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Classes of graphs with low complexity: the case of classes with bounded linear rankwidth*

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ABSTRACT

Classes with bounded rankwidth are MSO-transductions of trees and classes with bounded linear rankwidth are MSO-transductions of paths – a result that shows a strong link between the properties of these graph classes considered from the point of view of structural graph theory and from the point of view of finite model theory. We take both views on classes with bounded linear rankwidth and prove structural and model theoretic properties of these classes. The structural results we obtain are the following. 1) The number of unlabeled graphs of order n with linear rank-width at most r is at most $\left[(2^r+1)(r+1)!\,2^{\binom{r}{2}}3^{r+1}\right]^n$ 2) Graphs with linear rankwidth at most r are linearly χ-bounded. Actually, they have bounded c-chromatic number, meaning that they can be colored with $f(r)$ colors, each color inducing a cograph. 3) To the contrary, based on a Ramsey-like argument, we prove for every proper hereditary family $\mathscr{F}$ of graphs (like cographs) that there is a class with bounded rankwidth that does not have the property that graphs in it can be colored by a bounded number of colors, each inducing a subgraph in $\mathscr{F}$.

From the model theoretical side we obtain the following results: 1) A direct short proof that graphs with linear rankwidth at most r are first-order transductions of linear orders. This result could also be derived from Colcombet's theorem on first-order transduction of linear orders and the equivalence of linear rankwidth with linear cliquewidth. 2) For a class $\mathscr{C}$ with bounded linear rankwidth the following conditions are equivalent: a) $\mathscr{C}$ is stable, b) $\mathscr{C}$ excludes some half-graph as a semi-induced subgraph, c) $\mathscr{C}$ is a first-order transduction of a class with bounded pathwidth. These results open the perspective to study classes admitting low linear rankwidth covers.

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On devient jeune à soixante ans.
Malheureusement, c'est trop tard.

You become young when you're sixty.
Unfortunately, it's too late.

到60岁，我们才开始变得年轻。
不幸的是，为时晚矣。

Pablo Picasso

Contents

1. Introduction

A primary concern in many areas of mathematics is to classify structures (or classes of structures) according to their intrinsic complexity. In this paper we consider three approaches and their interplay to the notion of structural complexity: the model theoretic approach based on the standard dividing lines that are stability and dependence, the algebraic approach founding the notion of rankwidth and linear rankwidth, and a more classical graph theoretical approach based on colorings and decompositions of graphs.

A theory of sparse structures was initiated in [39], which mainly fits to the classification of monotone classes (i.e., classes that are closed under taking subgraphs). The theory has led to the nowhere dense/somewhere dense dichotomy that can be observed in several areas of graph theory, theoretical computer science, model theory, analysis, category theory and probability theory. Motivated by the connection with model theory – nowhere dense classes are monadically stable [1] and even have low VC-density [42] – and by a possible extension of first-order model-checking algorithms for bounded expansion classes [17, 18] and for nowhere dense classes [24], these notions were extended to classes that are obtained as first-order transductions of sparse classes, the *structurally sparse classes* [40, 19]. The central tool used in our approach is the transduction machinery, which establishes a fruitful bridge between graph theory and finite model theory. Informally, a first-order transduction is a way to interpret a structure in another structure, where the new structure is defined by means of first-order formulas with set parameters. Indeed, a standard approach of both model theory and computability theory is to determine the relative complexity of two structures by showing that the first interprets in the second, and is therefore not more complex than the second. In this context, important classes of structures are the class of finite linear orders and the class of element to finite set membership graphs (powerset graphs), as they define the two most important model theoretical dividing lines: *stability*, which corresponds to the impossibility to interpret arbitrarily large linear orders, and *dependence* (or *NIP*, for "Non-Independence Property"), which corresponds to the the impossibility to interpret arbitrarily large membership graphs. The versions of these properties where we allow set parameters are *monadic stability* and *monadic dependence*.

The use of first-order transductions naturally fits the study of hereditary classes (i.e., classes that are closed under taking induced subgraphs). If we consider classes that are obtained as first-order transductions of other classes, the natural tractability limit is the realm of monadically NIP structures, as non monadically NIP classes allow to interpret the whole class of finite graphs. In this world, typical well behaved monadically NIP but monadically unstable classes of graphs are classes with bounded rankwidth (like cographs) and classes with bounded linear rankwidth (like half-graphs). This justifies a specific study of these classes, as well as the classes that admit finite p-covers with bounded rankwidth [33] or classes that admit finite p-covers with bounded linear rankwidth (like unit interval graphs), as they naturally extend structurally bounded expansion classes, which admit finite p-covers with bounded shrubdepth [19]. However we do not know whether classes with such covers are monadically NIP. The whole framework is schematically pictured in Figure 1.

This paper consists of two parts. The first part sets the scene and builds the framework that supports our study. The second part roots our study in concrete problems. In particular, we consider classes with bounded linear rankwidth and show how model theoretic and structural properties of classes with bounded linear rankwidth allow to prove new properties of these classes. In particular we prove the following theorems (formal definitions will be given in Section 2).

Theorem 4.6. *Let $\mathscr{C}$ be a class of graphs with bounded linear rankwidth. Then the following are equivalent:*

1. *$\mathscr{C}$ is stable,*
2. *$\mathscr{C}$ is monadically stable,*
3. *$\mathscr{C}$ has 2-covers with bounded shrubdepth,*
4. *$\mathscr{C}$ is sparsifiable,*
5. *$\mathscr{C}$ excludes some semi-induced half-graph,*
6. *$\mathscr{C}$ is a first-order transduction of a class with bounded expansion (i.e. has structurally bounded expansion),*
7. *$\mathscr{C}$ is a first-order transduction of a class with bounded pathwidth (i.e. has structurally bounded pathwidth).*

And we deduce

Theorem 6.2. *Let $\mathscr{C}$ be a class with low linear rankwidth covers. Then the following are equivalent:*

1. *$\mathscr{C}$ is monadically stable,*
2. *$\mathscr{C}$ is stable,*
3. *$\mathscr{C}$ excludes a semi-induced half-graph,*
4. *$\mathscr{C}$ has structurally bounded expansion.*

From the graph theoretic point of view, we briefly discuss how classes with bounded rankwidth differ from classes with bounded linear rankwidth and give some lower bounds for χ-boundedness of graphs with bounded rankwidth and for graphs with bounded linear rankwidth. Then we prove upper bounds for graphs with bounded linear rankwidth.

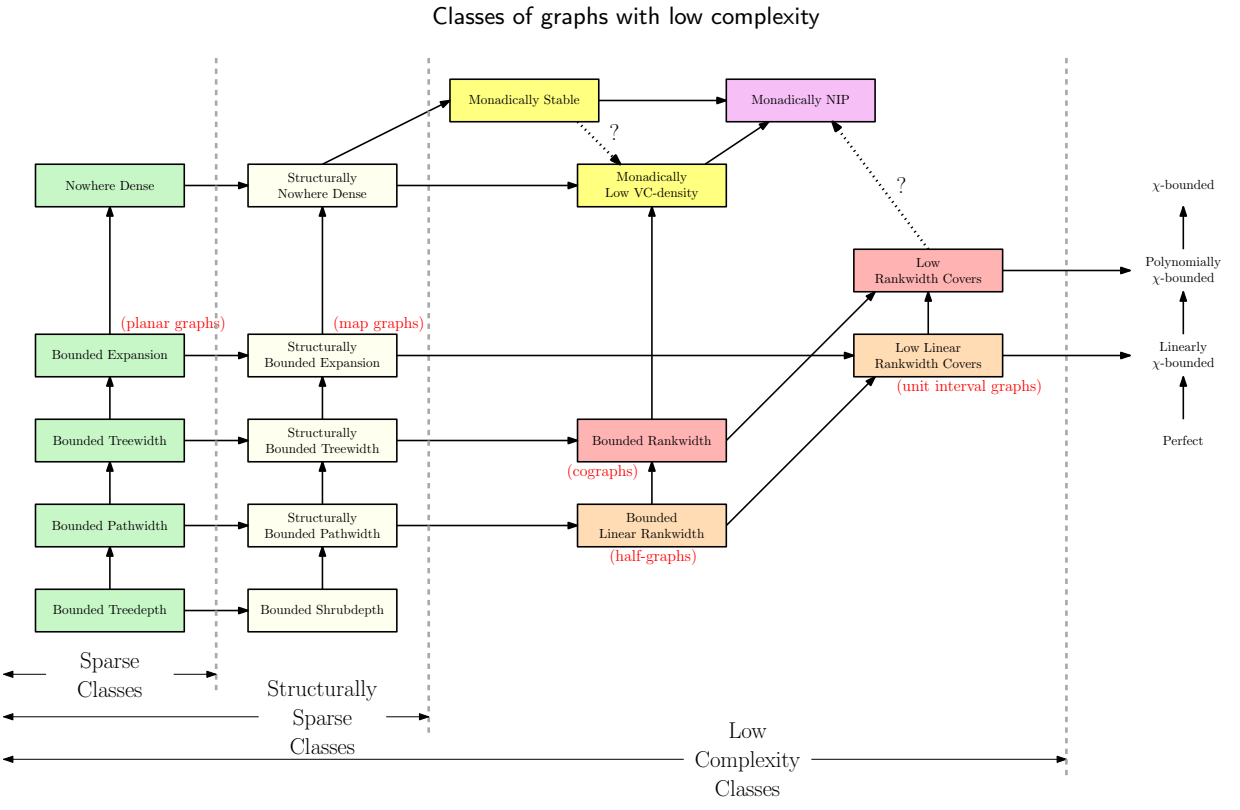

Figure 1: Inclusion map of graph classes. Some examples of classes are given in brackets.

Theorem 5.17. *Let* $f(r) = 2(2^r+1)(r+1)!\,2^{\binom{r+1}{2}}$. *The c-chromatic number of every graph G (that is the minimum order of a partition of* $V(G)$ *where each part induces a cograph) is bounded by* $f(\mathrm{lrw}(G))$, *where* $\mathrm{lrw}(G)$ *denotes the linear rank-width of G. Hence*

$$\chi(G) \leq f(\mathrm{lrw}(G))\,\omega(G).$$

Theorem 4.6 and a weaker form of Theorem 5.17 (Theorem 4.3) are proved in Section 4 by using the notion of linear NLC-width expression and Simon's factorization forest theorem.

The strong form of Theorem 5.17 is proved in Section 5 by a fine analysis of linear rankwidth decompositions. Along the way we also obtain an upper bound for the number of graphs with linear rankwidth at most r.

Theorem 5.15. *Unlabeled graphs with linear rankwidth at most r can be encoded using at most* $\binom{r}{2} + r\log_2 r + \log_2(12/e)r + O(\log_2 r)$ *bits per vertex. Precisely, the number of unlabelled graphs of order n with linear rankwidth at most r is at most*

$$\left[(2^r+1)(r+1)!\,2^{\binom{r}{2}}3^{r+1}\right]^n$$

.

2. Classes with low complexity

2.1. Structures and logic

A *signature* Σ is a finite set of relation and function symbols, each with a prescribed arity. In this paper we consider only signatures with relation symbols. A Σ-*structure* $\mathbf{A}$ consists of a finite *universe* (or *domain*) $V(\mathbf{A})$ and interpretations of the symbols in the signature: each relation symbol $R \in \Sigma$, say of arity k, is interpreted as a k-ary relation $R^{\mathbf{A}} \subseteq V(\mathbf{A})^k$. For a signature Σ, we consider standard first-order logic over Σ. If $\mathbf{A}$ is a structure and $X \subseteq V(\mathbf{A})$ then we denote by $\mathbf{A}[X]$ the *substructure* of $\mathbf{A}$ induced by X. The *Gaifman graph* of a structure $\mathbf{A}$ is the graph with vertex set $V(\mathbf{A})$

where two distinct elements $u, v \in \mathbf{A}$ are adjacent if and only if u and v appear together in some tuple in some relation of $\mathbf{A}$. For a formula $\varphi(x_1, \dots, x_k)$ with k free variables and a structure $\mathbf{A}$, we define

$$\varphi(\mathbf{A}) = \{(v_1, \dots, v_k) \in V(\mathbf{A})^k \ : \ \mathbf{A} \vDash \varphi(v_1, \dots, v_k)\}.$$

We usually write $\bar{x}$ for a tuple $(x_1, \dots, x_k)$ of variables and leave it to the context to determine the length of the tuple. The above equality then rewrites as $\varphi(\mathbf{A}) = \{\bar{v} \in V(\mathbf{A})^{|\bar{x}|} \ : \ \mathbf{A} \vDash \varphi(\bar{v})\}$. Also, for a formula $\varphi(\bar{x}, \bar{y})$ and $\bar{b} \in V(\mathbf{A})^{|\bar{y}|}$ we define

$$\varphi(\bar{b}, \mathbf{A}) = \{\bar{v} \in V(\mathbf{A})^{|\bar{x}|} \ : \ \mathbf{A} \vDash \varphi(\bar{v}, \bar{b})\}.$$

For signatures Σ, Σ^+ with $\Sigma \subset \Sigma^+$, the Σ-*reduct* of a Σ^+-structure $\mathbf{A}$ is the structure obtained from $\mathbf{A}$ by "forgetting" the relations in $\Sigma^+ \setminus \Sigma$. For a signature Σ, a *monadic lift* of a Σ-structure $\mathbf{A}$ is a Σ^+-structure $\Lambda(\mathbf{A})$ such that Σ^+ is the union of Σ and a set of unary relation symbols and $\mathbf{A}$ is the Σ-reduct of $\Lambda(\mathbf{A})$. Note that in the case of graphs, a monadic lift corresponds to a coloring of the vertices.

2.2. Graphs, colored graphs and trees.

Graphs can be viewed as finite structures over the signature consisting of a binary relation symbol E, interpreted as the edge relation, in the usual way. For graphs we follow the notations of [39]. In particular, for a graph G we denote by $|G|$ the *order* of G, that is the number of vertices of G, and by $\|G\|$ the *size* of G, that is the number of edges of G. A graph H is a *subgraph* of G, denoted $H \subseteq G$ if H can be obtained from G by deleting some vertices and edges. For a subset A of vertices of a graph G we denote by $G[A]$ the subgraph of G induced by G, that is the subgraph of G with vertex set A and same adjacencies as in G and we say that a graph H is an induced subgraph of G if it is isomorphic to some $G[A]$. We write $H \subseteq_i G$ if H is an induced subgraph of G. A class of graphs $\mathscr{C}$ is *monotone* if every subgraph of a graph in $\mathscr{C}$ also belongs to $\mathscr{C}$; it is *hereditary* if every induced subgraph of a graph in $\mathscr{C}$ also belongs to $\mathscr{C}$.

For a non-negative integer r, a $\leq r$-*subdivision* of a graph H is a graph obtained from H by subdividing each of its edges by at most k vertices (not necessarily the same number on each edge). An r-*subdivision* of H is a $\leq r$-subdivision of H where each edge is subdivided by exactly r vertices.

