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Wave finite element method for waveguides and periodic structures subjected to arbitrary loads

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Abstract

The wave finite element method has been developed for waveguides and periodic structures with advantages in the calculation time. However, this method cannot be applied easily if the structure is subjected to complex or density loads and this is the aim of this article. Based on the finite element method, the dynamic equation of one period of the structure is rewritten to obtain a relation between the responses (DOF and nodal loads) on the left and right boundaries. This relation presents an additive term which links to the loads applying on the period. Then, by using WFE technique, we can compute the wave mode of the period and the wave decomposition. Because of the periodicity, we can also obtain a relation between the response and the left and right ends of the structure. Afterwards, the response of the structure is calculated by using the wave decomposition to apply in the dynamic stiffness matrix (DSM) approach or the wave analysis (WA). For the DSM approach, this technique shows that the external loads have no contribution to the global matrix but they lead to a equivalent force in the dynamic equation. Meanwhile, the external loads create waves propagating to the left and right of the structure in the WA approach.

Keywords: Wave Finite Element, Dynamic, Vibration, Periodic structure, Waveguide

1. Introduction

The dynamic response of a periodic structure is a subject of numerous publications. Beside the classical finite element method, some numerical methods has been developed in order to reduce the calculation time. One of them is the wave finite element method (WFE), which has been developed originally in the seventies in order to describe the wave propagation along one-dimensional periodic elastic structures [1]. Afterwards, this method has been developed for more complex periodic structures [2–9]. Based on the periodicity, the wave mode is calculated from the dynamic stiffness matrix of one period of the structure, provided that it is not submitted to any external load [2, 10]. Then, the response is decomposed in order to compute the wave amplitudes by the dynamic stiffness matrix (DMS) or the wave analysis (WA) approaches. For a structure subjected to external loads, this method can be applied by considering with different parts as superelements [11, 12]. Another technique of WFE has been developed to deal with the loads by using the Fourier transform and the contour integration. Based on the solution of an infinite waveguide to a concentrated load, [13] the wave amplitudes of the response have been represented in terms of the wave amplitude of the loads yielded from the integration. This technique is efficient for waveguides and simple periodic structures [14–16]. However, it cannot be applied easily for periodic structures subjected to external loads at inner nodes of each period, which come from complex shapes. In addition, the integration step makes

23 more calculations.

24 In this article, we present a new development of WFE for periodic struc-
25 tures of any shape and load. To do that, the dynamic equation of one period is
26 rewritten by taking into account the loads at the inner and boundary loads.
27 Then, by using the same technique of the classical WFE, we can obtain a
28 recurrent relation between the responses (DOF and nodal loads) on the left
29 and right boundary of the period. The external loads appear naturally as a
30 additive term in this relation which is missing on the classical WFE. Then,
31 we use the wave basis of WFE to decompose the vector of responses and the
32 additive term. Finally, the responses of the global structure are assembled
33 in the both approaches: DMS and WA. The results show that the external
34 loads present global equivalent loads in the DMS approach or an addition
35 wave amplitudes in the WA analysis.

36 2. Basic framework

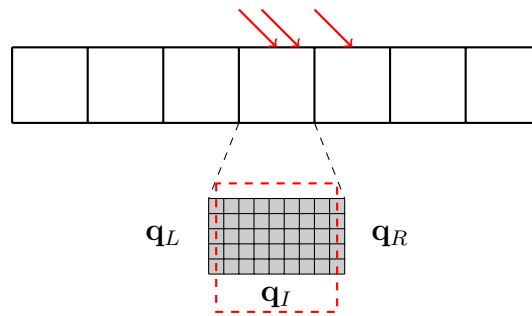


Figure 1: A periodic structure subjected to external loads

37 Let's consider a periodic structure as shown in Figure 1. Each period of
38 the structure contains the left (L), right (R) boundaries and inner (I) nodes.

39 By using the finite element method, the dynamic equation of the period can
 40 be written in matrix form

$$\tilde{\mathbf{D}}\mathbf{q} = \mathbf{F} \quad (1)$$

41 with $\tilde{\mathbf{D}} = \mathbf{K} + j\omega\mathbf{C} - \omega^2\mathbf{M}$ is the dynamic stiffness matrix, and \mathbf{q} , \mathbf{F} are the
 42 vectors of degrees of freedom (DOF) and nodal loads. We can rewrite the
 43 dynamic equation to separate the boundaries and inner DOF as follows

$$\begin{bmatrix} \tilde{\mathbf{D}}_{II} & \tilde{\mathbf{D}}_{IL} & \tilde{\mathbf{D}}_{IR} \\ \tilde{\mathbf{D}}_{LI} & \tilde{\mathbf{D}}_{LL} & \tilde{\mathbf{D}}_{LR} \\ \tilde{\mathbf{D}}_{RI} & \tilde{\mathbf{D}}_{RL} & \tilde{\mathbf{D}}_{RR} \end{bmatrix} \begin{bmatrix} \mathbf{q}_I \\ \mathbf{q}_L \\ \mathbf{q}_R \end{bmatrix} = \begin{bmatrix} \mathbf{F}_I \\ \mathbf{F}_L \\ \mathbf{F}_R \end{bmatrix} \quad (2)$$

44 We can rewrite the first row of equation (2) as follows

$$\mathbf{q}_I = \tilde{\mathbf{D}}_{II}^{-1} \left[\mathbf{F}_I - \tilde{\mathbf{D}}_{IL} \mathbf{q}_L - \tilde{\mathbf{D}}_{IR} \mathbf{q}_R \right] \quad (3)$$

45 Then, by substituting the aforementioned equation into the second and the
 46 third rows of equation (2), we obtain

$$\begin{aligned} \tilde{\mathbf{D}}_{LI}\tilde{\mathbf{D}}_{II}^{-1} \left[\mathbf{F}_I - \tilde{\mathbf{D}}_{IL} \mathbf{q}_L - \tilde{\mathbf{D}}_{IR} \mathbf{q}_R \right] + \tilde{\mathbf{D}}_{LL} \mathbf{q}_L + \tilde{\mathbf{D}}_{LR} \mathbf{q}_R &= \mathbf{F}_L \\ \tilde{\mathbf{D}}_{RI}\tilde{\mathbf{D}}_{II}^{-1} \left[\mathbf{F}_I - \tilde{\mathbf{D}}_{IL} \mathbf{q}_L - \tilde{\mathbf{D}}_{IR} \mathbf{q}_R \right] + \tilde{\mathbf{D}}_{RL} \mathbf{q}_L + \tilde{\mathbf{D}}_{RR} \mathbf{q}_R &= \mathbf{F}_R \end{aligned} \quad (4)$$

