Marching cubes method with connectivity

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Abstract

In this paper, we solve the topological problem of isosurfaces generated by the marching cubes method using the approach of combinatorial topology. For each marching cube, we examine the connectivity of polyhedral configuration in the sense of combinatorial topology. For the cubes where the connectivities are not considered, we modify the polyhedral configurations with the connectivity and construct polyhedral isosurfaces with the correct topologies.

1 Introduction

The marching cubes method proposed by Lorensen and Cline in 1987 [1] is a powerful technique for visualization of a 3-dimensional gray-scale digital image. It constructs polyhedral isosurfaces with triangle patches. Let us consider that voxels of a digital image are lattice points in a 3-dimensional space and that lattice points are called black points if their gray values are more than the isovalue and white points, otherwise. The locations of the vertices of triangles are decided by linear interpolation of gray values between black and white points. In a unit cubic region of $2 \times 2 \times 2$ points, the number of possible arrangements of black and white points is limited to 256. For each unit cubic region, the local polyhedral configuration iw predetermined as shown in Table 1.

It has been pointed out, however, that the classical marching cubes method causes the topological ambiguity of isosurfaces. If a set of black points at the joint of adjacent unit cubes are arranged as illustrated in Figure 1, two different configurations of triangles' edges are conceivable such as (a) and (b) of the figure. If the configurations in two adjacent unit cubes are not coincident at the joint, the isosurface cracks and have no topological structure equivalent to a closed surface [2, 3]. In order to avoid this problem, we need to consider the connectivity of black (or white) points at the joints of unit cubes. One of the solutions is to prepare all polyhedral configurations correspond-

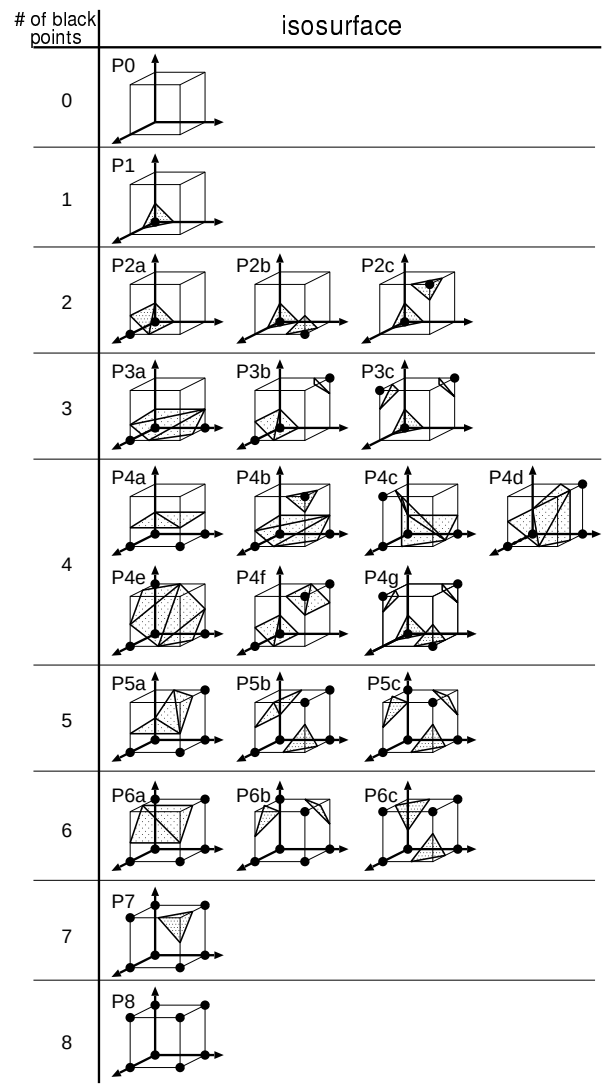

Table 1: The look-up table for generation of isosurfaces by the classical marching cubes method. Black points indicate the points whose gray values are more than the isovalue. The arrangements of black points which are obtained by rotations of those in the table are omitted in the table.

