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► **To cite this version:**

Elie Aïdékon, Yueyun Hu, Zhan Shi. An infinite-dimensional representation of the Ray-Knight theorems. *Science China Mathematics*, 2022, 10.1007/s11425-022-2068-0 . hal-03035292

HAL Id: hal-03035292

<https://hal.science/hal-03035292v1>

Submitted on 2 Dec 2020

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An infinite-dimensional representation of the Ray–Knight theorems

by

Elie Aïdékon¹, Yueyun Hu², and Zhan Shi³

Summary. The classical Ray–Knight theorems for Brownian motion determine the law of its local time process either at the first hitting time of a given value a by the local time at the origin, or at the first hitting time of a given position b by Brownian motion. We extend these results by describing the local time process jointly for all a and all b , by means of stochastic integral with respect to an appropriate white noise. Our result applies to μ -processes, and has an immediate application: a μ -process is the height process of a Feller continuous-state branching process (CSBP) with immigration (Lambert [10]), whereas a Feller CSBP with immigration satisfies a stochastic differential equation driven by a white noise (Dawson and Li [7]); our result gives an explicit relation between these two descriptions and shows that the stochastic differential equation in question is a reformulation of Tanaka’s formula.

Keywords. Ray–Knight theorem, μ -process, white noise, Tanaka’s formula.

2010 Mathematics Subject Classification. 60J65, 60J55.

1 Introduction

Let $(B_t)_{t \geq 0}$ be standard one-dimensional Brownian motion associated with its completed natural filtration $(\mathcal{B}_t)_{t \geq 0}$. Denote by $(\mathfrak{L}_t)_{t \geq 0}$ a continuous version of local times of (B_t) at position 0. Let $\mu \in \mathbb{R} \setminus \{0\}$. The μ -process $X := (X_t)_{t \geq 0}$ is defined as follows:

$$X_t := |B_t| - \mu \mathfrak{L}_t, \quad t \geq 0.$$

There are two important special cases of μ -processes: Brownian motion ($\mu = 1$, this is seen using Lévy’s identity), and the three-dimensional Bessel process ($\mu = -1$, seen by means of Lévy’s and Pitman’s identities).

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The μ -process, also referred to as perturbed reflecting Brownian motion, has attracted much attention in the nineties: Lévy's arc sine law, Ray–Knight theorems as well as pathwise uniqueness of doubly perturbed Brownian motion, see for example [12, 22, 3, 4, 18, 21, 5, 6, 14, 15].

The local time of the μ -process at suitable stopping times, as a process of the space variable, turns out to be a squared Bessel process. This is referred to as a Ray–Knight theorem. More precisely, let us fix $\mu > 0$ from now on, so the process $(X_t, t \geq 0)$ is recurrent on \mathbb{R} . Since X is a continuous semimartingale, we may define

$$L(t, r) := \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_0^t \mathbf{1}_{\{r \leq X_s \leq r + \varepsilon\}} ds, \quad t \geq 0, r \in \mathbb{R},$$

as the local time of X at time t and position r . Moreover, we may and will take a bicontinuous version of local times $L(\cdot, \cdot)$, see [17], Theorem VI.1.7. Let

$$(1.1) \quad \tau_t^r := \inf\{s \geq 0 : L(s, r) > t\},$$

be the inverse local time of X . Denote by

$$(1.2) \quad T_r := \inf\{t \geq 0 : X_t = r\},$$

the hitting time of r . The following Ray–Knight theorems were established by Carmona, Petit and Yor [4] (see also Yor [22], Chapter 9) and by Le Gall and Yor [12] respectively.

Theorem 1.1. *Fix $\mu > 0$.*

(i) ([4], [22]) *Let $a > 0$. The process $(L(\tau_a^0, -h), h \geq 0)$ is a squared Bessel process of dimension $(2 - \frac{2}{\mu})$, starting from a and absorbed at 0.*

(ii) ([12]) *Let $b < 0$. The process $(L(T_b, b+h), 0 \leq h \leq |b|)$ is a squared Bessel process of dimension $\frac{2}{\mu}$, starting from 0 and reflected at 0.*

In the special case $\mu = 1$: the process X is Brownian motion by Lévy's identity, so Theorem 1.1 boils down to the classical Ray–Knight theorem for Brownian motion, originally proved by Ray [16] and Knight [9] independently. Werner [21] gave an alternative proof of Theorem 1.1 using a result of Lamperti [11] on semi-stable Markov processes. Perman [14] gave another proof of (i) by establishing a path-decomposition result of X .

