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SHAPE DECOMPOSITION BY TOPOLOGY

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Shape decomposition is a fundamental method of extracting geometric features of objects. The decomposition of objects using geometric features is not sufficient for image representation with subpixel accuracy. This is because not only geometric structures but also topological structures of objects are variable and depend on the resolution of an image. In this paper, we define “dimensionalities” as new topological features. The term “dimensionality” comes from the dimensions of simplexes which are fundamental properties in classical combinatorial topology. First, we introduce a combinatorial shape description in 3-dimensional discrete space. Second, a method of shape decomposition using “dimensionalities” is presented.

1 Introduction

Shape decomposition is a fundamental method of extracting geometric features of objects [1]. A given object is divided into a collection of parts; the collection enables us to describe the given object as a collection of models which are precreated as fundamental shapes for the description of objects. The decomposition of objects into a collection of geometric parts is, however, insufficient for image representation with subpixel accuracy [2, 3]. This is because not only geometric properties but also topological properties of objects are variable and depend on the resolution of images. These topological properties are called “dimensionalities” in this paper. The term “dimensionality” comes from the dimensions of simplexes which are the basic properties in classical combinatorial topology [4].

In the first part of this paper, we introduce a combinatorial shape description in a 3-dimensional (3D) discrete space. We call the shape description a discrete complex which is taken from a term in combinatorial topology. We also show an algorithm for constructing a discrete complex from a given set of points in the 3D discrete space. A similar approach for discrete shape representation, which is called finite topology [5], exists. Since finite topology assumes that objects which are treated in a 3D discrete space are only 3D, dimensionalities of objects are only 3, whereas our shape description can include from 0 to 3 dimensionalities depending on the parts of represented objects. In the second part of this paper, a method of shape decomposition into n D parts of objects is presented using dimensionalities of each point. Each point has at least one kind of dimensionality, which can be from 0 to 3 in a 3D discrete

space. A simple example of decomposing an object in the 3D discrete space is also illustrated.

2 Combinatorial Shape Description

This section gives an overview of a 3D shape description in a 3D discrete space using combinatorial topology. The details of combinatorial topology in a 3D discrete space are described in reference [6]. Setting \mathbf{R}^3 to be a 3D Euclidean space, a 3D discrete space \mathbf{Z}^3 is defined as a set of all points in \mathbf{R}^3 having integer coordinates. Denoting the Euclidean distance between two points, \mathbf{x} and \mathbf{y} , by $\|\mathbf{x} - \mathbf{y}\|$, two types of neighborhoods of a point \mathbf{x} in \mathbf{Z}^3 are defined as

$$N_m(\mathbf{x}) = \{\mathbf{y} \mid \|\mathbf{x} - \mathbf{y}\| \leq p, \mathbf{y} \in \mathbf{Z}^3\}, \quad (1)$$

where $m = 6$ or 26 corresponding to p which is 1 or $\sqrt{3}$. They are called 6- and 26-neighborhoods, respectively. One of the two neighborhoods must be assumed in \mathbf{Z}^3 .

An n D discrete simplex is defined as a topological element which has an n D region in \mathbf{Z}^3 , depending on each neighborhood. A discrete simplex consists of several points in \mathbf{Z}^3 . If we embed these points of a discrete simplex in \mathbf{R}^3 , then we can create the convex hull of these points in \mathbf{R}^3 ; the convex hull is called the embedded simplex. The dimension of a discrete simplex depends on the dimension of the region of its embedded simplex. Note that embedded simplexes do not include their boundary; in topological terms, they are relatively open in \mathbf{R}^3 . The dimension can be from 0 to 3 in \mathbf{Z}^3 . For each neighborhood, 0D discrete simplexes are regarded as isolated points in \mathbf{Z}^3 . Discrete simplexes whose dimensions are more than 0 are obtained by the following recursive definition.

Definition 1 *An n D discrete simplex is defined as a set of points in \mathbf{Z}^3 , such that the embedded simplex becomes one of n D minimum nonzero regions in \mathbf{R}^3 which are bounded by $(n - 1)$ D embedded simplexes.*

We abbreviate n D discrete simplexes as n -simplexes. An n -simplex which consists of k points, $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k$, is represented by $[\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k]$.

