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# Network topology detection via uncertainty analysis of an identified static model

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**Abstract:** In this paper, we propose a methodology to detect the topology of a dynamic network that is based on the analysis of the uncertainty of the static characteristic of the matrix of transfer functions between the external excitations and the node signals.

*Keywords:* Dynamic network identification, topology detection

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## 1. INTRODUCTION

This paper contributes to the efforts of developing techniques for the identification of large-scale or interconnected systems. In these efforts, we can distinguish two main lines of research. The first line consists in determining a model of the network when the interconnection structure (i.e. the topology) of the network is known (see e.g. (Haber and Verhaegen, 2013; Van den Hof et al., 2013; Gevers and Bazanella, 2015)). The second line consists in determining the topology of the unknown network (see e.g. (Gonçalves and Warnick, 2008; Yuan et al., 2011; Materassi and Salapaka, 2012; Materassi et al., 2013)). This paper belongs to this second line of research.

We consider a dynamical network made up of  $N_{mod}$  nodes. A given node  $w_i$  of such a network can be written as a function of the other nodes  $w_k$  ( $k \neq i$ ) and as a function of known external excitations and unknown noise excitations (Van den Hof et al., 2013). In this paper, we are interested in determining the topology of such a network i.e. in determining for each possible pairs of nodes  $w_i$  and  $w_k$  whether there exists a non-zero causal transfer function  $G_{ik}(z)$  linking these two nodes. To achieve this objective, one can of course use the node measurements to identify these transfer functions in a direct manner (Van den Hof et al., 2013) or in an indirect way (i.e. by first estimating the transfer between the external excitations and the different nodes and then by back-computing the transfer between the nodes) (Yuan et al., 2011; Materassi and Salapaka, 2012). Due to the presence of the unknown noise excitations, even if a given transfer function  $G_{ik}(z)$  is identically zero, its estimate will not be equal to zero. Consequently, it may be difficult in practice to determine the exact topology of the network if we only look at the estimates of the different transfer

functions  $G_{ik}(z)$ . In order to avoid these issues, different sparsity-inducing approaches have been considered. In (Chiuso and Pillonetto, 2012; Materassi et al., 2013), one considers algorithms that favour sparse solutions. In (Yuan et al., 2011) and in another section of (Materassi et al., 2013), one considers algorithms that compare the quality of models identified when supposing that some of the transfer functions  $G_{ik}(z)$  are indeed equal to zero. In (Materassi et al., 2013), a forward selection approach is used while, in (Yuan et al., 2011), all possible topologies are tested and the quality of the identified models are compared using a criterion penalizing a large number of connections.

In this paper, we develop a topology detection method that is based on the analysis of the uncertainty of the identified network model. For this purpose, we will take inspiration from (Yuan et al., 2011) that shows that the topology of the network can be determined via the inversion of the static characteristic of the transfer matrix between the external excitations and the different nodes signals (the method in (Yuan et al., 2011) is thus an indirect approach). We will however not only consider the estimate of this static characteristic but also its uncertainty. Using this uncertainty, we will be able to derive an uncertainty interval for the static gain of each transfer function  $G_{ik}(z)$ . If this uncertainty interval contains zero, we will then conclude that  $G_{ik}(z)$  is equal to zero. We analyze the property of this decision rule and we show that we can increase our confidence in this decision rule by reducing the length of the confidence intervals, which can be achieved by an appropriate experiment design.

*Notations:* The matrix  $I_n$  denotes the identity matrix of dimension  $n$ . The matrix

$$\begin{pmatrix} X_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & X_n \end{pmatrix}$$

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will be denoted  $\text{diag}(X_1, \dots, X_n)$  when the elements  $X_i$  ( $i = 1, \dots, n$ ) are scalar quantities, while it will be denoted  $\text{bdiag}(X_1, \dots, X_n)$  when the elements  $X_i$  ( $i = 1, \dots, n$ ) are matrices.. For a matrix  $A$ ,  $A^T$  denotes the transpose of  $A$ . Finally,  $\otimes$  denotes the Kronecker product.

## 2. NETWORK DESCRIPTION

We consider a stable dynamic network made up of  $N_{\text{mod}}$  nodes that are each characterized by a scalar valued measurable signal  $w_i(t)$  ( $i = 1, \dots, N_{\text{mod}}$ ). The vector  $\bar{w}(t) = (w_1(t), w_2(t), \dots, w_{N_{\text{mod}}}(t))^T$  obeys the following equation (Van den Hof et al., 2013):

$$\bar{w}(t) = G_0(z) \bar{w}(t) + \bar{r}(t) + \bar{v}(t) \quad (1)$$

$$G_0(z) = \begin{pmatrix} 0 & G_{0,12}(z) & \dots & G_{0,1N_{\text{mod}}}(z) \\ G_{0,21}(z) & 0 & \dots & G_{0,2N_{\text{mod}}}(z) \\ \dots & \dots & \dots & \dots \\ G_{0,N_{\text{mod}}1}(z) & G_{0,N_{\text{mod}}2}(z) & \dots & 0 \end{pmatrix} \quad (2)$$

where  $\bar{r}(t) = (r_1(t), r_2(t), \dots, r_{N_{\text{mod}}}(t))^T$  is a vector of external excitation signals that can be freely chosen by the user e.g., for identification purposes ( $\bar{r}(t) = 0$  in normal operations) and where the vector  $\bar{v}(t) = (v_1(t), v_2(t), \dots, v_{N_{\text{mod}}}(t))^T$  represents the process noise acting on the network and is assumed to be a stationary stochastic process.

