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# An infinite-dimensional representation of the Ray–Knight theorems

by

Elie Aïdékon<sup>1</sup>, Yueyun Hu<sup>2</sup>, and Zhan Shi<sup>3</sup>

**Summary.** The classical Ray–Knight theorems for Brownian motion determine the law of its local time process either at the first hitting time of a given value  $a$  by the local time at the origin, or at the first hitting time of a given position  $b$  by Brownian motion. We extend these results by describing the local time process jointly for all  $a$  and all  $b$ , by means of stochastic integral with respect to an appropriate white noise. Our result applies to  $\mu$ -processes, and has an immediate application: a  $\mu$ -process is the height process of a Feller continuous-state branching process (CSBP) with immigration (Lambert [10]), whereas a Feller CSBP with immigration satisfies a stochastic differential equation driven by a white noise (Dawson and Li [7]); our result gives an explicit relation between these two descriptions and shows that the stochastic differential equation in question is a reformulation of Tanaka’s formula.

**Keywords.** Ray–Knight theorem,  $\mu$ -process, white noise, Tanaka’s formula.

**2010 Mathematics Subject Classification.** 60J65, 60J55.

## 1 Introduction

Let  $(B_t)_{t \geq 0}$  be standard one-dimensional Brownian motion associated with its completed natural filtration  $(\mathcal{B}_t)_{t \geq 0}$ . Denote by  $(\mathfrak{L}_t)_{t \geq 0}$  a continuous version of local times of  $(B_t)$  at position 0. Let  $\mu \in \mathbb{R} \setminus \{0\}$ . The  $\mu$ -process  $X := (X_t)_{t \geq 0}$  is defined as follows:

$$X_t := |B_t| - \mu \mathfrak{L}_t, \quad t \geq 0.$$

There are two important special cases of  $\mu$ -processes: Brownian motion ( $\mu = 1$ , this is seen using Lévy’s identity), and the three-dimensional Bessel process ( $\mu = -1$ , seen by means of Lévy’s and Pitman’s identities).

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The  $\mu$ -process, also referred to as perturbed reflecting Brownian motion, has attracted much attention in the nineties: Lévy's arc sine law, Ray–Knight theorems as well as pathwise uniqueness of doubly perturbed Brownian motion, see for example [12, 22, 3, 4, 18, 21, 5, 6, 14, 15].

The local time of the  $\mu$ -process at suitable stopping times, as a process of the space variable, turns out to be a squared Bessel process. This is referred to as a Ray–Knight theorem. More precisely, let us fix  $\mu > 0$  from now on, so the process  $(X_t, t \geq 0)$  is recurrent on  $\mathbb{R}$ . Since  $X$  is a continuous semimartingale, we may define

$$L(t, r) := \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_0^t \mathbf{1}_{\{r \leq X_s \leq r + \varepsilon\}} ds, \quad t \geq 0, r \in \mathbb{R},$$

as the local time of  $X$  at time  $t$  and position  $r$ . Moreover, we may and will take a bicontinuous version of local times  $L(\cdot, \cdot)$ , see [17], Theorem VI.1.7. Let

$$(1.1) \quad \tau_t^r := \inf\{s \geq 0 : L(s, r) > t\},$$

be the inverse local time of  $X$ . Denote by

$$(1.2) \quad T_r := \inf\{t \geq 0 : X_t = r\},$$

the hitting time of  $r$ . The following Ray–Knight theorems were established by Carmona, Petit and Yor [4] (see also Yor [22], Chapter 9) and by Le Gall and Yor [12] respectively.

**Theorem 1.1.** *Fix  $\mu > 0$ .*

(i) ([4], [22]) *Let  $a > 0$ . The process  $(L(\tau_a^0, -h), h \geq 0)$  is a squared Bessel process of dimension  $(2 - \frac{2}{\mu})$ , starting from  $a$  and absorbed at 0.*

(ii) ([12]) *Let  $b < 0$ . The process  $(L(T_b, b+h), 0 \leq h \leq |b|)$  is a squared Bessel process of dimension  $\frac{2}{\mu}$ , starting from 0 and reflected at 0.*

In the special case  $\mu = 1$ : the process  $X$  is Brownian motion by Lévy's identity, so Theorem 1.1 boils down to the classical Ray–Knight theorem for Brownian motion, originally proved by Ray [16] and Knight [9] independently. Werner [21] gave an alternative proof of Theorem 1.1 using a result of Lamperti [11] on semi-stable Markov processes. Perman [14] gave another proof of (i) by establishing a path-decomposition result of  $X$ .

The aim of this work is to describe the underlying Brownian motion, jointly for all  $a$  and  $b$ , in the local time processes in Theorem 1.1. We do this by means of Tanaka's formula and Walsh's stochastic integral with respect to a white noise  $W$ ; see Theorem 1.2 below. The idea of using Tanaka's formula to prove Ray–Knight theorems is not new, and can be found for example in Jeulin [8] (for diffusion processes) and in Norris, Rogers and Williams [13] (for Brownian motion with a local time drift); our main contribution is to show how the white noise  $W$  explicitly gives the Brownian part jointly for all  $a$  and  $b$  in Theorem 1.1.

The aforementioned white noise  $W$  is defined as follows. For any Borel function  $g : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$  such that  $\int_{\mathbb{R}_+} d\ell \int_{\mathbb{R}} g^2(\ell, x) dx < \infty$ , let

$$(1.3) \quad W(g) := \int_0^\infty g(L(t, X_t), X_t) \operatorname{sgn}(B_t) dB_t.$$

It is easily seen that  $W$  is a white noise on  $\mathbb{R}_+ \times \mathbb{R}$ ; indeed, by the occupation time formula (Exercise VI.1.15 in [17]),

$$\int_0^\infty g^2(L(t, X_t), X_t) dt = \int_{\mathbb{R}} dx \int_0^\infty g^2(L(t, x), x) d_t L(t, x) = \int_{\mathbb{R}} dx \int_0^\infty g^2(\ell, x) d\ell.$$

The exponential martingale for Brownian motion implies that

$$\mathbb{E} \left[ e^{W(g) - \frac{1}{2} \int_0^\infty g^2(L(t, X_t), X_t) dt} \right] = 1,$$

showing that  $W(g)$  is a centered Gaussian random variable with variance  $\int_{\mathbb{R}_+} \int_{\mathbb{R}} g^2(\ell, x) dx d\ell$ .

The main result of this work is the following theorem:

**Theorem 1.2.** *Fix  $\mu > 0$ . Let  $W$  be the white noise defined via (1.3).*

(i) *Almost surely for all  $a > 0$ ,*

$$(1.4) \quad L(\tau_a^0, -h) = a - 2 \int_{-h}^0 W([0, L(\tau_a^0, x)], dx) + \left(2 - \frac{2}{\mu}\right) h, \quad h \in [0, |I_{\tau_a^0}|],$$

where  $I_t := \inf_{0 \leq s \leq t} X_s$ ,  $t \geq 0$ , denotes the infimum process of  $X$ .