For a finite label set Γ, by a Γ-*colored* graph we mean a graph enriched by a unary predicate U_γ for each $\gamma \in \Gamma$. A rooted forest is an acyclic graph F together with a unary predicate $R \subseteq V(F)$ selecting one root in each connected component of F. A tree is a connected forest. The *depth* of a node x in a rooted forest F is the number of vertices in the unique path between x and the root of the connected component of x in F. In particular, x is a root of F if and only if F has depth 1 in F. The depth of a forest is the largest depth of any of its nodes. The *least common ancestor* of nodes x and y in a rooted tree is the common ancestor of x and y that has the largest depth.

2.3. Sparse graph classes

Treewidth, pathwidth and treedepth. Treewidth is an important width parameter of graphs that was introduced in [45] as part of the graph minors project. Pathwidth is a more restricted width measure that was introduced in [44]. The notion of treedepth was introduced in [35].

For our purposes it will be convenient to define these width measures in terms of intersection graphs. Let $S_1, \dots, S_n$ be a family of sets. The *intersection graph* defined by this family is the graph with vertex set $\{v_1, \dots, v_n\}$ and edge set $\{\{v_i, v_j\} \ : \ S_i \cap S_j \neq \emptyset\}$.

A *chordal graph* is the intersection graph of a family of subtrees of a tree. An *interval graph* is the intersection graph of a family of intervals. A *trivially perfect graph* is the intersection graph of a family of nested intervals. Alternatively, a trivially perfect graph is the comparability graph of a bounded-depth tree order.

The *treewidth* of a graph G is one less than the minimum clique number $\omega(H)$ of a chordal supergraph H of G [45], the *pathwidth* of a graph G is one less than the minimum clique number of an interval supergraph of G [6], and the *treedepth* of a graph G is the minimum clique number of a trivially perfect supergraph of G (direct from the definition):

$$\begin{aligned}
\mathrm{tw}(G) &= \min\{\omega(H) - 1 \ : \ H \text{ chordal and } H \supseteq G\},\\
\mathrm{pw}(G) &= \min\{\omega(H) - 1 \ : \ H \text{ interval graph and } H \supseteq G\},\\
\mathrm{td}(G) &= \min\{\omega(H) \ : \ H \text{ trivially perfect and } H \supseteq G\}.
\end{aligned}$$

A class $\mathscr{C}$ of graphs has *bounded treewidth, bounded pathwidth, or bounded treedepth*, respectively, if there is a bound $k \in \mathbb{N}$ such that every graph in $\mathscr{C}$ has treewidth, pathwidth, or treedepth, respectively, at most k.

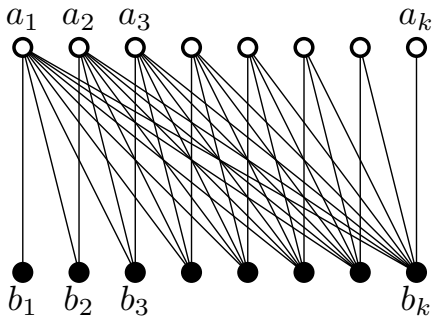

Figure 2: The half-graph H_k

Classes with bounded expansion. A graph H is a *depth-r topological minor* of a graph G if G contains a subgraph isomorphic to a $\leq 2r$-subdivision of H. A class $\mathscr{C}$ of graphs has *bounded expansion* if there is a function $f : \mathbb{N} \to \mathbb{N}$ such that $\frac{\|H\|}{|H|} \leq f(r)$ for every $r \in \mathbb{N}$ and every depth-r topological minor H of a graph from $\mathscr{C}$. Examples of classes with bounded expansion include the class of planar graphs, any class of graphs with bounded maximum degree, or more generally, any class of graphs that excludes a fixed topological minor. We lift the notion with bounded expansion to classes of structures over an arbitrary fixed signature, by requiring that their class of Gaifman graphs has bounded expansion. In particular, a class of colored graphs has bounded expansion if and only if the class of underlying uncolored graphs has bounded expansion. For an in-depth study of classes with bounded expansion we refer the reader to the monograph [39].

Nowhere dense classes. A class $\mathscr{C}$ is *nowhere dense* if there is a function $f : \mathbb{N} \to \mathbb{N}$ such that $\omega(H) \leq f(r)$ for every $r \in \mathbb{N}$ and every depth-r topological minor H of a graph from $\mathscr{C}$ [37, 38].

2.4. Monadic stability, monadic dependence, and low VC-density

The model theoretic approach of complexity is based on the study of properties rather than on the study of objects. This is witnessed by the fact that the central subjects of study in model theory are theories and that the actual structures are only considered as models of theories. Nevertheless, most notions defined on theories have their counterpart on models or on classes of models. One of the main goals of stability theory (also known as classification theory) is to classify the models of a given first-order theory according to some simple system of cardinal invariants. In this respect, elementary theories are *stable theories* and still reasonably well behaved theories are *NIP theories* (also called *dependent theories*). These notions can be translated to classes of structures as follows:

Definition 2.1. A class $\mathscr{C}$ of structures is *stable* if for every first-order formula $\varphi(\bar{x}, \bar{y})$ there exists an integer k such that for every structure $\mathbf{A} \in \mathscr{C}$ and for all tuples $\bar{a}_1, \dots, \bar{a}_\ell, \bar{b}_1, \dots, \bar{b}_\ell$ of elements of $\mathbf{A}$, if

$$\mathbf{A} \models \varphi(\bar{a}_i, \bar{b}_j) \quad \iff \quad i \leq j \tag{1}$$

for all $i, j \in [\ell]$, then $\ell \leq k$.

The graph H_k on vertices $a_1, \dots, a_k, b_1, \dots b_k$ with edges $\{a_i, b_j\}$ for $1 \leq i \leq j \leq k$ is called a *half-graph* or *ladder* of length k, see Figure 2.

Definition 2.2. A class $\mathscr{C}$ of structures is *dependent* (or *NIP*) if for every first-order formula $\varphi(\bar{x}, \bar{y})$ there exists an integer k such that for every structure $\mathbf{A} \in \mathscr{C}$ and for all tuples $\bar{a}_i$ ($i \in [\ell]$) and, $\bar{b}_I$ ($I \subseteq [\ell]$) of elements of $\mathbf{A}$, if

$$\mathbf{A} \models \varphi(\bar{a}_i, \bar{b}_I) \quad \iff \quad i \in I \tag{2}$$

for all $i \in [\ell]$ and all $I \subseteq [\ell]$, then $\ell \leq k$.

Note that every stable class is dependent.

A stronger notion of stability and of dependence arises when one allows to apply arbitrary monadic lifts to the structures in $\mathscr{C}$ before using the formula φ. These variants are called *monadic stability* and *monadic dependence*. The expressive power gained by the monadic lift is so strong that tuples of free variables can be replaced by single free variables in the above definitions [4].

Definition 2.3. A class $\mathscr{C}$ of Σ-structures is *monadically stable* if for every expansion Σ^+ of Σ by unary predicate symbols and every first-order Σ^+-formula $\varphi(x, y)$ there exists an integer k such that for every monadic lift $\mathbf{A}^+$ of any structure $\mathbf{A} \in \mathscr{C}$ and for all elements $a_1, \dots, a_\ell, b_1, \dots, b_\ell$ of $\mathbf{A}$, if

$$\mathbf{A}^+ \models \varphi(a_i, b_j) \quad \iff \quad i < j \tag{3}$$

for all $i, j \in [\ell]$, then $\ell \leq k$.

Definition 2.4. A class $\mathscr{C}$ of Σ-structures is *monadically dependent* (or *monadically NIP*) if for every expansion Σ^+ of Σ by unary predicate symbols and every first-order Σ^+-formula $\varphi(x, y)$ there exists an integer k such that for every monadic lift $\mathbf{A}^+$ of any structure $\mathbf{A} \in \mathscr{C}$ and for all elements a_i ($i \in [\ell]$) and b_I ($I \subseteq [\ell]$) of $\mathbf{A}$, if

$$\mathbf{A}^+ \models \varphi(a_i, b_I) \quad \iff \quad i \in I \tag{4}$$

for all $i \in [\ell]$ and all $I \subseteq [\ell]$, then $\ell \leq k$.

Note that every monadically stable class is monadically dependent.

For a formula $\varphi(\bar{x}, \bar{y})$, the *VC-density* $\mathrm{vc}^{\mathscr{C}}(\varphi)$ of a formula φ in a class $\mathscr{C}$ (containing arbitrarily large structures) is defined as

$$\mathrm{vc}^{\mathscr{C}}(\varphi) = \lim_{t\to\infty} \sup_{\mathbf{A}\in\mathscr{C}} \sup_{\substack{B\subseteq V(\mathbf{A})\\ |B|=t}} \frac{\log |\{\varphi(\bar{v}, \mathbf{A}) \cap B^{|\bar{x}|} \ : \ \bar{v} \in V(\mathbf{A})^{|\bar{y}|}\}|}{\log |B|}$$

The *VC-density* $\mathrm{vc}^{\mathscr{C}}$ of the class $\mathscr{C}$ is

$$\mathrm{vc}^{\mathscr{C}}(n) = \sup\{\mathrm{vc}^{\mathscr{C}}(\varphi) \ : \ \varphi(\bar{x}; \bar{y}) \text{ is a formula with } |\bar{y}| = n\}.$$

According to the Sauer-Shelah Lemma [46, 47], a class $\mathscr{C}$ is NIP if and only if $\mathrm{vc}^{\mathscr{C}}(\varphi) < \infty$ for every formula φ. However, it is possible for a NIP class (and even for a stable class) to have $\mathrm{vc}^{\mathscr{C}}(1) = \infty$. On the other hand, it is easily checked that (unless structures in $\mathscr{C}$ have bounded size) for every positive integer n we have $\mathrm{vc}^{\mathscr{C}}(n) \geq n$. A class $\mathscr{C}$ has *low VC-density* if $\mathrm{vc}^{\mathscr{C}}(n) = n$ for all integers n [25]. In particular, if a class $\mathscr{C}$ has low VC-density we have $\mathrm{vc}^{\mathscr{C}}(\varphi) < \infty$ for every formula φ thus $\mathscr{C}$ is NIP. We say that $\mathscr{C}$ has *monadically low VC-density* if the class $\{\mathbf{A}^+ \ : \ \mathbf{A}^+$ monadic lift of $\mathbf{A} \in \mathscr{C}\}$ has low VC-density. Note that every class with monadically low VC-density is monadically NIP.

Theorem 2.5. *Let $\mathscr{C}$ be a class of graphs.*

1. *If $\mathscr{C}$ is nowhere dense, then $\mathscr{C}$ is monadically stable ([Adler, Adler [1]; Podewski, Ziegler [43]).*
2. *If $\mathscr{C}$ is nowhere dense, then $\mathscr{C}$ has monadicallly low VC-density (Pilipczuk, Siebertz, and Toruńczyk [42]]).*

Theorem 2.6 ([Adler, Adler [1]; Podewski, Ziegler [43])**.** *Let $\mathscr{C}$ be a monotone class of graphs. If $\mathscr{C}$ is NIP, then $\mathscr{C}$ is nowhere dense.*

Corollary 2.1. *Let $\mathscr{C}$ be a monotone class of graphs. Then the following are equivalent.*

1. *$\mathscr{C}$ is nowhere dense,*
2. *$\mathscr{C}$ is stable,*
3. *$\mathscr{C}$ is monadically stable,*
4. *$\mathscr{C}$ is NIP,*
5. *$\mathscr{C}$ is monadically NIP,*
6. *$\mathscr{C}$ has low VC-density,*
7. *$\mathscr{C}$ has monadically low VC-density.*

2.5. Interpretations and transductions

In this paper, by an *interpretation* of Σ'-structures in Σ-structures we mean a transformation $\mathtt{I}$ defined by means of formulas $\varphi_R(\bar{x})$ (for each $R \in \Sigma'$ of arity $|\bar{x}|$) and a formula $\nu(x)$. For every Σ-structure $\mathbf{A}$, the Σ'-structure $\mathtt{I}(\mathbf{A})$ has domain $\nu(\mathbf{A})$ and the interpretation of each relation $R \in \Sigma'$ is given by $R^{\mathtt{I}(\mathbf{A})} = \varphi_R(\mathbf{A}) \cap \nu(\mathbf{A})^{|\bar{x}|}$. For some fixed interpretation $\mathtt{I}$ we often say that a structure $\mathbf{B}$ is an interpretation of $\mathbf{A}$ (it would be more precise to say that $\mathbf{B}$ is an interpretation *in* $\mathbf{A}$) if $\mathbf{B} = \mathtt{I}(\mathbf{A})$.

A *transduction* T is the composition $\mathtt{I} \circ \Lambda$ of a monadic lift Λ and an interpretation $\mathtt{I}$. It is easily checked that the composition of two transductions is again a transduction. Again, for some fixed transduction T we often say that a structure $\mathbf{B}$ is a transduction of $\mathbf{A}$ if $\mathbf{B} = \mathtt{I}(\mathbf{A})$.

Let $\mathscr{C}$ and $\mathscr{D}$ be classes of $\Sigma_{\mathscr{C}}$-structures and $\Sigma_{\mathscr{D}}$-structures, respectively. Let $\mathtt{I}$ be an interpretation of $\Sigma_{\mathscr{D}}$-structures in $\Sigma^+_{\mathscr{C}}$-structures, where $\Sigma^+_{\mathscr{C}} \setminus \Sigma_{\mathscr{C}}$ is a finite set of unary relation symbols. If, for every $\mathbf{B}$ in $\mathscr{D}$ there exists a lift $\mathbf{A}^+$ of some structure $\mathbf{A} \in \mathscr{C}$ such that $\mathbf{B} = \mathtt{I}(\mathbf{A}^+)$ we write

$$\mathscr{C} \xrightarrow{\ \mathtt{I}\ } \!\!\!\!\rightarrow \mathscr{D},$$

and we write

$$\mathscr{C} \longrightarrow\!\!\!\!\rightarrow \mathscr{D}$$

if there exists $\mathtt{I}$ such that $\mathscr{C} \xrightarrow{\ \mathtt{I}\ } \!\!\!\!\rightarrow \mathscr{D}$. In this case we call the class $\mathscr{D}$ a *transduction* of $\mathscr{C}$. The classes $\mathscr{C}$ and $\mathscr{D}$ are called *transduction-equivalent* if $\mathscr{C} \longrightarrow\!\!\!\!\rightarrow \mathscr{D}$ and $\mathscr{D} \longrightarrow\!\!\!\!\rightarrow \mathscr{C}$.

The definitions of interpretations and transductions given above naturally extend to any logic $\mathcal{L}$. We speak of an $\mathcal{L}$-interpretation if the formulas φ_R and ν in the above definition are $\mathcal{L}$-formulas, and of an $\mathcal{L}$-transduction if we combine a monadic lift with an $\mathcal{L}$-interpretation. If the logic $\mathcal{L}$ is not mentioned explicitly, we mean first-order logic FO. Another commonly considered logic is monadic second-order logic (MSO).