For the sequel, we will need the following *closed-loop* expression of (1):

$$\bar{w}(t) = T_0(z)\bar{r}(t) + \bar{v}_{cl}(t) \quad (3)$$

where the transfer matrix  $T_0(z) = (I_{N_{\text{mod}}} - G_0(z))^{-1}$  is stable since the network is assumed stable and where  $\bar{v}_{cl}(t) = T_0(z)\bar{v}(t)$ . Note that the stochastic process  $\bar{v}_{cl}(t)$  can always be modeled as

$$\bar{v}_{cl}(t) = H_0(z)\bar{e}(t) \quad (4)$$

for some matrix of transfer functions  $H_0(z)$  that is stable, inversely stable and that is also monic (i.e.  $H_0(\infty) = I_{N_{\text{mod}}}$ ) and for some white noise vector  $\bar{e}(t) = (e_1(t), e_2(t), \dots, e_{N_{\text{mod}}}(t))^T$  such that  $E\bar{e}(t)\bar{e}^T(t) = \Sigma_0 > 0$  and  $E\bar{e}(t)\bar{e}^T(t-\tau) = 0$  for all  $\tau \neq 0$ . It is important to note that  $T_0(z)$  and  $H_0(z)$  can be both very complex matrices of transfer functions.

*Remark.* The stochastic process  $v(t)$  in (1) can also be written as  $v(t) = D_0(z)\bar{e}(t)$  with a transfer matrix  $D_0(z)$  which is stable, inversely stable and monic. If  $G_0(z)$  is stable and if  $G_0(\infty) = 0$ , then the transfer matrix  $H_0(z)$  describing  $v_{cl}(t)$  is given by  $T_0(z)D_0(z)$ . If  $G_0(z)$  does not have these two properties,  $H_0(z)$  will have a more complex expression.

## 3. TOPOLOGY DETECTION PROBLEM

As mentioned in the introduction, in this paper, we wish to determine the topology of the network, i.e. we wish to determine which off-diagonal elements  $G_{0,ik}(z)$  of  $G_0(z)$

are nonzero transfer functions and which off-diagonal elements  $G_{0,ik}(z)$  are identically equal to zero. In other words, the topology detection problem consists in discriminating between the following hypotheses:

$$\begin{cases} \mathcal{H}_0 : G_{0,ik}(z) = 0 \\ \mathcal{H}_1 : G_{0,ik}(z) \neq 0 \end{cases} \quad (5)$$

for all off-diagonal elements  $G_{0,ik}(z)$  of  $G_0(z)$  in (1). In this paper, we will develop a topology detection procedure under the following assumption on  $G_0(z)$  that will generally hold in practice.

*Assumption 1.* Consider the network configuration described in Section 2. For any arbitrary transfer function  $G_{0,ik}(z)$  in  $G_0(z)$  (see (2)), if the static gain  $G_{0,ik}(1)$  of  $G_{0,ik}(z)$  is equal to zero, then the transfer function  $G_{0,ik}(z)$  is also equal to zero. ■

As shown in (Yuan et al., 2011), we can under this assumption derive the topology of the network by simply inspecting the inverse  $\mathcal{Q}_0$  of the static characteristic  $\mathcal{T}_0 = T_0(1)$  of the closed-loop transfer matrix  $T_0(z)$ . This inverse  $\mathcal{Q}_0$  is indeed given by:

$$\mathcal{Q}_0 = \mathcal{T}_0^{-1} = (I_{N_{\text{mod}}} - G_0(1)) \quad (6)$$

Consequently, we see that, under Assumption 1, any arbitrary transfer function  $G_{0,ik}(z)$  ( $i \neq k$ ) in  $G_0(z)$  (see (2)) is equal to zero if and only if

$$\mathcal{Q}_{0,ik} = 0 \quad (7)$$

where  $\mathcal{Q}_{0,ik}$  is the  $(i, k)$ -entry of the matrix  $\mathcal{Q}_0 \in \mathbf{R}^{N_{\text{mod}} \times N_{\text{mod}}}$ . In other words, under Assumption 1, (5) is equivalent to:

$$\begin{cases} \mathcal{H}_0 : \mathcal{Q}_{0,ik} = 0 \\ \mathcal{H}_1 : \mathcal{Q}_{0,ik} \neq 0 \end{cases} \quad (8)$$

## 4. ESTIMATE OF $\mathcal{T}_0$ AND ITS UNCERTAINTY

To be able to discriminate between  $\mathcal{H}_0$  and  $\mathcal{H}_1$ , since  $\mathcal{Q}_0$  is unknown, we will use an estimate  $\hat{\mathcal{Q}}$  of  $\mathcal{Q}_0$ . This estimate  $\hat{\mathcal{Q}}$  of  $\mathcal{Q}_0$  will be given as:

$$\hat{\mathcal{Q}} = \hat{\mathcal{T}}^{-1} \quad (9)$$

where  $\hat{\mathcal{T}}$  is an estimate of  $\mathcal{T}_0$ . It is very difficult to derive a reliable decision rule for the hypothesis test (8) based on such an estimate. Indeed, even if the actual off-diagonal entry  $\mathcal{Q}_{0,ik}$  of  $\mathcal{Q}_0$  is identically equal to zero, the off-diagonal entry  $\hat{\mathcal{Q}}_{ik}$  of  $\hat{\mathcal{Q}}$  will not be equal to zero. Consequently, in this paper, we will propose a decision rule for (8) by combining  $\hat{\mathcal{Q}}_{ik}$  with its uncertainty. For this purpose, we need to derive an estimate  $\hat{\mathcal{T}}$  of  $\mathcal{T}_0$  and its uncertainty. Since we are only interested in an estimate of the static characteristic of  $T_0(z)$  (and not an estimate of  $T_0(z)$ ), the identification problem can be strongly simplified.

