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CONTINUITY OF SINGULAR KÄHLER-EINSTEIN POTENTIALS

by

Vincent Guedj, Henri Guenancia & Ahmed Zeriahi

Abstract. — In this note, we investigate some regularity aspects for solutions of degenerate complex Monge-Ampère equations (DCMAE) on singular spaces.

First, we study the Dirichlet problem for DCMAE on singular Stein spaces, showing a general continuity result. A consequence of our results is that Kähler-Einstein potentials are continuous at isolated singularities.

Next, we establish the global continuity of solutions to DCMAE when the reference class belongs to the real Néron-Severi group. This yields in particular the continuity of Kähler-Einstein potentials on any irreducible Calabi-Yau variety.

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Introduction

Generalizing Yau’s solution of the Calabi conjecture [Yau78], singular Kähler-Einstein metrics on mildly singular Kähler varieties X have been constructed in [EGZ09] and further studied by many authors (see [GZ17, Bou18, Don18] and the references therein). These are honest Kähler forms $\omega_{\text{KE}} = \omega + dd^c \varphi_{\text{KE}}$ on the regular part X_{reg} of X , where $c_1(X) = \lambda[\omega]$ is proportional to a reference Kähler class $[\omega]$, s.t.

$$\text{Ric}(\omega_{\text{KE}}) = \lambda\omega_{\text{KE}}.$$

One constructs $\omega_{\text{KE}} = \omega + dd^c \varphi_{\text{KE}}$ by solving a degenerate complex Monge-Ampère equation

$$(0.1) \quad (\omega + dd^c \varphi_{\text{KE}})^n = e^{-\lambda\varphi_{\text{KE}}} \mu_X,$$

where $n = \dim_{\mathbb{C}} X$ and μ_X is an appropriate volume form.

Understanding the asymptotic behavior of φ_{KE} near X_{sing} is a major open problem. A very precise description has been obtained by Hein-Sun in [HS17], when the singularities are isolated and furthermore isomorphic to a smoothable strongly regular Calabi-Yau cone. It would be highly desirable to extend this description to more general contexts, but this seems presently out of reach.

It has been shown in [EGZ09] that φ_{KE} is always uniformly bounded across X_{sing} . The viscosity approach developed in [EGZ11] fails to establish continuity at the singularities (see [EGZ17]). In this article we use both local (Dirichlet problem analysis) and global (stability estimates) arguments to establish continuity properties of φ_{KE} , in various geometrical contexts that we now describe.

Isolated singularities. — Let X be a Stein complex space which is reduced and locally irreducible, of complex dimension $n \geq 1$, equipped with a hermitian metric whose fundamental form is a positive $(1,1)$ -form denoted by β ; we do not assume that β is closed here. Let $\Omega \Subset X$ be a strongly pseudoconvex domain in X , i.e. Ω admits a smooth defining function ρ which is strictly plurisubharmonic in a neighborhood of $\bar{\Omega}$.

Let ϕ be a continuous function on $\partial\Omega$ and $0 \leq f \in L^p(\Omega, \beta^n)$ for some $p > 1$. The Dirichlet problem with boundary datum ϕ and right hand side $\mu := f\beta^n$ consists in finding a bounded plurisubharmonic function U satisfying the following properties :

$$(0.2) \quad \begin{cases} (dd^c U)^n = \mu & \text{in } \Omega, \\ U|_{\partial\Omega} = \phi, & \text{on } \partial\Omega, \end{cases}$$

where the equation is here understood in the sense of Bedford-Taylor [BT76, BT82, Bed82], as we briefly recall in Section 1.2.

When X , ϕ and f are smooth and f is positive, this problem has been addressed by several authors (see notably [CH99, GL10]) following the fundamental work of Caffarelli-Kohn-Nirenberg-Spruck [CKNS85]. When f merely belongs to L^p , $p > 1$, a unique continuous solution has been constructed by Kolodziej [Kol98]. There is however not a very abundant literature dedicated to the case where X is singular. Let us mention [Wik09] which studies maximal plurisubharmonic functions, [PS10] which treats the case of isolated singularities with RHS smoothly degenerating to zero, [EGZ09] which deals with global Monge-Ampère equations on mildly singular compact Kähler varieties and [DLS17] which seeks for solutions in Cegrell finite energy classes.

Our first main result is the following.

Theorem A. — *The Dirichlet problem (0.2) admits a unique solution $U \in \text{PSH}(\Omega) \cap C^0(\bar{\Omega})$.*

The general strategy to prove Theorem A consists in producing the solution as the envelop of all subsolutions. This requires to approximate the data (ϕ, f) by smooth data and establish precise estimates along the regularization process. The main technical tool we use is the so-called stability estimate, cf Proposition 1.8.

We apply this local theory to establish the continuity of singular Kähler-Einstein potentials at isolated singularities as a consequence of Theorem A:

Corollary B. — *A Kähler-Einstein potential φ_{KE} , i.e. a solution of (0.1), is continuous at any isolated singularity of X .*

We show more generally that the results holds for Kähler-Einstein metrics on pairs, cf. Theorem 3.4.

Non-isolated singularities. — In order to deal with non-isolated singularities we require the reference class $[\omega]$ to belong to the real Néron-Severi group, cf Section 3.3 for the notation and Theorem 3.9.

Theorem C. — *Let (X, ω) be a compact normal Kähler space such that $[\omega] \in \widetilde{\text{NS}}_{\mathbb{R}}(X) \subset H^1(X, \text{PH}_X)$. Let $f \geq 0$ be a function in $L^p(X)$ for some $p > 1$ such that $\int_X f \omega^n = \int_X \omega^n$. Let $\varphi \in \text{PSH}(X, \omega) \cap L^\infty(X)$ be the unique solution of the equation*

$$(\omega + dd^c \varphi)^n = f \omega^n$$

such that $\sup_X \varphi = 0$. Then φ is continuous on X .

A few remarks are in order:

– This result does not rely on the previous Theorem A, since we allow here arbitrary singularities.

– The continuity of φ when $[\omega] \in \widetilde{\text{NS}}_{\mathbb{Q}}(X)$ is a (rational) Hodge class is a consequence of the extension result of [CGZ13] (some particular cases were obtained by Dinew-Zhang [DZ10]).

– If X has rational singularities (e.g. if X has klt singularities), the natural map $\beta : H^1(X, \text{PH}_X) \rightarrow H^2(X, \mathbb{R})$ is an injection and identifies $\widetilde{\text{NS}}_{\mathbb{R}}(X)$ with the usual Néron-Severi group $\text{NS}_{\mathbb{R}}(X)$.

The proof of Theorem C goes as follows. We approximate ω by a Kähler form ω_ε whose cohomology class is rational, and solve the equation $(\omega_\varepsilon + dd^c \varphi_\varepsilon)^n = c_\varepsilon f \omega^n$ where c_ε is a harmless mass adjustment constant. By the remark above, φ_ε is continuous. Finally, one can show that φ_ε converges to φ uniformly by a suitable application of the maximum principle.

We finally apply Theorem C to the case where X is an irreducible Calabi-Yau variety X (cf. Definition 3.10 and Corollary 3.11).

Corollary D. — *Let X be an irreducible Calabi-Yau variety, and let α be a Kähler class. Then, the unique singular Ricci-flat metric $\omega_{\text{KE}} \in \alpha$ has continuous potentials on X .*

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1. Preliminaries

1.1. Plurisubharmonic functions on complex spaces. — Let X be a reduced complex analytic space of pure dimension $n \geq 1$. We will denote by X_{reg} the complex manifold of regular points of X . The set

$$X_{\text{sing}} := X \setminus X_{\text{reg}}$$

of singular points is an analytic subset of X of complex codimension ≥ 1 .

By definition for each point $x_0 \in X$ there exists a neighborhood U of x_0 and a local embedding $j : U \hookrightarrow \mathbb{C}^N$ onto an analytic subset of \mathbb{C}^N for some $N \geq 1$.

Using these local embeddings, it is possible to define the spaces of smooth forms of given degree on X as smooth forms on X_{reg} that are locally on X the restriction of an ambient form on \mathbb{C}^N . The notion of currents on X is then defined by duality by their action on compactly supported smooth forms on X . The operators ∂ and $\bar{\partial}$, d , d^c and dd^c are then well defined by duality (see [Dem85] for a careful treatment).

In the same way one can define the analytic notions of holomorphic and plurisubharmonic functions. There are essentially two different notions :

Definition 1.1. — Let $u : X \rightarrow \mathbb{R} \cup \{-\infty\}$ be a given function.

1. We say that u is plurisubharmonic on X if it is locally the restriction of a plurisubharmonic function on a local embedding of X onto an analytic subset of \mathbb{C}^N .
2. We say that u is weakly plurisubharmonic on X if u is locally bounded from above on X and its restriction to the complex manifold X_{reg} is plurisubharmonic.

Fornaess and Narasimhan proved in [FN80] that u is plurisubharmonic on X if and only if for any analytic disc $h : \mathbb{D} \rightarrow X$, the restriction $u \circ h$ is subharmonic on \mathbb{D} or identically $-\infty$.

