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# Spectral sequences of a Morse shelling

Jean-Yves Welschinger

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## Abstract

We recently introduced a notion of tilings of the geometric realization of a finite simplicial complex and related those tilings to the discrete Morse theory of R. Forman, especially when they have the property to be shellable, a property shared by the classical shellable complexes. We now observe that every such tiling supports a quiver which is acyclic precisely when the tiling is shellable and then that every shelling induces two spectral sequences which converge to the (co)homology of the complex. Their first pages are free modules over the critical tiles of the tiling.

Keywords : spectral sequence, simplicial complex, discrete Morse theory, shellable complex, tilings.

MATHEMATICS SUBJECT CLASSIFICATION 2020: 55T99, 57Q70, 55U10, 52C22.

## 1 Introduction

We introduced in [6, 7] a notion of tilings of the geometric realization of a finite simplicial complex  $K$ . It is a partition by tiles, where a tile is a maximal simplex of  $K$  deprived of several of its codimension one faces together with possibly a unique face of higher codimension, see §2. When the tiles can be totally ordered in a way defining a filtration of  $K$  by subcomplexes, the tiling is said to be shellable, or rather Morse shellable, since this notion extends the classical notion of shellability [1, 9] and is related to discrete Morse theory [3, 7]. In particular, the boundary of any convex simplicial polytope is shellable by [1], but every closed triangulated surface is Morse shellable as well [7] and any finite product of closed manifolds of dimensions less than four carries Morse shellable triangulations [8]. Not all tilings are shellable though, for any product of a sphere and a torus of positive dimension carries Morse tiled triangulations which cannot be shelled, see Theorem 1.1 of [8].

We now observe that every Morse tiling supports a quiver which encodes the shellability of the tiling, see §3.2 and Remark 4.4.

**Theorem 1.1.** *A Morse tiling is shellable if and only if its quiver does not contain any oriented cycle.*

We then observe that every such shelling recovers the homology and cohomology of the complex, via two spectral sequences, see §4.

**Theorem 1.2.** *Any Morse shelling on a finite simplicial complex induces two spectral sequences which converge to its homology and cohomology respectively and whose first pages are free graded modules over the critical tiles.*

A critical Morse tile is a simplex which has been deprived of several facets together with its face of maximal codimension, see §2. This terminology originates from its relation to the discrete Morse theory of Robin Forman [3], for any Morse shelling encodes a class of compatible discrete Morse functions whose critical points are in one-to-one correspondance with the critical tiles of the shelling, preserving the index, see [7]. As in the case of discrete Morse complexes, the chain and cochain complexes appearing in the spectral sequences given by Theorem 1.2 have much lower ranks than the simplicial ones, see Remark 4.4.

Finally, the quiver of a Morse tiled simplicial complex  $K$  makes it possible to weaken the shellability condition into some partial shellability up to the order  $q \geq 0$ , see Definition 3.14. The latter implies Morse shellability of the  $q$ -skeleton of  $K$ , see Theorem 3.15, and provides spectral sequences which converge to the (co)homology of  $K$  in degrees less than  $q$ , see Theorem 4.1.

We compute in §2 the relative (co)homology of Morse tiles, showing that it does not vanish only for critical tiles. We introduce in §3 the quivers of Morse tilings and prove Theorem 1.1. We also define the notion of partial shellability and prove Theorem 3.15. We finally prove Theorem 1.2 in §4.

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## 2 Relative homology of a Morse tile

Let  $n$  be a non-negative integer. We recall that an  $n$ -simplex is the convex hull of  $n + 1$  points affinely independent in some real affine space or rather, abstractly, just a set of cardinality  $n + 1$  whose elements are vertices, see [5, 4]. A face of a simplex is the convex hull of a subset of its vertices and its dimension is the dimension of the affine space it spans. An open simplex (resp. face) is the relative interior of a simplex (resp. face) in its supporting affine space. The following definition has been given in [7].