(ii) *Almost surely for all  $b < 0$ ,*

$$(1.5) \quad L(T_b, b+h) = 2 \int_b^{b+h} W([0, L(T_b, x)], dx) + \frac{2}{\mu} h, \quad h \in [0, |b|].$$

The precise meaning of stochastic integrals with respect to  $W$  is given in Section 3. Indeed, we will show that almost surely (1.4) holds for any fixed  $a > 0$ , hence for all  $a$  belonging to a countable dense set of  $\mathbb{R}_+$ . By using the regularity of local times, we may and will choose a version of stochastic integral such that (1.4) holds simultaneously for all  $a > 0$ . The same remark applies to (1.5) as well as to Theorem 5.1 in Section 5.

It is not surprising, at least in the case (ii), that the local times of a  $\mu$ -process can be represented as solution of an SDE driven by a white noise. As a matter of fact, by duality (see [21]), the process  $(X_{T_b-t} - b, t \in [0, T_b])$  has the same law as the process  $|B_t| + \mu \mathfrak{L}_t$ , stopped when leaving  $|b|$  for the last time. On one hand, the process  $(|B_t| + \mu \mathfrak{L}_t, t \geq 0)$  is the height process of a Feller CSBP with immigration (see [10], remark p. 57 in Section 4). On the other hand, Bertoin and Le Gall showed in [1] that general CSBPs are related to flows of subordinators, constructed in [2] critical CSBPs without Gaussian coefficient as solutions of SDEs driven by compensated Poisson random measures. Dawson and Li [7] generalized this SDE to include a Gaussian coefficient and possible immigration. Applied

to our setting, it is shown that a Feller CSBP with immigration can be constructed as a solution of (1.5). Theorem 1.2 connects directly the local times of the  $\mu$ -process to equation (1.5), without making use of the framework of CSBPs, and in Section 4 we are going to see Theorem 1.1 as a consequence of Tanaka's formula for  $X$ .

The rest of the paper is as follows. In Section 2, we follow Walsh [19] by introducing the excursion filtration, then make an enlargement of the filtration à la Jeulin [8]. Section 3 is devoted to study of the martingale measure associated with the white noise  $W$ . In particular, stochastic integration with respect to  $W$  is defined. Theorems 1.2 and 1.1 are proved in Section 4. Section 5 presents analogous results for the  $\mu$ -process defined on  $\mathbb{R}$ .

## 2 The excursion filtration

We first introduce some notation which will be used throughout the paper.

**Notation 2.1.** Let  $x \in \mathbb{R}$ . We define the process  $X^{-,x}$  obtained by gluing the excursions of  $X$  below  $x$  as follows. Let, for  $t \geq 0$ ,

$$A_t^{-,x} := \int_0^t \mathbf{1}_{\{X_s \leq x\}} ds, \quad \alpha_t^{-,x} := \inf\{u > 0, A_u^{-,x} > t\},$$

with the usual convention  $\inf \emptyset := \infty$ . Define

$$X_t^{-,x} := X_{\alpha_t^{-,x}}, \quad t < A_\infty^{-,x} := \int_0^\infty \mathbf{1}_{\{X_s \leq x\}} ds.$$

Similarly, we define  $A_t^{+,x}$ ,  $\alpha_t^{+,x}$  and  $X^{+,x}$  by replacing  $X_s \leq x$  by  $X_s > x$ . When the process is denoted by  $X$  with some superscript, the analogous quantities keep the same superscript. For example,  $L^{+,x}(t, y)$  denotes the local time of  $X^{+,x}$  at position  $y$  and time  $t$ , and  $I_t^{+,x} = \inf_{0 \leq s \leq t} X_s^{+,x}$ .

**Remark 2.2.** Let  $x \in \mathbb{R}$ . One can reconstruct  $X$  from  $X^{-,x}$  and  $X^{+,x}$  by gluing the excursions of  $X^{-,x}$  and of  $X^{+,x}$ , indexed by their local time.

The following proposition is adapted from Section 8.5 of [22].

**Proposition 2.3.** Let  $x \leq 0$ .

(i) Define the filtration  $(\mathcal{F}_u^{+,x})_{u \geq 0}$  by  $\mathcal{F}_u^{+,x} := \sigma(X_s^{+,x}, s \in [0, u])$  and the process

$$\beta_u^{+,x} := \int_0^{\alpha_u^{+,x}} \mathbf{1}_{\{X_s > x\}} \operatorname{sgn}(B_s) dB_s, \quad u \geq 0.$$

Then  $\beta^{+,x}$  is  $(\mathcal{F}_u^{+,x})$ -Brownian motion and  $X^{+,x}$  is an  $(\mathcal{F}_u^{+,x})$ -semimartingale with decomposition

$$(2.1) \quad X_u^{+,x} = \beta_u^{+,x} - \frac{1-\mu}{\mu} I_u^{+,x} + \frac{1}{2} L^{+,x}(u, x), \quad u \geq 0.$$

(ii) Define the filtration  $(\mathcal{F}_u^{-,x})_{u \geq 0}$  by  $\mathcal{F}_u^{-,x} := \sigma(X_s^{-,x}, s \in [0, u])$  and the process

$$\beta_u^{-,x} := \int_0^{\alpha_u^{-,x}} \mathbf{1}_{\{X_s \leq x\}} \operatorname{sgn}(B_s) dB_s, \quad u \geq 0.$$

Then  $\beta^{-,x}$  is  $(\mathcal{F}_u^{-,x})$ -Brownian motion and  $X^{-,x}$  is an  $(\mathcal{F}_u^{-,x})$ -semimartingale with decomposition

$$(2.2) \quad X_u^{-,x} = x + \beta_u^{-,x} - \frac{1-\mu}{\mu}(I_u^{-,x} - x) - \frac{1}{2}L^{-,x}(u, x), \quad u \geq 0.$$

(iii) The Brownian motions  $\beta^{+,x}$  and  $\beta^{-,x}$  are independent.

*Proof.* By Tanaka's formula,

$$(X_t - x)^+ = (X_0 - x)^+ + \int_0^t \mathbf{1}_{\{X_s > x\}} dX_s + \frac{1}{2}L(t, x).$$

Take  $t = \alpha_u^{+,x}$ . We get

$$(2.3) \quad X_u^{+,x} = \int_0^{\alpha_u^{+,x}} \mathbf{1}_{\{X_s > x\}} dX_s + \frac{1}{2}L(\alpha_u^{+,x}, x).$$

Moreover,  $dX_s = d|B_s| - \mu d\mathfrak{L}_s = \operatorname{sgn}(B_s)dB_s + (1-\mu)d\mathfrak{L}_s$  by another application of Tanaka's formula. Also observe that  $I_t = -\mu\mathfrak{L}_t$ , hence

$$(2.4) \quad dX_s = \operatorname{sgn}(B_s)dB_s - \frac{1-\mu}{\mu}dI_s, \quad s \geq 0.$$

Therefore,

$$X_u^{+,x} = \int_0^{\alpha_u^{+,x}} \mathbf{1}_{\{X_s > x\}} \operatorname{sgn}(B_s) dB_s - \frac{1-\mu}{\mu} \int_0^{\alpha_u^{+,x}} \mathbf{1}_{\{X_s > x\}} dI_s + \frac{1}{2}L(\alpha_u^{+,x}, x).$$

We notice that  $L^{+,x}(u, r) = L(\alpha_u^{+,x}, r)$  for any  $r \in [x, \infty)$  and  $u \geq 0$ . On the other hand,  $\int_0^{\alpha_u^{+,x}} \mathbf{1}_{\{X_s > x\}} dI_s = I_{\alpha_u^{+,x} \wedge T_x}$  which is also the infimum of  $X^{+,x}$  on the time interval  $[0, u]$ . This yields (2.1). This equation also implies that  $\beta^{+,x}$  is adapted to  $\mathcal{F}^{+,x}$ . Moreover, from the definition of  $\beta_u^{+,x}$  and Proposition V.1.5 of [17],  $\beta_u^{+,x}$  is a  $(\mathcal{B}_{\alpha_u^{+,x}})$ -continuous martingale with  $\langle \beta^{+,x}, \beta^{+,x} \rangle_u = u$ , hence  $(\mathcal{B}_{\alpha_u^{+,x}})$ -Brownian motion. Since  $\mathcal{F}_u^{+,x} \subset \mathcal{B}_{\alpha_u^{+,x}}$ , we deduce that  $\beta_u^{+,x}$  is also  $(\mathcal{F}_u^{+,x})$ -Brownian motion. This proves (i).