For each of the 6- and 26-neighborhoods, n -simplexes are defined as shown in Fig. 1, where n is from 0 to 3. Let us consider discrete simplexes in the case of the 6-neighborhood. Embedded 1-simplexes in \mathbf{R}^3 are unit line segments because 1D regions in \mathbf{R}^3 are in lines; 1-simplexes become sets of two neighboring points which are regarded as endpoints of the line segments. Embedded 2-simplexes are unit squares bounded by four unit line segments which are 1-simplexes embedded in \mathbf{R}^3 ; 2-simplexes become sets of four points,

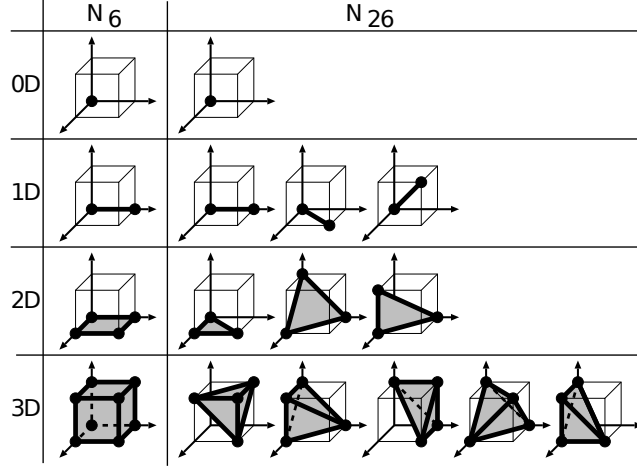


Figure 1: All possible discrete simplexes in \mathbf{Z}^3 , whose shapes depend on the 6- or 26-neighborhood. We ignore congruent ones that differ from the discrete simplexes in this figure by translation and/or rotation.

each of which neighbors two other points. Embedded 3-simplexes are unit cubes bounded by unit squares which are 2-simplexes embedded in \mathbf{R}^3 ; 3-simplexes consist of sets of 8 points in \mathbf{Z}^3 . Note that discrete simplexes shown in Fig. 1 are only counted if we ignore the congruent simplexes that differ from those in Fig. 1 by translation and rotation.

For an n -simplex, $[\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k]$, a function, $face([\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k])$, can be defined as a set of all discrete simplexes included in the n -simplex, whose dimensions are less than n . For instance, the faces of a 2-simplex are 0- and 1-simplexes. Using this function, discrete complexes, which are combinations of discrete simplexes, are defined in \mathbf{Z}^3 .

Definition 2 A discrete complex is defined by a finite set \mathbf{C} of discrete simplexes satisfying the following two conditions.

1. If $[a] \in \mathbf{C}$, $face([a]) \subseteq \mathbf{C}$.
2. If $[a_1], [a_2] \in \mathbf{C}$ and $|a_1| \cap |a_2| \neq \emptyset$, $[a_1] = [a_2]$,

where $[a]$ and $|a|$ represent a discrete simplex and its embedded simplex, respectively.

The dimension of a discrete complex is equivalent to the maximum dimension of its associated set of discrete simplexes. We abbreviate n D discrete complexes as n -complexes.

If every discrete simplex $[a]$ in \mathbf{C} whose dimension is less than n satisfies

$$[a] \in face([b]), \quad (2)$$

where $[b]$ is one of the n -simplexes in \mathbf{C} , \mathbf{C} is called pure. In addition to the pureness of complexes, \mathbf{C} is called connected if, for any two elements $[a]$ and $[b]$ in \mathbf{C} , there exists a path between them:

$$[a_1] = [a], [a_2], \dots, [a_n] = [b], \quad (3)$$

where $[a_i] \in \mathbf{C}$ for $i = 1, 2, \dots, n$ and $[a_i] \cap [a_{i+1}] \neq \emptyset$ for $i = 1, 2, \dots, n - 1$.

If we embed all discrete simplexes included in a discrete complex into \mathbf{R}^3 , a complex embedded in \mathbf{R}^3 , which we call a body, can be obtained.

Definition 3 *The body of a discrete complex \mathbf{C} is defined as*

$$\mathbf{B} = \bigcup_{[a] \in \mathbf{C}} |a|. \quad (4)$$

Obviously, bodies are sets in \mathbf{R}^3 , not in \mathbf{Z}^3 .

3 Construction of Discrete Complex

We assume that a subset \mathbf{V} in \mathbf{Z}^3 is given. Every point in \mathbf{V} is assigned a value of 1 while every point in the complement of \mathbf{V} is assigned a value of 0. We call points whose values are 1 and 0 1-points and 0-points, respectively. This section presents a method of constructing a discrete complex from \mathbf{V} .