Since we wish to characterize the uncertainty of  $\hat{\mathcal{T}}$ , we will nevertheless need to derive a model of  $H_0(z)$ . Using time series analysis (Box et al., 2015), this can be done in advance based on normal operation data (i.e., data  $\bar{w}(t)$  collected on (1) when  $r(t) = 0$ ). Using an AR structure allows to estimate a model  $\hat{H}(z) = \hat{A}(z)^{-1}$  of  $H_0(z)$  by solving a least-squares optimization problem. Referring to Theorem 3.1 in (Ljung and Wahlberg, 1992),  $\hat{H}(e^{j\omega})$  will tend to  $H_0(e^{j\omega})$  if the number of estimation data tends to infinity and if the order of the AR model increase at a suitable rate with this number of estimation data. In practice, the order of the AR model is chosen in such a way that the residuals  $\hat{H}^{-1}(z)\bar{w}(t)$  are whitened. The obtained AR model will be used to derive the estimate of  $\mathcal{T}_0$ .

The estimate of the static matrix  $\mathcal{T}_0$  will be determined via  $N_{mod}$  identification experiments leading each to an estimate of one column of  $\mathcal{T}_0$ . More precisely, the  $k^{th}$  experiment ( $k = 1, \dots, N_{mod}$ ) allows to determine an estimate  $\hat{\mathcal{T}}_k$  of the  $k^{th}$  column  $\mathcal{T}_{0,k}$  of  $\mathcal{T}_0$ . For this purpose, we apply to the network (1) an excitation vector  $\bar{r}(t)$  where all the elements except  $r_k(t)$  are zero and where  $r_k(t)$  is equal to the constant  $\alpha_k$  for all  $t$ :

$$\bar{r}(t) = \alpha_k \bar{m}_k \quad \forall t \quad (10)$$

where  $\bar{m}_k$  ( $k = 1, \dots, N_{mod}$ ) denotes a unit (column) vector of dimension  $N_{mod}$  for which the  $k^{th}$  entry is equal to 1 and the other entries are equal to zero. After the end of the transient, the vector  $\bar{w}(t)$  that is collected during such an experiment obeys the following relation:

$$\bar{w}(t) = \mathcal{T}_{0,k} \alpha_k + H_0(z)\bar{e}(t) \quad (11)$$

Consequently, an estimate  $\hat{\mathcal{T}}_k$  of  $\mathcal{T}_{0,k}$  (and an estimate  $\hat{\Sigma}$  of the covariance matrix  $\Sigma_0$  of  $\bar{e}$ ) can be obtained by considering the model  $\hat{H}(z)$  of  $H_0(z)$  and the following simple prediction error criterion (Ljung, 1999):

$$\min_{\hat{\mathcal{T}}_k, \hat{\Sigma}} \frac{1}{N} \sum_{t=N_{k,ss}}^{N_k+N_{k,ss}} \bar{e}_k^T(t, \hat{\mathcal{T}}_k) \hat{\Sigma}^{-1} \bar{e}_k(t, \hat{\mathcal{T}}_k) \quad (12)$$

$$\bar{e}_k(t, \hat{\mathcal{T}}_k) = \hat{H}^{-1}(z) (\bar{w}(t) - \hat{\mathcal{T}}_k \alpha_k) \quad (13)$$

where  $\hat{\mathcal{T}}_k$  is a column vector of dimension  $N_{mod}$ ,  $\hat{\Sigma}$  is a square matrix of dimension  $N_{mod}$ ,  $N_k + N_{k,ss}$  is the duration of the  $k^{th}$  experiment and  $N_{k,ss}$  is the moment where  $\bar{w}_H(t) = \hat{H}^{-1}(z)\bar{w}(t)$  reaches steady-state in this  $k^{th}$  experiment. For  $N_{k,ss} \leq t \leq N_k + N_{k,ss}$ ,  $\bar{e}_k(t, \hat{\mathcal{T}}_k) = \bar{w}_H(t) - \hat{\Phi} \hat{\mathcal{T}}_k$  with  $\hat{\Phi} = \alpha_k \hat{H}^{-1}(1)$ . Consequently, (12) is basically a least-squares optimization problem for which the procedure in (Ljung, 1999, page 485) can be used.

Using the framework of (Ljung and Wahlberg, 1992) described above,  $\hat{H}(e^{j\omega})$  can be made to converge to  $H_0(e^{j\omega})$  at a sufficiently fast rate so that the estimate  $\hat{\mathcal{T}}_k$  obtained via (12) is asymptotically normally distributed around  $\mathcal{T}_{0,k}$  with a covariance matrix  $P_{\hat{\mathcal{T}}_k} = (1/N_k) (\hat{\Phi}^T \hat{\Sigma}_0^{-1} \hat{\Phi})^{-1}$  (Ljung, 1999). Using the estimate  $\hat{\Sigma}$  of  $\Sigma_0$ ,  $P_{\hat{\mathcal{T}}_k}$  can thus be estimated as:

$$P_{\hat{\mathcal{T}}_k} \approx \frac{1}{\alpha_k^2 N_k} \left( \hat{H}(1) \hat{\Sigma} \hat{H}^T(1) \right) \quad (14)$$

