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GLOBAL WELL-POSEDNESS FOR THE DERIVATIVE NONLINEAR SCHRÖDINGER EQUATION

HAJER BAHOURI AND GALINA PERELMAN

ABSTRACT. This paper is dedicated to the study of the derivative nonlinear Schrödinger equation on the real line. The local well-posedness of this equation in the Sobolev spaces $H^s(\mathbb{R})$ is well understood since a couple of decades, while the global well-posedness is not completely settled. For the latter issue, the best known results up-to-date concern either Cauchy data in $H^{\frac{1}{2}}(\mathbb{R})$ with mass strictly less than 4π or general initial conditions in the weighted Sobolev space $H^{2,2}(\mathbb{R})$. In this article, we prove that the derivative nonlinear Schrödinger equation is globally well-posed for general Cauchy data in $H^{\frac{1}{2}}(\mathbb{R})$ and that furthermore the $H^{\frac{1}{2}}$ norm of the solutions remains globally bounded in time. One should recall that for $H^s(\mathbb{R})$, with $s < 1/2$, the associated Cauchy problem is ill-posed in the sense that uniform continuity with respect to the initial data fails. Thus, our result closes the discussion in the setting of the Sobolev spaces H^s . The proof is achieved by combining the profile decomposition techniques with the integrability structure of the equation.

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1. INTRODUCTION

This paper aims to investigating global well-posedness for the derivative nonlinear Schrödinger equation (DNLS) on the real line:

$$(1.1) \quad iu_t + u_{xx} = \pm i\partial_x(|u|^2u), \quad x \in \mathbb{R},$$

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with initial conditions

$$(1.2) \quad u|_{t=0} = u_0 \in H^s(\mathbb{R}), \quad s \geq 1/2.$$

The transformation $u(t, x) \rightarrow u(t, -x)$ maps solutions of (1.1) with sign $-$ to solutions of (1.1) with sign $+$. In what follows, we shall fix the sign $-$ in (1.1).

The DNLS equation was derived by Mio-Ogino-Minami-Takeda and Mjolhus in [25, 26] for studying the one-dimensional compressible magneto-hydrodynamic equation in the presence of the Hall effect and the propagation of circular polarized nonlinear Alfvén waves in magnetized plasmas¹.

The equation (1.1) is known to be completely integrable, and to admit an infinite number of conservation laws including the conservation of mass, momentum and energy:

$$(1.3) \quad M(u) \stackrel{\text{def}}{=} \int_{\mathbb{R}} |u|^2 dx,$$

$$(1.4) \quad P(u) \stackrel{\text{def}}{=} \text{Im} \int_{\mathbb{R}} \bar{u} u_x dx + \frac{1}{2} \int_{\mathbb{R}} |u|^4 dx,$$

$$(1.5) \quad E(u) \stackrel{\text{def}}{=} \int_{\mathbb{R}} \left(|u_x|^2 - \frac{3}{2} \text{Im}(|u|^2 u \bar{u}_x) + \frac{1}{2} |u|^6 \right) dx.$$

The problem of local and eventually global well-posedness for the DNLS equation has received a lot of attention over the last twenty years. The local well-posedness is fully understood in the scale of the Sobolev spaces $H^s(\mathbb{R})$: combining a gauge transformation with the Fourier restriction method, Takaoka proved in [32] that the corresponding Cauchy problem is locally well-posed in $H^s(\mathbb{R})$ for $s \geq 1/2$, improving the earlier $H^1(\mathbb{R})$ -result of Hayashi and Ozawa [28]. The Takaoka's result is optimal accordingly to the works [4, 33] where it was shown that the Cauchy problem (1.1)-(1.2) is ill-posed in $H^s(\mathbb{R})$ for $s < 1/2$, in the sense that uniform continuity with respect to the initial conditions fails. Note that DNLS is L^2 -critical being invariant under the scaling:

$$(1.6) \quad u(t, x) \longrightarrow u_\mu(t, x) \stackrel{\text{def}}{=} \sqrt{\mu} u(\mu^2 t, \mu x), \quad \mu > 0.$$

The $1/2$ derivative gap in the local well-posedness can be closed by leaving the $H^s(\mathbb{R})$ -scale and considering more general functional spaces, see for instance [11] and the references therein.

Concerning the question of global well-posedness, Hayashi and Ozawa [13] proved the global existence for H^1 solutions with initial data satisfying $\|u_0\|_{L^2(\mathbb{R})} < \sqrt{2\pi}$. By the sharp Gagliardo-Nirenberg inequality [35]

$$(1.7) \quad \|f\|_{L^6(\mathbb{R})}^6 \leq \frac{4}{\pi^2} \|f\|_{L^2(\mathbb{R})}^4 \|f_x\|_{L^2(\mathbb{R})}^2, \quad \forall f \in H^1(\mathbb{R}),$$

this smallness assumption allows to control the H^1 -norm of the solution by its energy and the mass². This result was extended to H^s data with $s > 1/2$ by Colliander-Keel-Staffilani-Takaoka-Tao [6]. More recently, Wu [36] and Guo-Wu [12] increased the upper bound $\sqrt{2\pi}$ to $\sqrt{4\pi}$ respectively for H^1 and $H^{\frac{1}{2}}$ solutions. In the H^1 setting, the result follows from the mass, momentum and energy conservation combined with the following Gagliardo-Nirenberg inequality [2]

$$(1.8) \quad \|f\|_{L^6(\mathbb{R})} \leq C_{GN} \|f\|_{L^4(\mathbb{R})}^{\frac{8}{9}} \|f_x\|_{L^2(\mathbb{R})}^{\frac{1}{9}}, \quad \forall f \in L^4(\mathbb{R}) \cap \dot{H}^1(\mathbb{R}),$$

¹The DNLS equation also appears as a model for ultrashort optical pulses [27]. For an outline on physical applications of this equation, one can consult [5, 16] and the references therein.

²Note that, under the gauge transformation $u = v e^{-\frac{3i}{4} \int_{-\infty}^x |v(y)|^2 dy}$, the energy (1.5) reduces to the energy of the focusing quintic nonlinear Schrödinger equation $iv_t + v_{xx} = -\frac{3}{16}|v|^4 v$.

where the optimal constant C_{GN} is given by $C_{GN} = 3^{\frac{1}{6}}(2\pi)^{-\frac{1}{9}}$. The extension to $H^{\frac{1}{2}}$ -solutions was achieved by using the I-method.

The both bounds $\sqrt{2\pi}$ and $\sqrt{4\pi}$ are related to the L^2 -norm of solitary wave solutions of the DNLS equation. These solutions can be written in the explicit form:

$$(1.9) \quad u_{E,c}(t, x) = e^{i\omega t + i\frac{c}{2}x - \frac{3}{4}i \int_{-\infty}^{x-ct} |\varphi_{E,c}(s)|^2 ds} \varphi_{E,c}(x - ct),$$

where $E > 0$, $c \in \mathbb{R}$, $\omega = E - \frac{c^2}{4}$ and

$$\varphi_{E,c}(y) = \frac{2\sqrt{2E}}{(c^2 + 4E)^{\frac{1}{4}}} \frac{1}{\sqrt{\cosh(2\sqrt{E}y) - \frac{c}{\sqrt{c^2+4E}}}},$$

is the unique positive even exponentially decaying solution of

$$(1.10) \quad -\varphi_{yy} + E\varphi + \frac{c}{2}\varphi^3 - \frac{3}{16}\varphi^5 = 0.$$

These solutions are usually referred to as the bright solitons³. In the limiting case $E = 0$, $c > 0$, the profile $\varphi_{E,c}$ reduces to

$$\varphi_{0,c}(y) = \frac{2\sqrt{c}}{\sqrt{1 + c^2y^2}},$$

solving

$$(1.11) \quad -\varphi_{yy} + \frac{c}{2}\varphi^3 - \frac{3}{16}\varphi^5 = 0.$$

The corresponding solution $u_{0,c}$ is called the algebraic soliton. One can easily check that

$$(1.12) \quad M(u_{E,c}) = 8 \arctan \sqrt{\frac{\sqrt{c^2 + 4E} + c}{\sqrt{c^2 + 4E} - c}} \leq 4\pi.$$

The values 2π and 4π correspond to the mass of the static bright solitons $u_{E,0}$ and the algebraic solitons $u_{0,c}$, their profiles $\varphi_{E,0}$ and $\varphi_{0,c}$ being the extremals⁴ of the Gagliardo-Nirenberg inequalities (1.7) and (1.8) respectively.

There is also a number of works [15, 16, 17, 29, 30] where the global well-posedness of the DNLS equation was studied by means of the inverse scattering techniques. The corresponding results get rid of the smallness assumption on the mass, but require more regularity and decay on the initial data. The most definite result is due to Jenkins, Liu, Perry and Sulem who proved in [15] that the Cauchy problem for the DNLS equation is globally well-posed for any initial data u_0 in $H^{2,2}(\mathbb{R}) = \{f \in H^2(\mathbb{R}) : x^2 f \in L^2(\mathbb{R})\}$. Let us also mention the work of Pelinovsky-Saalmann-Shimabukuro [29] that gives the global well-posedness for generic initial data in $H^2(\mathbb{R}) \cap H^{1,1}(\mathbb{R})$.

The purpose of this paper is to prove the global well-posedness of the DNLS equation for general initial data in H^s , $s \geq 1/2$. Our main result is the following:

Theorem 1. *For any initial data $u_0 \in H^{\frac{1}{2}}(\mathbb{R})$, the Cauchy problem (1.1)-(1.2) is globally well-posed, and the corresponding solution u satisfies*

$$(1.13) \quad \sup_{t \in \mathbb{R}} \|u(t)\|_{H^{\frac{1}{2}}(\mathbb{R})} < +\infty.$$

Remark 1.1. *Combining the conservation laws with the $H^{\frac{1}{2}}$ bound (1.13), it is possible to show that if the initial datum is in $H^s(\mathbb{R})$ for some $s > 1/2$, then the H^s -norm of the solution remains globally bounded in time as well. We are planning to address this issue in a subsequent paper.*

³Their orbital stability was studied in [7, 10, 23].

⁴the only extremals up to the symmetries of the DNLS equation.

The proof of Theorem 1 relies heavily on the complete integrability of the DNLS equation, but avoids a direct use of the inverse scattering transform that requires a localization of initial data and breaks down for the solutions we are considering in this article. Instead, we exploit as much as possible the conservation quantities, namely the conservation of the transmission coefficient of the corresponding spectral problem that remains well defined for L^2 data, as soon as we stay away from the spectrum, the property that has been already extensively used in the works of Killip-Visan, Killip-Visan-Zhang and Koch-Tataru [19, 20, 22] on the low regularity solutions of the cubic NLS and KdV equations on the real line⁵.

The structure of the paper is as follows. Section 2 is devoted to the preliminary results related to the integrable structure of the DNLS equation, that will be needed in the proof of Theorem 1. In the first subsection, we describe the zero curvature formulation of the DNLS equation and introduce the main elements of the inverse scattering analysis following the founding paper of Kaup-Newell [18]. In the second subsection, we study the properties of the corresponding spectral problem in the case of $H^{\frac{1}{2}}$ potentials. The last subsection is devoted to the Bäcklund transformation and its basic properties. In Section 3, we establish Theorem 1. We argue by contradiction. In the subsection 3.2, combining the profile decomposition techniques with the integrability structure of the equation, we show that if the theorem fails, it would imply the existence of solutions of a very special structure. We then use the Bäcklund transformation to show that in fact, such solutions cannot exist. This is done in the subsection 3.3. There are also two appendices: the first one contains a short introduction to the regularized determinants that play an important role in Section 2, and the second one is dedicated to the proof of a technical estimate related to the Bäcklund transformation.

Throughout this article, we shall use the following convention for the Fourier transform:

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ix\xi} f(x) dx.$$

We shall designate by \mathcal{C}_n the set of bounded operators A on $L^2(\mathbb{R}, \mathbb{C}^2)$ such that $|A|^n$ is of trace-class, endowed with the norm $\|A\|_n \stackrel{\text{def}}{=} [\text{Tr}(|A|^n)]^{\frac{1}{n}}$.

Finally, we mention that the letter C will be used to denote universal constants which may vary from line to line. If we need the implied constant to depend on parameters, we shall indicate this by subscripts. We also use the notation $A \lesssim B$ to denote the bound of the form $A \leq CB$, and $A \lesssim_{\alpha} B$ for $A \leq C_{\alpha}B$, where C_{α} depends only on α . For simplicity, we shall still denote by (u_n) any subsequence of (u_n) .

2. PRELIMINARY RESULTS IN CONNEXION WITH THE INTEGRABILITY STRUCTURE OF THE DNLS EQUATION

2.1. An overview of the scattering transform. In this subsection we recall briefly some basic facts about the inverse scattering transform for the DNLS equation, limiting ourselves to the case of Schwartz class solutions. The details can be found in [1, 15, 16, 17, 18, 24, 30, 34].

As was shown by Kaup-Newell [18], the DNLS equation arises as compatibility condition of the following linear system

$$(2.1) \quad \begin{aligned} \partial_x \psi &= \mathcal{U}(\lambda) \psi, \\ \partial_t \psi &= \Upsilon(\lambda) \psi, \end{aligned}$$

⁵See also the recent paper of Klaus-Schippa [21], where this property has been used to obtain low regularity a priori estimates for small mass solutions of the DNLS equation.

with

$$\begin{aligned}\mathcal{U}(\lambda) &= -i\sigma_3(\lambda^2 + i\lambda U), \quad U = \begin{pmatrix} 0 & u \\ \bar{u} & 0 \end{pmatrix}, \\ \Upsilon(\lambda) &= -i(2\lambda^4 - \lambda^2|u|^2)\sigma_3 + \begin{pmatrix} 0 & 2\lambda^3u - \lambda|u|^2u + i\lambda u_x \\ -2\lambda^3\bar{u} + \lambda|u|^2\bar{u} + i\lambda\bar{u}_x & 0 \end{pmatrix},\end{aligned}$$

where $\lambda \in \mathbb{C}$ is a (t, x) -independent spectral parameter, ψ is a \mathbb{C}^2 -valued function of (t, x, λ) , and σ_3 the Pauli matrix given by $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Namely, u satisfies the DNLS equation if and only if

$$\frac{\partial \mathcal{U}}{\partial t} - \frac{\partial \Upsilon}{\partial x} + [\mathcal{U}, \Upsilon] = 0,$$

which is referred to in the literature as the zero curvature representation of DNLS.

The scattering transform associated with the DNLS equation is defined via the first equation of (2.1) that we rewrite in the form

$$(2.2) \quad L_u(\lambda)\psi = 0,$$

with $L_u(\lambda) = i\sigma_3\partial_x - \lambda^2 - i\lambda U$. Given $u \in \mathcal{S}(\mathbb{R})$ (for the moment we ignore the time dependence), for any $\lambda \in \mathbb{C}$ with $\text{Im } \lambda^2 \geq 0$, there are unique solutions $\psi_1^-(x, \lambda)$, $\psi_2^+(x, \lambda)$ to (2.2), the so-called Jost solutions, satisfying

$$\begin{aligned}\psi_1^-(x, \lambda) &= e^{-i\lambda^2 x} \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} + o(1) \right], \quad \text{as } x \rightarrow -\infty, \\ \psi_2^+(x, \lambda) &= e^{i\lambda^2 x} \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix} + o(1) \right], \quad \text{as } x \rightarrow +\infty.\end{aligned}$$

The solutions ψ_1^-, ψ_2^+ are holomorphic functions of λ on $\Omega_+ = \{\lambda \in \mathbb{C} : \text{Im } \lambda^2 > 0\}$, C^∞ up to the boundary. Similarly, for $\lambda \in \mathbb{C}$ with $\text{Im } \lambda^2 \leq 0$, there are unique solutions $\psi_2^-(x, \lambda)$, $\psi_1^+(x, \lambda)$ to (2.2) satisfying

$$\begin{aligned}\psi_2^-(x, \lambda) &= e^{i\lambda^2 x} \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix} + o(1) \right], \quad \text{as } x \rightarrow -\infty, \\ \psi_1^+(x, \lambda) &= e^{-i\lambda^2 x} \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} + o(1) \right], \quad \text{as } x \rightarrow +\infty.\end{aligned}$$

For $\lambda \in \mathbb{R} \cup i\mathbb{R}$, this gives two pairs of linearly independent solutions: ψ_1^-, ψ_2^- and ψ_1^+, ψ_2^+ . We denote the corresponding transfer matrix by $t_u(\lambda) = \begin{pmatrix} a_u(\lambda) & c_u(\lambda) \\ b_u(\lambda) & d_u(\lambda) \end{pmatrix}$:

$$\begin{pmatrix} \psi_1^-(x, \lambda) & \psi_2^-(x, \lambda) \end{pmatrix} = \begin{pmatrix} \psi_1^+(x, \lambda) & \psi_2^+(x, \lambda) \end{pmatrix} t_u(\lambda).$$

The functions $\frac{1}{a_u}$, $\frac{1}{d_u}$ and $\frac{b_u}{a_u}$, $\frac{c_u}{d_u}$ are called transmission and reflection coefficients respectively. Thanks to the symmetry relations

$$(2.3) \quad \begin{aligned}\psi_1^-(x, \lambda) &= \sigma_3 \psi_1^-(x, -\lambda), & \psi_2^+(x, \lambda) &= -\sigma_3 \psi_2^+(x, -\lambda), \\ \psi_1^-(x, \lambda) &= -\sigma_1 \sigma_3 \overline{\psi_2^-(x, \bar{\lambda})}, & \psi_2^+(x, \lambda) &= \sigma_1 \sigma_3 \overline{\psi_1^+(x, \bar{\lambda})},\end{aligned}$$

where $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, one has, for all $\lambda \in \mathbb{R} \cup i\mathbb{R}$,

$$\begin{aligned}a_u(\lambda) &= a_u(-\lambda), & a_u(\lambda) &= \overline{d_u(\bar{\lambda})}, \\ b_u(-\lambda) &= -b_u(\lambda), & c_u(\lambda) &= -\overline{b_u(\bar{\lambda})}.\end{aligned}$$

Since $\det t_u(\lambda) = 1$, the above relations imply that

$$(2.4) \quad |a_u(\lambda)|^2 + |b_u(\lambda)|^2 = 1, \quad \forall \lambda \in \mathbb{R},$$

$$(2.5) \quad |a_u(\lambda)|^2 - |b_u(\lambda)|^2 = 1, \quad \forall \lambda \in i\mathbb{R}.$$

Observe also that $a_u(0) = 1$.