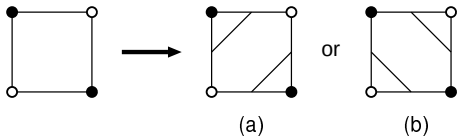

Figure 1: The topological ambiguity problem of the classical marching cubes method.

ing to all possible cases of point connectivities at the joint [2] as shown in Table 2. Although this solution enables us to avoid the ambiguity problem if an adequate configuration of triangles is chosen at each joint, it is incoherent from the viewpoint of the point connectivity; we consider the connectivity of points only at the joint of unit cubes, but not in each unit cube.

In order to discuss the connectivity of points in a polyhedral surface, we use the approach of combinatorial topology. We first discretize the marching cubes in Tables 1 and 2, and then try to obtain discrete marching cubes for generation of discrete polyhedra which are defined as closed surfaces in a 3-dimensional lattice space [4, 5]. Discrete polyhedra satisfy the following conditions;

1. all vertices of a polyhedron are lattice points,
2. any pair of adjacent vertices which are the endpoints of an edge of a polyhedron are m-neighboring where $m = 6, 18, 26$ [4, 5].

It has been proven that the topological structures of these discrete polyhedra are equivalent to those of 2-dimensional manifolds [6]. Therefore, discrete polyhedra are already guaranteed not to cause the topological ambiguity. Because of the conditions of discrete polyhedra, in addition, the normal vector of each polygon is calculated by integer arithmetic and the directions of all normal vectors are classified into exactly 13 types. Thus, much effect will be produced by postprocessing for visualization such as merging polygons, so that displaying a polyhedron consumes less time [7].

2 Discrete Polyhedral Surface and Its Connectivity

Let $\mathbf{Z}$ be the set of all integers. Then, $\mathbf{Z}^3$ ($= \mathbf{Z} \times \mathbf{Z} \times \mathbf{Z}$) denotes a set of lattice points whose coordinates are all integers. In $\mathbf{Z}^3$ we define three different neighborhoods of a lattice point $\boldsymbol{x} = (i, j, k)$ as follows:

$$\mathbf{N}_m(\boldsymbol{x}) = \{(p, q, r) \in \mathbf{Z}^3 \ : \ (i-p)^2 + (j-q)^2 + (k-r)^2 \leq t\}.$$

where $m = 6, 18, 26$ corresponding to $t = 1, 2, 3$. They are called 6-, 18- and 26-neighborhoods, respectively.