Combining the estimates of the columns of  $\mathcal{T}_0$  obtained in the  $N_{mod}$  experiments, we can deduce the following estimate  $\hat{\mathcal{T}}$  of  $\mathcal{T}_0$ :

$$\hat{\mathcal{T}} = \left( \hat{\mathcal{T}}_1, \hat{\mathcal{T}}_2, \dots, \hat{\mathcal{T}}_{N_{mod}} \right) \quad (15)$$

Let us also analyze the variance/uncertainty of this estimate. For this purpose, let us denote by  $\theta_0$  (resp.  $\hat{\theta}$ ) the vector of dimension  $N_{mod}^2$  made up of the vectorization of the columns of  $\mathcal{T}_0$  (resp.  $\hat{\mathcal{T}}$ ) i.e.

$$\theta_0 = \left( \mathcal{T}_{0,1}^T, \mathcal{T}_{0,2}^T, \dots, \mathcal{T}_{0,N_{mod}}^T \right)^T \quad (16)$$

$$\hat{\theta} = \left( \hat{\mathcal{T}}_1^T, \hat{\mathcal{T}}_2^T, \dots, \hat{\mathcal{T}}_{N_{mod}}^T \right)^T \quad (17)$$

From the reasoning above, it is clear that  $\hat{\theta}$  is (asymptotically) normally distributed around  $\theta_0$  with a covariance matrix  $P_{\theta}$  given by:

$$P_{\theta} = bdiag \left( P_{\mathcal{T}_1}, P_{\mathcal{T}_2}, \dots, P_{\mathcal{T}_{N_{mod}}} \right) \quad (18)$$

with  $P_{\mathcal{T}_k}$  ( $k = 1, \dots, N_{mod}$ ) defined in (14).

## 5. DECISION RULE FOR (8)

As we will show in the next section, using  $\hat{\theta}$  and its covariance matrix  $P_{\theta}$ , we will be able to derive, for each off-diagonal entry  $\mathcal{Q}_{0,ik}$  of  $\mathcal{Q}_0$ , an uncertainty interval  $\mathcal{I}_{ik} = [Q_{ik}^{min}, Q_{ik}^{max}]$  that contains  $\mathcal{Q}_{0,ik}$  (modulo a given probability level). Based on this uncertainty interval  $\mathcal{I}_{ik}$ , we propose the following decision rule for the hypothesis test (8) (equivalent to (5)):

$$\begin{cases} 0 \in \mathcal{I}_{ik} \implies \mathcal{H}_0 \\ 0 \notin \mathcal{I}_{ik} \implies \mathcal{H}_1 \end{cases} \quad (19)$$

When we assume that  $\mathcal{Q}_{0,ik} \in \mathcal{I}_{ik}$ , this decision rule has the following properties for an uncertainty interval  $\mathcal{I}_{ik}$  of length  $L_{ik} = Q_{ik}^{max} - Q_{ik}^{min}$ . If  $\mathcal{Q}_{0,ik} = 0$  (i.e., under  $\mathcal{H}_0$ ), this decision rule will always yield the right decision i.e., to decide  $\mathcal{H}_0$ . If  $\mathcal{Q}_{0,ik} \neq 0$  (i.e., under  $\mathcal{H}_1$ ), the properties of (19) will depend on the absolute value  $|\mathcal{Q}_{0,ik}|$  of the unknown  $\mathcal{Q}_{0,ik}$ . If  $|\mathcal{Q}_{0,ik}|$  is larger than the length  $L_{ik}$  of the interval  $\mathcal{I}_{ik}$ , the decision rule (19) will always yield the right decision i.e., to decide  $\mathcal{H}_1$ . If  $\mathcal{Q}_{0,ik} \neq 0$  and  $|\mathcal{Q}_{0,ik}| < L_{ik}$ , the decision rule (19) may yield, depending on the actual value of  $\mathcal{I}_{ik}$ , to a good or a false decision. It is thus clear that a small  $L_{ik}$  will increase the reliability of the decision rule (19). We will see in the sequel that the smaller  $P_{\theta}$ , the smaller the lengths  $L_{ik}$  of the intervals  $\mathcal{I}_{ik}$  will be. Consequently, the reliability of the decision rule (19) can be improved by an appropriate design of the identification experiments described in Section 4.

To be able to use the decision rule (19), we need a manner to determine the intervals  $\mathcal{I}_{ik}$  for each pair  $(i, k)$ . This is done in the next section.

*Remark.* One could wonder whether we could not also derive a decision rule based on the inverse of the AR model  $\hat{H}(z)$  of  $H_0(z)$  and its uncertainty. This is indeed the case, but only under more restrictive assumptions on the network described in Section 2. For this let us refer to the remark at the end of Section 2. If  $G_0(z)$  is stable and satisfies  $G_0(\infty) = 0$ , we have  $H_0^{-1}(1) = D_0^{-1}(1)\mathcal{Q}_0$ . Consequently, if, in addition,  $D_0(z)$  is assumed diagonal, the topology of the network can be determined by inspecting  $H_0^{-1}(1)$  and an alternative decision rule can thus be determined by considering the uncertainty of the identified model  $\hat{H}(z)$  of  $H_0(z)$ .