The function a_u extends analytically to Ω_+ since it can be expressed through the Wronskian of ψ_1^- and ψ_2^+ :

$$(2.6) \quad a_u(\lambda) = \det(\psi_1^-(x, \lambda), \psi_2^+(x, \lambda)).$$

The relation (2.6) also shows that the zeros of a_u in Ω_+ coincide with the values of λ for which the system (2.2) has a non trivial L^2 solution. In this case, we say that λ is an eigenvalue of the spectral problem (2.2) (or of the operator pencil $L_u(\lambda)$). These eigenvalues give rise to the bright solitons. For one-soliton solutions (1.9), one has

$$a_{u_{E,c}}(\lambda) = e^{-\frac{i}{2}\|u_{E,c}\|_{L^2(\mathbb{R})}^2} \frac{\lambda^2 - \zeta_{E,c}}{\lambda^2 - \bar{\zeta}_{E,c}} \quad \text{with} \quad \zeta_{E,c} = -\frac{c}{4} + i\frac{\sqrt{E}}{2}.$$

To determine the behavior of a_u at infinity, it is convenient to transform the Kaup-Newell spectral problem (2.2) into a more "familiar" Zakharov-Shabat spectral problem (the spectral problem associated with the cubic NLS) which is linear with respect to the spectral parameter. This can be done by means of the following transformation [18, 29]:

$$(2.7) \quad \tilde{\psi}(x) = \exp\left(\frac{1}{2i}\sigma_3 \int_x^\infty dy |u(y)|^2\right) \begin{pmatrix} 1 & 0 \\ -\bar{u}(x) & 2i\lambda \end{pmatrix} \psi(x).$$

One can easily check that ψ is a solution of (2.2) if and only if $\tilde{\psi}$ satisfies

$$(2.8) \quad i\sigma_3 \partial_x \tilde{\psi} - Q \tilde{\psi} = \zeta \tilde{\psi}, \quad Q = \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix}, \quad \zeta = \lambda^2,$$

with

$$q(x) = \frac{1}{2}u(x) \exp\left(-i \int_x^\infty dy |u(y)|^2\right), \quad r(x) = (i\bar{u}_x + \frac{1}{2}\bar{u}|u|^2)(x) \exp\left(i \int_x^\infty dy |u(y)|^2\right).$$

The equivalence between the systems (2.2) and (2.8) shows that

$$(2.9) \quad \lim_{|\lambda| \rightarrow \infty, \lambda \in \bar{\Omega}_+} a_u(\lambda) = e^{-\frac{i}{2}\|u\|_{L^2(\mathbb{R})}^2}.$$

Furthermore, denoting $\tilde{a}_u(\zeta) = e^{\frac{i}{2}\|u\|_{L^2(\mathbb{R})}^2} a_u(\sqrt{\zeta})$, one has the following asymptotic expansion as $|\zeta| \rightarrow +\infty$, $\text{Im} \zeta \geq 0$:

$$(2.10) \quad \ln \tilde{a}_u(\zeta) = \sum_{k \geq 1} E_k(u) \zeta^{-k}.$$

The coefficients E_k are polynomial with respect to u and its derivatives and can be determined recursively. They are homogeneous with respect to the scaling $u(x) \rightarrow u_\mu(x) \stackrel{\text{def}}{=} \sqrt{\mu}u(\mu x)$, $\mu > 0$, since

$$(2.11) \quad \tilde{a}_{u_\mu}(\zeta) = \tilde{a}_u\left(\frac{\zeta}{\mu}\right).$$

The first two among them coincide, up to a constant, with the momentum and energy previously introduced by (1.4)-(1.5):

$$E_1(u) = \frac{i}{4}P(u), \quad E_2(u) = -\frac{i}{8}E(u).$$

Note that $\tilde{a}_u(\zeta)$ is holomorphic in the open upper half plane \mathbb{C}_+ , C^∞ up to the boundary and verifies, in view of (2.4)-(2.5),

$$(2.12) \quad |\tilde{a}_u(\zeta)| \geq 1 \text{ for } \zeta < 0, \quad |\tilde{a}_u(\zeta)| \leq 1 \text{ for } \zeta > 0 \quad \text{and} \quad \tilde{a}_u(0) = e^{\frac{i}{2}\|u\|_{L^2(\mathbb{R})}^2}.$$

Furthermore, one can show that $|\tilde{a}_u(\zeta)|^2 \in 1 + \mathcal{S}(\mathbb{R})$. The analyticity of \tilde{a}_u allows to express the functionals E_k in terms of the zeros of \tilde{a}_u in \mathbb{C}_+ and of its trace on \mathbb{R} (by the so-called trace formulas). In the simplest case where (i) \tilde{a}_u does not vanish⁶ on \mathbb{R}_+ and (ii) \tilde{a}_u has only simple zeros ζ_1, \dots, ζ_N in \mathbb{C}_+ , one has the formulae

$$(2.13) \quad E_k(u) = -\frac{2i}{k} \sum_{j=1}^N \text{Im} \zeta_j^k + \frac{i}{2\pi} \int_{-\infty}^{\infty} d\xi \xi^{k-1} \ln |\tilde{a}_u(\xi)|^2, \quad \forall k \in \mathbb{N}^*,$$

that follow immediately from the representation

$$(2.14) \quad \tilde{a}_u(\zeta) = \prod_{j=1}^N \left(\frac{\zeta - \zeta_j}{\zeta - \bar{\zeta}_j} \right) \exp \left(\frac{1}{2i\pi} \int_{-\infty}^{\infty} \frac{d\xi}{\xi - \zeta} \ln |\tilde{a}_u(\xi)|^2 \right), \quad \forall \zeta \in \mathbb{C}_+.$$

Taking into account (2.12) and the continuity of \tilde{a}_u with respect to u , one can also deduce from (2.14) that

$$(2.15) \quad M(u) = 4 \sum_{j=1}^N \arg(\zeta_j) - \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{d\xi}{\xi} \ln |\tilde{a}_u(\xi)|^2,$$

with, all along this paper, $\arg(\zeta) \in [0, 2\pi[$.

It is known that the properties (i), (ii) hold generically: the subset of Schwartz functions u verifying the hypothesis (i) and (ii), that we shall denote all along this article by $\mathcal{S}_{reg}(\mathbb{R})$, is dense in $\mathcal{S}(\mathbb{R})$ [3, 15, 16, 17, 24].

In the above discussion, we have suppressed the time dependence. If we now restore it, assuming that $u(t)$ is a solution of the DNLS equation, then the time dependence of the scattering coefficients $a_{u(t)}(\lambda)$ and $b_{u(t)}(\lambda)$ can be deduced from the second equation of (2.1). By straightforward computations, one finds a particular simple linear evolution system:

$$(2.16) \quad \partial_t a_{u(t)}(\lambda) = 0, \quad \partial_t b_{u(t)}(\lambda) = -4i\lambda^4 b_{u(t)}(\lambda).$$

This provides a way of solving the DNLS equation as soon as the potential u can be recovered from the scattering coefficients a_u, b_u , which can be done if a_u has no zeros in $\overline{\Omega}_+$. In this case, one can reconstruct the potential u from the reflection coefficient $\frac{b_u}{a_u}$, by solving a suitable Riemann-Hilbert problem. This procedure can be also adapted to the general case but the set of scattering data needed to reconstruct the potential becomes more intricate, see [15, 16, 17] for the details. Note also that since a_u is time-independent, the expansion (2.10) produces an infinite number of polynomial conservation laws.

The use of the inverse scattering transform is restricted to the localized data: although the assumption $u \in \mathcal{S}(\mathbb{R})$ can be weakened (see [15, 16, 17, 29, 30]), even to define the scattering data $a_u(\lambda), b_u(\lambda)$, $\lambda \in \mathbb{R} \cup i\mathbb{R}$, one needs at least $u \in L^1(\mathbb{R})$. A way to overcome this difficulty and to keep a trace of the complete integrability for H^s solutions, is to exploit the conservation of $a_u(\lambda)$, for $\lambda \in \Omega_+$, that remains well defined via (2.6) for $u \in L^2(\mathbb{R})$. As we have already mentioned above, this idea goes back to the works of Killip-Visan-Zang, Killip-Visan and Koch-Tataru [19, 20, 22] on the NLS and KdV equations, and will play a crucial role in the proof of Theorem 1.

⁶It follows from Formula (2.9) that in that case, \tilde{a}_u has only a finite number of zeros in \mathbb{C}_+ , all of them being of finite multiplicity.

2.2. Study of the function a_u for $H^{\frac{1}{2}}$ potentials. In this section, we perform a detailed analysis of the function a_u in Ω_+ for u in $H^{\frac{1}{2}}(\mathbb{R})$. A convenient way to do it is to realize the Wronskian (2.6) as a regularized Fredholm determinant.

2.2.1. *Regularized determinant realization of a_u .* Consider

$$(2.17) \quad T_u(\lambda) \stackrel{\text{def}}{=} i\lambda(\mathcal{L}_0 - \lambda^2)^{-1}U,$$

where $\mathcal{L}_0 = i\sigma_3\partial_x$ and $U = \begin{pmatrix} 0 & u \\ \bar{u} & 0 \end{pmatrix}$. For any $u \in L^2(\mathbb{R})$, T_u is an holomorphic function of λ in Ω_+ with values in \mathcal{C}_2 , and

$$(2.18) \quad \|T_u(\lambda)\|_2^2 = \frac{|\lambda|^2}{\text{Im}(\lambda^2)} \|u\|_{L^2(\mathbb{R})}^2.$$

Indeed,

$$(2.19) \quad T_u(\lambda) = i\lambda \begin{pmatrix} 0 & -(D + \lambda^2)^{-1}u \\ (D - \lambda^2)^{-1}\bar{u} & 0 \end{pmatrix},$$

with $D = -i\partial_x$, and therefore

$$\|T_u(\lambda)\|_2^2 = \frac{|\lambda|^2}{2\pi} \left[\int_{\mathbb{R}^2} dpdp' \frac{|\hat{u}(p')|^2}{|p - \lambda^2|^2} + \int_{\mathbb{R}^2} dpdp' \frac{|\hat{u}(p')|^2}{|p + \lambda^2|^2} \right],$$

which readily leads to (2.18), by virtue of the Cauchy's residue theorem.

As well, we find that the trace of $T_u^2(\lambda)$ can be written explicitly as follows:

$$(2.20) \quad \text{Tr } T_u^2(\lambda) = 2i\lambda^2 \int_{\mathbb{R}} dp \frac{|\hat{u}(p)|^2}{p + 2\lambda^2}.$$

Using the explicit kernel of the free resolvent $(\mathcal{L}_0 - \lambda^2)^{-1}$:

$$(2.21) \quad (\mathcal{L}_0 - \lambda^2)^{-1}(x, y) = \begin{cases} \begin{pmatrix} ie^{-i\lambda^2(x-y)} & 0 \\ 0 & 0 \end{pmatrix} & \text{for } x < y \\ \begin{pmatrix} 0 & 0 \\ 0 & ie^{i\lambda^2(x-y)} \end{pmatrix} & \text{for } x > y, \end{cases}$$

one can also easily check that there exists a positive constant C such that, for any $p \geq 2$, there holds

$$(2.22) \quad \|T_u(\lambda)\| \stackrel{\text{def}}{=} \|T_u(\lambda)\|_{\mathcal{L}(L^2, L^2)} \leq C \frac{|\lambda| \|u\|_{L^p(\mathbb{R})}}{(\text{Im}(\lambda^2))^{1-\frac{1}{p}}}, \quad \forall \lambda \in \Omega_+, \quad u \in L^p(\mathbb{R}).$$

The key point will be the fact that the function a_u given by (2.6) can be expressed in terms of T_u as follows⁷:

$$(2.23) \quad a_u(\lambda) = \det_2(\text{I} - T_u(\lambda)), \quad \forall \lambda \in \Omega_+, \quad u \in L^2(\mathbb{R}).$$

We also define

$$(2.24) \quad a_u^{(4)}(\lambda) \stackrel{\text{def}}{=} \det_4(\text{I} - T_u(\lambda)).$$

Similarly to a_u , the function $a_u^{(4)}$ is holomorphic on Ω_+ , for any $u \in L^2(\mathbb{R})$. Since the matrix U is anti-diagonal, the identity (A.3) implies that the two functions a_u and $a_u^{(4)}$ are connected by the following relation:

$$(2.25) \quad a_u^{(4)}(\lambda) = a_u(\lambda) \exp\left(\frac{\text{Tr } T_u^2(\lambda)}{2}\right).$$

⁷See Appendix A for the definition of the regularized determinants \det_n and their basic properties.

Below we collect some general bounds on the functions a_u and $a_u^{(4)}$ that follow directly from the corresponding properties of the regularized determinants. The first bounds can be stated as follows:

Proposition 2.1. *There exists a positive constant C such that the following estimates hold*

$$(2.26) \quad |a_u(\lambda)| \leq e^{C \frac{|\lambda|^2}{\text{Im}(\lambda^2)} \|u\|_{L^2(\mathbb{R})}^2}, \quad |a_u^{(4)}(\lambda)| \leq e^{C \frac{|\lambda|^2}{\text{Im}(\lambda^2)} \|u\|_{L^2(\mathbb{R})}^2}, \quad \forall \lambda \in \Omega_+, u \in L^2(\mathbb{R}),$$

$$(2.27) \quad |a_{u_1}(\lambda) - a_{u_2}(\lambda)| \leq C e^{C \frac{|\lambda|^2}{\text{Im}(\lambda^2)} (\|u_1\|_{L^2(\mathbb{R})}^2 + \|u_2\|_{L^2(\mathbb{R})}^2)} \frac{|\lambda|}{\sqrt{\text{Im}(\lambda^2)}} \|u_1 - u_2\|_{L^2(\mathbb{R})}, \\ \forall \lambda \in \Omega_+, u_1, u_2 \in L^2(\mathbb{R}),$$

and

$$(2.28) \quad |a_u(\lambda) - 1| + |a_u^{(4)}(\lambda) - 1| \leq C e^{C \frac{|\lambda|^4}{(\text{Im}(\lambda^2))^2} \|u\|_{L^2(\mathbb{R})}^4} \\ \times \frac{|\lambda|^2}{\sqrt{\text{Im}(\lambda^2)}} \int_{\mathbb{R}} dp |\hat{u}(p)|^2 \left(\frac{1}{|p + 2\lambda^2|^{\frac{1}{2}}} + \frac{1}{|p - 2\lambda^2|^{\frac{1}{2}}} \right), \quad \forall \lambda \in \Omega_+, u \in L^2(\mathbb{R}).$$

Proof. The two first estimates (2.26) and (2.27) readily follow from the relations (A.5), (A.7) and (2.18), (2.25). In order to establish the last estimate, we start by observing that by virtue of (2.18), (2.20) and (2.25), we have

$$(2.29) \quad |a_u(\lambda) - 1| \leq e^{C \frac{|\lambda|^2}{\text{Im}(\lambda^2)} \|u\|_{L^2(\mathbb{R})}^2} (|a_u^{(4)}(\lambda) - 1| + |\text{Tr } T_u^2(\lambda)|) \\ \lesssim e^{C \frac{|\lambda|^2}{\text{Im}(\lambda^2)} \|u\|_{L^2(\mathbb{R})}^2} \left(|a_u^{(4)}(\lambda) - 1| + \frac{|\lambda|^2}{\sqrt{\text{Im}(\lambda^2)}} \int_{\mathbb{R}} dp \frac{|\hat{u}(p)|^2}{|p + 2\lambda^2|^{\frac{1}{2}}} \right).$$

It remains to control $a_u^{(4)} - 1$. To this end, we apply (A.6), which, in view of (2.18) and (2.22), leads to the following inequality

$$(2.30) \quad |a_u^{(4)}(\lambda) - 1| \leq C e^{C \frac{|\lambda|^4}{(\text{Im}(\lambda^2))^2} \|u\|_{L^2(\mathbb{R})}^4} \|T_u^2(\lambda)\|_2^2.$$

According to (2.19), the operator $T_u^2(\lambda)$ has the form

$$T_u^2(\lambda) = \lambda^2 \begin{pmatrix} (D + \lambda^2)^{-1} u (D - \lambda^2)^{-1} \bar{u} & 0 \\ 0 & (D - \lambda^2)^{-1} \bar{u} (D + \lambda^2)^{-1} u \end{pmatrix}.$$

Then, using the explicit kernel of $(D \pm \lambda^2)^{-1}$ (see (2.21)), we get by straightforward computations

$$\|(D + \lambda^2)^{-1} u (D - \lambda^2)^{-1} \bar{u}\|_2^2 \leq \frac{1}{\text{Im } \lambda^2} \|u\|_{L^2(\mathbb{R})}^2 \|(D + 2\lambda^2)^{-1} u\|_{L^\infty(\mathbb{R})}^2.$$

Since

$$\|(D + 2\lambda^2)^{-1} u\|_{L^\infty(\mathbb{R})}^2 \leq \|(p + 2\lambda^2)^{-1/4} \hat{u}\|_{L^2(\mathbb{R})}^2 \int_{\mathbb{R}} \frac{dp}{|p + 2\lambda^2|^{\frac{3}{2}}} \\ \lesssim \frac{1}{(\text{Im } \lambda^2)^{\frac{1}{2}}} \|(p + 2\lambda^2)^{-1/4} \hat{u}\|_{L^2(\mathbb{R})}^2,$$

we infer that

$$(2.31) \quad \|(D + \lambda^2)^{-1} u (D - \lambda^2)^{-1} \bar{u}\|_2^2 \lesssim \frac{1}{(\text{Im } \lambda^2)^{\frac{3}{2}}} \|u\|_{L^2(\mathbb{R})}^2 \|(p + 2\lambda^2)^{-1/4} \hat{u}\|_{L^2(\mathbb{R})}^2.$$

Similarly, we have

$$(2.32) \quad \|(D - \lambda^2)^{-1} \bar{u} (D + \lambda^2)^{-1} u\|_2^2 \lesssim \frac{1}{(\text{Im } \lambda^2)^{\frac{3}{2}}} \|u\|_{L^2(\mathbb{R})}^2 \|(p - 2\lambda^2)^{-1/4} \hat{u}\|_{L^2(\mathbb{R})}^2.$$

Combining the two latter inequalities together with (2.29)-(2.30), we get the desired bound (2.28). \square

Invoking the asymptotic formula (2.9) together with the stability estimate (2.27), we obtain the following corollary:

Corollary 2.1. *Let u be a function in $L^2(\mathbb{R})$. Then, for any $0 < \delta < \frac{\pi}{2}$, there holds*

$$(2.33) \quad \lim_{\lambda \rightarrow 0, \lambda \in \Gamma_\delta} a_u(\lambda) = 1,$$

and

$$(2.34) \quad \lim_{|\lambda| \rightarrow \infty, \lambda \in \Gamma_\delta} a_u(\lambda) = e^{-\frac{i}{2}\|u\|_{L^2(\mathbb{R})}^2},$$

where we denote $\Gamma_\delta \stackrel{\text{def}}{=} \{\lambda \in \Omega_+ : \delta < \arg(\lambda^2) < \pi - \delta\}$.