The aim of this work is to describe the underlying Brownian motion, jointly for all a and b , in the local time processes in Theorem 1.1. We do this by means of Tanaka's formula and Walsh's stochastic integral with respect to a white noise W ; see Theorem 1.2 below. The idea of using Tanaka's formula to prove Ray–Knight theorems is not new, and can be found for example in Jeulin [8] (for diffusion processes) and in Norris, Rogers and Williams [13] (for Brownian motion with a local time drift); our main contribution is to show how the white noise W explicitly gives the Brownian part jointly for all a and b in Theorem 1.1.

The aforementioned white noise W is defined as follows. For any Borel function $g : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$ such that $\int_{\mathbb{R}_+} d\ell \int_{\mathbb{R}} g^2(\ell, x) dx < \infty$, let

$$(1.3) \quad W(g) := \int_0^\infty g(L(t, X_t), X_t) \operatorname{sgn}(B_t) dB_t.$$

It is easily seen that W is a white noise on $\mathbb{R}_+ \times \mathbb{R}$; indeed, by the occupation time formula (Exercise VI.1.15 in [17]),

$$\int_0^\infty g^2(L(t, X_t), X_t) dt = \int_{\mathbb{R}} dx \int_0^\infty g^2(L(t, x), x) d_t L(t, x) = \int_{\mathbb{R}} dx \int_0^\infty g^2(\ell, x) d\ell.$$

The exponential martingale for Brownian motion implies that

$$\mathbb{E} \left[e^{W(g) - \frac{1}{2} \int_0^\infty g^2(L(t, X_t), X_t) dt} \right] = 1,$$

showing that $W(g)$ is a centered Gaussian random variable with variance $\int_{\mathbb{R}_+} \int_{\mathbb{R}} g^2(\ell, x) dx d\ell$.

The main result of this work is the following theorem:

Theorem 1.2. *Fix $\mu > 0$. Let W be the white noise defined via (1.3).*

(i) *Almost surely for all $a > 0$,*

$$(1.4) \quad L(\tau_a^0, -h) = a - 2 \int_{-h}^0 W([0, L(\tau_a^0, x)], dx) + \left(2 - \frac{2}{\mu}\right) h, \quad h \in [0, |I_{\tau_a^0}|],$$

where $I_t := \inf_{0 \leq s \leq t} X_s$, $t \geq 0$, denotes the infimum process of X .

(ii) *Almost surely for all $b < 0$,*

$$(1.5) \quad L(T_b, b+h) = 2 \int_b^{b+h} W([0, L(T_b, x)], dx) + \frac{2}{\mu} h, \quad h \in [0, |b|].$$

The precise meaning of stochastic integrals with respect to W is given in Section 3. Indeed, we will show that almost surely (1.4) holds for any fixed $a > 0$, hence for all a belonging to a countable dense set of \mathbb{R}_+ . By using the regularity of local times, we may and will choose a version of stochastic integral such that (1.4) holds simultaneously for all $a > 0$. The same remark applies to (1.5) as well as to Theorem 5.1 in Section 5.

It is not surprising, at least in the case (ii), that the local times of a μ -process can be represented as solution of an SDE driven by a white noise. As a matter of fact, by duality (see [21]), the process $(X_{T_b-t} - b, t \in [0, T_b])$ has the same law as the process $|B_t| + \mu \mathfrak{L}_t$, stopped when leaving $|b|$ for the last time. On one hand, the process $(|B_t| + \mu \mathfrak{L}_t, t \geq 0)$ is the height process of a Feller CSBP with immigration (see [10], remark p. 57 in Section 4). On the other hand, Bertoin and Le Gall showed in [1] that general CSBPs are related to flows of subordinators, constructed in [2] critical CSBPs without Gaussian coefficient as solutions of SDEs driven by compensated Poisson random measures. Dawson and Li [7] generalized this SDE to include a Gaussian coefficient and possible immigration. Applied

to our setting, it is shown that a Feller CSBP with immigration can be constructed as a solution of (1.5). Theorem 1.2 connects directly the local times of the μ -process to equation (1.5), without making use of the framework of CSBPs, and in Section 4 we are going to see Theorem 1.1 as a consequence of Tanaka's formula for X .