First, we create a discrete complex for each unit cube whose eight vertices are points in \mathbf{Z}^3 and side lengths are one. A discrete complex for each unit cube is created so that all points in the discrete complex are 1-points and that the region occupied by the discrete complex becomes maximum. The number of all possible patterns of 1- and 0-points in a unit cube is 23 if we ignore the congruent patterns that differ from them by rotation of the center of the unit cube. Tables 1 and 2 give discrete complexes for the 23 1-point patterns in a unit cube. Note that there can be several combinations of discrete simplexes for a 1-point pattern in a unit cube, even if the bodies of the discrete complexes for the 1-point pattern in a unit cube are equivalent, as shown in Fig. 2.

Next, we combine the discrete complexes for all unit cubes into a discrete complex without contradictions at the joints of two discrete complexes in adjacent unit cubes. According to reference [7], no contradiction occurs at any joint if we consider the 6-neighborhood. If we consider the 26-neighborhood, we join discrete complexes for all 1-point patterns, except for P4a, P5a, P6a, P6b and P7 in Table 2, first. Then, for each of these five patterns, we choose a discrete complex such that it can be joined to discrete complexes in adjacent unit cubes without contradictions among several candidates of discrete complexes whose combinations of discrete simplexes are different.

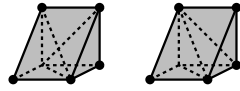


Figure 2: An example of two discrete complexes for a pattern, P6a, whose combinations of simplexes are not the same but whose bodies are equivalent.

Table 1: Discrete complexes in a unit cube corresponding to all 1- and 0-point patterns. The 6-neighborhood is assumed.

1-point number	discrete complex in a unit cube	1-point number	discrete complex in a unit cube
0	P0	5	P5a P5b P5c
1	P1	6	P6a P6b P6c
2	P2a P2b P2c	7	P7
3	P3a P3b P3c	8	P8
4	P4a P4b P4c P4d P4e P4f P4g		

a unit cube
 a 1-point

Table 2: Discrete complexes in a unit cube corresponding to all 1- and 0-point patterns. The 26-neighborhood is assumed.

1-point number	discrete complex in a unit cube	1-point number	discrete complex in a unit cube
0	P0	5	P5a P5b P5c
1	P1	6	P6a P6b P6c
2	P2a P2b P2c	7	P7
3	P3a P3b P3c	8	P8
4	P4a P4b P4c P4d P4e P4f P4g		

a unit cube
 a 1-point

The following two theorems are proved in reference [7].

Theorem 1 *For the 6-neighborhood, a discrete complex can be uniquely obtained from \mathbf{V} .*

Theorem 2 *For the 26-neighborhood, a discrete complex can be obtained from \mathbf{V} . Even if there exist several discrete complexes for \mathbf{V} , their bodies are always equivalent.*

Note that discrete complexes created from \mathbf{V} may not be pure, because, for some 1-point patterns in a unit cube, any discrete simplexes whose dimensions can be from 0 to 3 are created according to Tables 1 and 2. In other words, there can exist 0-, 1- and 2-simplexes which are not included in any 3-simplex in a created 3-complex. Figure 3 illustrates examples of 3-complexes for the 6- and 26-neighborhoods.