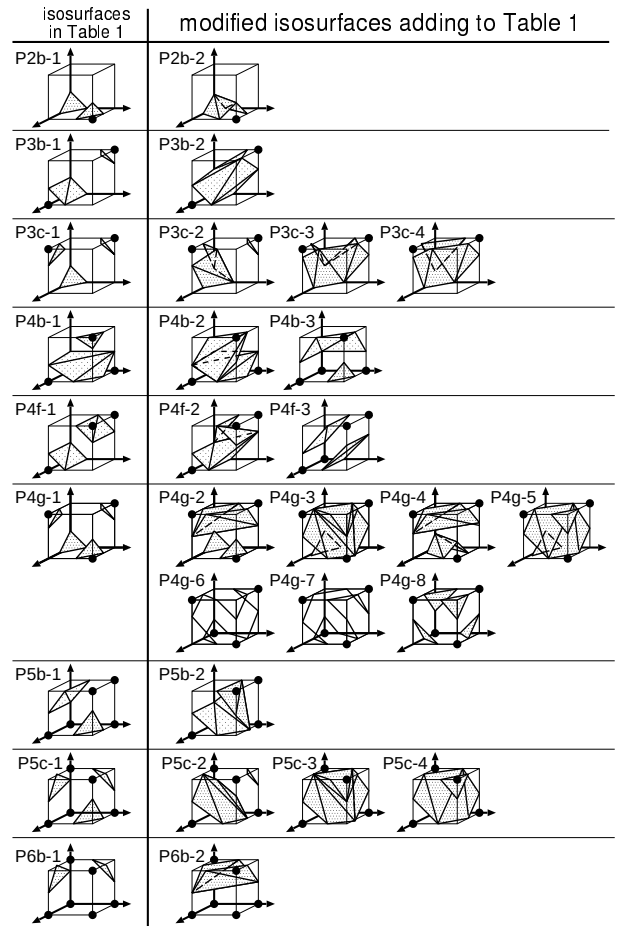

Table 2: Polyhedral surface configurations which we need to add to Table 1 for avoiding the topological ambiguities of isosurfaces.

Depending on each neighborhood, we define elements of 1-dimensional curves and 2-dimensional surfaces in $\mathbf{Z}^3$. These elements are called 1- and 2-simplexes, respectively [4]. Suppose we define 0-simplexes as isolated points in $\mathbf{Z}^3$. Then 1- and 2-simplexes are defined recursively as follows.

Definition 1 *An n-simplex for $n = 1, 2$ is defined as a set of k points in $\mathbf{Z}^3$,*

$$[\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_k] = \{\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_k\},$$

so that the embedded simplex of $[\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_k]$ is one of n-dimensional minimum nonzero regions in $\mathbf{R}^3$ which are bounded by embedded $(n-1)$-simplexes.

Let $\mathbf{R}$ be a set of real numbers; $\mathbf{R}^3$ denotes a 3-dimensional Euclidean space. In $\mathbf{R}^3$, The embedded simplex of $[\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_k]$ is defined as the convex hull

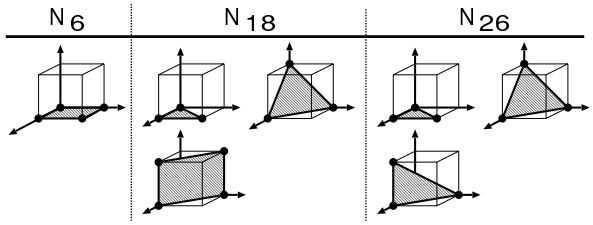

Table 3: All possible 2-dimensional discrete simplexes for the 6-, 18- and 26-neighborhood systems. Note that congruent simplexes which differ from those in this table by translations and rotations are omitted in the table.

of the k points, and is denoted by $|\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_k|$. If $[\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_k]$ is an n-simplex, $|\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_k|$ is also called the embedded n-simplex. According to Definition 1, a 1-simplex is defined as a set of two m-neighboring points in $\mathbf{Z}^3$, and a 2-simplex is defined as a set of points whose convex hull is bound by a set of 1-simplexes. Table 3 shows all 2-simplexes for each neighborhood system. The constructive definitions of discrete simplexes are also presented in reference [4].

If an s-simplex is included in an n-simplex where $s < n$, the s-simplex is called an s-face of the n-simplex. Using 0- and 1-faces, we can combine 2-simplexes so that combined 2-simplexes represent surfaces in $\mathbf{Z}^3$ which are called discrete surfaces.

Definition 2 *Let* $\mathbf{K}$ *be a set of* 2*-simplexes and* $[a]$ *and* $[b]$ *be any pair of* 2*-simplexes in* $\mathbf{K}$ *so that* $[a] \cap [b] \neq \emptyset$*. If* $[a] \cap [b]$ *is the common* 0*- or* 1*-face of* $[a]$ *and* $[b]$ *and every* 1*-face in* $\mathbf{K}$ *belongs to exactly two* 2*-simplexes,* $\mathbf{K}$ *is a discrete surface.*

If a discrete surface $\mathbf{K}$ consists of only 2-simplexes with m-neighborhood system and there is a path between two arbitrary elements $[a]$ and $[b]$ in $\mathbf{K}$ such as

$$[a_1] = [a],\ [a_2],\ \ldots,\ [a_k] = [b],$$

where $[a_i] \in \mathbf{K}$ and $[a_i] \cap [a_{i+1}] \neq \emptyset$ for $i = 1, 2, \ldots, k-1$, $\mathbf{K}$ is called m-connected.

