N.$$

The goal of this paper is to prove a Central Limit Theorem (CLT) for  $\text{Fluct}_N(\xi)$ , under some regularity assumptions on  $V$  and  $\xi$ .

### 1.1. Assumptions.

**(H1) - Regularity and growth of  $V$ :** The potential  $V$  is in  $C^p(\mathbb{R})$  and satisfies the growth condition

$$(1.5) \quad \liminf_{|x| \rightarrow \infty} \frac{V(x)}{2 \log |x|} > 1.$$

It is well-known, see e.g. [ST13], that if  $V$  satisfies (H1) with  $p \geq 0$ , then the *logarithmic potential energy* functional defined on the space of probability measures by

$$(1.6) \quad \mathcal{I}_V(\mu) = \int_{\mathbb{R} \times \mathbb{R}} -\log |x - y| d\mu(x) d\mu(y) + \int_{\mathbb{R}} V(x) d\mu(x)$$

has a unique global minimizer  $\mu_V$ , the aforementioned *equilibrium measure*. This measure has a compact support that we will denote by  $\Sigma_V$ , and  $\mu_V$  is characterized by the fact that there exists a constant  $c_V$  such that the function  $\zeta_V$  defined by

$$(1.7) \quad \zeta_V(x) := \int -\log |x - y| d\mu_V(y) + \frac{V(x)}{2} - c_V$$

satisfies the Euler-Lagrange conditions

$$(1.8) \quad \zeta_V \geq 0 \text{ in } \mathbb{R}, \quad \zeta_V = 0 \text{ on } \Sigma_V.$$

We will work under two additional assumptions: one deals with the possible form of  $\mu_V$  and the other one is a non-criticality hypothesis concerning  $\zeta_V$ .

**(H2) - Form of the equilibrium measure:** The support  $\Sigma_V$  of  $\mu_V$  is a finite union of  $n+1$  non-degenerate intervals

$$\Sigma_V = \bigcup_{0 \leq l \leq n} [\alpha_{l,-}; \alpha_{l,+}], \text{ with } \alpha_{l,-} < \alpha_{l,+}.$$

The points  $\alpha_{l,\pm}$  are called the *endpoints* of the support  $\Sigma_V$ . For  $x$  in  $\Sigma_V$ , we let

$$(1.9) \quad \sigma(x) := \prod_{l=0}^n \sqrt{|x - \alpha_{l,-}| |x - \alpha_{l,+}|}.$$

We assume that the equilibrium measure has a density with respect to the Lebesgue measure on  $\Sigma_V$  given by

$$(1.10) \quad \mu_V(x) = S(x)\sigma(x),$$

where  $S$  can be written as

$$(1.11) \quad S(x) = S_0(x) \prod_{i=1}^m (x - s_i)^{2k_i}, \quad S_0 > 0 \text{ on } \Sigma_V,$$

where  $m \geq 0$ , all the points  $s_i$ , called *singular points*<sup>2</sup>, belong to  $\Sigma_V$  and the  $k_i$  are natural integers.

**(H3) - Non-criticality of  $\zeta_V$ :** The function  $\zeta_V$  is positive on  $\mathbb{R} \setminus \Sigma_V$ .

We introduce the operator  $\Xi_V$ , which acts on  $C^1$  functions by

$$(1.12) \quad \Xi_V[\psi] := -\frac{1}{2}\psi V' + \int \frac{\psi(\cdot) - \psi(y)}{\cdot - y} d\mu_V(y).$$

## 1.2. Main result.

**Theorem 1** (Central limit theorem for fluctuations of linear statistics). *Let  $\xi$  be a function in  $C^r(\mathbb{R})$ , assume that (H1)-(H3) hold. We let*

$$k = \max_{i=1, \dots, m} 2k_i,$$

where the  $k_i$ 's are as in (1.11), and assume that,  $p$  (resp.  $r$ ) denoting the regularity of  $V$  (resp.  $\xi$ )

$$(1.13) \quad p \geq (3k + 5), \quad r \geq (2k + 3).$$

If  $n \geq 1$ , assume that  $\xi$  satisfies the  $n$  following conditions

$$(1.14) \quad \int_{\Sigma_V} \frac{\xi(y)y^d}{\sigma(y)} dy = 0 \quad \text{for } d = 0, \dots, n-1.$$

Moreover, if  $m \geq 1$ , assume that for all  $i = 1, \dots, m$

$$(1.15) \quad \int_{\Sigma_V} \frac{\xi(y) - R_{s_i, d}\xi(y)}{\sigma(y)(y - s_i)^d} dy = 0 \quad \text{for } d = 1, \dots, 2k_i,$$

where  $R_{x, d}\xi$  is the Taylor expansion of  $\xi$  to order  $d-1$  around  $x$  given by

$$R_{x, d}\xi(y) = \xi(x) + (y-x)\xi'(x) + \dots + \frac{(y-x)^{d-1}}{(d-1)!} \xi^{(d-1)}(x).$$

Then there exists a constant  $c_\xi$  and a function  $\psi$  of class  $C^2$  in some open neighborhood  $U$  of  $\Sigma_V$  such that  $\Xi_V[\psi] = \frac{\xi}{2} + c_\xi$  on  $U$ , and the fluctuation  $\text{Fluct}_N(\xi)$  converges in law as  $N \rightarrow \infty$  to a Gaussian distribution with mean

$$m_\xi = \left(1 - \frac{2}{\beta}\right) \int \psi' d\mu_V,$$

and variance

$$v_\xi = -\frac{2}{\beta} \int \psi \xi' d\mu_V.$$

It is proven in (B.32) that the variance  $v_\xi$  has the equivalent expression

$$(1.16) \quad v_\xi := \frac{2}{\beta} \left( \iint \left( \frac{\psi(x) - \psi(y)}{x - y} \right)^2 d\mu_V(x) d\mu_V(y) + \int V'' \psi^2 d\mu_V \right).$$

Let us note that  $\psi$ , hence also  $m_\xi$  and  $v_\xi$ , can be explicitly written in terms of  $\xi$ .

<sup>2</sup>Let us emphasize that a singular point  $s_i$  can be equal to an endpoint  $\alpha_{i, \pm}$ .

**1.3. Comments on the assumptions.** The growth condition (1.5) is standard and expresses the fact that the logarithmic repulsion is beaten at long distance by the confinement, thus ensuring that  $\mu_V$  has a compact support. Together with the non-criticality assumption (H3) on  $\zeta_V$ , it implies that the particles of the log-gas effectively stay within some neighborhood of  $\Sigma_V$ , up to very rare events.

The case  $n = 0$ , where the support has a single connected component, is called *one-cut*, whereas  $n \geq 1$  is a *multi-cut* situation. If  $m \geq 1$ , we are in a *critical case*.

The relationship between  $V$  and  $\mu_V$  is complicated in general, and we mention some examples where  $\mu_V$  is known to satisfy our assumptions.

- If  $V$  is real-analytic, then the assumptions are satisfied with  $n$  finite,  $m$  finite and  $S$  analytic on  $\Sigma_V$ , see [DKM98, Theorem 1.38], [DKM+99, Sec.1].
- If  $V$  is real-analytic, then for a “generic”  $V$  the assumptions are satisfied with  $n$  finite,  $m = 0$  and  $S$  analytic on  $\Sigma_V$ , see [KM00].
- If  $V$  is uniformly convex and smooth, then the assumptions are satisfied with  $n = 0$ ,  $m = 0$ , and  $S$  smooth on  $\Sigma_V$ , see e.g. [BdMPS95, Example 1].
- Examples of multi-cut, non-critical situations with  $n = 0, 1, 2$  and  $m = 0$ , are mentioned in [BdMPS95, Examples 3-4].
- An example of criticality at the edge of the support is given by  $V(x) = \frac{1}{20}x^4 - \frac{4}{15}x^3 + \frac{1}{5}x^2 + \frac{8}{5}x$ , for which the equilibrium measure, as computed in [CKI10, Example 1.2], is given by

$$\mu_V(x) = \frac{1}{10\pi} \sqrt{|x - (-2)||x - 2|(x - 2)^2} \mathbf{1}_{[-2,2]}(x).$$

- An example of criticality in the bulk of the support is given by  $V(x) = \frac{x^4}{4} - x^2$ , for which the equilibrium measure, as computed in [CK06], is

$$\mu_V(x) = \frac{1}{2\pi} \sqrt{|x - (-2)||x - 2|(x - 0)^2} \mathbf{1}_{[-2,2]}(x).$$

Following the terminology used in the literature [DKM+99, KM00, CK06], we may say that our assumptions allow the existence of singular points of type II (the density vanishes in the bulk) and III (the density vanishes at the edge faster than a square root). Assumption (H3) rules out the possibility of singular points of type I, also called “birth of a new cut”, for which the behavior might be quite different, see [Cla08, Mo08].

#### 1.4. Existing literature, strategy and perspectives.

**1.4.1. Connection to previous results.** The CLT for fluctuations of linear statistics in the context of  $\beta$ -ensembles was proven in the pioneering paper [Joh98] for polynomial potentials in the case  $n = 0, m = 0$ , and generalized in [Shc13] to real-analytic potentials in the possibly multi-cut, non-critical cases ( $n \geq 0, m = 0$ ), where a set of  $n$  necessary and sufficient conditions on a given test function in order to satisfy the CLT is derived. If these conditions are not fulfilled, the fluctuations are shown to exhibit oscillatory behaviour. Such results are also a by-product of the all-orders expansion of the partition function obtained in [BG13b] ( $n = 0, m = 0$ ) and [BG13a] ( $n \geq 0, m = 0$ ). A CLT for the fluctuations of linear statistics for test functions living at mesoscopic scales was recently obtained in [BL16].

1.4.2. *Motivation and strategy.* Our goal is twofold: on the one hand, we provide a simple proof of the CLT using a change of variables argument, retrieving the results cited above. On the other hand, our method allows to substantially relax the assumptions on  $V$ , in particular for the first time we are able to treat critical situations where  $m \geq 1$ .

Our method, which is adapted from the one introduced in [LS16] for two-dimensional log-gases, can be summarized as follows

- (1) We prove the CLT by showing that the Laplace transform of the fluctuations converges to the Laplace transform of the correct Gaussian law. This idea is already present in [Joh98] and many further works. Computing the Laplace transform of  $\text{Fluct}_N(\xi)$  leads to working with a new potential  $V + t\xi$  (with  $t$  small), and thus to considering the associated perturbed equilibrium measure.
- (2) Following [LS16], our method then consists in finding a change of variables (or a transport map) that pushes  $\mu_V$  onto the perturbed equilibrium measure. In fact we do not exactly achieve this, but rather we construct a transport map  $I + t\psi$ , which is a perturbation of identity, and consider the *approximate* perturbed equilibrium measure  $(I + t\psi)\#\mu_V$ . The map  $\psi$  is found by inverting the operator (1.12), which is well-known in this context, it appears e.g. in [BG13b, BG13a, Shc13, BFG13]. A CLT will hold if the function  $\xi$  is (up to constants) in the image of  $\Xi_V$ , leading to the conditions (1.14)–(1.15). The change of variables approach for one-dimensional log-gases was already used e.g. in [Shc14, BFG13], see also [GMS07, GS14] which deal with the non-commutative context.
- (3) The proof then leverages on the expansion of  $\log Z_{N,\beta}^V$  up to order  $N$  proven in [LS15], valid in the multi-cut and critical case, and whose dependency in  $V$  is explicit enough. This step replaces the a priori bound on the commutators used e.g. in [BG13b].

1.4.3. *More comments and perspectives.* Using the Cramér-Wold theorem, the result of Theorem 1 extends readily to any finite family of test functions satisfying the conditions ((1.14), (1.15)): the joint law of their fluctuations converges to a Gaussian vector, using the bilinear form associated to (1.16) in order to determine the covariance.

In the multi-cut case, the CLT results of [Shc13] or [BG13a] are stated under  $n$  necessary and sufficient conditions on the test function, and the non-Gaussian nature of the fluctuations if these conditions are not satisfied is explicitly described. In the critical cases, we only state sufficient conditions (1.15) under which the CLT holds. It would be interesting to prove that these conditions are necessary, and to characterize the behavior of the fluctuations for functions which do not satisfy (1.15).

Finally, we expect Theorem 1 to hold also at mesoscopic scales.

1.5. **The one-cut noncritical case.** In the case  $n = 0$  and  $m = 0$ , following the transport approach, we can obtain the convergence of the Laplace transform of fluctuations with an explicit rate, under the assumption that  $\xi$  is very regular (we have not tried to optimize in the regularity):

**Theorem 2** (Rate of convergence in the one-cut noncritical case). *Under the assumptions of Theorem 1, if in addition  $\mathfrak{n} = 0$ ,  $\mathfrak{m} = 0$ ,  $\mathfrak{p} \geq 6$  and  $\mathfrak{r} \geq 17$ , then we also have*

$$(1.17) \quad \left| \log \mathbf{E}_{\mathbb{P}_{N,\beta}^V} (\exp(s\text{Fluct}_N(\xi)) - sm_\xi - s^2 v_\xi) \right| \leq C \left( \frac{s}{N} \|\xi\|_{C^{17}(\mathbb{R})} + \frac{s^3}{N} \|\xi\|_{C^2(\mathbb{R})} + \frac{s^4}{N^2} \|\xi\|_{C^3(\mathbb{R})}^4 \right),$$

where the constant  $C$  depends only on  $V$ .

The assumed regularity on  $\xi$  allows to avoid using the result of [LS15] on the expansion of  $\log Z_{N,\beta}^V$ . Our transport approach also provides a functional relation on the expectation of fluctuations which allows by a bootstrap procedure to recover an expansion of  $\log Z_{N,\beta}^V$  (relative to a reference potential) to arbitrary powers of  $1/N$  in very regular cases, i.e the result of [BG13b] but without the analyticity assumption. All these results are presented in Appendix A.

**1.6. Some notation.** We denote by *P.V.* the principal value of an integral having a singularity at  $x_0$ , i.e.

$$(1.18) \quad P.V. \int f = \lim_{\varepsilon \rightarrow 0} \int_{-\infty}^{x_0 - \varepsilon} f + \int_{x_0 + \varepsilon}^{+\infty} f.$$

If  $\Phi$  is a  $C^1$ -diffeomorphism and  $\mu$  a probability measure, we denote by  $\Phi\#\mu$  the push-forward of  $\mu$  by  $\Phi$ , which is by definition such that for  $A \subset \mathbb{R}$  Borel,

$$(\Phi\#\mu)(A) := \mu(\Phi^{-1}(A)).$$

If  $A \subset \mathbb{R}$  we denote by  $\mathring{A}$  its interior.

For  $k \geq 0$ , and  $U$  some bounded domain in  $\mathbb{R}$ , we endow the spaces  $C^k(U)$  with the usual norm

$$\|\psi\|_{C^k(U)} := \sum_{j=0}^k \sup_{x \in U} |\psi^{(j)}(x)|.$$

If  $z$  is a complex number, we denote by  $\mathcal{R}(z)$  (resp.  $\mathcal{I}(z)$ ) its real (resp. imaginary) part.

For any probability measure  $\mu$  on  $\mathbb{R}$  we denote by  $h^\mu$  the logarithmic potential generated by  $\mu$ , defined as the map

$$(1.19) \quad x \in \mathbb{R}^2 \mapsto h^\mu(x) = \int -\log|x-y|d\mu(y).$$

## 2. EXPRESSING THE LAPLACE TRANSFORM OF THE FLUCTUATIONS

We start by the standard approach of reexpressing the Laplace transform of the fluctuations in terms the ratio of partition functions of a perturbed log-gas by that of the original one. This is combined with the energy splitting formula of [SS15] that separates fixed leading order terms from variable next order ones.