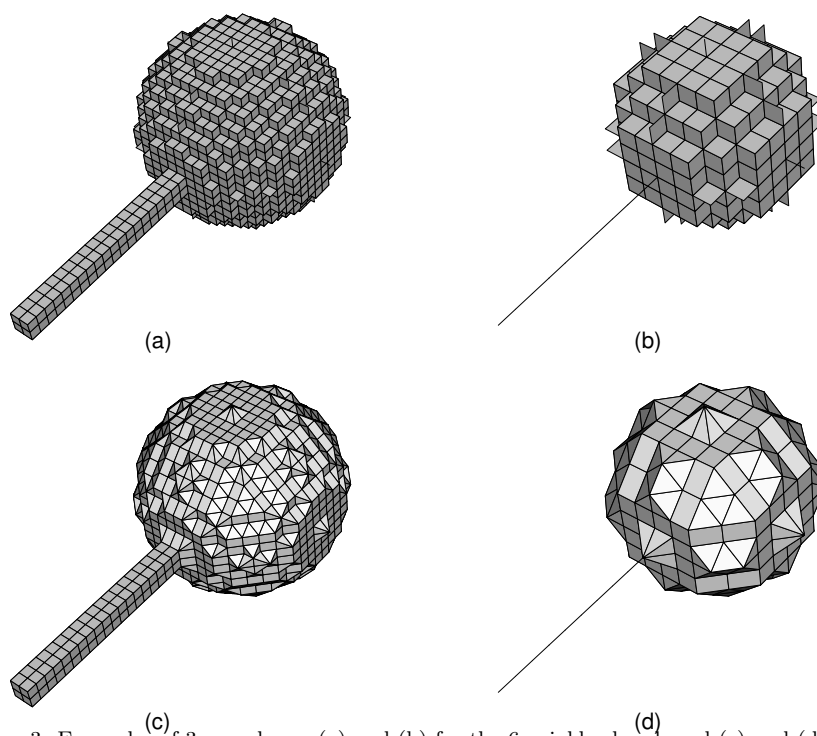


Figure 3: Examples of 3-complexes, (a) and (b) for the 6-neighborhood, and (c) and (d) for the 26-neighborhood. (a) and (c) are obtained from \mathbf{V}_1 , and (b) and (d) are obtained from \mathbf{V}_2 ; \mathbf{V}_2 can be obtained from \mathbf{V}_1 , such that the resolution of \mathbf{V}_2 is half that of \mathbf{V}_1 .

4 Shape Decomposition

If a discrete complex \mathbf{C} is obtained from a given subset \mathbf{V} in \mathbf{Z}^3 , we can divide \mathbf{C} into four sets as

$$\mathbf{C} = \mathbf{C}_0 \cup \mathbf{C}_1 \cup \mathbf{C}_2 \cup \mathbf{C}_3, \quad (5)$$

where \mathbf{C}_n is a set of all n -simplexes in \mathbf{C} . Obviously, \mathbf{C}_n can be obtained from \mathbf{C} without any calculations. Using these \mathbf{C}_n s, we can calculate \mathbf{P}_n where $n = 0, 1, \dots, 3$ consecutively as

$$\mathbf{P}_3 = \mathbf{C}_3 \cup \left(\bigcup_{[a] \in \mathbf{C}_3} \text{face}([a]) \right), \quad (6)$$

$$\mathbf{P}_2 = \mathbf{C}'_2 \cup \left(\bigcup_{[a] \in \mathbf{C}'_2} \text{face}([a]) \right) \quad \text{where} \quad \mathbf{C}'_2 = \mathbf{C}_2 \setminus \left(\bigcup_{[a] \in \mathbf{C}_3} \text{face}([a]) \right), \quad (7)$$

$$\mathbf{P}_1 = \mathbf{C}'_1 \cup \left(\bigcup_{[a] \in \mathbf{C}'_1} \text{face}([a]) \right) \quad \text{where} \quad \mathbf{C}'_1 = \mathbf{C}_1 \setminus \left(\bigcup_{[a] \in \mathbf{C}_2} \text{face}([a]) \right), \quad (8)$$

$$\mathbf{P}_0 = \mathbf{C}_0 \setminus \left(\bigcup_{[a] \in \mathbf{C}_1} \text{face}([a]) \right), \quad (9)$$

and we can decompose \mathbf{C} to

$$\mathbf{C} = \mathbf{P}_0 \cup \mathbf{P}_1 \cup \mathbf{P}_2 \cup \mathbf{P}_3. \quad (10)$$

The next theorem concerning \mathbf{P}_n is derived from equations (6) to (9).

Theorem 3 \mathbf{P}_n is a pure n -complex included in \mathbf{C} where $n = 0, 1, 2, 3$.

(Proof) It is obvious that \mathbf{P}_n is an n -complex because the maximum dimension of discrete simplexes in \mathbf{P}_n is n , and that \mathbf{P}_n is included in \mathbf{C} , according to equations (6) to (9). In addition, every discrete simplex in \mathbf{P}_n , whose dimension is less than n , is included in at least one n -simplex in \mathbf{P}_n . Thus, \mathbf{P}_n is a pure n -complex. (Q.E.D.)

Note that \mathbf{P}_n can be empty. For instance, if \mathbf{C} is a pure 3-complex, \mathbf{P}_0 , \mathbf{P}_1 and \mathbf{P}_2 are empty because all discrete simplexes in \mathbf{C}_0 , \mathbf{C}_1 and \mathbf{C}_2 are included in \mathbf{C}_3 as faces of 3-simplexes in \mathbf{C}_3 . In other words, if \mathbf{C} is a pure 3-complex,

$$\mathbf{P}_3 = \mathbf{C}. \quad (11)$$

Let \mathbf{B}_n be the body of \mathbf{P}_n where $n = 0, 1, 2, 3$; the body of \mathbf{C} , \mathbf{B} , is decomposed in a similar way to (10) as follows:

$$\mathbf{B} = \mathbf{B}_0 \cup \mathbf{B}_1 \cup \mathbf{B}_2 \cup \mathbf{B}_3. \quad (12)$$

According to Theorems 1 and 2, \mathbf{B} is directly created from \mathbf{V} and does not depend on \mathbf{C} . Thus, the next theorem is derived.