Remark 2.1. *Observe also that the stability estimate (2.27) combined with the $H^{\frac{1}{2}}$ continuity of the DNLS flow gives the conservation of $a_{u(t)}(\lambda)$ for $H^{\frac{1}{2}}(\mathbb{R})$ -solutions of DNLS.*

Assuming that the potential u is in $H^{\frac{1}{2}}(\mathbb{R})$, or more generally in $L^2(\mathbb{R}) \cap L^4(\mathbb{R})$, one gets:

Lemma 2.1. *There exists a positive constant C such that:*

$$(2.35) \quad |a_u^{(4)}(\lambda) - 1| \leq C e^{C \frac{|\lambda|^4}{(\text{Im}(\lambda^2))^2} \|u\|_{L^2(\mathbb{R})}^4} \frac{|\lambda|^4}{(\text{Im}(\lambda^2))^3} \|u\|_{L^4(\mathbb{R})}^4, \quad \forall \lambda \in \Omega_+, \forall u \in L^2(\mathbb{R}) \cap L^4(\mathbb{R}),$$

$$(2.36) \quad |a_u(\lambda) e^{\frac{i}{2}\|u\|_{L^2(\mathbb{R})}^2} - 1| \leq C e^{C \frac{|\lambda|^4}{(\text{Im}(\lambda^2))^2} \|u\|_{L^2(\mathbb{R})}^4} \frac{|\lambda|^2}{(\text{Im}(\lambda^2))^2} \|u\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R})}^2, \quad \forall \lambda \in \Omega_+, \forall u \in H^{\frac{1}{2}}(\mathbb{R}),$$

and

$$(2.37) \quad |a_{u_1}^{(4)}(\lambda) - a_{u_2}^{(4)}(\lambda)| \leq C e^{C \frac{|\lambda|^4}{(\text{Im}(\lambda^2))^2} (\|u_1\|_{L^2(\mathbb{R})}^4 + \|u_2\|_{L^2(\mathbb{R})}^4)} \frac{|\lambda|^{\frac{1}{2}}}{(\text{Im}(\lambda^2))^{\frac{3}{8}}} \|u_1 - u_2\|_{L^4(\mathbb{R})}^{\frac{1}{2}},$$

for all u_1, u_2 in $L^2(\mathbb{R}) \cap L^4(\mathbb{R})$ and all λ in Ω_+ .

Proof. To prove the first inequality, we use (A.3) which according to the fact that the matrix U is anti-diagonal gives

$$a_u^{(4)}(\lambda) = \det_6(\mathbf{I} - T_u(\lambda)) \exp\left(-\frac{\text{Tr} T_u^4(\lambda)}{4}\right).$$

Invoking (A.6), we deduce that there is a positive constant C such that

$$|a_u^{(4)}(\lambda) - 1| \leq C e^{C \|T_u(\lambda)\|_4^4} (\|T_u(\lambda)\|_4^4 \|T_u(\lambda)\|_2^2 + |\text{Tr} T_u^4(\lambda)|).$$

Then, taking advantage of (2.18) and (2.22), we infer that, for all u in $L^2(\mathbb{R}) \cap L^4(\mathbb{R})$, there holds

$$(2.38) \quad |a_u^{(4)}(\lambda) - 1| \leq C e^{C \frac{|\lambda|^4}{(\text{Im}(\lambda^2))^2} \|u\|_{L^2(\mathbb{R})}^4} \left(\frac{|\lambda|^4}{(\text{Im}(\lambda^2))^3} \|u\|_{L^4(\mathbb{R})}^4 + |\text{Tr} T_u^4(\lambda)| \right).$$

We next compute $\text{Tr} T_u^4(\lambda)$. In view of (2.19), we have

$$T_u^4(\lambda) = \lambda^4 \begin{pmatrix} A(\lambda) & 0 \\ 0 & B(\lambda) \end{pmatrix},$$

with

$$\begin{aligned} A(\lambda) &= (D + \lambda^2)^{-1} u (D - \lambda^2)^{-1} \bar{u} (D + \lambda^2)^{-1} u (D - \lambda^2)^{-1} \bar{u} \\ B(\lambda) &= (D - \lambda^2)^{-1} \bar{u} (D + \lambda^2)^{-1} u (D - \lambda^2)^{-1} \bar{u} (D + \lambda^2)^{-1} u. \end{aligned}$$

One can easily check that

$$\begin{aligned} \operatorname{Tr} B(\lambda) = \operatorname{Tr} A(\lambda) &= \frac{1}{(2\pi)^2} \int_{\mathbb{R}^4} dp dp_1 dp_2 dp_3 \frac{\hat{u}(p-p_1) \widehat{\bar{u}}(p_1-p_2) \hat{u}(p_2-p_3) \widehat{\bar{u}}(p_3-p)}{(p+\lambda^2)(p_1-\lambda^2)(p_2+\lambda^2)(p_3-\lambda^2)} \\ &= \frac{1}{(2\pi)^2} \int_{\mathbb{R}^3} dp dp_1 dp_2 \hat{u}(p-p_1) \widehat{\bar{u}}(p_2-p_1) \hat{u}(p_2) \widehat{\bar{u}}(p) \mathcal{I}(p, p_1, p_2, \lambda^2), \end{aligned}$$

where

$$\mathcal{I}(p, p_1, p_2, \lambda^2) = \int_{\mathbb{R}} \frac{dp_3}{(p+p_3+\lambda^2)(p_1+p_3-\lambda^2)(p_2+p_3+\lambda^2)(p_3-\lambda^2)}.$$

Applying the Cauchy's residue theorem, we get

$$\mathcal{I}(p, p_1, p_2, \lambda^2) = -2i\pi \left[\frac{1}{(p_2+2\lambda^2)(p-p_1+2\lambda^2)(p_2-p_1+2\lambda^2)} + \frac{1}{(p_2+2\lambda^2)(p-p_1+2\lambda^2)(p+2\lambda^2)} \right].$$

This implies that

$$\begin{aligned} \operatorname{Tr} T_u^4(\lambda) &= -\frac{i\lambda^4}{\pi} \int_{\mathbb{R}^3} dp dp_1 dp_2 \hat{u}(p-p_1) \widehat{\bar{u}}(p_2-p_1) \hat{u}(p_2) \widehat{\bar{u}}(p) \\ &\quad \times \left[\frac{1}{(p_2+2\lambda^2)(p-p_1+2\lambda^2)(p_2-p_1+2\lambda^2)} + \frac{1}{(p+2\lambda^2)(p_2+2\lambda^2)(p-p_1+2\lambda^2)} \right] \\ &= -\frac{2i\lambda^4}{\pi} \int_{\mathbb{R}^3} dp dp_1 dp_2 \frac{\hat{u}(p-p_1) \widehat{\bar{u}}(p_2-p_1) \hat{u}(p_2) \widehat{\bar{u}}(p)}{(p+2\lambda^2)(p_2+2\lambda^2)(p-p_1+2\lambda^2)}. \end{aligned}$$

Invoking Fourier-Plancherel formula, we deduce that

$$(2.39) \quad \operatorname{Tr} T_u^4(\lambda) = 4i\lambda^4 \int_{\mathbb{R}} dx \bar{u}(x) \left((D+2\lambda^2)^{-1} u(x) \right)^2 (D-2\lambda^2)^{-1} \bar{u}(x),$$

which, thanks to (2.21), shows that

$$|\operatorname{Tr} T_u^4(\lambda)| \lesssim \frac{|\lambda|^4}{(\operatorname{Im}(\lambda^2))^3} \|u\|_{L^4(\mathbb{R})}^4.$$

According to (2.38), this concludes the proof of (2.35).

Let us now go to the proof of (2.36). For that purpose, we start by combining (2.18) together with (2.25), which implies that

$$(2.40) \quad |a_u(\lambda) e^{\frac{i}{2}\|u\|_{L^2(\mathbb{R})}^2} - 1| \leq C e^{C \frac{|\lambda|^2}{\operatorname{Im}(\lambda^2)} \|u\|_{L^2(\mathbb{R})}^2} (|a_u^{(4)}(\lambda) - 1| + |\operatorname{Tr} T_u^2(\lambda) - i\|u\|_{L^2(\mathbb{R})}^2|).$$

Since by (2.20), we have

$$(2.41) \quad \operatorname{Tr} T_u^2(\lambda) - i\|u\|_{L^2(\mathbb{R})}^2 = -i \int_{\mathbb{R}} dp \frac{p|\hat{u}(p)|^2}{p+2\lambda^2},$$

we get

$$(2.42) \quad \left| \operatorname{Tr} T_u^2(\lambda) - i\|u\|_{L^2(\mathbb{R})}^2 \right| \leq \frac{1}{2\operatorname{Im}(\lambda^2)} \|u\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R})}^2.$$

Then invoking (2.35), (2.40), (2.42) together with the interpolation inequality

$$\|u\|_{L^4(\mathbb{R})}^4 \lesssim \|u\|_{L^2(\mathbb{R})}^2 \|u\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R})}^2,$$

we readily achieve the proof of the estimate (2.36).

Finally, to establish (2.37), we apply Estimate (A.7) with $n = 4$, which gives

$$\begin{aligned} |a_{u_1}^{(4)}(\lambda) - a_{u_2}^{(4)}(\lambda)| &\leq C e^{C(\|T_{u_1}(\lambda)\|_4^4 + \|T_{u_2}(\lambda)\|_4^4)} \|T_{(u_1-u_2)}(\lambda)\|_4 \\ &\leq C e^{C(\|T_{u_1}(\lambda)\|_4^4 + \|T_{u_2}(\lambda)\|_4^4)} \|T_{(u_1-u_2)}(\lambda)\|^{1/2} \|T_{(u_1-u_2)}(\lambda)\|_2^{1/2}, \end{aligned}$$

which completes the proof of the estimate, thanks to (2.18) and (2.22). \square

We will also need the following refinement of the above estimates, that we formulate in terms of $\ln \tilde{a}_u(\zeta)$ where, as above, $\tilde{a}_u(\zeta) = e^{\frac{i}{2}\|u\|_{L^2(\mathbb{R})}^2} a_u(\sqrt{\zeta})$. First note that by virtue of (2.34), $\ln \tilde{a}_u(\zeta)$ is an holomorphic function of ζ in \mathbb{C}_+ with $\text{Im } \zeta$ sufficiently large (depending on $\arg \zeta$), uniquely defined by the condition $\log \tilde{a}_u(\zeta) = o(1)$ as $|\zeta| \rightarrow \infty$. In addition, as soon as $\|T_u(\sqrt{\zeta})\| < 1$, it can be written as a convergent series:

$$(2.43) \quad \ln \tilde{a}_u(\zeta) = \frac{i}{2}\|u\|_{L^2(\mathbb{R})}^2 - \sum_{k=2}^{\infty} \frac{\text{Tr } T_u^k(\sqrt{\zeta})}{k}.$$

Denoting

$$\Phi_{0,u}(\zeta) \stackrel{\text{def}}{=} \frac{i}{2}\|u\|_{L^2(\mathbb{R})}^2 - \frac{1}{2} \text{Tr } T_u^2(\sqrt{\zeta}) = \frac{i}{2} \int_{\mathbb{R}} dp \frac{p|\hat{u}(p)|^2}{p+2\zeta},$$

we deduce that

$$(2.44) \quad \left| \ln \tilde{a}_u(\zeta) - \Phi_{0,u}(\zeta) + \frac{\text{Tr } T_u^4(\sqrt{\zeta})}{4} \right| \leq C \|T_u(\sqrt{\zeta})\|_2^2 \|T_u(\sqrt{\zeta})\|^4,$$

provided that $\|T_u(\sqrt{\zeta})\| \leq \frac{1}{2}$.

Note also that by (2.39), for any $0 \leq s \leq 1$, there holds

$$(2.45) \quad \left| \text{Tr } T_u^4(\sqrt{\zeta}) + \frac{i}{2\zeta} \|u\|_{L^4(\mathbb{R})}^4 \right| \leq C \frac{|\zeta|^2}{(\text{Im } \zeta)^{3+s}} \|u\|_{L^4(\mathbb{R})}^3 \|u\|_{\dot{H}^{\frac{1}{4}+s}(\mathbb{R})}, \quad \forall u \in H^{\frac{1}{4}+s}(\mathbb{R}), \quad \zeta \in \mathbb{C}_+.$$

Therefore, gathering the two latter estimates and taking into account (2.18) and (2.22), we obtain:

Lemma 2.2. *There exists a positive constant κ such that, for any $0 \leq s < \frac{1}{4}$, one has:*

$$\left| \ln \tilde{a}_u(\zeta) - \Phi_{0,u}(\zeta) - \frac{i}{8\zeta} \|u\|_{L^4(\mathbb{R})}^4 \right| \leq C_s \left(1 + \frac{|\zeta|}{\text{Im } \zeta} \|u\|_{L^2(\mathbb{R})}^2 \right) \frac{|\zeta|^2}{(\text{Im } \zeta)^{3+s}} \|u\|_{L^4(\mathbb{R})}^3 \|u\|_{\dot{H}^{\frac{1}{4}+s}(\mathbb{R})},$$

for all $u \in H^{\frac{1}{4}+s}(\mathbb{R})$ and all $\zeta \in \mathbb{C}_+$ satisfying $\frac{|\zeta|}{(\text{Im } \zeta)^{3/2}} \|u\|_{L^4(\mathbb{R})}^2 \leq \kappa$, with some positive constant C_s .

2.2.2. Resolvent estimates. Consider the resolvent $L_u^{-1}(\lambda)$. For any u in $L^2(\mathbb{R})$, $L_u^{-1}(\lambda)$ is a meromorphic function of λ in Ω_+ with values in the space of bounded operators on $L^2(\mathbb{R}, \mathbb{C}^2)$, whose poles coincide with the zeros of a_u . In addition, it admits the following estimate:

Proposition 2.2. *There exists a positive constant C such that, for all u in $L^2(\mathbb{R})$ and all λ in Ω_+ , we have*

$$(2.46) \quad \|L_u^{-1}(\lambda)\| \leq \exp \left(C \frac{|\lambda|^2}{\text{Im}(\lambda^2)} \|u\|_{L^2(\mathbb{R})}^2 \right) \frac{C}{|a_u(\lambda)| \text{Im}(\lambda^2)},$$

provided that $a_u(\lambda) \neq 0$.

Proof. Taking into account that

$$L_u^{-1}(\lambda) = (\text{I} - T_u(\lambda))^{-1} (\mathcal{L}_0 - \lambda^2)^{-1},$$

the result follows immediately from Proposition A.1 (3) and Identity (2.18). \square

2.2.3. *Bounds on the number of the eigenvalues of $L_u(\lambda)$.* We start by observing that due to (2.12) and (2.15), for any $u \in \mathcal{S}_{reg}$ and any $\theta \in]0, \pi[$, we have:

$$(2.47) \quad \#\{\zeta \in \mathbb{C}_+ : \tilde{a}_u(\zeta) = 0, \theta < \arg \zeta < \pi\} \leq \frac{\|u\|_{L^2(\mathbb{R})}^2}{4\theta}.$$

Furthermore, combining the density of \mathcal{S}_{reg} in L^2 together with the stability estimate (2.27) and Corollary 2.1, we infer that this inequality remains valid for $u \in L^2$. From (2.15) we also deduce:

Lemma 2.3. *Let u be a function of $\mathcal{S}_{reg}(\mathbb{R})$ and $\theta \in]0, \pi[$ such that $\tilde{a}_u(\zeta) \neq 0$, for all ζ in \mathbb{C}_+ with $\arg \zeta = \theta$. Then*

$$(2.48) \quad \#\{\zeta \in \mathbb{C}_+ : \tilde{a}_u(\zeta) = 0, \theta < \arg \zeta < \pi\} = \frac{1}{2i\pi} \int_0^{+\infty e^{i\theta}} \frac{\tilde{a}'_u(s)}{\tilde{a}_u(s)} ds + \frac{1}{4\pi} \|u\|_{L^2(\mathbb{R})}^2,$$

where, all along this paper, $\int_0^{+\infty e^{i\theta}} ds$ denotes the integral along the path $\gamma \stackrel{\text{def}}{=} \{z = \rho e^{i\theta}, \rho \in \mathbb{R}_+\}$.