**2.1. The next-order energy.** For any probability measure  $\mu$ , let us define,

$$(2.1) \quad F_N(\vec{X}_N, \mu) = - \iint_{\mathbb{R} \times \mathbb{R} \setminus \Delta} \log|x-y| \left( \sum_{i=1}^N \delta_{x_i} - \mu \right)(x) \left( \sum_{i=1}^N \delta_{x_i} - \mu \right)(y),$$

where  $\Delta$  denotes the diagonal in  $\mathbb{R} \times \mathbb{R}$ .

We have the following splitting formula for the energy, as introduced in [SS15] (we recall the proof in Section B.1).

**Lemma 2.1.** *For any  $\vec{X}_N \in \mathbb{R}^N$ , it holds that*

$$(2.2) \quad \mathcal{H}_N^V(\vec{X}_N) = N^2 \mathcal{I}_V(\mu_V) + 2N \sum_{i=1}^N \zeta_V(x_i) + F_N(\vec{X}_N, \mu_V).$$

Using this splitting formula (2.2), we may re-write  $\mathbb{P}_{N,\beta}^V$  as

$$(2.3) \quad d\mathbb{P}_{N,\beta}^V(\vec{X}_N) = \frac{1}{K_{N,\beta}(\mu_V, \zeta_V)} \exp \left( -\frac{\beta}{2} \left( F_N(\vec{X}_N, \mu_V) + 2N \sum_{i=1}^N \zeta_V(x_i) \right) \right) d\vec{X}_N,$$

with a next-order partition function  $K_{N,\beta}(\mu_V, \zeta_V)$  defined by

$$(2.4) \quad K_{N,\beta}(\mu_V, \zeta_V) := \int_{\mathbb{R}^N} \exp \left( -\frac{\beta}{2} \left( F_N(\vec{X}_N, \mu_V) + 2N \sum_{i=1}^N \zeta_V(x_i) \right) \right) d\vec{X}_N.$$

We extend this notation to  $K_{N,\beta}(\mu, \zeta)$  where  $\mu$  is a probability density and  $\zeta$  is a confinement potential.

**2.2. Perturbed potential and equilibrium measure.** Let  $\xi$  be in  $C^0(\mathbb{R})$  with compact support.

**Definition 2.2.** *For any  $t \in \mathbb{R}$ , we define*

- *The perturbed potential  $V_t$  as  $V_t := V + t\xi$ .*
- *The perturbed equilibrium measure  $\mu_t$  as the equilibrium measure associated to  $V_t$ . Since  $\xi$  has compact support,  $V_t$  satisfies the growth assumption (1.5) and thus  $\mu_t$  is well-defined. In particular,  $\mu_0$  coincides with  $\mu_V$ .*
- *The next-order confinement term  $\zeta_t := \zeta_{V_t}$ , as in (1.7).*
- *The next-order energy  $F_N(\vec{X}_N, \mu_t)$  as in (2.1).*
- *The next-order partition function  $K_{N,\beta}(\mu_t, \zeta_t)$  as in (2.4).*

**2.3. The Laplace transform of fluctuations as ratio of partition functions.**

**Lemma 2.3.** *For any  $s \in \mathbb{R}$  we have, letting  $t := \frac{-2s}{\beta N}$ ,*

$$(2.5) \quad \mathbf{E}_{\mathbb{P}_{N,\beta}^V} [\exp(s \text{Fluct}_N(\xi))] = \frac{K_{N,\beta}(\mu_t, \zeta_t)}{K_{N,\beta}(\mu_0, \zeta_0)} \exp \left( -\frac{\beta}{2} N^2 \left( \mathcal{I}_{V_t}(\mu_t) - \mathcal{I}_V(\mu_0) - t \int \xi d\mu_0 \right) \right).$$

*Proof.* First, we notice that, for any  $s$  in  $\mathbb{R}$

$$(2.6) \quad \mathbf{E}_{\mathbb{P}_{N,\beta}^V} [\exp(s \text{Fluct}_N(\xi))] = \frac{Z_{N,\beta}^{V_t}}{Z_{N,\beta}^V} \exp \left( -Ns \int \xi d\mu_V \right)$$



Using the splitting formula (2.2) and the definition of  $K_{N,\beta}$  as in (2.4) we see that for any  $t$

$$(2.7) \quad K_{N,\beta}(\mu_t, \zeta_t) = Z_{N,\beta}^{V_t} \exp\left(\frac{\beta}{2} N^2 \mathcal{I}_{V_t}(\mu_t)\right),$$

thus combining (2.6) and (2.7), with  $t := \frac{-2s}{\beta N}$  we obtain (2.5).  $\square$

**2.4. Comparison of partition functions.** If  $\mu$  is a probability density, we denote by  $\text{Ent}(\mu)$  the entropy function given by  $\text{Ent}(\mu) := \int_{\mathbb{R}} \mu \log \mu$ . The following asymptotic expansion is proven [LS15, Corollary 1.5] (cf. [LS15, Remark 4.3]) and valid in a general multi-cut critical situation.

**Lemma 2.4.** *Let  $\mu$  be a probability density on  $\mathbb{R}$ . Assume that  $\mu$  has the form (1.10), (1.11) with  $S_0$  in  $C^2(\Sigma)$ , and that  $\zeta$  is some Lipschitz function on  $\mathbb{R}$  satisfying*

$$\zeta = 0 \text{ on } \Sigma, \quad \zeta > 0 \text{ on } \mathbb{R} \setminus \Sigma, \quad \int_{\mathbb{R}} e^{-\beta N \zeta(x)} dx < \infty \text{ for } N \text{ large enough.}$$

Then, with the notation of (2.4) and for some  $C_\beta$  depending only on  $\beta$ , we have

$$(2.8) \quad \log K_{N,\beta}(\mu, \zeta) = \frac{\beta}{2} N \log N + C_\beta N - N \left(1 - \frac{\beta}{2}\right) \text{Ent}(\mu) + N o_N(1).$$

## 2.5. Additional bounds.

### 2.5.1. Exponential moments of the next-order energy.

**Lemma 2.5.** *We have, for some constant  $C$  depending on  $\beta$  and  $V$*

$$(2.9) \quad \left| \log \mathbf{E}_{\mathbb{P}_{N,\beta}^V} \left[ \exp\left(\frac{\beta}{4} \left(F_N(\vec{X}_N, \mu_V) + N \log N\right)\right) \right] \right| \leq CN.$$

*Proof.* This follows e.g. from [SS15, Theorem 6], but we can also deduce it from Lemma 2.4. We may write

$$\begin{aligned} & \mathbf{E}_{\mathbb{P}_{N,\beta}^V} \left[ \exp\left(\frac{\beta}{4} F_N(\vec{X}_N, \mu_V)\right) \right] \\ &= \frac{1}{K_{N,\beta}(\mu_V, \zeta_V)} \int \exp\left(-\frac{\beta}{4} \left(F_N(\vec{X}_N, \mu_V) - 2N \sum_{i=1}^N 2\zeta_V(x_i)\right)\right) d\vec{X}_N \\ &= \frac{K_{N,\beta}(\mu_V, 2\zeta_V)}{K_{N,\beta}(\mu_V, \zeta_V)}. \end{aligned}$$

Taking the log and using (2.8) to expand both terms up to order  $N$  yields the result.  $\square$

**2.5.2. The next-order energy controls the fluctuations.** The following result is a consequence of the analysis of [SS15, PS14], we give the proof in Section B.2 for completeness. It shows that  $F_N$  controls  $\text{fluct}_N$ . Here  $|\text{Supp } \xi|$  denotes the diameter of the support of  $\xi$ .

**Proposition 2.6.** *If  $\xi$  is compactly supported and Lipschitz, we have, for some universal constant  $C$*

$$(2.10) \quad \left| \int \xi d\text{fluct}_N \right| \leq C |\text{Supp } \xi|^{\frac{1}{2}} \|\nabla \xi\|_{L^\infty} \left( F_N(\vec{X}_N, \mu_V) + N \log N + C(\|\mu_V\|_{L^\infty} + 1)N \right)^{1/2}.$$

2.5.3. *Confinement bound.* We will also need the following bound on the confinement. The proof is very simple and identical to the proof of Lemma 3.3 of [LS16].

**Lemma 2.7.** *For any fixed open neighborhood  $U$  of  $\Sigma$ ,*

$$\mathbb{P}_{N,\beta}^V \left( \vec{X}_N \in U^N \right) \geq 1 - \exp(-cN)$$

where  $c > 0$  depends on  $U$  and  $\beta$ .

Lemma 2.7 is the only place where we use the non-degeneracy assumption (H3) on the next-order confinement term  $\zeta_V$ .

### 3. INVERTING THE OPERATOR AND DEFINING THE APPROXIMATE TRANSPORT

The goal of this section is to find transport maps  $\phi_t$  for  $t$  small enough such that the transported measure  $\phi_t \# \mu_0$  approximates the equilibrium measures  $\mu_t$ . Since the equilibrium measures are characterized by (1.7) with equality on the support, it is natural to seek  $\phi_t$  such that the quantity

$$\int -\log |\phi_t(x) - \phi_t(y)| d\mu_0(y) + \frac{1}{2} V_t(\phi_t(x))$$

is close to a constant.

#### 3.1. Preliminaries.

**Lemma 3.1.** *We have the following*

- The non-vanishing function  $S_0$  in (1.11) is in  $C^{\mathfrak{p}-3-2k}(\Sigma_V)$ .
- There exists an open neighborhood  $U$  of  $\Sigma_V$  and a non-vanishing function  $M$  in  $C^{\mathfrak{p}-3-2k}(U \setminus \hat{\Sigma}_V)$  such that

$$(3.1) \quad \zeta'_V(x) = M(x)\sigma(x) \prod_{i=1}^m (x - s_i)^{2k_i}.$$

In particular, (3.1) quantifies how fast  $\zeta'_V$  vanishes near an endpoint of the support. We postpone the proof to Section B.3.

3.2. **The approximate equilibrium measure equation.** In the following, we let

- $U$  be an open neighborhood of  $\Sigma_V$  such that (3.1) holds.
- $B$  be the open ball of radius  $\frac{1}{2}$  in  $C^2(U)$ .

We define a map  $\mathcal{F}$  from  $[-1, 1] \times B$  to  $C^1(U)$  by setting  $\phi := \text{Id} + \psi$  and

$$(3.2) \quad \mathcal{F}(t, \psi) := \int -\log |\phi(\cdot) - \phi(y)| d\mu_V(y) + \frac{1}{2} V_t \circ \phi(\cdot),$$

**Lemma 3.2.** *The map  $\mathcal{F}$  takes values in  $C^1(U)$  and has continuous partial derivatives in both variables. Moreover there exists  $C$  depending only on  $V$  such that for all  $(t, \psi)$  in  $[-1, 1] \times B$  we have*

$$(3.3) \quad \left\| \mathcal{F}(t, \psi) - \mathcal{F}(0, 0) - \frac{t}{2} \xi + \Xi_V[\psi] \right\|_{C^1(U)} \leq Ct^2 \|\psi\|_{C^2(U)}^2.$$

The proof is postponed to Section B.4.

### 3.3. Inverting the operator.

**Lemma 3.3.** *Let  $\psi$  be defined by*

$$(3.4) \quad \psi(x) = -\frac{1}{2\pi^2 S(x)} \left( \int_{\Sigma} \frac{\xi(y) - \xi(x)}{\sigma(y)(y-x)} dy \right) \quad \text{for } x \text{ in } \Sigma_V,$$

$$(3.5) \quad \psi(x) = \frac{\int \frac{\psi(y)}{x-y} d\mu_V(y) + \frac{\xi}{2} + c_\xi}{\int \frac{1}{x-y} d\mu_V(y) - \frac{1}{2} V'(x)} \quad \text{for } x \in U \setminus \Sigma_V,$$

then  $\psi$  is in  $C^l(U)$  with  $l = (p-3-3k) \wedge (r-1-2k)$  and

$$(3.6) \quad \|\psi\|_{C^l(U)} \leq C \|\xi\|_{C^r(\mathbb{R})}$$

for some constant  $C$  depending only on  $V$ , and there exists a constant  $c_\xi$  such that

$$\Xi_V[\psi] = \frac{\xi}{2} + c_\xi \text{ in } U,$$

with  $\Xi_V$  as in (1.12).

The proof of Lemma 3.3 is postponed to Section B.5. We may extend  $\psi$  to  $\mathbb{R}$  in such a way that  $\psi$  is in  $C^l(\mathbb{R})$  with compact support.

**3.4. Approximate transport and equilibrium measure.** We let  $\psi$  be the function defined in Lemma 3.3, and  $c_\xi$  be such that

$$\Xi_V[\psi] = \frac{\xi}{2} + c_\xi \text{ on } U.$$

**Definition 3.4.** For  $t \in [-t_{\max}, t_{\max}]$ , where  $t_{\max} = \left(2\|\psi\|_{C^1(U)}\right)^{-1}$ ,

- We let  $\psi_t$  be given by  $\psi_t := t\psi$ .
- We let  $\phi_t$  be the approximate transport, defined by  $\phi_t := \text{Id} + \psi_t$ .
- We let  $\tilde{\mu}_t$  be the approximate equilibrium measure, defined by  $\tilde{\mu}_t := \phi_t \# \mu_V$ .
- We let  $\tilde{\zeta}_t$  be the approximate confining term  $\tilde{\zeta}_t := \zeta_V \circ \phi_t^{-1}$ .
- We let  $\mathbb{P}_{N,\beta}^{(t)}$  be the probability measure

$$(3.7) \quad d\mathbb{P}_{N,\beta}^{(t)}(\vec{X}_N) = \frac{1}{K_{N,\beta}(\tilde{\mu}_t, \tilde{\zeta}_t)} \exp\left(-\frac{\beta}{2} \left( F_N(\vec{X}_N, \tilde{\mu}_t) + 2N \sum_{i=1}^N \tilde{\zeta}_t(x_i) \right)\right) d\vec{X}_N,$$

where  $K_{N,\beta}(\tilde{\mu}_t, \tilde{\zeta}_t)$  is as in (2.4).

Finally, we let  $\tau_t$  be defined by

$$(3.8) \quad \tau_t := \mathcal{F}(t, \psi_t) - \mathcal{F}(0, 0) - \tilde{c}_t.$$

This quantifies how close  $\tilde{\mu}_t$  is from satisfying the Euler-Lagrange equation for  $V_t$  and thus how well  $\tilde{\mu}_t$  approximates the real equilibrium measure  $\mu_t$ . We also define the extension  $\hat{\tau}_t$  of  $\tau_t \circ \phi_t^{-1}$  to  $\mathbb{R}^2$  by

$$(3.9) \quad \hat{\tau}_t(x, y) = \chi(x, y) \tau_t \circ \phi_t^{-1}(x),$$

where  $\chi$  is equal to one in a fixed neighborhood of  $\text{supp}(\mu_V)$  included in  $U$  and is in  $C_c^\infty(\mathbb{R}^2)$ .