**Definition 2.1** (Definition 2.4 of [7, 8]). *A Morse tile  $T$  of dimension  $n$  and order  $k \in \{0, \dots, n + 1\}$  is an  $n$ -simplex  $\bar{T}$  deprived of  $k$  of its codimension one faces together with, if  $k \geq 1$ , one possibly empty face  $\mu$  of higher codimension. It is critical of index  $k$  iff  $\dim \mu = k - 1$ , while a closed simplex is critical of vanishing index. The simplex  $\bar{T}$  is called the underlying simplex while  $\mu$  is called its Morse face.*

A tile with empty Morse face is said to be *basic* and a basic tile of order  $k > 0$  contains a unique open face of dimension  $k - 1$  and none of lower dimension. Critical tiles are thus the ones for which this peculiar face has been removed, see [7]. Note that critical tiles of maximal index are open simplices. The tiles which are not critical are said to be *regular*.

Recall that a finite simplicial complex is a finite collection of simplices which contains all faces of its elements and such that the intersection of any two of them is a common face, possibly empty. Such a simplicial complex  $K$  inherits a topology whose homology and cohomology can be computed out of the simplicial chain and cochain complexes  $C_*(K; \mathbb{Z}) = \bigoplus_{\sigma \in K} \mathbb{Z}_\sigma$  and  $C^*(K; \mathbb{Z}) = \bigoplus_{\sigma \in K} \text{hom}(\mathbb{Z}_\sigma, \mathbb{Z})$ . These complexes are graded by the dimension of the simplices and for every  $\sigma \in K$ ,  $\mathbb{Z}_\sigma$  denotes the infinite cyclic group whose generators are the two orientations on  $\sigma$ , that is the kernel of the augmentation morphism  $a\sigma + b\bar{\sigma} \in \mathbb{Z}\sigma \oplus \mathbb{Z}\bar{\sigma} \mapsto a + b \in \mathbb{Z}$ , see [5].

**Definition 2.2.** *The relative homology (resp. cohomology) of a Morse tile  $T$  with coefficients in the Abelian group  $G$  is the homology  $H_*(\bar{T}, \bar{T} \setminus T; G)$  (resp.  $H^*(\bar{T}, \bar{T} \setminus T; G)$ ) of the chain complex  $C_*(\bar{T}; \mathbb{Z}) \otimes G / C_*(\bar{T} \setminus T; \mathbb{Z}) \otimes G$  (resp.  $\ker(\text{hom}(C^*(\bar{T}; \mathbb{Z}), G) \rightarrow \text{hom}(C^*(\bar{T} \setminus T; \mathbb{Z}), G))$ ).*

**Proposition 2.3.** *The relative homology and cohomology of regular Morse tiles vanish while they are isomorphic to the coefficient group  $G$  and concentrated in degree  $k$  in the case of critical tiles of index  $k$ .*

*Proof.* Let  $T$  be a Morse tile of dimension  $n$  and order  $k$ , with underlying simplex  $\bar{T}$  and Morse face  $\mu$ . Let  $\sigma$  be the union of the  $k$  facets that have been removed from  $\bar{T}$ , so that  $T = \bar{T} \setminus (\sigma \cup \mu)$ . Let  $\theta$  be the convex hull of the  $k$  vertices that are not contained in those facets, so that  $\theta \subset \mu$ . The tile  $T$  is then regular iff  $\theta \neq \mu$ .

If  $T$  is regular, then  $\mu$  contains a vertex  $v$  that does not belong to  $\theta$ , so that  $\sigma \cup \mu$  is a cone with apex  $v$ . The inclusions  $v \rightarrow \sigma \cup \mu \xrightarrow{i} \bar{T}$  then induce isomorphisms in homology and cohomology, these being concentrated in degree zero, see [5]. The short exact sequences

$$0 \rightarrow C_*(\sigma \cup \mu; G) \xrightarrow{i_*} C_*(\bar{T}; G) \rightarrow C_*(\bar{T}, \sigma \cup \mu; G) \rightarrow 0 \quad (1)$$

and

$$0 \rightarrow C^*(\bar{T}, \sigma \cup \mu; G) \rightarrow C^*(\bar{T}; G) \xrightarrow{i^*} C^*(\sigma \cup \mu; G) \rightarrow 0 \quad (2)$$

induce long exact sequences in homology from which we deduce that  $H_*(\bar{T}, \sigma \cup \mu; G)$  and  $H^*(\bar{T}, \sigma \cup \mu; G)$  vanish.