The rest of the paper is as follows. In Section 2, we follow Walsh [19] by introducing the excursion filtration, then make an enlargement of the filtration à la Jeulin [8]. Section 3 is devoted to study of the martingale measure associated with the white noise W . In particular, stochastic integration with respect to W is defined. Theorems 1.2 and 1.1 are proved in Section 4. Sections 5 presents analogous results for the μ -process defined on \mathbb{R} .

2 The excursion filtration

We first introduce some notation which will be used throughout the paper.

Notation 2.1. Let $x \in \mathbb{R}$. We define the process $X^{-,x}$ obtained by gluing the excursions of X below x as follows. Let, for $t \geq 0$,

$$A_t^{-,x} := \int_0^t \mathbf{1}_{\{X_s \leq x\}} ds, \quad \alpha_t^{-,x} := \inf\{u > 0, A_u^{-,x} > t\},$$

with the usual convention $\inf \emptyset := \infty$. Define

$$X_t^{-,x} := X_{\alpha_t^{-,x}}, \quad t < A_\infty^{-,x} := \int_0^\infty \mathbf{1}_{\{X_s \leq x\}} ds.$$

Similarly, we define $A_t^{+,x}$, $\alpha_t^{+,x}$ and $X^{+,x}$ by replacing $X_s \leq x$ by $X_s > x$. When the process is denoted by X with some superscript, the analogous quantities keep the same superscript. For example, $L^{+,x}(t, y)$ denotes the local time of $X^{+,x}$ at position y and time t , and $I_t^{+,x} = \inf_{0 \leq s \leq t} X_s^{+,x}$.

Remark 2.2. Let $x \in \mathbb{R}$. One can reconstruct X from $X^{-,x}$ and $X^{+,x}$ by gluing the excursions of $X^{-,x}$ and of $X^{+,x}$, indexed by their local time.

The following proposition is adapted from Section 8.5 of [22].

Proposition 2.3. Let $x \leq 0$.

(i) Define the filtration $(\mathcal{F}_u^{+,x})_{u \geq 0}$ by $\mathcal{F}_u^{+,x} := \sigma(X_s^{+,x}, s \in [0, u])$ and the process

$$\beta_u^{+,x} := \int_0^{\alpha_u^{+,x}} \mathbf{1}_{\{X_s > x\}} \operatorname{sgn}(B_s) dB_s, \quad u \geq 0.$$

Then $\beta^{+,x}$ is $(\mathcal{F}_u^{+,x})$ -Brownian motion and $X^{+,x}$ is an $(\mathcal{F}_u^{+,x})$ -semimartingale with decomposition

$$(2.1) \quad X_u^{+,x} = \beta_u^{+,x} - \frac{1-\mu}{\mu} I_u^{+,x} + \frac{1}{2} L^{+,x}(u, x), \quad u \geq 0.$$

(ii) Define the filtration $(\mathcal{F}_u^{-,x})_{u \geq 0}$ by $\mathcal{F}_u^{-,x} := \sigma(X_s^{-,x}, s \in [0, u])$ and the process

$$\beta_u^{-,x} := \int_0^{\alpha_u^{-,x}} \mathbf{1}_{\{X_s \leq x\}} \operatorname{sgn}(B_s) dB_s, \quad u \geq 0.$$

Then $\beta^{-,x}$ is $(\mathcal{F}_u^{-,x})$ -Brownian motion and $X^{-,x}$ is an $(\mathcal{F}_u^{-,x})$ -semimartingale with decomposition

$$(2.2) \quad X_u^{-,x} = x + \beta_u^{-,x} - \frac{1-\mu}{\mu}(I_u^{-,x} - x) - \frac{1}{2}L^{-,x}(u, x), \quad u \geq 0.$$

(iii) The Brownian motions $\beta^{+,x}$ and $\beta^{-,x}$ are independent.