**Lemma 3.5.** *The following holds*

- The map  $\psi_t$  satisfies

$$\Xi_V[\psi_t] = \frac{t}{2}\xi + \tilde{c}_t, \text{ for } \tilde{c}_t := tc_\xi.$$

- The map  $\phi_t$  is a  $C^1$ -diffeomorphism which coincides with the identity outside a compact support independent of  $t \in [-t_{\max}, t_{\max}]$ .
- The error  $\tau_t$  is a  $O(t^2)$ , more precisely

$$(3.10) \quad \|\tau_t\|_{C^1(U)} \leq Ct^2 \|\psi\|_{C^2(U)}^2$$

$$(3.11) \quad \|\hat{\tau}_t\|_{C^1(\mathbb{R}^2)} \leq Ct^2 \|\psi\|_{C^2(U)}^2.$$

- On  $\phi_t(\Sigma_V)$ , we have

$$(3.12) \quad \tilde{\zeta}_t = h^{\tilde{\mu}_t} + \frac{V_t}{2} - \tilde{c}_t - c_V - \tau_t \circ \phi_t^{-1}.$$

*Proof.* The first two points are straightforward, the bound (3.10) follows from combining (3.3) with the conclusions of Lemma 3.2, and then (3.11) is an easy consequence.

For (3.12), let us first recall that

$$\mathcal{F}(t, \psi_t) = \int -\log |\phi_t(\cdot) - \phi_t(y)| d\mu_0(y) + \frac{1}{2}V_t \circ \phi_t,$$

which, with the notation of (1.19), yields

$$\mathcal{F}(t, \psi_t) = h^{\tilde{\mu}_t} \circ \phi_t + \frac{1}{2}V_t \circ \phi_t.$$

On the other hand, by definition of  $\tau_t$  as in (3.8), we have

$$\mathcal{F}(t, \psi_t) = \mathcal{F}(0, 0) + \tilde{c}_t + \tau_t.$$

Finally, we know that, on  $\Sigma_V$

$$\mathcal{F}(0, 0) = \zeta_V + c_V.$$

We thus see that

$$\zeta_V + c_V + \tilde{c}_t + \tau_t = h^{\tilde{\mu}_t} \circ \phi_t + \frac{1}{2}V_t \circ \phi_t.$$

Since, by definition,  $\tilde{\zeta}_t = \zeta_V \circ \phi_t^{-1}$ , we get (3.12).  $\square$

#### 4. STUDY OF THE LAPLACE TRANSFORM

The next goal is to compare the partition functions associated to  $\mu_t$  and  $\mu_0 = \mu_V$ . We split the comparison into two steps: first, we compare  $K_{N,\beta}(\mu_t, \zeta_t)$  with  $K_{N,\beta}(\tilde{\mu}_t, \tilde{\zeta}_t)$  using the bounds, obtained in the previous section, showing that  $\tilde{\mu}_t$  is a good approximation to  $\bar{\mu}_t$ , and then we compare  $K_{N,\beta}(\tilde{\mu}_t, \tilde{\zeta}_t)$  and  $K_{N,\beta}(\mu_0, \zeta_0)$  using the transport  $\phi_t$ , as in [LS16].

##### 4.1. Energy comparison: from $\mu_t$ to $\tilde{\mu}_t$ .

**Lemma 4.1.** *We have*

$$(4.1) \quad \int_{\mathbb{R}^2} |\nabla h^{\mu_t - \tilde{\mu}_t}|^2 \leq Ct^4 \|\psi\|_{C^2(U)}^4,$$

$$(4.2) \quad \int_{\mathbb{R}} \zeta_t d\tilde{\mu}_t + \int_{\mathbb{R}} \tilde{\zeta}_t d\mu_t \leq Ct^4 \|\psi\|_{C^2(U)}^4,$$

where  $C$  is universal.

*Proof.* For  $t$  small enough,  $\phi_t(U)$  contains some fixed open neighborhood of  $\Sigma_V$ , which itself contains the support of  $\mu_t$ . Integrating by parts we thus get

$$\begin{aligned}
(4.3) \quad \frac{1}{2\pi} \int_{\mathbb{R}^2} |\nabla h^{\mu_t - \tilde{\mu}_t}|^2 &= 2\pi \int h^{\mu_t - \tilde{\mu}_t} d(\mu_t - \tilde{\mu}_t) \\
&= \int (\zeta_t - \tilde{\zeta}_t - \tau_t \circ \phi_t^{-1}) d(\mu_t - \tilde{\mu}_t) \\
&= - \int \zeta_t d\tilde{\mu}_t - \int \tilde{\zeta}_t d\mu_t - \int_{\mathbb{R}} \tau_t \circ \phi_t^{-1} d(\mu_t - \tilde{\mu}_t) \\
&\leq - \int \tau_t \circ \phi_t^{-1} d(\mu_t - \tilde{\mu}_t).
\end{aligned}$$

In the first equality, we have re-written  $h^{\mu_t}$  and  $h^{\tilde{\mu}_t}$  using the confining terms  $\zeta_t$  and  $\tilde{\zeta}_t$ , see (1.7) and (3.12), discarding the constants which disappear when integrated against  $d(\mu_t - \tilde{\mu}_t)$ . In the second equality, we have used the fact that  $\zeta_t$  vanishes on the support of  $\mu_t$  and  $\tilde{\zeta}_t$  on the support of  $\tilde{\mu}_t$ . Finally, the last inequality is due to the fact that  $\zeta_t$  and  $\tilde{\zeta}_t$  are nonnegative on  $\mathbb{R}$ . Using (3.9) and (3.11), we may thus write

$$\begin{aligned}
\frac{1}{2\pi} \|\nabla h^{\mu_t - \tilde{\mu}_t}\|_{L^2(\mathbb{R}^2)}^2 &\leq \left| \int_{\mathbb{R}^2} \tau_t \circ \phi_t^{-1} d(\mu_t - \tilde{\mu}_t) \delta_{\mathbb{R}} \right| \leq \|\nabla \hat{\tau}_t\|_{L^2(\mathbb{R}^2)} \|\nabla h^{\mu_t - \tilde{\mu}_t}\|_{L^2(\mathbb{R}^2)} \\
&\leq Ct^2 \|\psi\|_{C^2(U)}^2 \|\nabla h^{\mu_t - \tilde{\mu}_t}\|_{L^2(\mathbb{R}^2)},
\end{aligned}$$

which proves (4.1). Coming back to (4.3), we also obtain

$$0 \leq - \int \zeta_t d\tilde{\mu}_t - \int \tilde{\zeta}_t d\mu_t + O(t^4 \|\psi\|_{C^2(U)}^4),$$

which in turn implies (4.2). □

**Lemma 4.2** (Energy comparison : from  $\mu_t$  to  $\tilde{\mu}_t$ ). *For any  $\vec{X}_N \in (\phi_t(U))^N$ , we have*

$$\begin{aligned}
(4.4) \quad &\left| \left( F_N(\vec{X}_N, \mu_t) + 2N \sum_{i=1}^N \zeta_t(x_i) \right) - \left( F_N(\vec{X}_N, \tilde{\mu}_t) + 2N \sum_{i=1}^N \tilde{\zeta}_t(x_i) \right) \right| \\
&\leq C \left( Nt^2 \|\psi\|_{C^2(U)}^2 (F_N(\vec{X}_N, \tilde{\mu}_t) + N \log N)^{1/2} + N^2 t^4 \|\psi\|_{C^2(U)}^4 \right).
\end{aligned}$$

*Proof.* By the definition (2.1) of the next-order energy, we may write

$$\begin{aligned}
(4.5) \quad F_N(\vec{X}_N, \mu_t) - F_N(\vec{X}_N, \tilde{\mu}_t) &= N^2 \int_{\mathbb{R} \times \mathbb{R}} -\log|x-y| d(\tilde{\mu}_t - \mu_t)(x) d(\tilde{\mu}_t - \mu_t)(y) \\
&\quad + 2N \int_{\mathbb{R} \times \mathbb{R}} -\log|x-y| d(\tilde{\mu}_t - \mu_t)(x) \left( \sum_{i=1}^N \delta_{x_i} - N\tilde{\mu}_t \right)(y) \\
&= N^2 \int_{\mathbb{R}^2} |\nabla h^{\mu_t - \tilde{\mu}_t}|^2 + 2N \int_{\mathbb{R}} h^{\tilde{\mu}_t - \mu_t} \left( \sum_{i=1}^N \delta_{x_i} - N\tilde{\mu}_t \right).
\end{aligned}$$

On the other hand, using that  $\tilde{\zeta}_t$  vanishes on the support of  $\tilde{\mu}_t$ , we get

$$(4.6) \quad \sum_{i=1}^N (\zeta_t(x_i) - \tilde{\zeta}_t(x_i)) = N \int_{\mathbb{R}} (\zeta_t - \tilde{\zeta}_t) d\tilde{\mu}_t + \int_{\mathbb{R}} (\zeta_t - \tilde{\zeta}_t) \left( \sum_{i=1}^N \delta_{x_i} - N\tilde{\mu}_t \right) \\ = N \int_{\mathbb{R}} \zeta_t d\tilde{\mu}_t + \int_{\mathbb{R}} (\zeta_t - \tilde{\zeta}_t) \left( \sum_{i=1}^N \delta_{x_i} - N\tilde{\mu}_t \right).$$

Combining (4.5) and (4.6), we obtain

$$\left( F_N(\vec{X}_N, \mu_t) + 2N \sum_{i=1}^N \zeta_t(x_i) \right) - \left( F_N(\vec{X}_N, \tilde{\mu}_t) + 2N \sum_{i=1}^N \tilde{\zeta}_t(x_i) \right) \\ = N^2 \int_{\mathbb{R}^2} |\nabla h^{\mu_t - \tilde{\mu}_t}|^2 + 2N^2 \int_{\mathbb{R}} \zeta_t d\tilde{\mu}_t + 2N \int_{\mathbb{R}} (h^{\tilde{\mu}_t - \mu_t} + \zeta_t - \tilde{\zeta}_t) \left( \sum_{i=1}^N \delta_{x_i} - N\tilde{\mu}_t \right).$$

From (1.7), (3.12) (see also the notation (1.19)), we have

$$h^{\tilde{\mu}_t - \mu_t} + \zeta_t - \tilde{\zeta}_t = \tau_t \circ \phi_t^{-1} + \text{constant},$$

hence we find

$$(4.7) \quad \left( F_N(\vec{X}_N, \mu_t) + 2N \sum_{i=1}^N \zeta_t(x_i) \right) - \left( F_N(\vec{X}_N, \tilde{\mu}_t) + 2N \sum_{i=1}^N \tilde{\zeta}_t(x_i) \right) \\ = N^2 \int_{\mathbb{R}^2} |\nabla h^{\mu_t - \tilde{\mu}_t}|^2 + 2N^2 \int_{\mathbb{R}} \zeta_t d\tilde{\mu}_t + 2N \int \tau_t \circ \phi_t^{-1} \left( \sum_{i=1}^N \delta_{x_i} - N\tilde{\mu}_t \right).$$

By the results of Lemma 4.1, the first two terms in the right-hand side of (4.7) are  $O(N^2 t^4)$ , while the last term is bounded, using (3.10) and Proposition 2.6, by

$$N \int \tau_t \circ \phi_t^{-1} \left( \sum_{i=1}^N \delta_{x_i} - N\tilde{\mu}_t \right) = O\left(N t^2 (F_N(\vec{X}_N, \tilde{\mu}_t) + N \log N)^{1/2}\right),$$

which concludes the proof.  $\square$

**Lemma 4.3.** *We have, for any fixed  $s \in \mathbb{R}$ , with  $t = \frac{-2s}{\beta N}$*

$$(4.8) \quad \left| \log \frac{K_{N,\beta}(\tilde{\mu}_t, \tilde{\zeta}_t)}{K_{N,\beta}(\mu_t, \zeta_t)} \right| \leq C N t^2 \sqrt{N} \|\psi\|_{C^2(U)}^2 + C t^4 N^2 \|\psi\|_{C^2(U)}^4 \\ = O\left(s^2 N^{-1/2} \|\psi\|_{C^2}^2 + s^4 N^{-2} \|\psi\|_{C^2}^4\right).$$

*Proof.* By definition of the next-order partition functions we may write

$$\frac{K_{N,\beta}(\tilde{\mu}_t, \tilde{\zeta}_t)}{K_{N,\beta}(\mu_t, \zeta_t)} = \int_{\mathbb{R}^N} \exp\left(-\frac{\beta}{2} \left( \left( F_N(\vec{X}_N, \mu_t) + 2N \sum_{i=1}^N \zeta_t(x_i) \right) \right. \right. \\ \left. \left. - \left( F_N(\vec{X}_N, \tilde{\mu}_t) + 2N \sum_{i=1}^N \tilde{\zeta}_t(x_i) \right) \right) \right) d\vec{X}_N.$$

The result follows from combining (2.9) and (4.4), and using Lemma 2.7 to argue that the particles  $\vec{X}_N$  may be assumed to all belong to the neighborhood  $U$  for  $t$  small enough, except for an event of exponentially small probability.  $\square$

**4.2. Energy comparison: from  $\tilde{\mu}_t$  to  $\mu_0$ .** Let us define

$$\text{fluct}_N^{(t)} = \sum_{i=1}^N \delta_{x_i} - N\tilde{\mu}_t \quad \text{Fluct}_N^{(t)}(\xi) = \int \xi d\text{fluct}_N^{(t)}.$$

For any  $\psi$ , let us define the following quantity (that may be called *anisotropy* by analogy with [LS16])

$$(4.9) \quad \mathbf{A}^{(t)}[\vec{X}_N, \psi] = \iint_{\mathbb{R} \times \mathbb{R}} \frac{\psi(x) - \psi(y)}{x - y} d\text{fluct}_N^{(t)}(x) d\text{fluct}_N^{(t)}(y).$$

**Lemma 4.4.** *Assume  $\psi \in C^2(\mathbb{R})$ . For any  $\vec{X}_N \in U^N$ , letting  $\Phi_t(\vec{X}_N) = (\phi_t(x_1), \dots, \phi_t(x_N))$ , we have*

$$(4.10) \quad \left| F_N(\Phi_t(\vec{X}_N), \tilde{\mu}_t) - F_N(\vec{X}_N, \mu_0) - \sum_{i=1}^N \log \phi_t'(x_i) + \frac{t}{2} \mathbf{A}^{(0)}[\vec{X}_N, \psi] \right| \leq Ct^2 \left( F_N(\vec{X}_N, \mu_0) + N \log N \right).$$

*Proof.* Since by definition  $\tilde{\mu}_t = \phi_t \# \mu_0$  we may write

$$\begin{aligned} & F_N(\Phi_t(\vec{X}_N), \tilde{\mu}_t) - F_N(\vec{X}_N, \mu_0) \\ &= - \iint_{\mathbb{R} \times \mathbb{R} \setminus \Delta} \log |x - y| \left( \sum_{i=1}^N \delta_{\phi_t(x_i)} - N\tilde{\mu}_t \right)(x) \left( \sum_{i=1}^N \delta_{\phi_t(x_i)} - N\tilde{\mu}_t \right)(y) \\ & \quad + \iint_{\mathbb{R} \times \mathbb{R} \setminus \Delta} \log |x - y| d\text{fluct}_N(x) d\text{fluct}_N(y) \\ &= - \iint_{\mathbb{R} \times \mathbb{R} \setminus \Delta} \log \frac{|\phi_t(x) - \phi_t(y)|}{|x - y|} d\text{fluct}_N(x) d\text{fluct}_N(y) \\ &= - \iint_{\mathbb{R} \times \mathbb{R}} \log \frac{|\phi_t(x) - \phi_t(y)|}{|x - y|} d\text{fluct}_N(x) d\text{fluct}_N(y) + \sum_{i=1}^N \log \phi_t'(x_i). \end{aligned}$$