If  $T$  is critical, the result is well known for  $k \in \{0, n\}$ . When  $k = 0$ , the relative (co)homology of  $T$  is the homology of a simplex while when  $k = n$ , it is the (co)homology of a simplex relative to its boundary. Otherwise, we deduce by excision that the relative (co)homology of the pair  $(\sigma \cup \mu, \sigma)$  is isomorphic to the (co)homology of  $\mu$  relative to its boundary, so that it is likewise isomorphic to the coefficient group  $G$  and concentrated in degree  $k - 1$ . From the long exact sequence of this pair  $(\sigma \cup \mu, \sigma)$ , we deduce that the reduced homology and cohomology groups  $\tilde{H}_*(\sigma \cup \mu; G)$  and  $\tilde{H}^*(\sigma \cup \mu; G)$  get isomorphic

to  $H_*(\sigma \cup \mu, \sigma; G)$  and  $H^*(\sigma \cup \mu, \sigma; G)$ , so that they are isomorphic to  $G$  and concentrated in degree  $k - 1$  as well. The result then again follows from (1) and (2), for the reduced homology and cohomology of  $\bar{T}$  vanish.  $\square$

### 3 Quiver and shellability of a tiling

#### 3.1 Acyclic quivers

**Definition 3.1.** A quiver is a quadruple  $Q = (Q_V, Q_A, s, t)$ , where  $Q_V, Q_A$  are finite sets whose elements are vertices and arrows, and where  $s, t : Q_A \rightarrow Q_V$  are maps called source and target.

A quiver is thus a finite directed graph which may contain several arrows between two vertices or even a loop arrow from a vertex to itself.

**Example 3.2.** Let  $(V, \leq)$  be a finite partially ordered set. It supports a quiver having  $V$  as set of vertices and containing an arrow for each pair  $(i, j) \in V^2$  such that  $i \leq j$ , where  $j$  is the source and  $i$  the target.

**Definition 3.3.** Let  $Q = (Q_V, Q_A, s, t)$  be a quiver. A path of  $Q$  is a  $p$ -tuple of arrows  $\pi = (a_1, \dots, a_p) \in Q_A^p$  such that for every  $j \in \{1, \dots, p - 1\}$ ,  $t(a_j) = s(a_{j+1})$ . Its source (resp. target)  $s(\pi)$  (resp.  $t(\pi)$ ) is  $s(a_1)$  (resp.  $t(a_p)$ ) and its length is  $p$ .

Every vertex of a quiver also defines a path of length zero. An oriented cycle is then a path of positive length with same source and target. A quiver is said to be *acyclic* iff it contains no oriented cycle.

**Lemma 3.4.** A quiver  $Q$  without loop is acyclic iff the relation " $i \leq j$  iff  $Q$  contains a path from  $j$  to  $i$ " defines a partial order on its vertices.

*Proof.* The relation  $\leq$  is reflexive thanks to the existence of paths of length zero and transitive by concatenation of paths. Now, if  $i, j$  are vertices such that  $i \leq j$  and  $j \leq i$ , we get by concatenation an oriented cycle containing  $i$  and  $j$ . This forces  $i = j$  under the hypothesis that  $Q$  is acyclic. Conversely, since  $Q$  contains no loop by hypothesis, any oriented cycle contains at least two vertices  $i \neq j$ . The latter satisfy  $i \leq j$  and  $j \leq i$ , which contradicts the antisymmetry of the order.  $\square$

**Definition 3.5.** A grading of a quiver  $Q = (Q_V, Q_A, s, t)$  is an injection  $\text{gr} : Q_V \rightarrow \mathbb{Z}$  such that for every path  $\pi$  of positive length,  $\text{gr} \circ s(\pi) > \text{gr} \circ t(\pi)$ . An arrow  $a \in Q_A$  gets then graded by the difference  $\text{gr} \circ s(a) - \text{gr} \circ t(a)$ .