Proof. This lemma is a straightforward consequence of the analyticity of \tilde{a}_u and of the asymptotics (2.10). Indeed, given u in $\mathcal{S}_{reg}(\mathbb{R})$, denote by ζ_j , $j = 1, \dots, N$, the zeros of $\tilde{a}_u(\zeta)$ in the upper half plane and by n the number of the zeros in the angle $\{\theta < \arg \zeta < \pi\}$. Then, taking advantage of Formula (2.14), we easily get

$$\begin{aligned} \int_0^{+\infty e^{i\theta}} ds \frac{\tilde{a}'_u(s)}{\tilde{a}_u(s)} &= \sum_{j=1}^N \int_0^{+\infty e^{i\theta}} ds \left(\frac{1}{s - \zeta_j} - \frac{1}{s - \bar{\zeta}_j} \right) - \frac{1}{2i\pi} \int_{-\infty}^{\infty} \frac{d\xi}{\xi} \log |\tilde{a}_u(\xi)|^2 \\ &= 2i\pi n - 2i \sum_{j=1}^N \arg \zeta_j - \frac{1}{2i\pi} \int_{-\infty}^{\infty} \frac{d\xi}{\xi} \log |\tilde{a}_u(\xi)|^2 \\ &= 2i\pi n - \frac{i}{2} \|u\|_{L^2(\mathbb{R})}^2, \end{aligned}$$

which completes the proof of (2.48). \square

In the proof of Theorem 1, we will also need the following L^2 version of Lemma 2.3.

Lemma 2.4. *Let u be a function in L^2 and $\theta \in]0, \pi[$ such that $\tilde{a}_u(\zeta) \neq 0$, for all ζ belonging to the ray $e^{i\theta}\mathbb{R}_+$. Then,*

$$(2.49) \quad \frac{1}{2i\pi} \int_0^{+\infty e^{i\theta}} ds \frac{\tilde{a}'_u(s)}{\tilde{a}_u(s)} + \frac{1}{4\pi} \|u\|_{L^2(\mathbb{R})}^2 \in \mathbb{N}.$$

Proof. First observe that in view of Corollary 2.1, the integral $\int_0^{+\infty e^{i\theta}} ds \frac{\tilde{a}'_u(s)}{\tilde{a}_u(s)}$ makes sense. In

order to establish (2.49), we proceed by approximation: let $(u_n)_{n \in \mathbb{N}}$ be a sequence of $\mathcal{S}_{reg}(\mathbb{R})$ converging to u in L^2 . The stability estimate (2.27) together with Corollary 2.1 ensures that, for n sufficiently large, \tilde{a}_{u_n} does not vanish on the ray $e^{i\theta}\mathbb{R}_+$, and one has

$$\int_0^{+\infty e^{i\theta}} ds \frac{\tilde{a}'_{u_n}(s)}{\tilde{a}_{u_n}(s)} \xrightarrow{n \rightarrow \infty} \int_0^{+\infty e^{i\theta}} ds \frac{\tilde{a}'_u(s)}{\tilde{a}_u(s)}.$$

Since by virtue of (2.48),

$$\frac{1}{2i\pi} \int_0^{+\infty} e^{i\theta} ds \frac{\tilde{a}'_{u_n}(s)}{\tilde{a}_{u_n}(s)} + \frac{1}{4\pi} \|u\|_{L^2(\mathbb{R})}^2 \in \mathbb{N},$$

Lemma 2.4 follows by passing to the limit $n \rightarrow \infty$. \square

We next show that the real parts of the zeros of $\tilde{a}_u(\zeta)$ are low-bounded uniformly with respect to u in bounded sets of $H^{\frac{1}{2}}$.

Lemma 2.5. *Let $u \in H^{\frac{1}{2}}(\mathbb{R})$. There exists a positive constant C depending only on $\|u\|_{H^{\frac{1}{2}}(\mathbb{R})}$, such that the function $\tilde{a}_u(\zeta)$ has no zeros in the region $\{\zeta \in \mathbb{C}_+ : \operatorname{Re} \zeta \leq -C\}$.*

Proof. Let $\zeta_0 \in \mathbb{C}_+$ be a zero of \tilde{a}_u . Then, there exists a function ψ in $H^1(\mathbb{R})$, with $\|\psi\|_{L^2(\mathbb{R})} = 1$, such that

$$(2.50) \quad i\sigma_3 \partial_x \psi = \lambda_0^2 \psi + i\lambda_0 \begin{pmatrix} 0 & u \\ \bar{u} & 0 \end{pmatrix} \psi,$$

where $\lambda_0 = \sqrt{\zeta_0} \in \mathbb{C}_{++} = \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda > 0, \operatorname{Im} \lambda > 0\}$, which shows that, for all $s > 0$,

$$\|\partial_x \psi\|_{L^2(\mathbb{R})} \leq |\lambda_0|^2 + C_s |\lambda_0| \|u\|_{H^{\frac{1}{2}}(\mathbb{R})} \|\psi\|_{H^s(\mathbb{R})}.$$

From this inequality we readily deduce that

$$(2.51) \quad \|\psi\|_{H^1(\mathbb{R})} \lesssim_{\|u\|_{H^{\frac{1}{2}}(\mathbb{R})}} (1 + |\zeta_0|).$$

Furthermore, taking the imaginary part of the scalar product of the identity (2.50) with ψ , we get

$$2\operatorname{Re}(\lambda_0) \operatorname{Im}(\lambda_0) = -\operatorname{Re} \lambda_0 \left\langle \begin{pmatrix} 0 & u \\ \bar{u} & 0 \end{pmatrix} \psi, \psi \right\rangle,$$

which ensures that

$$(2.52) \quad \operatorname{Im} \lambda_0 \lesssim_s \|u\|_{H^{\frac{1}{2}}(\mathbb{R})} \|\psi\|_{H^1(\mathbb{R})}^s \lesssim_{s, \|u\|_{H^{\frac{1}{2}}(\mathbb{R})}} (1 + |\zeta_0|)^s,$$

for any $s > 0$. Consequently,

$$(2.53) \quad \operatorname{Re} \zeta_0 = (\operatorname{Re} \lambda_0)^2 - (\operatorname{Im} \lambda_0)^2 \geq -C_{s, \|u\|_{H^{\frac{1}{2}}(\mathbb{R})}} (1 + |\zeta_0|)^{2s}.$$

Since by (2.36), ζ_0 satisfies: $|\zeta_0| \lesssim_{\|u\|_{H^{\frac{1}{2}}(\mathbb{R})}} (1 + |\operatorname{Re} \zeta_0|)$, the inequality (2.53) implies that

$$\operatorname{Re} \zeta_0 \geq -C_{\|u\|_{H^{\frac{1}{2}}(\mathbb{R})}}.$$

This completes the proof of the lemma. \square

2.2.4. Some additional results. Here we derive some consequences of Lemma 2.2 that will play an important role in the proof of Theorem 1. The first one relates the $\dot{H}^{\frac{1}{2}}$ norm of the potential u to the L^1 norm of the function $\operatorname{Im} \ln \tilde{a}_u$ on the imaginary half-axis $i\mathbb{R}_+$. Denoting

$$(2.54) \quad \varphi_u(\rho) = \operatorname{Im}(\ln \tilde{a}_u(i\rho)),$$

we have

Proposition 2.3. *Let $u \in H^{\frac{1}{2}}(\mathbb{R})$. Then the function φ_u belongs to $L^1([\|u\|_{L^4(\mathbb{R})}^4/\kappa^2, +\infty[)$, where κ is the constant introduced in Lemma 2.2, and for any $R \geq \frac{1}{\kappa^2}$ the following estimate holds:*

$$(2.55) \quad \|u\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R})}^2 \lesssim_{R, \|u\|_{L^2}} \|\varphi_u\|_{L^1([R\|u\|_{L^4}^4, +\infty])} + \|u\|_{L^4(\mathbb{R})}^4.$$

Proof. By scaling argument, it is enough to prove the proposition assuming that $\|u\|_{L^4(\mathbb{R})}^4 = 1$.

Denoting $\varphi_{0,u}(\rho) = \frac{1}{2} \int_{\mathbb{R}} \frac{p^2 |\hat{u}(p)|^2}{p^2 + 4\rho^2} dp$, we deduce from Lemma 2.2 that

$$(2.56) \quad |\varphi_u(\rho) - \varphi_{0,u}(\rho)| \leq C_s \rho^{-1-s} \left(1 + \|u\|_{L^2(\mathbb{R})}^2\right) \|u\|_{\dot{H}^{\frac{1}{4}+s}(\mathbb{R})},$$

provided that $\rho \geq \frac{1}{\kappa^2}$. Observing that

$$\|\varphi_{0,u}\|_{L^1([R,+\infty])} = \frac{1}{4} \int_{\mathbb{R}} |p| |\hat{u}(p)|^2 \left(\frac{\pi}{2} - \arctan \frac{2R}{|p|}\right) dp,$$

we deduce that for any $R > 0$,

$$(2.57) \quad \|\varphi_{0,u}\|_{L^1(\mathbb{R}_+)} \lesssim \|u\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R})}^2 \lesssim_R \|u\|_{L^2(\mathbb{R})}^2 + \|\varphi_{0,u}\|_{L^1([R,+\infty])}.$$

Therefore, invoking the estimate (2.56) and integrating with respect to ρ , we infer that the function φ_u belongs to $L^1([1/\kappa^2, +\infty])$, and that we have, for any $R \geq \frac{1}{\kappa^2}$,

$$\|u\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R})}^2 \lesssim_{R, \|u\|_{L^2(\mathbb{R})}} 1 + \|\varphi_u\|_{L^1([R,+\infty])},$$

which completes the proof of the proposition. \square

We conclude this subsection by the following rigidity result concerning the zero-free case.

Lemma 2.6. *Let $u \in H^{\frac{1}{2}}(\mathbb{R})$ be such that the corresponding function \tilde{a}_u has no zeros in \mathbb{C}_+ . Then*

$$(2.58) \quad \varphi_u(\rho) \geq 0, \quad \forall \rho \geq 0.$$

If in addition, $\varphi_u(\rho_0) = 0$ for some $\rho_0 > 0$, then

$$(2.59) \quad \tilde{a}_u(\zeta) = 1, \quad \forall \zeta \in \mathbb{C}_+.$$

Proof. In order to establish the lemma, we proceed by approximation: let $(u_n)_{n \in \mathbb{N}}$ be a sequence in \mathcal{S}_{reg} that converges to u in $H^{\frac{1}{2}}(\mathbb{R})$. We denote by ζ_j^n , $j = 1, \dots, N_n$, the zeros of $\tilde{a}_{u_n}(\zeta)$ in \mathbb{C}_+ . Combining Corollary 2.1 with the fact that \tilde{a}_u does not vanish on \mathbb{C}_+ , and taking into account the stability estimate (2.27), we infer that

$$(2.60) \quad \sup_{j=1, \dots, N_n} \frac{\text{Im } \zeta_j^n}{|\text{Re } \zeta_j^n|} \xrightarrow{n \rightarrow \infty} 0.$$

Therefore, invoking (2.14), we deduce that, for n sufficiently large,

$$(2.61) \quad \begin{aligned} \varphi_{u_n}(\rho) &= \sum_{\substack{1 \leq j \leq N_n \\ \text{Re } \zeta_j^n < 0}} \text{Im} \ln \left(\frac{i\rho - \zeta_j^n}{i\rho - \bar{\zeta}_j^n} \right) + \sum_{\substack{1 \leq j \leq N_n \\ \text{Re } \zeta_j^n > 0}} \text{Im} \ln \left(\frac{i\rho - \zeta_j^n}{i\rho - \bar{\zeta}_j^n} \right) \\ &\quad - \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{d\xi}{\xi^2 + \rho^2} \xi \ln |\tilde{a}_{u_n}(\xi)|^2, \quad \forall \rho > 0. \end{aligned}$$

In view of (2.12), we have

$$-\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{d\xi}{\xi^2 + \rho^2} \xi \ln |\tilde{a}_{u(n)}(\xi)|^2 \geq 0.$$

Note also that, by virtue of (2.60), $\operatorname{Im} \ln \left(\frac{i\rho - \zeta_j^n}{i\rho - \bar{\zeta}_j^n} \right)$ has the same sign as $\operatorname{Re} \zeta_j^n$ and

$$\sup_{\substack{j=1, \dots, N_n, \\ \rho > 0}} \left| \operatorname{Im} \ln \left(\frac{i\rho - \zeta_j^n}{i\rho - \bar{\zeta}_j^n} \right) \right| \xrightarrow{n \rightarrow \infty} 0.$$

Then, taking into account the bound

$$(2.62) \quad \#\{\zeta_j^n, \operatorname{Re}(\zeta_j^n) < 0\} \lesssim \|u^{(n)}\|_{L^2(\mathbb{R})}^2 \lesssim \|u\|_{L^2(\mathbb{R})}^2,$$

that readily follows from (2.47), we conclude that for any $\rho > 0$,

$$(2.63) \quad \varphi_{u_n}(\rho) = \underbrace{\sum_{\substack{1 \leq j \leq N_n \\ \operatorname{Re} \zeta_j^n > 0}} \operatorname{Im} \ln \left(\frac{i\rho - \zeta_j^n}{i\rho - \bar{\zeta}_j^n} \right)}_{\geq 0} - \underbrace{\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{d\xi}{\xi^2 + \rho^2} \xi \ln |\tilde{a}_{u_n}(\xi)|^2}_{\geq 0} + o(1), \quad n \rightarrow +\infty,$$

which gives (2.58) after passing to the limit $n \rightarrow +\infty$.

Assume now that $\varphi_u(\rho_0) = 0$, for some $\rho_0 > 0$. It follows then from (2.63) that

$$(2.64) \quad \int_{-\infty}^{\infty} \frac{d\xi}{\xi^2 + \rho_0^2} \xi \ln |\tilde{a}_{u_n}(\xi)|^2 \xrightarrow{n \rightarrow +\infty} 0,$$

and

$$(2.65) \quad \sum_{\substack{1 \leq j \leq N_n \\ \operatorname{Re}(\zeta_j^n) > 0}} \operatorname{Im} \ln \left(\frac{i\rho_0 - \zeta_j^n}{i\rho_0 - \bar{\zeta}_j^n} \right) \xrightarrow{n \rightarrow +\infty} 0.$$

Recall that

$$\|u_n\|_{L^2(\mathbb{R})}^2 = 4 \sum_{j=1}^{N_n} \arg(\zeta_j^n) - \underbrace{\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{d\xi}{\xi} \ln |\tilde{a}_{u_n}(\xi)|^2}_{\geq 0},$$

which, thanks to (2.60), implies that, for all n sufficiently large:

$$(2.66) \quad - \int_{-\infty}^{\infty} \frac{d\xi}{\xi} \ln |\tilde{a}_{u_n}(\xi)|^2 \lesssim \|u\|_{L^2(\mathbb{R})}^2,$$

and

$$(2.67) \quad \sum_{\substack{1 \leq j \leq N_n \\ \operatorname{Re} \zeta_j^n > 0}} \frac{\operatorname{Im} \zeta_j^n}{\operatorname{Re} \zeta_j^n} \lesssim \|u\|_{L^2(\mathbb{R})}^2.$$

Invoking (2.66) and applying Cauchy-Schwarz inequality, we readily gather that

$$\left| \int_{-\infty}^{\infty} \frac{d\xi}{\xi - i\rho} \ln |\tilde{a}_{u_n}(\xi)|^2 \right|^2 \lesssim \|u\|_{L^2(\mathbb{R})}^2 \left(\sup_{\xi \in \mathbb{R}} \frac{\xi^2 + \rho_0^2}{\xi^2 + \rho^2} \right) \int_{-\infty}^{\infty} \frac{d\xi}{\xi^2 + \rho^2} \xi \ln |\tilde{a}_{u_n}(\xi)|^2,$$

which according to (2.64) ensures that

$$(2.68) \quad \int_{-\infty}^{\infty} \frac{d\xi}{\xi - i\rho} \ln |\tilde{a}_{u_n}(\xi)|^2 \xrightarrow{n \rightarrow +\infty} 0, \quad \forall \rho > 0.$$

Applying again Cauchy-Schwarz inequality, we easily get, for any $\rho > 0$,

$$\sum_{\substack{1 \leq j \leq N_n \\ \operatorname{Re} \zeta_j^n > 0}} \frac{\operatorname{Im} \zeta_j^n}{|i\rho - \bar{\zeta}_j^n|} \lesssim_\rho \left(\sum_{\substack{1 \leq j \leq N_n \\ \operatorname{Re} \zeta_j^n > 0}} \frac{\operatorname{Im} \zeta_j^n}{\operatorname{Re} \zeta_j^n} \right)^{1/2} \left(\sum_{\substack{1 \leq j \leq N_n \\ \operatorname{Re} \zeta_j^n > 0}} \frac{\operatorname{Im} \zeta_j^n \operatorname{Re} \zeta_j^n}{|i\rho - \bar{\zeta}_j^n|^2} \right)^{1/2}.$$

Combining (2.60), (2.65) together with (2.67), we infer that for all $\rho > 0$,

$$(2.69) \quad \sum_{\substack{1 \leq j \leq N_n \\ \operatorname{Re} \zeta_j^n > 0}} \frac{\operatorname{Im} \zeta_j^n}{|i\rho - \bar{\zeta}_j^n|} \lesssim_\rho \left(\sum_{\substack{1 \leq j \leq N_n \\ \operatorname{Re} \zeta_j^n > 0}} \frac{\operatorname{Im} \zeta_j^n}{\operatorname{Re} \zeta_j^n} \right)^{1/2} \left(\sum_{\substack{1 \leq j \leq N_n \\ \operatorname{Re} \zeta_j^n > 0}} \operatorname{Im} \ln \left(\frac{i\rho_0 - \zeta_j^n}{i\rho_0 - \bar{\zeta}_j^n} \right) \right)^{1/2} \xrightarrow{n \rightarrow +\infty} 0.$$

Finally, in view of (2.60) and (2.62), we have

$$(2.70) \quad \sum_{\substack{1 \leq j \leq N_n \\ \operatorname{Re} \zeta_j^n < 0}} \frac{\operatorname{Im} \zeta_j^n}{|i\rho - \bar{\zeta}_j^n|} \leq \sum_{\substack{1 \leq j \leq N_n \\ \operatorname{Re} \zeta_j^n < 0}} \frac{\operatorname{Im} \zeta_j^n}{|\operatorname{Re} \zeta_j^n|} \xrightarrow{n \rightarrow +\infty} 0.$$