## 6. DETERMINATION OF THE UNCERTAINTY INTERVALS

### 6.1 Introduction

Using the statistical properties of  $\hat{\theta}$  (see Section 4), it is clear that the following ellipsoid  $U$  is a  $\beta\%$ -confidence region for the modeling error  $\delta_0 = \theta_0 - \hat{\theta}$ :

$$U := \left\{ \delta \in \mathbf{R}^{N_{mod}} \mid \delta^T P_{\hat{\theta}}^{-1} \delta < \chi \right\} \quad (20)$$

where  $Pr(\chi^2(N_{mod}^2) < \chi) = \beta$ . This ellipsoid  $U$  can be considered as an uncertainty set for the unknown modeling error  $\delta_0$ . From now onwards, we will therefore assume that  $\delta_0 = \theta_0 - \hat{\theta} \in U$  or, in other words, that  $\theta_0 \in \{\theta \mid \theta = \hat{\theta} + \delta \text{ and } \delta \in U\}$ . If, for a given vector  $\delta \in U$ , we denote by  $T(\delta)$ , the matrix obtained from the de-vectorization of the vector  $\theta := \hat{\theta} + \delta$ , we have that  $\hat{T} = T(0)$  and  $\mathcal{T}_0 = T(\delta_0)$ . Moreover, since  $\mathcal{Q}_0 = T(\delta_0)^{-1}$  and  $\delta_0 \in U$ , we have also that the unknown  $(i, k)$ -entry  $\mathcal{Q}_{0,ik}$  of  $\mathcal{Q}_0$  lies in the following set  $\mathcal{I}_{ik}^{orig} = \{Q_{ik}(\delta) \in \mathbf{R} \mid Q(\delta) = T(\delta)^{-1} \text{ and } \delta \in U\}$  where  $Q_{ik}(\delta)$  is the  $(i, k)$ -entry of  $Q(\delta)$ .

The set  $\mathcal{I}_{ik}^{orig}$  defined in the previous paragraph is an interval since the matrix inversion is a continuous operation. Consequently, this interval could be used in the decision rule (19) since  $\mathcal{I}_{ik}^{orig}$  contains  $\mathcal{Q}_{0,ik}$  (modulo a certain probability level). However, this will not be possible since we cannot compute an explicit expression for  $\mathcal{I}_{ik}^{orig}$ . To show this, let us observe that  $\mathcal{I}_{ik,orig}$  is also equal to  $[c_{ik}^{opt,orig} - \sqrt{\rho_{ik}^{opt,orig}}, c_{ik}^{opt,orig} + \sqrt{\rho_{ik}^{opt,orig}}]$  where  $c_{ik}^{opt,orig}$  and  $\rho_{ik}^{opt,orig}$  are the solutions of the following optimization problem:

$$\min_{c_{ik}, \rho_{ik}} \rho_{ik} \quad (21)$$

$$\text{s.t. } (Q_{ik}(\delta) - c_{ik})^2 < \rho_{ik} \quad \forall \delta \in U \quad (22)$$

As usual in robustness analysis (Zhou and Doyle, 1998), we cannot determine exactly the solutions of the above optimization problem. However, we can use convex relaxation to replace the constraint (22) by an alternative constraint that is linear in the decision variables  $c_{ik}$  and  $\rho_{ik}$  and that implies (22). If we denote by  $c_{ik}^{opt}$  and  $\rho_{ik}^{opt}$  the solutions of the convex optimization problem consisting in minimizing

$\rho_{ik}$  under the convex constraint mentioned above. Then, the computable interval  $\mathcal{I}_{ik} = [c_{ik}^{opt} - \sqrt{\rho_{ik}^{opt}}, c_{ik}^{opt} + \sqrt{\rho_{ik}^{opt}}]$  is such that  $\mathcal{I}_{ik}^{orig} \subseteq \mathcal{I}_{ik}$  and can thus be used in the decision rule (19). In the next subsections, we show how we can derive a convex constraint that implies (22).

### 6.2 LFT description of $Q_{ik}(\delta)$

To be able to determine a convex alternative for (22), a first step is to rewrite the quantity  $Q_{ik}(\delta)$  in an Linear Fractional Transform (LFT) in the variable  $\delta$ . As we will see below, this can be done by first rewriting  $T(\delta)$  as an LFT in  $\delta$ . For an arbitrary  $\delta \in U$ , the mapping  $\bar{y} = T(\delta)\bar{u}$  can be expressed in the LFT framework as follows:

$$\bar{p} = \underbrace{I_{N_{mod}} \otimes \delta}_{=\Delta(\delta)} \bar{q} \quad (23)$$

$$\begin{pmatrix} \bar{q} \\ \bar{y} \end{pmatrix} = \underbrace{\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}}_{=M} \begin{pmatrix} \bar{p} \\ \bar{u} \end{pmatrix} \quad (24)$$

where  $M$  is given by:

$$M = \left( \begin{array}{c|c} \mathbf{0} & I_{N_{mod}} \\ \hline \frac{M_{11} \mid M_{12}}{M_{21} \mid M_{22}} & \hat{T} \end{array} \right) = \left( \begin{array}{c|c} \mathbf{0} & I_{N_{mod}} \\ \hline (\bar{m}_1^T, \dots, \bar{m}_{N_{mod}}^T) \otimes I_{N_{mod}} & \hat{T} \end{array} \right)$$

with  $\bar{m}_k$  ( $k = 1, \dots, N_{mod}$ ) and  $\hat{T} = T(0)$  as defined in Section 4. In the sequel, we will use  $\bar{y} = \mathcal{F}(M, \Delta(\delta))\bar{u}$  as a shorthand notation for the LFT (23)-(24).