The proof of (ii) is similar. Tanaka's formula applied to  $(X_{t+T_x} - x)^-$  with  $t = \alpha_u^{-,x} - T_x$  implies that

$$X_u^{-,x} = x + \int_{T_x}^{\alpha_u^{-,x}} \mathbf{1}_{\{X_s \leq x\}} \operatorname{sgn}(B_s) dB_s - \frac{1-\mu}{\mu} \int_{T_x}^{\alpha_u^{-,x}} \mathbf{1}_{\{X_s \leq x\}} dI_s - \frac{1}{2}L(\alpha_u^{-,x}, x).$$

We observe that  $L(\alpha_u^{-,x}, x) = L^{-,x}(u, x)$  and  $\int_{T_x}^{\alpha_u^{-,x}} \mathbf{1}_{\{X_s \leq x\}} dI_s = I_{\alpha_u^{-,x}} - x$  while  $I_{\alpha_u^{-,x}} = I_u^{-,x}$  which gives (2.2). We conclude as for (i). The statement (iii) is a consequence of Knight's theorem on orthogonal martingales.  $\square$

The following result is well-known. It has been proved in Section 8.5 of [22] when  $\mu \in (0, 2)$ , in [21] and in [15]. Here, following [22], we choose to see it as a consequence of Proposition 2.3.

**Corollary 2.4.** *Let  $x \leq 0$ . The processes  $X^{+,x}$  and  $X^{-,x}$  are independent.*

*Proof.* By Proposition 2.3 (iii), the martingale parts of  $X^{+,x}$  and  $X^{-,x}$ , namely  $\beta^{+,x}$  and  $\beta^{-,x}$ , are independent. It remains to see that  $X^{+,x}$  is measurable with respect to  $\beta^{+,x}$  and  $X^{-,x}$  with respect to  $\beta^{-,x}$ , which was established by Chaumont and Doney [5] and Davis [6].  $\square$

The excursion filtration, introduced by Walsh [19], is defined as

$$\mathcal{E}_x^+ := \mathcal{F}_\infty^{+,x} = \sigma(X_s^{+,x}, s \geq 0), \quad x \in \mathbb{R}.$$

Similarly we define  $\mathcal{E}_x^- := \mathcal{F}_\infty^{-,x} = \sigma(X_s^{-,x}, s \geq 0)$  for  $x \in \mathbb{R}$ . It is routine to check, using the time-changes  $\alpha^{-,x}$  and  $\alpha^{+,x}$ , that  $\mathcal{E}_x^-$  is increasing in  $x$  whereas  $\mathcal{E}_x^+$  is decreasing. <sup>4</sup>

Define, for  $u \geq 0$ ,

$$\mathcal{G}_u^{+,x} := \sigma(\mathcal{F}_u^{+,x}, \mathcal{E}_x^-), \quad \mathcal{G}_u^{-,x} := \sigma(\mathcal{F}_u^{-,x}, \mathcal{E}_x^+).$$

The idea of such an enlargement of filtrations goes back at least to Jeulin [8].

**Corollary 2.5.** *Consider a random function  $g(\ell, y) = g(\ell, y, \omega)$  such that the process  $t \mapsto g(L(t, X_t), X_t)$  is  $(\mathcal{B}_t)$ -progressively measurable and  $\mathbb{E}[\int_{\mathbb{R}_+} d\ell \int_{\mathbb{R}} g(\ell, y)^2 dy] < \infty$ . Fix  $x \leq 0$ .*

(i) *The process  $u \mapsto g(L^{+,x}(u, X_u^{+,x}), X_u^{+,x})$  is  $(\mathcal{G}_u^{+,x})$ -progressive and almost surely,*

$$(2.5) \quad \int_0^\infty g(L(t, X_t), X_t) \mathbf{1}_{\{X_t > x\}} \operatorname{sgn}(B_t) dB_t = \int_0^\infty g(L^{+,x}(u, X_u^{+,x}), X_u^{+,x}) d\beta_u^{+,x}.$$

(ii) *The process  $u \mapsto g(L^{-,x}(u, X_u^{-,x}), X_u^{-,x})$  is  $(\mathcal{G}_u^{-,x})$ -progressive and almost surely,*

$$(2.6) \quad \int_0^\infty g(L(t, X_t), X_t) \mathbf{1}_{\{X_t \leq x\}} \operatorname{sgn}(B_t) dB_t = \int_0^\infty g(L^{-,x}(u, X_u^{-,x}), X_u^{-,x}) d\beta_u^{-,x}.$$

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<sup>4</sup>We will be implicitly working with a right-continuous (and complete) version of the filtrations  $(\mathcal{E}_x^-)_{x \in \mathbb{R}}$  and  $(\mathcal{E}_x^+)_{x \in \mathbb{R}}$  — if necessary, by means of the procedure of usual augmentation, as described in Section I.4 of Revuz and Yor [17]. In our work, we only study *continuous* martingales, which are also martingales with respect to augmented filtrations. The same remark applies to the  $\mu$ -process defined on  $\mathbb{R}$  in Section 5.

*Proof.* We prove (i). The process  $u \mapsto g(L^{+,x}(u, X_u^{+,x}), X_u^{+,x})$  is  $(\mathcal{B}_{\alpha_u^{+,x}})$ -progressive (Proposition V.1.4, [17]). Therefore it is also progressive with respect to  $(\mathcal{G}_u^{+,x})$  because the latter filtration is larger. We prove now (2.5). By a time-change (Proposition V.1.4, [17]),

$$\int_0^{\alpha_t^{+,x}} g(L(s, X_s), X_s) \mathbf{1}_{\{X_s > x\}} \operatorname{sgn}(B_s) dB_s = \int_0^t g(L^{+,x}(u, X_u^{+,x}), X_u^{+,x}) d\beta_u^{+,x}.$$

Letting  $t \rightarrow \infty$  yields (2.5). Statement (ii) is proved similarly.  $\square$

### 3 The martingale measure associated with $W$

Recall the definition of the white noise  $W$  in (1.3). For any Borel set  $A$  of  $\mathbb{R}_+$  with finite Lebesgue measure and  $r \geq 0$ , we define

$$(3.1) \quad M_r(A) := W(\mathbf{1}_{A \times [-r, 0]}).$$

**Proposition 3.1.** *In the setting of Walsh [20],  $(M_r, r \geq 0)$  is a continuous martingale measure with respect to the filtration  $(\mathcal{E}_{-r}^+, r \geq 0)$ .*