If u is weakly plurisubharmonic on X , u is plurisubharmonic on X_{reg} , hence upper semi-continuous on X_{reg} . Since no assumption is made on u at singular points, it is natural to extend u to X by the following formula :

$$(1.1) \quad u^*(x) := \limsup_{X_{\text{reg}} \ni y \rightarrow x} u(y), \quad x \in X.$$

The function u^* is upper semi-continuous, locally integrable on X and satisfies $dd^c u^* \geq 0$ in the sense of currents on X . By Demailly [Dem85], the two notions are equivalent when X is locally irreducible. More precisely we will need the following result :

Theorem 1.2. — [Dem85] Assume that X is a locally irreducible analytic space and $u : X \rightarrow \mathbb{R} \cup \{-\infty\}$ is a weakly plurisubharmonic function on X , then the function u^* defined by (1.1) is plurisubharmonic on X .

Observe that since u is plurisubharmonic on X_{reg} , we have $u^* = u$ on X_{reg} . Hence u^* is the upper semi-continuous extension of $u|_{X_{\text{reg}}}$ to X . We note the following important consequence :

Corollary 1.3. — Let $\mathcal{U} \subset \text{PSH}(X)$ be a non empty family of plurisubharmonic functions which is locally bounded from above on X . Then its upper envelope

$$U := \sup\{u ; u \in \mathcal{U}\},$$

is a well-defined Borel function whose upper semi-continuous regularization U^* is plurisubharmonic on X .

This result is an extension of a classical result of P. Lelong concerning envelopes of plurisubharmonic functions on open sets in \mathbb{C}^n (see [Lel61, GZ17]).

Following [FN80] we say that X is Stein if it admits a \mathcal{C}^2 -smooth strongly plurisubharmonic exhaustion. We will use the following definition :

Definition 1.4. — A domain $\Omega \Subset X$ is strongly pseudoconvex if it admits a negative \mathcal{C}^2 -smooth strongly plurisubharmonic exhaustion, i.e. a function ρ strongly plurisubharmonic in a neighborhood Ω' of $\overline{\Omega}$ such that $\Omega := \{x \in \Omega'; \rho(x) < 0\}$ and for any $c < 0$,

$$\Omega_c := \{x \in \Omega'; \rho(x) < c\} \Subset \Omega$$

is relatively compact.

Our complex spaces will be assumed to be reduced, locally irreducible of dimension $n \geq 1$. We denote by $\text{PSH}(X)$ the set of plurisubharmonic functions on X . Finally, we fix a smooth, strictly positive $(1,1)$ -form β on X .

1.2. Monge-Ampère operator on singular spaces. — The complex Monge-Ampère measure $(dd^c u)^n$ of a smooth psh function in a domain of \mathbb{C}^n is the Radon measure

$$(dd^c u)^n = c \det \left(\frac{\partial^2 u}{\partial z_i \partial \bar{z}_j} \right) dV_{\text{eucl}},$$

where $c > 0$ is a normalizing constant. The definition has been extended to any bounded psh function by Bedford-Taylor, who laid down the foundations of *pluripotential theory* in [BT76, BT82] (see [GZ17] for a recent treatment).

The complex Monge-Ampère operator has been defined and studied on complex spaces by Bedford in [Bed82] and later by Demailly in [Dem85]. It turns out that if $u \in \text{PSH}(X) \cap L_{\text{loc}}^\infty(X)$, the Monge-Ampère measure $(dd^c u)^n$ is well defined on X_{reg} and can be extended to X as a Borel measure with zero mass on X_{sing} .

Thus all standard properties of the complex Monge-Ampère operator acting on $\text{PSH}(X) \cap L_{\text{loc}}^\infty(X)$ extend to this setting (see [Bed82, Dem85]). In particular :

Proposition 1.5. — Let $\Omega \Subset X$ a relatively compact open set and $u, v \in \text{PSH}(\Omega) \cap L^\infty(\Omega)$. Assume that $\liminf_{x \rightarrow \zeta} (u(x) - v(x)) \geq 0$ for any $\zeta \in \partial\Omega$. Then

$$\int_{\{u < v\}} (dd^c v)^n \leq \int_{\{u < v\}} (dd^c u)^n.$$

In particular, if $(dd^c u)^n \leq (dd^c v)^n$ weakly on Ω , then $v \leq u$ on Ω .

The proof relies on Stokes formula (see [Bed82]). As a first step towards proving Theorem A, we treat the following simpler Dirichlet problem with *bounded density* and *zero boundary value*. This will be a key result to prove the subextension property (Lemma 1.7), on which relies, in turn, the central stability estimate (Proposition 1.8).

Lemma 1.6. — Let $\Omega \Subset X$ be a strongly pseudoconvex domain and $0 \leq f \in L^\infty(\Omega)$ a bounded density. There exists a unique function $w \in \text{PSH}(\Omega) \cap L^\infty(\Omega)$ such that

$$(dd^c w)^n = f \beta^n \text{ on } \Omega, \text{ with } w|_{\partial\Omega} = 0.$$

Proof. — Up to scaling ρ , one can assume that $dd^c \rho \geq \beta$. The function $v_A := A\rho$ is plurisubharmonic and smooth in a neighborhood of $\overline{\Omega}$, with $v_A|_{\partial\Omega} = 0$. Moreover $(dd^c v_A)^n \geq A^n \beta^n \geq f \beta^n$ in Ω if $A \geq \|f\|_\infty^{1/n}$, hence v_A is a subsolution to the Dirichlet problem if A is large enough.

We consider the envelope of subsolutions $w := \sup\{v; v \in \mathcal{S}\}$, where

$$\mathcal{S} := \{v; v \in \text{PSH}(\Omega) \cap L^\infty(\Omega), (dd^c v)^n \geq f\beta^n \text{ and } v|_{\partial\Omega} \leq 0\}.$$

It follows from the maximum principle that $w \leq 0$, while $w \geq v_A$ since $v_A \in \mathcal{S}$.

A classical argument guarantees that w is plurisubharmonic and again a subsolution to the Dirichlet problem i.e. $(dd^c w)^n \geq f\beta^n$ on Ω and $w = 0$ in $\partial\Omega$ (see the proof of Lemma 2.3 where this is explained in a slightly more general context).

It remains to show that $(dd^c w)^n = f\beta^n$ in Ω_{reg} . This follows from a balayage process. Choose a chart (B, z) in a neighborhood of given regular point $a \in \Omega_{\text{reg}}$ such that there exists an isomorphism $F : B \rightarrow \mathbb{B}$ from a neighborhood of \bar{B} onto a neighborhood of the closed euclidean unit ball $\bar{\mathbb{B}}$ with $F(B) = \mathbb{B}$. We consider the Dirichlet problem

$$(1.2) \quad (dd^c u)^n = f\beta^n, \text{ on } B \text{ and } u|_{\partial B} = w.$$

Observe that w is a subsolution. Using a result of Cegrell [Ceg84], we can find a solution $u_B \in \text{PSH}(B) \cap L^\infty(B)$ to this problem (see also [GZ17, Theorem 5.17]). It follows from the comparison principle (Proposition 1.5) that $w \leq u_B$ on B .

The function u defined on Ω by $u = u_B$ on B and $u = w$ on $\Omega \setminus B$ is a plurisubharmonic subsolution to the Dirichlet problem (1.2), since the smooth volume form β^n puts no mass on ∂B . Therefore $u \leq w$ on Ω . This yields $u = w = u_B$ on B , hence $(dd^c w)^n = f\beta^n$ in B . Since $B \subset \Omega_{\text{reg}}$ was arbitrary, it follows that $(dd^c w)^n = f\beta^n$ on Ω_{reg} , hence on the whole Ω since none of the two measures puts any mass on Ω_{sing} . \square

1.3. Stability estimates. —

1.3.1. Subextension. — We now explain a useful subextension property of plurisubharmonic functions with finite Monge-Ampère mass. In the sequel we set for a bounded open subset $\Omega \subset X$

$$\mathcal{E}^0(\Omega) = \left\{ \phi \in \text{PSH}(\Omega) \cap L^\infty(\Omega), \phi|_{\partial\Omega} = 0 \text{ and } \int_{\Omega} (dd^c \phi)^n < +\infty \right\}.$$

Let now $\Omega \Subset \tilde{\Omega} \Subset X$ be two strongly pseudoconvex domains.

Lemma 1.7. — *Given $u \in \mathcal{E}^0(\Omega)$ we extend it to $\tilde{\Omega}$ by 0 outside Ω and consider*

$$\tilde{u} := \sup\{v \in \text{PSH}(\tilde{\Omega}); v \leq \mathbf{1}_{\Omega} u \text{ in } \tilde{\Omega}\}.$$

Then $\tilde{u} \in \mathcal{E}^0(\tilde{\Omega})$, $\tilde{u} \leq u$ in Ω and $\int_{\tilde{\Omega}} (dd^c \tilde{u})^n \leq \int_{\Omega} (dd^c u)^n$.