Theorem 4 \mathbf{B}_n is uniquely extracted from \mathbf{B} depending on the given subset \mathbf{V} in \mathbf{Z}^3 , where $n = 0, 1, 2, 3$.

(Proof) If we are concerned with the 6-neighborhood, a discrete complex \mathbf{C} is uniquely constructed from \mathbf{V} according to Theorem 1. Because every \mathbf{C}_n is uniquely determined from \mathbf{C} , every \mathbf{P}_n is also uniquely calculated following the equations. Thus, \mathbf{B}_n is uniquely obtained. If we are concerned with the 26-neighborhood, \mathbf{B} is uniquely obtained from \mathbf{V} even if several discrete complexes can be constructed from \mathbf{V} according to Theorem 2. If \mathbf{B} has isolated points, these points must be included in \mathbf{B}_0 , and \mathbf{B}_0 must not include any points other than these isolated points; \mathbf{B}_0 is obtained uniquely. If \mathbf{B} has a series of line segments, these line segments must be included in \mathbf{B}_1 , and \mathbf{B}_1 must not include any points other than the points on these line segments; \mathbf{B}_1 is also obtained uniquely. Similarly, if \mathbf{B} has thin-wall-like parts, these parts must be included in \mathbf{B}_2 , and \mathbf{B}_2 must not include any points other than points in the thin-wall-like parts; \mathbf{B}_2 is also obtained uniquely. The remaining parts of \mathbf{B} , which are not included in either \mathbf{B}_0 , \mathbf{B}_1 or \mathbf{B}_2 , are also uniquely obtained and the closure of the remaining parts becomes \mathbf{B}_3 . (Q.E.D.)

Let \mathbf{V}_n be a subset of \mathbf{V} , whose points are included in \mathbf{B}_n ; \mathbf{V} is also decomposed in a similar way to (10) as follows:

$$\mathbf{V} = \mathbf{V}_0 \cup \mathbf{V}_1 \cup \mathbf{V}_2 \cup \mathbf{V}_3. \quad (13)$$

Obviously, \mathbf{V}_n can be determined from either \mathbf{B}_n or \mathbf{P}_n where $n = 0, 1, 2, 3$. Because \mathbf{B}_n and \mathbf{P}_n are uniquely extracted from \mathbf{V} according to Theorem 4, \mathbf{V}_n is also uniquely determined from \mathbf{V} .

5 Algorithm of Shape Decomposition

The previous section shows that we can decompose a given subset \mathbf{V} in \mathbf{Z}^3 into the n D parts \mathbf{V}_n where $n = 0, 1, 2, 3$ if a discrete complex \mathbf{C} is already constructed from \mathbf{V} . In this section we demonstrate an algorithm for obtaining each \mathbf{V}_n from \mathbf{V} directly without converting \mathbf{C} from \mathbf{V} .

Let $\mathbf{C}(\mathbf{x})$ be an n -complex constructed in the union of eight unit cubes, all of which include the point \mathbf{x} ; $\mathbf{C}(\mathbf{x})$ can be decomposed into pure discrete complexes such as (10). Assuming that $\mathbf{P}_m(\mathbf{x})$ is a set of nonempty pure m -complexes in $\mathbf{C}(\mathbf{x})$ and that every m -simplex in $\mathbf{P}_m(\mathbf{x})$ includes \mathbf{x} as one of the vertices of the m -simplex, it is said that \mathbf{x} has the dimensionality m , where $m \leq n$. If \mathbf{x} is included both in a 2-simplex in $\mathbf{P}_2(\mathbf{x})$ and in a 3-simplex in $\mathbf{P}_3(\mathbf{x})$, the dimensionalities of \mathbf{x} are 2 and 3. Using the dimensionalities of points, objects of interest, which are given as subsets of \mathbf{Z}^3 , can be decomposed following Algorithm 1.