*Proof.* Since  $W$  is a white noise, it suffices to show that  $M_r$  is measurable with respect to  $\mathcal{E}_{-r}^+$  and that  $M_s - M_r$  is independent of  $\mathcal{E}_{-r}^+$  for any  $0 \leq r < s$ . The first statement comes from (2.5) applied to  $x = -r$  and  $g(\ell, y) = \mathbf{1}_{A \times [-r, 0]}(\ell, y)$  for a Borel set  $A$  with finite Lebesgue measure. The second statement comes from (2.6) applied to  $x = -r$  and  $g(\ell, y) = \mathbf{1}_{A \times [-s, -r]}(\ell, y)$ . Since the processes  $X^{\cdot, -r}$  and  $\beta^{\cdot, -r}$  are independent of  $\mathcal{E}_{-r}^+$ , the proposition follows.  $\square$

We are going to extend (1.3), seen as an equality for deterministic functions  $g$ , to random functions. To this end, we first recall the construction by Walsh in [20] of stochastic integral with respect to the martingale measure  $M$ . A (random) function  $f$  is said to be elementary if it is of the form  $f(\ell, x) := Z \mathbf{1}_{[a, b]}(x) \mathbf{1}_A(\ell)$ , where  $a < b \leq 0$ ,  $A \subset \mathbb{R}_+$  is a Borel set of finite Lebesgue measure, and  $Z$  is a bounded  $\mathcal{E}_b^+$ -measurable real-valued random variable. Denote by  $f \cdot M$  the stochastic integral with respect to  $M$ :

$$f \cdot M := Z (M_{|a|}(A) - M_{|b|}(A)) = Z W(\mathbf{1}_{A \times [a, b]}).$$

A simple function is a (finite) linear combination of elementary functions. We extend by linearity the definition of  $f \cdot M$  to simple functions  $f$  and furthermore by isometry to any  $f \in \mathcal{L}^2$ , where  $\mathcal{L}^2$  denotes the space of  $(\mathcal{E}_{-r}^+, r \geq 0)$ -predictable and square-integrable functions, defined as the closure of the space of simple functions under the norm:

$$\|f\| := \left[ \mathbb{E} \left( \int_{\mathbb{R}_+} d\ell \int_{\mathbb{R}_-} f^2(\ell, x) dx \right) \right]^{1/2}.$$

For any  $f \in \mathcal{L}^2$ ,  $f \cdot M$  is a centered random variable with  $\mathbb{E}[(f \cdot M)^2] = \|f\|^2$ . We write  $f \cdot M \equiv \int_{\mathbb{R}_+ \times \mathbb{R}_-} f(\ell, x) W(d\ell, dx)$  and for any  $r \geq 0$ ,  $(f \mathbf{1}_{\mathbb{R}_+ \times [-r, 0]}) \cdot M \equiv$



$\int_{\mathbb{R}_+ \times [-r, 0]} f(\ell, x) W(d\ell, dx)$ . The latter, if furthermore  $f$  is of form  $f(\ell, x) = \mathbf{1}_{\{0 \leq \ell \leq \sigma_x\}} \eta_x$ , will be re-written as  $\int_{-r}^0 \eta_x W([0, \sigma_x], dx)$ . By the construction of stochastic integral and Proposition 3.1,  $\int_{\mathbb{R}_+ \times [-r, 0]} f(\ell, x) W(d\ell, dx)$  is a continuous martingale with respect to the filtration  $(\mathcal{E}_{-r}^+, r \geq 0)$ , of quadratic variation process  $\int_{\mathbb{R}_+ \times [-r, 0]} f^2(\ell, x) d\ell dx$ .

**Proposition 3.2.** *Take  $g \in \mathcal{L}^2$  such that  $s \mapsto g(L(s, X_s), X_s)$  admits a version which is progressive with respect to the Brownian filtration  $(\mathcal{B}_s)$ .<sup>5</sup> Then*

$$g \cdot M = \int_0^\infty g(L(s, X_s), X_s) \operatorname{sgn}(B_s) dB_s \quad \text{a.s.}$$

*Proof.* By definition of  $\mathcal{L}^2$ , there exists a sequence of simple functions  $g_n$  such that  $\|g - g_n\| \rightarrow 0$  as  $n \rightarrow \infty$ . By isometry,  $g \cdot M - g_n \cdot M \rightarrow 0$  in  $L^2$ . Since  $g_n$  is a simple function,  $g_n$  is of the form

$$g_n(\ell, x) = \sum_{k,j=1}^\infty Z_{k,j}^n \mathbf{1}_{[a_k^n, a_{k-1}^n)}(x) \mathbf{1}_{A_j^n}(\ell),$$

where for each  $n$ ,  $0 = a_0^n > \dots > a_k^n > a_{k+1}^n > \dots$  is a decreasing sequence such that  $a_k^n \rightarrow -\infty$  as  $k \rightarrow \infty$ ,  $(A_j^n)_{j \geq 1}$  is a collection of (nonrandom) pairwise disjoint Borel subsets of  $\mathbb{R}_+$  with finite Lebesgue measures, and for any  $k, j \geq 1$ ,  $Z_{k,j}^n$  is a bounded  $\mathcal{E}_{a_{k-1}^n}^+$ -measurable random variable. Moreover for all large  $k, j$ ,  $Z_{k,j}^n = 0$ , which means the above double sum runs in fact over a finite index set of  $k$  and  $j$ .

Note that for a.e.  $z \leq 0$ ,  $g(\cdot, z)$  is measurable with respect to  $\mathcal{E}_z^+$  (as  $g_n$  is). We may (and will) take a version of  $g$  such that  $g(\cdot, z)$  is measurable with respect to  $\mathcal{E}_z^+$  for all  $z \leq 0$ .

By applying (1.3) and (3.1), we deduce from the linearity of the integral that

$$g_n \cdot M = \sum_{k,j=1}^\infty Z_{k,j}^n \int_0^\infty \mathbf{1}_{[a_k^n, a_{k-1}^n)}(X_s) \mathbf{1}_{A_j^n}(L(s, X_s)) \operatorname{sgn}(B_s) dB_s.$$

Note that we can (and we will) take  $(a_k^n, a_{k-1}^n)$  instead of  $[a_k^n, a_{k-1}^n)$  without changing the value of  $g_n \cdot M$ . By (2.5) and (2.6) respectively,

$$\begin{aligned} & \int_0^\infty \mathbf{1}_{(a_k^n, a_{k-1}^n)}(X_s) \mathbf{1}_{A_j^n}(L(s, X_s)) \operatorname{sgn}(B_s) dB_s \\ (3.2) \quad &= \int_0^\infty \mathbf{1}_{(a_k^n, a_{k-1}^n)}(X_u^{-,x}) \mathbf{1}_{A_j^n}(L^{-,x}(u, X_u^{-,x})) d\beta_u^- \end{aligned}$$

$$(3.3) \quad = \int_0^\infty \mathbf{1}_{(a_k^n, a_{k-1}^n)}(X_u^{+,y}) \mathbf{1}_{A_j^n}(L^{+,y}(u, X_u^{+,y})) d\beta_u^{+,y}$$

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<sup>5</sup>By *version* we mean a  $(\mathcal{B}_s)$ -progressive process  $(h_s)$  such that  $\int_0^\infty \mathbf{1}_{\{g(L(s, X_s), X_s) \neq h_s\}} ds = 0$  a.s.