We want to emphasize that in this paper our focus is not the study of what graph class $\mathscr{D}$ is *produced* by a transduction from a class $\mathscr{C}$, it is rather the study of how to *encode* $\mathscr{D}$ *in* $\mathscr{C}$. In particular when we say that a class $\mathscr{D}$ is a transduction of a class $\mathscr{C}$, we do not need to use all the graphs in $\mathscr{C}$, nor to verify properties of the monadic lifts. For example, when $\mathscr{D}$ is a transduction of $\mathscr{C}$, then every subclass of $\mathscr{D}$ is also a transduction of $\mathscr{C}$.

The definition of monadic stability and monadic dependence can naturally be given in terms of transductions. Let $\mathscr{H}$ denote the class of *half-graphs*, that is the class of the bipartite graphs H_k with vertex set $\{a_1, \dots, a_k, b_1, \dots, b_k\}$ and edges $\{a_i, b_j\}$ for every $1 \le i \le j \le k$ (see Figure 2). Let $\mathscr{G}$ denote the class of all finite graphs. We have

$$\begin{aligned} \mathscr{C} \text{ is monadically stable} &\iff \mathscr{C} \not\longrightarrow\!\!\!\!\rightarrow \mathscr{H}. \\ \mathscr{C} \text{ is monadically NIP} &\iff \mathscr{C} \not\longrightarrow\!\!\!\!\rightarrow \mathscr{G}. \end{aligned}$$

Lemma 2.7 ([3])**.** *A stable class $\mathscr{C}$ is monadically unstable if and only if $\mathscr{C}$ has a transduction to the class of all* 1*-subdivisions of complete bipartite graphs.*

In particular, if a stable class is not monadically stable it is not monadically NIP as there is an easy transduction from the class of all 1-subdivisions of complete bipartite graphs to the class of all finite graphs. As monadically stable classes are monadically NIP we deduce the following corollary.

Corollary 2.2. *A class $\mathscr{C}$ is monadically stable if and only if it is both stable and monadically NIP.*

We use the term of *structurally* xxx for classes that are transductions of classes that are xxx. For instance, a class has structurally bounded treewidth if it is the transduction of a class with bounded treewidth.

The following characterizations of classes with bounded treewidth, pathwidth, rankwidth, linear rankwidth, and shrubdepth show the deep connections between these width measures and logical transductions (and at this point will serve as a definition of the notions of rankwidth, linear rankwidth and shrubdepth).

1. A class $\mathscr{C}$ of graphs has bounded treewidth (pathwidth, respectively) if and only if there exists an MSO-transduction T such that the incidence graph of every $G \in \mathscr{C}$ is the result of applying T to some tree (path, respectively) ([10] (see also [11], Theorem 7.47)).

2. A class $\mathscr{C}$ of graphs has bounded rankwidth (linear rankwidth, respectively) if and only if there exists an MSO-transduction T such that every $G \in \mathscr{C}$ is the result of applying T to some tree (path, respectively). ([10] (see also [11], Theorem 7.47)).

3. A class $\mathscr{C}$ of graphs has bounded rankwidth (linear rankwidth, respectively) if and only if there exists an FO-transduction T such that every $G \in \mathscr{C}$ is the result of applying T to some tree order (linear order, respectively) ([9]).

4. A class $\mathscr{C}$ of graphs has bounded shrubdepth if and only if there exist an FO-transduction T and a height h such that every $G \in \mathscr{C}$ is the result of applying T to some tree of depth at most h ([21, 20]).

We can rewrite properties (3) and (4) as follows:

$$\begin{aligned} \mathscr{C} \text{ has bounded rankwidth} &\iff \mathscr{Y}^{\leq} \longrightarrow\!\!\!\!\!\rightarrow \mathscr{C}, \\ \mathscr{C} \text{ has bounded linear rankwidth} &\iff \mathscr{L}^{\leq} \longrightarrow\!\!\!\!\!\rightarrow \mathscr{C}, \\ \mathscr{C} \text{ has bounded shrubdepth} &\iff \exists n \ \ \mathscr{Y}_n \longrightarrow\!\!\!\!\!\rightarrow \mathscr{C}, \end{aligned}$$

where $\mathscr{Y}^{\leq}$ denotes the class of all finite tree orders, $\mathscr{L}^{\leq}$ denotes the class of all linear orders, and $\mathscr{Y}_n$ denotes the class of trees with depth at most n.

Note that in the characterizations above $\mathscr{Y}^{\leq}$ can be replaced by the class of trivially perfect graphs (or by the larger class of cographs) and $\mathscr{L}^{\leq}$ can be replaced by the class of transitive tournaments or by the class of half-graphs.

Remark 2.8. Since the class of all graphs does not have bounded rankwidth, we deduce that if $\mathscr{C}$ has bounded rankwidth we have $\mathscr{C} \not\longrightarrow\!\!\!\!\!\rightarrow \mathscr{G}$. Hence every class with bounded rankwidth is monadically NIP.

In particular, Corollary 2.2 implies the following:

Remark 2.9. A class with bounded rankwidth is monadically stable if and only if it is stable.

2.6. Weakly sparse classes

It appears that a basic property that makes a graph class dense is that graphs in it contain arbitrarily large complete bipartite graphs $K_{t,t}$ with partitions of equal size (bicliques). Indeed, forbidding a biclique as a subgraph (or, equivalently, forbidding a clique and a biclique as induced subgraphs) is known to have a strong consequence on classes with low complexity. We call a class $\mathscr{C}$ *weakly sparse* if it excludes some biclique $K_{t,t}$ as a subgraph.

Theorem 2.10. *Let $\mathscr{C}$ be a weakly sparse class of graphs.*

1. *If $\mathscr{C}$ has bounded shrubdepth, then $\mathscr{C}$ has bounded treedepth [19].*

2. *If $\mathscr{C}$ has bounded linear rankwidth, then $\mathscr{C}$ has bounded pathwidth [27].*

3. *If $\mathscr{C}$ has bounded rankwidth, then $\mathscr{C}$ has bounded treewidth [27].*

We call a class *sparsifiable* if it is transduction-equivalent to a weakly sparse class.

The assumption that a class is weakly sparse allows frequently to work with induced subgraph instead of subgraphs. For instance:

Theorem 2.11 (Dvořák [15])**.** *Let $\mathscr{C}$ be a hereditary weakly sparse graph class. Then*

1. *$\mathscr{C}$ has bounded expansion if and only if there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that for every graph H, if the $\leq k$-subdivision of H belongs to $\mathscr{C}$ then the average degree of H is at most $f(k)$ (for all non-negative integers k).*

2. *$\mathscr{C}$ is nowhere dense if and only if there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that the class $\mathscr{C}$ contains no $\leq k$-subdivision of a complete graphs of order greater than $f(k)$ (for all non-negative integers k).*

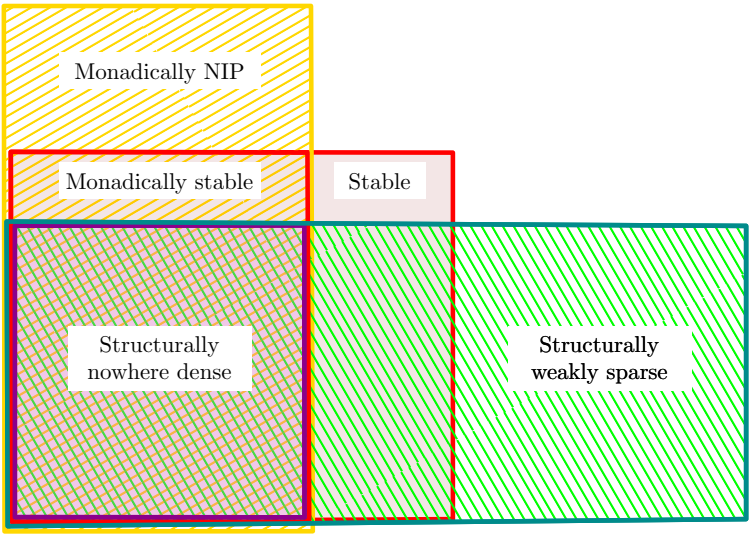

Figure 3: A class is monadically stable if and only if it is both monadically NIP and stable; it is structurally nowhere dense if and only if it is both monadically NIP and structurally weakly sparse. No class is currently known, which is monadically stable but not structurally nowhere dense.

Corollary 2.3. *Let $\mathscr{C}$ be a monadically NIP class. Then $\mathscr{C}$ is nowhere dense if and only if it is weakly sparse.*

Proof. If $\mathscr{C}$ is nowhere dense, then there exists a number t such that no graph in $\mathscr{C}$ contains a ≤ 1-subdivision of a complete graph K_t as a subgraph. In particular, no graph in $\mathscr{C}$ contains $K_{t,\binom{t}{2}}$ as a subgraph, hence $\mathscr{C}$ is weakly sparse.

Conversely, assume towards a contradiction that the class $\mathscr{C}$ is weakly sparse and not nowhere dense. According to Theorem 2.11 we can find arbitrarily large induced q-subdivisions of complete graphs for some integer q. It is then easy to interpret (in a monadic lift) arbitrary graphs, contradicting the hypothesis that $\mathscr{C}$ is monadically NIP. □

Corollary 2.4. *Every sparsifiable monadically NIP class of graphs is structurally nowhere dense.*

2.7. Decompositions and covers

For $p \in \mathbb{N}$, a *p-cover* of a structure $\mathbf{A}$ is a family $\mathcal{U}_{\mathbf{A}}$ of subsets of $V(\mathbf{A})$ such that every set of at most p elements of $\mathbf{A}$ is contained in some $U \in \mathcal{U}_{\mathbf{A}}$. If $\mathscr{C}$ is a class of structures, then a *p-cover* of $\mathscr{C}$ is a family $\mathcal{U} = (\mathcal{U}_{\mathbf{A}})_{\mathbf{A}\in\mathscr{C}}$, where $\mathcal{U}_{\mathbf{A}}$ is a p-cover of $\mathbf{A}$. A 1-cover is simply called a *cover*. A p-cover $\mathcal{U}$ is *finite* if $\sup\{|\mathcal{U}_{\mathbf{A}}| \,:\, \mathbf{A} \in \mathscr{C}\}$ is finite. Let $\mathscr{C}[\mathcal{U}]$ denote the class structures $\{\mathbf{A}[U] \,:\, \mathbf{A} \in \mathscr{C}, U \in \mathcal{U}_{\mathbf{A}}\}$. For a class $\mathscr{W}$ we say that a cover $\mathcal{U}$ is a $\mathscr{W}$-cover if $\mathscr{C}[\mathcal{U}] \subseteq \mathscr{W}$. If $\mathscr{W}$ is a class of bounded treedepth, bounded shrubdepth, etc., we call a $\mathscr{W}$-cover a bounded treedepth cover, bounded shrubdepth cover, etc. The class $\mathscr{C}$ admits *low treedepth covers*, *low shrubdepth covers*, etc. if and only if for every $p \in \mathbb{N}$ there is a finite p-cover $\mathcal{U}_p$ of $\mathscr{C}$ with bounded treedepth, shrubdepth, etc.

Theorem 2.12 ([36, 19])**.** *A class of graphs has bounded expansion if and only if it has low treedepth covers.*

The following notion of *shrubdepth* has been proposed in [21] as a dense analogue of treedepth. Originally, shrubdepth was defined using the notion of *tree-models*. We present an equivalent definition based on the notion of *connection models*, introduced in [21] under the name of *m-partite cographs* with bounded depth.

A *connection model* with labels from Γ is a rooted labeled tree T where each leaf u is labeled by a label $\gamma(u) \in \Gamma$, and each non-leaf node x is labeled by a binary relation $C(x) \subset \Gamma \times \Gamma$. If $C(x)$ is symmetric for all non-leaf nodes x, then such a model defines a graph G on the leaves of T, in which two distinct leaves u and v are connected by an edge if and only if $(\gamma(u), \gamma(v)) \in C(x)$, where x is the least common ancestor of u and v. We say that T is a *connection model* of the resulting graph G. A class of graphs $\mathscr{C}$ has *bounded shrubdepth* if there are a number $h \in \mathbb{N}$ and a finite set of labels Γ such that every graph $G \in \mathscr{C}$ has a connection model of depth at most h using labels from Γ.

A *cograph* is a graph that has a connection model (called a *cotree*) with a labels set Γ containing only a single label. Cographs are perfect graphs, that is, graphs in which the chromatic number of every induced subgraph equals the clique number of that subgraph.

Theorem 2.13 ([19])**.** *A class of graphs has structurally bounded expansion if and only if it has low shrubdepth covers.*

Lemma 2.14 ([19])**.** *Every class that admits* 2*-covers of bounded shrubdepth is sparsifiable.*

2.8. χ-boundedness

Recall that a class of graphs $\mathscr{C}$ is *χ-bounded* if there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that for every graph G in $\mathscr{C}$ we have $\chi(G) \leq f(\omega(G))$ [29]. If f is polynomial (resp. linear) then the class is said to be *polynomially χ-bounded* (resp. *linearly χ-bounded*). A prototypical example of χ-bounded class is the class of *perfect graphs*, which is the class of graphs G, such that all induced subgraphs of G have their chromatic number equal to their clique number.

The *c-chromatic number* of a graph G is the minimum size of a partition $V_1, \dots, V_k$ of the vertex set of G such that $G[V_i]$ is a cograph for each $i \in \{1, \dots, k\}$ [23]. We denote by $c(G)$ the c-chromatic number of G. As cographs are perfect [5] we have the following general inequality for every graph G:

$$\chi(G) \leq c(G)\,\omega(G).$$

Lemma 2.15. *Every class with bounded shrubdepth has bounded c-chromatic number.*

Proof. Let $h \in \mathbb{N}$ and let Γ be a finite set such that every graph $G \in \mathscr{C}$ has a connection model of depth at most h using labels from Γ, and let $\gamma \in \Gamma$. It is easily checked that the subgraph of G induced by the vertices with label γ has a connection model using only the label γ. It follows that this induced subgraph is a cograph, hence the c-chromatic number of G is at most $|\Gamma|$. □

Corollary 2.5. *Every class $\mathscr{C}$ that admits* 1*-covers of bounded shrubdepth has bounded c-chromatic number, and hence is linearly χ-bounded.*

Proof. Indeed, if $\mathscr{C}$ admits a 1-cover of bounded shrubdepth then $\mathscr{C}$ has bounded c-chromatic number, thus is linearly χ-bounded. □

3. Rankwidth and linear rankwidth

We now turn to the study of classes of bounded rankwidth and linear rankwidth. After recalling several equivalent definitions of these width measures, we prove that for every proper hereditary family $\mathscr{F}$ of graphs (like cographs), there is a class $\mathscr{C}$ with bounded rankwidth such that for every integer k there is a graph $G \in \mathscr{C}$ such that all vertex colorings with k colors contain a monochromatic induced subgraph not in $\mathscr{F}$.