Discrete surfaces derived by Definition 2 are classified into the following two classes: a class of polyhedral surfaces which are closed and a class of surfaces which are infinitely spread in $\mathbf{Z}^3$ such as planes. In this paper, we focus on discrete surfaces in the first class, and call such discrete surfaces discrete polyhedral surfaces, hereafter. The rigorous definition of discrete surfaces in the sense of combinatorial topology is given in the reference [4].

3 Discretized Marching Cubes

Discretizing Tables 1 and 2 to obtain discretized marching cubes, we examine a polyhedral configuration for each cube from the viewpoint of connectivity.

3.1 Discretization of Marching Cubes

We first binarize the gray value of each voxel in a 3-dimensional digital image to be either 1 or 0 using a threshold v. Each voxel corresponds to a point in $\mathbf{Z}^3$, and points whose values are 1 or 0 are called 1- or 0-points, respectively. We then interpolate an isosurface whose isovalue is nearly 1 such as 0.99 by referring to Tables 1 and 2, and obtain Table 4. Since Table 2 contains more than one polyhedral configurations for a single arrangement of 1-points, Table 4 also contains more than one polyhedral configurations for such an arrangement. Because of the constraints such that the vertices of generated polyhedra are lattice points, there are some arrangements for which no oriented polyhedral face is constructed such as P2b, P3b, P3c, etc.

3.2 Cube Classification by Connectivity

All polyhedral configurations in Table 4 can be classified into four types by connectivities. If every pair of m-neighboring 1-points constitutes an edges of polyhedra in a unit cube, such a polyhedral configuration is classified into a type of m-connectivity where $m = 6, 18, 26$. If a polyhedral configuration is classified into neither of the types of m-connectivity, we classify it into the fourth type. Table 5 shows the results of the classification. From Table 5, we conclude that

1. polyhedral configurations with 6-connectivity are not always considered for the arrangements of 1-points in Table 4;

2. a polyhedral configuration with 18-connectivity is considered for every arrangement of 1-points in Table 4 except for P3b, P4b, P4c and P4d;

3. a polyhedral configuration with 26-connectivity is considered for every arrangement of 1-points in Table 4 except for P2c and P5a.

4 Discrete Marching Cubes with Connectivity

Table 5 shows that polyhedral configurations with 6-connectivity are not always considered. This is because 2-simplexes for 6-neighborhood system are only square as illustrated in Table 3 and they are not suitable for the representation of an isosurface interpolated in a unit cube since they are located on the boundary of a unit cube and not inside a unit cube. Thus, in this section, we only consider 18- and 26-connectivities and present discrete marching cubes with 18- and 26-connectivities.