Proof. By Tanaka's formula,

$$(X_t - x)^+ = (X_0 - x)^+ + \int_0^t \mathbf{1}_{\{X_s > x\}} dX_s + \frac{1}{2}L(t, x).$$

Take $t = \alpha_u^{+,x}$. We get

$$(2.3) \quad X_u^{+,x} = \int_0^{\alpha_u^{+,x}} \mathbf{1}_{\{X_s > x\}} dX_s + \frac{1}{2}L(\alpha_u^{+,x}, x).$$

Moreover, $dX_s = d|B_s| - \mu d\mathfrak{L}_s = \operatorname{sgn}(B_s)dB_s + (1-\mu)d\mathfrak{L}_s$ by another application of Tanaka's formula. Also observe that $I_t = -\mu\mathfrak{L}_t$, hence

$$(2.4) \quad dX_s = \operatorname{sgn}(B_s)dB_s - \frac{1-\mu}{\mu}dI_s, \quad s \geq 0.$$

Therefore,

$$X_u^{+,x} = \int_0^{\alpha_u^{+,x}} \mathbf{1}_{\{X_s > x\}} \operatorname{sgn}(B_s) dB_s - \frac{1-\mu}{\mu} \int_0^{\alpha_u^{+,x}} \mathbf{1}_{\{X_s > x\}} dI_s + \frac{1}{2}L(\alpha_u^{+,x}, x).$$

We notice that $L^{+,x}(u, r) = L(\alpha_u^{+,x}, r)$ for any $r \in [x, \infty)$ and $u \geq 0$. On the other hand, $\int_0^{\alpha_u^{+,x}} \mathbf{1}_{\{X_s > x\}} dI_s = I_{\alpha_u^{+,x} \wedge T_x}$ which is also the infimum of $X^{+,x}$ on the time interval $[0, u]$. This yields (2.1). This equation also implies that $\beta^{+,x}$ is adapted to $\mathcal{F}^{+,x}$. Moreover, from the definition of $\beta_u^{+,x}$ and Proposition V.1.5 of [17], $\beta_u^{+,x}$ is a $(\mathcal{B}_{\alpha_u^{+,x}})$ -continuous martingale with $\langle \beta^{+,x}, \beta^{+,x} \rangle_u = u$, hence $(\mathcal{B}_{\alpha_u^{+,x}})$ -Brownian motion. Since $\mathcal{F}_u^{+,x} \subset \mathcal{B}_{\alpha_u^{+,x}}$, we deduce that $\beta_u^{+,x}$ is also $(\mathcal{F}_u^{+,x})$ -Brownian motion. This proves (i).

The proof of (ii) is similar. Tanaka's formula applied to $(X_{t+T_x} - x)^-$ with $t = \alpha_u^{-,x} - T_x$ implies that

$$X_u^{-,x} = x + \int_{T_x}^{\alpha_u^{-,x}} \mathbf{1}_{\{X_s \leq x\}} \operatorname{sgn}(B_s) dB_s - \frac{1-\mu}{\mu} \int_{T_x}^{\alpha_u^{-,x}} \mathbf{1}_{\{X_s \leq x\}} dI_s - \frac{1}{2}L(\alpha_u^{-,x}, x).$$

We observe that $L(\alpha_u^{-,x}, x) = L^{-,x}(u, x)$ and $\int_{T_x}^{\alpha_u^{-,x}} \mathbf{1}_{\{X_s \leq x\}} dI_s = I_{\alpha_u^{-,x}} - x$ while $I_{\alpha_u^{-,x}} = I_u^{-,x}$ which gives (2.2). We conclude as for (i). The statement (iii) is a consequence of Knight's theorem on orthogonal martingales. \square

The following result is well-known. It has been proved in Section 8.5 of [22] when $\mu \in (0, 2)$, in [21] and in [15]. Here, following [22], we choose to see it as a consequence of Proposition 2.3.