Let us consider the same  $\delta \in U$  as well as the inverse mapping  $\bar{u} = Q(\delta)\bar{y} = T^{-1}(\delta)\bar{y}$ . This inverse mapping can also be expressed in the LFT framework with (23) and:

$$\begin{pmatrix} \bar{q} \\ \bar{u} \end{pmatrix} = \underbrace{\begin{pmatrix} N_{11} \mid N_{12} \\ N_{21} \mid N_{22} \end{pmatrix}}_{=N} \begin{pmatrix} \bar{p} \\ \bar{y} \end{pmatrix} \quad (25)$$

where  $N$  can be derived from  $M$ :

$$N = \left( \begin{array}{c|c} N_{11} \mid N_{12} \\ \hline N_{21} \mid N_{22} \end{array} \right) = \left( \begin{array}{c|c} M_{11} - M_{12}M_{22}^{-1}M_{21} \mid M_{12}M_{22}^{-1} \\ \hline -M_{22}^{-1}M_{21} \mid M_{22}^{-1} \end{array} \right)$$

In other words, we have  $\bar{u} = Q(\delta)\bar{y} = \mathcal{F}(N, \Delta(\delta))\bar{y}$ . This LFT of the inverse mapping can be adapted to each entry of the matrix  $Q(\delta)$ . In particular, for the same  $\delta \in U$ , the scalar mapping  $u_i = Q_{ik}(\delta)y_k$  can be rewritten as the LFT made up of (23) and:

$$\begin{pmatrix} \bar{q} \\ u_i \end{pmatrix} = \underbrace{\begin{pmatrix} N_{11} \mid N_{12}^{:,k} \\ N_{21}^{i,:} \mid N_{22}^{ik} \end{pmatrix}}_{=N_{ik}} \begin{pmatrix} \bar{p} \\ y_k \end{pmatrix} \quad (26)$$

where  $N_{22}^{ik}$  represents the  $(i, k)$ -entry of the matrix  $N_{22}$ , where  $N_{12}^{:,k}$  represents the  $k^{\text{th}}$  column of the matrix  $N_{12}$  and finally where  $N_{21}^{i,:}$  represents the  $i^{\text{th}}$  row of the matrix  $N_{21}$ . Consequently, we have that  $Q_{ik}(\delta) = \mathcal{F}(N_{ik}, \Delta(\delta))$  for any  $\delta \in U$ .

### 6.3 Set of multipliers related to the uncertainty set $U$

Another crucial ingredient to determine a convex alternative for (22) is to associate, with the set  $U$ , a so-called set of multipliers (Safonov, 1980).

*Definition 1.* Consider the set  $U$  defined in (20). We define the set of multipliers  $\mathcal{A}_U$  as a set of affinely parametrized Hermitian matrices  $A$  (of dimension  $N_{mod}(N_{mod}^2 + 1) \times N_{mod}(N_{mod}^2 + 1)$ ) that all have the following property:

$$\begin{pmatrix} I_{N_{mod}} \\ I_{N_{mod}} \otimes \delta \end{pmatrix}^T A \begin{pmatrix} I_{N_{mod}} \\ I_{N_{mod}} \otimes \delta \end{pmatrix} \geq 0 \quad \forall \delta \in U \quad (27)$$

In other words,  $A \in \mathcal{A}_U \implies (27)$ .  $\blacksquare$

We have derived an extensive parametrization of the set of multipliers  $\mathcal{A}_U$  corresponding to  $U$  in our previous contribution (Barenthin et al., 2008, Proposition 2). That the parametrization of  $\mathcal{A}_U$  is extensive is important since the more extensive the parametrization of the set of multipliers, the tighter the embedding  $\mathcal{I}_{ik}$  of  $\mathcal{I}_{ik}^{orig}$  will be (Safonov, 1980).

#### 6.4 LMI optimization to determine the uncertainty interval

We have now all the elements to derive the convex alternative for (22). We will for this purpose use a result in (Dinh et al., 2014) and adapt it to our situation.

*Proposition 1.* Consider the LFT  $\mathcal{F}(N_{ik}, \Delta(\theta))$  for  $Q_{ik}(\theta)$  (see (23)-(26)) and the set of multipliers  $\mathcal{A}_U$  associated to  $U$  (see Definition 1). Then, the constraint (22) holds for given  $c_{ik}$  and  $\rho_{ik}$  if we can find a matrix  $A \in \mathcal{A}_U$  such that the following LMI constraint holds:

$$\left( \begin{array}{c|c} -1 & \mathcal{X}(c_{ik}) \\ \hline \mathcal{X}(c_{ik})^T & \mathcal{K}^T A \mathcal{K} + \begin{pmatrix} 0 & 0 \\ 0 & -\rho_{ik} \end{pmatrix} \end{array} \right) < 0 \quad (28)$$

with  $\mathcal{X}(c_{ik}) = \begin{pmatrix} N_{21}^{i,:} & N_{22}^{i,k} - c_{ik} \end{pmatrix}$  and  $\mathcal{K} = \begin{pmatrix} N_{11} & N_{12}^{:,k} \\ I & 0 \end{pmatrix}$ .  $\blacksquare$

*Proof.* First, let us observe that the matrix inequality (28) is linear in the decision variables  $A$ ,  $c_{ik}$  and  $\rho_{ik}$ . Consequently (28) is indeed an LMI. Subsequently, using the Schur complements, (28) is equivalent to:

$$\mathcal{K}^T A \mathcal{K} + \begin{pmatrix} 0 & 0 \\ 0 & -\rho_{ik} \end{pmatrix} + \mathcal{X}(c_{ik})^T \mathcal{X}(c_{ik}) < 0 \quad (29)$$

Let us now consider the LFT (23)-(26) for a given  $\delta \in U$  and for  $y_k = 1$  and let us consider the corresponding signals  $\bar{p}$ ,  $\bar{q}$  and  $u_i = Q_{ik}(\delta)$ . Let us then pre- and post-multiply the constraint (29) with  $(\bar{p}^T, 1)$  and  $(\bar{p}^T, 1)^T$ , respectively. Using (26), this yields:

$$\begin{pmatrix} \bar{q} \\ \bar{p} \end{pmatrix}^T A \begin{pmatrix} \bar{q} \\ \bar{p} \end{pmatrix} + (u_i - c_{ik})^2 < \rho_{ik} \quad (30)$$