with  $x = a_{k-1}^n$  and  $y = a_k^n$ . Similarly,

$$(3.4) \quad \begin{aligned} & \int_0^\infty g(L(s, X_s), X_s) \mathbf{1}_{(a_k^n, a_{k-1}^n)}(X_s) \operatorname{sgn}(B_s) dB_s \\ &= \int_0^\infty g(L^{-,x}(u, X_u^{-,x}), X_u^{-,x}) \mathbf{1}_{(a_k^n, a_{k-1}^n)}(X_u^{-,x}) d\beta_u^{-,x}, \end{aligned}$$

$$(3.5) \quad = \int_0^\infty g(L^{+,y}(u, X_u^{+,y}), X_u^{+,y}) \mathbf{1}_{(a_k^n, a_{k-1}^n)}(X_u^{+,y}) d\beta_u^{+,y}$$

with  $x = a_{k-1}^n$  and  $y = a_k^n$  as in (3.2) and (3.3). Write

$$I(g) := \int_0^\infty g(L(s, X_s), X_s) \operatorname{sgn}(B_s) dB_s.$$

Then

$$g_n \cdot M - I(g) = \sum_{k=1}^\infty \Delta_n(k),$$

where, from (3.3) and (3.5),

$$\begin{aligned} \Delta_n(k) &:= \sum_{j=1}^\infty Z_{k,j}^n \int_0^\infty \mathbf{1}_{(a_k^n, a_{k-1}^n)}(X_u^{+,y}) \mathbf{1}_{A_j^n}(L^{+,y}(u, X_u^{+,y})) d\beta_u^{+,y} \\ &\quad - \int_0^\infty g(L^{+,y}(u, X_u^{+,y}), X_u^{+,y}) \mathbf{1}_{(a_k^n, a_{k-1}^n)}(X_u^{+,y}) d\beta_u^{+,y}, \end{aligned}$$

where as before  $y = a_k^n$ . Since  $Z_{k,j}^n$  is  $\mathcal{E}_{a_{k-1}^n}^+$ -measurable, hence  $\mathcal{E}_{a_k^n}^+$ -measurable, we see that the sum over  $j$  in the definition of  $\Delta_n(k)$  is  $\mathcal{E}_{a_k^n}^+$ -measurable. For the last integral in  $\Delta_n(k)$ , we use the fact that for  $z \leq 0$ ,  $g(\cdot, z)$  is measurable with respect to  $\mathcal{E}_z^+$ . Note that  $X_u^{+,y}$  and  $L^{+,y}(\cdot, \cdot)$  are measurable with respect to  $\mathcal{E}_y^+$ . Since  $X_u^{+,y} \geq y$  and  $\mathcal{E}_z^+$  decreases on  $z$ , we deduce that  $g(\cdot, X_u^{+,y})$  is  $\mathcal{E}_y^+$ -measurable, and so is  $g(L^{+,y}(u, X_u^{+,y}), X_u^{+,y})$ . It follows that  $\Delta_n(k)$  is measurable with respect to  $\mathcal{E}_{a_k^n}^+$ .

Now we prove that  $(\Delta_n(k))_{k \geq 1}$  is a martingale difference sequence with respect to the filtration  $(\mathcal{E}_{a_k^n}^+)_{k \geq 1}$ . Indeed, using (3.2) and (3.4) instead of (3.3) and (3.5), one can also write

$$\begin{aligned} \Delta_n(k) &= \sum_{j=1}^\infty Z_{k,j}^n \int_0^\infty \mathbf{1}_{(a_k^n, a_{k-1}^n)}(X_u^{-,x}) \mathbf{1}_{A_j^n}(L^{-,x}(u, X_u^{-,x})) d\beta_u^{-,x} \\ &\quad - \int_0^\infty g(L^{-,x}(u, X_u^{-,x}), X_u^{-,x}) \mathbf{1}_{(a_k^n, a_{k-1}^n)}(X_u^{-,x}) d\beta_u^{-,x}, \end{aligned}$$

with  $x = a_{k-1}^n$ . Recall from Proposition 2.3 that  $\beta^{-,x}$  is  $(\mathcal{F}_u^{-,x})$ -Brownian motion, which is independent of  $\mathcal{E}_x^+$  by Corollary 2.4. Then  $\beta^{-,x}$  can be seen as  $(\mathcal{G}_u^{-,x})$ -Brownian motion.

By Corollary 2.5 (ii),  $g(L^{-,x}(u, X_u^{-,x}), X_u^{-,x})$  is progressive with respect to the filtration  $(\mathcal{G}_u^{-,x})$ , while

$$\mathbf{1}_{(a_k^n, a_{k-1}^n)}(X_u^{-,x}) g_n(L^{-,x}(u, X_u^{-,x}), X_u^{-,x}) = \sum_{j=1}^{\infty} Z_{k,j}^n \mathbf{1}_{(a_k^n, a_{k-1}^n)}(X_u^{-,x}) \mathbf{1}_{A_j^n}(L^{-,x}(u, X_u^{-,x}))$$

is  $(\mathcal{G}_u^{-,x})$ -progressive as well. Therefore, one can write

$$(3.6) \quad \Delta_n(k) = \int_0^{\infty} \mathbf{1}_{(a_k^n, a_{k-1}^n)}(X_u^{-,x}) (g_n - g)(L^{-,x}(u, X_u^{-,x}), X_u^{-,x}) d\beta_u^{-,x},$$

with  $x = a_{k-1}^n$ . It follows that (since  $\mathcal{G}_0^{-,x} = \mathcal{E}_x^+$ )

$$\mathbb{E}[\Delta_n(k) \mid \mathcal{E}_{a_{k-1}^n}^+] = 0.$$

In other words, the process  $j \rightarrow \sum_{k=1}^j \Delta_n(k)$  is a martingale and we have

$$\mathbb{E}[(g_n \cdot M - I(g))^2] = \sum_{k=1}^{\infty} \mathbb{E}[\Delta_n(k)^2].$$

From (3.6), we get

$$\begin{aligned} \mathbb{E}[\Delta_n(k)^2] &= \mathbb{E} \left[ \int_0^{\infty} \mathbf{1}_{(a_k^n, a_{k-1}^n)}(X_u^{-,x}) ((g_n - g)(L^{-,x}(u, X_u^{-,x}), X_u^{-,x}))^2 du \right] \\ &= \mathbb{E} \left[ \int_{a_k^n}^{a_{k-1}^n} (g_n(\ell, z) - g(\ell, z))^2 d\ell dz \right], \end{aligned}$$

where in the second equality we have used the occupation time formula. It follows that

$$\mathbb{E}[(g_n \cdot M - I(g))^2] = \|g_n - g\|^2$$

which goes to 0 as  $n \rightarrow \infty$ . Thus we get that  $g \cdot M = I(g)$ .  $\square$