Using that by definition  $\phi_t = \text{Id} + t\psi$  where  $\psi$  is in  $C_c^2(\mathbb{R})$ , we get by the chain rule

$$\log \frac{|\phi_t(x) - \phi_t(y)|}{|x - y|} = t \frac{\psi(x) - \psi(y)}{x - y} + t^2 \varepsilon_t(x, y),$$

with  $\|\varepsilon_t\|_{C^2(\mathbb{R} \times \mathbb{R})}$  uniformly bounded in  $t$ . Applying Proposition 2.6 twice, we get that

$$\left| \iint \varepsilon_t(x, y) d\text{fluct}_N(x) d\text{fluct}_N(y) \right| \leq Ct^2 \left( F_N(\vec{X}_N, \mu_0) + N \log N \right),$$

which yields the result.  $\square$

**4.3. Comparison of partition functions I: using the transport.** In this section and the following one, we will write  $\mathbf{A}$  instead of  $\mathbf{A}^{(0)}[\vec{X}_N, \psi]$

**Proposition 4.5.** *We have, for any  $t$  small enough*

$$(4.11) \quad \frac{K_{N,\beta}(\tilde{\mu}_t, \tilde{\zeta}_t)}{K_{N,\beta}(\mu_0, \zeta_0)} = \exp\left(N\left(1 - \frac{\beta}{2}\right)(\text{Ent}(\mu_0) - \text{Ent}(\tilde{\mu}_t))\right) \mathbf{E}_{\mathbb{P}_{N,\beta}^{(0)}}\left(\exp\left(\frac{\beta}{2}t\mathbf{A} + t^2\text{Error}_1(\vec{X}_N) + t\text{Error}_2(\vec{X}_N)\right)\right),$$

with error terms bounded by

$$(4.12) \quad |\text{Error}_1(\vec{X}_N)| \leq C\left(F_N(\vec{X}_N, \mu_0) + N \log N\right),$$

$$(4.13) \quad |\text{Error}_2(\vec{X}_N)| \leq C\left(F_N(\vec{X}_N, \mu_0) + N \log N\right)^{1/2}.$$

*Proof.* By a change of variables and in view of (4.10), we may write

$$(4.14) \quad \begin{aligned} K_{N,\beta}(\tilde{\mu}_t, \tilde{\zeta}_t) &= \int \exp\left(-\frac{\beta}{2}\left(F_N(\Phi_t(\vec{X}_N), \tilde{\mu}_t) + 2N \sum_{i=1}^N \tilde{\zeta}_t \circ \phi_t(x_i)\right) + \sum_{i=1}^N \log \phi'_t(x_i)\right) d\vec{X}_N \\ &= \int \exp\left(-\frac{\beta}{2}\left(F_N(\Phi_t(\vec{X}_N), \tilde{\mu}_t) + 2N \sum_{i=1}^N \zeta_0(x_i)\right) + \sum_{i=1}^N \log \phi'_t(x_i)\right) d\vec{X}_N, \end{aligned}$$

since  $\zeta_0 = \tilde{\zeta}_t \circ \phi_t$  by definition. Using Lemma 4.4 we may write

$$(4.15) \quad \begin{aligned} \frac{K_{N,\beta}(\tilde{\mu}_t, \tilde{\zeta}_t)}{K_{N,\beta}(\mu_0, \zeta_0)} &= \frac{1}{K_{N,\beta}(\mu_0, \zeta_0)} \int_{\mathbb{R}^N} \exp\left(-\frac{\beta}{2}\left(F_N(\vec{X}_N, \mu_0) + 2N \sum_{i=1}^N \zeta(x_i)\right) \right. \\ &\quad \left. + \left(1 - \frac{\beta}{2}\right) \sum_{i=1}^N \log \phi'_t(x_i) + \frac{\beta}{2}t\mathbf{A} + t^2\text{Error}_1(\vec{X}_N)\right) d\vec{X}_N \\ &= \mathbf{E}_{\mathbb{P}_{N,\beta}^{(0)}}\left(\exp\left(\left(1 - \frac{\beta}{2}\right) \sum_{i=1}^N \log \phi'_t(x_i) + \frac{\beta}{2}t\mathbf{A} + t^2\text{Error}_1(\vec{X}_N)\right)\right), \end{aligned}$$

where the  $\text{Error}_1$  term is bounded as in (4.12). On the other hand, since  $\phi_t$  is regular enough, using Proposition 2.6 we may write

$$\sum_{i=1}^N \log \phi'_t(x_i) = N \int_{\mathbb{R}} \log \phi'_t d\mu_0 + t\text{Error}_2(\vec{X}_N)$$

with an  $\text{Error}_2$  term as in (4.13). Finally, since by definition  $\phi_t \# \mu_0 = \tilde{\mu}_t$  we may observe that  $\phi'_t = \frac{\mu_0}{\tilde{\mu}_t \circ \phi_t}$  and thus

$$(4.16) \quad \int_{\mathbb{R}} \log \phi'_t d\mu_0 = \int_{\mathbb{R}} \log \mu_0 d\mu_0 - \int_{\mathbb{R}} \log \mu_t \circ \phi_t d\mu_0 = \text{Ent}(\mu_0) - \text{Ent}(\tilde{\mu}_t).$$

This yields (4.11). □



#### 4.4. Comparison of partition functions II: the anisotropy is small.

**Proposition 4.6.** *For any  $s$ , we have*

$$(4.17) \quad \log \mathbf{E}_{\mathbb{P}_{N,\beta}^V} \left( \exp \left( \frac{-s}{N} \mathbf{A} \right) \right) = o_N(1).$$

*Proof.* Applying Cauchy-Schwarz to (4.11) we may write

$$(4.18) \quad \begin{aligned} & \mathbf{E}_{\mathbb{P}_{N,\beta}^V} \left( \exp \left( \frac{\beta}{4} t \mathbf{A} \right) \right)^2 \\ & \leq \mathbf{E}_{\mathbb{P}_{N,\beta}^V} \left( \exp \left( \frac{\beta}{2} t \mathbf{A} + t^2 \text{Error}_1 + t \text{Error}_2 \right) \right) \mathbf{E}_{\mathbb{P}_{N,\beta}^V} \left( \exp \left( -t^2 \text{Error}_1 - t \text{Error}_2 \right) \right) \\ & \leq \frac{K_{N,\beta}(\tilde{\mu}_t, \tilde{\zeta}_t)}{K_{N,\beta}(\mu_0, \zeta_0)} \exp \left( \left( 1 - \frac{\beta}{2} \right) N \left( \text{Ent}(\tilde{\mu}_t) - \text{Ent}(\mu_0) \right) \right) \mathbf{E}_{\mathbb{P}_{N,\beta}^V} \left( \exp \left( -t^2 \text{Error}_1 - t \text{Error}_2 \right) \right). \end{aligned}$$

In view of (2.9) we get, for  $t$  small enough,

$$(4.19) \quad \log \mathbf{E}_{\mathbb{P}_{N,\beta}^V} \left( \exp(t \text{Error}_1) \right) \leq CtN, \quad \log \mathbf{E}_{\mathbb{P}_{N,\beta}^V} \left( \exp(t \text{Error}_2) \right) \leq CtN.$$

Inserting (2.8) into (4.18) we obtain that for  $t$  small enough,

$$(4.20) \quad \log \mathbf{E}_{\mathbb{P}_{N,\beta}^V} \left( \exp \left( \frac{\beta}{4} t \mathbf{A} \right) \right) \leq C(Nt^2 + N^{1/2}t) + \delta_N,$$

for some sequence  $\{\delta_N\}_N$  with  $\lim_{N \rightarrow \infty} \delta_N = 0$ . Applying this to  $t = 4\varepsilon/\beta$  with  $\varepsilon$  small and using Hölder's inequality, we deduce

$$\log \mathbf{E}_{\mathbb{P}_{N,\beta}^V} \left( \exp \left( \frac{-s}{N} \mathbf{A} \right) \right) \leq \frac{|s|}{N\varepsilon} \log \mathbf{E}_{\mathbb{P}_{N,\beta}^V} \left( \exp(\varepsilon \mathbf{A}) \right) \leq C|s|\varepsilon + \frac{|s|}{\varepsilon} \delta_N.$$

In particular, choosing  $\varepsilon = \sqrt{\delta_N}$ , we get (4.17). □

#### 4.5. Conclusion: proof of Theorem 1.

*Proof.* Combining (4.11) for  $t = -\frac{2s}{\beta N}$  (where  $s$  is independent of  $N$ ) and (4.17) we find

$$(4.21) \quad \log \frac{K_{N,\beta}(\tilde{\mu}_{\frac{-2s}{\beta N}})}{K_{N,\beta}(\mu_0)} = \left( 1 - \frac{\beta}{2} \right) N \left( \text{Ent}(\mu_0) - \text{Ent}(\tilde{\mu}_{\frac{-2s}{\beta N}}) \right) + o_N(1).$$

Using again (4.16) and  $\phi'_t = 1 + t\psi'$ , we may rewrite this as

$$(4.22) \quad \log \frac{K_{N,\beta}(\tilde{\mu}_{\frac{-2s}{\beta N}})}{K_{N,\beta}(\mu_0)} = - \left( 1 - \frac{\beta}{2} \right) \frac{2s}{\beta} \int \psi' d\mu_0 + o_N(1).$$

Combining (4.8) and (4.22) and sending  $N$  to  $+\infty$  we obtain,

$$(4.23) \quad \log \frac{K_{N,\beta}(\tilde{\mu}_{\frac{-2s}{\beta N}})}{K_{N,\beta}(\mu_0)} = - \left( 1 - \frac{\beta}{2} \right) \frac{2s}{\beta} \int \psi' d\mu_0 + o_N(1),$$

with an error  $o_N(1)$  uniform for  $s$  in a compact set of  $\mathbb{R}$ .

To conclude, we need the following relation, whose proof is given in Section B.6.

**Lemma 4.7.**

$$(4.24) \quad \mathcal{I}_{V_t}(\mu_t) - \mathcal{I}_V(\mu_0) = t \int \xi d\mu_0 + \frac{t^2}{2} \int \xi' \psi d\mu_0 + O(t^3 \|\xi\|_{C^2(U)} + t^4 \|\psi\|_{C^2(U)}^4),$$

where the  $O$  only depends on  $V$ .

Combining (2.5) with (4.23) and (4.24) we obtain,

$$\log \mathbf{E}_{\mathbb{P}_{N,\beta}^V} (\exp(s \text{Fluct}_N(\xi))) = - \left(1 - \frac{\beta}{2}\right) \frac{2s}{\beta} \int \psi' d\mu_V - \frac{s^2}{\beta} \int_{\mathbb{R}} \xi' \psi d\mu_V + o_N(1),$$

with an error  $o_N(1)$  uniform for  $s$  in a compact set of  $\mathbb{R}$ .

Thus the Laplace transform of  $\text{Fluct}_N(\xi)$  converges (uniformly on compact sets) to that of a Gaussian of mean  $m_\xi$  and variance  $v_\xi$ , which implies convergence in law and proves the main theorem.  $\square$

## APPENDIX A. THE ONE-CUT REGULAR CASE

In the one-cut noncritical case, every regular enough function is in the range of the operator  $\Xi$ , so that the map  $\psi$  can always be built. This allows to bootstrap the approach used for proving Theorem 1. In this appendix, we expand on how we can proceed in this simpler setting without referring to the result of [LS15] but assuming more regularity of  $\xi$ , and retrieve the findings of [BG13b] (but without assuming analyticity), as well as a rate of convergence for the Laplace transform of the fluctuations.

**A.1. The bootstrap argument.** Let us first explain the main computational point for the bootstrap argument: by (4.14) and in view of Lemma 4.4, we may write

$$(A.1) \quad \frac{d}{dt} \Big|_{t=0} \log K_{N,\beta}(\tilde{\mu}_t, \tilde{\zeta}_t) = \mathbf{E}_{\mathbb{P}_{N,\beta}^{(0)}} \left[ -\frac{\beta}{2} \mathbf{A}^{(0)}[\vec{X}_N, \psi] + \left(1 - \frac{\beta}{2}\right) \frac{d}{dt} \Big|_{t=0} \sum_{i=1}^N \log \phi'_t(x_i) \right].$$

Differentiating (2.5) with respect to  $t$  and using Lemma 4.7 we thus obtain

$$\frac{-\beta N}{2} \mathbf{E}_{\mathbb{P}_{N,\beta}^{(0)}} [\text{Fluct}_N^{(0)}(\xi)] = \mathbf{E}_{\mathbb{P}_{N,\beta}^{(0)}} \left[ -\frac{\beta}{2} \mathbf{A}^{(0)}[\vec{X}_N, \psi] + \left(1 - \frac{\beta}{2}\right) \frac{d}{dt} \Big|_{t=0} \sum_{i=1}^N \log \phi'_t(x_i) \right].$$

This is true as well for all  $t \in [-t_{\max}, t_{\max}]$ , i.e.

$$(A.2) \quad \mathbf{E}_{\mathbb{P}_{N,\beta}^{(t)}} [\text{Fluct}_N^{(t)}(\xi)] = -\frac{2}{\beta N} \mathbf{E}_{\mathbb{P}_{N,\beta}^{(t)}} \left[ -\frac{\beta}{2} \mathbf{A}^{(t)}[\vec{X}_N, \psi] + \left(1 - \frac{\beta}{2}\right) \frac{d}{dt} \sum_{i=1}^N \log \phi'_t(x_i) \right].$$

We may in addition write that

$$(A.3) \quad \frac{d}{dt} \sum_{i=1}^N \log \phi'_t(x_i) = N \int \frac{d}{dt} \log \phi'_t d\tilde{\mu}_t + \text{Fluct}_N^{(t)} \left( \frac{d}{dt} \log \phi'_t \right)$$

so that

$$(A.4) \quad \mathbf{E}_{\mathbb{P}_{N,\beta}^{(t)}} [\text{Fluct}_N^{(t)}(\xi)] = -\frac{2}{\beta} \left(1 - \frac{\beta}{2}\right) \int \frac{d}{dt} \log \phi'_t d\tilde{\mu}_t - \frac{2}{\beta N} \mathbf{E}_{\mathbb{P}_{N,\beta}^{(t)}} \left[ -\frac{\beta}{2} \mathbf{A}^{(t)}[\vec{X}_N, \psi] + \left(1 - \frac{\beta}{2}\right) \text{Fluct}_N^{(t)} \left( \frac{d}{dt} \log \phi'_t \right) \right].$$

This provides a functional equation which gives the expectation of the fluctuation in terms of a constant term plus a lower order expectation of another fluctuation and the  $\mathbf{A}$  term (which

itself can be written as a fluctuation, as noted below), allowing to expand it in powers of  $1/N$  recursively.

## A.2. Improved control on the fluctuations.

**Lemma A.1.** *Under the assumptions of Theorem 1 and assuming in addition*

$$(A.5) \quad \mathfrak{p} \geq 3\mathfrak{k} + 6 \quad \mathfrak{r} \geq 2\mathfrak{k} + 4$$

we have for any  $t$  in  $(-t_{\max}, t_{\max})$  and<sup>3</sup>  $s$  in  $\mathbb{R}$

$$(A.6) \quad \log \mathbf{E}_{\mathbb{P}_{N,\beta}^{(0)}} \left[ \exp \left( s \text{Fluct}_N^{(t)}(\xi) \right) \right] \\ \leq C \left( s \|\xi\|_{C^{2\mathfrak{k}+4}(U)} + s^2 \|\xi\|_{C^{2\mathfrak{k}+3}(U)}^2 + \frac{s^3}{N} \|\xi\|_{C^2(U)} + \frac{s^4}{N^2} \|\xi\|_{C^{2\mathfrak{k}+3}(U)} + \frac{s^4}{N^2} \|\xi\|_{C^{2\mathfrak{k}+3}(U)}^4 \right)$$

where  $C$  depends only on  $V$ .