Since  $\bar{p} = (I_{N_{mod}} \otimes \delta) \bar{q}$  and  $u_i = Q_{ik}(\delta)$ , we can rewrite (30) as follows:

$$\bar{q}^T \begin{pmatrix} I_{N_{mod}} \\ I_{N_{mod}} \otimes \delta \end{pmatrix}^T A \begin{pmatrix} I_{N_{mod}} \\ I_{N_{mod}} \otimes \delta \end{pmatrix} \bar{q} + (Q_{ik}(\delta) - c_{ik})^2 < \rho_{ik} \quad (31)$$

The above reasoning can be done for any value of  $\delta \in U$ . In other words, for the matrix  $A \in \mathcal{A}_U$  for which (28) holds, (31) holds true for all  $\delta \in U$ . Consequently, using Definition 1, we have therefore that the existence of  $A \in \mathcal{A}_U$  such that (28) holds implies that, for each  $\delta \in U$ ,  $(Q_{ik}(\delta) - c_{ik})^2 < \rho_{ik}$ ; which is the desired result.  $\blacksquare$

Using Proposition 1, we can now compute the uncertainty interval  $\mathcal{I}_{ik}$  containing  $\mathcal{Q}_{0,ik}$  for each pair  $(i, k)$  such that  $i \neq k$ . Indeed, this interval  $\mathcal{I}_{ik}$  is given by  $\mathcal{I}_{ik} = [c_{ik}^{opt} - \sqrt{\rho_{ik}^{opt}}, c_{ik}^{opt} + \sqrt{\rho_{ik}^{opt}}]$  where  $c_{ik}^{opt}$  and  $\rho_{ik}^{opt}$  are the solutions of the LMI optimization problem consisting in finding the smallest value of  $\rho_{ik}$  for which we can find a scalar  $c_{ik}$  and a matrix  $A \in \mathcal{A}_U$  such that the LMI constraint (28) holds. In this LMI optimization problem, when we speak of finding a matrix  $A \in \mathcal{A}_U$ , we more precisely mean finding the free parameters in the affine structure of the matrix  $A$ .

*Remark.* The results above only hold if  $Q(\delta) = T(\delta)^{-1}$  is a valid operation for all  $\delta \in U$ . Since  $Q(\delta) = \mathcal{F}(N, \Delta(\delta))$  i.e.,  $Q(\delta) = N_{22} + N_{21} \Delta(\delta) (I - N_{11} \Delta(\delta))^{-1} N_{12}$ , the operation  $Q(\delta) = T(\delta)^{-1}$  is valid for a given  $\delta \in U$  if and only if the determinant of  $I - N_{11} \Delta(\delta)$  is not equal to zero. We have thus to verify that  $\det(I - N_{11} \Delta(\delta)) \neq 0$  for all  $\delta \in U$  or equivalently that the loop  $[N_{11} \Delta(\delta)]$  is stable for all  $\delta \in U$  (Zhou and Doyle, 1998). This is a classical robustness analysis problem that can e.g., be treated using the tools in (Barenthin et al., 2008).

## 7. NUMERICAL ILLUSTRATION

Let us consider a network (1) with  $N_{mod} = 3$  nodes and

$$G_0(z) \triangleq \begin{pmatrix} 0 & 0 & G_{0,13}(z) \\ G_{0,21}(z) & 0 & 0 \\ G_{0,31}(z) & G_{0,32}(z) & 0 \end{pmatrix} \quad (32)$$

with  $G_{0,21}(z) = \frac{0.095z^{-1}}{1-0.905z^{-1}}$ ,  $G_{0,31}(z) = \frac{0.181z^{-1}}{1-0.819z^{-1}}$  and  $G_{0,32}(z) = \frac{0.259z^{-1}}{1-0.741z^{-1}}$  and  $G_{0,13}(z) = 0.3 G_{0,32}(z)$ . Moreover,  $\bar{v}(t) = (I_3 \otimes \frac{1}{1-0.7z^{-1}}) \bar{e}(t)$  ( $i = 1, 2, 3$ ) with  $\bar{e}(t)$  a white noise vector of covariance matrix  $\Sigma_0 = 0.1I_3$ . Since  $G_0(z)$  is stable and  $G_0(\infty) = 0$ , the transfer matrix  $H_0(z)$  in (4) is given by  $T_0(z)(I_3 \otimes \frac{1}{1-0.7z^{-1}})$  (see the remark at the end of Section 2).

In this network, we see that the transfer functions  $G_{0,ik}$  ( $i \neq k$ ) that are equal to zero are the transfer function  $G_{0,12}(z)$  and  $G_{0,23}(z)$ . Let us compute the inverse  $\mathcal{Q}_0$  of the static gain matrix  $\mathcal{T}_0$  of the transfer matrix  $T_0(z) = (I_{N_{mod}} - G_0(z))^{-1}$ :

$$\mathcal{Q}_0 = \begin{pmatrix} 1 & 0 & -0.3 \\ -1 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix}. \quad (33)$$

We indeed observe that  $\mathcal{Q}_{0,12}$  and  $\mathcal{Q}_{0,23}$  are the sole entries  $\mathcal{Q}_{0,ik}$  in  $\mathcal{Q}_0$  that are equal to zero.