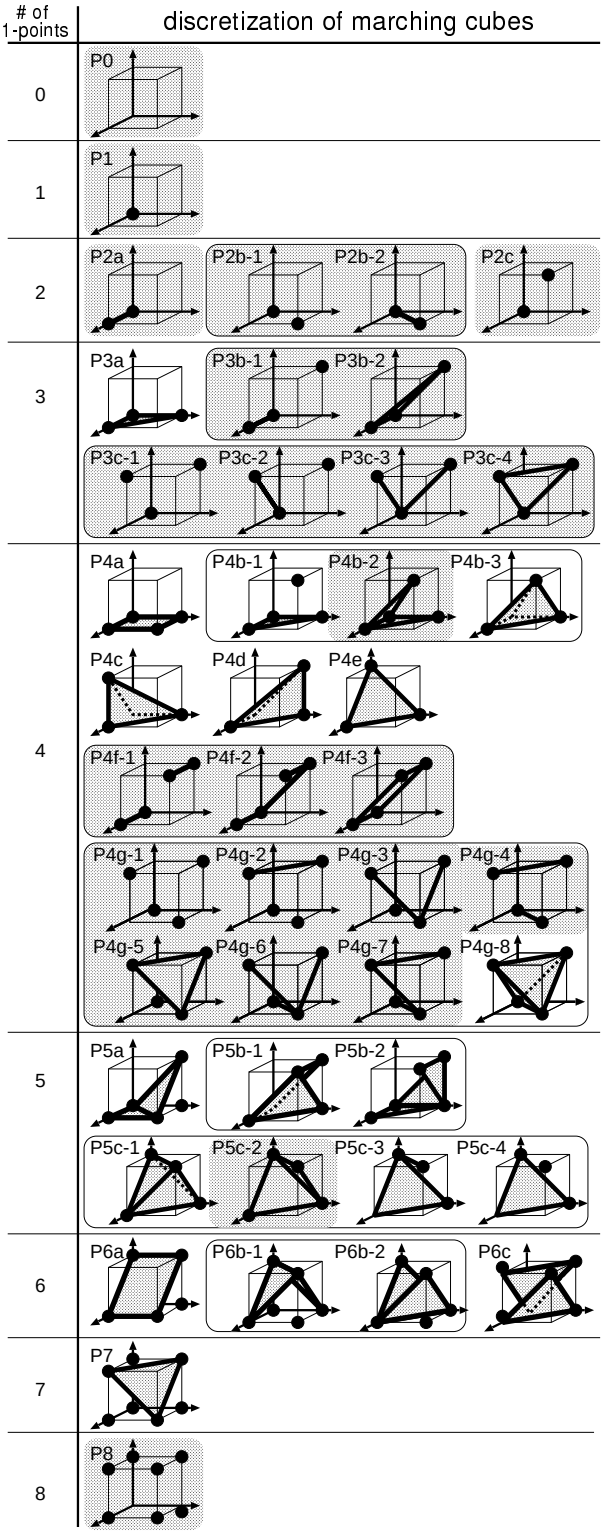

Table 4: Discrete marching cubes derived from Tables 1 and 2. Polyhedral configurations which are not used for the generation of isosurfaces are hatched in the table.

$\mathbf{N}_6$	$\mathbf{N}_{18}$	$\mathbf{N}_{26}$	else
P0	P0	P0	
P1	P1	P1	
P2a	P2a	P2a	
P2b-1	P2b-2	P2b-2	
P2c	P2c		
	P3a	P3a	
P3b-1		P3b-2	
P3c-1	P3c-4	P3c-4	P3c-2, P3c-3
P4a	P4a	P4a	
		P4b-3	P4b-1, P4b-2
		P4c	
		P4d	
	P4e	P4e	
P4f-1	P4f-3	P4f-3	P4f-2
P4g-1	P4g-8	P4g-8	P4g-2, P4g-3, ..., P4g-7
	P5a		
	P5b-1	P5b-1	P5b-2
	P5c-1	P5c-1	P5c-2, P5c-3, P5c-4
	P6a	P6a	
	P6b-1	P6b-1	P6b-2
	P6c	P6c	
	P7	P7	
P8	P8	P8	

Table 5: Classification of all discrete marching cubes in Table 4 by m-connectivities where $m = 6, 18, 26$.

Since Table 5 shows that polyhedral configurations with 18-connectivity for P3b, P4b, P4c and P4d are not prepared in Table 4, we determine them as illustrated in Figure 2 (a). Similarly, we determine polyhedral configurations with 26-connectivity for P2c and P5a in Figure 2 (b).

To construct m-connected discrete polyhedral surfaces where $m = 18, 26$ from a 3-dimensional digital image, we only use the polyhedral configurations in Table 6 since we can ignore discrete simplexes which are not oriented and less than two dimensions. Table 6 is coincident with the table which is used in the references [5, 8], and the following theorem was derived in the references.