As an immediate consequence of the latter estimates (2.68)-(2.70), we obtain according to (2.14) that $\tilde{a}_{u_n}(i\rho) \xrightarrow{n \rightarrow +\infty} 1$, for any $\rho > 0$. Therefore, $\tilde{a}_u \equiv 1$ on $i\mathbb{R}_+$. The analyticity of \tilde{a}_u ensures then that $\tilde{a}_u \equiv 1$ on \mathbb{C}_+ . \square

Remark 2.2. *It follows from Corollary 2.1, that $\tilde{a}_u \equiv 1$ on \mathbb{C}_+ implies $\|u\|_{L^2}^2 \in 4\pi\mathbb{N}$. Let us also mention that the set of potentials $u \in H^{\frac{1}{2}}(\mathbb{R})$ verifying (2.59) is not trivial: it contains the algebraic solitons $u_{0,c}$. We shall denote this set by \mathcal{A} .*

2.3. Bäcklund transformation. In this paragraph, we introduce the Bäcklund transformation for the Kaup-Newell spectral problem (2.2) in the form needed for the proof of Theorem 1, following closely [29] (see also [14] and the references therein). Given $u \in \mathcal{S}(\mathbb{R})$, $\lambda \in \mathbb{C}_{++}$ and $\eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}$ a non zero smooth solution of the Kaup-Newell spectral problem $L_u(\lambda)\eta = 0$, one defines the Bäcklund transformation $\mathcal{B}_\lambda(\eta)$ by

$$(2.71) \quad \mathcal{B}_\lambda(\eta)u \stackrel{\text{def}}{=} G_\lambda(\eta) \left[G_\lambda(\eta)u - \mathcal{S}_\lambda(\eta) \right],$$

where

$$(2.72) \quad G_\lambda(\eta) = \frac{d_{\bar{\lambda}}(\eta)}{d_\lambda(\eta)}, \quad \mathcal{S}_\lambda(\eta) = 2i(\lambda^2 - \bar{\lambda}^2) \frac{\eta_1 \bar{\eta}_2}{d_\lambda(\eta)},$$

with $d_\lambda(\eta) = \lambda|\eta_1|^2 + \bar{\lambda}|\eta_2|^2$. Since η depends implicitly of u , the transformation (2.71) is nonlinear with respect to the function u . One can easily check that $\mathcal{B}_\lambda(\eta)u \in \mathcal{S}(\mathbb{R})$. Observe also that

$$(2.73) \quad |G_\lambda(\eta)| = 1 \quad \text{and} \quad |\mathcal{S}_\lambda(\eta)| \leq 4 \operatorname{Im} \lambda.$$

Moreover, by straightforward computations, one can check that (see Appendix B for the proof)

$$(2.74) \quad \left| \frac{d}{dx} G_\lambda(\eta)(x) \right| \leq 8(\operatorname{Im} \lambda)^2 + 4 \operatorname{Im}(\lambda) |u(x)|.$$

The key property of the Bäcklund transformation (2.71) is that it allows to add or to remove eigenvalues of the Kaup-Newell spectral problem without changing the scattering coefficient $b_u(\lambda)$. In particular, assume that $a_u(\lambda)$ has a simple zero $\lambda_1 \in \mathbb{C}_{++}$ and let $\eta \in L^2(\mathbb{R}, \mathbb{C}^2) \setminus \{0\}$ be the

corresponding eigenfunction: $L_u(\lambda_1)\eta = 0$. Then the scattering coefficients associated to the potential $u^{(1)} \stackrel{\text{def}}{=} \mathcal{B}_{\lambda_1}(\eta)u$ are given by (see for instance [14, 29])⁸

$$(2.75) \quad a_{u^{(1)}}(\lambda) = a_u(\lambda) \frac{\lambda_1^2}{\lambda_1} \frac{\lambda^2 - \bar{\lambda}_1^2}{\lambda^2 - \lambda_1^2}, \quad b_{u^{(1)}}(\lambda) = b_u(\lambda).$$

Thus, $a_{u^{(1)}}$ does not vanish at $\pm\lambda_1$.

3. PROOF OF THE MAIN THEOREM

3.1. Strategy of proof. We start by a brief overview of the main ideas involved in the proof of Theorem 1. By the local well-posedness result of Takaoka, Theorem 1 amounts to showing that any $H^{\frac{1}{2}}(\mathbb{R})$ solution u of the DNLS equation, defined on a time interval I , satisfies

$$(3.1) \quad \sup_{t \in I} \|u(t)\|_{H^{\frac{1}{2}}(\mathbb{R})} < +\infty.$$

To establish Property (3.1), we combine the integrability structure of DNLS with the profile decomposition techniques, proceeding by contradiction. Namely, assuming that there exists u_0 in $H^{\frac{1}{2}}(\mathbb{R})$ generating a solution $u \in C([0, T[, H^{\frac{1}{2}}(\mathbb{R}))$ of the DNLS equation that verifies

$$\sup_{0 \leq t < T} \|u(t)\|_{H^{\frac{1}{2}}(\mathbb{R})} = +\infty,$$

we take a sequence $(t_n)_{n \in \mathbb{N}} \subset [0, T[$ such that $\|u(t_n)\|_{H^{\frac{1}{2}}(\mathbb{R})} \rightarrow +\infty$, as n goes to infinity. Then, setting

$$U_n(x) = \frac{1}{\sqrt{\mu_n}} u(t_n, \frac{x}{\mu_n}) \quad \text{with} \quad \mu_n = \|u(t_n)\|_{H^{\frac{1}{2}}(\mathbb{R})}^2,$$

we start by analyzing the profile decomposition of the sequence (U_n) with respect to the Sobolev embedding $H^{\frac{1}{2}}(\mathbb{R}) \hookrightarrow L^p(\mathbb{R})$, for $2 < p < \infty$. Using the conservation of a_u , we show that this decomposition contains at least one non-zero profile, that the number of profiles is bounded by $\frac{\|u_0\|_{L^2(\mathbb{R})}^2}{4\pi}$, and that all of them belong to the set \mathcal{A} . This rigidity property is established in Section 3.2 and relies heavily on the results of Section 2.2. With such a decomposition at hand, we then show, making use of the Bäcklund transformation and of Lemmas 2.3, 2.4, that up to a subsequence and a suitable regularization, the function a_{U_n} admits a zero z_n satisfying $\text{Re}(z_n^2) = c_0 < 0$. To conclude the proof, it remains to invoke the scaling property (2.11) that implies that $a_u(z_n \sqrt{\mu_n}) = 0$, which is in contradiction with our assumption $\|u(t_n)\|_{H^{\frac{1}{2}}(\mathbb{R})} \xrightarrow{n \rightarrow \infty} +\infty$, since, by Lemma 2.5, any zero z of a_u satisfies $\text{Re}(z^2) \geq -C$, for some positive constant C depending only on $\|u_0\|_{H^{\frac{1}{2}}(\mathbb{R})}$.

3.2. Rigidity type results.

3.2.1. Profile decompositions. The first step in the proof of Theorem 1 consists in establishing the following profile decomposition for solutions violating the bound (3.1), assuming that such solutions exist.

Theorem 2. *Assume that for some $u_0 \in H^{\frac{1}{2}}(\mathbb{R}) \setminus \{0\}$, the solution $u \in C([0, T^*[, H^{\frac{1}{2}}(\mathbb{R}))$ of the DNLS equation with initial data $u(0) = u_0$ verifies $\sup_{0 \leq t < T^*} \|u(t)\|_{H^{\frac{1}{2}}(\mathbb{R})} = +\infty$. Let $(t_n)_{n \in \mathbb{N}}$ be such that $\|u(t_n)\|_{H^{\frac{1}{2}}(\mathbb{R})} \xrightarrow{n \rightarrow \infty} +\infty$, and set $U_n(x) = \frac{1}{\sqrt{\mu_n}} u(t_n, \frac{x}{\mu_n})$ with $\mu_n = \|u(t_n)\|_{H^{\frac{1}{2}}(\mathbb{R})}^2$. Then*

⁸According to (2.15), we thus have $\tilde{a}_{u^{(1)}}(\zeta) = \tilde{a}_u(\zeta) \frac{\zeta - \bar{\zeta}_1}{\zeta - \zeta_1}$.

there exist an integer⁹ $1 \leq L_0 \leq \frac{\|u_0\|_{L^2(\mathbb{R})}^2}{4\pi}$, a family of functions $(V^{(\ell)})_{1 \leq \ell \leq L_0}$ in $\mathcal{A} \setminus \{0\}$ and a family of orthogonal cores¹⁰ $(\underline{y}^{(\ell)})_{\ell \geq 1}$, in the sense that for all $\ell \neq \ell'$, we have $|y_n^{(\ell)} - y_n^{(\ell')}| \xrightarrow{n \rightarrow \infty} \infty$, such that, up to a subsequence,

$$U_n(y) = \sum_{\ell=1}^{L_0} V^{(\ell)}(y - y_n^{(\ell)}) + r_n(y),$$

where

$$\lim_{n \rightarrow \infty} \|r_n\|_{L^p(\mathbb{R})} = 0,$$

for all $2 < p < \infty$.

Furthermore,

$$(3.2) \quad \|\chi(D)U_n\|_{L^2(\mathbb{R})}^2 = \sum_{\ell=1}^{L_0} \|\chi(D)V^{(\ell)}\|_{L^2(\mathbb{R})}^2 + \|\chi(D)r_n\|_{L^2(\mathbb{R})}^2 + o(1), \quad n \rightarrow \infty,$$

for any function $\chi \in \langle p \rangle^{1/2} L^\infty(\mathbb{R})$.

We begin the proof of Theorem 2 with the following proposition.

Proposition 3.1. *With the previous notations, there exist a sequence of profiles $(V^{(\ell)})_{\ell \geq 1}$ in $H^{\frac{1}{2}}(\mathbb{R})$ which are not all zero and may be regarded as ordered by decreasing $H^{\frac{1}{2}}$ norm, and a sequence of orthogonal cores $(\underline{y}^{(\ell)})_{\ell \geq 1}$ such that, up to a subsequence extraction, we have for all $L \geq 1$,*

$$(3.3) \quad U_n(y) = \sum_{\ell=1}^L V^{(\ell)}(y - y_n^{(\ell)}) + r_n^L(y),$$

where

$$(3.4) \quad \limsup_{n \rightarrow \infty} \|r_n^L\|_{L^p(\mathbb{R})} \xrightarrow{L \rightarrow \infty} 0,$$

for all $2 < p < \infty$. In addition,

$$(3.5) \quad \|\chi(D)U_n\|_{L^2(\mathbb{R})}^2 = \sum_{\ell=1}^L \|\chi(D)V^{(\ell)}\|_{L^2(\mathbb{R})}^2 + \|\chi(D)r_n^L\|_{L^2(\mathbb{R})}^2 + o(1), \quad n \rightarrow \infty,$$

for any $\chi \in \langle p \rangle^{1/2} L^\infty(\mathbb{R})$ and any $L \geq 1$.

Proof. Since the sequence $(U_n)_{n \in \mathbb{N}}$ is bounded in $H^{\frac{1}{2}}(\mathbb{R})$, following the work of P. Gérard [9], (see Proposition 4, [9]), we infer that there exist a sequence of profiles $(V^{(\ell)})_{\ell \geq 1}$ in $H^{\frac{1}{2}}(\mathbb{R})$ and a sequence of orthogonal cores $(\underline{y}^{(\ell)})_{\ell \geq 1}$ such that, up to a subsequence extraction, the properties (3.3)-(3.5) are satisfied. To conclude the proof of the proposition, it remains to show that the profile decomposition (3.3) includes at least one profile $V^{(\ell)} \neq 0$. To this end, we will use Proposition 2.3. Since $\mu_n \xrightarrow{n \rightarrow \infty} +\infty$, combining the estimate (2.55) with the conservation of a_u , we deduce that

$$\|u(t_n)\|_{L^4(\mathbb{R})}^4 \geq c\mu_n \quad \forall n \in \mathbb{N},$$

for some positive constant c depending on the initial data. Therefore, $\|U_n\|_{L^4(\mathbb{R})}^4 \geq c$, which ensures the existence of at least one non zero profile. \square

⁹In particular, $\|u_0\|_{L^2(\mathbb{R})}^2 \geq 4\pi$.

¹⁰Following the terminology of Patrick Gérard in [9], we designate by a core $\underline{y}^{(\ell)}$ any real sequence $(y_n^{(\ell)})_{n \in \mathbb{N}}$.

Remark 3.1. Note that thanks to (3.5), we have

$$\sum_{\ell=1}^{\infty} \|V^{(\ell)}\|_{L^2(\mathbb{R})}^2 \leq \|u_0\|_{L^2(\mathbb{R})}^2,$$

and

$$\sum_{\ell=1}^{\infty} \|V^{(\ell)}\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R})}^2 \leq 1.$$

3.2.2. *Factorization of a_u .* Proposition 3.1 reduces the proof of Theorem 2 to showing that for all ℓ , $V^{(\ell)} \in \mathcal{A}$, which, in view of Remarks 2.2, 3.1 will also imply that the number of (non-zero) profiles in (3.3) is bounded by $\frac{\|u_0\|_{L^2(\mathbb{R})}^2}{4\pi}$. To prove this property, we will need the following structural result for a_{U_n} .

Proposition 3.2. *With the notations of Proposition 3.1, we have for all $0 < \delta < \frac{\pi}{2}$,*

$$(3.6) \quad \limsup_{n \rightarrow \infty} |a_{U_n}^{(4)}(\lambda) - \prod_{\ell=1}^L a_{V^{(\ell)}}^{(4)}(\lambda)| \xrightarrow{L \rightarrow \infty} 0,$$

uniformly with respect to $\lambda \in \Gamma_\delta$, $\text{Im}(\lambda^2) \geq \delta$.

Proof. Setting $U_n^L(y) = \sum_{\ell=1}^L V^{(\ell)}(y - y_n^{(\ell)})$ and applying (2.37), we get for all $L \geq 1$ and all n sufficiently large,

$$(3.7) \quad |a_{U_n^L}^{(4)}(\lambda) - a_{U_n}^{(4)}(\lambda)| \lesssim \|u_0\|_{L^2, \delta} \|r_n^L\|_{L^4(\mathbb{R})}^{\frac{1}{2}}.$$

Furthermore, (2.26) and (A.9) ensure that for all L and all n ,

$$(3.8) \quad |a_{U_n^L}^{(4)}(\lambda) - \prod_{\ell=1}^L a_{V^{(\ell)}}^{(4)}(\lambda)| \lesssim \|u_0\|_{L^2, \delta, L} \sum_{\substack{1 \leq \ell, \ell' \leq L \\ \ell \neq \ell'}} \|T_{V^{(\ell)}(\cdot - y_n^\ell)}(\lambda) T_{V^{(\ell')}(\cdot - y_n^{\ell'})}(\lambda)\|.$$

Observing that, for all f, g in $L^2(\mathbb{R})$ and all λ in \mathbb{C}_{++} , there holds

$$\|T_f(\lambda) T_g(\lambda)\|^2 \leq C \frac{|\lambda|^4}{(\text{Im}(\lambda^2))^2} \int_{\mathbb{R}^2} e^{-2 \text{Im}(\lambda^2)|x-y|} |f(x)|^2 |g(y)|^2 dx dy,$$

we readily gather that

$$(3.9) \quad \|T_{V^{(\ell)}(\cdot - y_n^{(\ell)})}(\lambda) T_{V^{(\ell')}(\cdot - y_n^{(\ell')})}(\lambda)\| \longrightarrow 0 \quad \text{as } |y_n^{(\ell)} - y_n^{(\ell')}| \rightarrow \infty,$$

uniformly with respect to $\lambda \in \Gamma_\delta$, $\text{Im}(\lambda^2) \geq \delta$. Invoking (3.7), (3.8), and taking into account (3.4), we get (3.6). \square

As a corollary of the above proposition, we obtain¹¹ :

Corollary 3.1. *There exists a positive constant $C_{\|u_0\|_{L^2}}$ such that*

$$(3.10) \quad \limsup_{n \rightarrow \infty} \left| \sum_{\ell=1}^L \varphi_{V^{(\ell)}}(\rho) + \varphi_{0, r_n^L}(\rho) \right| \xrightarrow{L \rightarrow \infty} 0$$

for any $\rho \geq C_{\|u_0\|_{L^2}}$.