3.1. Definitions

Classes with bounded rankwidth and classes with bounded linear rankwidth enjoy several characterizations. In particular, for a class $\mathscr{C}$ the following are equivalent:

1. $\mathscr{C}$ has bounded rankwidth,
2. $\mathscr{C}$ has bounded cliquewidth,
3. $\mathscr{C}$ has bounded NLC-width,
4. $\mathscr{Y}^{\leq} \longrightarrow\!\!\!\!\!\rightarrow \mathscr{C}$,

as well as the following:

1. $\mathscr{C}$ has bounded linear rankwidth,
2. $\mathscr{C}$ has bounded linear cliquewidth,
3. $\mathscr{C}$ has bounded linear NLC-width,
4. $\mathscr{C}$ has bounded neighborhood-width,
5. $\mathscr{L}^{\leq} \longrightarrow\!\!\!\!\!\rightarrow \mathscr{C}$.

Cliquewidth and linear cliquewidth. Graphs of bounded treewidth have bounded average degree and therefore the application of treewidth is (mostly) limited to sparse graph classes. *Cliquewidth* was introduced in [12] with the aim to extend hierarchical decompositions also to dense graphs. However, there is no known polynomial-time algorithm to determine whether the cliquewidth of an input graph is at most k for fixed $k \geq 4$. A notable application of cliquewidth is the extension of Courcelle's Theorem for testing MSO properties in cubic time (or linear time if a clique decomposition is given) on graph classes of bounded cliquewidth [13]. The notion of linear cliquewidth has been introduced in [28]. We denote by $\mathrm{cw}(G)$ the cliquewidth of a graph G and by $\mathrm{lcw}(G)$ the linear cliquewidth of G.

NLC-width and linear NCL-width. The notions of NLC-width and linear NLC-width were introduced in [49] and [28]. Before giving the definition of linear NLC-width we recall some terminology of formal language theory. An *alphabet* is a finite set Ω, whose members are called *letters* (or *symbols*). A *word* (or *string*) of length n over the alphabet Ω is a sequence of n letters from Ω, and we denote by Ω^* (resp. Ω^+) the set of all words (resp. of all non-empty words) over Ω.

Definition 3.1. For $k \in \mathbb{N}$, let V be a finite set, and let $\Omega_k(V)$ be the alphabet whose letters are quadruples (v, c, e, r), where

- $v \in V$,
- $c \in [k]$,
- $e \subseteq [k]$, and
- $r : [k] \to [k]$.

For a letter $a = (v, c, e, r) \in \Omega_k(V)$ we write v_a, c_a, e_a and r_a for v, c, e and r, respectively.

Let k be a positive integer. We say that a word $\alpha \in \Omega_k(V)^+$ is *admissible* if no two letters a and b of α have the same v-value. We denote by $\mathfrak{A}_k(V)$ the set of all admissible words in Ω_k^+.

Definition 3.2. A *linear NLC-expression of width k over V* is a word in $\mathfrak{A}_k(V)$. With linear NLC-expressions α of width k over V we recursively associate a colored graph $\Xi(\alpha)$ whose vertices are the v-values of the letters of α, colored by colors from $[k]$ as follows.

- If $|\alpha| = 1$, then $\Xi(\alpha)$ is the single vertex graph, with vertex v_α colored c_α.
- If $\alpha = \alpha' a$, where $|a| = 1$, then $\Xi(\alpha)$ is the graph obtained from $\Xi(\alpha')$ by adding the vertex v_a with color c_a, connecting v_a to all vertices w of $\Xi(\alpha')$ that have a color in e_a, and finally, changing the color of each vertex with color i to color $r_a(i)$.

The linear NLC-width of a graph G is the minimum integer k such that G is identical to the graph $\Xi(\alpha)$ for some $\alpha \in \mathfrak{A}_k(V(G))$.

It is clear that the vertex set of $\Xi(\alpha)$ can be identified with the letters of α. and that for every subword β of α the graph $\Xi(\beta)$ is the subgraph of $\Xi(\alpha)$ induced by the v-values of the letters of β.

We have [28]:

$$\text{linear NLC-width}(G) \leq \mathrm{lcw}(G) \leq \text{linear NLC-width}(G) + 1. \tag{5}$$

Neighborhood-width. The *neighborhood-width* of a graph is the smallest integer k, such that there is a linear order $v_1, \ldots, v_n$ on the vertex set of G such that for every vertex v_j the vertices v_i with $i \leq j$ can be divided into at most k subsets, each members having the same neighborhood with respect to the vertices v_k with $k > j$. The neighbourhood-width of a graph differs from its linear clique-width or linear NLC-width at most by one [26].

Rankwidth and linear rankwidth. The notion of *rankwidth* was introduced in [41] as an efficient approximation to cliquewidth. For a graph G and a subset $X \subseteq V(G)$ we define the *cut-rank* of X in G, denoted $\rho_G(X)$, as the rank of the $|X| \times |V(G) \setminus X|$ 0-1 matrix A_X over the binary field $\mathbb{F}_2$, where the entry of A_X on the i-th row and j-th column is 1 if and only if the i-th vertex in X is adjacent to the j-th vertex in $V(G) \setminus X$. If $X = \emptyset$ or $X = V(G)$, then we define $\rho_G(X)$ to be zero.

A *subcubic* tree is a tree where every node has degree 1 or 3. A *rank decomposition* of a graph G is a pair (T, L), where T is a subcubic tree with at least two nodes and L is a bijection from $V(G)$ to the set of leaves of T. For an edge $e \in E(T)$, the connected components of $T - e$ induce a partition (X, Y) of the set of leaves of T. The *width* of an edge e of (T, L) is $\rho_G(L^{-1}(X))$. The width of (T, L) is the maximum width over all edges of T (and at least 0). The *rankwidth* $\mathrm{rw}(G)$ of G is the minimum width over all rank decompositions of G. When the graph has at most one vertex then there is no rank decomposition and the rankwidth is defined to be 0.

Cliquewidth and rankwidth are functionally related [41]: For every graph G we have

$$\mathrm{rw}(G) \leq \mathrm{cw}(G) \leq 2^{\mathrm{rw}(G)+1} - 1. \tag{6}$$

Hence, a class $\mathscr{C}$ of graphs has bounded cliquewidth if and only if $\mathscr{C}$ has bounded rankwidth.

The *linear rankwidth* of a graph is a linearized variant of rankwidth, similarly as pathwidth is a linearized variant of treewidth. Let G be an n-vertex graph and let $v_1, \ldots, v_n$ be an order of $V(G)$. The *width* of this order is $\max_{1 \leq i \leq n-1} \rho_G(\{v_1, \ldots, v_i\})$. The *linear rankwidth* of G, denoted $\mathrm{lrw}(G)$, is the minimum width over all linear orders of G. If G has less than 2 vertices we define the linear rankwidth of G to be zero. An alternative way to define the linear rankwidth is to define a linear rank decomposition (T, L) to be a rank decomposition such that T is a caterpillar and then define linear rankwidth as the minimum width over all linear rank decompositions. Recall that a caterpillar is a tree in which all the vertices are within distance 1 of a central path.

It was proved in [26] that the linear cliquewidth and the linear rankwidth of a graph are bound to each other: Precisely, for every graph G we have

$$\mathrm{lrw}(G) \leq \text{linear NLC-width}(G) \leq \mathrm{lcw}(G) \leq 2^{\mathrm{lrw}(G)}. \tag{7}$$

A linear ordering witnessing $\mathrm{lrw}(G) \leq k$ (or deciding $\mathrm{lrw}(G) > k$) for fixed k can be computed in time $O(n^3)$ [30].

3.2. Lexicographic product

We denote by $G \bullet H$ the lexicographic product of G and H, that is the graph with vertex set $V(G) \times V(H)$ where (u, v) is adjacent to (u', v') is u is adjacent to u' in G or $u = u'$ and v is adjacent to v' in H. Note that this operation, though non-commutative, is associative. By $G \oplus H$ we denote the operation of forming the disjoint union of G and H and connecting all vertices of the copy of G to all vertices of the copy of H.

Lemma 3.3. *For all graphs G, H we have*

$$\mathrm{rw}((G \bullet H) \oplus K_1) = \max(\mathrm{rw}(G \oplus K_1), \mathrm{rw}(H \oplus K_1)).$$

Proof. Let (Y_G, L_G) and (Y_H, L_H) be rank decompositions of $G \oplus K_1$ and $H \oplus K_1$, respectively, of minimum width. Assume the leaves of Y_G are $V(G) \cup \{\alpha\}$ and the leaves of Y_H are $V(H) \cup \{\beta\}$. Consider $|G|$ copies of Y_H and glue these copies on Y_G by identifying each leaf of Y_G that is a vertex of G with the vertex β of the associated copy. The obtained tree Y together with the naturally inherited mapping L from the vertices of $(G \bullet H) \oplus K_1$ to the leaves of Y is a rank decomposition of $(G \bullet H) \oplus K_1$ (see Figure 4).

Now consider any edge of this rank decomposition of $(G \bullet H) \oplus K_1$. There are two cases:

- Assume the edge is within the rank decomposition Y_G of $G \oplus K_1$. Let A, B be the induced partition of the vertices of $(G \bullet H) \oplus K_1$. This partition corresponds to a partition A', B' of $G \oplus K_1$. Let $p \colon A \to A'$ be the natural projection. We may assume that the vertex α belongs to B in $(G \bullet H) \oplus K_1$ (hence to B' in $G \oplus K_1$). For every vertex $v \in B$ we have $N_{(G \bullet H) \oplus K_1}(v) \cap A = (N_{G \oplus K_1}(p(v)) \cap A') \times V(H)$. Hence the cut-rank of A in $(G \bullet H) \oplus K_1$ equals the cut-rank of A' in $G \oplus K_1$.

- Otherwise, the edge is within the rank decomposition of a copy of $H \oplus K_1$. Let A, B be the induced partition of the vertices of $(G \bullet H) \oplus K_1$, where $B \subseteq \{v_0\} \times B'$ for some $v_0 \in V(G)$ and some $B' \subseteq V(H)$. Then all vertices $v \in (\{v_0\} \times V(H)) \setminus B$ have the neighborhood $(\{v_0\} \times N_H(v)) \cap B$ on B, while the vertices $v \in A \setminus (\{v_0\} \times V(H))$ have the same neighborhood in B, which is $\{v_0\} \times N_{H \oplus K_1}(\beta)$. It follows that the cut-rank of A in $(G \bullet H) \oplus K_1$ equals the cut-rank of B' in $H \oplus K_1$.

It follows that $\mathrm{rw}((G \bullet H) \oplus K_1) \leq \max(\mathrm{rw}(G \oplus K_1), \mathrm{rw}(H \oplus K_1))$. The reverse inequality follows from the fact that $G \oplus K_1$ and $H \oplus K_1$ are both induced subgraphs of $(G \bullet H) \oplus K_1$. □

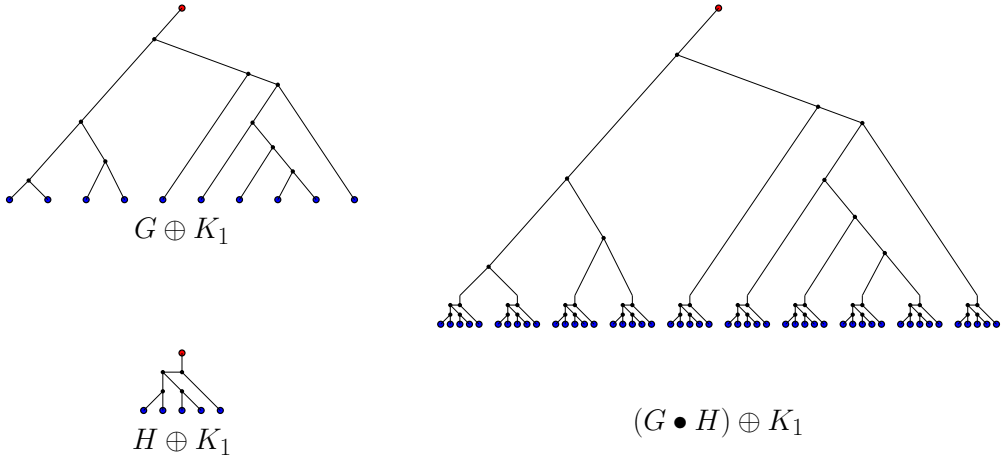

Figure 4: Rank decomposition of $(G \bullet H) \oplus K_1$ from the rank decompositions of $G \oplus K_1$ and $H \oplus K_1$.

Remark 3.4. The *substitution* operation as defined in [2, 8] can be expressed as the composition of lexicographic products and extraction of induced subgraphs. As taking induced subgraphs preserves rankwidth, one obtains as a corollary that closing a class by substitution increases the rankwidth by at most one.

For a class $\mathscr{C}$, let $\mathscr{C} \oplus K_1$ denote the class $\{G \oplus K_1 \ : \ G \in \mathscr{C}\}$, and let $\mathscr{C}^\bullet$ denote the closure of $\mathscr{C}$ under lexicographic product. As a direct consequence of the previous lemma we have

Corollary 3.1. *For every class of graphs $\mathscr{C}$ with bounded rankwidth we have*

$$\mathrm{rw}(\mathscr{C}) \leq \mathrm{rw}(\mathscr{C}^\bullet) = \mathrm{rw}(\mathscr{C} \oplus K_1) \leq \mathrm{rw}(\mathscr{C}) + 1. \tag{8}$$

(Indeed, $G \oplus K_1 \subseteq_i G \bullet H$ if H contains at least one edge.)

We remark that a stronger version of Corollary 3.1 holds for cliquewidth (the cliquewidth does not increase when going to the closure under lexicographic product), which follows from Lemma 3.4 of [14].

By substituting each vertex of $V(G)$ in the linear order witnessing $\mathrm{lrw}(G)$ by the linear order of $V(H)$ witnessing $\mathrm{lrw}(H)$ we similarly obtain the following results.

Lemma 3.5. *For all graphs G, H we have*

$$\mathrm{lrw}(G \bullet H) \leq \mathrm{lrw}(G) + \mathrm{lrw}(H).$$

Proof. Let $<_1$ be a linear order of $V(G)$ witnessing $\mathrm{lrw}(G)$ and let $<_2$ be a linear order of $V(H)$ witnessing $\mathrm{lrw}(H)$. Let $<$ be the lexicographic order on $V = V(G) \times V(H)$ defined by $<_1, <_2$, i.e., $(u, v) < (u', v')$ if $u <_1 u'$ or ($u = u'$ and $v <_2 v'$). Let $t = (u_t, v_t) \in V$ and let $(u, v) \leq t$. We have

$$N_{G \bullet H}((u, v)) \cap V^{>t} = \big((N_G(u) \cap V(G)^{>u_t}) \times V(H)\big) \cup \big(\{u_t\} \times (N_H(v) \cap V(H)^{>v_t})\big).$$

It follows that the vector space spanned by the sets $N_{G \bullet H}((u, v)) \cap V^{>t}$ is in the sum of the vector space spanned by the sets $(N_G(u) \cap V(G)^{>u_t}) \times V(H)$ (which has dimension at most $\mathrm{lrw}(G)$) and of the vector space spanned by the sets $\{u_t\} \times (N_H(v) \cap V(H)^{>v_t})$ (which has dimension at most $\mathrm{lrw}(H)$). Hence the claim follows. □

3.3. Ramsey properties of rankwidth

In this section we prove that the class of all graphs with rankwidth at most $r + 1$ is "Ramsey" for the class of all graphs with rankwidth at most r, in the following sense.

