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MOMENTS OF q -JACOBI POLYNOMIALS AND q -ZETA VALUES

FRÉDÉRIC CHAPOTON, CHRISTIAN KRATTENTHALER[†], AND JIANG ZENG

ABSTRACT. We explore some connections between moments of rescaled little q -Jacobi polynomials, q -analogues of values at negative integers for some Dirichlet series, and the q -Eulerian polynomials of wreath products of symmetric groups.

INTRODUCTION

This article is about a connection between three kinds of objects, namely

- (A) q -analogues of Dirichlet series and their values at negative integers,
- (B) basic hypergeometric polynomials and their sequences of moments,
- (C) weighted enumeration of elements in coloured symmetric groups.

Let us give more details on these three points in order.

The point (A) is about a q -analogue of the Dirichlet series

$$L(s, c, r) = \sum_{\substack{m \geq 1 \\ m \equiv c \pmod{r}}} \frac{1}{m^s}, \quad (0.1)$$

where c, r are fixed integers. This is the Riemann zeta function when $(c, r) = (1, 1)$. For general c and r , the summands do not form a multiplicative sequence, so there is no Euler product. One defines as in [3] a q -analogue of this Dirichlet series as an operator

$$L_q(s, c, r) = \sum_{\substack{m \geq 1 \\ m \equiv c \pmod{r}}} \frac{1}{[m]_q^s} F_m, \quad (0.2)$$

where $[m]_q = (q^m - 1)/(q - 1)$ is the usual q -integer and F_m is the formal Frobenius operator, acting on formal power series in z with no constant term and coefficients in $\mathbb{Q}(q)$, defined by

$$F_m(f)(q, z) = f(q^m, z^m).$$

Whenever the Dirichlet series $L(s, c, r)$ factorizes as an Euler product, then so does the operator $L_q(s, c, r)$ as a product of commuting operators.

One then introduces some q -analogues of the values of $L(s, c, r)$ at non-positive integers, namely $L_q(-n, c, r)(z)$ for $n \geq 0$. As images of the formal power series $f(z) = z$, these are formal power series in the variable z with coefficients in $\mathbb{Q}(q)$. As we will see, these are in fact rational functions in q and z .

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The point (B) is about the little q -Jacobi polynomials, a system of orthogonal polynomials in one variable. This is one of the families in the Askey–Wilson scheme of basic hypergeometric orthogonal polynomials (cf. [7]). The little q -Jacobi polynomials, orthogonal with respect to the variable x , depend on the variable q and two further parameters. For each choice of integers (c, r) , by an appropriate choice of these parameters and some affine change in the variable x , one obtains a system of orthogonal polynomials involving the variables q and z . Their sequence of moments, which are evaluations of the associated linear functional at the monomials x^n , are therefore rational functions in q and z .

The point (C) is about the complex reflection groups $G(r, n)$ defined as the wreath product of a symmetric group S_n by a cyclic group \mathbb{Z}_r . The elements of these groups can be seen as coloured permutation matrices, where non-zero entries contain a root of unity of order dividing r . By using two combinatorial statistics on these elements, one can refine the number $r^n n!$ of elements of $G(r, n)$ into a polynomial in two variables q and z , with positive integer coefficients. In this context, the parameter c is absent.

The aim of this article is to show that (A), (B) and (C) all give essentially the same rational function in q and z . More precisely, the rational functions from (A) and (B) are essentially the quotients of the polynomial from (C) by simple denominators. The part (C) is involved only when the parameter c equals 1.

The relationship between (C) and (A) is merely a reformulation of the results by Biagioli and the third author in [1]. The relationship between (A) and (B) is a (q, z) -analogue of well-known results about Bernoulli numbers and Euler numbers, for which we refer to [6].

1. PRELIMINARIES

1.1. Orthogonal polynomials. In this subsection we recall some fundamental results of the theory of orthogonal polynomials [4, 10]. Let \mathbb{K} be a field.

Definition 1.1. *Let $\varphi : \mathbb{K}[x] \rightarrow \mathbb{K}$ be a linear functional. A sequence of polynomials $\{p_n(x)\}_{n \geq 0}$ in $\mathbb{K}[x]$ is said to be orthogonal with respect to the linear functional φ if:*

- (i) $p_n(x)$ is of degree n , for $n = 0, 1, \dots$;
- (ii) $\varphi(p_n(x)p_{n'}(x)) = K_n \delta_{n,n'}$, $K_n \neq 0$, for $n = 0, 1, \dots$.

The sequence $\{\mu_n\}_{n \geq 0}$ with $\mu_n = \varphi(x^n)$ for $n \geq 0$ is called the moment sequence associated with φ .