Theorem 1 *Referring to Table 6, we can uniquely construct m-connected discrete polyhedral surfaces from a finite subset $\mathbf{V} \subset \mathbf{Z}^3$ with the additional procedures as illustrated in Figure 3.*

5 Conclusions

In this paper, we discretized marching cubes in Tables 1 and 2, obtained Table 4, and examined the connectivity of discrete simplexes for each cube in Table

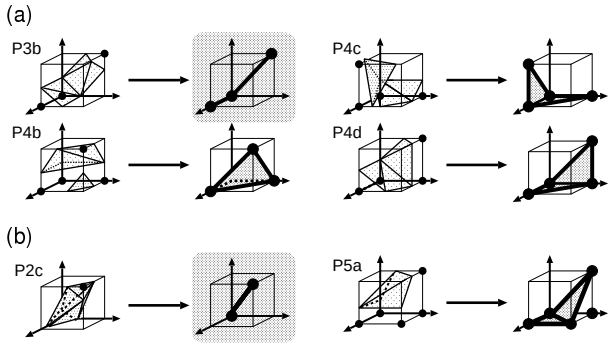

Figure 2: Polyhedral configurations with 18-connectivity (a) and 26-connectivity (b), which are not prepared in Tables 1, 2 and 4. Configurations which are not used for construction of discrete polyhedral surfaces are hatched in the figure.

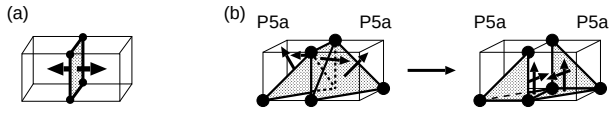

Figure 3: Two additional procedures for construction of m-connected discrete polyhedral surfaces; (a) if two adjacent unit cubes have a common 2-simplexes, they are ignored; (b) if two adjacent unit cubes have the polyhedral configurations P5a for 18-connectivity, the 2-simplexes are replaced such as the illustration.

4 in the sense of combinatorial topology. From the examination, we clarify that the addition polyhedral configurations in Figure 2 are necessary for preserving the correct topological structures of polyhedral isosurfaces. Consequently, we obtained Table 6 for construction of m-connected discrete polyhedral surfaces from a 3-dimensional digital image.

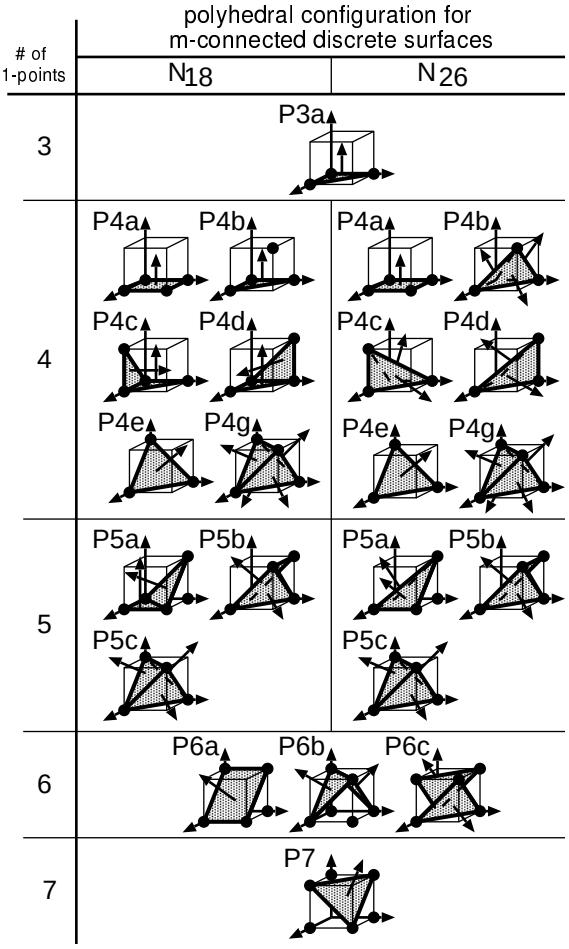

Table 6: The look-up table for generation of m-connected discrete surfaces where $m = 18, 26$ by marching cubes method.

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