**Remark 3.3.** Fix  $b < 0$ . Similarly to (3.1) we may define a martingale measure  $\widehat{M}$  by

$$\widehat{M}_r(A) := W(\mathbf{1}_{A \times [b, b+r]}),$$

for any Borel set  $A \subset \mathbb{R}_+$  of finite Lebesgue measure and  $r \geq 0$ . The analogs of Propositions 3.1 and 3.2 hold for  $\widehat{M}$ . Specifically,  $(\widehat{M}_r, r \geq 0)$  is a martingale measure with respect to the filtration  $(\mathcal{E}_{b+r}^-, r \geq 0)$ . Moreover, we may define in a similar way the stochastic integral  $f \cdot \widehat{M}$  for any  $f \in \widehat{\mathcal{L}}^2$ , where  $\widehat{\mathcal{L}}^2$  denotes the space of  $(\mathcal{E}_{b+r}^-)_{r \geq 0}$ -predictable functions  $f$  such that  $\mathbb{E}[\int_{\mathbb{R}_+} d\ell \int_{[b, \infty)} f^2(\ell, x) dx] < \infty$ . Then for any  $g \in \widehat{\mathcal{L}}^2$

such that  $s \mapsto g(L(s, X_s), X_s)$  admits a version which is progressive with respect to the Brownian filtration  $(\mathcal{B}_s)$ , we have

$$(3.7) \quad g \cdot \widehat{M} = \int_0^\infty g(L(s, X_s), X_s) dX_s - \frac{1-\mu}{\mu} \int_{-\infty}^0 g(0, x) dx, \quad \text{a.s.}$$

With a slight abuse of notation, we shall write  $g \cdot \widehat{M} \equiv \int_{\mathbb{R}_+ \times [b, \infty)} g(\ell, x) W(d\ell, dx)$  and  $(g \mathbf{1}_{\mathbb{R}_+ \times [b, t)}) \cdot \widehat{M} \equiv \int_{\mathbb{R}_+ \times [b, t)} g(\ell, x) W(d\ell, dx)$  for any  $t \geq b$ . Then  $\int_{\mathbb{R}_+ \times [b, b+r)} g(\ell, x) W(d\ell, dx)$ , is an  $(\mathcal{E}_{b+r}^-)$ -continuous martingale with quadratic variation process  $\int_{\mathbb{R}_+ \times [b, b+r)} g^2(\ell, x) d\ell dx$  for  $r \geq 0$ .

## 4 Proofs of Theorems 1.2 and 1.1

By Tanaka's formula, for any  $r \geq 0$  and  $x \in \mathbb{R}$ ,

$$(4.1) \quad (X_r - x)^- = (-x)^- - \int_0^r \mathbf{1}_{\{X_s \leq x\}} dX_s + \frac{1}{2} L(r, x).$$

*Part (i) of Theorems 1.2 and 1.1:* Applying (4.1) to  $r = \tau_a^0$  gives that  $L(\tau_a^0, x) = 2 \int_0^{\tau_a^0} \mathbf{1}_{\{X_s \leq x\}} dX_s$  for all  $x \in \mathbb{R}$ . Let  $h \geq 0$ . Taking  $x = -h$  and  $x = 0$ , and using the fact that  $L(\tau_a^0, 0) = a$ , we obtain that

$$(4.2) \quad \begin{aligned} L(\tau_a^0, -h) &= a - 2 \int_0^{\tau_a^0} \mathbf{1}_{\{-h < X_s \leq 0\}} dX_s \\ &= a - 2 \int_0^{\tau_a^0} \mathbf{1}_{\{-h < X_s \leq 0\}} \text{sgn}(B_s) dB_s + \left(2 - \frac{2}{\mu}\right) \min(h, |I_{\tau_a^0}|), \end{aligned}$$

where the second equality follows from (2.4) and the fact that  $\int_0^{\tau_a^0} \mathbf{1}_{\{-h < X_s \leq 0\}} dI_s = \int_0^{\tau_a^0} \mathbf{1}_{\{I_s > -h\}} dI_s = I_{\min(T_{-h}, I_{\tau_a^0})} = -\min(h, |I_{\tau_a^0}|)$ .

To deal with the stochastic integral with respect to  $(B_s)$  in (4.2), we shall use Proposition 3.2. First we remark that for  $x \leq 0$ ,  $L(\tau_a^0, x)$  is measurable with respect to  $\mathcal{E}_x^+$ . In fact, let  $u := A_{\tau_a^0}^{+, x}$ . Then  $\alpha_u^{+, x} = \tau_a^0$  as  $\tau_a^0$  is an increasing time for  $A^{+, x}$ . Therefore  $L(\tau_a^0, x) = L^{+, x}(A_{\tau_a^0}^{+, x}, x)$ . Since  $\{A_{\tau_a^0}^{+, x} \geq t\} = \{L(\alpha_t^{+, x}, 0) \leq a\} = \{L^{+, x}(t, 0) \leq a\}$ , we obtain that  $L(\tau_a^0, x)$  is measurable with respect to  $\mathcal{E}_x^+$ .

Let  $g(\ell, x) := \mathbf{1}_{\{0 \leq \ell \leq L(\tau_a^0, x)\} \cap \{-h < x \leq 0\}}$ . Using the continuity of local times  $L(\tau_a^0, x)$  on  $x$  and the fact that  $\mathbb{E}[\int_{\mathbb{R}_+ \times \mathbb{R}} g^2(\ell, x) d\ell dx] < \infty$ , we get that  $g \in \mathcal{L}^2$ .

Observe that a.s.,  $g(L(s, X_s), X_s) = \mathbf{1}_{\{0 \leq s \leq \tau_a^0, -h < X_s \leq 0\}}$ , ds-a.e. This follows from the fact that  $\int_0^\infty \mathbf{1}_{\{L(s, X_s) \leq L(\tau_a^0, X_s), s > \tau_a^0\}} ds = \int_{-\infty}^\infty dx \int_{\tau_a^0}^\infty \mathbf{1}_{\{L(s, x) \leq L(\tau_a^0, x)\}} d_s L(s, x) = 0$ , by the occupation time formula. Then  $s \mapsto g(L(s, X_s), X_s)$  admits a version which is  $(\mathcal{B}_s)$ -progressive and we are entitled to apply Proposition 3.2 to see that

$$\int_0^{\tau_a^0} \mathbf{1}_{\{-h < X_s \leq 0\}} \text{sgn}(B_s) dB_s = \int_{\mathbb{R}_+ \times (-h, 0]} \mathbf{1}_{\{0 \leq \ell \leq L(\tau_a^0, x)\}} W(d\ell, dx) = \int_{-h}^0 W([0, L(\tau_a^0, x)], dx).$$

In view of (4.2), this yields Theorem 1.2 (i) for each fixed  $a > 0$  and  $h \in [0, |I_{\tau_a^0}|]$ . Since the processes are continuous in  $h$  and càdlàg in  $a$ , they coincide except on a null set.

Theorem 1.1 (i) follows quite simply from Theorem 1.2 (i): for given  $a > 0$ , since  $h \mapsto \int_{-h}^0 W([0, L(\tau_a^0, x)], dx)$  is an  $(\mathcal{E}_{-h}^+)_{h \geq 0}$ -continuous martingale with quadratic variation process  $\int_{-h}^0 L(\tau_a^0, x) dx$ , it follows from the Dambis–Dubins–Schwarz theorem that there exists  $(\mathcal{E}_{-h}^+)$ -Brownian motion  $\gamma$  such that  $\int_{-h}^0 W([0, L(\tau_a^0, x)], dx) = \int_0^h \sqrt{L(\tau_a^0, s)} d\gamma_s$ . Going back to (4.2), we see that for all  $0 \leq h \leq \inf\{s \geq 0 : L(\tau_a^0, -s) = 0\} = |I_{\tau_a^0}|$ ,

$$L(\tau_a^0, -h) = a - 2 \int_0^h \sqrt{L(\tau_a^0, s)} d\gamma_s + \left(2 - \frac{2}{\mu}\right)h,$$

proving Theorem 1.1 (i).