47 In **the other hand**, we can write

$$\begin{bmatrix} \mathbf{D}_{LI}\mathbf{F}_I \\ \mathbf{D}_{RI}\mathbf{F}_I \end{bmatrix} + \begin{bmatrix} \mathbf{D}_{LL} & \mathbf{D}_{LR} \\ \mathbf{D}_{RL} & \mathbf{D}_{RR} \end{bmatrix} \begin{bmatrix} \mathbf{q}_L \\ \mathbf{q}_R \end{bmatrix} = \begin{bmatrix} \mathbf{F}_L \\ \mathbf{F}_R \end{bmatrix} \quad (5)$$

48 where

$$\begin{aligned} \mathbf{D}_{LL} &= \tilde{\mathbf{D}}_{LL} - \tilde{\mathbf{D}}_{LI}\tilde{\mathbf{D}}_{II}^{-1}\tilde{\mathbf{D}}_{IL} & \mathbf{D}_{LR} &= \tilde{\mathbf{D}}_{LR} - \tilde{\mathbf{D}}_{LI}\tilde{\mathbf{D}}_{II}^{-1}\tilde{\mathbf{D}}_{IR} \\ \mathbf{D}_{RL} &= \tilde{\mathbf{D}}_{RL} - \tilde{\mathbf{D}}_{RI}\tilde{\mathbf{D}}_{II}^{-1}\tilde{\mathbf{D}}_{IL} & \mathbf{D}_{RR} &= \tilde{\mathbf{D}}_{RR} - \tilde{\mathbf{D}}_{RI}\tilde{\mathbf{D}}_{II}^{-1}\tilde{\mathbf{D}}_{IR} \\ \mathbf{D}_{LI} &= \tilde{\mathbf{D}}_{LI}\tilde{\mathbf{D}}_{II}^{-1} & \mathbf{D}_{RI} &= \tilde{\mathbf{D}}_{RI}\tilde{\mathbf{D}}_{II}^{-1} \end{aligned} \quad (6)$$

49 We see that equation (5) presents a relation between the responses (DOF
50 and nodal loads) at the left and right boundaries of a period. It contains
51 a term of \mathbf{F}_I which is zero when the period is free. For two consecutive
52 connected periods (n) and ($n + 1$), the right boundary of (n) is also the left
53 boundary of ($n + 1$). Thus, we have

$$\begin{aligned}\mathbf{q}_R^{(n)} &= \mathbf{q}_L^{(n+1)} \\ \mathbf{F}_R^{(n)} + \mathbf{F}_L^{(n+1)} &= -\mathbf{F}_{\partial R}^{(n)}\end{aligned}\tag{7}$$

54 where $\mathbf{F}_{\partial R}^{(n)}$ is the external nodal load at the right boundary R of the period
55 (n). By combining equations (5) and (7), we obtain

$$\begin{bmatrix} \mathbf{q}_L^{(n+1)} \\ -\mathbf{F}_L^{(n+1)} \end{bmatrix} = \begin{bmatrix} \mathbf{D}_{qI}\mathbf{F}_I^{(n)} \\ \mathbf{D}_{fI}\mathbf{F}_I^{(n)} + \mathbf{F}_{\partial R}^{(n)} \end{bmatrix} + \mathbf{S} \begin{bmatrix} \mathbf{q}_L^{(n)} \\ -\mathbf{F}_L^{(n)} \end{bmatrix}\tag{8}$$

56 where

$$\mathbf{S} = \begin{bmatrix} -\mathbf{D}_{LR}^{-1}\mathbf{D}_{LL} & -\mathbf{D}_{LR}^{-1} \\ \mathbf{D}_{RL} - \mathbf{D}_{RR}\mathbf{D}_{LR}^{-1}\mathbf{D}_{LL} & -\mathbf{D}_{RR}\mathbf{D}_{LR}^{-1} \end{bmatrix}\tag{9}$$

57 and

$$\begin{bmatrix} \mathbf{D}_{qI} \\ \mathbf{D}_{fI} \end{bmatrix} = \begin{bmatrix} -\mathbf{D}_{LR}^{-1}\mathbf{D}_{LI} \\ \mathbf{D}_{RI} - \mathbf{D}_{RR}\mathbf{D}_{LR}^{-1}\mathbf{D}_{LI} \end{bmatrix}\tag{10}$$

58 We can rewrite equation (8) as follows

$$\mathbf{u}^{(n+1)} = \mathbf{S}\mathbf{u}^{(n)} + \mathbf{b}^{(n)}\tag{11}$$

59 where

$$\mathbf{u}^{(n)} = \begin{bmatrix} \mathbf{q}_L^{(n)} \\ -\mathbf{F}_L^{(n)} \end{bmatrix}, \quad \mathbf{b}^{(n)} = \begin{bmatrix} \mathbf{D}_{qI}\mathbf{F}_I^{(n)} \\ \mathbf{D}_{fI}\mathbf{F}_I^{(n)} + \mathbf{F}_{\partial R}^{(n)} \end{bmatrix}\tag{12}$$

60 Equation (11) presents a relation between the periods (n) and $(n + 1)$. Here
 61 $\mathbf{b}^{(n)}$ presents the external loads at the inner nodes \mathbf{F}_I and at the right bound-
 62 ary $\mathbf{F}_{\partial R}$ of the period (n) . When the period is free, $\mathbf{b}^{(n)} = 0$ and we meet
 63 the expression of \mathbf{S} of the classical WFE.

64 In the other hand, equation (11) presents also a recurrent relation with
 65 regard to n . Therefore, we can develop a relation between $\mathbf{u}^{(n)}$ and $\mathbf{u}^{(1)}$ as
 66 follows

$$\mathbf{u}^{(n)} = \mathbf{S}^{n-1} \mathbf{u}^{(1)} + \sum_{k=1}^{n-1} \mathbf{S}^{n-k-1} \mathbf{b}^{(k)} \quad (13)$$

67 Similarly, we can get the relation between $\mathbf{u}^{(N+1)}$ and $\mathbf{u}^{(n)}$

$$\mathbf{u}^{(N+1)} = \mathbf{S}^{N-n+1} \mathbf{u}^{(n)} + \sum_{k=n}^N \mathbf{S}^{N-k} \mathbf{b}^{(k)} \quad (14)$$

68 Equations (13) and (14) are the relations between the period (n) and the
 69 left and the right ends of a structure of N periods. Next, we will develop
 70 these expressions with the help of the wave decomposition.