*Proof.* Note that in view of Lemma 3.3, the assumption (A.5) ensures that the transport map  $\psi$  is in  $C^3(U)$ . By (4.14) and in view of Lemma 4.4, we may write

$$(A.7) \quad \frac{d}{dt} \Big|_{t=0} \log K_{N,\beta}(\tilde{\mu}_t, \tilde{\zeta}_t) = \mathbf{E}_{\mathbb{P}_{N,\beta}^{(0)}} \left[ -\frac{\beta}{2} \mathbf{A}^{(0)}[\vec{X}_N, \psi] + \left(1 - \frac{\beta}{2}\right) \frac{d}{dt} \Big|_{t=0} \sum_{i=1}^N \log \phi'_t(x_i) \right].$$

Similarly, we have for all  $t$ ,

$$(A.8) \quad \frac{d}{dt} \log K_{N,\beta}(\tilde{\mu}_t, \tilde{\zeta}_t) = \mathbf{E}_{\mathbb{P}_{N,\beta}^{(t)}} \left[ -\frac{\beta}{2} \mathbf{A}^{(t)}[\vec{X}_N, \psi] + \left(1 - \frac{\beta}{2}\right) \frac{d}{dt} \sum_{i=1}^N \log \phi'_t(x_i) \right].$$

Indeed,  $V_t$  has the same regularity as  $V$  and  $\tilde{\mu}_t$  the same as  $\mu_0$ .

Next, we express the anisotropy term as a fluctuation, by writing

$$(A.9) \quad \mathbf{A}^{(t)}[\vec{X}_N, \psi] = \int g(x) d\text{fluct}_N^{(t)}(x),$$

where we let

$$(A.10) \quad g(x) := \int \hat{\psi}(x, y) d\text{fluct}_N^{(t)}(y), \quad \hat{\psi}(x, y) := \frac{\psi(x) - \psi(y)}{x - y}.$$

It is clear that

$$(A.11) \quad \|\hat{\psi}\|_{C^2(U \times U)} \leq \|\psi\|_{C^3(U)}.$$

Using Proposition 2.6 twice, we can thus write

$$\|\nabla g\|_{L^\infty} \leq \left| \int \nabla_x \hat{\psi}(x, y) d\text{fluct}_N^{(t)}(y) \right| \leq C \|\nabla_x \nabla_y \hat{\psi}\|_{L^\infty} \left( F_N(\vec{X}_N, \tilde{\mu}_t) + N \log N + CN \right)^{\frac{1}{2}}$$

and

$$|\mathbf{A}^{(t)}[\vec{X}_N, \psi]| = \left| \int g(x) d\text{fluct}_N^{(t)}(x) \right| \leq C \|\nabla g\|_{L^\infty} \left( F_N(\vec{X}_N, \tilde{\mu}_t) + N \log N + CN \right)^{\frac{1}{2}} \\ \leq C \|\hat{\psi}\|_{C^2(U \times U)} \left( F_N(\vec{X}_N, \tilde{\mu}_t) + N \log N + CN \right).$$

<sup>3</sup>In this statement,  $s$  and  $t$  are not related.

In view of (2.9) and (A.11), we deduce that

$$(A.12) \quad \left| \mathbf{E}_{\mathbb{P}_{N,\beta}^{(t)}} \left[ -\frac{\beta}{2} \mathbf{A}^{(t)}[\vec{X}_N, \psi] \right] \right| \leq CN \|\psi\|_{C^3(U)}.$$

For the term  $\log \phi'_t$  we use (A.3) and in view of Proposition 2.6, since  $\phi_t = \text{Id} + t\psi$  is regular enough, we may write

$$(A.13) \quad \left| \int \frac{d}{dt} \log \phi'_t d\text{fluct}_N^{(t)} \right| \leq C \|\psi\|_{C^2(U)} \left( F_N(\vec{X}_N, \tilde{\mu}_t) + N \log N + CN \right)^{\frac{1}{2}}.$$

We conclude from (A.8), using again (2.9) that

$$(A.14) \quad \left| \frac{d}{dt} \log K_{N,\beta}(\tilde{\mu}_t, \tilde{\zeta}_t) \right| \leq CN \|\psi\|_{C^3(U)}.$$

Integrating this relation between 0 and  $-\frac{2s}{\beta N}$ , and combining with (4.8), we find that, for  $t = \frac{-2s}{\beta N}$ ,

$$(A.15) \quad \left| \log \frac{K_{N,\beta}(\mu_t, \zeta_t)}{K_{N,\beta}(\mu_0, \zeta_0)} \right| \leq Cs \|\psi\|_{C^3(U)}.$$

Inserting this, (4.8) and (4.24) into (2.5), we deduce that

$$(A.16) \quad \left| \log \mathbf{E}_{\mathbb{P}_{N,\beta}^{(0)}} [\exp(s \text{Fluct}_N(\xi))] \right| \\ \leq C \left( s \|\psi\|_{C^3(U)} + s^2 \|\psi\|_{C^0(U)} \|\xi\|_{C^1(U)} + \frac{s^3}{N} \|\xi\|_{C^2(U)} + \frac{s^4}{N^2} \|\psi\|_{C^2(U)} \right. \\ \left. + \frac{s^2}{\sqrt{N}} \|\psi\|_{C^2(U)}^2 + \frac{s^4}{N^2} \|\psi\|_{C^2(U)}^4 \right).$$

In view of (3.6), it yields the result for the expectation under  $\mathbb{P}_{N,\beta}^{(0)}$ , and then this can be generalized from  $\mathbb{P}_{N,\beta}^{(0)}$  to  $\mathbb{P}_{N,\beta}^{(t)}$  for  $t$  in  $(-t_{\max}, t_{\max})$  because  $\tilde{\mu}_t$  has the same regularity as  $\mu_0$ .  $\square$

Assuming from now on that  $\mathbf{n} = 0$  and  $\mathbf{m} = 0$  (so that every regular function is in the range of  $\Xi$ ) we can upgrade this control of exponential moments into the control of a weak norm of  $\text{Fluct}_N^{(t)}$ . Here we use the Sobolev spaces  $H^\alpha(\mathbb{R})$ .

**Lemma A.2.** *Under the same assumptions, for  $\alpha \geq 8$  we have*

$$(A.17) \quad \left| \mathbf{E}_{\mathbb{P}_{N,\beta}^{(t)}} \left[ \|\text{fluct}_N^{(t)}\|_{H^{-\alpha}}^2 \right] \right| \leq C,$$

where  $C$  depends only on  $V$ .

*Proof.* The proof is inspired by [AKM17], in particular we start from [AKM17, Prop. D.1] which states that

$$(A.18) \quad \|u\|_{H^{-\alpha}(\mathbb{R})}^2 \leq C \int_0^1 r^{\alpha-1} \|u * \Phi(r, \cdot)\|_{L^2(\mathbb{R})}^2 dr$$

where  $\Phi(r, \cdot)$  is the standard heat kernel, i.e.  $\Phi(r, x) = \frac{1}{\sqrt{4\pi r}} e^{-\frac{|x|^2}{4r}}$ . It follows that

$$(A.19) \quad \mathbf{E}_{\mathbb{P}_{N,\beta}^{(t)}} \left[ \|\text{fluct}_N^{(t)}\|_{H^{-\alpha}(\mathbb{R})}^2 \right] \leq C \int_0^1 r^{\alpha-1} \mathbf{E}_{\mathbb{P}_{N,\beta}^{(t)}} \left[ \|\text{fluct}_N^{(t)} * \Phi(r, \cdot)\|_{L^2(\mathbb{R})}^2 \right] dr.$$

On the other hand we may easily check that, letting  $\xi_{x,r} := \Phi(r, x - \cdot)$ , we have

$$(A.20) \quad \mathbf{E}_{\mathbb{P}_{N,\beta}^{(t)}} \left[ \|\text{fluct}_N^{(t)} * \Phi(r, \cdot)\|_{L^2(\mathbb{R})}^2 \right] = \int \mathbf{E}_{\mathbb{P}_{N,\beta}^{(t)}} \left[ \left( \text{Fluct}_N^{(t)}(\xi_{x,r}) \right)^2 \right] dx.$$

Applying the result of Lemma A.1 to  $\xi_{x,r}$  gives us a control on the second moment of  $\text{Fluct}_N^{(t)}[\xi_{x,r}]$  of the form

$$\mathbf{E}_{\mathbb{P}_{N,\beta}^{(t)}} \left[ \left( \text{Fluct}_N^{(t)}(\xi_{x,r}) \right)^2 \right] \leq C \left( \|\xi_{x,r}\|_{C^4(U)} + \|\xi_{x,r}\|_{C^3(U)}^2 \right).$$

Inserting into (A.19) and (A.20), we are led to

$$\mathbf{E}_{\mathbb{P}_{N,\beta}^{(t)}} \left[ \|\text{fluct}_N^{(t)}\|_{H^{-\alpha}(\mathbb{R})}^2 \right] \leq C \int_0^1 \int r^{\alpha-1} \left( \|\xi_{x,r}\|_{C^4(U)} + \|\xi_{x,r}\|_{C^3(U)}^2 \right) dx dr.$$

Since  $U$  is bounded, we may check that this right-hand side can be bounded by  $C \int_0^1 r^{\alpha-1} r^{-7} dr$ , which converges if  $\alpha > 7$ .  $\square$

**A.3. Proof of Theorem 2.** For any test function  $\phi(x, y)$  we may write

$$\int \phi(x, y) d\text{fluct}_N^{(t)}(x) d\text{fluct}_N^{(t)}(y) \leq \|\phi\|_{C^{2\alpha}(U \times U)} \|\text{fluct}_N^{(t)}\|_{H^{-\alpha}(\mathbb{R})}^2$$

and so by the result of Lemma A.2, we find

$$(A.21) \quad \left| \mathbf{E}_{\mathbb{P}_{N,\beta}^{(t)}} \left( \int \phi(x, y) d\text{fluct}_N^{(t)}(x) d\text{fluct}_N^{(t)}(y) \right) \right| \leq C \|\phi\|_{C^{2\alpha}(U \times U)}.$$

We may now bootstrap the result of Lemma A.1 by returning to (A.9) and, using (A.21), writing that

$$(A.22) \quad \left| \mathbf{E}_{\mathbb{P}_{N,\beta}^{(t)}} \left[ \mathbf{A}^{(t)}[\vec{X}_N, \psi] \right] \right| \leq C \|\psi\|_{C^{2\alpha+1}(U)}.$$

On the other hand, by differentiating (A.6) applied with  $\xi = \frac{d}{dt} \log \phi'_t$ , we have

$$(A.23) \quad \left| \mathbf{E}_{\mathbb{P}_{N,\beta}^{(t)}} \left[ \int \frac{d}{dt} \log \phi'_t d\text{fluct}_N^{(t)} \right] \right| \leq C \|\psi\|_{C^5(U)}$$

Inserting (4.16) and (A.22) and (A.23), (A.3) into (A.8), and integrating between 0 and  $t = -2s/N\beta$ , we obtain

$$\log \frac{K_{N,\beta}(\tilde{\mu}_t, \tilde{\zeta}_t)}{K_{N,\beta}(\mu_0, \zeta_0)} = \left( 1 - \frac{\beta}{2} \right) N (\text{Ent}(\tilde{\mu}_t) - \text{Ent}(\mu_0)) + \frac{s}{N} O(\|\xi\|_{C^{2\alpha+1}(U)}).$$

Using again (4.16) and  $\phi'_t = 1 + t\psi'$ , we may rewrite this as

$$\log \frac{K_{N,\beta}(\tilde{\mu}_{\frac{-2s}{\beta N}}, \tilde{\zeta}_{\frac{-2s}{\beta N}})}{K_{N,\beta}(\mu_0, \zeta_0)} = - \left( 1 - \frac{\beta}{2} \right) \frac{2s}{\beta} \int \psi' d\mu_0 + O\left(\frac{s}{N} \|\xi\|_{C^{2\alpha+1}(U)}\right)$$

Combining this with (4.8), (2.5) with (4.23) and (4.24) we obtain

$$(A.24) \quad \left| \log \mathbf{E}_{\mathbb{P}_{N,\beta}^V} (\exp(s\text{Fluct}_N(\xi)) + \left(1 - \frac{\beta}{2}\right) \frac{2s}{\beta} \int \psi' d\mu_V + \frac{s^2}{\beta} \int_{\mathbb{R}} \xi' \psi d\mu_V) \right| \\ \leq C \left( \frac{s}{N} \|\xi\|_{C^{2\alpha+1}} + \frac{s^3}{N} \|\xi\|_{C^2} + \frac{s^4}{N^2} \|\xi\|_{C^3}^4 \right).$$

with  $C$  depending only on  $V$ . This proves Theorem 2.

**A.4. Iteration and expansion of the partition function to arbitrary order.** Let  $V, W$  be two  $C^\infty$  potentials, such that the associated equilibrium measures  $\mu_V, \mu_W$  satisfy our assumptions with  $n = 0, m = 0$ . In this section, we explain how to iterate the procedure described above to obtain a relative expansion of the partition function, namely an expansion of  $\log Z_{N,\beta}^W - \log Z_{N,\beta}^V$  to any order of  $1/N$ . Up to applying an affine transformation to one of the gases, whose effect on the partition function is easy to compute, we may assume that  $\mu_V$  and  $\mu_W$  have the same support  $\Sigma$ , which is a line segment.

Since  $V, W$  are  $C^\infty$  and  $\mu_V, \mu_W$  have the same support and a density of the same form (1.10) which is  $C^\infty$  on the interior of  $\Sigma$ , the optimal transportation map (or monotone rearrangement)  $\phi$  from  $\mu_V$  to  $\mu_W$  is  $C^\infty$  on  $\Sigma$  and can be extended as a  $C^\infty$  function with compact support on  $\mathbb{R}$ . We let  $\psi := \phi - \text{Id}$ , which is smooth, and for  $t \in [0, 1]$  the map  $\phi_t := \text{Id} + t\psi$  is a  $C^\infty$ -diffeomorphism, by the properties of optimal transport. We let  $\tilde{\mu}_t := \phi_t \# \mu_V$  as before.