¹¹using the notations of page 14

Proof. For any $\rho \geq C_{\|u_0\|_{L^2}}$, with a suitable constant $C_{\|u_0\|_{L^2}}$, we can write

$$\varphi_{U_n}(\rho) = \operatorname{Im} \left(\ln a_{U_n}^{(4)}(\sqrt{i\rho}) \right) + \varphi_{0,U_n}(\rho).$$

Invoking Proposition 3.2 together with (2.35) and Remark 3.1, we infer that for any $\rho \geq C_{\|u_0\|_{L^2}}$,

$$(3.11) \quad \limsup_{n \rightarrow \infty} \left| \ln a_{U_n}^{(4)}(\sqrt{i\rho}) - \sum_{\ell=1}^L \ln a_{V^{(\ell)}}^{(4)}(\sqrt{i\rho}) \right| \xrightarrow{L \rightarrow \infty} 0.$$

Furthermore, it follows from (3.5) that for all $L \geq 1$ and all $\rho > 0$,

$$(3.12) \quad \varphi_{0,U_n}(\rho) = \frac{1}{2} \int_{\mathbb{R}} \frac{p^2 |\hat{U}_n(p)|^2}{p^2 + 4\rho^2} dp = \sum_{\ell=1}^L \varphi_{0,V^{(\ell)}}(\rho) + \varphi_{0,r_n^L}(\rho) + o(1), \quad n \rightarrow \infty.$$

Observe also that due to the scaling property (2.11) and the bound (2.36), we have

$$(3.13) \quad \varphi_{U_n}(\rho) \xrightarrow{n \rightarrow \infty} 0, \quad \forall \rho > 0,$$

which together with (3.11) and (3.12) gives (3.10). \square

3.2.3. End of the proof of the rigidity type theorem. Here we complete the proof of Theorem 2. This will be done by combining Lemma 2.6 and Corollary 3.1. In order to apply Lemma 2.6, we first need to check that:

Lemma 3.1. *For each profile $V^{(\ell)}$ involved in the decomposition (3.3), the spectral coefficient $a_{V^{(\ell)}}$ does not vanish on \mathbb{C}_{++} .*

Proof. We proceed by contradiction, assuming that there exist $\ell_0 \geq 1$, $\lambda_0 \in \mathbb{C}_{++}$ and $\psi_0 \in H^1(\mathbb{R})$ such that $\|\psi_0\|_{L^2(\mathbb{R})} = 1$ and

$$(3.14) \quad L_{V^{(\ell_0)}}(\lambda_0)\psi_0 = 0.$$

Then we have

$$(3.15) \quad L_{U_n}(\lambda_0)\psi_0(\cdot - y_n^{(\ell_0)}) = \mathcal{R}_n(y),$$

where

$$\mathcal{R}_n(y) = -i\lambda_0 \begin{pmatrix} 0 & \sum_{\substack{\ell \neq \ell_0 \\ 1 \leq \ell \leq L}} V^{(\ell)}(y - y_n^{(\ell)}) + r_n^L(y) \\ \sum_{\substack{\ell \neq \ell_0 \\ 1 \leq \ell \leq L}} V^{(\ell)}(y - y_n^{(\ell)}) + r_n^L(y) & 0 \end{pmatrix} \psi_0(y - y_n^{(\ell_0)}).$$

The scaling property (2.11) and the estimate (2.35) ensure that $|a_{U_n}(\lambda_0)| \geq \frac{1}{2}$, for n large enough. It then follows from Proposition 2.2 that the operator $L_{U_n}(\lambda_0)$ is invertible and

$$\|L_{U_n}^{-1}(\lambda_0)\| \leq C_{\lambda_0, \|u_0\|_{L^2}},$$

which implies that

$$(3.16) \quad \|\psi_0\|_{L^2(\mathbb{R})} \leq C_{\lambda_0, \|u_0\|_{L^2}} \|\mathcal{R}_n\|_{L^2(\mathbb{R})}.$$

Consider \mathcal{R}_n . The orthogonality condition between the cores ensures that, for all $\ell \neq \ell_0$, there holds

$$\|V^{(\ell)}(\cdot - y_n^{(\ell)})\psi_0(\cdot - y_n^{(\ell_0)})\|_{L^2(\mathbb{R})} \xrightarrow{n \rightarrow +\infty} 0,$$

which together with the fact that, for all $2 < p < \infty$, $\limsup_{n \rightarrow \infty} \|r_n^L\|_{L^p(\mathbb{R})} \xrightarrow{L \rightarrow \infty} 0$ allows us to conclude that

$$(3.17) \quad \|\mathcal{R}_n\|_{L^2(\mathbb{R})} \xrightarrow{n \rightarrow +\infty} 0.$$

Combining (3.16), (3.17) and taking into account the fact that $\|\psi_0\|_{L^2(\mathbb{R})} = 1$, we get a contradiction. \square

We are now in position to finish the proof of Theorem 2. From Lemmas 2.6 and 3.1, we have for all $\ell \geq 1$,

$$\varphi_{V^{(\ell)}}(\rho) \geq 0, \quad \forall \rho > 0.$$

Recalling that

$$\varphi_{0,r_n^L}(\rho) = \frac{1}{2} \int_{\mathbb{R}} \frac{p^2 |\hat{r}_n^L(p)|^2}{p^2 + 4\rho^2} dp \geq 0,$$

we deduce from Corollary 3.1 that for all $\rho \geq C_{\|u_0\|_{L^2}}$,

$$(3.18) \quad \varphi_{V^{(\ell)}}(\rho) = 0, \quad \forall \ell \geq 1,$$

and

$$(3.19) \quad \limsup_{n \rightarrow \infty} \varphi_{0,r_n^L}(\rho) \xrightarrow{L \rightarrow \infty} 0.$$

In view of Lemma 2.6, Identity (3.18) implies that $a_{V^{(\ell)}} \equiv 1$ on \mathbb{C}_{++} for all $\ell \geq 1$. Accordingly to Remarks 2.2 and 3.1, this ensures that the number of non zero profiles in the decomposition (3.3) is finite and bounded by $\frac{\|u_0\|_{L^2(\mathbb{R})}^2}{4\pi}$. Denoting this number by L_0 and setting $r_n = r_n^{L_0}$, we get

$$\lim_{n \rightarrow \infty} \|r_n\|_{L^p(\mathbb{R})} = 0,$$

for all $2 < p < \infty$, which concludes the proof of Theorem 2.

3.2.4. A key result. Our aim now is to show that the profile decomposition given by Theorem 2 is in contradiction with the conservation of a_u . To this end, we will need the following result that ensures the closeness of the functions a_{U_n} and a_{r_n} .

Proposition 3.3. *Let $(V^{(\ell)})_{1 \leq \ell \leq L}$ be a finite family of functions in \mathcal{A} . For $\underline{\varphi} = (\varphi_\ell)$, $\underline{y} = (y_\ell)$ in \mathbb{R}^L , we denote $u_{\underline{\varphi}, \underline{y}}(y) = \sum_{\ell=1}^L e^{i\varphi_\ell} V^{(\ell)}(y - y_\ell)$. Then, for all $0 < \delta < \frac{\pi}{2}$ and all $m > 0$, we have,*

$$(3.20) \quad a_{u_{\underline{\varphi}, \underline{y}}+r}(\lambda) - a_r(\lambda) \longrightarrow 0,$$

as $\min_{\ell \neq \ell'} |y_\ell - y_{\ell'}| \rightarrow \infty$ and $\|r\|_{L^4} \rightarrow 0$, $r \in L^2(\mathbb{R}) \cap L^4(\mathbb{R})$ with $\|r\|_{L^2} \leq m$, uniformly with respect to $\lambda \in \Gamma_\delta$ and $\underline{\varphi} \in \mathbb{R}^L$.

Proof. In order to establish the result, we shall consider separately the cases $|\lambda| \ll 1$ and $|\lambda| \gtrsim 1$, reducing the proof of Proposition 3.3 to the two following lemmas:

Lemma 3.2. *Under the assumptions of Proposition 3.3, for all $0 < \delta < \frac{\pi}{2}$ and all $m > 0$, we have*

$$a_{u_{\underline{\varphi}, \underline{y}}+r}(\lambda) - a_r(\lambda) \xrightarrow{\lambda \rightarrow 0, \lambda \in \Gamma_\delta} 0$$

uniformly with respect to $\underline{\varphi}, \underline{y} \in \mathbb{R}^L$ and $r \in L^2(\mathbb{R})$ satisfying $\|r\|_{L^2(\mathbb{R})}^2 \leq m$.

Lemma 3.3. For all $0 < \delta < \frac{\pi}{2}$ and all $\alpha > 0$, we have

$$a_{u_{\underline{\varphi}, \underline{y}}+r}(\lambda) - a_r(\lambda) \longrightarrow 0,$$

as $\min_{\ell \neq \ell'} |y_\ell - y_{\ell'}| \rightarrow \infty$ and $\|r\|_{L^4(\mathbb{R})} \rightarrow 0$ with $\|r\|_{L^2(\mathbb{R})}^2 \leq m$, uniformly with respect to $\underline{\varphi}$ in \mathbb{R}^L and λ in $\Gamma_\delta \cap \{|\lambda| \geq \alpha\}$.

We start with the proof of the first lemma:

Proof of Lemma 3.2. First, recall that in view of (2.20), (2.25) and (2.26), we have

$$|a_{u_{\underline{\varphi}, \underline{y}}+r}(\lambda) - a_r(\lambda)| \lesssim_{\delta, m} |\operatorname{Tr} T_{u_{\underline{\varphi}, \underline{y}}+r}^2(\lambda) - \operatorname{Tr} T_r^2(\lambda)| + |a_{u_{\underline{\varphi}, \underline{y}}+r}^{(4)}(\lambda) - a_r^{(4)}(\lambda)|.$$

Then, note that it follows from (2.20) that, for all $\lambda \in \Gamma_\delta$, all $\underline{\varphi}, \underline{y} \in \mathbb{R}^L$ and $r \in L^2(\mathbb{R})$ with $\|r\|_{L^2(\mathbb{R})}^2 \leq m$, there holds

$$|\operatorname{Tr} T_{u_{\underline{\varphi}, \underline{y}}+r}^2(\lambda) - \operatorname{Tr} T_r^2(\lambda)| \lesssim_{\delta, m} |\lambda| \sum_{\ell=1}^L \|(p + 2\lambda^2)^{-1/2} \hat{V}^{(\ell)}\|_{L^2(\mathbb{R})} \xrightarrow{\lambda \rightarrow 0, \lambda \in \Gamma_\delta} 0.$$

To estimate $a_{u_{\underline{\varphi}, \underline{y}}+r}^{(4)}(\lambda) - a_r^{(4)}(\lambda)$, we use (A.10). Approximating the potentials $V^{(\ell)}$ by functions of $L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ and using that

$$\|T_f(\lambda)T_g(\lambda)\| \leq C \frac{|\lambda|^2}{\sqrt{\operatorname{Im}(\lambda^2)}} \|f\|_{L^1} \|g\|_{L^2}, \quad \forall \lambda \in \mathbb{C}_{++}, f \in L^1(\mathbb{R}), g \in L^2(\mathbb{R}),$$

one easily checks that

$$\|T_{u_{\underline{\varphi}, \underline{y}}}^2(\lambda)\| + \|T_{u_{\underline{\varphi}, \underline{y}}}(\lambda)T_r(\lambda)\| \xrightarrow{\lambda \in \Gamma_\delta, \lambda \rightarrow 0} 0,$$

uniformly with respect to $\underline{\varphi}, \underline{y} \in \mathbb{R}^L$ and $r \in L^2(\mathbb{R})$, $\|r\|_{L^2(\mathbb{R})}^2 \leq m$. Therefore, applying (A.10), we get

$$a_{u_{\underline{\varphi}, \underline{y}}+r}^{(4)}(\lambda) - a_r^{(4)}(\lambda) \xrightarrow{\lambda \in \Gamma_\delta, \lambda \rightarrow 0} 0,$$

uniformly with respect to $\underline{\varphi}, \underline{y} \in \mathbb{R}^L$ and $r \in L^2(\mathbb{R})$ with $\|r\|_{L^2(\mathbb{R})}^2 \leq m$, which achieves the proof of the lemma. \square

Proof of Lemma 3.3. We start by observing that according to (2.20), (2.25), (2.26) and the fact that $V^{(\ell)} \in \mathcal{A}$, we have for all $\lambda \in \Gamma_\delta$, all $\underline{\varphi}, \underline{y} \in \mathbb{R}^L$, and $r \in L^2(\mathbb{R})$ with $\|r\|_{L^2(\mathbb{R})}^2 \leq m$

$$\begin{aligned} |a_{u_{\underline{\varphi}, \underline{y}}+r}(\lambda) - a_r(\lambda)| &\lesssim_{\delta, m} \underbrace{\left| a_r^{(4)}(\lambda) - 1 \right|}_{\mathcal{J}_1} + \underbrace{\left| a_{u_{\underline{\varphi}, \underline{y}}+r}^{(4)}(\lambda) - \prod_{\ell=1}^L a_{V^{(\ell)}}^{(4)}(\lambda) \right|}_{\mathcal{J}_2} + \\ &\quad + \underbrace{\left| \operatorname{Tr} \left(T_{u_{\underline{\varphi}, \underline{y}}+r}^2(\lambda) - T_r^2(\lambda) - \sum_{\ell=1}^L T_{V^{(\ell)}}^2(\lambda) \right) \right|}_{\mathcal{J}_3}. \end{aligned}$$

By virtue of Estimate (2.35), we have

$$|\mathcal{J}_1| \lesssim_{\delta, \alpha, m} \|r\|_{L^4(\mathbb{R})}^4,$$

for all $\lambda \in \Gamma_\delta$, $|\lambda| \geq \alpha$ and all $r \in L^2(\mathbb{R}) \cap L^4(\mathbb{R})$ with $\|r\|_{L^2(\mathbb{R})}^2 \leq m$.

We next address \mathcal{J}_2 . Arguing as in the proof of Proposition 3.2, we get

$$|\mathcal{J}_2| \lesssim_{\delta, \alpha, m} \|r\|_{L^4(\mathbb{R})}^{\frac{1}{2}} + \sum_{\substack{1 \leq \ell, \ell' \leq L \\ \ell \neq \ell'}} \|T_{e^{i\varphi_\ell V^{(\ell)}(\cdot - y_\ell)}(\lambda)} T_{e^{i\varphi_{\ell'} V^{(\ell')}(\cdot - y_{\ell'})}(\lambda)}\|,$$

with

$$\sum_{\substack{1 \leq \ell, \ell' \leq L \\ \ell \neq \ell'}} \|T_{e^{i\varphi_\ell V^{(\ell)}(\cdot - y_\ell)}(\lambda)} T_{e^{i\varphi_{\ell'} V^{(\ell')}(\cdot - y_{\ell'})}(\lambda)}\| \xrightarrow{\min_{\ell \neq \ell'} |y_\ell - y_{\ell'}| \rightarrow \infty} 0,$$

uniformly with respect to $\lambda \in \Gamma_\delta$, $|\lambda| \geq \alpha$, and $\underline{\varphi} \in \mathbb{R}^L$.

In order to end the proof of the lemma, it remains to estimate \mathcal{J}_3 :

$$\mathcal{J}_3 = 2 \sum_{\substack{1 \leq \ell, \ell' \leq L \\ \ell \neq \ell'}} \operatorname{Tr}(T_{e^{i\varphi_\ell V^{(\ell)}(\cdot - y_\ell)}(\lambda)} T_{e^{i\varphi_{\ell'} V^{(\ell')}(\cdot - y_{\ell'})}(\lambda)}) + 2 \sum_{1 \leq \ell \leq L} \operatorname{Tr}(T_{e^{i\varphi_\ell V^{(\ell)}(\cdot - y_\ell)}(\lambda)} T_r(\lambda)).$$

Clearly,

$$|\operatorname{Tr}(T_f(\lambda) T_g(\lambda))| \leq 2|\lambda|^2 \int_{\mathbb{R}^2} e^{-2\operatorname{Im}(\lambda^2)|x-y|} |f(x)| |g(y)| dx dy, \quad \forall \lambda \in \mathbb{C}_{++}, f \in L^2(\mathbb{R}), g \in L^2(\mathbb{R}),$$

which readily implies that

$$\sum_{\substack{1 \leq \ell, \ell' \leq L \\ \ell \neq \ell'}} \operatorname{Tr}(T_{e^{i\varphi_\ell V^{(\ell)}(\cdot - y_\ell)}(\lambda)} T_{e^{i\varphi_{\ell'} V^{(\ell')}(\cdot - y_{\ell'})}(\lambda)}) \xrightarrow{\min_{\ell \neq \ell'} |y_\ell - y_{\ell'}| \rightarrow \infty} 0,$$

uniformly with respect to $\lambda \in \Gamma_\delta$ and $\underline{\varphi} \in \mathbb{R}^L$, and

$$\sum_{1 \leq \ell \leq L} \operatorname{Tr}(T_{e^{i\varphi_\ell V^{(\ell)}(\cdot - y_\ell)}(\lambda)} T_r(\lambda)) \xrightarrow{\|r\|_{L^2} \leq m, \|r\|_{L^4} \rightarrow 0} 0,$$

uniformly with respect to $\lambda \in \Gamma_\delta$ and $\underline{y}, \underline{\varphi} \in \mathbb{R}^L$. This completes the proof of the lemma and therefore, the proof of Proposition 3.3 as well. \square

3.3. End of the proof of Theorem 1. Assume that $u_0 \in H^{\frac{1}{2}}(\mathbb{R})$, with $\|u_0\|_{L^2(\mathbb{R})}^2 \geq 4\pi$, is such that the solution $u \in C([0, T^*[, H^{\frac{1}{2}}(\mathbb{R}))$ of the DNLS equation with initial data $u(0) = u_0$ verifies $\sup_{0 \leq t < T^*} \|u(t)\|_{H^{\frac{1}{2}}(\mathbb{R})} = +\infty$. For a sequence $(t_n)_{n \in \mathbb{N}} \subset [0, T^*[$ such that $\|u(t_n)\|_{H^{\frac{1}{2}}(\mathbb{R})} \rightarrow +\infty$, as n goes to infinity, set as above

$$U_n(x) = \frac{1}{\sqrt{\mu_n}} u(t_n, \frac{x}{\mu_n}) \quad \text{with} \quad \mu_n = \|u(t_n)\|_{H^{\frac{1}{2}}(\mathbb{R})}^2.$$

Let us next take a sequence of functions $(u_0^{(k)})_{k \in \mathbb{N}}$ in $\mathcal{S}_{\text{reg}}(\mathbb{R})$ that converges in $H^{\frac{1}{2}}$ to u_0 . Denoting by $u^{(k)}(t)$ the solution of the DNLS equation with initial data $u^{(k)}(0) = u_0^{(k)}$, we have

$$u^{(k)} \xrightarrow{k \rightarrow +\infty} u \text{ in } C([0, T], H^{\frac{1}{2}}(\mathbb{R})), \quad \forall T < T^*.$$

Consequently, the functions $U_n^{(k)} \stackrel{\text{def}}{=} \frac{1}{\sqrt{\mu_n}} u^{(k)}\left(t_n, \frac{\cdot}{\mu_n}\right)$ belong to $\mathcal{S}_{\text{reg}}(\mathbb{R})$ and satisfy for any integer n

$$(3.21) \quad U_n^{(k)} \xrightarrow{k \rightarrow +\infty} U_n \text{ in } H^{\frac{1}{2}}(\mathbb{R}).$$

Since for any fixed $0 < \theta < \pi$, the function $\tilde{a}_{u_0}(\zeta)$ admits at most a finite number of zeros in the angles $\{\zeta \in \mathbb{C} : \theta \leq \arg \zeta < \pi\}$, there exists $\frac{\pi}{2} < \theta_0 < \pi$ such that

$$(3.22) \quad \tilde{a}_{u_0}(\zeta) \neq 0, \quad \forall \zeta, \theta_0 \leq \arg \zeta < \pi.$$

We denote by $\zeta_{n,j}^{(k)}$, $j = 1, \dots, I_k$ the zeros of $\tilde{a}_{U_n^{(k)}}(\zeta)$ in the angle $\{\zeta \in \mathbb{C} : \theta_0 \leq \arg \zeta < \pi\}$ (because of the scaling property (2.11), the number of zeros I_k does not depend on n).