71 **Remark:** $\mathbf{F}_{\partial R}^{(n)}$ in equation (7) is the external nodal load at the right
 72 boundary but it is not included on the left end of the period. This will
 73 explain the different expressions for the left and right boundary in the DSM
 74 and WA approaches presented in section 3.

75 3. Wave decomposition

76 We are looking for wave modes $\{(\mu_j, \phi_j)\}$ which are the eigenvalues and
 77 eigenvectors of the matrix \mathbf{S} such that $\mathbf{S}\phi_j = \mu_j\phi_j$. Due to the symplectic
 78 nature [11] of the matrix \mathbf{S} , its eigenvalues come in pairs as $(\mu_j, 1/\mu_j)$. Then,

79 the eigenvectors are separated to ϕ_i for the eigenvalue less than 1 and ϕ_i^* for
80 the rest. We call ϕ_i the left (to right) wave and ϕ_i^* the right (to left) wave. **If**
81 **we note $\Phi = [\phi_1 \cdots \phi_n]$ and $\Phi^* = [\phi_1^* \cdots \phi_n^*]$, we call $\{\Phi, \Phi^*\}$ the wave mode**
82 **of the transformation \mathbf{S} . We can also separate the components of the wave**
83 **mode corresponding to \mathbf{q} and \mathbf{F} as follows**

$$\Phi = \begin{bmatrix} \Phi_q \\ \Phi_F \end{bmatrix} \quad \Phi^* = \begin{bmatrix} \Phi_q^* \\ \Phi_F^* \end{bmatrix} \quad (15)$$

84 **It is important to note that the wave mode is symplectic orthogonal basis.**
85 **We can normalize this basis** by considering the weighting matrix given by

$$\begin{aligned} \Psi &= \Phi^{*T} \mathbf{J} \Phi = \Phi_q^{*T} \Phi_F - \Phi_F^{*T} \Phi_q \\ \Psi^* &= \Phi^T \mathbf{J} \Phi^* = \Phi_q^T \Phi_F^* - \Phi_F^T \Phi_q^* \end{aligned} \quad (16)$$

86 where

$$\mathbf{J} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{I} & \mathbf{0} \end{bmatrix} \quad (17)$$

87 We can verify that $\Psi^* = -\Psi^T$ and they are diagonal matrices. Therefore,
88 we can define a normalized basis by multiplying on the right side with $\Psi^{-1/2}$
89 (e.i. the basis is given by $\Phi \Psi^{-1/2}$ and $\Phi^* \Psi^{-1/2}$).

90 Now we can decompose each vector of equation (11) in **this normalized**
91 **basis** as follows

$$\begin{aligned} \mathbf{u}^{(n)} &= \Phi \mathbf{Q}^{(n)} - \Phi^* \mathbf{Q}^{*(n)} \\ \mathbf{b}^{(n)} &= \Phi \mathbf{Q}_B^{(n)} - \Phi^* \mathbf{Q}_B^{*(n)} \end{aligned} \quad (18)$$

92 It is remarkable that the sign minus in these decompositions yields a more
93 convenient expression of the wave amplitudes as follows

$$\mathbf{Q}^{(n)} = \Phi^{*T} \mathbf{J} \mathbf{u}^{(n)}, \quad \mathbf{Q}^{*(n)} = \Phi^T \mathbf{J} \mathbf{u}^{(n)} \quad (19)$$

$$\mathbf{Q}_B^{(n)} = \Phi^{*T} \mathbf{J} \mathbf{b}^{(n)}, \quad \mathbf{Q}_B^{*(n)} = \Phi^T \mathbf{J} \mathbf{b}^{(n)} \quad (20)$$

94 In the other hand, from equation (12) we obtain

$$\begin{aligned}\Phi^{*T} \mathbf{J} \mathbf{b}^{(n)} &= (\Phi_q^{*T} \mathbf{D}_{fI} - \Phi_F^{*T} \mathbf{D}_{qI}) \mathbf{F}_I^{(n)} + \Phi_q^{*T} \mathbf{F}_{\partial R}^{(n)} \\ \Phi^T \mathbf{J} \mathbf{b}^{(n)} &= (\Phi_q^T \mathbf{D}_{fI} - \Phi_F^T \mathbf{D}_{qI}) \mathbf{F}_I^{(n)} + \Phi_q^T \mathbf{F}_{\partial R}^{(n)}\end{aligned}\quad (21)$$

95 Thereafter, by substituting \mathbf{D}_{fI} and \mathbf{D}_{qI} in equation (10) into the afore-
96 mentioned equation and then combining the results with equation (20), we
97 obtain

$$\begin{aligned}\mathbf{Q}_B^{(n)} &= [(\Phi_F^{*T} - \Phi_q^{*T} \mathbf{D}_{RR}) \mathbf{D}_{LR}^{-1} \mathbf{D}_{LI} + \Phi_q^{*T} \mathbf{D}_{RI}] \mathbf{F}_I^{(n)} + \Phi_q^{*T} \mathbf{F}_{\partial R}^{(n)} \\ \mathbf{Q}_B^{*(n)} &= [(\Phi_F^T - \Phi_q^T \mathbf{D}_{RR}) \mathbf{D}_{LR}^{-1} \mathbf{D}_{LI} + \Phi_q^T \mathbf{D}_{RI}] \mathbf{F}_I^{(n)} + \Phi_q^T \mathbf{F}_{\partial R}^{(n)}\end{aligned}\quad (22)$$

98 We have also a relation between the components of the wave basis (see [10])

$$\begin{aligned}\Phi_F &= \mathbf{D}_{RR} \Phi_q + \mathbf{D}_{RL} \Phi_q \boldsymbol{\mu}^* \\ \Phi_F^* &= \mathbf{D}_{RR} \Phi_q^* + \mathbf{D}_{RL} \Phi_q^* \boldsymbol{\mu}\end{aligned}\quad (23)$$

99 By substituting the aforementioned equation into equation (22), we obtain

$$\begin{aligned}\mathbf{Q}_B^{(n)} &= (\boldsymbol{\mu} \Phi_q^{*T} \mathbf{D}_{LI} + \Phi_q^{*T} \mathbf{D}_{RI}) \mathbf{F}_I^{(n)} + \Phi_q^{*T} \mathbf{F}_{\partial R}^{(n)} \\ \mathbf{Q}_B^{*(n)} &= (\boldsymbol{\mu}^* \Phi_q^T \mathbf{D}_{LI} + \Phi_q^T \mathbf{D}_{RI}) \mathbf{F}_I^{(n)} + \Phi_q^T \mathbf{F}_{\partial R}^{(n)}\end{aligned}\quad (24)$$

100 We see that the term \mathbf{b}^n in equation (11), which links to the external loads
101 on each period can be decomposed in the wave basis with the amplitudes
102 calculated by equation (24). Now we will build the relation of these wave
103 amplitudes with the amplitude of the response $\mathbf{u}^{(n)}$.