This statement is proved in [CZ03] for domains in \mathbb{C}^n . The proof follows the same idea with a twist relying on [FN80] and Lemma 1.6 as we explain below.

Proof. — Let $\tilde{\rho}$ be a plurisubharmonic defining function of $\tilde{\Omega}$. Since u is bounded in $\Omega \Subset \tilde{\Omega}$, one can find $A \gg 1$ such that $A\tilde{\rho} \leq u \leq 0$ in Ω . Thus \tilde{u} is a well defined bounded plurisubharmonic function on $\tilde{\Omega}$ such that $A\tilde{\rho} \leq \tilde{u} \leq 0$ in $\tilde{\Omega}$ and $\tilde{u} \leq u$ in Ω .

We need to estimate the Monge-Ampère mass of \tilde{u} . The main idea is to construct a subsolution with at most the same Monge-Ampère mass by solving the complex Monge-Ampère equation $(dd^c v)^n = \mathbf{1}_{\Omega} (dd^c u)^n$ on $\tilde{\Omega}$. We are not yet able to solve such an equation unless the right hand side is a measure with bounded density thanks to Lemma 1.6, so we proceed by approximation.

Assume first that u is plurisubharmonic in a Stein neighborhood Ω' of $\tilde{\Omega}$. A result of Fornæss-Narasimhan [FN80] ensures the existence of a sequence (u_j) of smooth plurisubharmonic functions on Ω' which decreases to u . As the measure $\mu_j := (dd^c u_j)^n = f_j \beta^n$ has bounded density $f_j \in L^\infty(\Omega)$, it follows from Lemma 1.6 that one can solve the Dirichlet problem $(dd^c v_j)^n = \mathbf{1}_\Omega (dd^c u_j)^n$ on $\tilde{\Omega}$ with boundary value $v_j = 0$ in $\partial\tilde{\Omega}$. Since $v_j \leq 0 \leq u_j$ in $\partial\Omega$, it follows from Proposition 1.5 on Ω , that $v_j \leq u_j$ in Ω . Moreover $\int_{\tilde{\Omega}} (dd^c v_j)^n = \int_{\Omega} (dd^c u_j)^n$. By definition $v_j \leq \tilde{u}_j \leq 0$ in $\tilde{\Omega}$ and $v_j = \tilde{u}_j = 0$ in $\partial\tilde{\Omega}$. Proposition 1.5 thus yields $\int_{\tilde{\Omega}} (dd^c \tilde{u}_j)^n \leq \int_{\tilde{\Omega}} (dd^c v_j)^n = \int_{\Omega} (dd^c u_j)^n$.

Since $(u_j)_j$ decreases to u in a neighborhood of $\tilde{\Omega}$ and $A\rho \leq u$ in Ω , the sequence (\tilde{u}_j) decreases to a bounded plurisubharmonic function v such that $A\tilde{\rho} \leq v \leq 0$ in $\tilde{\Omega}$ and $v \leq u$ in Ω . Moreover $\int_{\tilde{\Omega}} (dd^c v)^n \leq \int_{\tilde{\Omega}} (dd^c u)^n < \infty$ hence $v \in \mathcal{E}^0(\tilde{\Omega})$. Observe that $v \leq \tilde{u}$ in $\tilde{\Omega}$ and $v = \tilde{u} = 0$ in $\partial\tilde{\Omega}$. Hence $\int_{\tilde{\Omega}} (dd^c \tilde{u})^n \leq \int_{\tilde{\Omega}} (dd^c v)^n \leq \int_{\Omega} (dd^c u)^n < \infty$.

To treat the general case we let $(\Omega_j)_j$ be an exhaustive sequence of bounded strongly pseudoconvex domains in Ω and set

$$w_j := \sup\{v \in \text{PSH}(\tilde{\Omega}) ; v < 0 \text{ and } v \leq u \text{ on } \Omega_j\}.$$

Since $A\tilde{\rho} \leq u$ in Ω , it follows that w_j is a bounded plurisubharmonic function in $\tilde{\Omega}$ such that $A\tilde{\rho} \leq \tilde{u} \leq w_j \leq \mathbf{1}_{\Omega_j} u$ in $\tilde{\Omega}$. Moreover (w_j) decreases to a bounded psh functions w such that $A\tilde{\rho} \leq \tilde{u} \leq w \leq \mathbf{1}_\Omega u$ in $\tilde{\Omega}$. Hence $w \leq \tilde{u}$ in $\tilde{\Omega}$. This proves that $w = \tilde{u}$ in $\tilde{\Omega}$.

The previous case ensures that $\int_{\tilde{\Omega}} (dd^c w_j)^n \leq \int_{\tilde{\Omega}_j} (dd^c u)^n \leq \int_{\Omega} (dd^c u)^n$, hence

$$\int_{\tilde{\Omega}} (dd^c \tilde{u})^n \leq \liminf_{j \rightarrow +\infty} \int_{\tilde{\Omega}} (dd^c w_j)^n \leq \int_{\Omega} (dd^c u)^n.$$

This shows that $\tilde{u} \in \mathcal{E}^0(\tilde{\Omega})$ and $\int_{\tilde{\Omega}} (dd^c \tilde{u})^n \leq \int_{\Omega} (dd^c u)^n$. \square

1.3.2. Stability. — We now state an important stability estimate that we shall use repeatedly in the sequel:

Proposition 1.8. — *Let $\Omega \Subset X$ a relatively compact open set and $u, v \in \text{PSH}(\Omega) \cap L^\infty(\Omega)$. Let β be a smooth positive $(1, 1)$ -form on X and let $f \in L^p(\Omega)$ for some $p > 1$. Assume that*

$$(dd^c u)^n = f \beta^n \quad \text{on } \Omega.$$

Then for any $0 < \gamma < 1/(nq + 1)$, there exists a constant $C = C(\gamma, \|f\|_p) > 0$ such that

$$\sup_{\Omega} (v - u)_+ \leq \sup_{\partial\Omega} (v - u)_+^* + C \|(v - u)_+\|_1^\gamma.$$

where $\|(v - u)_+\|_1 := \int_{\Omega} (v - u)_+ \beta^n$, and q is the conjugate exponent of p , i.e. $\frac{1}{p} + \frac{1}{q} = 1$.

Here $w_+ := \sup\{w, 0\}$ and w^* means the upper semi-continuous extension of a bounded function w on Ω to the boundary i.e. $w(\zeta) := \limsup_{z \rightarrow \zeta} w(z)$.

This result is proved in [GKZ08, Theorem 1.1] when the ambient space is $X = \mathbb{C}^n$ (see also [EGZ09, Proposition 3.3] for a similar result in the context of compact Kähler varieties). The proof relies on capacity estimates and the use of the comparison principle (Proposition 1.5). It can be reproduced identically in our present context, once the following volume-capacity comparison has been established:

Lemma 1.9. — For any $u \in \text{PSH} \cap L^\infty(\Omega)$ such that $\int_\Omega (dd^c u)^n \leq 1$ and $u|_{\partial\Omega} = 0$,

$$(1.3) \quad \int_\Omega e^{-u} \beta^n \leq C = C(n, \Omega).$$

In particular, we have the following inequality for any Borel set $K \subset \Omega$

$$(1.4) \quad \text{Vol}_\beta(K) \leq C \exp\left(-\text{Cap}(K, \Omega)^{-1/n}\right).$$

Here $\text{Vol}_\beta(K) := \int_K \beta^n$ and $\text{Cap}(K, \Omega)$ denotes the Monge-Ampère capacity.

Sketch of proof of Lemma 1.9. — The fact that (1.4) follows from (1.3) is standard by choosing u to be a multiple of the extremal function $h_{K, \Omega}$ of K , which satisfies $\int_\Omega (dd^c h_{K, \Omega}^*)^n = \text{Cap}(K, \Omega)$, cf [GZ17, Theorems 4.32 & 4.34].

The proof of (1.3) goes roughly as follows. Choose a strongly pseudoconvex domain $\tilde{\Omega}$ (with strictly psh exhaustion function $\tilde{\rho}$) containing $\bar{\Omega}$ in its interior, and consider the subextension \tilde{u} of u constructed in Lemma 1.7. Recall that $\tilde{u} \in \mathcal{E}^0(\tilde{\Omega})$ and $\int_{\tilde{\Omega}} (dd^c \tilde{u})^n \leq 1$. Since $\tilde{u} \leq u$ on Ω , one has $\int_\Omega e^{-u} \beta^n \leq \int_{\tilde{\Omega}} e^{-\tilde{u}} \beta^n$ hence it is sufficient to bound that last quantity.

For any element $v \in \mathcal{E}^0(\tilde{\Omega})$, one has $\int_{\tilde{\Omega}} (-v)^n (dd^c \tilde{\rho})^n \leq c_n \|\tilde{\rho}\|_\infty^n \int_{\tilde{\Omega}} (dd^c v)^n$ for some universal constant c_n . In particular, the set $\Sigma := \{v \in \mathcal{E}^0(\tilde{\Omega}); \int_{\tilde{\Omega}} (dd^c v)^n \leq 1\}$ is relatively compact in $\text{PSH}(\tilde{\Omega})$ for the L^1_{loc} topology and one can check that for any $v \in \Sigma$, v has a well-defined Monge-Ampère and $\int_\Omega (dd^c v)^n \leq 1$, cf [Zer01, Corollary 4.4].