Corollary 2.4. *Let $x \leq 0$. The processes $X^{+,x}$ and $X^{-,x}$ are independent.*

Proof. By Proposition 2.3 (iii), the martingale parts of $X^{+,x}$ and $X^{-,x}$, namely $\beta^{+,x}$ and $\beta^{-,x}$, are independent. It remains to see that $X^{+,x}$ is measurable with respect to $\beta^{+,x}$ and $X^{-,x}$ with respect to $\beta^{-,x}$, which was established by Chaumont and Doney [5] and Davis [6]. \square

The excursion filtration, introduced by Walsh [19], is defined as

$$\mathcal{E}_x^+ := \mathcal{F}_\infty^{+,x} = \sigma(X_s^{+,x}, s \geq 0), \quad x \in \mathbb{R}.$$

Similarly we define $\mathcal{E}_x^- := \mathcal{F}_\infty^{-,x} = \sigma(X_s^{-,x}, s \geq 0)$ for $x \in \mathbb{R}$. It is routine to check, using the time-changes $\alpha^{-,x}$ and $\alpha^{+,x}$, that \mathcal{E}_x^- is increasing in x whereas \mathcal{E}_x^+ is decreasing.⁴

Define, for $u \geq 0$,

$$\mathcal{G}_u^{+,x} := \sigma(\mathcal{F}_u^{+,x}, \mathcal{E}_x^-), \quad \mathcal{G}_u^{-,x} := \sigma(\mathcal{F}_u^{-,x}, \mathcal{E}_x^+).$$

The idea of such an enlargement of filtrations goes back at least to Jeulin [8].

Corollary 2.5. *Consider a random function $g(\ell, y) = g(\ell, y, \omega)$ such that the process $t \mapsto g(L(t, X_t), X_t)$ is (\mathcal{B}_t) -progressively measurable and $\mathbb{E}[\int_{\mathbb{R}_+} d\ell \int_{\mathbb{R}} g(\ell, y)^2 dy] < \infty$. Fix $x \leq 0$.*

(i) *The process $u \mapsto g(L^{+,x}(u, X_u^{+,x}), X_u^{+,x})$ is $(\mathcal{G}_u^{+,x})$ -progressive and almost surely,*

$$(2.5) \quad \int_0^\infty g(L(t, X_t), X_t) \mathbf{1}_{\{X_t > x\}} \operatorname{sgn}(B_t) dB_t = \int_0^\infty g(L^{+,x}(u, X_u^{+,x}), X_u^{+,x}) d\beta_u^{+,x}.$$

(ii) *The process $u \mapsto g(L^{-,x}(u, X_u^{-,x}), X_u^{-,x})$ is $(\mathcal{G}_u^{-,x})$ -progressive and almost surely,*

$$(2.6) \quad \int_0^\infty g(L(t, X_t), X_t) \mathbf{1}_{\{X_t \leq x\}} \operatorname{sgn}(B_t) dB_t = \int_0^\infty g(L^{-,x}(u, X_u^{-,x}), X_u^{-,x}) d\beta_u^{-,x}.$$

⁴We will be implicitly working with a right-continuous (and complete) version of the filtrations $(\mathcal{E}_x^-)_{x \in \mathbb{R}}$ and $(\mathcal{E}_x^+)_{x \in \mathbb{R}}$ — if necessary, by means of the procedure of usual augmentation, as described in Section I.4 of Revuz and Yor [17]. In our work, we only study *continuous* martingales, which are also martingales with respect to augmented filtrations. The same remark applies to the μ -process defined on \mathbb{R} in Section 5.

Proof. We prove (i). The process $u \mapsto g(L^{+,x}(u, X_u^{+,x}), X_u^{+,x})$ is $(\mathcal{B}_{\alpha_u^{+,x}})$ -progressive (Proposition V.1.4, [17]). Therefore it is also progressive with respect to $(\mathcal{G}_u^{+,x})$ because the latter filtration is larger. We prove now (2.5). By a time-change (Proposition V.1.4, [17]),

$$\int_0^{\alpha_t^{+,x}} g(L(s, X_s), X_s) \mathbf{1}_{\{X_s > x\}} \operatorname{sgn}(B_s) dB_s = \int_0^t g(L^{+,x}(u, X_u^{+,x}), X_u^{+,x}) d\beta_u^{+,x}.$$

Letting $t \rightarrow \infty$ yields (2.5). Statement (ii) is proved similarly. \square