Sometimes the polynomials $\{p_n(x)\}$ are also said to be orthogonal with respect to the sequence of moments $\{\mu_n\}_{n \geq 0}$.

Let us write OPS as a shorthand for *orthogonal polynomial system*.

Theorem 1.1 (FAVARD'S THEOREM). *A sequence of polynomials $\{p_n(x)\}_{n \geq 0}$ in $\mathbb{K}[x]$ is a monic OPS if and only if there is a sequence $\{b_n\}_{n \geq 0}$ and a non-zero sequence $\{\lambda_n\}_{n \geq 0}$ such that $p_0(x) = 1$, $p_1(x) = x - b_0$ and*

$$p_{n+1}(x) = (x - b_n)p_n(x) - \lambda_n p_{n-1}(x) \quad \text{for } n \geq 1. \quad (1.1)$$

Theorem 1.2. *Let the polynomials $(p_n(x))_{n \geq 0}$ satisfy (1.1). Then, if we fix $\mu_0 := \lambda_0 \neq 0$, the functional φ with respect to which this OPS is orthogonal is unique. Furthermore, for $Q_n(x) = \alpha^{-n} p_n(\alpha x + \beta)$, $\alpha \neq 0$, we have*

$$Q_{n+1}(x) = \left(x - \frac{b_n - \beta}{\alpha} \right) Q_n(x) - \frac{\lambda_n}{\alpha^2} Q_{n-1}(x), \quad \text{for } n \geq 1, \quad (1.2)$$

and, if $(p_n(x))_{n \geq 0}$ is the OPS with respect to the moments (μ_n) , then $(Q_n(x))$ is the OPS with respect to the moments ν_n given by

$$\nu_n = \varphi \left(\left(\frac{x - \beta}{\alpha} \right)^n \right) = \alpha^{-n} \sum_{j=0}^n \binom{n}{j} (-\beta)^{n-j} \mu_j, \quad \text{for } n \geq 0. \quad (1.3)$$

Theorem 1.3. *The generating function of the moments $\{\varphi(x^n)\}$ has the continued fraction expansion*

$$\sum_{n \geq 0} \varphi(x^n) t^n = \frac{\lambda_0}{1 - b_0 t - \frac{\lambda_1 t^2}{1 - b_1 t - \frac{\lambda_2 t^2}{\ddots}}}. \quad (1.4)$$

There is also an associated formula for Hankel determinants of the sequence of moments, see [10, 8, 9]

1.2. Wreath product of a symmetric group by a cyclic group. Let $r \geq 1$ and $n \geq 1$ be integers. Let S_n be the symmetric group on $\{1, \dots, n\}$. A permutation $\sigma \in S_n$ will be denoted by $\sigma = \sigma(1) \cdots \sigma(n)$. The wreath product $\mathbb{Z}_r \wr S_n$ of \mathbb{Z}_r by S_n is the set

$$G(r, n) := \{(c_1, \dots, c_n; \sigma) \mid c_i \in \{0, \dots, r-1\}, \sigma \in S_n\}. \quad (1.5)$$

Using a fixed primitive r -th root of unity ξ , one can see the elements in this set as square matrices, starting from the permutation matrix for σ and replacing the non-zero entry in column i by ξ^{c_i} .

This group is therefore also called the *group of r -coloured permutations*. We will represent its elements as

$$\gamma = [\gamma(1), \dots, \gamma(n)] = [\sigma(1)^{c_1}, \dots, \sigma(n)^{c_n}].$$

We denote by

$$\text{col}(\gamma) := \sum_{i=1}^n c_i,$$

the *colour weight* of any $\gamma \in G(r, n)$. For example, if $\gamma = [4^1, 3^0, 2^4, 1^2] \in G(5, 4)$ then $\text{col}(\gamma) = 7$.

We endow the set of possible values for the $\gamma(i)$ with the following total order:

$$n^{r-1} < \dots < n^1 < \dots < 1^{r-1} < \dots < 1^1 < 0 < 1^0 < \dots < n^0.$$

The 0 is inserted here to separate the “positive” values i^0 from the “negative” values i^c with $c \geq 1$. It will also be used in the statistics that we are going to define now.

The *descent set* of $\gamma \in G(r, n)$ is defined by

$$\text{Des}_G(\gamma) := \{i \in \{0, \dots, n-1\} \mid \gamma(i) > \gamma(i+1)\}, \quad (1.6)$$

where $\gamma(0) := 0$, and its cardinality is denoted by $\text{des}_G(\gamma)$.