104 By substituting equation (18) into equations (13), we obtain

$$\Phi \mathbf{Q}^{(n)} - \Phi^* \mathbf{Q}^{*(n)} = \mathbf{S}^{n-1} (\Phi \mathbf{Q}^{(1)} - \Phi^* \mathbf{Q}^{*(1)}) + \sum_{k=1}^{n-1} \mathbf{S}^{n-k-1} (\Phi \mathbf{Q}_B^{(n)} - \Phi^* \mathbf{Q}_B^{*(n)})\quad (25)$$

105 In addition, by definition we have $\mathbf{S}^k \Phi = \Phi \mu^k$ and $\mathbf{S}^k \Phi^* = \Phi^* \mu^{*k}$. Thus, by
 106 multiplying the aforementioned equation with $\Phi^{*T} \mathbf{J}$ on the left side and then
 107 combining with equation (19), we obtain

$$\mathbf{Q}^{(n)} = \mu^{n-1} \mathbf{Q} + \sum_{k=1}^{n-1} \mu^{n-k-1} \mathbf{Q}_B^{(k)} \quad (26)$$

108 where $\mathbf{Q} = \mathbf{Q}^{(1)}$.

109 In a similar way for equation (14), we obtain the following result

$$\mathbf{Q}^{(N+1)} = \mu^{*N-n+1} \mathbf{Q}^{*(n)} + \sum_{k=n}^N \mu^{*N-k} \mathbf{Q}_B^{*(k)} \quad (27)$$

110 In addition, $\mu^* = \mu^{-1}$. Thus, we have

$$\mathbf{Q}^{*(n)} = \mu^{N+1-n} \mathbf{Q}^* - \sum_{k=n}^N \mu^{k+1-n} \mathbf{Q}_B^{*(k)} \quad (28)$$

111 where $\mathbf{Q}^* = \mathbf{Q}^{*(N+1)}$.

112 We see that the two wave amplitudes $\mathbf{Q}^{(n)}$ and $\mathbf{Q}^{*(n)}$ depend on the
 113 wave amplitudes of the first and the last periods (\mathbf{Q} and \mathbf{Q}^*) and the wave
 114 amplitudes of the external loads $\mathbf{Q}_B^{(k)}$ and $\mathbf{Q}_B^{*(k)}$ which are calculated by
 115 equation (24). Next, we will use these relations to obtain the responses by
 116 using two different approaches.

117 4. Analysis of a complete structure

118 4.1. DSM approach

119 **The principle of the DSM approach is to establish** the relation between
 120 the DOFs and nodal loads of the left end $\mathbf{u}^{(1)}$ and the right ends $\mathbf{u}^{(N+1)}$ by
 121 using the wave decomposition. This relation is also yielded from the stiffness

122 matrix of the whole structure with the help of FEM and this is the reason of
 123 the name of the approach.

124 For the first and the last period of the structure, we obtain the following
 125 results from equations (26) and (28)

$$\begin{aligned}\mathbf{u}^{(N+1)} &= \Phi \boldsymbol{\mu}^N \mathbf{Q} - \Phi^* \mathbf{Q}^* + \Phi \sum_{k=1}^N \boldsymbol{\mu}^{N-k} \mathbf{Q}_B^{(k)} \\ \mathbf{u}^{(1)} &= \Phi \mathbf{Q} - \Phi^* \boldsymbol{\mu}^N \mathbf{Q}^* + \Phi^* \sum_{k=1}^N \boldsymbol{\mu}^k \mathbf{Q}_B^{*(k)}\end{aligned}\quad (29)$$

126 Then by multiplying the aforementioned equation with $\Phi^{*T} \mathbf{J}$ for the first line
 127 and $\Phi^T \mathbf{J}$ for the second line, we obtain

$$\begin{aligned}\Phi^{*T} \mathbf{J} \mathbf{u}^{(N+1)} &= \boldsymbol{\mu}^N \mathbf{Q} + \sum_{k=1}^N \boldsymbol{\mu}^{N-k} \mathbf{Q}_B^{(k)} \\ \Phi^T \mathbf{J} \mathbf{u}^{(1)} &= \boldsymbol{\mu}^N \mathbf{Q}^* - \sum_{k=1}^N \boldsymbol{\mu}^k \mathbf{Q}_B^{*(k)}\end{aligned}\quad (30)$$

128 In addition, we have $\mathbf{Q} = \Phi^{*T} \mathbf{J} \mathbf{u}^{(1)}$ and $\mathbf{Q}^* = \Phi^T \mathbf{J} \mathbf{u}^{(N+1)}$. Thus, by using
 129 the definition of $\mathbf{u}^{(1)}$ and $\mathbf{u}^{(N+1)}$ in equation (12), we obtain

$$\begin{aligned}\boldsymbol{\mu}^N \Phi_q^{*T} \mathbf{F}_L^{(1)} - \Phi_q^{*T} \mathbf{F}_L^{(N+1)} &= -\boldsymbol{\mu}^N \Phi_F^{*T} \mathbf{q}_L^{(1)} + \Phi_F^{*T} \mathbf{q}_L^{(N+1)} + \sum_{k=1}^N \boldsymbol{\mu}^{N-k} \mathbf{Q}_B^{(k)} \\ -\Phi_q^T \mathbf{F}_L^{(1)} + \boldsymbol{\mu}^N \Phi_q^T \mathbf{F}_L^{(N+1)} &= \Phi_F^T \mathbf{q}_L^{(1)} - \boldsymbol{\mu}^N \Phi_F^T \mathbf{q}_L^{(N+1)} - \sum_{k=1}^N \boldsymbol{\mu}^k \mathbf{Q}_B^{*(k)}\end{aligned}\quad (31)$$

130 Otherwise, we have $\mathbf{F}_L^{(N+1)} = -\mathbf{F}_R^{(N)}$ and $\mathbf{q}_L^{(N+1)} = \mathbf{q}_R^{(N)}$. Therefore, we
 131 can rewrite the aforementioned equation to obtain the following result (see
 132 Appendix A).

$$\begin{bmatrix} \mathbf{F}_L^{(1)} \\ \mathbf{F}_R^{(N)} \end{bmatrix} = \mathbf{D}_T \begin{bmatrix} \mathbf{q}_L^{(1)} \\ \mathbf{q}_R^{(N)} \end{bmatrix} + \mathbf{F}_T \quad (32)$$

133 where $\mathbf{D}_T, \mathbf{F}_T$ are the reduced dynamic stiffness and the external loads, which
 134 are calculated by equations (A.4) and (A.5).