Now, take a resolution $p : \hat{\Omega} \rightarrow \tilde{\Omega}$, choose a strictly positive $(1, 1)$ -form $\hat{\beta} \geq p^* \beta$ on $\hat{\Omega}$ and fix a compact set $K \subset \tilde{\Omega}$. By the property above, functions $\hat{v} \in p^* \Sigma$ have Monge-Ampère mass at most one, hence $\sup_{x \in p^{-1}(K)} v(\hat{v}, x) \leq 1$. By covering $p^{-1}(K)$ with finitely many coordinate charts, one can use [Zer01, Corollary 3.2] to get a constant $C_K > 0$ such that $\int_{p^{-1}(K)} e^{-\hat{v}} \hat{\beta}^n \leq C_K$ for any $\hat{v} \in p^* \Sigma$. The conclusion now follows by choosing $K = \bar{\Omega}$ and $\hat{v} = p^* \tilde{u}$. \square

2. Dirichlet problem on singular complex spaces

In this Section, we will work in the following

Setting 2.1. — Let X be a Stein space of dimension $n \geq 1$, reduced and locally irreducible. We fix a smooth positive $(1, 1)$ -form β on X , not necessarily closed. Let $\Omega \Subset X$ be a pseudoconvex domain with C^2 -smooth exhaustion function ρ , let $f \in L^p(\Omega)$ for some $p > 1$ and let $\mu := f \beta^n$. Finally, let $\phi \in C^0(\partial\Omega)$.

By the comparison principle Proposition 1.5, if the Dirichlet problem (0.2) admits a solution, it dominates any subsolution. We will now prove that the maximal subsolution is psh, hence coincides with the solution in that case.

Definition 2.2. — A plurisubharmonic function $v \in \text{PSH}^\infty(\Omega) := \text{PSH}(\Omega) \cap L^\infty(\Omega)$ is a subsolution to the Dirichlet problem (0.2) with data (ϕ, μ) if the following two conditions are satisfied :

1. $v^*(\zeta) := \limsup_{\Omega \ni z \rightarrow \zeta} v(z) \leq \phi(\zeta)$, for any $\zeta \in \partial\Omega$,

2. $(dd^c v)^n \geq \mu$ weakly in Ω .

It is natural to consider the family $\mathcal{S}_{\phi, \mu} = \mathcal{S}_{\phi, \mu}(\Omega)$ of all subsolutions to the Dirichlet problem (0.2) with data (ϕ, μ) and its upper envelope when it exists.

2.1. The subsolution property. — We start with the following standard observation.

Lemma 2.3. — *In Setting 2.1, assume that the Dirichlet problem (0.2) with data (ϕ, μ) admits a subsolution $v_0 \in \mathcal{S}_{\phi, \mu}(\Omega)$. Then the upper envelope of subsolutions*

$$U := U_{\phi, \mu} := \sup\{v ; v \in \mathcal{S}_{\phi, \mu}(\Omega)\}$$

is a subsolution to the Dirichlet problem (0.2), i.e. $U_{\phi, \mu} \in \mathcal{S}_{\phi, \mu}(\Omega)$. Moreover, if $v_0 = \phi$ on $\partial\Omega$ then

$$(2.1) \quad \lim_{z \rightarrow \zeta} U(z) = \phi(\zeta), \quad \text{for any } \zeta \in \partial\Omega.$$

The proof of the lemma will actually not make use of the assumption $f \in L^p(\Omega)$, but we chose to stick to Setting 2.1 throughout this section for clarity.

Proof. — Observe that for any $u \in \mathcal{S}_{\phi, \mu}(\Omega)$ we have $u \leq M_\phi := \max_{\partial\Omega} \phi$ on Ω . Therefore, the envelope $U := U_{\phi, \mu}$ is a well-defined bounded function on Ω and U^* is a bounded plurisubharmonic function on Ω by Corollary 1.3. It follows then from a topological lemma of Choquet and the comparison principle Proposition 1.5, that U^* satisfies the inequality $(dd^c U^*)^n \geq \mu$ in Ω (see [GZ17, Theorem 5.12, Step 1]). It remains to show that $U = U^* \in \text{PSH}(\Omega)$ and that the boundary values of U are $\leq \phi$ (resp. $= \phi$ in the second case).

Claim. Let $U_\phi := U_{\phi, 0}$ be the maximal plurisubharmonic function on Ω with boundary values $\leq \phi$. Then, $U_\phi = U_\phi^* \in \text{PSH}(\Omega)$ and it satisfies

$$(2.2) \quad \lim_{z \rightarrow \zeta} U_\phi(z) = \phi(\zeta), \quad \text{for any } \zeta \in \partial\Omega.$$

For the time being, assume that the claim holds and let us finish the proof of Lemma 2.3. Since we have

$$(2.3) \quad v_0 \leq U \leq U_\phi \quad \text{on } \Omega$$

the claim would imply $v_0 \leq U^* \leq U_\phi$ on Ω . Thus, one would get $U^* \in \mathcal{S}_{\phi, \mu}(\Omega)$, hence $U^* \leq U$ on Ω , and finally $U^* = U$ on Ω . The boundary condition follows from (2.3).

It now remains to prove the claim. To establish (2.2), we fix $\varepsilon > 0$ and let ψ be a \mathcal{C}^2 function on $\partial\Omega$ such that $\psi - \varepsilon \leq \phi \leq \psi$ on $\partial\Omega$. By definition

$$U_\psi - \varepsilon \leq U_\phi \leq U_\phi^* \leq U_\psi^* \quad \text{on } \Omega,$$

reducing the problem to the case when ϕ is \mathcal{C}^2 on $\partial\Omega$. In this case we can extend ϕ as a \mathcal{C}^2 function in a neighborhood of $\bar{\Omega}$. Choosing $A > 1$ large enough, we can assume that $v_\phi := A\rho + \phi$ is plurisubharmonic and \mathcal{C}^2 in a neighborhood of $\bar{\Omega}$ and $v_\phi = \phi$ on $\partial\Omega$. Hence $v_\phi \leq U_\phi$. Similarly we choose $B > 1$ large enough so that $w_\phi := -B\rho + \phi$ is a \mathcal{C}^2 plurisuperharmonic function in a neighborhood of $\bar{\Omega}$ such that $w_\phi = \phi$ on $\partial\Omega$. It

follows from the maximum principle that $v_\phi \leq U_\phi \leq U_\phi^* \leq w_\phi$ on Ω . This implies that U_ϕ^* is a subsolution hence $U_\phi^* \leq U_\phi$ and finally $U_\phi = U_\phi^* \in \text{PSH}(\Omega)$. At the same time, we get that U_ϕ satisfies (2.2). The claim and lemma follow. \square

2.2. Existence of a solution. — We now show that $U_{\phi,\mu}$ is a solution of the Dirichlet problem (0.2) using Lemma 2.3 and a balayage process in Ω_{reg} , adapting the proofs of [BT76, Theorem 8.3] and [GZ17, Theorem 5.16].

Theorem 2.4. — *In Setting 2.1, we have $\mathcal{S}_{\phi,\mu}(\Omega) \neq \emptyset$ and its upper envelope*

$$(2.4) \quad U := \sup\{v; v \in \mathcal{S}_{\phi,\mu}(\Omega)\}$$

satisfies $U \in \text{PSH}(\Omega) \cap \mathcal{C}(\Omega)$. Moreover, U admits a continuous extension to $\overline{\Omega}$ and it solves the Dirichlet problem (0.2).

When $f \equiv 0$, a similar result has been obtained by Wikström [Wik09, Theorem 1.8].

Proof. — We proceed in several steps.

Step 1: Assume first that $\phi \in \mathcal{C}^2(\partial\Omega)$ and $\mu = f\beta^n$ with $f \in L^\infty(\Omega)$. We claim that there exists a subsolution v plurisubharmonic and smooth in a neighborhood of $\overline{\Omega}$ such that $v = \phi$ on $\partial\Omega$. Indeed consider any smooth extension of ϕ to a neighborhood of $\overline{\Omega}$. For $A > 1$ large enough, the function $v_\phi = v_\phi^A := A\rho + \phi$ is plurisubharmonic and smooth in a neighborhood of $\overline{\Omega}$ and satisfies $(dd^c v_\phi)^n \geq f\beta^n$ in Ω , since f is bounded from above. Since $v_\phi = \phi$ on $\partial\Omega$ we can apply Lemma 2.3 to conclude that $(dd^c U)^n \geq \mu$ on Ω and

$$\lim_{z \rightarrow \zeta} U(z) = \phi(\zeta) \quad \text{for any } \zeta \in \partial\Omega.$$

The identity $(dd^c U)^n = \mu$ in Ω can be obtained by a balayage argument exactly as in the proof of Lemma 1.6.