The *major index* is defined to be the sum of descent positions:

$$\text{maj}(\gamma) = \sum_{i \in \text{Des}_G(\gamma)} i,$$

and the *flag-major index* is defined by

$$\text{fmaj}(\gamma) := r \cdot \text{maj}(\gamma) + \text{col}(\gamma).$$

For example, for $\gamma = [4^1, 3^0, 2^4, 1^2] \in G(5, 4)$ we have $\text{Des}_G(\gamma) = \{0, 2\}$, $\text{des}_G(\gamma) = 2$, $\text{maj}(\gamma) = 2$, and $\text{fmaj}(\gamma) = 17$.

Biagioli and the third author [1] defined the generating polynomials for $G(r, n)$ with respect to the bi-statistic $(\text{des}, \text{fmaj})$:

$$G_{r,n}(Z, q) = \sum_{\gamma \in G(r,n)} Z^{\text{des}_G(\gamma)} q^{\text{fmaj}(\gamma)}, \quad (1.7)$$

and they proved the following identity. We refer the reader to Subsection 1.3 for the meaning of the q -notations.

Theorem 1.2 (CARLITZ–MACMAHON IDENTITY FOR $G(r, n)$). *Let r and n be positive integers. Then*

$$\frac{G_{r,n}(Z, q)}{(Z; q^r)_{n+1}} = \sum_{k \geq 0} Z^k [rk + 1]_q^n. \quad (1.8)$$

The above formula gives a nice generalization of identities of Carlitz [2] for the symmetric group (corresponding to the case where $r = 1$), and of Chow and Gessel [5] for the hyperoctahedral group (corresponding to the case where $r = 2$).

1.3. Little q -Jacobi polynomials. We use the standard q -notations from [7], among which

$$[x]_q = \frac{1 - q^x}{1 - q},$$

the q -Pochhammer symbol

$$(a; q)_n = (1 - a)(1 - aq) \dots (1 - aq^{n-1}),$$

and the convenient shorthand

$$(a, b; q)_n = (a; q)_n (b; q)_n.$$

We furthermore need the q -binomial theorem [7, p. 16]

$${}_1\Phi_0(a; -; q, z) = \sum_{k=0}^{\infty} \frac{(a; q)_k}{(q; q)_k} z^k = \frac{(az; q)_{\infty}}{(z; q)_{\infty}} \quad (1.9)$$

and the q -Chu–Vandermonde formula [7, p. 17]

$${}_2\Phi_1(q^{-n}, b; c; q, q) = \sum_{k=0}^{\infty} \frac{(q^{-n}; q)_k (b; q)_k}{(c; q)_k (q; q)_k} q^k = \frac{(c/b; q)_n b^n}{(c; q)_n}. \quad (1.10)$$

The little q -Jacobi polynomials [7, p. 482] have the explicit representation

$$p_n(x; a, b \mid q) = {}_2\Phi_1\left(\begin{matrix} q^{-n}, abq^{n+1} \\ aq \end{matrix}; q, qx\right) = \sum_{k=0}^{\infty} \frac{(q^{-n}; q)_k (abq^{n+1}; q)_k}{(aq; q)_k (q; q)_k} (qx)^k, \quad (1.11)$$

and are orthogonal with respect to the inner product defined by

$$\int_0^1 f(x)g(x)d_q w(x) = \sum_{k=0}^{\infty} f(q^k)g(q^k)w(q^k),$$

where

$$w(x) = \frac{(aq, bq; q)_{\infty}}{(abq^2, q; q)_{\infty}} \cdot \frac{(qx; q)_{\infty}}{(bqx; q)_{\infty}} x^{\alpha+1}$$

with $a = q^{\alpha}$.

Let $p_n(x)$ be the monic little q -Jacobi polynomials, i.e.,

$$p_n(x; a, b \mid q) = \frac{(-1)^n q^{-\binom{n}{2}} (abq^{n+1}; q)_n}{(aq; q)_n} p_n(x).$$

Then the normalized recurrence relation [7, p. 483] reads

$$xp_n(x) = p_{n+1}(x) + (A_n + C_n)p_n(x) + A_{n-1}C_n p_{n-1}(x), \quad (1.12)$$

where

$$A_n = q^n \frac{(1 - aq^{n+1})(1 - abq^{n+1})}{(1 - abq^{2n+1})(1 - abq^{2n+2})},$$

$$C_n = aq^n \frac{(1 - q^n)(1 - bq^n)}{(1 - abq^{2n})(1 - abq^{2n+1})}.$$

By the q -binomial theorem (1.9), the n th moment is

$$\mu_n = \int_0^1 x^n d_q w(x) = \frac{(aq; q)_n}{(abq^2; q)_n}, \quad \text{for } n = 0, 1, 2, \dots \quad (1.13)$$

We can also verify (1.13) by using the explicit formula (1.11) and the q -Chu–Vandermonde formula (1.10): namely, for $n \geq 1$, we have

$$\int_0^1 p_n(x; a, b \mid q) d_q w(x) = 0. \quad (1.14)$$

We can now prove the connection between (B) and (A).