Theorem 3.6. *For all integers r, m and every graph G with rankwidth at most r there exists a graph $G' = G^{\bullet m}$ with rankwidth $r + 1$ and with the property that every m-coloring of G' contains an induced monochromatic copy of G.*

Proof. We define inductively graphs $G^{\bullet i}$ for $i \geq 1$: $G^{\bullet 1} = G$ and, for $i \geq 1$ we let $G^{\bullet(i+1)} = G^{\bullet i} \bullet G = G \bullet G^{\bullet i}$. According to Corollary 3.1 we have $\mathrm{rw}(\{G^{\bullet i} \ : \ i \in \mathbb{N}\}) \leq r + 1$.

We prove by induction on m that in every m-partition of $G' = G^{\bullet m}$ one class induces a subgraph with a copy of G. If $m = 1$ the result is straightforward. Let $m > 1$. Consider a partition $V_1, \ldots, V_m$ of the vertex set of $G^{\bullet m}$. If all the copies of $G^{\bullet(m-1)}$ forming $G^{\bullet m}$ contain a vertex in V_m, then $G^{\bullet m}[V_m]$ contains an induced copy of G. Otherwise, there is a copy of $G^{\bullet(m-1)}$ in $G^{\bullet m}$ whose vertex set is covered by $V_1, \ldots, V_{m-1}$. By induction hypothesis $G^{\bullet(m-1)}[V_i]$ contains an induced copy of G for some i. □

Corollary 3.2. *Let $\mathscr{F}$ be a proper hereditary class of graphs. Then there exists a class $\mathscr{C}$ with bounded rankwidth such that for every integer m there is $G \in \mathscr{C}$ with the property that for every partition of $V(G)$ into m classes, one class induces a graph not in $\mathscr{F}$.*

Corollary 3.3. *The class of graphs with rankwidth at most 2 does not have the property that its graphs can be vertex partitioned into a bounded number of cographs, the class of graphs with rankwidth at most 3 does not have the property that its graphs can be vertex partitioned into a bounded number of circle graphs, etc.*

Proof. This follows from Theorem 3.6 by noticing that $\mathrm{rw}(P_4) = 1$ (where P_4 denotes the path on 4 vertices) and P_4 is a forbidden induced subgraph for cographs , and that $\mathrm{rw}(W_5) = 2$ (where $W_5 = C_5 \oplus K_1$ denotes the wheel on 6 vertices) and W_5 is not a circle graph. □

3.4. Lower bounds for χ-boundedness

Dvořák and Kráľ [16] proved that classes with bounded rankwidth are χ-bounded. This result has been strengthened by Bonamy and Pilipczuk [7] who proved that classes with bounded rankwidth are polynomially χ-bounded. We give here a lower bound on the degrees of the involved polynomials. We write $\chi_f(G)$ for the *fractional chromatic number* of a graph G, which is defined as $\chi_f(G) = \inf\left\{\frac{\chi(G \bullet K_n)}{n} \ : \ n \in \mathbb{N}\right\}$.

Theorem 3.7. *For $r \in \mathbb{N}$, let P_r be a polynomial such that for every graph G with rankwidth at most r we have $\chi(G) \leq P_r(\omega(G))$. Then* $\deg P_r \in \Omega(\log r)$.

Proof. As shown in [22] for all graphs G and H we have $\chi(G \bullet H) = \chi(G \bullet K_{\chi(H)})$. Furthermore we have $\chi(G \bullet K_{\chi(H)}) \geq \chi(H)\chi_f(G)$. We deduce that $\chi(G \bullet H) \geq \chi_f(G)\chi(H)$. Hence for every integer n we have $\chi(G^{\bullet n}) \geq \chi_f(G)^n$. As $\omega(G^{\bullet n}) = \omega(G)^n$ we have $\chi(G^{\bullet n}) \geq \omega(G^{\bullet n})^{\frac{\log \chi_f(G)}{\log \omega(G)}}$ and hence

$$\deg P_r \geq \sup_{\mathrm{rw}(G \oplus K_1) \leq r} \frac{\log \chi_f(G)}{\log \omega(G)}.$$

For a sufficiently large integer n there exists a triangle-free graph G_n of order n with $\chi_f(G_n) \geq \frac{1}{9}\sqrt{\frac{n}{\log n}}$ (see [31]). As the rankwidth of a graph of order n is at most $\lceil n/3 \rceil$ we have $n > \mathrm{rw}(G_n \oplus K_1)$ thus

$$\deg P_r \geq \left(\frac{1}{2\log 2} - o(1)\right) \log r.$$

□

Linear rankwidth. We give a short proof in Section 4 (Corollary 4.1) that classes with bounded linear rankwidth are linearly χ-bounded using the equivalence between classes with bounded linear rankwidth and classes with bounded linear NLC-width. We improve the obtained upper bound of the χ/ω ratio in Section 5 using a more technical analysis of linear rank-width (Theorem 5.17), leading to an order of magnitude of $2^{O(r^2)}$. We now prove that the ratio χ/ω can be as large as α^r for some constant $\alpha > 1$ and for graphs with arbitrarily large linear rankwidth r and clique number ω.

From Lemma 3.5 we deduce $\mathrm{lrw}(C_5^{\bullet n}) \leq 2n$. As $\omega(C_5^{\bullet n}) = 2^n$ and as $\chi(C_5^{\bullet n}) \geq \chi(C_5)\chi_f(C_5)^{n-1} = 3(5/2)^{n-1}$ we deduce

$$\frac{\chi(C_5^{\bullet n})}{\omega(C_5^{\bullet n})} \geq (6/5)(5/4)^n \geq (6/5)(5/4)^{\mathrm{lrw}(C_5^{\bullet n})/2}.$$

As $6/5 > \sqrt{5}/2$, for every integer r we have:

$$\lim_{t\to\infty} \sup_{\substack{\mathrm{lrw}(G)\le r \\ \omega(G)\ge t}} \frac{\chi(G)}{\omega(G)} \ge \left(\frac{\sqrt{5}}{2}\right)^r. \tag{9}$$

4. Linear NLC-width

In this section we prove that classes with bounded linear NLC-width (and hence classes of bounded linear rankwdith) are linearly χ-bounded, and if they are stable, then they are transduction equivalent to classes of bounded pathwidth. We prove the result using Simon's factorization forest theorem.

4.1. Simon's factorization forest theorem

A *semigroup* is an algebra with one associative binary operation, usually denoted as multiplication. An *idempotent* in a semigroup is an element e with $ee = e$. Given an alphabet Ω we denote by Ω^+ the semigroup of all non-empty finite words over Ω, with concatenation as product.

Fix an alphabet Ω and a semigroup morphism $h\colon \Omega^+ \to T$, where T is a finite semigroup. A *factorization tree* is an ordered rooted tree (that is: a rooted plane tree) in which each node is either a leaf labeled by a letter, or an internal node. The value of a node is the word obtained by reading the descendant leaves below from left to right. The value of a factorization tree is the value of the root of the tree. A *factorization tree* of a word $\alpha \in \Omega^+$ is a factorization tree of value α. The *depth* of the tree is defined as usual, with the convention that the depth of a single leaf is 1. A factorization tree is *Ramseyan* (for h) if every node 1) is a leaf, or 2) has two children, or, 3) the values of its children are all mapped by h to the same idempotent of T.

Theorem 4.1 (Simon's Factorization Forest Theorem [32, 48])**.** *For every alphabet Ω, every finite semigroup T, and every semigroup morphism $h\colon \Omega^+ \to T$, every word $\alpha \in \Omega^+$ has a Ramseyan factorization tree of depth at most $3|T|$.*

The existence of an upper bound expressed only in terms of $|T|$ was first proved by Simon [48]. The improved upper bound of $3|T|$ is due to Kufleitner [32].

4.2. Application to classes with bounded linear NLC-width

In the following we consider the semigroup Γ_k on functions $r\colon [k] \to [k]$. Obviously, $h\colon \Omega_k(V)^+ \to \Gamma_k$ induced by $h(a) = r_a$ for $a \in \Omega_k(V)$ is a semigroup homomorphism (recall Definition 3.1). An idempotent of Γ_k is a function r that satisfies that if $r(i) = j$, then $r(j) = j$. We call $\alpha \in \Omega_k(V)^+$ an idempotent if $h(\alpha)$ is an idempotent in Γ_k.

For $\alpha \in \mathfrak{L}_k(V)$ (recall Definition 3.2) and for a letter a of α and $v = v_a$ define $\mathrm{col}_\alpha(v)$ as the color of the vertex v in $\Xi(\alpha)$. Note that if $\alpha\beta \in \mathfrak{L}_k(V)$ then $\mathrm{col}_{\alpha\beta}(v) = h(\beta)(\mathrm{col}_\alpha(v))$.

Fix $\alpha \in \mathfrak{L}_k(V)$. According to Theorem 4.1, there exists an ordered rooted tree Y that is a Ramseyan factorization tree of α for h with depth at most $3|\Gamma_k|$. For the rest of this section fix such a tree Y.

For a node x of Y we denote by $\bar{\mathsf{x}}$ the value of x, which is a subword of α. Note that the leaves of Y are naturally identified with the letters of α. If x, y are two nodes of Y, note that:

- x is an ancestor of y in Y if and only if $\bar{\mathsf{y}}$ is a subword of $\bar{\mathsf{x}}$,
- x is *to the left of* y in Y if and only if $\bar{\mathsf{x}}$ and $\bar{\mathsf{y}}$ are disjoint and $\bar{\mathsf{x}}$ appears before $\bar{\mathsf{y}}$ in α,
- x is *immediately to the left of* y in Y (meaning that they are consecutive children of a same node, with x at the left of y) if and only if $\bar{\mathsf{x}}\bar{\mathsf{y}}$ is a subword of α.

For a word $\beta = b_1 \cdots b_n$ (where the b_i's are letters), for a leaf z of Y with $\bar{\mathsf{z}} = b_p$, and for $1 \le p \le n$ we define

$$\mathrm{recol}_{\beta,\mathsf{z}} = r_{b_{p-1}} \circ \cdots \circ r_{b_1} = h(b_1 \cdots b_{p-1}),$$
$$\mathrm{eset}_\beta(\mathsf{z}) = \mathrm{recol}^{-1}_{\beta,\mathsf{z}}(e_{b_p}).$$

Lemma 4.2. *Let $\mathsf{z}_1, \mathsf{z}_2$ be two leaves of Y such that the letters of $\bar{\mathsf{z}}_1$ and $\bar{\mathsf{z}}_2$ appear in this order in α, let $\mathsf{x} = \mathsf{z}_1 \wedge \mathsf{z}_2$ be their least common ancestor in Y, and let y_1 (resp. y_2) be the children of x that are ancestors of z_1 and z_2, respectively. Then v_{z_1} and v_{z_2} are adjacent in $\Xi(\alpha)$ if*

- y_1 *is not immediately to the left of* y_2 *in* Y *and* $\mathrm{col}_{\bar{\mathsf{y}}_1}(v_{\mathsf{z}_1}) \in \mathrm{eset}_{\bar{\mathsf{x}}}(\mathsf{z}_2)$,
- *or* y_1 *is immediately to the left of* y_2 *in* Y *and* $\mathrm{col}_{\bar{\mathsf{y}}_1}(v_{\mathsf{z}_1}) \in \mathrm{eset}_{\bar{\mathsf{y}}_2}(\mathsf{z}_2)$.

Proof. Assume that y_1 is immediately to the left of y_2 in Y, and let $\bar{x} = \eta\bar{\mathsf{y}}_1\bar{\mathsf{y}}_2\eta'$. Let $\bar{\mathsf{y}}_2 = b_1 \dots b_p$ with $b_p = \bar{\mathsf{z}}_2$. The color of v_{z_1} in $\Xi(\eta\bar{\mathsf{y}}_1)$ is the same as in $\Xi(\bar{\mathsf{y}}_1)$ — that is $\mathrm{col}_{\bar{\mathsf{y}}_1}(v_{\mathsf{z}_1})$ — as $\bar{\mathsf{z}}_1$ is in $\bar{\mathsf{y}}_1$. The color of v_{z_1} at the point where v_{z_2} is created is thus $h(b_1 \cdots b_{p-1})(\mathrm{col}_{\bar{\mathsf{y}}_1}(v_{\mathsf{z}_1})) = \mathrm{recol}_{\bar{\mathsf{y}}_2,\mathsf{z}_2}(\mathrm{col}_{\bar{\mathsf{y}}_1}(v_{\mathsf{z}_1}))$. Thus v_{z_1} and v_{z_2} are adjacent if $\mathrm{col}_{\bar{\mathsf{y}}_1}(v_{\mathsf{z}_1}) \in \mathrm{eset}_{\bar{\mathsf{y}}_2}(\mathsf{z}_2)$.

Now assume that y_1 is not immediately to the left of y_2 in Y, and let $\bar{x} = \eta\bar{\mathsf{y}}_1\eta'\bar{\mathsf{y}}_2\eta''$. In this case x has more than two children, hence, all its children have the same h-value, which is an idempotent. In particular, $h(\eta') = h(\eta\bar{\mathsf{y}}_1\eta')$. The color of v_{z_1} at the point where v_{z_2} is created is

$$\begin{aligned} h(\eta' b_1 \cdots b_{p-1})(\mathrm{col}_{\bar{\mathsf{y}}_1}(v_{\mathsf{z}_1})) &= h(b_1 \cdots b_{p-1})\big(h(\eta')(\mathrm{col}_{\bar{\mathsf{y}}_1}(v_{\mathsf{y}_2}))\big) \\ &= h(b_1 \cdots b_{p-1})\big(h(\eta\bar{\mathsf{y}}_1\eta')(\mathrm{col}_{\bar{\mathsf{y}}_1}(v_{\mathsf{z}_1}))\big) \\ &= h(\eta\bar{\mathsf{y}}_1\eta' b_1 \cdots b_{p-1})(\mathrm{col}_{\bar{\mathsf{y}}_1}(v_{\mathsf{z}_1})). \end{aligned}$$

Thus v_{z_1} and v_{z_2} are adjacent if $\mathrm{col}_{\bar{\mathsf{y}}_1}(v_{\mathsf{z}_1}) \in \mathrm{eset}_{\bar{\mathsf{x}}}(\mathsf{z}_2)$. □

We can now prove our first main theorem.

Theorem 4.3. *Let* $f(k) = (k2^{k+1})^{3k^k}$ *and* $g(k) = 3k^k$. *Every graph with linear NLC-width at most* k *can be vertex partitioned into* $f(k)$ *cographs with a cotree of depth at most* $g(k)$.