To show the continuity of U on $\overline{\Omega}$ we first observe that U can be extended as a plurisubharmonic function on a neighborhood Ω' of $\overline{\Omega}$. Indeed since v_ϕ is plurisubharmonic on a neighborhood Ω' of $\overline{\Omega}$ and $v_\phi \leq U$ on Ω with $v_\phi = \phi = U$ on $\partial\Omega$, the function V defined by $V = U$ on Ω and $V := v_\phi$ on $\Omega' \setminus \Omega$ is plurisubharmonic on Ω' . We can always assume that Ω' is a Stein space. By Fornaess-Narasimhan approximation Theorem [FN80, Theorem 5.5] there exists a decreasing sequence (V_j) of smooth plurisubharmonic functions on Ω' which converges to V pointwise on Ω' . Consider now $v_j := V_j$ on $\overline{\Omega}$. Then $v_j \in \text{PSH}(\Omega) \cap \mathcal{C}^0(\overline{\Omega})$ and (v_j) decreases to U on $\overline{\Omega}$. In particular, $v_j \rightarrow U$ in L^1 by the monotone convergence theorem. It follows from Proposition 1.8 that

$$(2.5) \quad \sup_{\Omega} (v_j - U) \leq \sup_{\partial\Omega} (v_j - U) + C \|v_j - U\|_1^\gamma,$$

where $0 < \gamma < 1/(nq + 1)$. Since (v_j) decreases to $U = \phi$ on $\partial\Omega$, Dini's lemma ensures that the convergence is uniform on $\partial\Omega$. It follows therefore from (2.5) that (v_j) uniformly converges to U on $\overline{\Omega}$, hence U is continuous on $\overline{\Omega}$.

Step 2 : Assume $\phi \in C^0(\partial\Omega)$ and $\mu = f\beta^n$ with $f \in L^\infty(\Omega)$. Take a decreasing sequence of smooth functions $(\phi_j)_{j \in \mathbb{N}^*}$ converging to ϕ on $\partial\Omega$ so that $\phi_j - 1/j \leq \phi \leq \phi_j$ on $\partial\Omega$ for any $j \geq 1$. Then from the definitions we see that

$$U_{\phi_j, \mu} - 1/j \leq U_{\phi, \mu} \leq U_{\phi_j, \mu}, \text{ on } \bar{\Omega}.$$

This implies that the sequence $(U_{\phi_j, \mu})$ uniformly converges to $U_{\phi, \mu}$ on $\bar{\Omega}$. Hence $U = U_{\phi, \mu}$ is continuous on $\bar{\Omega}$, $U = \phi$ on $\partial\Omega$ and satisfies the equation $(dd^c U)^n = \mu$ on Ω thanks to [Dem09, Corollary 3.6].

Step 3 : Assume $\phi \in C^0(\partial\Omega)$ and $\mu = f\beta^n$ with $f \in L^p(\Omega)$, with $p > 1$. Define $f_j := \min\{f, j\}$ for $j \in \mathbb{N}$. The bounded densities (f_j) increase to f in $L^p(\Omega)$. Set $U_j := U_{\phi, \mu_j}$ where $\mu_j := f_j\beta^n$ for $j \in \mathbb{N}$. By the comparison principle Proposition 1.5, $(U_j)_{j \in \mathbb{N}}$ is a decreasing sequence of bounded plurisubharmonic functions on Ω that are continuous up to the boundary.

We first show that the U_j 's are uniformly bounded. Let $v := U_{\phi, 0}$ denote the maximal plurisubharmonic function on Ω with boundary value ϕ (whose properties have been established the proof of Lemma 2.3). The comparison principle Proposition 1.5 ensures that $U_j \leq v$ on Ω . It follows from Proposition 1.8 that for all $j \in \mathbb{N}$,

$$\|v - U_j\|_{L^\infty} \leq C \|v - U_j\|_{L^1}^\gamma,$$

where $0 < \gamma < 1$ and $C > 0$ is a uniform constant depending only on a uniform bound of $\|f_j\|_p \leq \|f\|_p$. This implies in particular that $\|v - U_j\|_{L^\infty}^{1-\gamma} \leq C'$, where $C' > 0$ is a uniform constant, hence $\sup_j \|U_j\|_{L^\infty} < +\infty$.

The sequence $(U_j)_{j \in \mathbb{N}}$ of continuous functions on $\bar{\Omega}$ therefore decreases to a bounded function V such that $V|_\Omega$ is plurisubharmonic and $V|_{\partial\Omega} = \phi$ since $U_j|_{\partial\Omega} = \phi$ for all j . Moreover $(dd^c V)^n = \mu$, as follows from the continuity property of the complex Monge-Ampère operator along monotone sequences, cf [GZ17, Theorem 3.18]. Thus V is a subsolution to (0.2). By Lemma 2.3, the function $U = U_{\mu, \phi}$ is a subsolution to (0.2). Since U is the maximal subsolution we have $V \leq U$. On the other hand $(dd^c U)^n \geq (dd^c V)^n$ so Proposition 1.5 ensures $U \leq V$ on Ω . Therefore $U = V$ and $(dd^c U)^n = \mu$. In particular, U_j converges to U in L^1 by the monotone convergence theorem.

The continuity of U again follows from the stability estimate (Proposition 1.8) : there exists a constant $C > 0$ such that for all $j \in \mathbb{N}$

$$\sup_{\Omega} (U_j - U) \leq C \|U_j - U\|_1^\gamma,$$

showing that (U_j) uniformly converges to U on $\bar{\Omega}$, hence U is continuous on $\bar{\Omega}$ and solves the Dirichlet problem (0.2). \square

3. Continuity of Kähler-Einstein potentials

3.1. Generalities. — Let X be a normal compact space. We say that X is Kähler if it admits a Kähler form in the following sense

Definition 3.1. — A Kähler form ω on a normal complex space X can be defined as a Kähler form on X_{reg} which extends to a smooth, closed $(1, 1)$ -form under local embeddings $X \xrightarrow{\text{loc}} \mathbb{C}^N$. In particular, one can cover X with open subsets U_α , find embeddings $j_\alpha : U_\alpha \xrightarrow{\text{loc}} \mathbb{C}^N$ as well as smooth, strictly psh functions φ_α defined on a neighborhood of $j_\alpha(U_\alpha)$ such that $\omega|_{U_\alpha} = j_\alpha^* dd^c \varphi_\alpha$.

The study of complex Monge-Ampère equations in this context has been initiated in [EGZ09], providing a way of constructing singular Kähler-Einstein metrics and extending Yau's fundamental solution to the Calabi conjecture [Yau78]. More precisely, it is proven there that given a Kähler metric ω on X and a non-negative function $f \in L^p(X)$ for some $p > 1$ satisfying $\int_X f \omega^n = \int \omega^n$, then the equation

$$(3.1) \quad (\omega + dd^c \varphi^n) = f \cdot \omega^n$$

has a unique solution $\varphi \in \text{PSH}(X, \omega) \cap L^\infty(X)$ such that $\sup_X \varphi = 0$.

Let us now explain the relation between the equation (3.1) above and the existence of singular Kähler-Einstein metrics.

We choose a pair (X, D) consisting of an n -dimensional compact Kähler variety X and a divisor $D = \sum a_i D_i$ with $a_i \in [0, 1] \cap \mathbb{Q}$. We assume that there exists an integer $m \geq 1$ such that $m(K_X + D)$ is a line bundle.

Given a hermitian metric h on $K_X + D$ and the singular metric $e^{-\phi_D}$ (only unique up to a positive multiple), one constructs a measure $\mu_{(X,D),h}$ on X as follows. If U is any open set where $m(K_X + D)$ admits a trivialization σ on U_{reg} , then the expression

$$\frac{i^{n^2} (\sigma \wedge \bar{\sigma})^{\frac{1}{m}} e^{-\phi_D}}{|\sigma|_{h^{\otimes m}}^{\frac{2}{m}}}$$

defines a measure on U_{reg} which is independent of m as well as the choice of σ and can thus be patched to a measure on X_{reg} . Its extension by 0 on X_{sing} is by definition $\mu_{(X,D),h}$. We recall the following properties satisfied by the measure $\mu := \mu_{(X,D),h}$, cf [EGZ09, Lemma 6.4].

- The Ricci curvature of μ on X_{reg} is equal to $-i\Theta(h) + [D]$.
- The mass $\int_X d\mu$ is finite if and only if (X, D) has klt singularities.
- If μ has finite mass, then the density f of μ wrt ω^n (i.e. $\mu = f \cdot \omega^n$) satisfies $f \in L^p(X)$ for some $p > 1$.

From now on, we work in the following

Setting 3.2. — Let (X, D) be a pair where X is a compact normal Kähler space and D is an effective \mathbb{Q} -divisor. Assume that (X, D) has klt singularities, pick a Kähler metric ω and a hermitian metric h on $K_X + D$, normalized so that $\int_X d\mu_{(X,D),h} = \int_X \omega^n$. We assume either

- $K_X + D$ is ample and $\omega = i\Theta(h)$; or
- $K_X + D \equiv 0$ and h satisfies $i\Theta(h) = 0$; or
- $K_X + D$ is anti-ample and $\omega = -i\Theta(h)$.