Following the procedure in Section 4, we identify an AR model  $\hat{H}(z) = \hat{A}^{-1}(z)$  of  $H_0(z)$  using 10000 samples of  $\bar{w}(t)$  collected on (1) with  $\bar{r}(t) = 0$ . A polynomial matrix  $\hat{A}(z)$  having entries of degree 5 is sufficient to whiten  $\hat{H}^{-1}(z)\bar{w}(t) = \hat{A}(z)\bar{w}(t)$ . This AR model is then used to subsequently identify the columns of  $\mathcal{T}_0$  via three identification experiments and the identification criterion (12). In these three experiments, we choose  $\alpha_k = 1$  and  $N_k = 800$  ( $k = 1, 2, 3$ ). This allows to deduce an estimate  $\hat{\mathcal{T}}$  of  $\mathcal{T}_0$  (see (15)) and we then obtain the following estimate of  $\mathcal{Q}_0$ :

$$\hat{Q} = \hat{\mathcal{T}}^{-1} = \begin{pmatrix} 1.1523 & -0.1243 & -0.3011 \\ -1.0373 & 1.0850 & -0.0276 \\ -0.9523 & -0.9678 & 0.9739 \end{pmatrix} \quad (34)$$

We observe that  $\hat{Q}_{12} = -0.1243$  and  $\hat{Q}_{23} = -0.0276$  are smaller than the other  $\hat{Q}_{ik}$  ( $i \neq k$ ), but not extremely small. To have an idea of the uncertainty of the estimates  $\hat{Q}_{ik}$  ( $i \neq k$ ), the estimate of the covariance matrix  $P_\theta$  of  $\hat{\theta}$  (see (17) and (18)) is used to determine the uncertainty ellipsoid  $U$  (the confidence level is chosen equal to 95% and  $\chi$  is thus chosen equal to 16.919). In our case,  $U$  indeed contains  $\theta_0$ . We then use the procedure described in Section 6 to derive the uncertainty intervals  $\mathcal{I}_{ik}$  corresponding to each non-diagonal entry  $\hat{Q}_{ik}$  of  $\hat{Q}$ . In Table 1, we give these intervals  $\mathcal{I}_{ik} = [Q_{ik}^{min}, Q_{ik}^{max}]$  together with their length  $L_{ik}$  and the true coefficients  $Q_{0,ik}$ . We observe that, for all pairs  $(i, k)$ , the true coefficient  $Q_{0,ik}$  lies in its confidence interval  $\mathcal{I}_{ik}$ . Even though the lengths  $L_{ik}$  of these intervals are relatively large with respect to  $|Q_{0,ik}|$ , the decision rule (19) allows to correctly conclude that the true unknown non-diagonal entries  $Q_{0,13}$ ,  $Q_{0,21}$ ,  $Q_{0,31}$ ,  $Q_{0,32}$  are all non-zero since 0 is not an element of their confidence interval and that both  $Q_{0,12}$  and  $Q_{0,23}$  are equal to zero since 0 lies in both  $\mathcal{I}_{12}$  and  $\mathcal{I}_{23}$ .

We have repeated the above procedure 500 times with different realizations of the noise vector  $\bar{e}$  in each step (i.e. the identification of  $\hat{H}(z)$  and of  $\hat{\mathcal{T}}$ ) and we have observed that  $\theta_0$  lies in  $U$  in 93.8% of these Monte Carlo simulations (which is close to the prescribed confidence level of 95%) and also that all these Monte Carlo simulations allowed to correctly determine the topology of the network using (19). If  $\alpha_k = 0.5$  ( $k = 1, 2, 3$ ) instead of  $\alpha_k = 1$  ( $k = 1, 2, 3$ ), the correct topology is determined in 42.4% of the cases. Indeed, if  $\alpha_k$  is decreased, the covariance matrix  $P_\theta$  (see (14)) is increased and, consequently,  $U$  has a larger volume and the size  $L_{ik}$  of the intervals are therefore larger and, as discussed below (19), the decision rule (19) is then less reliable. What we in fact observe is that other intervals than  $\mathcal{I}_{12}$  and  $\mathcal{I}_{23}$  contain zero and, consequently, (19) wrongly conclude that the corresponding non-diagonal entries of  $Q_0$  are zero.

Table 1.

$(i, k)$	$Q_{0,ik}$	$Q_{ik}^{min}$	$Q_{ik}^{max}$	$L_{ik}$
(1, 2)	0	-0.4142	0.1271	0.5413
(1, 3)	-0.3	-0.5036	-0.1151	0.3885
(2, 1)	-1	-1.4553	-0.7351	0.7203
(2, 3)	0	-0.2193	0.1764	0.3957
(3, 1)	-1	-1.2891	-0.6738	0.6153
(3, 2)	-1	-1.2417	-0.7425	0.4992

## 8. CONCLUDING REMARKS

In this paper, we derive a topology detection procedure that is based on a decision rule involving uncertainty intervals. We have seen that the decision rule can become unreliable if the length  $L_{ik}$  of the intervals is large. More precisely, the decision rule (19) is, when we observe that 0 lies in  $\mathcal{I}_{ik}$ , not really in state of distinguishing between elements  $Q_{0,ik}$  that are really zero and elements  $Q_{0,ik}$  that have a nonzero value such that  $|Q_{0,ik}| < L_{ik}$ . If we

assume that a nonzero  $Q_{0,ik}$  cannot be smaller than a given small threshold  $\varepsilon$ , the decision rule (19) will always lead to the right decision if each interval  $\mathcal{I}_{ik}$  containing zero has a length  $L_{ik} < \varepsilon$ . In future work, we will design the experimental conditions of the identification experiment of Section 4 to achieve this objective.

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