135 Equation (32) is the relation between the nodal loads and DOF at the
 136 left and the right ends of the structure. When $\mathbf{F}_T = 0$ (which means $\mathbf{F}_I^{(k)} =$
 137 $\mathbf{F}_B^{(k)} = 0$), we obtain the same relation for the free loaded structure [2]. We
 138 can combine this relation with the DSM of the rest of the structure to get
 139 the reduced DSM of the whole structure or apply the boundary conditions
 140 on the left and the right ends of the structure to calculate the response.

141 4.2. WA approach

142 The WA approach seeks to calculate the response via the wave amplitudes
 143 from the boundary condition at the right and left ends of the structure.
 144 Indeed, by combining equation (18) and (26), we obtain

$$\begin{aligned}
 \mathbf{q}_L^{(n)} &= \Phi_q \mu^{n-1} \mathbf{Q} - \Phi_q^* \mu^{N+1-n} \mathbf{Q}^* + \Phi_q \sum_{k=1}^{n-1} \mu^{n-k-1} \mathbf{Q}_B^{(k)} + \Phi_q^* \sum_{k=n}^N \mu^{k+1-n} \mathbf{Q}_B^{*(k)} \\
 -\mathbf{F}_L^{(n)} &= \Phi_F \mu^{n-1} \mathbf{Q} - \Phi_F^* \mu^{N+1-n} \mathbf{Q}^* + \Phi_F \sum_{k=1}^{n-1} \mu^{n-k-1} \mathbf{Q}_B^{(k)} + \Phi_F^* \sum_{k=n}^N \mu^{k+1-n} \mathbf{Q}_B^{*(k)}
 \end{aligned} \tag{33}$$

145 The aforementioned equation presents the responses in terms of wave am-
 146 plitudes $\{\mathbf{Q}, \mathbf{Q}^*\}$. We note that $\mathbf{Q}_B^{(k)}, \mathbf{Q}_B^{*(k)}$ are calculated from the external
 147 loads by equation (23). Therefore, if the boundary conditions are given on
 148 $\mathbf{u}^{(1)}$ and $\mathbf{u}^{(N+1)}$, we can compute directly $\{\mathbf{Q}, \mathbf{Q}^*\}$ from these conditions.

149 Otherwise, if the periodic structure is linked to the other parts of the
 150 structure, we can combine this wave decomposition with the dynamic equa-
 151 tion of these parts. From the dynamic stiffness matrix at the left and right

152 ends, we can write (see Appendix B)

$$-\mathbf{F}_L^{(1)} = \mathbb{D}\mathbf{q}_L^{(1)} + \mathbb{D}_q\mathbf{q}_0 + \mathbb{D}_F\mathbf{F}_0 + \mathbf{F}_{\partial R}^{(0)} \quad (34)$$

$$\mathbf{F}_L^{(N+1)} = \mathbb{D}^*\mathbf{q}_L^{(N+1)} + \mathbb{D}_q^*\mathbf{q}_0^* + \mathbb{D}_F^*\mathbf{F}_0^* \quad (35)$$

153 where $\mathbf{F}_{\partial R}^{(0)}$ is the external loads at the left end of the periodic structure. We
 154 note that the second line of equation (35) does not have the equivalent term
 155 because it is already taken into account at $\mathbf{b}^{(N)}$ defined in equation (12). By
 156 substituting equation (33) with $n = 1$ and $n = N + 1$ into the aforementioned
 157 equations, we obtain

$$\begin{aligned} (\Phi_F - \mathbb{D}\Phi_q) \mathbf{Q} &= (\Phi_F^* - \mathbb{D}\Phi_q^*) \left(\mu^N \mathbf{Q}^* - \sum_{k=1}^N \Phi_q^* \mu^k \mathbf{Q}_B^{*(k)} \right) + \mathbb{D}_q\mathbf{q}_0 + \mathbb{D}_F\mathbf{F}_0 + \mathbf{F}_{\partial R}^{(0)} \\ (\Phi_F^* + \mathbb{D}^*\Phi_q^*) \mathbf{Q}^* &= (\Phi_F^* + \mathbb{D}^*\Phi_q^*) \left(\mu^N \mathbf{Q} + \sum_{k=1}^N \Phi_q \mu^k \mathbf{Q}_B^{(k)} \right) + \mathbb{D}_q^*\mathbf{q}_0^* + \mathbb{D}_F^*\mathbf{F}_0^* \end{aligned} \quad (36)$$

158 Then we can rewrite equation (36) as follows

$$\begin{aligned} \mathbf{Q} &= \mathbb{C} \left[\mu^N \mathbf{Q}^* - \sum_{k=1}^N \mu^k \mathbf{Q}_B^{*(k)} \right] + \mathbb{F}, \\ \mathbf{Q}^* &= \mathbb{C}^* \left[\mu^N \mathbf{Q} + \sum_{k=1}^N \mu^{N-k} \mathbf{Q}_B^{(k)} \right] + \mathbb{F}^* \end{aligned} \quad (37)$$

159 where

$$\begin{aligned} \mathbb{C} &= -[\mathbb{D}\Phi_q - \Phi_F]^{-1} [\mathbb{D}\Phi_q^* - \Phi_F^*] \\ \mathbb{F} &= -[\mathbb{D}\Phi_q - \Phi_F]^{-1} [\mathbb{D}_q\mathbf{q}_0 + \mathbb{D}_F\mathbf{F}_0 + \mathbf{F}_{\partial R}^{(0)}] \\ \mathbb{C}^* &= -[\mathbb{D}^*\Phi_q^* + \Phi_F^*]^{-1} [\mathbb{D}^*\Phi_q + \Phi_F] \\ \mathbb{F}^* &= -[\mathbb{D}^*\Phi_q + \Phi_F]^{-1} [\mathbb{D}_q^*\mathbf{q}_0^* + \mathbb{D}_F^*\mathbf{F}_0^*] \end{aligned} \quad (38)$$

160 Finally, we deduce from equation (37) to obtain

$$\begin{bmatrix} \mathbf{I}_n & -\mathbf{C}\boldsymbol{\mu}^N \\ -\mathbf{C}^*\boldsymbol{\mu}^N & \mathbf{I}_n \end{bmatrix} \begin{bmatrix} \mathbf{Q} \\ \mathbf{Q}^* \end{bmatrix} = \begin{bmatrix} \mathbb{F} \\ \mathbb{F}^* \end{bmatrix} + \sum_{k=1}^N \begin{bmatrix} \mathbf{C}\boldsymbol{\mu}^k \mathbf{Q}_B^{*(k)} \\ \mathbf{C}^*\boldsymbol{\mu}^{N-k} \mathbf{Q}_B^{(k)} \end{bmatrix} \quad (39)$$

161 Equation (39) permits to calculate the wave amplitudes $\{\mathbf{Q}, \mathbf{Q}^*\}$ from
 162 the boundary conditions and external loads of the whole structure. Then the
 163 response is calculated by using the wave decomposition in equation (33).