The key ingredient in the proof of Theorem 1 is given by the following lemma, the proof of which is postponed to the end of this paragraph.

Lemma 3.4. *With the above notations, we have*

$$(3.23) \quad \liminf_{k \rightarrow \infty} I_k \geq 1.$$

Moreover, numbering the zeros $\zeta_{n,j}^{(k)}$ so that

$$\operatorname{Re}(\zeta_{n,1}^{(k)}) = \min_{1 \leq j \leq I_k} \operatorname{Re}(\zeta_{n,j}^{(k)}),$$

one has

$$(3.24) \quad \liminf_{n \rightarrow \infty} \liminf_{k \rightarrow \infty} \operatorname{Re}(\zeta_{n,1}^{(k)}) = c_0 < 0.$$

Admitting for a while Lemma 3.4, let us achieve the proof of Theorem 1. Accordingly to the lemma, for all k sufficiently large and all $n \in \mathbb{N}$, the function $\tilde{a}_{U_n^{(k)}}$ has at least one zero in $\{\zeta \in \mathbb{C} : \theta_0 \leq \arg \zeta < \pi\}$. By (2.11), for all $j = 1, \dots, I_k$,

$$\tilde{a}_{u_0^{(k)}}(\zeta_{n,j}^{(k)} \mu_n) = 0.$$

In view of Lemma 2.5, this implies that for all k sufficiently large and all n ,

$$\operatorname{Re} \zeta_{n,1}^{(k)} \geq -\frac{C}{\mu_n},$$

for some positive constant C depending only on $\|u_0\|_{H^{\frac{1}{2}}(\mathbb{R})}$. Since $\mu_n \xrightarrow{n \rightarrow +\infty} +\infty$, this contradicts the property (3.24).

Thus, to complete the proof of the theorem, we need to establish Lemma 3.4. To this end, let us start by observing that in view of Theorem 2, we have

$$(3.25) \quad U_n^{(k)}(y) = \sum_{\ell=1}^{L_0} V^{(\ell)}(y - y_n^{(\ell)}) + r_n^{(k)}(y), \quad r_n^{(k)} = r_n + U_n^{(k)} - U_n,$$

with

$$(3.26) \quad \min_{\ell \neq \ell'} |y_n^{(\ell)} - y_n^{(\ell')}| \xrightarrow{n \rightarrow +\infty} +\infty \quad \text{and} \quad \lim_{k \rightarrow +\infty} \|r_n^{(k)}\|_{L^p(\mathbb{R})} \xrightarrow{n \rightarrow +\infty} 0,$$

for all $2 < p < \infty$.

Combining (3.22) with Corollary 2.1 and taking into account the stability estimate (2.27) and the scaling property (2.11), we infer that there exist $C \geq 1$ and $K \in \mathbb{N}$ such that for all $k \geq K$, all $n \in \mathbb{N}$ and all $\zeta \in \mathbb{C}_+$ with $\arg \zeta = \theta_0$ we have

$$(3.27) \quad \frac{1}{C} \leq \left| \frac{1}{\tilde{a}_{U_n^{(k)}}(\zeta)} \right| \leq C.$$

Invoking Proposition 3.3, we deduce that

$$(3.28) \quad \limsup_{k \rightarrow +\infty} \sup_{\substack{\zeta \in \mathbb{C}_+ \\ \arg \zeta = \theta_0}} \left| 1 - \frac{\tilde{a}_{r_n^{(k)}}(\zeta)}{\tilde{a}_{U_n^{(k)}}(\zeta)} \right| \xrightarrow{n \rightarrow +\infty} 0.$$

It follows then that, for all n and k large enough, $\tilde{a}_{r_n^{(k)}}$ does not vanish on the ray $e^{i\theta_0}\mathbb{R}_+^*$ and therefore, we can apply Lemmas 2.3, 2.4, which gives

$$I_k \geq \frac{1}{2i\pi} \int_0^{+\infty} e^{i\theta_0} \left(\frac{\tilde{a}'_{U_n^{(k)}}(s)}{\tilde{a}_{U_n^{(k)}}(s)} - \frac{\tilde{a}'_{r_n^{(k)}}(s)}{\tilde{a}_{r_n^{(k)}}(s)} \right) ds + \frac{1}{4\pi} \left(\|U_n^{(k)}\|_{L^2(\mathbb{R})}^2 - \|r_n^{(k)}\|_{L^2(\mathbb{R})}^2 \right).$$

In view of (3.5) and (3.28), we have respectively

$$\lim_{k \rightarrow +\infty} \left(\|U_n^{(k)}\|_{L^2(\mathbb{R})}^2 - \|r_n^{(k)}\|_{L^2(\mathbb{R})}^2 \right) \xrightarrow{n \rightarrow +\infty} \sum_{\ell=1}^{L_0} \|V^{(\ell)}\|_{L^2(\mathbb{R})}^2,$$

and

$$\limsup_{k \rightarrow +\infty} \left| \int_0^{+\infty} e^{i\theta_0} \left(\frac{\tilde{a}'_{U_n^{(k)}}(s)}{\tilde{a}_{U_n^{(k)}}(s)} - \frac{\tilde{a}'_{r_n^{(k)}}(s)}{\tilde{a}_{r_n^{(k)}}(s)} \right) ds \right| \xrightarrow{n \rightarrow +\infty} 0,$$

which shows that

$$\liminf_{k \rightarrow \infty} I_k \geq \frac{1}{4\pi} \sum_{\ell=1}^{L_0} \|V^{(\ell)}\|_{L^2(\mathbb{R})}^2 \geq 1.$$

To establish (3.24), we argue by contradiction assuming that

$$(3.29) \quad \liminf_{n \rightarrow \infty} \liminf_{k \rightarrow \infty} \operatorname{Re} \zeta_{n,1}^{(k)} \geq 0.$$

Since $\frac{\pi}{2} < \theta_0 < \pi$, this means that

$$(3.30) \quad \lim_{n \rightarrow \infty} \limsup_{k \rightarrow \infty} \max_{j=1, \dots, I_k} |\zeta_{n,j}^{(k)}| = 0.$$

Observe also that since \tilde{a}_u does not vanish in the angle $\{\zeta \in \mathbb{C} : \theta_0 \leq \arg \zeta < \pi\}$, it follows from the stability estimate (2.27) and Corollary 2.1 that

$$(3.31) \quad \max_{j=1, \dots, I_k} \frac{\operatorname{Im} \zeta_{n,j}^{(k)}}{|\operatorname{Re} \zeta_{n,j}^{(k)}|} \xrightarrow{k \rightarrow \infty} 0.$$

We shall now eliminate one by one all the zeros $\zeta_{n,j}^{(k)}$ by applying recursively the Bäcklund transformation that we have introduced in Paragraph 2.3. Namely, consider the family of Schwartz class functions $(U_{n,j}^{(k)})_{j=0, \dots, I_k}$ defined by

$$\begin{aligned} U_{n,0}^{(k)} &= U_n^{(k)}, \\ U_{n,j}^{(k)} &= \mathcal{B}_{\lambda_{n,j}^{(k)}}(\eta_{n,j}^{(k)}) U_{n,j-1}^{(k)}, \quad j = 1, \dots, I_k, \end{aligned}$$

or explicitly,

$$U_{n,j}^{(k)} = G_{\lambda_{n,j}^{(k)}}^2(\eta_{n,j}^{(k)}) U_{n,j-1}^{(k)} - G_{\lambda_{n,j}^{(k)}}(\eta_{n,j}^{(k)}) S_{\lambda_{n,j}^{(k)}}(\eta_{n,j}^{(k)}), \quad j = 1, \dots, I_k,$$

where $\lambda_{n,j}^{(k)} = \sqrt{\zeta_{n,j}^{(k)}} \in \mathbb{C}_{++}$, and $\eta_{n,j}^{(k)} \in L^2(\mathbb{R}, \mathbb{C}^2) \setminus \{0\}$, $L_{U_{n,j-1}^{(k)}}(\lambda_{n,j}^{(k)}) \eta_{n,j}^{(k)} = 0$. It then follows from (2.15) and (2.75) that

$$(3.32) \quad \|U_{n,j}^{(k)}\|_{L^2(\mathbb{R})}^2 = \|U_n^{(k)}\|_{L^2(\mathbb{R})}^2 - 4 \sum_{i=1}^j \arg(\zeta_{n,i}^{(k)})$$

(see also Appendix B, Remark B.1). Since $|G_{\lambda_{n,j}^{(k)}}(\eta_{n,j}^{(k)})| = 1$, the above relation ensures that there exists a positive constant C such that for all n, k, j , there holds

$$(3.33) \quad \|S_{\lambda_{n,j}^{(k)}}(\eta_{n,j}^{(k)})\|_{L^2(\mathbb{R})} \leq C.$$

Combining this bound with the estimate (2.73), we readily gather by an obvious induction that, for all $2 < p < \infty$,

$$\|U_{n,j}^{(k)}\|_{L^p(\mathbb{R})} \leq \|U_n^{(k)}\|_{L^p(\mathbb{R})} + C \sum_{i=1}^j (\operatorname{Im} \lambda_{n,i}^{(k)})^{1-\frac{2}{p}}.$$

Let us now consider the functions $\mathcal{W}_n^{(k)} \stackrel{\text{def}}{=} U_{n,I_k}^{(k)}$. Clearly,

$$(3.34) \quad \begin{aligned} \mathcal{W}_n^{(k)} &= G_{\lambda_{n,I_k}^{(k)}}^2(\eta_{n,I_k}^{(k)}) \cdots G_{\lambda_{n,1}^{(k)}}^2(\eta_{n,1}^{(k)}) U_n^{(k)} \\ &\quad - \sum_{j=1}^{I_k} G_{\lambda_{n,I_k}^{(k)}}^2(\eta_{n,I_k}^{(k)}) \cdots G_{\lambda_{n,j+1}^{(k)}}^2(\eta_{n,j+1}^{(k)}) G_{\lambda_{n,j}^{(k)}}(\eta_{n,j}^{(k)}) \mathcal{S}_{\lambda_{n,j}^{(k)}}(\eta_{n,j}^{(k)}). \end{aligned}$$

We claim that there exists a family $(\beta_{n,\ell}^{(k)})_{1 \leq \ell \leq L_0}$ of complex numbers of modulus 1 such that, for all $2 < p < \infty$, there holds

$$(3.35) \quad \mathcal{W}_n^{(k)} = \sum_{\ell=1}^{L_0} \beta_{n,\ell}^{(k)} V^{(\ell)}(\cdot - y_n^{(\ell)}) + \mathcal{R}_n^{(k)}, \quad \limsup_{k \rightarrow \infty} \|\mathcal{R}_n^{(k)}\|_{L^p(\mathbb{R})} \xrightarrow{n \rightarrow +\infty} 0.$$

Indeed, setting

$$\beta_{n,\ell}^{(k)} = \underbrace{(G_{\lambda_{n,I_k}^{(k)}}^2(\eta_{n,I_k}^{(k)}) \cdots G_{\lambda_{n,1}^{(k)}}^2(\eta_{n,1}^{(k)}))}_{\mathcal{G}_n^{(k)}}(y_n^{(\ell)}),$$

we obviously obtain a family of complex numbers of modulus 1. Then, taking advantage of (3.25), we deduce that

$$\mathcal{R}_n^{(k)} = \mathcal{R}_{n,1}^{(k)} + \mathcal{R}_{n,2}^{(k)} + \mathcal{R}_{n,3}^{(k)},$$

with

$$\begin{aligned} \mathcal{R}_{n,1}^{(k)}(y) &= \mathcal{G}_n^{(k)}(y) r_n^{(k)}(y), \\ \mathcal{R}_{n,2}^{(k)}(y) &= - \sum_{j=1}^{I_k} G_{\lambda_{n,I_k}^{(k)}}^2(\eta_{n,I_k}^{(k)}) \cdots G_{\lambda_{n,j+1}^{(k)}}^2(\eta_{n,j+1}^{(k)}) G_{\lambda_{n,j}^{(k)}}(\eta_{n,j}^{(k)}) \mathcal{S}_{\lambda_{n,j}^{(k)}}(\eta_{n,j}^{(k)}) \\ \mathcal{R}_{n,3}^{(k)}(y) &= \sum_{\ell=1}^{L_0} (\mathcal{G}_n^{(k)}(y) - \mathcal{G}_n^{(k)}(y_n^{(\ell)})) V^{(\ell)}(y - y_n^{(\ell)}). \end{aligned}$$

Since $|\mathcal{G}_n^{(k)}| = 1$, (3.26) ensures that, for all $2 < p < \infty$,

$$\lim_{k \rightarrow \infty} \|\mathcal{R}_{n,1}^{(k)}\|_{L^p(\mathbb{R})} \xrightarrow{n \rightarrow +\infty} 0.$$

For $\mathcal{R}_{n,2}^{(k)}$, we have

$$\|\mathcal{R}_{n,2}^{(k)}\|_{L^p(\mathbb{R})} \leq \sum_{j=1}^{I_k} \|\mathcal{S}_{\lambda_{n,j}^{(k)}}(\eta_{n,j}^{(k)})\|_{L^p(\mathbb{R})} \lesssim I_k \max_{1 \leq j \leq I_k} (\operatorname{Im} \lambda_{n,j}^{(k)})^{1-\frac{2}{p}}.$$

Note that by (2.47), I_k is bounded independently of k . Therefore, taking into account (3.30), we deduce from the above estimate that

$$\limsup_{k \rightarrow \infty} \|\mathcal{R}_{n,2}^{(k)}\|_{L^p(\mathbb{R})} \xrightarrow{n \rightarrow +\infty} 0.$$

Finally, in order to investigate the term $\mathcal{R}_{n,3}^{(k)}$, we shall make use of the pointwise estimate (2.74) which gives:

$$\left| \frac{d}{dy} \mathcal{G}_n^{(k)}(y) \right| \leq 16 \sum_{j=1}^{I_k} \left((\operatorname{Im} \lambda_{n,j}^{(k)})^2 + \operatorname{Im}(\lambda_{n,j}^{(k)}) |U_{n,j-1}^{(k)}(y)| \right).$$

Clearly, for all j , we have

$$|U_{n,j}^{(k)}(y)| \leq |U_n^{(k)}(y)| + \sum_{j=1}^{I_k} |\mathcal{S}_{\lambda_{n,j}^{(k)}}(n_{n,j}^{(k)}(y))| \leq |U_n^{(k)}(y)| + 4 \sum_{j=1}^{I_k} \operatorname{Im}(\lambda_{n,j}^{(k)}).$$

Therefore, combining the two last inequalities, we obtain:

$$(3.36) \quad \left| \frac{d}{dy} \mathcal{G}_n^{(k)}(y) \right| \lesssim \left(\sum_{j=1}^{I_k} |\lambda_{n,j}^{(k)}| \right)^2 + |U_n^{(k)}(y)| \sum_{j=1}^{I_k} |\lambda_{n,j}^{(k)}|,$$

which implies that, for all $1 \leq \ell \leq L_0$,

$$|\mathcal{G}_n^{(k)}(y + y_n^{(\ell)}) - \mathcal{G}_n^{(k)}(y_n^{(\ell)})| \lesssim \left(\sum_{j=1}^{I_k} |\lambda_{n,j}^{(k)}| \right)^2 |y| + \sum_{j=1}^{I_k} |\lambda_{n,j}^{(k)}| |y|^{\frac{1}{2}} \|U_n^{(k)}\|_{L^2(\mathbb{R})}.$$

In view of (3.30), this ensures that, for all $1 \leq \ell \leq L_0$ and all $R > 0$,

$$\limsup_{k \rightarrow +\infty} \sup_{y \in [-R, R]} |\mathcal{G}_n^{(k)}(y + y_n^{(\ell)}) - \mathcal{G}_n^{(k)}(y_n^{(\ell)})| \xrightarrow{n \rightarrow +\infty} 0.$$

Since $|G_n^{(k)}(y)| = 1$, this allows us to conclude that

$$\limsup_{k \rightarrow \infty} \|\mathcal{R}_{n,3}^{(k)}\|_{L^p(\mathbb{R})} \xrightarrow{n \rightarrow +\infty} 0,$$

for all $2 \leq p < \infty$, which ends the proof of (3.35).