**Lemma 3.6.** A quiver is gradable iff it is acyclic.

*Proof.* Let  $Q = (Q_V, Q_A, s, t)$  be a quiver. If it is gradable with grading  $\text{gr} : Q_V \rightarrow \mathbb{Z}$ , then the relation " $i \leq j$  iff  $\text{gr}(i) \leq \text{gr}(j)$ " defines a total order on  $Q_V$  which strictly decreases along paths of positive length, so that  $Q$  cannot contain any oriented cycle. Conversely, if  $Q$  is acyclic, then the relation " $i \leq j$  iff  $Q$  contains a path from  $j$  to  $i$ " defines a partial

order on  $Q_V$  by Lemma 3.4. We then get a grading on  $Q$  by sending any minimal element  $i_0$  of  $Q_V$  to zero and then by induction, for every  $j \geq 0$ , by sending any minimal element  $i_{j+1}$  of  $Q_V \setminus \{i_0, \dots, i_j\}$  to  $j + 1 \in \mathbb{Z}$ .  $\square$

**Example 3.7.** *Let  $V$  be a finite subset of  $\mathbb{Z}$ . Then, the quiver  $Q = (Q_V, Q_A, s, t)$  given by Example 3.2, where  $Q_A$  contains exactly one arrow for each pair  $(i, j) \in Q_V^2$  such that  $i \leq j$ , is graded by the inclusion  $V \hookrightarrow \mathbb{Z}$ .*

## 3.2 Shellability of a Morse tiling

The following definition has been given in [7], see also [6, 8].

**Definition 3.8** (Definition 2.8 of [7] and 2.5 of [8]). *A subset  $S$  of the geometric realization of a finite simplicial complex  $K$  is Morse tileable iff it admits a partition by Morse tiles such that for every  $j \geq 0$ , the union of tiles of dimension greater than  $j$  is closed in  $S$ . Such a partition is called a Morse tiling and the collection of simplices underlying these tiles, together with their faces, is the underlying simplicial complex  $\overline{S}$  of  $S$ .*

A special class of Morse tilings consists of the following shellable ones and our purpose in this section is to characterize when a tiling is shellable, proving Theorems 1.1.

**Definition 3.9** (Definition 2.14 of [7] and 2.10 of [8]). *A Morse tiled set  $S$  is shellable iff it admits a filtration  $\emptyset = S_0 \subset S_1 \subset \dots \subset S_N = S$  by closed subsets such that for every  $j \in \{1, \dots, N\}$ ,  $S_j \setminus S_{j-1}$  consists of a single tile.*

When the tiling uses only basic tiles, this notion of shelling recovers the classical one [1, 9], see Theorem 2.15 and Remark 2.16 of [7].

**Example 3.10.** *1) The boundary of every convex simplicial polytope is shellable by [1], so that it is Morse shellable, the shelling using only basic tiles.*

*2) By Theorem 1.3 of [7], every closed triangulated surface is Morse shellable.*

*3) By Theorem 1.2 of [8], any finite product of closed manifolds of dimensions  $\leq 3$  carries Morse shellable triangulations.*

More examples of Morse shellings can be found in [8]. We now observe that a Morse tiling supports a quiver which encodes the obstruction to shell it.