Proof. Let κ be a coloring of the nodes x of Y with color in $\{1, 2\}$ such that two consecutive children of a node have a different color. For a letter z of α, color v_{z} by the vector of values $(\kappa(\mathsf{x}), \mathrm{col}_{\bar{\mathsf{x}}}(v_{\mathsf{z}}), \mathrm{eset}_{\bar{\mathsf{x}}}(\mathsf{z}))$ for x ancestor of z (ordered in increasing distance to the root). (This gives a vector of at most $3|\Gamma_k|$ triples.) Consider a monochromatic subset of vertices A. Let $v_{\mathsf{z}_1}, v_{\mathsf{z}_2}$ be distinct vertices of A, let $\mathsf{x} = \mathsf{z}_1 \wedge \mathsf{z}_2$, and let y_1 and y_2 be the children of x that are ancestors of z_1 and z_2, respectively. As A is monochromatic and as y_1 and y_2 are at the same height in Y we have in particular $\kappa(\mathsf{y}_1) = \kappa(\mathsf{y}_2)$ hence y_1 and y_2 are not consecutive children of x. As A is monochromatic we also have $\mathrm{col}_{\bar{\mathsf{y}}_1}(v_{\mathsf{z}_1}) = \mathrm{col}_{\bar{\mathsf{y}}_2}(v_{\mathsf{z}_2})$ and $\mathrm{eset}_{\bar{\mathsf{x}}}(\mathsf{z}_1) = \mathrm{eset}_{\bar{\mathsf{x}}}(\mathsf{z}_2)$. Hence, we can label the internal nodes of Y with 0 and 1 in such a way that two vertices in A are adjacent if and only if the label at their least common ancestor in Y is 1. In particular, A induces a cograph with cotree height at most $g(k) = 3|\Gamma_k| = 3k^k$.

The colors we used are vectors of at most $g(k)$ triples. Each triple consists of $\kappa(\mathsf{x})$ (2 possible values), $\mathrm{col}_{\bar{\mathsf{x}}}(v_{\mathsf{z}})$ (k possible values) and $\mathrm{eset}_{\bar{\mathsf{x}}}(\mathsf{z})$ (2^k possible values). Altogether, this gives at most $(2k2^k)^{g(k)} = f(k)$ colors. □

Corollary 4.1. *Classes with bounded linear NLC-width are linearly* χ*-bounded.*

Towards the goal of characterizing stable classes of bounded linear NLC-width, we observe that the following configuration leads to semi-induced half-graphs. We call H_k *semi-induced* in G if we can find in G vertices $a_1, \dots, a_k$ and $b_1, \dots, b_k$ such that $\{a_i, b_j\} \in E(G)$ if and only if $1 \le i \le j \le k$. Observe that we make no statement about edges between the a_i or between the b_i.

Lemma 4.4. *Assume there exist a node* z *and leaves* $\mathsf{x}_1, \mathsf{y}_1, \mathsf{x}_2, \mathsf{y}_2, \dots, \mathsf{x}_\ell, \mathsf{y}_\ell$ *of* Y *(in left-right order) such that* x *is the least common ancestor of each pair of these leaves, and that there exist* $c_x, c_y \in [k]$ *and* $e_x, e_y \subseteq [k]$ *with* $c_x \in e_y, c_y \notin e_x$, *and, for each* $1 \le i \le \ell$, $\mathrm{col}_{\bar{\mathsf{z}}}(v_{\mathsf{x}_i}) = c_x$, $\mathrm{eset}_{\bar{\mathsf{z}}}(\mathsf{x}_i) = e_x$, $\mathrm{col}_{\bar{\mathsf{z}}}(v_{\mathsf{y}_i}) = c_y$, *and* $\mathrm{eset}_{\bar{\mathsf{z}}}(\mathsf{v}_{\mathsf{y}_i}) = e_y$. *Then* $\Xi(\alpha)$ *contains a semi-induced half-graph of order at least* $\lfloor \ell/3 \rfloor$.

Proof. By taking at least a third of the indices we can assume that no two letters appear in consecutive children of β. Then it follows directly from Lemma 4.2 that these vertices semi-induce a half-graph. □

Theorem 4.5. *Let* $\mathscr{C}$ *be a class with bounded linear NLC-width. If the graphs in* $\mathscr{C}$ *exclude some semi-induced half-graph, then* $\mathscr{C}$ *is a transduction of a class with bounded pathwidth.*

Proof. We first construct an interval graph H, where each node δ of Y corresponds to an interval I_δ. The descendent relation of Y is the containment relation in the set of intervals.

Now consider an internal node δ of Y and a 4-tuple $(c_1, e_1, c_2, e_2) \in [k] \times 2^{[k]} \times [k] \times 2^{[k]}$ with $c_1 \in e_2$ and $c_2 \notin e_1$, such that at least one descendent z_1 of δ is such that $\mathrm{col}_{\bar{\delta}}(z_1) = c_1$ and $\mathrm{eset}_{\bar{\delta}}(z_1) = e_1$ and at least one descendent z_2 of δ is such that $\mathrm{col}_{\bar{\delta}}(z_2) = c_2$ and $\mathrm{eset}_{\bar{\delta}}(z_2) = e_2$. We consider new intervals coming from the split of the I_δ into subintervals (we keep the interval I_δ, as well as the new intervals arising from the split): These subintervals are obtained by considering the children of δ in order. The subintervals are of three types:

- type (1) intervals subsume the intervals of consecutive children of δ with at least one descendant z with $\mathrm{col}_{\bar{\delta}}(z) = c_1$ and $\mathrm{eset}_{\bar{\delta}}(z) = e_1$, but no descendant z with $\mathrm{col}_{\bar{\delta}}(z) = c_2$ and $\mathrm{eset}_{\bar{\delta}}(z) = e_2$;
- type (2) intervals subsume the intervals of consecutive children of δ with at least one descendant z with $\mathrm{col}_{\bar{\delta}}(z) = c_2$ and $\mathrm{eset}_{\bar{\delta}}(z) = e_2$, but no descendant z with $\mathrm{col}_{\bar{\delta}}(z) = c_1$ and $\mathrm{eset}_{\bar{\delta}}(z) = e_1$;
- type (1 + 2) intervals contain the interval of a single child of δ with both a descendent z_1 with $\mathrm{col}_{\bar{\delta}}(z_1) = c_1$ and $\mathrm{eset}_{\bar{\delta}}(z_1) = e_1$ and a descendent z_2 with $\mathrm{col}_{\bar{\delta}}(z_2) = c_2$ and $\mathrm{eset}_{\bar{\delta}}(z_2) = e_2$.

The division of I_δ into subintervals is done in such a way that no two consecutive subintervals are both of type (1) or both of type (2). Note that such a division into subintervals exists. Furthermore, for all new subintervals I_γ and $I_{\gamma'}$ (obtained from the split of I_γ) that are direct neighbors, we add a new interval $I_{\gamma,\gamma'}$ subsuming the two intervals I_γ and $I_{\gamma'}$. This finishes the construction of the graph H.

Assume that the number of subintervals into which we divided I_δ is N. Then we can select, among the descendants of the distinct children of δ some vertices $\alpha_1, \beta_1, \dots, \alpha_n, \beta_n$ (with $n \geq N/4$) such that $\mathrm{col}_{\bar{\delta}}(\alpha_i) = c_1$, $\mathrm{eset}_{\bar{\delta}}(\alpha_i) = e_1$, $\mathrm{col}_{\bar{\delta}}(\beta_i) = c_2$, and $\mathrm{eset}_{\bar{\delta}}(\beta_i) = e_2$. We deduce from Lemma 4.4 applied to δ and $\alpha_1, \beta_1, \dots, \alpha_n, \beta_n$ and the assumption that $\mathscr{C}$ excludes some semi-induced half-graph, that I_δ is divided into a bounded number of subintervals. Now, it is immediate from the definition of pathwidth as one less than the minimum clique number of an interval supergraph of H that H has bounded pathwidth (depending on the NLC-width of G and the bound on the length of the largest semi-induced half-graph in G) as desired.

We now add colors to the vertices of H (that will be used by the transduction to reconstruct the graph G). First, we assign each vertex representing an interval I_δ all associated 4-tuples (c_1, e_1, c_2, e_2). For each vertex u we add the information $\mathrm{col}_{\bar{\delta}}(u)$ and $e_1 = \mathrm{eset}_{\bar{\delta}}(u)$ for each predecessor δ of u. Finally, recall that each split of I_γ into subintervals is into at most N parts. We add additional colors to number these intervals (in their left-to-right order) as $1, \dots, N$.

Let us show how to reconstruct the edges of G from the colored graph H. Let u, v be vertices, let δ be their least common ancestor in Y and let δ_u and δ_v be the children of δ such that u is a descendant of δ_u and such that v is a descendant of δ_v. We aim to apply Lemma 4.2 to decode whether u and v are adjacent. The problem is that we do not know in what order u and v appear below δ. Assume first that δ_u and δ_v are not direct neighbors (this can be checked using the vertex representing the interval I_{δ_u,δ_v}). Let $c_1 = \mathrm{col}_{\bar{\delta}}(u)$, $e_1 = \mathrm{eset}_{\bar{\delta}}(u)$, $c_2 = \mathrm{col}_{\bar{\delta}}(v)$, and $e_2 = \mathrm{eset}_{\bar{\delta}}(v)$. The values of $\mathrm{col}_{\bar{\delta}}$ and $\mathrm{eset}_{\bar{\delta}}$ for u and v are available from the predicates at these vertices. If $c_1 \in e_2$ and $c_2 \in e_1$, then the order of δ_u and δ_v does not matter, and we can conclude that u and v are adjacent. Similarly, if $c_1 \notin e_2$ and $c_2 \notin e_1$ then u and v are non-adjacent.

In the last case, without loss of generality, we can assume $c_1 \in e_2$ and $c_2 \notin e_1$. Observe that in this case the two vertices u and v cannot belong to a same subinterval of I_δ. Then from the numbering marks associated to the subintervals that contain u and v we deduce which of u and v is smaller than the other and hence the we can derive the adjacency between u and v.

If δ_u and δ_v are direct neighbors we argue analogously, referring to the values of $\mathrm{col}_{\bar{\delta}_u}$, $\mathrm{col}_{\bar{\delta}_v}$, $\mathrm{eset}_{\bar{\delta}_u}$ and $\mathrm{eset}_{\bar{\delta}_v}$, which are also known from the predicates at these vertices.

To conclude, observe that the above reconstruction can easily be done by a first-order formula. □

From this we deduce.

Theorem 4.6. *Let $\mathscr{C}$ be a class of graphs with linear rankwidth at most r. Then the following are equivalent:*

1. *$\mathscr{C}$ is stable,*
2. *$\mathscr{C}$ is monadically stable,*
3. *$\mathscr{C}$ is sparsifiable,*
4. *$\mathscr{C}$ has 2-covers with bounded shrubdepth,*
5. *$\mathscr{C}$ has structurally bounded expansion,*
6. *$\mathscr{C}$ is a transduction of a class with bounded pathwidth,*
7. *$\mathscr{C}$ excludes some semi-induced half-graph.*

5. Linear rankwidth

In this section we present a second proof for the result that classes with bounded linear rankwidth are linearly χ-bounded and thereby provide improved constants.

5.1. Notation

For sets $M, N \subseteq V(G)$ we define $M \oplus N$ as the symmetric difference of M and N, that is, $v \in M \oplus N$ if and only if $v \in M \cup N$ but $v \notin M \cap N$. For $t \in V$, we define $V^{>t} := \{v : v > t\}$, $V^{<t} := \{v : v < t\}$ and $V^{\leq t} := \{v : v \leq t\}$. For $v \in V$ we denote by $N(v)$ the neighborhood of $v \in G$ (where v not included). We let $N^{<t}(v) := N(v) \cap V^{<t}$ and define similarly $N^{>t}$ and $N^{\leq t}$. For $M \subseteq V(G)$ we define $N_{\oplus}(M) := \bigoplus_{v \in M} N(v)$ and $N_{\oplus}^{>t}(M) := N_{\oplus}(M) \cap V^{>t}$.

Remark 5.1. If $t < t'$, then $N_{\oplus}^{>t}(M) = N_{\oplus}^{>t}(N)$ implies $N_{\oplus}^{>t'}(M) = N_{\oplus}^{>t'}(N)$.

For $t \in V$ the closure of $\{N^{>t}(v) : v \leq t\}$ under $\oplus$ is a vector space over $\oplus$ and scalar multiplication with 0 and 1, where $0 \cdot M = \emptyset$ and $1 \cdot M = M$.

For $t \in V$, we call an inclusion-minimal subset $B \subseteq V_{\leq t}$ a *neighbor basis for $V^{>t}$* if for every $v \leq t$ there exists $B' \subseteq B$ such that $N^{>t}(v) = N_{\oplus}^{>t}(B')$. In other words, B is a neighbor basis for $V^{>t}$ if $\{N^{>t}(v) : v \in B\}$ forms a basis for the space spanned by $\{N^{>t}(v) : v \leq t\}$.

The following is immediate by the definition of linear rankwidth.

Remark 5.2. As G has linear rankwidth at most r, for every $t \in V$ every neighbor basis for $V^{>t}$ has at most r vertices.

5.2. Activity intervals and active basis

For $t \in V$ we define the *active basis B_t at t* as the set of all vertices smaller or equal to t, whose neighborhood in $V^{>t}$ is not in the vector space generated by the neighborhoods in $V^{>t}$ of smaller vertices, that is:

$$B_t = \{v \leq t \ : \ (\nexists B \subseteq V^{<v})\ N^{>t}(v) = N_{\oplus}^{>t}(B)\}. \tag{10}$$

Note that this is the lexicographically least neighborhood basis of $V^{>t}$.

Remark 5.3. If the linear order of $V(G)$ is given, the set of all neighborhood basis B_t for $t \in V(G)$ can be computed in quadratic time, by iteratively considering t in increasing order and maintaining the set of at most 2^r neighborhoods in $V^{>t}$.

To each $v \in V$ we associate its *activity interval I_v* defined as the interval $[v, \tau(v)]$ starting at v and ending at the minimum vertex $\tau(v) \geq v$ such that $v \notin B_{\tau(v)}$. Note that $\tau(v)$ is well defined as we have $B_{\max V} = \emptyset$.

We extend the definitions of the activity intervals and of the τ function to all non-empty subsets M of $V(G)$ by

$$I_M := \bigcap_{v \in M} I_v \quad \text{and} \quad \tau(M) = \min_{v \in M} \tau(v). \tag{11}$$

Note that either $I_M = \emptyset$ or $I_M = [\max M, \tau(M)]$. We call a set M *active* if $|I_M| > 1$, that is, if $\max M < \tau(M)$. We call a vertex v *active* if the singleton set $\{v\}$ is active.

For every $v \in V$, as $v \notin B_{\tau(v)}$, there exists a unique $F_0(v) \subseteq B_{\tau(v)}$ with

$$N^{>\tau(v)}(v) = N_{\oplus}^{>\tau(v)}(F_0(v)). \tag{12}$$

According to (10) and as $v \notin B_{\tau(v)}$ we have $F_0(v) \subseteq V^{<v}$ hence $\max F_0(v) < v$. Moreover, $\tau(v) \geq v$ by definition and, if $F_0(v)$ is a non-empty subset of $B_{\tau(v)}$ then every vertex x in $F_0(v)$ is such that $\tau(x) > \tau(v)$ hence $\tau(F_0(v)) > \tau(v)$. Altogether we have that if $F_0(v) \neq \emptyset$ then we have:

$$\max F_0(v) < v \leq \tau(v) < \tau(F_0(v)). \tag{13}$$

Hence, in this case, the set $F_0(v)$ is active.