Definition 3.3. — In the Setting 3.2 above, a Kähler-Einstein metric is a solution $\omega_{\text{KE}} := \omega + dd^c \varphi_{\text{KE}}$ of the Monge-Ampère equation

$$(3.2) \quad (\omega + dd^c \varphi_{\text{KE}})^n = e^{-\lambda \varphi_{\text{KE}}} \mu_{(X,D),h}$$

where $\lambda = -1, 0, 1$ according to whether we are in the first, second or third case. It satisfies

$$(3.3) \quad \text{Ric}(\omega_{\text{KE}}) = \lambda \omega_{\text{KE}} + [D]$$

in the weak sense.

By the results [EGZ09] recalled above, (3.2) admits a unique solution ω_{KE} whenever $\lambda \in \{-1, 0\}$. Its potential φ_{KE} is globally bounded on X and ω_{KE} is a honest Kähler-Einstein metric on $X_{\text{reg}} \setminus \text{Supp}(D)$ and it has cone singularities along D generically [Gue13]. For $\lambda = +1$ we refer the reader to [BBE⁺19, Bou18].

Understanding the asymptotic behavior of φ_{KE} (resp. ω_{KE}) near X_{sing} is a major open problem. A breakthrough has been obtained in [HS17] for special type of Calabi-Yau varieties, but a lack of local models prevent us from a good understanding in more general situations. We discuss below two different tools that allow one to show -in many contexts- global continuity of the Kähler-Einstein potential φ_{KE} , a statement that was overlooked in [EGZ11], cf [EGZ17].

3.2. Isolated singularities. —

Theorem 3.4. — Let (X, D) be a pair as in Setting 3.2. Then the potential φ_{KE} of any Kähler-Einstein metric in the sense of Definition 3.3 is continuous near an isolated singularity of X .

Proof. — We work near an isolated singular point a . We let B denote a small strictly pseudoconvex neighborhood of a in X , isomorphic to the trace of a ball in some local embedding in \mathbb{C}^N , and let ρ denote a local smooth potential for $\omega = dd^c \rho$ in B .

Recall from [EGZ17] that the Kähler-Einstein potential φ_{KE} is continuous in $B \setminus \{a\}$. We can apply the local theory developed in Section 2 to obtain that $\psi = \rho + \varphi_{\text{KE}}$ is the unique solution of the Dirichlet problem

$$(dd^c \psi)^n = e^{-\lambda \varphi_{\text{KE}}} \mu_{(X,D),h} \text{ in } B, \quad \text{with } \psi|_{\partial B} = (\rho + \varphi_{\text{KE}})|_{\partial B}.$$

Recall that $\mu_{(X,D),h}$ has an L^p density wrt ω^n since (X, D) is klt. It follows therefore from Theorem 2.4 that ψ is continuous, hence so is φ_{KE} at point a . \square

3.3. Hodge classes vs transcendental classes. — Let X be a compact Kähler space. In $H^2(X, \mathbb{R})$, an important subgroup is made up by the first Chern class of line bundles L on X via the map

$$c_1 : H^1(X, \mathcal{O}_X^*) \rightarrow H^2(X, \mathbb{R})$$

induced by the exponential exact sequence

$$(3.4) \quad 0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_X \xrightarrow{e^{2\pi i \cdot}} \mathcal{O}_X^* \longrightarrow 0$$

and the sheaf injection $\mathbb{Z} \hookrightarrow \mathbb{R}$.

Definition 3.5. — The group $\text{Im}(c_1) \subset H^2(X, \mathbb{R})$ is called Néron-Severi group of X and denoted by $\text{NS}(X)$. One defines the vector spaces $\text{NS}_{\mathbb{Q}}(X) := \text{NS}(X) \otimes \mathbb{Q} \subset H^2(X, \mathbb{R})$ and $\text{NS}_{\mathbb{R}}(X) := \text{NS}(X) \otimes \mathbb{R} \subset H^2(X, \mathbb{R})$.

A Kähler metric ω (cf Definition 3.1) is canonically attached to an element in $H^0(X, \mathcal{C}_X^\infty / \text{PH}_X)$ where \mathcal{C}_X^∞ (resp. PH_X) is the subsheaf of continuous functions on X that are local restrictions of smooth functions (resp. pluriharmonic functions) under local embeddings $X \xrightarrow[\text{loc}]{\hookrightarrow} \mathbb{C}^N$. It can be proved that a pluriharmonic function is locally the real part of a holomorphic function, [BEG13, §4.6.1]. The exact sequences

$$\begin{aligned} 0 \longrightarrow \text{PH}_X \longrightarrow \mathcal{C}_X^\infty \longrightarrow \mathcal{C}_X^\infty / \text{PH}_X \longrightarrow 0 \\ 0 \longrightarrow \mathbb{R} \longrightarrow \mathcal{O}_X \xrightarrow{\text{Im}(\cdot)} \text{PH}_X \longrightarrow 0 \end{aligned}$$

yield a map

$$H^0(X, \mathcal{C}_X^\infty / \text{PH}_X) \xrightarrow{[\cdot]} H^1(X, \text{PH}_X) \xrightarrow{\beta} H^2(X, \mathbb{R})$$

Definition 3.6. — A class $\alpha \in H^1(X, \text{PH}_X)$ is called Kähler if there exists a Kähler metric ω on X such that $\alpha = [\omega]$. The Kähler cone of X is the set $\mathcal{K}_X \subset H^1(X, \text{PH}_X)$ made out of Kähler classes. It is an open, convex set.

Remark 3.7. — When X has rational singularities, it is proved in [GK20, Remark 3.2 (2)] that β is an injection and that we have an exact sequence

$$0 \longrightarrow H^1(X, \text{PH}_X) \xrightarrow{\beta} H^2(X, \mathbb{R}) \longrightarrow H^2(X, \mathcal{O}_X) \longrightarrow 0.$$

In particular, under those assumptions, one can alternatively view the Kähler cone $\mathcal{K}_X \subset H^2(X, \mathbb{R})$ without any ambiguity.

Finally, the logarithm map yields an exact sequence

$$0 \longrightarrow \mathbb{R}/\mathbb{Z} \xrightarrow{e^{2\pi i \cdot}} \mathcal{O}_X^* \xrightarrow{\log|\cdot|} \text{PH}_X \longrightarrow 0$$

hence a map

$$H^1(X, \mathcal{O}_X^*) \xrightarrow{\gamma} H^1(X, \text{PH}_X)$$

such that $\beta \circ \gamma = c_1$.

Definition 3.8. — One set $\widetilde{\text{NS}}(X) := \text{Im}(\gamma) \subset H^1(X, \text{PH}_X)$. One also defines the vector spaces $\widetilde{\text{NS}}_{\mathbb{Q}}(X) := \widetilde{\text{NS}}(X) \otimes \mathbb{Q} \subset H^1(X, \text{PH}_X)$ and $\widetilde{\text{NS}}_{\mathbb{R}}(X) := \widetilde{\text{NS}}(X) \otimes \mathbb{R} \subset H^1(X, \text{PH}_X)$.

The relation between the two groups introduced is that

$$(3.5) \quad \text{NS}(X) = \beta(\widetilde{\text{NS}}(X))$$

and when X has rational singularities, then Remark 3.7 implies that β induces an isomorphism between the two groups and, in particular, between their tensorization by \mathbb{Q} or \mathbb{R} .

A generalization of Kodaira theorem shows that a line bundle $L \rightarrow X$ on a normal compact space X such that $\gamma(L) \in \mathcal{K}_X$ is automatically ample, cf e.g. [EGZ09, Proposition 5.12]. In particular, there exists an embedding $f : X \hookrightarrow \mathbb{P}^N$ such that $f^* \mathcal{O}_{\mathbb{P}^N}(1) = L$ and $[f^* \omega_{\text{FS}}] = \gamma(L)$.

One can slightly extend the result above by asserting that if $\widetilde{\text{NS}}_{\mathbb{R}}(X) \cap \mathcal{K}_X \neq \emptyset$, then X is projective. Indeed, \mathcal{K}_X is open and $\widetilde{\text{NS}}_{\mathbb{Q}}(X)$ is dense in $\widetilde{\text{NS}}_{\mathbb{R}}(X)$, so $\widetilde{\text{NS}}_{\mathbb{Q}}(X) \cap \mathcal{K}_X \neq \emptyset$. Since \mathcal{K}_X is a cone, we further deduce $\widetilde{\text{NS}}(X) \cap \mathcal{K}_X \neq \emptyset$ and we can apply the result above.

Coming back to Equation (3.1), it is proved in [CGZ13] that if $[\omega] \in \widetilde{\text{NS}}_{\mathbb{Q}}(X)$, then the potential solution φ can be approximated by a decreasing sequence of smooth ω -psh functions, hence [EGZ09, Theorem 2.1] ensures that φ is globally continuous on X .