*Part (ii) of Theorems 1.2 and 1.1:* Let  $b < 0$  and  $T_b := \inf\{t \geq 0 : X_t = b\}$ . For  $h \in [0, |b|]$ , we get from (4.1) that

$$\begin{aligned} L(T_b, b+h) &= 2h + 2 \int_0^{T_b} \mathbf{1}_{\{X_s \leq b+h\}} dX_s \\ (4.3) \qquad &= 2 \int_0^{T_b} \mathbf{1}_{\{X_s \leq b+h\}} \operatorname{sgn}(B_s) dB_s + \frac{2}{\mu}h, \end{aligned}$$

where the second equality follows from (2.4) again and the fact that  $\int_0^{T_b} \mathbf{1}_{\{X_s \leq b+h\}} dI_s = \int_0^{T_b} \mathbf{1}_{\{I_s \leq b+h\}} dI_s = -h$ .

The main difference with Part (i) is the measurability. As a matter of fact, for any  $x \geq b$ ,  $L(T_b, x)$  is  $\mathcal{E}_x^-$ -measurable: observe that  $L(T_b, x) = L^{-,x}(A_{T_b}^{-,x}, x)$  and for any  $t \geq 0$ ,  $\{A_{T_b}^{-,x} > t\} = \{T_b > \alpha_t^{-,x}\} = \{\inf_{0 \leq s \leq \alpha_t^{-,x}} X_s > b\} = \{\inf_{0 \leq s \leq t} X_s^{-,x} > b\}$  is  $\mathcal{E}_x^-$ -measurable.

Let  $g(\ell, x) := \mathbf{1}_{\{0 < \ell \leq L(T_b, x)\} \times \{b \leq x \leq b+h\}}$ . We can check as in (i) that we may apply (3.7) to get that

$$\int_0^{T_b} \mathbf{1}_{\{X_s \leq b+h\}} \operatorname{sgn}(B_s) dB_s = \int_{\mathbb{R}_+ \times [b, b+h]} \mathbf{1}_{\{0 \leq \ell \leq L(T_b, b+h)\}} W(d\ell, dx) = \int_b^{b+h} W([0, L(T_b, x)], dx),$$

proving, in view of (4.3), Theorem 1.2 (ii). Furthermore by Remark 3.3, the process  $h \rightarrow \int_b^{b+h} W([0, L(T_b, x)], dx)$  is an  $(\mathcal{E}_{b+h}^-)$ -continuous martingale with quadratic variation process  $\int_b^{b+h} L(T_b, x) dx$ . This easily yields Theorem 1.1 (ii).  $\square$

## 5 Extension to the two-sided $\mu$ -process

In this Section, we shall explore the strong Markov property at the hitting times of a  $\mu$ -process defined on  $\mathbb{R}$  and present an analogue of Theorem 1.2. This result, apart from its own interest, will be useful in a forthcoming work on the duality of Jacobi stochastic flows.

Let  $(B_t)_{t \in \mathbb{R}}$  be a two-sided Brownian motion, which means that for  $t \leq 0$ ,  $B_t = B'_{-t}$ , where  $B'$  is a standard Brownian motion independent of  $(B_t)_{t \geq 0}$ . Denote by  $(\mathfrak{L}'_t)_{t \geq 0}$  the local time process at position zero of  $B'$ .

Recall that  $X_t = |B_t| - \mu \mathfrak{L}_t$ , for  $t \geq 0$ . For  $t \leq 0$ , we let  $X_t := |B'_{-t}| + \mu \mathfrak{L}'_{-t}$ . We call  $(X_t)_{t \in \mathbb{R}}$  a two-sided  $\mu$ -process. Fix  $\mu > 0$ . Notice that  $X_t \rightarrow \infty$  as  $t \rightarrow -\infty$ , and  $T_r < 0$  when  $r > 0$ .

We naturally extend the notation  $T_r := \inf\{t \in \mathbb{R} : X_t = r\}$  for  $r \in \mathbb{R}$ ,

$$L(t, x) := \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_{-\infty}^t \mathbf{1}_{\{x \leq X_s \leq x + \varepsilon\}} ds, \quad x \in \mathbb{R},$$

the local time accumulated by  $(X_t, t \in \mathbb{R})$  at position  $x$  up to time  $t$ , and

$$\tau_a^x := \inf\{t \in \mathbb{R} : L(t, x) > a\}, \quad a \geq 0,$$

the inverse local time at position  $x$ . We define now for bounded Borel functions  $g$  with compact support,

$$(5.1) \quad W(g) := \int_{-\infty}^{\infty} g(L(t, X_t), X_t) \operatorname{sgn}(B_t) dB_t.$$

The stochastic integral has to be understood as an integral with respect to the Brownian motion  $B^{(r)} := (B_{t+T_r}, t \geq 0)$  where  $r$  is any positive real such that  $g(\ell, x) = 0$  for all  $x \geq r$  (that  $B^{(r)}$  is a standard Brownian motion comes from the fact that  $(B'_{T_r+t}, t \in [0, |T_r|])$  is distributed as  $(B_t, t \in [0, T_{-r}])$ ). We will see in the following theorem that  $W$  defines a white noise.

Similarly to Notation 2.1, for  $x \in \mathbb{R}$ , we can consider the process  $(X_u^{-,x}, u \geq 0)$  obtained by gluing the excursions of  $X$  below  $x$ : that is we set for  $t \in \mathbb{R}$ ,  $A_t^{-,x} := \int_{-\infty}^t \mathbf{1}_{\{X_s \leq x\}} ds$ ,  $\alpha_u^{-,x} := \inf\{t \in \mathbb{R} : A_t^{-,x} > u\}$ , and  $X_u^{-,x} := X_{\alpha_u^{-,x}}$  for  $u \geq 0$ . The excursion filtration  $\mathcal{E}^-$  is defined as  $\mathcal{E}_x^- := \sigma(X_u^{-,x}, u \geq 0)$ , for all  $x \in \mathbb{R}$ .

For  $\mathcal{E}_x^+$ , we let  $A_t^{+,x} := t$  if  $t \leq T_x$ , and for  $t > T_x$ ,  $A_t^{+,x} := T_x + \int_{T_x}^t \mathbf{1}_{\{X_s > x\}} ds$ . For  $s \in \mathbb{R}$ , we set  $\alpha_s^{+,x} := \inf\{t \in \mathbb{R}, A_t^{+,x} > s\}$ , and  $X_s^{+,x} := X_{\alpha_s^{+,x}}$ . The excursion filtration  $\mathcal{E}^+$  is defined as  $\mathcal{E}_x^+ := \sigma(X_s^{+,x}, s \in \mathbb{R})$ , for all  $x \in \mathbb{R}$ . Note that  $\mathcal{E}_x^+$  is decreasing in  $x$  whereas  $\mathcal{E}_x^-$  is increasing.