We can integrate (A.8) to obtain

$$\log \frac{K_{N,\beta}(\mu_W, \zeta_W)}{K_{N,\beta}(\mu_V, \zeta_V)} \\ = \int_0^1 \mathbf{E}_{\mathbb{P}_{N,\beta}^{(t)}} \left[ -\frac{\beta}{2} \mathbf{A}^{(t)}[\vec{X}_N, \psi] + \left(1 - \frac{\beta}{2}\right) N \int \frac{d}{dt} \log \phi_t' d\tilde{\mu}_t + \left(1 - \frac{\beta}{2}\right) \int \frac{d}{dt} \log \phi_t' d\text{fluct}_N^{(t)} \right] dt \\ = N \left(1 - \frac{\beta}{2}\right) (\text{Ent}(\mu_W) - \text{Ent}(\mu_V)) \\ + \int_0^1 \mathbf{E}_{\mathbb{P}_{N,\beta}^{(t)}} \left[ -\frac{\beta}{2} \mathbf{A}^{(t)}[\vec{X}_N, \psi] + \left(1 - \frac{\beta}{2}\right) \text{Fluct}_N \left[ \int \frac{d}{dt} \log \phi_t' d\text{fluct}_N^{(t)} \right] \right] dt.$$

The integral on the right-hand side is of order 1, and we claim that the terms in the integral can actually be computed and expanded up to an error  $O(1/N)$  using the previous lemma. This is clear for the term  $\mathbf{E}_{\mathbb{P}_{N,\beta}^{(t)}} \left[ \text{Fluct}_N^{(t)} \left( \frac{d}{dt} \log \phi_t' \right) \right]$  which can be computed up to an error  $O(1/N)$  by the result of Theorem 2. The term  $\mathbf{E}_{\mathbb{P}_{N,\beta}^{(t)}} \left[ -\frac{\beta}{2} \mathbf{A}^{(t)}[\vec{X}_N, \psi] \right]$  can on the other hand be deduced from the knowledge of the covariance structure of the fluctuations. Let  $\mathcal{F}$  denote the Fourier transform. In view of (A.9), using the identity

$$\frac{\psi(x) - \psi(y)}{x - y} = \int_0^1 \psi'(sx + (1-s)y) ds$$

and the Fourier inversion formula we may write

$$(A.25) \quad \mathbf{E}_{\mathbb{P}_{N,\beta}^{(t)}} \left[ \mathbf{A}^{(t)}[\vec{X}_N, \psi] \right] = \mathbf{E}_{\mathbb{P}_{N,\beta}^{(t)}} \left[ \iint_{\mathbb{R} \times \mathbb{R}} \int_0^1 \psi'(sx + (1-s)y) ds d\text{fluct}_N^{(t)}(x) d\text{fluct}_N^{(t)}(y) \right] \\ = \int \int_0^1 \lambda \mathcal{F}(\psi)(\lambda) \mathbf{E}_{\mathbb{P}_{N,\beta}^{(t)}} \left[ \text{Fluct}_N^{(t)}(e^{is\lambda}) \text{Fluct}_N^{(t)}(e^{i(1-s)\lambda}) \right] ds d\lambda.$$

On the other hand, let  $\varphi_{s,\lambda}$  be the map associated to  $e^{is\lambda}$  by Lemma 3.3. Separating the real part and the imaginary part we may use the results of the previous subsection to  $e^{is\lambda}$  and obtain

$$\mathbf{E}_{\mathbb{P}_{N,\beta}^{(t)}} \left[ \text{Fluct}_N^{(t)}(e^{is\lambda}) \right] = \left( 1 - \frac{2}{\beta} \right) \int \varphi'_{s,\lambda} d\tilde{\mu}_t + O\left(\frac{1}{N}\right).$$

By polarization of the expression for the variance (see (1.16)) and linearity

$$\mathbf{E}_{\mathbb{P}_{N,\beta}^{(t)}} \left[ \text{Fluct}_N^{(t)}(e^{is\lambda}) \text{Fluct}_N^{(t)}(e^{i(1-s)\lambda}) \right] = \mathbf{E}_{\mathbb{P}_{N,\beta}^{(t)}} \left[ \text{Fluct}_N^{(t)}(e^{is\lambda}) \right] \mathbf{E}_{\mathbb{P}_{N,\beta}^{(t)}} \left[ \text{Fluct}_N^{(t)}(e^{i(1-s)\lambda}) \right] \\ + \frac{2}{\beta} \left( \iint \left( \frac{\varphi_{s,\lambda}(u) - \varphi_{s,\lambda}(v)}{u-v} \right) \left( \frac{\varphi_{(1-s),\lambda}(u) - \varphi_{(1-s),\lambda}(v)}{u-v} \right) d\tilde{\mu}_t(u) d\tilde{\mu}_t(v) \right. \\ \left. + \int V_t'' \varphi_{s,\lambda} \varphi_{(1-s),\lambda} d\tilde{\mu}_t \right) + O\left(\frac{1}{N}\right).$$

Letting  $N \rightarrow \infty$ , we may then find the expansion up to  $O(1/N)$  of  $\mathbf{E}_{\mathbb{P}_{N,\beta}^{(t)}} \left[ -\frac{\beta}{2} \mathbf{A}^{(t)}[\vec{X}_N, \psi] \right]$ . Inserting it into the integral gives a relative expansion to order  $1/N$  of the (logarithm of the) partition function  $\log K_{N,\beta}$ . This procedure can then be iterated to yield a relative expansion to arbitrary order of  $1/N$  as desired.

## APPENDIX B. AUXILIARY PROOFS

### B.1. Proof of Lemma 2.1.

*Proof.* Denoting  $\Delta$  the diagonal in  $\mathbb{R} \times \mathbb{R}$  we may write

$$\mathcal{H}_N^V(\vec{X}_N) = \sum_{i \neq j} -\log |x_i - x_j| + N \sum_{i=1}^N V(x_i) \\ = \iint_{\Delta^c} -\log |x - y| \left( \sum_{i=1}^N \delta_{x_i} \right)(x) \left( \sum_{i=1}^N \delta_{x_i} \right)(y) + N \int_{\mathbb{R}} V(x) \left( \sum_{i=1}^N \delta_{x_i} \right)(x).$$

Writing  $\sum_{i=1}^N \delta_{x_i}$  as  $N\mu_V + \text{fluct}_N$  we get

$$(B.1) \quad \mathcal{H}_N^V(\vec{X}_N) = N^2 \iint_{\Delta^c} -\log |x - y| d\mu_V(x) d\mu_V(y) + N^2 \int_{\mathbb{R}} V d\mu_V \\ + 2N \iint_{\Delta^c} -\log |x - y| d\mu_V(x) d\text{fluct}_N(y) + N \int_{\mathbb{R}} V d\text{fluct}_N \\ + \iint_{\Delta^c} -\log |x - y| d\text{fluct}_N(x) d\text{fluct}_N(y).$$

We now recall that  $\zeta_V$  was defined in (1.7), and that  $\zeta_V = 0$  in  $\Sigma_V$ . With the help of this we may rewrite the medium line in the right-hand side of (B.1) as

$$\begin{aligned} 2N \iint_{\Delta^c} -\log|x-y| d\mu_V(x) d\text{fluct}_N(y) + N \int_{\mathbb{R}} V d\text{fluct}_N \\ = 2N \int_{\mathbb{R}} \left( -\log|\cdot| * d\mu_V(x) + \frac{V}{2} \right) d\text{fluct}_N = 2N \int_{\mathbb{R}} (\zeta_V + c) d\text{fluct}_N \\ = 2N \int_{\mathbb{R}} \zeta_V d\left( \sum_{i=1}^N \delta_{x_i} - N\mu_V \right) = 2N \sum_{i=1}^N \zeta_V(x_i). \end{aligned}$$

The last equalities are due to the facts that  $\zeta_V$  vanishes on the support of  $\mu_V$  and that  $\text{fluct}_N$  has a total mass 0 since  $\mu_V$  is a probability measure. We may also notice that since  $\mu_V$  is absolutely continuous with respect to the Lebesgue measure, we may include the diagonal back into the domain of integration. By that same argument, one may recognize in the first line of the right-hand side of (B.1) the quantity  $N^2 \mathcal{I}_V(\mu_V)$ .  $\square$

**B.2. Proof of Proposition 2.6.** We follow the energy approach introduced in [SS15, PS14], which views the energy as a Coulomb interaction in the plane, after embedding the real line in the plane. We view  $\mathbb{R}$  as identified with  $\mathbb{R} \times \{0\} \subset \mathbb{R}^2 = \{(x, y), x \in \mathbb{R}, y \in \mathbb{R}\}$ . Let us denote by  $\delta_{\mathbb{R}}$  the uniform measure on  $\mathbb{R} \times \{0\}$ , i.e. such that for any smooth  $\varphi(x, y)$  (with  $x \in \mathbb{R}, y \in \mathbb{R}$ ) we have

$$\int_{\mathbb{R}^2} \varphi \delta_{\mathbb{R}} = \int_{\mathbb{R}} \varphi(x, 0) dx.$$

Given  $(x_1, \dots, x_N)$  in  $\mathbb{R}^N$ , we identify them with the points  $(x_1, 0), \dots, (x_N, 0)$  in  $\mathbb{R}^2$ . For a fixed  $\vec{X}_N$  and a given probability density  $\mu$  we introduce the electric potential  $H_N^\mu$  by

$$(B.2) \quad H_N^\mu = (-\log|\cdot|) * \left( \sum_{i=1}^N \delta_{(x_i, 0)} - N\mu\delta_{\mathbb{R}} \right).$$

Next, we define versions of this potential which are truncated hence regular near the point charges. For that let  $\delta_x^{(\eta)}$  denote the uniform measure of mass 1 on  $\partial B(x, \eta)$  (where  $B$  denotes an Euclidean ball in  $\mathbb{R}^2$ ). We define  $H_{N, \eta}^\mu$  in  $\mathbb{R}^2$  by

$$(B.3) \quad H_{N, \eta}^\mu = (-\log|\cdot|) * \left( \sum_{i=1}^N \delta_{(x_i, 0)}^{(\eta)} - N\mu\delta_{\mathbb{R}} \right).$$

These potentials make sense as functions in  $\mathbb{R}^2$  and are harmonic outside of the real axis. Moreover,  $H_{N, \eta}^\mu$  solves

$$(B.4) \quad -\Delta H_{N, \eta}^\mu = 2\pi \left( \sum_{i=1}^N \delta_{(x_i, 0)}^{(\eta)} - N\mu\delta_{\mathbb{R}} \right).$$

**Lemma B.1.** *For any probability density  $\mu$ ,  $\vec{X}_N$  in  $\mathbb{R}^N$  and  $\eta$  in  $(0, 1)$ , we have*

$$(B.5) \quad F_N(\vec{X}_N, \mu) \geq \frac{1}{2\pi} \int_{\mathbb{R}^2} |\nabla H_{N, \eta}^\mu|^2 + N \log \eta - 2N^2 \|\mu\|_{L^\infty} \eta.$$



*Proof.* First we notice that  $\int_{\mathbb{R}^2} |\nabla H_{N,\eta}|^2$  is a convergent integral and that

$$(B.6) \quad \int_{\mathbb{R}^2} |\nabla H_{N,\eta}|^2 = 2\pi \iint -\log|x-y| d\left(\sum_{i=1}^N \delta_{x_i}^{(\eta)} - N\mu\delta_{\mathbb{R}}\right)(x) d\left(\sum_{i=1}^N \delta_{x_i}^{(\eta)} - N\mu\delta_{\mathbb{R}}\right)(y).$$

Indeed, we may choose  $R$  large enough so that all the points of  $\vec{X}_N$  are contained in the ball  $B_R = B(0, R)$ . By Green's formula and (B.4), we have

$$(B.7) \quad \int_{B_R} |\nabla H_{N,\eta}|^2 = \int_{\partial B_R} H_{N,\eta} \frac{\partial H_N}{\partial \nu} + 2\pi \int_{B_R} H_{N,\eta} \left(\sum_{i=1}^N \delta_{x_i}^{(\eta)} - N\mu\delta_{\mathbb{R}}\right).$$

In view of the decay of  $H_N$  and  $\nabla H_N$ , the boundary integral tends to 0 as  $R \rightarrow \infty$ , and so we may write

$$\int_{\mathbb{R}^2} |\nabla H_{N,\eta}|^2 = 2\pi \int_{\mathbb{R}^2} H_{N,\eta} \left(\sum_{i=1}^N \delta_{x_i}^{(\eta)} - N\mu\right)$$

and thus (B.6) holds. We may next write

$$(B.8) \quad \begin{aligned} & \iint -\log|x-y| d\left(\sum_{i=1}^N \delta_{x_i}^{(\eta)} - N\mu\delta_{\mathbb{R}}\right)(x) d\left(\sum_{i=1}^N \delta_{x_i}^{(\eta)} - N\mu\delta_{\mathbb{R}}\right)(y) \\ & \quad - \iint_{\Delta^c} -\log|x-y| d\text{fluct}_N(x) d\text{fluct}_N(y) \\ & = -\sum_{i=1}^N \log \eta + \sum_{i \neq j} \iint -\log|x-y| \left(\delta_{x_i}^{(\eta)} \delta_{x_j}^{(\eta)} - \delta_{x_i} \delta_{x_j}\right) + 2N \sum_{i=1}^N \iint -\log|x-y| \left(\delta_{x_i} - \delta_{x_i}^{(\eta)}\right) \mu. \end{aligned}$$

Let us now observe that  $\int -\log|x-y| \delta_{x_i}^{(\eta)}(y)$ , the potential generated by  $\delta_{x_i}^{(\eta)}$  is equal to  $\int -\log|x-y| \delta_{x_i}$  outside of  $B(x_i, \eta)$ , and smaller otherwise. Since its Laplacian is  $-2\pi \delta_{x_i}^{(\eta)}$ , a negative measure, this is also a superharmonic function, so by the maximum principle, its value at a point  $x_j$  is larger or equal to its average on a sphere centered at  $x_j$ . Moreover, outside  $B(x_i, \eta)$  it is a harmonic function, so its values are equal to its averages. We deduce from these considerations, and reversing the roles of  $i$  and  $j$ , that for each  $i \neq j$ ,

$$-\int \log|x-y| \delta_{x_i}^{(\eta)} \delta_{x_j}^{(\eta)} \leq -\int \log|x-y| \delta_{x_i} \delta_{x_j}^{(\eta)} \leq -\int \log|x-y| \delta_{x_i} \delta_{x_j}.$$

We may also obviously write

$$\int -\log|x-y| \delta_{x_i} \delta_{x_j} - \int -\log|x-y| \delta_{x_i}^{(\eta)} \delta_{x_j}^{(\eta)} \leq -\log|x_i - x_j| \mathbf{1}_{|x_i - x_j| \leq 2\eta}.$$

We conclude that the second term in the right-hand side of (B.8) is nonpositive, equal to 0 if all the balls are disjoint, and bounded below by  $\sum_{i \neq j} \log|x_i - x_j| \mathbf{1}_{|x_i - x_j| \leq 2\eta}$ . Finally, by the above considerations, since  $\int -\log|x-y| \delta_{x_i}^{(\eta)}$  coincides with  $\int -\log|x-y| \delta_{x_i}$  outside  $B(x_i, \eta)$ , we may rewrite the last term in the right-hand side of (B.8) as

$$2N \sum_{i=1}^N \int_{B(x_i, \eta)} (-\log|x-x_i| + \log \eta) d\mu\delta_{\mathbb{R}}.$$

But we have that

$$(B.9) \quad \int_{B(0,\eta)} (-\log|x| + \log\eta) \delta_{\mathbb{R}} = \eta$$

so if  $\mu \in L^\infty$ , this last term is bounded by  $2\|\mu\|_{L^\infty} N^2 \eta$ . Combining with all the above results yields the proof.  $\square$

*Proof of Proposition 2.6.* We now apply Lemma B.1 for  $\mu_V$  with  $\eta = \frac{1}{2N}$ . We obtain

$$(B.10) \quad \frac{1}{2\pi} \int_{\mathbb{R}^2} |\nabla H_{N,\eta}^\mu|^2 \leq F_N(\vec{X}_N, \mu_V) + N \log N + C(\|\mu_V\|_{L^\infty} + 1)N.$$