In particular, the previous discussion applies to show that if (X, D) is a klt pair such that $K_X + D$ has a sign, then a solution φ_{KE} of (3.2) is continuous on X as soon as $\lambda \neq 0$. That is, singular Kähler-Einstein metrics ω_{KE} with positive or negative Ricci curvature have continuous potentials.

The remaining case of interest is when the curvature is zero (i.e. $\lambda = 0$) and $[\omega]$ is an arbitrary Kähler class in $H^1(X, \text{PH}_X)$. The following results deals with the case $[\omega] \in \widetilde{\text{NS}}_{\mathbb{R}}(X)$.

Theorem 3.9. — *Let (X, ω) be a compact normal Kähler space such that $[\omega] \in \widetilde{\text{NS}}_{\mathbb{R}}(X) \subset H^1(X, \text{PH}_X)$. Let $f \geq 0$ be a function in $L^p(X)$ for some $p > 1$ such that $\int_X f \omega^n = \int_X \omega^n$. Let $\varphi \in \text{PSH}(X, \omega) \cap L^\infty(X)$ be the unique solution of the equation*

$$(\omega + dd^c \varphi)^n = f \omega^n$$

such that $\sup_X \varphi = 0$. Then φ is continuous on X .

Generalizing the celebrated Beauville-Bogomolov decomposition, it was recently proved in [BGL20] that compact Kähler spaces X with klt singularities such that K_X is numerically trivial admit a finite cover $X' \rightarrow X$, unramified in codimension one and such that $X' = T \times \prod Y_i \times \prod Z_j$ where T is a (smooth) torus, Y_i are irreducible Calabi-Yau varieties and Z_j are irreducible holomorphic symplectic varieties, cf. also [Dru18, G GK19, HP19, CGGN20] for related anterior results. We will focus on the irreducible Calabi-Yau factors, whose definition we now recall.

Definition 3.10. — *Let X be a compact, normal Kähler space of dimension $n \geq 3$. We say that X is an irreducible Calabi-Yau variety if X has canonical singularities, trivial canonical bundle and is such that for any finite quasi-étale cover $X' \rightarrow X$, we have $h^0(X', \Omega_{X'}^{[p]}) = 0$ for any $1 \leq p \leq n - 1$.*

Since there are no reflexive two-forms on an irreducible Calabi-Yau variety X , it is automatically projective. From Theorem 3.9, we deduce the following

Corollary 3.11. — *Let X be an irreducible Calabi-Yau variety and let $\alpha \in \mathcal{K}_X$ be a Kähler class. Then, the unique singular Ricci-flat metric $\omega \in \alpha$ has continuous potentials on X .*

Proof of Corollary 3.11. — Let $p : \tilde{X} \rightarrow X$ be a resolution of singularities. Since X has rational singularities and \tilde{X} is compact Kähler, we have for any $p \leq n$

$$h^p(X, \mathcal{O}_X) = h^p(\tilde{X}, \mathcal{O}_{\tilde{X}}) = h^0(\tilde{X}, \Omega_{\tilde{X}}^p).$$

By [KS21], we have $p_*\Omega_{\tilde{X}}^p = \Omega_X^{[p]}$, and therefore

$$h^p(X, \mathcal{O}_X) = h^0(X, \Omega_X^{[p]}) = 0$$

for $p = 1, 2$. The exact sequence (3.4) shows that the integral first Chern class yields an isomorphism $H^1(X, \mathcal{O}_X^*) \simeq H^2(X, \mathbb{Z})$ so, in particular, one has

$$(3.6) \quad \text{NS}_{\mathbb{R}}(X) = H^2(X, \mathbb{R}).$$

By (3.5) and the remark below it, it follows that $\widetilde{\text{NS}}_{\mathbb{R}}(X) = H^1(X, \text{PH}_X)$. The corollary is now proved. \square

Proof of Theorem 3.9. — It follows from [EGZ09, Theorem 4.1] that there is a unique solution $u \in \text{PSH}(X, \omega) \cap L^\infty(X)$ of the equation

$$(3.7) \quad (\omega + dd^c u)^n = e^u \cdot f\omega^n$$

We claim it is continuous. If that claim holds, we can apply it to φ which is the solution of the equation

$$(\omega + dd^c u)^n = e^u \cdot (fe^{-\varphi})\omega^n$$

This is indeed legitimate since $fe^{-\varphi} \in L^p(X)$ as φ is bounded.

We are now going to prove the claim. Since $\widetilde{\text{NS}}_{\mathbb{Q}}(X)$ is dense in $\widetilde{\text{NS}}_{\mathbb{R}}(X)$, one can find a family of Kähler metrics ω_ε such that

- $[\omega_\varepsilon] \in \widetilde{\text{NS}}_{\mathbb{Q}}(X)$
- $[\omega_\varepsilon] \rightarrow [\omega]$ in $H^1(X, \text{PH}_X)$.
- $\omega_\varepsilon \rightarrow \omega$ smoothly on X .

Indeed, $H^1(X, \text{PH}_X)$ is a finite-dimensional vector space and it is the image of the surjective map $[\cdot] : H^0(X, \mathcal{C}_X^\infty / \text{PH}_X) \rightarrow H^1(X, \text{PH}_X)$. Clearly, $[\omega]$ is a non-zero element, hence one can find smooth $(1, 1)$ -forms $\omega_1, \dots, \omega_r$ with local potentials such that $([\omega], [\omega_1], \dots, [\omega_r])$ is a basis of $H^1(X, \text{PH}_X)$. In order to uniformize the notation, we set $\omega_0 := \omega$.

By assumption, $[\omega]$ is a limit of rational classes. This means that one can find a $(r+1)$ -tuple $\lambda_\varepsilon = (\lambda_{i,\varepsilon})_{0 \leq i \leq r}$ of real numbers converging to $(1, 0, \dots, 0)$ and such that $\sum_{i=0}^r \lambda_{i,\varepsilon} [\omega_i]$ is a rational class. If one sets $\omega_\varepsilon := \sum_{i=0}^r \lambda_{i,\varepsilon} \omega_i$, then ω_ε is a Kähler metric for ε small enough and it clearly satisfies the desired conditions stated above. In particular, there is no loss of generality in assuming that we have the following set of inequalities

$$(3.8) \quad (1 - \varepsilon)\omega \leq \omega_\varepsilon \leq (1 + \varepsilon)\omega$$

We consider the unique function $u_\varepsilon \in \text{PSH}(X, \omega) \cap L^\infty(X)$ solution of the equation

$$(3.9) \quad (\omega_\varepsilon + dd^c u_\varepsilon)^n = e^{u_\varepsilon} \cdot f\omega^n$$

As we explained above, $u_\varepsilon \in \mathcal{C}^0(X)$ since $[\omega_\varepsilon] \in \widetilde{\text{NS}}_{\mathbb{Q}}(X)$. Moreover, by Jensen's inequality, it follows easily that

$$(3.10) \quad \sup_X u_\varepsilon \leq C$$

for some $C > 0$ independent of ε . Moreover, it follows from (3.8) that the function $v_\varepsilon := (1 - \varepsilon)u + n \log(1 - \varepsilon) - \varepsilon \|u\|_\infty$ is ω_ε -psh and satisfies

$$\begin{aligned} (\omega_\varepsilon + dd^c v_\varepsilon)^n &\geq (1 - \varepsilon)^n (\omega + dd^c u)^n \\ &= e^{(1-\varepsilon)u + n \log(1-\varepsilon) + \varepsilon u} \cdot f \omega^n \\ &\geq e^{v_\varepsilon} \cdot f \omega^n. \end{aligned}$$

This shows that v_ε is a subsolution of (3.9) and we get

$$(3.11) \quad v_\varepsilon = (1 - \varepsilon)u + n \log(1 - \varepsilon) - \varepsilon \|u\|_\infty \leq u_\varepsilon$$

In particular, we have a uniform lower bound for u_ε independent of ε . Combined with (3.10) we obtain, up to adjusting C , the following control

$$(3.12) \quad \|u_\varepsilon\|_\infty \leq C$$

Finally, we consider $w_\varepsilon := \frac{1}{1+\varepsilon}u_\varepsilon - n \log(1 + \varepsilon) - \frac{\varepsilon}{1+\varepsilon}\|u_\varepsilon\|_\infty$. By (3.8), the function w_ε is ω -psh and satisfies

$$\begin{aligned} (\omega + dd^c w_\varepsilon)^n &\geq (1 + \varepsilon)^{-n} (\omega_\varepsilon + dd^c u_\varepsilon)^n \\ &= e^{\frac{1}{1+\varepsilon}u_\varepsilon - n \log(1+\varepsilon) + \frac{\varepsilon}{1+\varepsilon}u_\varepsilon} \cdot f \omega^n \\ &\geq e^{w_\varepsilon} \cdot f \omega^n. \end{aligned}$$

This shows that w_ε is a subsolution of (3.7) and we get

$$(3.13) \quad w_\varepsilon = \frac{1}{1+\varepsilon}u_\varepsilon - n \log(1 + \varepsilon) - \frac{\varepsilon}{1+\varepsilon}\|u_\varepsilon\|_\infty \leq u.$$