Theorem 1.4. For integers $r \geq 1$, the n th moment μ_n of the shifted little q -Jacobi polynomials $p_n(q^{-c}(1 + (q-1)x); Zq^{-r}, 1 \mid q^r)$ is

$$\mu_n = (1 - Z) \sum_{k \geq 0} ([rk + c]_q)^n Z^k. \quad (1.15)$$

For $c = 1$, we have

$$\mu_n = \frac{G_{r,n}(Z, q)}{(Zq^r; q^r)_n}. \quad (1.16)$$

Proof. By (1.3), the n th moment of $p_n(q^{-c}(1 + (q-1)x); a, b \mid q^r)$ is

$$\nu_n = q^{nc}(q-1)^{-n} \sum_{j=0}^n \binom{n}{j} (-q^c)^{j-n} \frac{(aq^r; q^r)_j}{(abq^{2r}; q^r)_j}.$$

Substituting a by Zq^{-r} and b by 1, we get

$$\begin{aligned} \nu_n &= q^{nc}(q-1)^{-n} \sum_{j=0}^n \binom{n}{j} (-q^c)^{j-n} \frac{1-Z}{1-Zq^{rj}} \\ &= (1-Z)(q-1)^{-n} \sum_{k \geq 0} Z^k \sum_{j=0}^n \binom{n}{j} (-1)^{n-j} q^{(rk+c)j} \\ &= (1-Z) \sum_{k \geq 0} ([rk + c]_q)^n Z^k. \end{aligned}$$

The last statement follows from (1.8). \square

Theorem 1.5. The generating function for the moments μ_n in (1.15) has the continued fraction expansion

$$\sum_{n \geq 0} \mu_n t^n = \frac{1}{1 - b_0 t - \frac{\lambda_1 t^2}{1 - b_1 t - \frac{\lambda_2 t^2}{\ddots}}}. \quad (1.17)$$

where the coefficients b_n and λ_n are given by

$$\lambda_n = \frac{Zq^{2r(n-1)+2c} [rn]_q^2 (1 - Zq^{r(n-1)})^2}{(1 - Zq^{2rn})(1 - Zq^{r(2n-1)})^2 (1 - Zq^{r(2n-2)})} \quad (1.18)$$

and

$$b_n = \frac{q^c}{q-1} \left(\frac{q^{rn}(1 - Zq^{rn})^2}{(1 - Zq^{2rn})(1 - Zq^{r(2n+1)})} + \frac{Zq^{r(n-1)}(1 - q^{rn})^2}{(1 - Zq^{r(2n-1)})(1 - Zq^{2rn})} - q^{-c} \right). \quad (1.19)$$

Proof. This follows by combining (1.12) and Theorems 1.4, 1.2 and 1.3 with $\alpha = (q-1)/q^c$ and $\beta = q^{-c}$. \square

2. ZETA OPERATORS AT NEGATIVE INTEGERS

We define the q -difference operator on the formal power series f in z by

$$\Delta_z(f) = \frac{f(qz) - f(z)}{q - 1}. \quad (2.1)$$

Note that $\Delta_z(z^m) = [m]_q z^m$. This implies that repeated application of the operator Δ_z creates the sequence of values at negative integers for the q -analogues of Dirichlet series. Indeed, for $n \geq 0$, we have

$$L_q(-n, c, r)(z) = \sum_{\substack{m \geq 1 \\ m \equiv c \pmod{r}}} [m]_q^n z^m. \quad (2.2)$$

and therefore

$$\Delta_z(L_q(-n, c, r)(z)) = L_q(-n - 1, c, r)(z). \quad (2.3)$$

Computing the initial value for $n = 0$, one finds

$$L_q(0, c, r)(z) = \frac{z^c}{1 - z^r}. \quad (2.4)$$

By induction using (2.3), the expression $L_q(-n, c, r)(z)$ is a rational function in q and z with denominator $(z^r, q^r)_{n+1}$.

The general relation between (A) and (B) is therefore, by comparison of (2.2) with (1.15), using (2.4), that

$$L_q(-n, c, r)(z)/L_q(0, c, r)(z) = \mu_n \Big|_{Z=z^r}. \quad (2.5)$$

For $c = 1$, comparison with (1.8) reveals the combinatorial expression

$$L_q(-n, c, r)(z) = z \frac{G_{r,n}(z^r, q)}{(z^r, q^r)_{n+1}}, \quad (2.6)$$

which makes the precise connection between (A) and (C).

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