**Theorem 5.1.** Fix  $\mu > 0$ . Equation (5.1) defines a white noise on  $\mathbb{R}_+ \times \mathbb{R}$ .

(i) Almost surely for all  $a > 0$ ,  $r \in \mathbb{R}$  and  $h \in [0, |I_{\tau_a^r} - r|]$ ,

$$(5.2) \quad L(\tau_a^r, r - h) = a - 2 \int_{r-h}^r W([0, L(\tau_a^r, x)], dx) + \left(2 - \frac{2}{\mu}\right) h,$$

where  $I_t := \inf_{-\infty < s \leq t} X_s, t \in \mathbb{R}$ , denotes the infimum process of  $X$ .

(ii) Almost surely for all  $a > 0$ ,  $r \in \mathbb{R}$  and  $h \geq 0$ ,

$$(5.3) \quad L(\tau_a^r, r + h) = a + 2 \int_r^{r+h} W([0, L(\tau_a^r, x)], dx) + \frac{2}{\mu} h.$$

The stochastic integral  $\int_{r-h}^r W([0, L(\tau_a^r, x)], dx)$  is the stochastic integral with respect to the martingale measure  $M_h^{(r)}(\bullet) := W(\mathbf{1}_{\bullet \times [r-h, r]})$ ,  $h \geq 0$ , associated to the filtration  $(\mathcal{E}_{r-h}^+)_{h \geq 0}$ . The stochastic integral  $\int_r^{r+h} W([0, L(\tau_a^r, x)], dx)$  is the stochastic integral with respect to the martingale measure  $\widehat{M}_h^{(r)}(\bullet) := W(\mathbf{1}_{\bullet \times [r, r+h]})$ ,  $h \geq 0$ , associated to the filtration  $(\mathcal{E}_{r+h}^-)_{h \geq 0}$ .

*Proof.* Notice that for any  $r \in \mathbb{R}$ ,  $X^{(r)} := (X_{T_r+t} - r, t \geq 0)$  is distributed as  $(X_t, t \geq 0)$ . Therefore we can apply Theorem 1.2 to  $X^{(r)}$ . As in (1.3), we define  $W^{(r)}$  the white noise associated to  $X^{(r)}$ : for any Borel function  $g : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$  such that  $\int_{\mathbb{R}_+} dl \int_{\mathbb{R}} g^2(\ell, x) dx < \infty$ ,

$$W^{(r)}(g) := \int_0^\infty g(L^{(r)}(t, X_t^{(r)}), X_t^{(r)}) \operatorname{sgn}(B_t^{(r)}) dB_t^{(r)},$$

where  $L^{(r)}(\cdot, \cdot)$  denote the local times of  $X^{(r)}$ . Let  $\tau_a^{x, (r)}$  be the associated inverse local times. By Theorem 1.2 (i) applied to  $X^{(r)}$ , for  $a > 0$ ,

$$(5.4) \quad L^{(r)}(\tau_a^{0, (r)}, -h) = a - 2 \int_{-h}^0 W^{(r)}([0, L^{(r)}(\tau_a^{0, (r)}, x)], dx) + \left(2 - \frac{2}{\mu}\right) h, \quad h \in [0, |I_{\tau_a^{0, (r)}}^{(r)}|],$$

where  $I_t^{(r)} := \inf_{0 \leq s \leq t} X_s^{(r)}$ ,  $t \geq 0$ , is the infimum process of  $X^{(r)}$ . Notice that  $\tau_a^{0, (r)} = \tau_a^r - T_r$ ,  $L^{(r)}(t, x) = L(t + T_r, r + x)$  for  $x \leq 0$ , and for  $t \geq 0$ ,  $I_t^{(r)} = I_{t+T_r} - r$  where  $I_t := \inf_{-\infty < s \leq t} X_s$ . Moreover, for any  $x \leq y$ ,

$$W^{(r)}(\mathbf{1}_{A \times [x, y]}) = \int_0^\infty \mathbf{1}_{A \times [x, y]}(L^{(r)}(t, X_t^{(r)}), X_t^{(r)}) \operatorname{sgn}(B_t^{(r)}) dB_t^{(r)}$$

where we recall that  $B_t^{(r)} = B_{t+T_r}$ ,  $t \geq 0$ . We deduce that if  $x \leq y \leq 0$  and  $A$  is a bounded Borel set of  $\mathbb{R}_+$ ,

$$(5.5) \quad \begin{aligned} W^{(r)}(\mathbf{1}_{A \times [x, y]}) &= \int_0^\infty \mathbf{1}_{A \times [x, y]}(L(t + T_r, X_{t+T_r}), X_{t+T_r} - r) \operatorname{sgn}(B_t^{(r)}) dB_t^{(r)} \\ &= \int_{T_r}^\infty \mathbf{1}_{A \times [r+x, r+y]}(L(t, X_t), X_t) \operatorname{sgn}(B_t) dB_t \\ &= W(\mathbf{1}_{A \times [r+x, r+y]}). \end{aligned}$$

We deduce that (5.1) defines a white noise on  $\mathbb{R}_+ \times (-\infty, r)$ , hence on  $\mathbb{R}_+ \times \mathbb{R}$  since  $r$  can be made arbitrary large. Equation (5.4) becomes

$$L(\tau_a^r, r - h) = a - 2 \int_{r-h}^r W([0, L(\tau_a^r, x)], dx) + \left(2 - \frac{2}{\mu}\right) h, \quad h \in [0, |I_{\tau_a^r} - r|].$$

It is (5.2). We prove now (5.3). Let  $r' > r + h$  arbitrary. Using Tanaka's formula applied to  $(X_{T_{r'}+t}, t \geq 0)$ , we have for any  $x \in \mathbb{R}$  and  $t \geq T_{r'}$ ,

$$(5.6) \quad (X_t - x)^- = (r' - x)^- - \int_{T_{r'}}^t \mathbf{1}_{\{X_s \leq x\}} dX_s + \frac{1}{2}(L(t, x) - L(T_{r'}, x)).$$

Taking  $t = \tau_a^r$ , and  $x = r$  then  $x = r + h$ , we get

$$L(\tau_a^r, r + h) = 2h + a + 2 \int_{T_{r'}}^{\tau_a^r} \mathbf{1}_{\{r < X_s \leq r+h\}} dX_s = a + 2 \int_{T_{r'}}^{\tau_a^r} \mathbf{1}_{\{r < X_s \leq r+h\}} \operatorname{sgn}(B_s) dB_s + \frac{2}{\mu} h$$

where the second equality follows from (2.4) and the fact that  $\int_{T_{r'}}^{\tau_a^r} \mathbf{1}_{\{r < X_s \leq r+h\}} dI_s = \int_{T_{r'}}^{\tau_a^r} \mathbf{1}_{\{I_s \leq r+h\}} dI_s = -h$ . By Proposition 3.2 applied to  $(X^{(r')}, B^{(r')})$ ,

$$\begin{aligned} \int_{T_{r'}}^{\tau_a^r} \mathbf{1}_{\{r < X_s \leq r+h\}} \operatorname{sgn}(B_s) dB_s &= \int_0^{\tau_a^{r-r', (r')}} \mathbf{1}_{\{r-r' < X_s^{(r')} \leq r+h-r'\}} \operatorname{sgn}(B_s^{(r')}) dB_s^{(r')} \\ &= \int_{r-r'}^{r+h-r'} W^{(r')}([0, L(\tau_a^r, x)], dx) \\ &= \int_r^{r+h} W([0, L(\tau_a^r, x)], dx) \end{aligned}$$

by (5.5) applied to  $r = r'$ . This proves (5.3) and completes the proof of the theorem.  $\square$

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