Remark 5.4. Assume that M is an active set and let $v \in M$.

1. If $\tau(v) > \tau(M)$, then $v \in B_{\tau(M)}$.
2. If $\tau(v) = \tau(M)$, then $F_0(v) \subseteq B_{\tau(M)}$.

Proof. As M is active we have $I_M = [\max M, \tau(M)]$. In particular, if $\tau(v) > \tau(M)$ we have $\tau(M) \in I_v$ (since $v \leq \max M$) thus $v \in B_{\tau(M)}$. If $\tau(v) = \tau(M)$, then by definition of F_0 we have $F_0(v) \subseteq B_{\tau(v)} = B_{\tau(M)}$. □

5.3. The F-tree

We define a mapping F extending F_0, that will define a rooted tree on the set Z consisting of all active sets, all singleton sets $\{v\}$ for $v \in V(G)$, and $\emptyset$ (which will be the root of the tree and the unique fixed point of F). Before we define F we make one more observation.

Lemma 5.5. *Let $u, v \in V(G)$ be active. If $\tau(u) = \tau(v)$, then $u = v$.*

Proof. Let $t = \tau(u) = \tau(v)$ and let t' be the predecessor of t in the linear order. Assume for contradiction that $u \neq v$. By definition of F_0 we have $N^{>t}(u) = N^{>t}(F_0(u))$ and $N^{>t}(v) = N^{>t}(F_0(v))$. We have $N^{>t'}(u) \neq N^{>t'}(F_0(u))$ as otherwise $\tau(u) \leq t'$. As $N^{>t'}(u) \oplus N^{>t}(u) \subseteq \{t\}$ and $N^{>t'}(F_0(u)) \oplus N^{>t}(F_0(u)) \subseteq \{t\}$, we have $N^{>t'}(F_0(u)) = N^{>t'}(u) \oplus \{t\}$. Similarly, we have $N^{>t'}(F_0(v)) = N^{>t'}(v) \oplus \{t\}$. Assume without loss of generality that $u < v$. Then $N^{>t'}(v) = N^{>t'}(\{u\}) \oplus N^{>t'}(F_0(u)) \oplus N^{>t'}(F_0(v))$. As $\max(\{u\} \cup F_0(u) \cup F_0(v)) < v$ we deduce that $\tau(v) \leq t'$, contradicting $\tau(v) = t$. □

Corollary 5.1. *For each active set $M \subseteq V(G)$ there exists exactly one $v_M \in M$ with $\tau(v_M) = \tau(M)$.*

The mapping $F : Z \to Z$ is defined as

$$F(M) = \begin{cases} \emptyset & \text{if } M = \emptyset, \\ M \oplus \{v_M\} \oplus F_0(v_M) & \text{for the unique } v_M \in M \\ & \text{with } \tau(v_M) = \tau(M), \text{ otherwise.} \end{cases} \tag{14}$$

Remark 5.6. If the linear order on $V(G)$ is given then F-mapping on Z can be computed in quadratic time. Indeed, the computation of all the active basis can be done in quadratic time, and each time a vertex v leaves the current active basis B_t one can compute $F_0(v)$ by checking the space of the 2^r neighborhoods in $V^{>t}$ generated by B_t. (Note that $|Z| \leq 2^r|V(G)|$.)

The following lemma shows for every active set M, either $F(M) = \emptyset$ or $F(M)$ is active, and thus $F(M) \in Z$ and F is well defined. Furthermore, the lemma shows that $I_{F(M)} \supset I_M$.

Lemma 5.7. *Let $M \in Z$. Then $F(M) \subseteq B_{\tau(M)}$ and furthermore, either $F(M) = \emptyset$, or $\max F(M) \leq \max M < \tau(M) < \tau(F(M))$ and hence $F(M)$ is active.*

Proof. The statement is obvious if $M = \emptyset$. For $M = \{v\}$, the statement is immediate from the definition of $F_0(v)$ and (13). For all other $M \in Z$, according to remark 5.4 we have for each $v \in M$ either $v \in B_{\tau(M)}$ if $\tau(v) > \tau(M)$, or $F_0(v) \subseteq B_{\tau(M)}$ if $\tau(v) = \tau(M)$. This implies $F(M) \subseteq B_{\tau(M)}$. Finally, if $F(M) \neq \emptyset$, then $\max F(M) \leq \max M < \tau(M) < \tau(F(M))$ follows from the fact that these inequalities hold for all $v \in M$ with $\tau(v) > \tau(M)$ and for $F_0(v)$ for the unique $v \in M$ with $\tau(v) = \tau(M)$ according to (13). □

The mapping F guides the process of iterative referencing and ensures that, for an active set M, if $t \geq \tau(M)$, then the set $N_{\oplus}^{>t}(M)$ can be rewritten as $N_{\oplus}^{>t}(F(M))$. This property is stated in the next lemma.

Lemma 5.8. *Let $M \in Z \setminus \{\emptyset\}$ and let $w \in V(G)$. If $w > \tau(M)$, then*

$$w \in N_{\oplus}(M) \Leftrightarrow w \in N_{\oplus}(F(M)).$$

Proof. If $M = \{v\}$ for $v \in V(G)$, then this follows from (12). Otherwise, M is an active set. Let $t = \tau(M)$ and let $v \in M$ be the unique element with $\tau(v) = t$. Then we have $N_{\oplus}^{>t}(F_0(v)) = N_{\oplus}^{>t}(v)$, and hence

$$\begin{aligned} N_{\oplus}^{>t}(F(M)) &= N_{\oplus}^{>t}(\{v\}) \oplus N_{\oplus}^{>t}(F(M) \oplus \{v\}) \\ &= N_{\oplus}^{>t}(F_0(v)) \oplus N_{\oplus}^{>t}(F(M) \oplus \{v\}) \\ &= N_{\oplus}^{>t}(M). \end{aligned}$$

□

This lemma can be applied repeatedly to $M, F(M)$, etc. until $F^k(M) = \emptyset$, or until for some given $w \in V(G)$ we have $\tau(F^k(M)) \geq w$. This justifies to introduce, for distinct vertices u and v the value

$$\xi(u, v) := \min\{k \geq 0 \ : \ F^k(\{u\}) = \emptyset \text{ or } F^k(\{u\}) \neq \emptyset \text{ and } v \in I_{F^k(\{u\})}\}, \tag{15}$$

where we let $F^0(M) = M$ by convention.

As a direct consequence of the previous lemma we have

Corollary 5.2. *For $u < v$ in $V(G)$ we have*

$$\{u, v\} \in E(G) \iff v \in N_{\oplus}(F^{\xi(u,v)}(\{u\})).$$

Proof. We claim that for all $0 \leq k \leq \xi(u, v)$ and $u < v$, u and v are adjacent if and only if $v \in N_{\oplus}(F^k(\{u\}))$. We proceed by induction on k.

If $k = 0$, then the statement is $\{u, v\} \in E(G) \Leftrightarrow v \in N_{\oplus}(u)$, which trivially holds. Assume $k \geq 1$. By Lemma 5.7 we have $v > \tau(F^{k-1}(\{u\}))$. Moreover, $F^{k-1}(\{u\}) \in Z \setminus \{\emptyset\}$. Hence by Lemma 5.8 we have $v \in F^{k-1}(\{u\}) \iff v \in F^k(\{u\})$. As $\{u, v\} \in E(G) \iff v \in N_{\oplus}(F^{k-1}(\{u\}))$ by induction hypothesis, we deduce $\{u, v\} \in E(G) \iff v \in N_{\oplus}(F^k(\{u\}))$. □

The monotonicity property of F (i.e. the property $\tau(F(M)) > \tau(M)$ if $F(M) \neq \emptyset$) implies that F defines a rooted tree, *the F-tree*, with vertex set Z, root $\emptyset$ and edges $\{M, F(M)\}$. Here the monotonicity guarantees that the graph is acyclic and it is connected because $\emptyset$ is the only fixed point of F. The following lemma shows that the F-tree has bounded height. Recall that r denotes the linear rankwidth of G.

Lemma 5.9. *For every $M \in Z$ we have $F^{r+1}(M) = \emptyset$.*

Proof. If $M = \emptyset$, the statement is obvious, so assume $M \neq \emptyset$. It is sufficient to prove that for every active set M we have $F^r(M) = \emptyset$, as this implies $F^{r+1}(\{v\}) = \emptyset$ also for all $v \in V(G)$. Let M be an active set and let $t \in I_M$. Then every $v \in M$ is in B_t, so $M \subseteq B_t$.

Assume $i \geq 1$ is such that $F^i(M) \neq \emptyset$. As $\max F(M) \leq \max M$ and $\tau(F(M)) > \tau(M)$ by Lemma 5.7, we get

$$\max F^i(M) \leq \max M \leq t < \tau(M) \leq \tau(F^{i-1}(M)) < \tau(F^i(M)).$$

As $\tau(F^i(M)) = \min_{v \in F^i(M)} \tau(v)$, we have $F^i(M) \subseteq B_t$. Hence, considering the sequence $M, F(M), \ldots, F^i(M)$, each iteration of F removes the unique element with minimum τ value. It follows that the union of the sets has cardinality at least $i + 1$. As $|B_t| \leq r$, we have $i < r$ and hence $F^r(M) = \emptyset$. □

For distinct vertices u, v, let $u \wedge v$ denote the greatest common ancestor of u and v in the F-tree, i.e. the first common vertex on the paths to the root. If $u \wedge v$ is not the root of the F-tree then there exist ℓ_u and ℓ_v such that $u \wedge v = F^{\ell_u}(\{u\}) = F^{\ell_v}(\{v\}) \neq \emptyset$, hence both u and v belong to $I_{u\wedge v}$. Thus we have $\tau(u \wedge v) > u$ and $\tau(u \wedge v) > v$. In other words, we have $\xi(u, v) \leq \ell_u$ and $\xi(v, u) \leq \ell_v$.

5.4. The activity interval graph

Let H be the intersection graph of the intervals I_v for $v \in V(G)$. Note that we may identify $V(H)$ with $V(G)$ as $\min I_v = v$ for all v $V(G)$.

Lemma 5.10. *In the intersection graph H of the intervals I_u at most $r + 2$ intervals intersect in each point (hence $\omega(H) \leq r + 2$).*

Proof. Consider any vertex t with $t \in I_u$ for some u. The case $u \in B_t$ gives a maximum of r intervals intersecting in t. Otherwise $t = \tau(u)$, which gives at most two possibilities for u: either u is inactive (and $u = t$), or u is active (and u is uniquely determined, according to Lemma 5.5). Thus at most $r + 2$ intervals intersect at point t. □

As mentioned in the proof of the above lemma, every clique of H contains at most one inactive vertex. It follows that there is a coloring $\gamma : V(G) \to [r + 2]$ with the following properties:

(1) for every $u \in V(G)$ we have $\gamma(u) = r + 2$ if and only if u is inactive;

(2) for all distinct $u, v \in V(G)$ we have

$$I_u \cap I_v \neq \emptyset \quad \Longrightarrow \quad \gamma(u) \neq \gamma(v). \tag{16}$$

We extend this coloring to sets as follows: for $M \subseteq V(G)$ we let

$$\Gamma(M) := \{\gamma(v) \ : \ v \in M\}. \tag{17}$$

This coloring allows to define, for each $v \in V(G)$

$$\begin{aligned} \mathrm{Class}(v) &:= \big(\gamma(v), \Gamma(F(\{v\})), \dots, \Gamma(F^r(\{v\}))\big), \\ \mathrm{NCol}(v) &:= \{\gamma(u) \ : \ u \in N(v) \text{ and } v \in I_u\} \end{aligned}$$

Note that all u with $v \in I_u$ define a clique of H (because all I_u contain v) and hence have distinct γ-colors.

Lemma 5.11. *Let $v \in V(G)$. Every $u \in B_v$ can be defined as the maximum vertex $x \leq v$ with $\gamma(x) = \gamma(u)$.*

Proof. By assumption we have $u \leq v$. Assume towards a contradiction that there exists $x \in V(G)$ with $u < x \leq v$ and $\gamma(x) = \gamma(u)$. As $u \in B_v$ we have $\tau(u) > v$, hence $x \in I_u$. It follows that $I_x \cap I_u \neq \emptyset$, in contradiction to $\gamma(x) = \gamma(u)$. □

Towards the aim of bounding the number of graphs of linear rankwidth at most r, we give a bound on the number of colors that can appear.

Lemma 5.12. *Let $f(r) := 2(2^r + 1)(r + 1)!\, 2^{\binom{r+1}{2}}$. The number of* $\mathrm{Class}(v)$ *for $v \in V(G)$ can be bounded by $(2^r + 1)(r + 1)!\, 2^{\binom{r}{2}}$ and the number of pairs* $(\mathrm{Class}(v), \mathrm{NCol}(v))$ *for $v \in V(G)$ can be bounded by $f(r)$.*

Proof. Let $v \in V(G)$. From the fact that $\gamma(v) = r + 2$ if and only if v is inactive, that images by F only contain active vertices, as well as from Lemma 5.7 we deduce:

- If $\gamma(v) = r + 2$, then there exists a linear order on $[r + 1]$ colors such that for $1 \leq i \leq r$, the set $\Gamma(F^i(v))$ is a subset of the first $r + 1 - i$ colors of $[r + 1]$.
- If $\gamma(v) \leq r + 1$, then there exists a linear order on $[r + 1] \setminus \{\gamma(v)\}$ such that for $1 \leq i \leq r$, the set $\Gamma(F^i(v))$ is a subset of the first $r - i$ colors of $[r + 1] \setminus \{\gamma(v)\}$.

Thus the number of distinct $\mathrm{Class}(v)$ for $v \in V(G)$ is bounded by

$$(r + 1)!\, 2^r 2^{r-1} \dots 2 + (r + 1)r!\, 2^{r-1} \dots 2 = (2^r + 1)(r + 1)!\, 2^{\binom{r}{2}}.$$

Furthermore, the number of distinct $\mathrm{NCol}(v)$ for $v \in V(G)$ is at most 2^{r+1}. □

5.5. Encoding the graph in the linear order

We first make use of Corollary 5.2 to encode G by a first-order formula using only the newly added colors and the order $<$ on $V(G)$. More precisely, for $v \in V(G)$, let

$$\mathrm{ICol}(v) := \{\gamma(u) \ : \ v \in I_u\}.$$

Let $\mathcal{L}$ be the structure over signature $\Lambda \cup \{<\}$, where Λ is the set of all colors of the form $(\mathrm{Class}(v), \mathrm{NCol}(v), \mathrm{ICol}(v))$, with the same elements as G and $<$ interpreted as in G. Every element v of $\mathcal{L}$ is equipped with the color $(\mathrm{Class}(v), \mathrm{NCol}(v), \mathrm{ICol}(v))$. The following lemma gives a new proof of the result of [9].