Let  $\xi$  be a smooth compactly supported test function in  $\mathbb{R}$ . We may extend it to a smooth compactly supported test function in  $\mathbb{R}^2$  coinciding with  $\xi(x)$  for any  $(x, y)$  such that  $|y| \leq 1$  and equal to 0 for  $|y| \geq 2$ . Letting  $\#I$  denote the number of balls  $B(x_i, \eta)$  intersecting the support of  $\xi$ , we have

$$(B.11) \quad \left| \int \left( \text{fluct}_N - \left( \sum_{i=1}^N \delta_{x_i}^{(\eta)} - N\mu_V \right) \right) \xi \right| = \left| \int \left( \sum_{i=1}^N (\delta_{x_i} - \delta_{x_i}^{(\eta)}) \right) \xi \right| \\ \leq \#I \eta \|\nabla \xi\|_{L^\infty} = \frac{1}{2} \frac{\#I}{N} \|\nabla \xi\|_{L^\infty}.$$

But in view of (B.4), we also have

$$(B.12) \quad \left| \int \left( \sum_{i=1}^N \delta_{x_i}^{(\eta)} - N\mu_V \right) \xi \right| = \frac{1}{2\pi} \left| \int_{\mathbb{R}^2} \nabla H_{N,\eta}^{\mu_V} \cdot \nabla \xi \right| \\ \leq |\text{Supp } \xi|^{\frac{1}{2}} \|\nabla \xi\|_{L^\infty} \|\nabla H_{N,\eta}^{\mu_V}\|_{L^2(\text{Supp } \xi)}.$$

Combining (B.10), (B.11) and (B.12), we obtain

$$(B.13) \quad \left| \int \xi \text{fluct}_N \right| \\ \leq C \|\nabla \xi\|_{L^\infty} \left( \frac{\#I}{N} + |\text{Supp } \xi|^{\frac{1}{2}} \left( F_N(\vec{X}_N, \mu_V) + N \log N + C(\|\mu_V\|_{L^\infty} + 1)N \right)^{\frac{1}{2}} \right).$$

Bounding  $\#I$  by  $N$  yields the result.  $\square$

### B.3. Proof of Lemma 3.1.

*Proof.* Since  $\mu_V$  minimizes the logarithmic potential energy (1.6), for any bounded continuous function  $h$  we have

$$(B.14) \quad \iint \frac{h(x) - h(y)}{x - y} d\mu_V(x) d\mu_V(y) = \int V'(x) h(x) d\mu_V(x).$$

Of course, an identity like (B.14) extends to complex-valued functions, and applying it to  $h = \frac{1}{z - \cdot}$  for some fixed  $z \in \mathbb{C} \setminus \Sigma_V$  leads to

$$(B.15) \quad G(z)^2 - G(z) V'(\mathcal{R}(z)) + L(z) = 0,$$

where  $G$  is the usual Stieltjes transform of  $\mu_V$

$$(B.16) \quad G(z) = \int \frac{1}{z - y} d\mu_V(y),$$

and  $L$  is defined by

$$(B.17) \quad L(z) = \int \frac{V'(\mathcal{R}(z)) - V'(y)}{z - y} d\mu_V(y).$$

Solving (B.15) for  $G$  yields

$$(B.18) \quad G(z) = \frac{1}{2} \left( V'(\mathcal{R}(z)) - \sqrt{V'(\mathcal{R}(z))^2 - 4L(z)} \right).$$

As is well-known,  $-\frac{1}{\pi}\mathcal{I}(G(x + i\varepsilon))$  converges towards the density  $\mu_V(x)$  as  $\varepsilon \rightarrow 0^+$ , hence we have for  $x$  in  $\Sigma_V$

$$(B.19) \quad \mu_V(x)^2 = S(x)^2 \sigma^2(x) = -\frac{1}{(2\pi)^2} (V'(x)^2 - 4L(x)).$$

This proves that  $\mu_V$  has regularity  $C^{p-2}$  at any point where it does not vanish. Assuming the form (1.11) for  $S$ , we also deduce that the function  $S_0$  has regularity at least  $C^{p-3-2k}$  on  $\Sigma_V$ .

Applying (B.18) on  $\mathbb{R} \setminus \Sigma$ , we obtain

$$\frac{1}{2}V'(x) - \int \frac{1}{x-y} d\mu_V(y) = \frac{1}{2}\sqrt{V'(x)^2 - 4L(x)},$$

and the left-hand side is equal to  $\zeta'(x)$ .

Using (1.11), (B.19) and the fact that  $V$  is regular, we may find a neighborhood  $U$  small enough such that  $\zeta'$  does not vanish on  $U \setminus \Sigma_V$  and on which we can write  $\zeta'$  as in (3.1).  $\square$

#### B.4. Proof of Lemma 3.2.

*Proof.* We first prove that the image of  $F$  is indeed contained in  $C^1(U)$ .

For  $(t, \psi) = (0, 0)$ , we have indeed  $\mathcal{F}(0, 0) = \zeta_V + c$  and  $\zeta_V$  is in  $C^1(\mathbb{R})$  by the regularity assumptions on  $V$ . We may also write

$$\mathcal{F}(t, \psi) = \mathcal{F}(0, 0) - \int \log \frac{|\phi(\cdot) - \phi(y)|}{|\cdot - y|} d\mu_V(y) + \frac{1}{2}(V_t \circ \phi - V \circ \phi),$$

and since  $\|\psi\|_{C^2(U)} \leq 1/2$ , the second and third terms are also in  $C^1(U)$ .

Next, we compute the partial derivatives of  $\mathcal{F}$  at a fixed point  $(t_0, \psi_0) \in [-1, 1] \times B$ . It is easy to see that

$$\frac{\partial \mathcal{F}}{\partial t} \Big|_{(t_0, \psi_0)} = \frac{1}{2} \xi \circ \phi_0,$$

and the map  $(t_0, \psi_0) \mapsto \xi \circ \phi_0$  is indeed continuous.

The Fréchet derivative of  $F$  with respect to the second variable can be computed as follows

$$\begin{aligned} \mathcal{F}(t_0, \psi_0 + \psi_1) &= - \int \log \left| (\phi_0(\cdot) - \phi_0(y)) + (\psi_1(\cdot) - \psi_1(y)) \right| d\mu_V(y) + \frac{1}{2} V_{t_0} \circ (\phi_0 + \psi_1) \\ &= \mathcal{F}(t_0, \psi_0) - \int \log \left| 1 + \frac{\psi_1(\cdot) - \psi_1(y)}{\phi_0(\cdot) - \phi_0(y)} \right| d\mu_V(y) + \frac{1}{2} (V_{t_0} \circ (\phi_0 + \psi_1) - V_{t_0} \circ \phi_0) \\ &= \mathcal{F}(t_0, \psi_0) - \int \frac{\psi_1(\cdot) - \psi_1(y)}{\phi_0(\cdot) - \phi_0(y)} d\mu_V(y) + \frac{1}{2} \psi_1 V'_{t_0} \circ \phi_0 + \varepsilon_{t_0, \psi_0}(\psi_1), \end{aligned}$$

where  $\varepsilon_{t_0, \psi_0}(\psi_1)$  is given by

$$\begin{aligned} \varepsilon_{t_0, \psi_0}(\psi_1) &= - \int \left[ \log \left| 1 + \frac{\psi_1(\cdot) - \psi_1(y)}{\phi_0(\cdot) - \phi_0(y)} \right| - \frac{\psi_1(\cdot) - \psi_1(y)}{\phi_0(\cdot) - \phi_0(y)} \right] d\mu_V(y) \\ &\quad + \frac{1}{2} (V_{t_0} \circ (\phi_0 + \psi_1) - V_{t_0} \circ \phi_0 - \psi_1 V'_{t_0} \circ \phi_0) . \end{aligned}$$

By differentiating twice inside the integral we get the bound

$$\|\varepsilon_{t_0, \psi_0}(\psi_1)\|_{C^1(U)} \leq C(t_0, \psi_0) \|\psi_1\|_{C^2(U)}^2,$$

with a constant depending on  $V$ . It implies that

$$\frac{\partial \mathcal{F}}{\partial \psi} \Big|_{(t_0, \psi_0)} [\psi_1] = - \int \frac{\psi_1(\cdot) - \psi_1(y)}{\phi_0(\cdot) - \phi_0(y)} d\mu_V(y) + \frac{1}{2} \psi_1 V'_{t_0} \circ \phi_0 ,$$

and we can check that this expression is also continuous in  $(t_0, \psi_0)$ . In particular, we may observe that

$$(B.20) \quad \frac{\partial \mathcal{F}}{\partial \psi} \Big|_{(0,0)} [\psi] = -\Xi_V[\psi].$$

Finally, we prove the bound (3.3). For any fixed  $(t, \psi) \in [-1, 1] \times B$ , we write

$$\mathcal{F}(t, \psi) - \mathcal{F}(0, 0) = \int_0^1 \frac{d\mathcal{F}(st, s\psi)}{ds} ds = \int_0^1 \left( t \frac{\partial \mathcal{F}}{\partial t} \Big|_{(st, s\psi)} + \frac{\partial \mathcal{F}}{\partial \psi} \Big|_{(st, s\psi)} [\psi] \right) ds ,$$

we get

$$(B.21) \quad \begin{aligned} \|\mathcal{F}(t, \psi) - \mathcal{F}(0, 0) - \frac{t}{2} \xi + \Xi_V[\psi]\|_{C^1(U)} &\leq \int_0^1 \left( \frac{t}{2} \|\xi \circ \phi_s - \xi\|_{C^1(U)} \right. \\ &\quad \left. + \left\| \frac{\partial \mathcal{F}}{\partial \psi} \Big|_{(st, s\psi)} [\psi] - \frac{\partial \mathcal{F}}{\partial \psi} \Big|_{(0,0)} [\psi] \right\|_{C^1(U)} \right) ds, \end{aligned}$$

with  $\phi_s = \text{Id} + s\psi$ . It is straightforward to check that

$$\|\xi \circ \phi_s - \xi\|_{C^1(U)} \leq C \|\xi\|_{C^2(U)} \|\psi\|_{C^1(U)} .$$

To control the second term inside the integral we write

$$\begin{aligned} \frac{\partial \mathcal{F}}{\partial \psi} \Big|_{(st, s\psi)} [\psi] - \frac{\partial \mathcal{F}}{\partial \psi} \Big|_{(0,0)} [\psi] \\ = - \int \left( \frac{\psi(\cdot) - \psi(y)}{\phi_s(\cdot) - \phi_s(y)} - \frac{\psi(\cdot) - \psi(y)}{\cdot - y} \right) d\mu_V(y) + \frac{1}{2} (V'_{st} \circ \phi_s - V') \psi \end{aligned}$$

and we obtain

$$\begin{aligned} \left\| \frac{\partial \mathcal{F}}{\partial \psi} \Big|_{(st, s\psi)} [\psi] - \frac{\partial \mathcal{F}}{\partial \psi} \Big|_{(0,0)} [\psi] \right\|_{C^1(U)} \\ \leq \int \left\| \frac{\psi(\cdot) - \psi(y)}{\phi_s(\cdot) - \phi_s(y)} - \frac{\psi(\cdot) - \psi(y)}{\cdot - y} \right\|_{C^1(U)} d\mu_V(y) \\ \quad + \|(V'_{st} \circ \phi_s - V') \psi\|_{C^1(U)} \end{aligned}$$

We now use that

$$\begin{aligned}
\left\| \left( \frac{\psi(\cdot) - \psi(y)}{\phi_s(\cdot) - \phi_s(y)} - \frac{\psi(\cdot) - \psi(y)}{\cdot - y} \right) \right\|_{C^1(U)} &= \left\| \left( \frac{\psi(\cdot) - \psi(y)}{\cdot - y} \right) \left( \frac{\cdot - y}{\phi_s(\cdot) - \phi_s(y)} - 1 \right) \right\|_{C^1(U)} \\
&\leq C \|\psi\|_{C^2(U)} \left\| \frac{\cdot - y}{\phi_s(\cdot) - \phi_s(y)} - 1 \right\|_{C^1(U)} \\
&= Cs \|\psi\|_{C^2(U)} \left\| \frac{\psi(\cdot) - \psi(y)}{\phi_s(\cdot) - \phi_s(y)} \right\|_{C^1(U)} \\
&\leq C \|\psi\|_{C^2(U)}^2 \left\| \frac{\cdot - y}{\phi_s(\cdot) - \phi_s(y)} \right\|_{C^1(U)} \\
&\leq C \|\psi\|_{C^2(U)}^2.
\end{aligned}$$

In the second and the fourth line, we used Leibniz formula. In the last line we used that  $s(\psi(\cdot) - \psi(y))/(\cdot - y)$  is uniformly bounded by  $1/2$  in  $C^2(U)$  so its composition with the function  $x \rightarrow 1/(1+x)$  is bounded in  $C^2(U)$ . We conclude by checking that

$$\|(V'_{st} \circ \phi_s - V')\psi\|_{C^1(U)} \leq C \left( \|V\|_{C^3(U)} \|\psi\|_{C^1(U)} + t \|\psi\|_{C^2(U)} \right) \|\psi\|_{C^0(U)}.$$

□

### B.5. Proof of Lemma 3.3.

*Proof.* First, we solve the equation  $\Xi_V[\psi] = \frac{1}{2}\xi + c_\xi$  in  $\mathring{\Sigma}_V$ , where  $\Xi_V$  is operator defined in (1.12). For  $x$  in  $\mathring{\Sigma}_V$ , we have the following Schwinger-Dyson equation

$$(B.22) \quad \frac{V'(x)}{2} = P.V. \int \frac{1}{x-y} d\mu_V(y).$$

In particular, for  $x$  in  $\mathring{\Sigma}_V$ , it implies

$$(B.23) \quad \Xi_V[\psi](x) := P.V. \int_{\Sigma_V} \frac{\psi(y)}{y-x} \mu_V(y) dy,$$

and we might thus try to solve

$$(B.24) \quad P.V. \int_{\Sigma_V} \frac{\psi(y)}{y-x} \mu_V(y) dy = \frac{1}{2}\xi + c_\xi.$$

Equation (B.24) is a singular integral equation, we refer to [Mus92, Chap. 10-11-12] for a detailed treatment. In particular, it is known that if the conditions (1.14) are satisfied, then there exists a solution  $\psi_0$  to

$$(B.25) \quad P.V. \int_{\Sigma_V} \frac{\psi_0(y)}{y-x} dy = \frac{1}{2}\xi + c_\xi \text{ on } \mathring{\Sigma}_V,$$

which is explicitly given by the formula

$$(B.26) \quad \psi_0(x) = -\frac{\sigma(x)}{2\pi^2} P.V. \int_{\Sigma_V} \frac{\xi(y)}{\sigma(y)(y-x)} dy.$$

Since we have, for  $x$  in  $\mathring{\Sigma}_V$

$$P.V. \int_{\Sigma_V} \frac{1}{\sigma(y)(y-x)} dy = 0,$$

we may re-write (B.26) as

$$(B.27) \quad \psi_0(x) = -\frac{\sigma(x)}{2\pi^2} \int_{\Sigma_V} \frac{\xi(y) - \xi(x)}{\sigma(y)(y-x)} dy \text{ on } \mathring{\Sigma}_V,$$

where the integral is now a definite Riemann integral. From (B.27) we deduce that the map  $\frac{\psi_0}{\sigma}$  is of class  $C^{r-1}$  in  $\mathring{\Sigma}_V$  and extends readily to a  $C^{r-1}$  function on  $\Sigma_V$ .