Algorithm 1

input: Finite nonempty subset \mathbf{V} in \mathbf{Z}^3 .

output: Decomposed objects $\mathbf{V}_0, \mathbf{V}_1, \dots, \mathbf{V}_3$.

begin

1. Set $\mathbf{V}_m = \emptyset$ where $m = 0, 1, 2, 3$.
2. For each point \mathbf{x} in \mathbf{V} ,
 - 2.1 For each of eight unit cubes which include \mathbf{x} , look up the discrete complex in Table 1 or 2.
 - 2.2 Combine eight discrete complexes in the eight unit cubes into a discrete complex, $\mathbf{C}(\mathbf{x})$.
 - 2.3 Decompose $\mathbf{C}(\mathbf{x})$ into $\mathbf{P}_m(\mathbf{x})$ s where $m = 0, 1, 2, \dots, 3$.
 - 2.4 For $m = 0, 1, 2, 3$, if $\mathbf{x} \in \mathbf{P}_m(\mathbf{x})$, then add \mathbf{x} to \mathbf{V}_m

end

According to Tables 1 and 2, the number of possible patterns of discrete complexes in a unit cube is 23 both for the 6-neighborhood and for the 26-neighborhood, if we ignore discrete complexes which differ from those in the tables by rotation and/or translation. However, in Algorithm 1, we can precalculate discrete complexes for all 256 patterns of 1- and 0-points in a unit cube. This preprocessing for all 1- and 0-point patterns accelerates the computation of Step 2.1 in Algorithm 1.

Using Algorithm 1, we can decompose the 3-complex in Fig. 3 (d) into 1D and 3D parts. Every point in the 1D and 3D parts has dimensionality 1 and 3 respectively, and thus the joining point of the 1D and 3D parts has both dimensionalities, 1 and 3. Since every point in Fig. 3 (c) has dimensionality 3, the 3-complex cannot be decomposed into more than one part and contains no part except for the 3D part. The different results of decomposition between (c) and (d) in Fig. 3 is caused by the different resolutions of their object images.

If we compare Fig. 3 (a) with (c) and similarly (b) with (d), the results of shape decomposition are different, even if both are obtained from the same \mathbf{V}_1 and \mathbf{V}_2 . This is caused by the different neighborhoods. According to Fig. 1, there are several shapes of discrete simplexes for the 26-neighborhood, whereas there is only one discrete simplex in each dimension for the 6-neighborhood. This implies that we can combine discrete simplexes for the 26-neighborhood more flexibly than for the 6-neighborhood and leads to the relation:

$$\mathbf{V}_3[6] \subseteq \mathbf{V}_3[26], \quad (14)$$

where $\mathbf{V}_3[m]$ is the 3D parts of \mathbf{V} if we adopt the m -neighborhood. In Fig. 3, the relation (14) holds between (a) and (c), and between (b) and (d), respectively.

6 Conclusions

In this paper we introduced a combinatorial shape description in \mathbf{Z}^3 called a discrete complex which comes from a term in classical combinatorial topology. An algorithm for constructing a discrete complex from a given subset in \mathbf{Z}^3 was also presented. If a discrete complex is constructed from a given subset in \mathbf{Z}^3 , the given subset is automatically decomposed into n D parts using topological structures of the discrete complex. We also showed an algorithm that directly extracts n D parts of the given finite subset in \mathbf{Z}^3 from the given subset, using the dimensionalities of each point. These n D parts are uniquely obtained.

References

- [1] A. Held and K. Abe, "On the decomposition of binary shapes into meaningful parts", *Pattern Recognition*, vol. 27, no. 5, pp. 637-647, 1994.
- [2] C. A. Berenstein, L. N. Kanal, D. Lavine and E. C. Olson, "A geometric approach to subpixel registration accuracy", *Computer Vision, Graphics, and Image Processing*, vol. 40, pp. 334-360, 1987.
- [3] A. Imiya, "Subpixel superresolution and inversion of image pyramid", in *Proceedings of the Sixth International Conference on Artificial Intelligence and Information-Control Systems of Robots*, 1994, pp. 47-58.
- [4] P. S. Aleksandrov, *Combinatorial Topology, Vol. 1*, Graylock Press, Rochester, N.Y., 1956.
- [5] V. A. Kovalevsky, "Finite topology as applied to image analysis", *Computer Vision, Graphics, and Image Processing*, vol. 46, pp. 141-161, 1989.
- [6] Y. Kenmochi, A. Imiya, and A. Ichikawa, "Discrete combinatorial geometry", to appear in *Pattern Recognition*.
- [7] Y. Kenmochi, A. Imiya, and A. Ichikawa, "Boundary extraction of discrete objects", to appear in *Computer Vision and Image Understanding*.