**Definition 3.11.** *The quiver of a Morse tiled set  $S$  has the tiles of  $S$  as vertices and contains an arrow  $T \rightarrow T'$  iff  $\overline{T} \cap T' \neq \emptyset$ . The arrows are labelled by the dimension of the face  $\overline{T} \cap T'$  and the vertices by the order of the corresponding tiles.*

In particular, the label of the target of an arrow cannot exceed the label of the arrow itself by more than one, see §2.

**Example 3.12.** *1) If  $S = T_0 \sqcup \dots \sqcup T_{n+1}$  is the shelling of  $\partial\Delta_{n+1}$  using one  $n$ -dimensional basic tile of each order, where  $\Delta_{n+1}$  denotes the standard  $(n + 1)$ -simplex, then its quiver is the graph with vertices  $T_0, \dots, T_{n+1}$  and arrows  $T_j \rightarrow T_i$  for every  $0 \leq i \leq j \leq n + 1$ , see Corollary 4.2 of [6].*

2) If  $S = T_1 \sqcup T_1 \sqcup T_1$  is the non-shellable tiling of  $\partial\Delta_2$  using three basic tiles of order and dimension one, then its quiver is the oriented boundary of a triangle, see Example 3.4 of [8].

We now prove Theorem 1.1.

*Proof of Theorem 1.1.* Let us first assume that a Morse tiled set  $S$  is shelled and denote by  $T_1, \dots, T_N$  the shelling order on the tiles of  $S$ . Let  $a : T_j \rightarrow T_i$  be an arrow of its quiver, so that  $\overline{T}_j \cap T_i \neq \emptyset$ . Since by definition the union  $\cup_{k \leq j} T_k$  is closed in  $S$ , this forces  $i \leq j$  so that the shelling order of  $S$  induces a grading of its quiver by Definition 3.5. We deduce from Lemma 3.6 that the latter is acyclic.

Conversely, if the quiver of  $S$  is acyclic, it is gradable by Lemma 3.6 and any grading provides a total order on the tiles of  $S$ . We may then label them  $T_1, \dots, T_N$  in increasing order. Then, for every  $k \in \{1, \dots, N\}$  and every  $1 \leq i \leq k < j \leq N$ ,  $\overline{T}_i \cap T_j = \emptyset$ , for otherwise the quiver of  $S$  would contain an arrow  $T_i \rightarrow T_j$  by Definition 3.11 which is impossible by Definition 3.5. We deduce that  $S_k = \cup_{i \leq k} T_i$  is closed in  $S$ , so that the filtration  $(S_k)_{k \in \{1, \dots, N\}}$  is a Morse shelling of  $S$ .  $\square$

**Remark 3.13.** 1) *The acyclicity condition in Theorem 1.1 may be compared to the criterion given in Theorem 9.3 of [3], up to which a discrete vector field on a simplicial complex is the gradient vector field of a discrete Morse function iff it contains no non-stationary closed path. Indeed, by Theorem 3.12 of [7], a Morse tiling on a finite simplicial complex encodes a class of compatible discrete vector fields and in the case of a shellable one, these are gradient vector fields of discrete Morse functions whose critical points are in one-to-one correspondance with the critical tiles, preserving the indices, see Theorem 1.2 of [7].*

2) *By Theorem 1.2 of [8], every product of a sphere and a torus of positive dimension carries Morse tiled triangulations using only basic tiles. These tilings cannot be shelled, so that their quivers all contain oriented cycles, as in the second part of Example 3.12.*

We also deduce from Theorem 1.1 criteria of partial shellability of a Morse tiling. In particular.

**Definition 3.14.** *A Morse tiled set is shellable up to the order  $q \geq 0$  iff its quiver does not contain any oriented cycle whose arrows have labels  $\leq q$ .*

**Theorem 3.15.** *The  $q$ -skeleton  $S^{(q)}$  of a Morse tiled set  $S$  which is shellable up to the order  $q \geq 0$  is itself Morse shellable and carries a filtration  $\emptyset = S_0 \subset S_1 \cap S^{(q)} \subset \dots \subset S_N \cap S^{(q)} = S^{(q)}$  by closed subset such that for every  $j \in \{1, \dots, N\}$ ,  $S_j \setminus S_{j-1}$  consists of a single tile of order  $\leq q + 1$  of  $S$ .*

The  $q$ -skeleton of a Morse tiled set  $S$  is the intersection with  $S$  of the  $q$ -skeleton of its underlying simplicial complex  $\overline{S}$ . Any filtration given by theorem 3.15 is called a *partial shelling*.