164 5. Examples

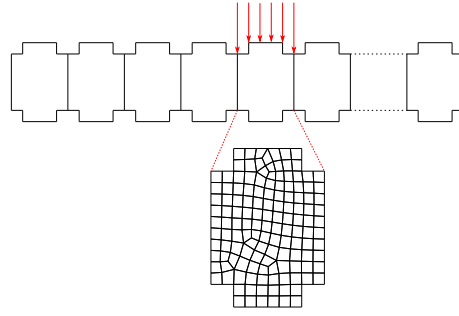


Figure 2: Fixed-free periodic beam subjected to dynamic loads

165 Let's consider a beam of width 0.1m, thickness varied between 0.1-0.14m
 166 and length of 2m as shown in Figure 2. The material parameter is given
 167 by the Young's modulus of 30GPa and the mass density of 2200kg/m³. The
 168 beam is fixed at the left boundary and it is subjected to a pressure in an
 169 interval between 0.3m and 0.4m . In this structure, the boundary conditions
 170 are the following

$$\mathbf{q}^{(1)} = 0, \quad \mathbf{F}^{(N+1)} = 0 \quad (40)$$

171 We use the DMS approach presented in section 3 to calculate the beam
 172 responses. The beam is made of 20 substructures which are squares of di-

173 mension $a = 0.1\text{m}$ and the same thickness. With the external load at the
 174 third substructure, we have $\mathbf{Q}_B^{(k)} = \mathbf{Q}_B^{*(k)} = 0, \forall k \neq 4$. By using the finite
 175 element method, we obtain the DMS of the substructure with the mesh as
 176 shown in Figure 2. Then, by using the WFE, we can combine equations (32)
 177 and (40), we obtain

$$\begin{bmatrix} \mathbf{F}_L^{(1)} \\ \mathbf{0} \end{bmatrix} = \mathbf{D}_T \begin{bmatrix} \mathbf{0} \\ \mathbf{q}_R^{(N+1)} \end{bmatrix} + \mathbf{F}_T \quad (41)$$

178 Thus, we can calculate $\mathbf{q}_R^{(N+1)}$ from the second line of the aforementioned
 equation and then the force $\mathbf{F}_L^{(1)}$ from the first row.

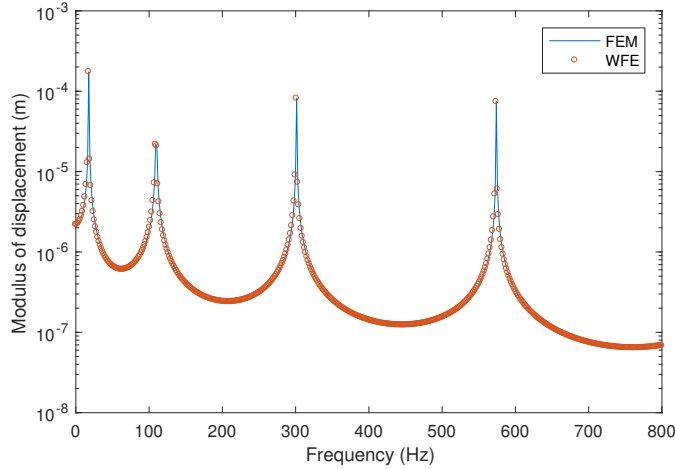


Figure 3: Response of the beam at the upper corner

179

180 Figure 3 shows the results obtained by WFE and the classical FEM for
 181 the displacement in the frequency domain at the upper right corner of the
 182 beam. We see that the WFE has the same quality as the FEM. Moreover,
 183 the calculation time for WFE is 14.0s while 120.2s for FEM, equivalent to
 184 88% of time reduction.

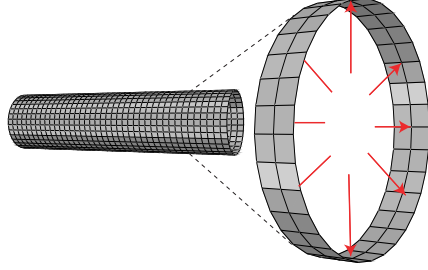


Figure 4: Fixed-fixed pipeline subjected to dynamic pressures

185 We take another example of a pipeline under pressure as shown in Figure
 186 4. The pipeline is a cylinder of radius 1m and thickness 5mm, made of steel
 187 with Young modulus 210GPa and Poisson coefficient of 0.3. The cylinder
 188 has two fixed ends and it is subjected to a dynamic pressure. We can use
 189 the wave analysis by substituting the boundary condition into equation (38)
 190 and obtain

$$\begin{aligned}
 \Phi_q \mathbf{Q} - \Phi_q^* \mu^N \mathbf{Q}^* + \Phi_q^* \sum_{k=1}^N \mu^k \mathbf{Q}_B^{*(k)} &= 0 \\
 \Phi_q \mu^N \mathbf{Q} - \Phi_q^* \mathbf{Q}^* + \Phi_q \sum_{k=1}^N \mu^{N-k} \mathbf{Q}_B^{(k)} &= 0
 \end{aligned} \tag{42}$$

191 Then we can calculate \mathbf{Q} and \mathbf{Q}^* from the aforementioned equation. The
 192 response is calculated then by equation (33).

193 Figure 5 presents a comparison of the results between the finite element
 194 method and the wave finite element method. The calculation time is reduced
 195 from 1277,7s with FEM to 332,8s with WFE, that means a reduction of 76%
 196 of calculation time.