3 The martingale measure associated with W

Recall the definition of the white noise W in (1.3). For any Borel set A of \mathbb{R}_+ with finite Lebesgue measure and $r \geq 0$, we define

$$(3.1) \quad M_r(A) := W(\mathbf{1}_{A \times [-r, 0]}).$$

Proposition 3.1. *In the setting of Walsh [20], $(M_r, r \geq 0)$ is a continuous martingale measure with respect to the filtration $(\mathcal{E}_{-r}^+, r \geq 0)$.*

Proof. Since W is a white noise, it suffices to show that M_r is measurable with respect to \mathcal{E}_{-r}^+ and that $M_s - M_r$ is independent of \mathcal{E}_{-r}^+ for any $0 \leq r < s$. The first statement comes from (2.5) applied to $x = -r$ and $g(\ell, y) = \mathbf{1}_{A \times [-r, 0]}(\ell, y)$ for a Borel set A with finite Lebesgue measure. The second statement comes from (2.6) applied to $x = -r$ and $g(\ell, y) = \mathbf{1}_{A \times [-s, -r]}(\ell, y)$. Since the processes $X^{-, -r}$ and $\beta^{-, -r}$ are independent of \mathcal{E}_{-r}^+ , the proposition follows. \square

We are going to extend (1.3), seen as an equality for deterministic functions g , to random functions. To this end, we first recall the construction by Walsh in [20] of stochastic integral with respect to the martingale measure M . A (random) function f is said to be elementary if it is of the form $f(\ell, x) := Z \mathbf{1}_{[a, b]}(x) \mathbf{1}_A(\ell)$, where $a < b \leq 0$, $A \subset \mathbb{R}_+$ is a Borel set of finite Lebesgue measure, and Z is a bounded \mathcal{E}_b^+ -measurable real-valued random variable. Denote by $f \cdot M$ the stochastic integral with respect to M :

$$f \cdot M := Z (M_{|a|}(A) - M_{|b|}(A)) = Z W(\mathbf{1}_{A \times [a, b]}).$$

A simple function is a (finite) linear combination of elementary functions. We extend by linearity the definition of $f \cdot M$ to simple functions f and furthermore by isometry to any $f \in \mathcal{L}^2$, where \mathcal{L}^2 denotes the space of $(\mathcal{E}_{-r}^+, r \geq 0)$ -predictable and square-integrable functions, defined as the closure of the space of simple functions under the norm:

$$\|f\| := \left[\mathbb{E} \left(\int_{\mathbb{R}_+} d\ell \int_{\mathbb{R}_-} f^2(\ell, x) dx \right) \right]^{1/2}.$$

For any $f \in \mathcal{L}^2$, $f \cdot M$ is a centered random variable with $\mathbb{E}[(f \cdot M)^2] = \|f\|^2$. We write $f \cdot M \equiv \int_{\mathbb{R}_+ \times \mathbb{R}_-} f(\ell, x) W(d\ell, dx)$ and for any $r \geq 0$, $(f \mathbf{1}_{\mathbb{R}_+ \times [-r, 0]}) \cdot M \equiv$

$\int_{\mathbb{R}_+ \times [-r, 0]} f(\ell, x) W(d\ell, dx)$. The latter, if furthermore f is of form $f(\ell, x) = \mathbf{1}_{\{0 \leq \ell \leq \sigma_x\}} \eta_x$, will be re-written as $\int_{-r}^0 \eta_x W([0, \sigma_x], dx)$. By the construction of stochastic integral and Proposition 3.1, $\int_{\mathbb{R}_+ \times [-r, 0]} f(\ell, x) W(d\ell, dx)$ is a continuous martingale with respect to the filtration $(\mathcal{E}_{-r}^+, r \geq 0)$, of quadratic variation process $\int_{\mathbb{R}_+ \times [-r, 0]} f^2(\ell, x) d\ell dx$.