*Proof.* Let  $S$  be a Morse tiled set which is shellable up to the order  $q \geq 0$  and let  $\tilde{S}$  be the union of tiles of order  $\leq q+1$  of  $S$ . Then,  $\tilde{S}$  contains the  $q$ -skeleton of  $S$  since a tile of order greater than  $q+1$  does not contain any open face of dimension  $\leq q$ , see §2. By Definition 3.14, the quiver  $Q$  whose vertices are the tiles of  $\tilde{S}$  and which contains an arrow  $T \rightarrow T'$  iff  $\bar{T} \cap T'$  is non-empty of dimension  $\leq q$ , is acyclic. By Lemma 3.6, it is then gradable and any grading provides a total order on the tiles of  $\tilde{S}$ , so that we may label them  $T_1, \dots, T_N$  in increasing order. Then, for every  $1 \leq i \leq k < j \leq N$ ,  $\bar{T}_i \cap T_j \cap S^{(q)} = \emptyset$ , for otherwise  $Q$  would contain an arrow  $T_i \rightarrow T_j$  which is impossible by Definition 3.5. We deduce that  $\tilde{S}_k^{(q)} = (\cup_{i \leq k} T_i) \cap S^{(q)}$  is closed in  $S^{(q)}$ , so that it provides the desired filtration by closed subsets of  $S^{(q)}$ . Now, by Theorem 2.18 of [7], the  $q$ -skeleton of every Morse tile is Morse shellable. Choosing such a Morse shelling on each  $T_i \cap S^{(q)}$ ,  $i \in \{1, \dots, N\}$ , we get the result by concatenation of these shelling orders.  $\square$

## 4 Spectral sequences

We finally observe that any Morse shelling on a finite simplicial complex provides a way to compute its (co)homology via two spectral sequences.

**Theorem 4.1.** *Let a Morse tiled simplicial complex be shellable up to the order  $q \geq 0$ . Then, any such partial shelling induces a spectral sequence which converges to the homology (resp. cohomology) of its  $q$ -skeleton and whose first page in gradings less than  $q$  is the relative homology (resp. cohomology) of its critical tiles.*

By Proposition 2.3, only the critical tiles of indices less than  $q$  contribute to the spectral sequences given by Theorem 4.1 in gradings less than  $q$ .

**Remark 4.2.** *The inclusion  $i$  of the  $q$ -skeleton  $K^{(q)}$  into its ambient simplicial complex  $K$  induces an homomorphism  $i_* : H_*(K^{(q)}; G) \rightarrow H_*(K; G)$  (resp.  $i^* : H^*(K; G) \rightarrow H^*(K^{(q)}; G)$ ) in homology (resp. cohomology) which is surjective (resp. injective) in degree  $q$  and bijective in degrees less than  $q$ , see [5], so that the spectral sequences given by Theorem 4.1 converge to the (co)homology of the simplicial complex in degrees less than  $q$ .*