For  $d = 0, \dots, r-1$  and for  $x \in \Sigma_V$ , we compute that

$$\left(\frac{\psi_0}{\sigma}\right)^{(d)}(x) = -\frac{d!}{2\pi^2} \int_{\Sigma_V} \frac{\xi(y) - R_{s_i, d+1}\xi(y)}{\sigma(y)(y-s_i)^{d+1}} dy.$$

In particular, if conditions (1.15) hold, in view of Lemma 3.1 the map

$$\psi(x) := \frac{\psi_0(x)}{S(x)\sigma(x)}$$

extends to a function of class  $(\mathfrak{p} - 3 - 2\mathfrak{k}) \wedge (r - 1 - \mathfrak{k})$ , hence  $C^2$  on  $\Sigma_V$ , and in view of (B.25) it satisfies  $\Xi_V[\psi] = \frac{\xi}{2} + c_\xi$  on  $\Sigma_V$ .

Now, we define  $\psi$  outside  $\Sigma_V$ . By definition, for  $x$  outside  $\Sigma_V$ , the equation

$$\Xi_V[\psi](x) = \frac{1}{2}\xi(x) + c_\xi$$

can be written as

$$\psi(x) \int \frac{1}{x-y} d\mu_V(y) - \int \frac{\psi(y)}{x-y} d\mu_V(y) - \frac{1}{2}\psi(x)V'(x) = \frac{1}{2}\xi(x) + c_\xi,$$

and thus the choice (3.5) ensures that  $\Xi_V[\psi] = \frac{1}{2}\xi + c_\xi$ . Moreover,  $\psi$  is clearly of class  $C^{r \wedge (\mathfrak{p}-1)}$  on  $\mathbb{R} \setminus \Sigma_V$ . It remains to check that  $\psi$  has the desired regularity at the endpoints of  $\Sigma_V$ . For a given endpoint  $\alpha$  we consider  $\tilde{\psi}$  the Taylor development of order  $l := (\mathfrak{p} - 3 - 2\mathfrak{k}) \wedge (r - 1 - \mathfrak{k})$  at  $\alpha$  of  $\psi$ . We can write (3.5) as

$$\begin{aligned} \frac{\int \frac{\psi(y)}{x-y} d\mu_V(y) + \frac{\xi(x)}{2} + c_\xi}{\int \frac{1}{x-y} d\mu_V(y) - \frac{1}{2}V'(x)} &= \frac{-\int \frac{\tilde{\psi}(x) - \tilde{\psi}(y)}{x-y} d\mu_V(y) + \tilde{\psi}(x) \int \frac{1}{x-y} d\mu_V(y) + \frac{\xi(x)}{2} + c_\xi}{\int \frac{1}{x-y} d\mu_V(y) - \frac{1}{2}V'(x)} \\ &= \tilde{\psi}(x) + \frac{\frac{\xi(x)}{2} + c_\xi - \Xi_V[\tilde{\psi}](x)}{\int \frac{1}{x-y} d\mu_V(y) - \frac{1}{2}V'(x)}. \end{aligned}$$

As  $\Xi_V[\psi] = \frac{\xi}{2} + c_\xi$  on  $\Sigma_V$ , the numerator on the right hand side of the last equation and its first  $l$  derivatives vanish at  $\alpha$ . From Lemma (3.1) we conclude that  $\psi$  is of class  $l - \mathfrak{k} = (\mathfrak{p} - 3 - 3\mathfrak{k}) \wedge (r - 1 - 2\mathfrak{k})$  at  $\alpha$ , hence  $C^2$  from (1.13).  $\square$

**B.6. Proof of Lemma 4.7.** Using definition (1.6) we can write  $\mathcal{I}_{V_t}(\mu_t)$  in the following form

$$\mathcal{I}_{V_t}(\mu_t) = \int h^{\mu_t} d\mu_t + \int V_t d\mu_t.$$

To prove Lemma 4.7, we introduce the auxiliary quantity

$$\mathcal{I}(\tilde{\mu}_t) := \int h^{\tilde{\mu}_t} d\tilde{\mu}_t + \int V_t d\tilde{\mu}_t,$$

and we first prove that  $\mathcal{I}(\tilde{\mu}_t)$  is close to  $\mathcal{I}_{V_t}(\mu_t)$ .

**Claim 1.** *We have*

$$(B.28) \quad \mathcal{I}_{V_t}(\mu_t) = \mathcal{I}(\tilde{\mu}_t) + O\left(t^4 \|\psi\|_{C^2(U)}^4\right).$$

*Proof.* Let us write

$$(B.29) \quad \begin{aligned} \mathcal{I}_{V_t}(\mu_t) &= \int h^{\mu_t} d\mu_t + \int V_t d\mu_t \\ &= \int h^{\tilde{\mu}_t} d\tilde{\mu}_t + \int (h^{\mu_t} + h^{\tilde{\mu}_t}) d(\mu_t - \tilde{\mu}_t) + \int V_t d\mu_t. \end{aligned}$$

We have used the fact that, integrating by parts twice,

$$\int h^{\mu_t} d\tilde{\mu}_t = \int h^{\tilde{\mu}_t} d\mu_t.$$

We have, using the definition of  $\zeta_t, \tilde{\zeta}_t$  and (3.10)

$$\int (h^{\mu_t} + h^{\tilde{\mu}_t}) d(\mu_t - \tilde{\mu}_t) = \int \left( \zeta_t - \frac{1}{2}V_t - c_t + \tilde{\zeta}_t - \frac{1}{2}V_t - \tilde{c}_t + O(t^2 \|\psi\|_{C^2(U)}^2) \right) d(\mu_t - \tilde{\mu}_t).$$

In view of (4.1), (4.2), we thus get

$$(B.30) \quad \int (h^{\mu_t} + h^{\tilde{\mu}_t}) d(\mu_t - \tilde{\mu}_t) + \int V_t d\mu_t = O(t^4 \|\psi\|_{C^2(U)}^4) + \int V_t d\tilde{\mu}_t.$$

Combining (B.29) and (B.30) yields the result.  $\square$

We may now compare  $\mathcal{I}(\tilde{\mu}_t)$  and  $\mathcal{I}_V(\mu_V)$  using the transport map.

**Claim 2.** *We have*

$$(B.31) \quad \begin{aligned} \mathcal{I}(\tilde{\mu}_t) &= \mathcal{I}_V(\mu_V) + t \int \xi d\mu_V \\ &\quad + \frac{t^2}{2} \left( \iint \left( \frac{\psi(x) - \psi(y)}{x - y} \right)^2 d\mu_V(x) d\mu_V(y) + \int V'' \psi^2 d\mu_V + 2 \int \xi' \psi d\mu_V \right) \\ &\quad + O(t^3 \|\xi\|_{C^2(U)}). \end{aligned}$$

*Proof.* We may write

$$\begin{aligned} \mathcal{I}(\tilde{\mu}_t) &= - \int \log |\phi_t(x) - \phi_t(y)| d\mu_0(x) d\mu_0(y) + \int V \circ \phi_t d\mu_0 + t \int \xi \circ \phi_t d\mu_0 \\ &= \int h^{\mu_0} d\mu_0 - \iint \log \left| 1 + t \frac{\psi(x) - \psi(y)}{x - y} \right| d\mu_0(x) d\mu_0(y) + \int V \circ \phi_t d\mu_0 + t \int \xi \circ \phi_t d\mu_0. \end{aligned}$$

By a Taylor expansion, we obtain

$$\begin{aligned} \mathcal{I}(\tilde{\mu}_t) &= \mathcal{I}_V(\mu_0) - t \iint \frac{\psi(x) - \psi(y)}{x - y} d\mu_0(x) d\mu_0(y) + \frac{t^2}{2} \iint \left( \frac{\psi(x) - \psi(y)}{x - y} \right)^2 d\mu_0(x) d\mu_0(y) \\ &\quad + t \int V' \psi d\mu_V + \frac{t^2}{2} \int V'' \psi^2 d\mu_V + t \int \xi d\mu_V + t^2 \int \xi' \psi d\mu_0 + O(t^3 \|\xi\|_{C^2(\mathbb{R})}). \end{aligned}$$

Let us recall that by definition  $\mu_0 = \mu_V$ . By (B.14) we have

$$\iint \frac{\psi(x) - \psi(y)}{x - y} d\mu_V(x) d\mu_V(y) = \int V' \psi d\mu_V,$$

hence we obtain (B.31).  $\square$

To conclude the proof of Lemma 4.7 it remains to prove the following identity.

**Claim 3.**

$$(B.32) \quad \int \xi' \psi d\mu_V = - \iint \left( \frac{\psi(x) - \psi(y)}{x - y} \right)^2 d\mu_V(x) d\mu_V(y) - \int V'' \psi^2 d\mu_V.$$

*Proof.* By definition of  $\psi$  we have

$$\frac{1}{2}(\xi + c_\xi) = \int \frac{\psi(x) - \psi(y)}{x - y} d\mu_V(y) - \frac{1}{2} \psi V' ,$$

and thus

$$\xi' = 2 \int \frac{\psi(y) - \psi(x) - \psi'(x)(y - x)}{(x - y)^2} d\mu_V(y) - \psi' V' - \psi V'' .$$

Integrating both sides against  $\psi \mu_V$  yields

$$\begin{aligned} \int \xi' \psi d\mu_V = 2 \iint \frac{(\psi(y) - \psi(x) - \psi'(x)(y - x))\psi(x)}{(x - y)^2} d\mu_V(y) d\mu_V(x) \\ - \int \psi \psi' V' d\mu_V - \int V'' \psi^2 d\mu_V. \end{aligned}$$

Using (B.14) for the second term we obtain

$$\begin{aligned} \int \xi' \psi d\mu_V = 2 \iint \frac{(\psi(y) - \psi(x) - \psi'(x)(y - x))\psi(x)}{(y - x)^2} d\mu_V(y) d\mu_V(x) \\ - \iint \frac{\psi \psi'(y) - \psi \psi'(x)}{y - x} d\mu_V(x) d\mu_V(y) - \int V'' \psi^2 d\mu_V. \end{aligned}$$

We may then combine the first two terms in the right-hand side to obtain (B.32).  $\square$

## REFERENCES

- [AKM17] S. Armstrong, T. Kuusi, and J.-C. Mourrat. Quantitative stochastic homogenization and large-scale regularity. <https://arxiv.org/abs/1705.05300>, 2017.
- [BdMPS95] A. Boutet de Monvel, L. Pastur, and M. Shcherbina. On the statistical mechanics approach in the random matrix theory: integrated density of states. *Journal of statistical physics*, 79(3):585–611, 1995.
- [BFG13] F. Bekerman, A. Figalli, and A. Guionnet. Transport maps for  $\beta$ -matrix models and universality. *Communications in Mathematical Physics*, 338(2):589–619, 2013.
- [BG13a] G. Borot and A. Guionnet. Asymptotic expansion of  $\beta$  matrix models in the multi-cut regime. *arXiv preprint arXiv:1303.1045*, 2013.
- [BG13b] G. Borot and A. Guionnet. Asymptotic expansion of  $\beta$  matrix models in the one-cut regime. *Communications in Mathematical Physics*, 317(2):447–483, 2013.
- [BL16] F. Bekerman and A. Lodhia. Mesoscopic central limit theorem for general  $\beta$ -ensembles. *arXiv preprint arXiv:1605.05206*, 2016.
- [CK06] T. Claeys and A.B.J. Kuijlaars. Universality of the double scaling limit in random matrix models. *Communications on Pure and Applied Mathematics*, 59(11):1573–1603, 2006.
- [CKI10] T. Claeys, I. Krasovsky, and A. Its. Higher-order analogues of the Tracy-Widom distribution and the Painlevé ii hierarchy. *Communications on pure and applied mathematics*, 63(3):362–412, 2010.
- [Cla08] T. Claeys. Birth of a cut in unitary random matrix ensembles. *International Mathematics Research Notices*, 2008:rnm166, 2008.
- [DKM98] P. Deift, T. Kriecherbauer, and K. T.-R. McLaughlin. New results on the equilibrium measure for logarithmic potentials in the presence of an external field. *Journal of approximation theory*, 95(3):388–475, 1998.



- [DKM<sup>+</sup>99] P. Deift, T. Kriecherbauer, K. T.-R. McLaughlin, S. Venakides, and X. Zhou. Uniform asymptotics for polynomials orthogonal with respect to varying exponential weights and applications to universality questions in random matrix theory. *Communications on Pure and Applied Mathematics*, 52(11):1335–1425, 1999.
- [For10] P. Forrester. *Log-gases and random matrices*. Princeton University Press, 2010.
- [GMS07] A. Guionnet and E. Maurel-Segala. Second order asymptotics for matrix models. *The Annals of Probability*, 35(6):2160–2212, 2007.
- [GS14] A. Guionnet and D. Shlyakhtenko. Free monotone transport. *Inventiones mathematicae*, 197(3):613–661, 2014.
- [Joh98] K. Johansson. On fluctuations of eigenvalues of random Hermitian matrices. *Duke Mathematical Journal*, 91(1):151–204, 1998.
- [KM00] A. Kuijlaars and K. McLaughlin. Generic behavior of the density of states in random matrix theory and equilibrium problems in the presence of real analytic external fields. *Communications on Pure and Applied Mathematics*, 53(6):736–785, 2000.
- [LS15] T. Leblé and S. Serfaty. Large deviation principle for empirical fields of Log and Riesz gases. *To appear in Inventiones Mathematicae*, 2015.
- [LS16] T. Leblé and S. Serfaty. Fluctuations of two-dimensional Coulomb gases. *arXiv preprint arXiv:1609.08088*, 2016.
- [Mo08] M.Y. Mo. The Riemann–Hilbert approach to double scaling limit of random matrix eigenvalues near the “birth of a cut” transition. *International Mathematics Research Notices*, 2008.
- [Mus92] N. I. Muskhelishvili. *Singular integral equations*. Dover Publications, Inc., New York, 1992.
- [PS14] M. Petrache and S. Serfaty. Next order asymptotics and renormalized energy for Riesz interactions. *Journal of the Institute of Mathematics of Jussieu*, pages 1–69, 2014.
- [Shc13] M. Shcherbina. Fluctuations of linear eigenvalue statistics of  $\beta$  matrix models in the multi-cut regime. *Journal of Statistical Physics*, 151(6):1004–1034, 2013.
- [Shc14] M. Shcherbina. Change of variables as a method to study general  $\beta$ -models: bulk universality. *Journal of Mathematical Physics*, 55(4):043504, 2014.
- [SS15] É. Sandier and S. Serfaty. 1D log gases and the renormalized energy: crystallization at vanishing temperature. *Probability Theory and Related Fields*, 162(3-4):795–846, 2015.
- [ST13] E.B. Saff and V. Totik. *Logarithmic potentials with external fields*, volume 316. Springer Science & Business Media, 2013.

(F. Berman) DEPARTMENT OF MATHEMATICS, MASSACHUSETTS INSTITUTE OF TECHNOLOGY, 77 MASSACHUSETTS AVE, CAMBRIDGE, MA 02139-4307 USA.

*E-mail address:* `bekerman@mit.edu`

(T. Leblé) COURANT INSTITUTE, NEW YORK UNIVERSITY, 251 MERCER ST, NEW YORK, NY 10012, USA.

*E-mail address:* `thomasl@cims.nyu.edu`

(S. Serfaty) COURANT INSTITUTE, NEW YORK UNIVERSITY, 251 MERCER ST, NEW YORK, NY 10012, USA.

*E-mail address:* `serfaty@cims.nyu.edu`