Let us now consider $\tilde{a}_{\mathcal{W}_n^{(k)}}$. By (2.75), it has the following form

$$(3.37) \quad \tilde{a}_{\mathcal{W}_n^{(k)}}(\zeta) = \tilde{a}_{U_n^{(k)}}(\zeta) \prod_{j=1}^{I_k} \frac{\zeta - \bar{\zeta}_{n,j}^{(k)}}{\zeta - \zeta_{n,j}^{(k)}}.$$

Due to (3.31), we have

$$(3.38) \quad \sup_{\substack{\zeta \in \mathbb{C}_+ \\ \arg \zeta = \theta_0}} \left| 1 - \prod_{j=1}^{I_k} \frac{\zeta - \bar{\zeta}_{n,j}^{(k)}}{\zeta - \zeta_{n,j}^{(k)}} \right| \xrightarrow{k \rightarrow +\infty} 0,$$

which in view of (3.37) and (3.27) implies that for all k sufficiently large, all $n \in \mathbb{N}$ and all $\zeta \in \mathbb{C}_+$ with $\arg \zeta = \theta_0$, we have

$$(3.39) \quad \frac{1}{2C} \leq \left| \frac{1}{\tilde{a}_{\mathcal{W}_n^{(k)}}(\zeta)} \right| \leq 2C.$$

This bound together with (3.35) and (3.26) allows us to apply Proposition 3.3 and Lemmas 2.3, 2.4 to the sequence $(\mathcal{W}_n^{(k)})$, repeating the argument we have used above to prove (3.23). Taking

into account that by construction, the function $\tilde{a}_{\mathcal{W}_n^{(k)}}(\zeta)$ has no zero in the angle $\{\zeta \in \mathbb{C}_+ : \theta_0 \leq \arg \zeta < \pi\}$, we obtain

$$0 \geq 1 + \frac{1}{2i\pi} \int_0^{+\infty} e^{i\theta_0} \left(\frac{\tilde{a}'_{\mathcal{W}_n^{(k)}}(s)}{\tilde{a}_{\mathcal{W}_n^{(k)}}(s)} - \frac{\tilde{a}'_{\mathcal{R}_n^{(k)}}(s)}{\tilde{a}_{\mathcal{R}_n^{(k)}}(s)} \right) ds + \frac{1}{4\pi} \left(\|\mathcal{W}_n^{(k)}\|_{L^2(\mathbb{R})}^2 - \sum_{\ell=1}^{L_0} \|V^{(\ell)}\|_{L^2(\mathbb{R})}^2 - \|\mathcal{R}_n^{(k)}\|_{L^2(\mathbb{R})}^2 \right),$$

with

$$\limsup_{k \rightarrow +\infty} \left| \int_0^{+\infty} e^{i\theta_0} \left(\frac{\tilde{a}'_{\mathcal{W}_n^{(k)}}(s)}{\tilde{a}_{\mathcal{W}_n^{(k)}}(s)} - \frac{\tilde{a}'_{\mathcal{R}_n^{(k)}}(s)}{\tilde{a}_{\mathcal{R}_n^{(k)}}(s)} \right) ds \right| \xrightarrow{n \rightarrow +\infty} 0,$$

and

$$\limsup_{k \rightarrow +\infty} \left| \|\mathcal{W}_n^{(k)}\|_{L^2(\mathbb{R})}^2 - \sum_{\ell=1}^{L_0} \|V^{(\ell)}\|_{L^2(\mathbb{R})}^2 - \|\mathcal{R}_n^{(k)}\|_{L^2(\mathbb{R})}^2 \right| \xrightarrow{n \rightarrow +\infty} 0,$$

which gives a contradiction and therefore, concludes the proof of lemma. Thus the proof of Theorem 1 is achieved.

APPENDIX A. REGULARIZED DETERMINANTS

In this appendix, we review the basic properties of the regularized determinants $\det_n(\mathbf{I} - A)$ for A in \mathcal{C}_n , the set of bounded operators¹² A on $L^2(\mathbb{R})$ such that $|A|^n$ is of trace-class, endowed with the norm $\|A\|_n \stackrel{\text{def}}{=} [\text{Tr}(|A|^n)]^{\frac{1}{n}}$. We refer to the monograph of Simon [31] and the references therein for further details and the proofs.

To introduce the regularized determinants, let us start by defining, for any bounded operator A on $L^2(\mathbb{R})$,

$$R_n(A) = \mathbf{I} - (\mathbf{I} - A) \exp \left(\sum_{k=1}^{n-1} \frac{A^k}{k} \right).$$

Clearly,

$$(A.1) \quad R_n(A) = A^n h_n(A),$$

where h_n is an entire function¹³ on \mathbb{C} . This shows that $R_n(A)$ belongs to \mathcal{C}_1 if A is in \mathcal{C}_n , which justifies the following definition:

Definition A.1. For any operator A in \mathcal{C}_n , $n \geq 2$, we define

$$(A.2) \quad \det_n(\mathbf{I} - A) \stackrel{\text{def}}{=} \det(\mathbf{I} - R_n(A)) = \det \left((\mathbf{I} - A) \exp \left(\sum_{k=1}^{n-1} \frac{A^k}{k} \right) \right).$$

Remark A.1. The above formula deserves some comments:

- For any $n \geq 2$, if A is in \mathcal{C}_{n-1} , then¹⁴

$$(A.3) \quad \det_n(\mathbf{I} - A) = \det_{n-1}(\mathbf{I} - A) \exp \left(\frac{\text{Tr}(A^{n-1})}{n-1} \right).$$

- Note also that, for all A in \mathcal{C}_n such that $\|A\| < 1$ (or more generally $\|A^p\| < 1$, for some p), one has:

$$(A.4) \quad \det_n(\mathbf{I} - A) = \exp \left(-\text{Tr} \sum_{k=n}^{\infty} \frac{A^k}{k} \right).$$

¹²For our purpose, we focus here on L^2 -framework, but all the results are available on separable Hilbert spaces.

¹³Explicitly, $h_n(z) = z^{-n} \left(1 - (1-z) e^{\sum_{k=1}^{n-1} \frac{z^k}{k}} \right)$.

¹⁴with the convention $\det_1(\mathbf{I} - A) = \det(\mathbf{I} - A)$

In the following proposition, we summarize some useful properties of the regularized determinants:

Proposition A.1. *With the previous notations, for any $n \geq 1$ there exists a positive constant C_n such that the following holds.*

(1) For all $A \in \mathcal{C}_n$,

$$(A.5) \quad |\det_n(I - A)| \leq \exp(C_n \|A\|_n^n),$$

$$(A.6) \quad |\det_n(I - A) - 1| \leq C_n \|A^n\|_1 \exp(C_n \|A\|_n^n).$$

(2) For all A, B in \mathcal{C}_n ,

$$(A.7) \quad |\det_n(I - A) - \det_n(I - B)| \leq \|A - B\|_n \exp(C_n (\|A\|_n + \|B\|_n + 1)^n).$$

(3) Let $A \in \mathcal{C}_n$. Then $I - A$ is invertible if and only if $\det_n(I - A) \neq 0$, and furthermore, one has

$$(A.8) \quad \|(I - A)^{-1}\| \leq \frac{C_n}{|\det_n(I - A)|} \exp(C_n \|A\|_n^n).$$

We conclude this appendix by the following result that we have used in the proof of Theorem 1.

Proposition A.2. *For any $m \geq 0$, there exists a positive constant C_m such that for all A, B in \mathcal{C}_2 satisfying $\|A\|_2 + \|B\|_2 \leq m$ we have*

$$(A.9) \quad |\det_4(I - A - B) - \det_4(I - A)\det_4(I - B)| \leq C_m(\|AB\| + \|BA\|),$$

and

$$(A.10) \quad |\det_4(I - A - B) - \det_4(I - B)| \leq C_m(\|A^2\| + \|AB\|).$$

Proof. In view of the definition (A.1), we have

$$\|R_4(A + B)\|_1 + \|R_4(A)\|_1 + \|R_4(B)\|_1 \leq C_m,$$

and

$$\|R_4(A + B) - R_4(A) - R_4(B) + R_4(A)R_4(B)\|_1 \leq C_m(\|AB\| + \|BA\|).$$

Combining these inequalities with the bound (A.7) and invoking the identity

$$\det(I + C)\det(I + D) = \det(I + C + D + CD), \quad \forall C, D \in \mathcal{C}_1,$$

we get (A.9).

To prove Estimate (A.10), we start by observing that $R_4(A + B)$ can be written in the form

$$R_4(A + B) = R_4(B) + f(B)A + \mathcal{G}(A, B),$$

where $f(z) = z^3 h_4(z)$ and the remainder $\mathcal{G}(A, B)$ admits the bound:

$$\|\mathcal{G}(A, B)\|_1 \leq C_m(\|A^2\| + \|AB\|).$$

Furthermore, we have

$$\|f(B)AR_4(B)\|_1 \leq C_m\|AB\|.$$

Therefore,

$$\|R_4(A + B) - R_4(B) - f(B)A + f(B)AR_4(B)\|_1 \leq C_m(\|A^2\| + \|AB\|),$$

which implies

$$(A.11) \quad |\det(I - R_4(A + B)) - \det(I - R_4(B))\det(I - f(B)A)| \leq C_m(\|A^2\| + \|AB\|).$$

To conclude the proof of (A.10), it remains to observe that

$$|\det(I - f(B)A) - 1| = |\det(I - Af(B)) - 1| \leq C_m\|AB\|.$$

□

APPENDIX B. PROOF OF THE ESTIMATE (2.74)

According to (2.72), we have

$$(B.1) \quad \frac{d}{dx}G_\lambda(\eta) = (\lambda^2 - \bar{\lambda}^2) \frac{[|\eta_1|^2 \frac{d}{dx}(|\eta_2|^2) - |\eta_2|^2 \frac{d}{dx}(|\eta_1|^2)]}{d_\lambda^2(\eta)}.$$

Since, in view of (2.2),

$$\begin{aligned} \partial_x \eta_1 &= -i\lambda^2 \eta_1 + \lambda u \eta_2 \\ \partial_x \eta_2 &= i\lambda^2 \eta_2 - \lambda \bar{u} \eta_1, \end{aligned}$$

we infer that

$$|\eta_1|^2 \frac{d}{dx}(|\eta_2|^2) = 2|\eta_1|^2 \operatorname{Re}(i\lambda^2 |\eta_2|^2 - \lambda \bar{u} \eta_1 \bar{\eta}_2),$$

and

$$|\eta_2|^2 \frac{d}{dx}(|\eta_1|^2) = 2|\eta_2|^2 \operatorname{Re}(-i\lambda^2 |\eta_1|^2 + \lambda u \eta_2 \bar{\eta}_1).$$

Therefore,

$$\begin{aligned} |\eta_1|^2 \frac{d}{dx}(|\eta_2|^2) - |\eta_2|^2 \frac{d}{dx}(|\eta_1|^2) &= 2i(\lambda^2 - \bar{\lambda}^2) |\eta_1|^2 |\eta_2|^2 \\ &\quad - u \bar{\eta}_1 \eta_2 d_{\bar{\lambda}}(\eta) - \bar{u} \eta_1 \bar{\eta}_2 d_\lambda(\eta), \end{aligned}$$

which implies that

$$(B.2) \quad \frac{d}{dx}G_\lambda(\eta) = -\frac{i}{2}G_\lambda(\eta) \left(|\mathcal{S}_\lambda(\eta)|^2 - uG_\lambda(\eta)\overline{\mathcal{S}_\lambda(\eta)} - \overline{uG_\lambda(\eta)}\mathcal{S}_\lambda(\eta) \right).$$

Taking into account that $|G_\lambda(\eta)| = 1$ and $|\mathcal{S}_\lambda(\eta)| \leq 4 \operatorname{Im} \lambda$, we obtain

$$\left| \frac{d}{dx}G_\lambda(\eta)(x) \right| \leq 8(\operatorname{Im} \lambda)^2 + 4 \operatorname{Im}(\lambda)|u(x)|.$$

Remark B.1. Assuming that $\eta \in L^2(\mathbb{R}, \mathbb{C}^2) \setminus \{0\}$, one easily deduces from (B.2) that

$$\|\mathcal{B}_\lambda(\eta)u\|_{L^2(\mathbb{R})}^2 = \|u\|_{L^2(\mathbb{R})}^2 - 8 \arg \lambda.$$

REFERENCES

- [1] M.-J. Ablowitz and H. Segur, Solitons and the inverse scattering transform, *Siam Studies in Applied Mathematics*, 1981.
- [2] M. Agueh, Sharp Gagliardo-Nirenberg inequalities and mass transport theory, *Journal of Dynamics and Differential Equations*, **18**, pages 1069-1093, 2006.
- [3] R. Beals and R.-R. Coifman, Scattering and inverse scattering for first order systems, *Communications on Pure and Applied Mathematics*, **37**, pages 39-90, 1984.
- [4] H. Biagioni and F. Linares, Ill-posedness for the derivative Schrödinger and generalized Benjamin-Ono equations, *Transactions of the American Mathematical Society*, **353**, pages 3649-3659, 2001.
- [5] D. Champeaux-Laveder, T. Passot, P.-L. Sulem, Remarks on the parallel propagation of small-amplitude dispersive, *Alfven waves Nonlinear Processes in Geophysics*, **6**, pages 169-178, 1999.
- [6] J. Colliander, M. Keel, G. Staffilani, H. Takaoka and T. Tao, Refined global well-posedness result for Schrödinger equations with derivative, *SIAM Journal on Mathematical Analysis*, **34**, pages 64-86, 2002.
- [7] M. Colin and M. Ohta, Stability of solitary waves for derivative nonlinear Schrödinger equation, *Annales de l'Institut Henri Poincaré, Analyse nonlinéaire*, **23**, pages 753-764, 2006.
- [8] P. Deift and R. Killip, On the absolutely continuous spectrum of one-dimensional Schrödinger operators with square summable potentials, *Communications in Mathematical Physics*, **203**, pages 341-347, 1999.
- [9] P. Gérard, Description du défaut de compacité de l'injection de Sobolev, *ESAIM Control, Optimisation and Calculus of Variations*, **3**, pages 213-233, 1998.
- [10] B. Guo and Yaping Wu, Orbital stability of solitary waves for the nonlinear derivative Schrödinger equation, *Journal of Differential Equations*, **123**, pages 35-55, 1994.
- [11] A. Grünrock, Bi- and trilinear Schrödinger estimates in one space dimension with applications to cubic NLS and DNLS, *International Mathematics Research Notices*, **41**, pages 2525-2558, 2005.

- [12] Z. Guo and Yifei Wu, Global well-posedness for the derivative nonlinear Schrödinger equation in $H^{\frac{1}{2}}$, *Discrete and Continuous Dynamical Systems*, **1**, pages 64-86, 2017.
- [13] N. Hayashi and T. Ozawa, On the derivative nonlinear Schrödinger equation, *Physica D*, **55**, pages 14-36, 1992.
- [14] J. He, L. Wang and S.Xu, The Darboux transformation of the derivative nonlinear Schrödinger equation, *Communications in Mathematical Physics*, **44**, pages 567-576, 2011.
- [15] R. Jenkins, J. Liu, P. Perry and C. Sulem, Global existence for the derivative nonlinear Schrödinger equation with arbitrary spectral singularities, to appear in *Analysis and PDE*.
- [16] R. Jenkins, J. Liu, P. Perry and C. Sulem, The derivative nonlinear Schrödinger equation: global well-posedness and soliton resolution, *Quarterly Journal of Pure and Applied Mathematics*, **78**, pages 33-73, 2020.
- [17] R. Jenkins, J. Liu, P. Perry and C. Sulem, Global well-posedness for the derivative nonlinear Schrödinger equation, *Communications on Partial Differential Equations*, **43**, pages 1151-1195, 2018.
- [18] D.-J. Kaup and A.-C. Newell, An exact solution for a derivative nonlinear Schrödinger equation, *Journal of Mathematical Physics*, **19**, pages 798-801, 1978.
- [19] R. Killip, M. Visan, KdV is well-posed in H^{-1} , *Annals of Mathematics*, **190**, pages 249-305, 2019.
- [20] R. Killip, M. Visan and X. Zhang, Low regularity conservation laws for integrable PDE, *Geometric and Functional Analysis*, **28**, pages 1062-1090, 2018.
- [21] F. Klaus and R. Schippa, A priori estimates for the derivative nonlinear Schrödinger equation, *arXiv:2007.13161*.
- [22] H. Koch and D. Tataru, Conserved energies for the cubic nonlinear Schrödinger equation in one dimension, *Duke Mathematical Journal*, **167**, pages 3207-3313, 2018.
- [23] S. Kwon and Y. Wu, Orbital stability of solitary waves for derivative nonlinear Schrödinger equation, *Journal d'Analyse Mathématique*, **135**, pages 473-486, 2018.
- [24] J.-H. Lee, Global solvability of the derivative nonlinear Schrödinger equation, *Transactions of the American Mathematical Society*, **314**, pages 107-118, 1989.
- [25] W. Mio, T. Ogino, K. Minami, S. Takeda, Modified nonlinear Schrödinger for Alfvén waves propagating along the magnetic field in cold plasma, *Journal of the Physical Society of Japan*, **41**, pages 265-271, 1976.
- [26] E. Mjølhus, On the modulational instability of hydromagnetic waves parallel to the magnetic field, *Journal of Plasma Physics*, **16**, pages 321-334, 1976.
- [27] J. Moses, B.-A. Malomed, and F.-W. Wise, Self-steepening of ultrashort optical pulses without self-phase-modulation, *Physical Review A*, **76**, 2007.
- [28] T. Ozawa, On the nonlinear Schrödinger equations of derivative type, *Indiana University Mathematics Journal*, **45**, pages 137-163, 1996.
- [29] D.-E. Pelinovsky, A. Saalman and Y. Shimabukuro, The derivative NLS equation: global existence with solitons, *Dynamics of PDE*, **14**, pages 271-294, 2017.
- [30] D.-E. Pelinovsky and Y. Shimabukuro, Existence of global solutions to the derivative NLS equation with the inverse scattering method, *International Mathematics Research Notices*, pages 5663-5728, 2017.
- [31] B. Simon, Trace ideals and their applications, Second edition. *Mathematical surveys and Monographs*, **120**, American Mathematical Society, Providence, 2005.
- [32] H. Takaoka, Well-posedness for the one-dimensional nonlinear Schrödinger equation with the derivative nonlinearity, *Advances in Differential Equations*, **4**, pages 561-580, 1999.
- [33] H. Takaoka, Global well-posedness for Schrödinger equations with derivative in a nonlinear term and data in low-order Sobolev spaces, *Electronic Journal of Differential Equations*, **42**, pages 1-23, 2001.
- [34] T. Tsuchida and M. Wadati, Complete integrability of derivative nonlinear Schrödinger-type equations, *Inverse Problems*, **15**, pages 1363-1373, 1999.
- [35] M.-I. Weinstein, Nonlinear Schrödinger equations and sharp interpolation estimates, *Communications in Mathematical Physics*, **87**, pages 567-576, 1983.
- [36] Y. Wu, Global well-posedness on the derivative nonlinear Schrödinger equation revisited, *Analysis and PDE*, **8**, pages 1101-1112, 2015.

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