*Proof.* Let  $K$  be a Morse tiled simplicial complex which is shellable up to the order  $q \geq 0$  and let  $G$  be any group of coefficients. By Theorem 3.15, the  $q$ -skeleton of  $K$  inherits from any partial shelling a filtration  $\emptyset = K_0 \subset K_1 \subset \dots \subset K_N = K^{(q)}$  by subcomplexes such that for every  $p \in \{1, \dots, N\}$ ,  $K_p \setminus K_{p-1} = T_p \cap K^{(q)}$  where  $T_p$  is a tile of order  $\leq q+1$  of  $K$ . This filtration induces a filtration  $0 \subset C_*(K_1, G) \subset \dots \subset C_*(K_N, G) = C_*(K^{(q)}, G)$  of simplicial chain complexes and thus a spectral sequence which converges to the homology of  $K^{(q)}$ , see chapter XV of [2] for instance. The zero-th page of this spectral sequence is by definition  $E_{p, *-p}^0 = C_*(K_p, K_{p-1})$ ,  $p \in \{1, \dots, N\}$ , where we omit from now on to mention the group of coefficients  $G$ . Its first page is  $E_{p, *-p}^1 = H_*(K_p, K_{p-1})$ ,  $p \in \{1, \dots, N\}$  and its limit term is the homology of  $K^{(q)}$  under the form  $E_{p, *-p}^\infty = H_*(K^{(q)})_p / H_*(K^{(q)})_{p-1}$ ,  $p \in \{1, \dots, N\}$ , where  $H_*(K^{(q)})_p$  denotes the image of the inclusion homomorphism  $H_*(K_p) \rightarrow H_*(K^{(q)})$ . These images indeed provide the induced filtration



$0 \subset H_*(K^{(q)})_1 \subset \cdots \subset H_*(K^{(q)})_N = H_*(K^{(q)})$  of the total homology of the  $q$ -skeleton. Moreover, the filtration being of finite length  $N$ , the spectral sequence degenerates at the  $N$ -th page, so that  $E^\infty = E^N$ . Now, for every  $p \in \{1, \dots, N\}$ , we deduce by excision an isomorphism  $H_*(K_p, K_{p-1}) \cong H_*(\overline{T}_p \cap K^{(q)}, (\overline{T}_p \setminus T_p) \cap K^{(q)})$ , see [5], and then by composition with the inclusion homomorphism, an isomorphism  $H_*(K_p, K_{p-1}) \cong H_*(\overline{T}_p, \overline{T}_p \setminus T_p)$  as long as  $* < q$ , see Remark 4.2. The homology part of the statement then follows from Proposition 2.3.

The proof of the cohomology part is similar and obtained from the filtration  $0 \subset C^*(K_N, K_{N-1}, G) \subset \cdots \subset C^*(K_N, K_0, G) = C^*(K^{(q)}, G)$  of cochain complexes. It induces a spectral sequence which converges to the cohomology of  $K^{(q)}$  and whose zero-th page is by definition  $E_0^{p,*-p} = C^*(K_N, K_p)/C^*(K_N, K_{p+1}) = C^*(K_{p+1}, K_p)$ ,  $p \in \{0, \dots, N-1\}$ , where the last isomorphism is given by restriction of the cochains to  $K_{p+1}$  and provides the short exact sequence of the triad  $(K_N, K_{p+1}, K_p)$ , and where we again omit to mention the coefficient group  $G$  from now on. Its first page is thus  $E_1^{p,*-p} = H^*(K_{p+1}, K_p)$ ,  $p \in \{0, \dots, N-1\}$  and its limit term is the cohomology of  $K^{(q)}$  under the form  $E_\infty^{p,*-p} = H^*(K^{(q)})_p/H^*(K^{(q)})_{p+1}$ ,  $p \in \{0, \dots, N-1\}$ , where  $H^*(K^{(q)})_p$  denotes the image of the inclusion homomorphism  $H^*(K^{(q)}, K_p) \rightarrow H^*(K^{(q)})$ . Now as before, we deduce by excision and Remark 4.2, for every  $p \in \{0, \dots, N-1\}$ , an isomorphism  $H^*(K_{p+1}, K_p) \cong H^*(\overline{T}_{p+1}, \overline{T}_{p+1} \setminus T_{p+1})$  as long as  $* < q$ , so that the result follows from Proposition 2.3.  $\square$

We now deduce.