197 6. Conclusions

198 This article presents a new technique of WFE for periodic structures

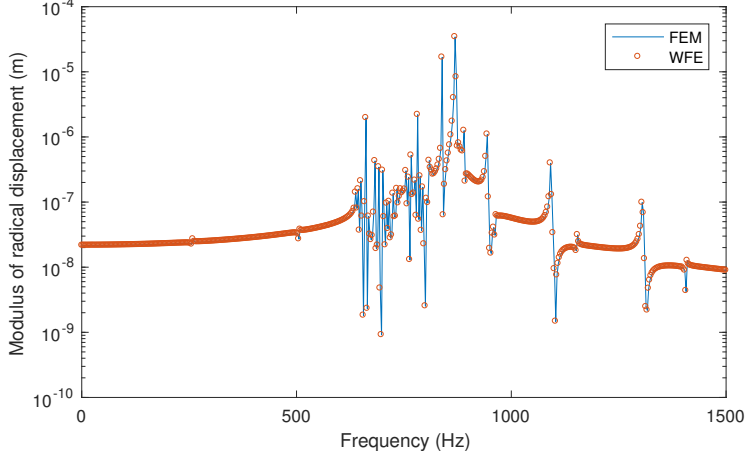


Figure 5: Response of the pipeline

199 subjected to external loads. By using the recurrent relation between the left
 200 and right boundaries of one period, we obtain a general formulation of the
 201 response in terms of loads. For the DMS approach, the external loads is
 202 represented by a global force in the reduced dynamic equation. For the WA
 203 approach, the loads on each period create the waves to the left and right sides
 204 of the structures. Moreover, the numerical application proves the efficiency
 205 of WFE in term of calculation time in comparing with classical FEM.

206 Appendix A. Calculation of DSM approach

207 We can rewrite equation (37) in the matrix form as follows

$$\begin{aligned}
 \begin{bmatrix} \boldsymbol{\mu}^N \boldsymbol{\Phi}_q^{*T} & \boldsymbol{\Phi}_q^{*T} \\ \boldsymbol{\Phi}_q^T & \boldsymbol{\mu}^N \boldsymbol{\Phi}_q^T \end{bmatrix} \begin{bmatrix} \mathbf{F}_L^{(1)} \\ \mathbf{F}_R^{(N)} \end{bmatrix} &= \begin{bmatrix} -\boldsymbol{\mu}^N \boldsymbol{\Phi}_F^{*T} & \boldsymbol{\Phi}_F^{*T} \\ \boldsymbol{\Phi}_F^T & -\boldsymbol{\mu}^N \boldsymbol{\Phi}_F^T \end{bmatrix} \begin{bmatrix} \mathbf{q}_L^{(1)} \\ \mathbf{q}_R^{(N)} \end{bmatrix} \\
 &+ \sum_{k=1}^N \begin{bmatrix} \boldsymbol{\mu}^{N-k} \mathbf{Q}_B^{(k)} \\ \boldsymbol{\mu}^k \mathbf{Q}_B^{*(k)} \end{bmatrix} \quad (\text{A.1})
 \end{aligned}$$

208 Moreover, by substituting equation (24) into the aforementioned equation,
 209 we obtain

$$\sum_{k=1}^N \begin{bmatrix} \boldsymbol{\mu}^{N-k} \mathbf{Q}_B^{(k)} \\ \boldsymbol{\mu}^k \mathbf{Q}_B^{*(k)} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\mu}^{N-k+1} \boldsymbol{\Phi}_q^{*T} & \boldsymbol{\mu}^{N-k} \boldsymbol{\Phi}_q^{*T} \\ \boldsymbol{\mu}^{k-1} \boldsymbol{\Phi}_q^T & \boldsymbol{\mu}^k \boldsymbol{\Phi}_q^T \end{bmatrix} \begin{bmatrix} \mathbf{D}_{LI} & \mathbf{0} \\ \mathbf{D}_{RI} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{F}_I^{(k)} \\ \mathbf{F}_{\partial R}^{(k)} \end{bmatrix} \quad (\text{A.2})$$

210 In the other hand, by using equation (23) we obtain the following result
 211 (see [2])

$$\begin{bmatrix} \boldsymbol{\mu}^N \boldsymbol{\Phi}_F^{*T} & \boldsymbol{\Phi}_F^{*T} \\ \boldsymbol{\Phi}_F^T & \boldsymbol{\mu}^N \boldsymbol{\Phi}_F^T \end{bmatrix} = \begin{bmatrix} \boldsymbol{\mu}^N \boldsymbol{\Phi}_q^{*T} & \boldsymbol{\Phi}_q^{*T} \\ \boldsymbol{\Phi}_q^T & \boldsymbol{\mu}^N \boldsymbol{\Phi}_q^T \end{bmatrix} \begin{bmatrix} \mathbf{D}_{LL} & \mathbf{0} \\ \mathbf{0} & \mathbf{D}_{RR} \end{bmatrix} + \begin{bmatrix} \boldsymbol{\mu}^{N-1} \boldsymbol{\Phi}_q^{*T} & \boldsymbol{\mu} \boldsymbol{\Phi}_q^{*T} \\ \boldsymbol{\mu} \boldsymbol{\Phi}_q^T & \boldsymbol{\mu}^{N-1} \boldsymbol{\Phi}_q^T \end{bmatrix} \begin{bmatrix} \mathbf{D}_{RL} & \mathbf{0} \\ \mathbf{0} & \mathbf{D}_{LR} \end{bmatrix} \quad (\text{A.3})$$

212 By substituting equations (A.2), (A.3) into equation (A.1), we obtain the
 213 result in equation (32) with the global stiffness matrix \mathbf{D}_T and global external
 214 force \mathbf{F}_T calculated by

$$\begin{aligned} \mathbf{D}_T &= \begin{bmatrix} \mathbf{D}_{LL} & \mathbf{0} \\ \mathbf{0} & \mathbf{D}_{RR} \end{bmatrix} + \begin{bmatrix} \boldsymbol{\Phi}_q^{*-T} \boldsymbol{\mu}^N \boldsymbol{\Phi}_q^{*T} & \mathbf{I} \\ \mathbf{I} & \boldsymbol{\Phi}_q^{-T} \boldsymbol{\mu}^N \boldsymbol{\Phi}_q^T \end{bmatrix}^{-1} \times \\ &\quad \begin{bmatrix} \boldsymbol{\Phi}_q^{*-T} \boldsymbol{\mu}^{N-1} \boldsymbol{\Phi}_q^{*T} & \boldsymbol{\Phi}_q^{*-T} \boldsymbol{\mu} \boldsymbol{\Phi}_q^{*T} \\ \boldsymbol{\Phi}_q^{-T} \boldsymbol{\mu} \boldsymbol{\Phi}_q^T & \boldsymbol{\Phi}_q^{-T} \boldsymbol{\mu}^{N-1} \boldsymbol{\Phi}_q^T \end{bmatrix} \begin{bmatrix} \mathbf{D}_{RL} & \mathbf{0} \\ \mathbf{0} & \mathbf{D}_{LR} \end{bmatrix} \quad (\text{A.4}) \\ \mathbf{F}_T &= \begin{bmatrix} \boldsymbol{\Phi}_q^{*-T} \boldsymbol{\mu}^N \boldsymbol{\Phi}_q^{*T} & \mathbf{I} \\ \mathbf{I} & \boldsymbol{\Phi}_q^{-T} \boldsymbol{\mu}^N \boldsymbol{\Phi}_q^T \end{bmatrix}^{-1} \times \\ &\quad \sum_{k=1}^N \begin{bmatrix} \boldsymbol{\Phi}_q^{*-T} \boldsymbol{\mu}^{N-k-1} \boldsymbol{\Phi}_q^{*T} & \boldsymbol{\Phi}_q^{*-T} \boldsymbol{\mu}^{N-k} \boldsymbol{\Phi}_q^{*T} \\ \boldsymbol{\Phi}_q^{-T} \boldsymbol{\mu}^{k+1} \boldsymbol{\Phi}_q^T & \boldsymbol{\Phi}_q^{-T} \boldsymbol{\mu}^k \boldsymbol{\Phi}_q^T \end{bmatrix} \begin{bmatrix} \mathbf{D}_{LI} & \mathbf{0} \\ \mathbf{D}_{RI} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{F}_I^{(k)} \\ \mathbf{F}_{\partial R}^{(k)} \end{bmatrix} \quad (\text{A.5}) \end{aligned}$$