Combining (3.11)-(3.13) and (3.12), we see that

$$\|u - u_\varepsilon\|_\infty \leq -n \log(1 - \varepsilon) + 2\varepsilon \max(\|u\|_\infty, \|u_\varepsilon\|_\infty) = O(\varepsilon).$$

That is, u is the uniform limit of the continuous functions u_ε when $\varepsilon \rightarrow 0$. In particular, it is continuous. \square

References

- [BBE⁺19] R. BERMAN, S. BOUCKSOM, P. EYSSIDIEUX, V. GUEDJ & A. ZERIAHI – “Kähler-Einstein metrics and the Kähler-Ricci flow on log Fano varieties”, *J. Reine Angew. Math.* **751** (2019), p. 27–89.
- [Bed82] E. BEDFORD – “The operator $(dd^c)^n$ on complex spaces”, in *Seminar Pierre Lelong-Henri Skoda (Analysis), 1980/1981, and Colloquium at Wimereux, May 1981*, Lecture Notes in Math., vol. 919, Springer, Berlin-New York, 1982, p. 294–323.
- [BEG13] S. BOUCKSOM, P. EYSSIDIEUX & V. GUEDJ – “Introduction”, in *An introduction to the Kähler-Ricci flow*, Lecture Notes in Math., vol. 2086, Springer, Cham, 2013, p. 1–6.
- [BGL20] B. BAKKER, H. GUENANCIA & C. LEHN – “Algebraic approximation and the decomposition theorem for Kähler Calabi-Yau varieties”, Preprint [arXiv:2012.00441](https://arxiv.org/abs/2012.00441), December 2020.

- [Bou18] S. BOUCKSOM – “Variational and non-archimedean aspects of the Yau-Tian-Donaldson conjecture”, in *Proceedings of the ICM—Rio de Janeiro 2018. Vol. II. Invited lectures*, World Sci. Publ., Hackensack, NJ, 2018, p. 591–617.
- [BT76] E. BEDFORD & B. TAYLOR – “The Dirichlet problem for a complex Monge-Ampère equation”, *Invent. Math.* **37** (1976), no. 1, p. 1–44.
- [BT82] ———, “A new capacity for plurisubharmonic functions”, *Acta Math.* **149** (1982), no. 1-2, p. 1–40.
- [Ceg84] U. CEGRELL – “On the Dirichlet problem for the complex Monge-Ampère operator”, *Math. Z.* **185** (1984), no. 2, p. 247–251.
- [CGGN20] B. CLAUDON, P. GRAF, H. GUENANCIA & P. NAUMANN – “Kähler spaces with zero first Chern class: Bochner principle, fundamental groups, and the Kodaira problem”, Preprint [arXiv:2008.13008](https://arxiv.org/abs/2008.13008), 2020.
- [CGZ13] D. COMAN, V. GUEDJ & A. ZERIAHI – “Extension of plurisubharmonic functions with growth control”, *J. Reine Angew. Math.* **676** (2013), p. 33–49.
- [CH99] P. CHERRIER & A. HANANI – “Le problème de Dirichlet pour les équations de Monge-Ampère en métrique hermitienne”, *Bull. Sci. Math.* **123** (1999), no. 7, p. 577–597.
- [CKNS85] L. CAFFARELLI, J. J. KOHN, L. NIRENBERG & J. SPRUCK – “The Dirichlet problem for nonlinear second-order elliptic equations. II. Complex Monge-Ampère, and uniformly elliptic, equations”, *Comm. Pure Appl. Math.* **38** (1985), no. 2, p. 209–252.
- [CZ03] U. CEGRELL & A. ZERIAHI – “Subextension of plurisubharmonic functions with bounded Monge-Ampère mass”, *C. R. Math. Acad. Sci. Paris* **336** (2003), no. 4, p. 305–308.
- [Dem85] J.-P. DEMAILLY – “Mesures de Monge-Ampère et caractérisation géométrique des variétés algébriques affines”, *Mém. Soc. Math. France (N.S.)* (1985), no. 19, p. 124.
- [Dem09] J.-P. DEMAILLY – “Complex Analytic and Differential Geometry”, September 2009, OpenContent Book, freely available from the author’s web site.
- [DLS17] N. Q. DIEU, T. V. LONG & O. SANPHET – “Monge-Ampère operators on complex varieties in \mathbb{C}^n ”, Preprint [arXiv:1702.07454](https://arxiv.org/abs/1702.07454), 2017.
- [Don18] S. DONALDSON – “Some recent developments in Kähler geometry and exceptional holonomy”, in *Proceedings of the ICM—Rio de Janeiro 2018. Vol. I. Plenary lectures*, World Sci. Publ., Hackensack, NJ, 2018, p. 425–451.
- [Dru18] S. DRUEL – “A decomposition theorem for singular spaces with trivial canonical class of dimension at most five”, *Invent. Math.* **211** (2018), no. 1, p. 245–296.
- [DZ10] S. DINEW & Z. ZHANG – “On stability and continuity of bounded solutions of degenerate complex Monge-Ampère equations over compact Kähler manifolds”, *Adv. Math.* **225** (2010), no. 1, p. 367–388.
- [EGZ09] P. EYSSIDIEUX, V. GUEDJ & A. ZERIAHI – “Singular Kähler-Einstein metrics”, *J. Amer. Math. Soc.* **22** (2009), p. 607–639.
- [EGZ11] ———, “Viscosity solutions to degenerate complex Monge-Ampère equations”, *Comm. Pure Appl. Math.* **64** (2011), no. 8, p. 1059–1094.
- [EGZ17] ———, “Corrigendum: Viscosity solutions to complex Monge-Ampère equations”, *Comm. Pure Appl. Math.* **70** (2017), no. 5, p. 815–821.
- [FN80] J. E. FORNÆSS & R. NARASIMHAN – “The Levi problem on complex spaces with singularities”, *Math. Ann.* **248** (1980), no. 1, p. 47–72.
- [GGK19] D. GREB, H. GUENANCIA & S. KEBEKUS – “Klt varieties with trivial canonical class: holonomy, differential forms, and fundamental groups”, *Geom. Topol.* **23** (2019), p. 2051–2124.

- [GK20] P. GRAF & T. KIRSCHNER – “Finite quotients of three-dimensional complex tori”, *Ann. Inst. Fourier (Grenoble)* **70** (2020), no. 2, p. 881–914.
- [GKZ08] V. GUEDJ, S. KOŁODZIEJ & A. ZERIAHI – “Hölder continuous solutions to Monge-Ampère equations”, *Bull. Lond. Math. Soc.* **40** (2008), no. 6, p. 1070–1080.
- [GL10] B. GUAN & Q. LI – “Complex Monge-Ampère equations and totally real submanifolds”, *Adv. Math.* **225** (2010), no. 3, p. 1185–1223.
- [Gue13] H. GUENANCIA – “Kähler-Einstein metrics with cone singularities on klt pairs”, *Internat. J. Math.* **24** (2013), no. 5, p. 1350035, 19.
- [GZ17] V. GUEDJ & A. ZERIAHI – *Degenerate complex Monge-Ampère equations*, EMS Tracts in Mathematics, vol. 26, European Mathematical Society (EMS), Zürich, 2017.
- [HP19] A. HÖRING & T. PETERNELL – “Algebraic integrability of foliations with numerically trivial canonical bundle”, *Invent. Math.* **216** (2019), no. 2, p. 395–419.
- [HS17] H.-J. HEIN & S. SUN – “Calabi-Yau manifolds with isolated conical singularities”, *Publ. Math. Inst. Hautes Études Sci.* **126** (2017), p. 73–130.
- [Koł98] S. KOŁODZIEJ – “The complex Monge-Ampère operator”, *Acta Math.* **180** (1998), no. 1, p. 69–117.
- [KS21] S. KEBEKUS & C. SCHNELL – “Extending holomorphic forms from the regular locus of a complex space to a resolution of singularities”, *J. Amer. Math. Soc.* **34** (2021), p. 315–368.
- [Lel61] P. LELONG – “Fonctions plurisousharmoniques et fonctions analytiques de variables réelles”, *Ann. Inst. Fourier (Grenoble)* **11** (1961), p. 515–562.
- [PS10] D. H. PHONG & J. STURM – “The Dirichlet problem for degenerate complex Monge-Ampère equations”, *Comm. Anal. Geom.* **18** (2010), no. 1, p. 145–170.
- [Wik09] F. WIKSTRÖM – “The Dirichlet problem for maximal plurisubharmonic functions on analytic varieties in \mathbb{C}^n ”, *Internat. J. Math.* **20** (2009), no. 4, p. 521–528.
- [Yau78] S.-T. YAU – “On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation. I.”, *Commun. Pure Appl. Math.* **31** (1978), p. 339–411.
- [Zer01] A. ZERIAHI – “Volume and capacity of sublevel sets of a Lelong class of plurisubharmonic functions”, *Indiana Univ. Math. J.* **50** (2001), no. 1, p. 671–703.

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