*Proof of Theorem 1.2.* Theorem 1.2 follows from Theorem 4.1 combined with Proposition 2.3.  $\square$

**Example 4.3.** 1) Any shelling of the boundary of a convex simplicial polytope given by [1] provides a Morse shelling which uses only basic tiles among which a single closed simplex and a single open one. The spectral sequences given by Theorem 1.2 degenerate at the first page and compute the (co)homology of a sphere.

2) The Morse shellings of any closed connected triangulated surface given by Theorem 1.3 of [7] can also be chosen to use a single closed simplex and a single open one by construction. Then, the spectral sequences given by Theorem 1.2 again degenerate at the first page and compute the (co)homology of the surface as the number of critical tiles of each index it uses.

3) In the case of the Morse shelled octahedron depicted in Figure 1 -taken out from [8]-, where the numbers appearing in Figure 1 indicate the shelling order, the spectral sequences given by Theorem 1.2 degenerate at the second page. Indeed, the (co)homology classes of the pair of critical tiles with shelling orders 6 and 7, which have consecutive indices one and two, get killed already at this second page.

**Remark 4.4.** 1) It follows from Theorem 4.1 that the  $k$ -th Betti number of a Morse tiled simplicial complex which is shellable up to the order  $q > k$  is bounded from above by the number of critical tiles of index  $k$  of the tiling. In the case of a Morse shellable complex, this recovers Corollary 1.5 of [7], which has been deduced from discrete Morse theory.

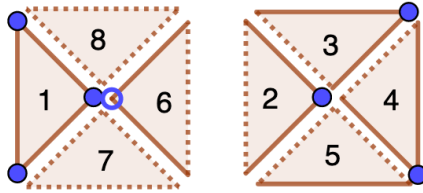


Figure 1: A Morse shelling on the octahedron.

2) Theorem 1.2 provides a way to compute the (co)homology of a Morse shellable simplicial complex using much smaller (co)chain complexes than the simplicial ones, as the discrete Morse complexes introduced by R. Forman would do, see [3]. For instance, performing a large number of barycentric subdivisions on such a simplicial complex increases by a lot the dimensions of its simplicial (co)chain complexes, whereas by Theorem 1.1 of [7], these inherit Morse shellings using a constant number of critical tiles.

3) In fact, the spectral sequences given by Theorem 1.2 can be computed using discrete Morse theory in the following way. By Theorem 1.2 of [7], any Morse shelling of a finite simplicial complex  $K$  encodes a class of compatible discrete Morse functions whose critical points are in one-to-one correspondance with the critical tiles of the shelling, preserving the index. Moreover, any discrete Morse function  $f$  induces a cellular decomposition of  $K$  using one cell for each critical point of  $f$ , see §7 of [3]. Each page of the spectral sequence can then be computed using this decomposition for a compatible discrete Morse function. Indeed, each cell of the decomposition provides a (co)cycle representative of the relative (co)homology of the corresponding critical tile and thus a representative of the corresponding (co)homology class in the first page. It remains a (co)cycle representative of this (co)homology class as long as the differential of the corresponding page maps it to a shelling degree where the boundary of the cell contains a simplex, which must then be a critical point of degree one less of the Morse function. The latter sits inside a critical tile of the shelling. Then, whether or not this homology class survives to higher pages depends on whether or not this boundary component can be annihilated by another cell, which is encoded by the discrete Morse complex.

It is however not needed to choose a compatible discrete Morse function to compute the spectral sequences, as the case of the octahedron given in Example 4.3 shows. They can be computed directly from the shelling.

4) By Theorem 8.10 of [3], the differentials of the Morse complex of a discrete Morse function can also be computed as a finite sum of gradient paths between critical points of consecutive indices. If the Morse function is chosen to be compatible with a Morse shelling, then any such gradient path provides an oriented path in the quiver of the shelling, whose source and target are critical tiles of consecutive indices.

5) The term quiver has been chosen in Definition 3.11 rather than oriented graph, since representations of the quiver of a Morse tiling on a simplicial complex seem to correspond to some sheaves on the complex, a relation which might be of interest to investigate.

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