215 with $[\cdot]^{-T}$ denotes for the inverse of the transpose matrix.

216 **Appendix B. Calculation of WA approach**

217 For the left end of the structure, we can write the dynamic equation as
 218 follows

$$\begin{bmatrix} (\mathbf{D}_1)_{LL} & (\mathbf{D}_1)_{LI} & (\mathbf{D}_1)_{LB} \\ (\mathbf{D}_1)_{IL} & (\mathbf{D}_1)_{II} & (\mathbf{D}_1)_{IB} \\ (\mathbf{D}_1)_{BL} & (\mathbf{D}_1)_{BI} & (\mathbf{D}_1)_{BB} \end{bmatrix} \begin{bmatrix} \mathbf{q}_L^{(1)} \\ \mathbf{q}_{1I} \\ \mathbf{q}_0 \end{bmatrix} = \begin{bmatrix} -\mathbf{F}_{\partial R}^{(0)} - \mathbf{F}_L^{(1)} \\ \mathbf{F}_0 \\ \mathbf{F}_{1B} \end{bmatrix} \quad (\text{B.1})$$

219

$$\begin{bmatrix} (\mathbf{D}_2)_{RR} & (\mathbf{D}_2)_{RI} & (\mathbf{D}_2)_{RB} \\ (\mathbf{D}_2)_{IR} & (\mathbf{D}_2)_{II} & (\mathbf{D}_2)_{IB} \\ (\mathbf{D}_2)_{BR} & (\mathbf{D}_2)_{BI} & (\mathbf{D}_2)_{BB} \end{bmatrix} \begin{bmatrix} \mathbf{q}_L^{(N+1)} \\ \mathbf{q}_{2I} \\ \mathbf{q}_0^* \end{bmatrix} = \begin{bmatrix} \mathbf{F}_L^{(N+1)} \\ \mathbf{F}_0^* \\ \mathbf{F}_{2B} \end{bmatrix} \quad (\text{B.2})$$

220 From the first line of the aforementioned equations, we can write

$$-\mathbf{F}_{\partial R}^{(0)} - \mathbf{F}_L^{(1)} = (\mathbf{D}_1)_{LL}\mathbf{q}_L^{(1)} + (\mathbf{D}_1)_{LI}\mathbf{q}_{1I} + (\mathbf{D}_1)_{LB}\mathbf{q}_0 \quad (\text{B.3})$$

$$\mathbf{F}_L^{(N+1)} = (\mathbf{D}_2)_{RR}\mathbf{q}_L^{(N+1)} + (\mathbf{D}_2)_{RI}\mathbf{q}_{2I} + (\mathbf{D}_2)_{LB}\mathbf{q}_0^* \quad (\text{B.4})$$

221 In addition, from the second line of equations (B.1) and (B.2), we have

$$\mathbf{q}_{1I} = (\mathbf{D}_1)_{LI}^{-1} \left[\mathbf{F}_0 - (\mathbf{D}_1)_{IL}\mathbf{q}_L^{(1)} - (\mathbf{D}_1)_{IB}\mathbf{q}_0 \right] \quad (\text{B.5})$$

$$\mathbf{q}_{2I} = (\mathbf{D}_2)_{RI}^{-1} \left[\mathbf{F}_0^* - (\mathbf{D}_2)_{IR}\mathbf{q}_L^{(N+1)} - (\mathbf{D}_2)_{IB}\mathbf{q}_0^* \right] \quad (\text{B.6})$$

222 Then, by combining equations (B.5) and (B.6), we obtain

$$-\mathbf{F}_L^{(1)} = \mathbb{D}\mathbf{q}_L^{(1)} + \mathbb{D}_q\mathbf{q}_0 + \mathbb{D}_F\mathbf{F}_0 + \mathbf{F}_{\partial R}^{(0)} \quad (\text{B.7})$$

$$\mathbf{F}_L^{(N+1)} = \mathbb{D}^*\mathbf{q}_L^{(N+1)} + \mathbb{D}_q^*\mathbf{q}_0^* + \mathbb{D}_F^*\mathbf{F}_0^* \quad (\text{B.8})$$

223 where

$$\mathbb{D} = (\mathbf{D}_1)_{LL} - (\mathbf{D}_1)_{LI}(\mathbf{D}_1)_{II}^{-1}(\mathbf{D}_1)_{IL} \quad (\text{B.9})$$

$$\mathbb{D}_q = (\mathbf{D}_1)_{LB} - (\mathbf{D}_1)_{LI}(\mathbf{D}_1)_{II}^{-1}(\mathbf{D}_1)_{IL} \quad (\text{B.10})$$

$$\mathbb{D}_F = (\mathbf{D}_1)_{LI}(\mathbf{D}_1)_{II}^{-1} \quad (\text{B.11})$$

224 and

$$\mathbb{D}^* = (\mathbf{D}_2)_{RR} - (\mathbf{D}_2)_{RI}(\mathbf{D}_2)_{II}^{-1}(\mathbf{D}_2)_{IR} \quad (\text{B.12})$$

$$\mathbb{D}_q^* = (\mathbf{D}_2)_{RB} - (\mathbf{D}_2)_{RI}(\mathbf{D}_2)_{II}^{-1}(\mathbf{D}_1)_{IR} \quad (\text{B.13})$$

$$\mathbb{D}_F^* = (\mathbf{D}_2)_{RI}(\mathbf{D}_2)_{II}^{-1} \quad (\text{B.14})$$

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