Lemma 5.13. *There exists an $\exists\forall$-first-order formula $\varphi(x, y)$ over the vocabulary $\Lambda \cup \{<\}$ such that for all $u, v \in V(G)$ we have*

$$\mathcal{L} \models \varphi(u, v) \iff \{u, v\} \in E(G).$$

Proof. By symmetry, we can assume that $u < v$. According to Corollary 5.2 for distinct $u, v \in V(G)$ we have

$$\{u, v\} \in E(G) \iff \begin{cases} v \in N_{\oplus}(F^{\xi(u,v)}(u)) & \text{if } u < v \\ u \in N_{\oplus}(F^{\xi(v,u)}(v)) & \text{if } u > v. \end{cases}$$

Note that we can extract any color from Λ, i.e. we can define $\gamma(x) \in \Gamma(F^i(y))$ and $\gamma(x) \in \mathrm{ICol}(y)$. For example, $\gamma(x) \in \Gamma(F^i(\{y\}))$ is a big disjunction over all possible colorings $\Lambda(x) = (\mathrm{Class}(x), \mathrm{NCol}(x), \mathrm{ICol}(x))$ and $\Lambda(y) = (\mathrm{Class}(y), \mathrm{NCol}(y), \mathrm{ICol}(y))$ satisfying that $\mathrm{Class}(x)$ has in its first component an element from the ith component of $\mathrm{Class}(y)$.

We first define formulas $\psi^i(x, y)$ such that for all $u, v \in V(G)$

$$\mathcal{G} \models \psi^i(u, v) \Leftrightarrow v \in F^i(\{u\}).$$

Let $C = \Gamma(F^i(\{u\}))$. According to Lemma 5.11, for $a \in C$, the element of $F^i(\{u\}) \subseteq B_u$ with color a is the maximal element $w < u$ such that $\gamma(w) = a$. The formula can express that $y < x$ is maximal with $\gamma(y) = a$ by $(y < x) \wedge (\gamma(y) = a) \wedge \forall z\,((z > y) \wedge (z < x) \to \gamma(z) \neq a)$. Here, for convenience, we use $\gamma(z) = a$ as an atom. Note that $\psi^i(x, y)$ is a $\forall$-formula.

We now define formulas $\alpha^k(x, y)$ such that for all $u, v \in V(G)$ with $u < v$ we have

$$\mathcal{G} \models \alpha^k(u, v) \Leftrightarrow k = \xi(u, v).$$

Observe that $v \in I_{F^k(\{u\})}$ if and only if for every $x \in F^k(\{u\})$ we have $x \leq v$, $a \in \mathrm{ICol}(v)$ (i.e. there exists some y with $\gamma(y) = a$ and $v \in I_y$) and there exists no z with $x < z \leq v$ with $\gamma(z) = a$ (hence $\min I_y \leq x$, which implies that I_y and I_x intersects thus $x = y$ as $\gamma(x) = \gamma(y)$). We restrict ourselves to the case $u < v$ and obtain

$$\begin{aligned} u < v \wedge v \in I_{F^k(\{u\})} \iff\ & u < v \wedge \Gamma(F^k(\{u\})) \subseteq \mathrm{ICol}(v) \\ & \wedge \forall x\,(x \in F^k(\{u\}) \to x \leq v \wedge \gamma(x) \notin \mathrm{ICol}(v))\,. \end{aligned}$$

Then $\xi(u, v)$ for $u < v$ is the minimum integer k such that $F^k(\{u\}) = \emptyset$, or $v \in I_{F^k(\{u\})}$ and this is easy to state as a $\forall$-formula. Finally, if we have determined $\xi(u, v)$, with the help of the formulas ψ^i we can determine whether $\{u, v\} \in E(G)$ as in the proof of Corollary 5.2 by existentially quantifying the elements of $F(\{u\}), F^2(\{u\}), \ldots, F^{\xi(u,v)}(\{u\})$ and expressing whether $v \in N_{\oplus}(F^{\xi(u,v)}(\{u\}))$. Indeed, for every $x \in F^{\xi(u,v)}(\{u\})$ we have $v \in I_{F^{\xi(u,v)}(\{u\})} \subseteq I_x$, hence the adjacency of x and y is encoded in $\mathrm{NCol}(v)$.

This information can hence be retrieved by an $\exists\forall$-formula, as claimed. □

Lemma 5.14. *Let $f'(r) := (2^r + 1)(r + 1)!\, 2^{\binom{r}{2}} 3^{r+1}$. The number of triples $(\mathrm{Class}(v), \mathrm{NCol}(v), \mathrm{ICol}(v))$ for $v \in V(G)$ can be bounded by $f'(r)$.*

Proof. By Lemma 5.12, the number of distinct $\mathrm{Class}(v)$ for $v \in V(G)$ is bounded by $(2^r + 1)(r + 1)!\, 2^{\binom{r}{2}}$. The number of pairs $(\mathrm{NCol}(v), \mathrm{ICol}(v))$ is at most 3^{r+1} (for each color a in $[r + 1]$ either $a \notin \mathrm{ICol}(v)$ or $a \in \mathrm{ICol}(v) \setminus \mathrm{NCol}(v)$ or $a \in \mathrm{NCol}(v)$). □

As a corollary we conclude an upper bound on the number of graphs of bounded linear rankwidth.

Theorem 5.15. *Unlabeled graphs with linear rankwidth at most r can be encoded using at most $\binom{r}{2} + r\log_2 r + r\log_2(12/e) + O(\log_2 r)$ bits per vertex. Precisely, the number of unlabelled graphs of order n with linear rankwidth at most r is at most $\left[(2^r+1)(r+1)!\,2^{\binom{r}{2}}3^{r+1}\right]^n$.*

Proof. According to Stirling's approximation formula we have

$$\log_2(r!) = r + 1 + r\log_2 r - r\log_2 e + O(\log_2 r).$$

As $\log_2[(r+1)!] = \log_2(r+1) + \log_2(r!)$ we have $\log_2\left[(2^r+1)(r+1)!\,2^{\binom{r}{2}}3^{r+1}\right] = \binom{r}{2} + r\log_2 r + r\log_2(12/e) + O(\log_2 r)$. □

Remark 5.16. The encoding can be computed in linear time if the linear order on G is given.

5.6. Partition into cographs

Theorem 5.17. *Let $f(r) = 2(2^r+1)(r+1)!\,2^{\binom{r+1}{2}}$. The c-chromatic number of every graph G (that is the minimum order of a partition of $V(G)$ where each part induces a cograph) is bounded by $f(\mathrm{lrw}(G))$. Hence*

$$\chi(G) \le f(\mathrm{lrw}(G))\,\omega(G). \tag{18}$$

Proof. Let $u \sim v$ hold if and only if $\mathrm{Class}(u) = \mathrm{Class}(v)$ and $\mathrm{NCol}(u) = \mathrm{NCol}(v)$. As proved in Lemma 5.12 there are at most $f(r)$ equivalence classes for the relation $\sim$.

Let X be an equivalence class for $\sim$, and let u, v be distinct elements in X. Let $k = \xi(u,v)$ and let $\ell = \xi(v,u)$.

If $F^k(u) = \emptyset$, then $F^k(v) = \emptyset$ as $\mathrm{Class}(v) = \mathrm{Class}(u)$. Otherwise, $F^k(\{u\}) \neq \emptyset$, thus $F^k(\{v\}) \neq \emptyset$. As $v \in I_{F^k(\{u\})}$ and $v \in I_{F^k(\{v\})}$ we deduce that $F^k(\{u\})$ and $F^k(\{v\})$ are both included in B_v. As the vertices of a given color in B_v are uniquely determined we deduce $F^k(\{u\}) = F^k(\{v\})$. Similarly, we argue that $F^\ell(\{u\}) = F^\ell(\{v\})$. It follows that $F^k(\{u\}) = F^\ell(\{u\}) = u \wedge v$.

Hence, if $x \wedge y = u \wedge v$ for $x, y \in X$, then we have $x \wedge y = F^k(\{x\}) = F^k(\{u\})$. As $\mathrm{NCol}(u) = \mathrm{NCol}(v)$, we deduce that for all $x, y \in X$ with $x \wedge y = u \wedge v$ we have $y \in N_\oplus(F^k(\{x\}))$ or for all $x, y \in X$ with $x \wedge y = u \wedge v$ we have $y \notin N_\oplus(F^k(\{x\}))$. Then it follows from Corollary 5.2 that at each inner vertex of F on X we either define a join or a union. Hence, $G[X]$ is a cograph with cotree F restricted to X of height at most $r+2$. □

Remark 5.18. The partition can be computed in quadratic time if the ordering of the vertex set is given.

The function $f(r)$ is most probably far from being optimal. This naturally leads to the following question.

Problem 5.19. Estimate the growth rate of function $g : \mathbb{N} \to \mathbb{R}$ defined by

$$g(r) = \sup\left\{ \frac{\chi(G)}{\omega(G)} \;:\; \mathrm{lrw}(G) \le r \right\}. \tag{19}$$

Remark 5.20. One may wonder whether bounding $\chi(G)$ by an affine function of $\omega(G)$ could decrease the coefficient of $\omega(G)$. In other words, is the ratio χ/ω be asymptotically much smaller (as $\omega \to \infty$) than its supremum? Note that if $\mathrm{lrw}(G) = r$ and $n \in \mathbb{N}$, then the graph G_n obtained as the join of n copies of G satisfies $\mathrm{lrw}(G_n) \le r+1$, $\omega(G_n) = n\omega(G)$ and $\chi(G_n) = n\chi(G)$. Thus

$$g(r-1) \le \limsup_{\omega\to\infty}\left\{ \frac{\chi(G)}{\omega(G)} \,\middle|\, \mathrm{lrw}(G) \le r \text{ and } \omega(G) \ge \omega \right\} \le g(r).$$

Problem 5.21. Is the ratio $\chi(G)/\omega(G)$ bounded by a polynomial function of the neighborhood-width of G (equivalently, of the linear cliquewidth or of the linear NLC-width of G)?

6. Conclusion, further works, and open problems

In this paper, several aspects of classes with bounded linear-rankwidth have been studied, both from (structural) graph theoretical and the model theoretical points of view.

On the one hand, it appeared that graphs with bounded linear rankwidth do not form a "prime" class, in the sense that they can be further decomposed/covered using pieces in classes with bounded embedded shrubdepth. As an immediate corollary we obtained that classes with bounded linear rankwidth are linearly χ-bounded. Of course, the χ/ω bound obtained in Theorem 5.17 is most probably very far from being optimal.

On the other hand, considering how graphs with linear rank-width at most r are encoded in a linear order or in a graph with bounded pathwidth with marginal "quantifier-free" use of a compatible linear order improved our understanding of this class in the first-order transduction framework.

Classes with bounded rankwidth seem to be much more complex than expected and no simple extension of the results obtained from classes with bounded linear rankwidth seems to hold. In particular, these classes seem to be "prime" in the sense that you cannot even partition the vertex set into a bounded number of parts, each inducing a graph is a simple hereditary class like the class of cographs (see Corollary 3.2). However, the following conjecture seems reasonable to us.

Conjecture 6.1. Let $\mathscr{C}$ be a class of graphs of bounded rankwidth. Then $\mathscr{C}$ has structurally bounded treewidth if and only if $\mathscr{C}$ is stable.

We believe that our study of classes with bounded linear rankwidth might open the perspective to study classes admitting low linear rankwidth covers. Let us elaborate on this. As a consequence of Theorem 4.6 we have the following:

Theorem 6.2. *Let $\mathscr{C}$ be a class with low linear rankwidth covers. Then the following are equivalent:*

1. *$\mathscr{C}$ is monadically stable,*
2. *$\mathscr{C}$ is stable,*
3. *$\mathscr{C}$ excludes a semi-induced half-graph,*
4. *$\mathscr{C}$ has structurally bounded expansion.*

Proof. Clearly 1 $\Rightarrow$ 2 $\Rightarrow$ 3. For 3 $\Rightarrow$ 4, let p be an integer and consider a depth-p cover $\mathcal{U}$ of $G \in \mathscr{C}$ with linear rankwidth at most r. If $\mathscr{C}$ excludes some semi-induced half-graph we deduce by Theorem 4.6 that each $U \in \mathcal{U}$ induces a subgraph that is a fixed transduction of a graph with pathwidth at most $C(r)$, hence, of a class that has depth-p covers with bounded shrubdepth. Considering the intersection of the two covers, we get that $\mathscr{C}$ has depth-p covers with bounded shrubdepth, hence, has structurally bounded expansion. Thus 3 $\Rightarrow$ 4. Finally, 4 $\Rightarrow$ 1 is implied by Theorem 2.5. □

The next example illustrates again the concept of simple transductions and as a side product will provide us with some examples of classes of graphs admitting low linear rankwidth covers.

Example 6.3. We consider the following graph classes, introduced in [34]. Let n, m be integers. The graph $H_{n,m}$ has vertex set $V = \{v_{i,j} \ : \ (i,j) \in [n] \times [m]\}$. In this graph, two vertices $v_{i,j}$ and $v_{i',j'}$ with $i \leq i'$ are adjacent if $i' = i+1$ and $j' \leq j$. The graph $\widetilde{H}_{n,m}$ is obtained from $H_{n,m}$ by adding all the edges between vertices having same first index (that is between $v_{i,j}$ and $v_{i,j'}$ for every $i \in [n]$ and all distinct $j, j' \in [m]$.

First note that for fixed $a \in \mathbb{N}$ the classes $\mathscr{H}_a = \{H_{a,m} \ : \ m \in \mathbb{N}\}$ and $\widetilde{\mathscr{H}_a} = \{\widetilde{H}_{a,m} \ : \ m \in \mathbb{N}\}$ have bounded linear rank-width as they can be obtained as interpretations of a-colored linear orders: we consider the linear order on $\{v_{i,j} \ : \ (i,j) \in [a] \times [m]\}$ defined by $v_{i,j} < v_{i',j'}$ if $j < j'$ or $(j = j')$ and $(i < i')$. We color $v_{i,j}$ by color i. Then the graphs in $\mathscr{H}_a$ are obtained by the interpretation stating that $x < y$ are adjacent if the color of x is one less than the color of y, and if there is no z between x and y with the same color as x. The graphs in $\widetilde{\mathscr{H}_a}$ are obtained by further adding all the edges between vertices with same color.

Following the lines of [33, Theorem 9] we deduce from Example 6.3:

Proposition 6.4. *The class of unit interval graphs and the class of bipartite permutation graphs admit low linear rank-width colorings.*

As we have shown above, classes with low linear rankwidth covers generalize structurally bounded expansion classes. Among the first problems to be solved on these class, two arise very naturally:

Problem 6.5. Is it true that every first-order transduction of a class with low linear rankwidth covers has again low linear rankwidth covers?

As a stronger form of this problem, one can also wonder whether classes with low linear rankwidth covers enjoy a form of quantifier elimination, as structurally bounded expansion class do.

Problem 6.6. Is it true that every class with low linear-rankwidth covers is monadically NIP?

Note that it is easily checked that a positive answer to Problem 6.5 would imply a positive answer to Problem 6.6.

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