Proposition 3.2. *Take $g \in \mathcal{L}^2$ such that $s \mapsto g(L(s, X_s), X_s)$ admits a version which is progressive with respect to the Brownian filtration (\mathcal{B}_s) .⁵ Then*

$$g \cdot M = \int_0^\infty g(L(s, X_s), X_s) \operatorname{sgn}(B_s) dB_s \quad \text{a.s.}$$

Proof. By definition of \mathcal{L}^2 , there exists a sequence of simple functions g_n such that $\|g - g_n\| \rightarrow 0$ as $n \rightarrow \infty$. By isometry, $g \cdot M - g_n \cdot M \rightarrow 0$ in L^2 . Since g_n is a simple function, g_n is of the form

$$g_n(\ell, x) = \sum_{k,j=1}^\infty Z_{k,j}^n \mathbf{1}_{[a_k^n, a_{k-1}^n)}(x) \mathbf{1}_{A_j^n}(\ell),$$

where for each n , $0 = a_0^n > \dots > a_k^n > a_{k+1}^n > \dots$ is a decreasing sequence such that $a_k^n \rightarrow -\infty$ as $k \rightarrow \infty$, $(A_j^n)_{j \geq 1}$ is a collection of (nonrandom) pairwise disjoint Borel subsets of \mathbb{R}_+ with finite Lebesgue measures, and for any $k, j \geq 1$, $Z_{k,j}^n$ is a bounded $\mathcal{E}_{a_{k-1}^n}^+$ -measurable random variable. Moreover for all large k, j , $Z_{k,j}^n = 0$, which means the above double sum runs in fact over a finite index set of k and j .

Note that for a.e. $z \leq 0$, $g(\cdot, z)$ is measurable with respect to \mathcal{E}_z^+ (as g_n is). We may (and will) take a version of g such that $g(\cdot, z)$ is measurable with respect to \mathcal{E}_z^+ for all $z \leq 0$.

By applying (1.3) and (3.1), we deduce from the linearity of the integral that

$$g_n \cdot M = \sum_{k,j=1}^\infty Z_{k,j}^n \int_0^\infty \mathbf{1}_{[a_k^n, a_{k-1}^n)}(X_s) \mathbf{1}_{A_j^n}(L(s, X_s)) \operatorname{sgn}(B_s) dB_s.$$

Note that we can (and we will) take (a_k^n, a_{k-1}^n) instead of $[a_k^n, a_{k-1}^n)$ without changing the value of $g_n \cdot M$. By (2.5) and (2.6) respectively,

$$\begin{aligned} & \int_0^\infty \mathbf{1}_{(a_k^n, a_{k-1}^n)}(X_s) \mathbf{1}_{A_j^n}(L(s, X_s)) \operatorname{sgn}(B_s) dB_s \\ (3.2) \quad &= \int_0^\infty \mathbf{1}_{(a_k^n, a_{k-1}^n)}(X_u^{-,x}) \mathbf{1}_{A_j^n}(L^{-,x}(u, X_u^{-,x})) d\beta_u^- \end{aligned}$$

$$(3.3) \quad = \int_0^\infty \mathbf{1}_{(a_k^n, a_{k-1}^n)}(X_u^{+,y}) \mathbf{1}_{A_j^n}(L^{+,y}(u, X_u^{+,y})) d\beta_u^{+,y}$$

⁵By *version* we mean a (\mathcal{B}_s) -progressive process (h_s) such that $\int_0^\infty \mathbf{1}_{\{g(L(s, X_s), X_s) \neq h_s\}} ds = 0$ a.s.

with $x = a_{k-1}^n$ and $y = a_k^n$. Similarly,

$$(3.4) \quad \begin{aligned} & \int_0^\infty g(L(s, X_s), X_s) \mathbf{1}_{(a_k^n, a_{k-1}^n)}(X_s) \operatorname{sgn}(B_s) dB_s \\ &= \int_0^\infty g(L^{-,x}(u, X_u^{-,x}), X_u^{-,x}) \mathbf{1}_{(a_k^n, a_{k-1}^n)}(X_u^{-,x}) d\beta_u^{-,x}, \end{aligned}$$

$$(3.5) \quad = \int_0^\infty g(L^{+,y}(u, X_u^{+,y}), X_u^{+,y}) \mathbf{1}_{(a_k^n, a_{k-1}^n)}(X_u^{+,y}) d\beta_u^{+,y}$$

with $x = a_{k-1}^n$ and $y = a_k^n$ as in (3.2) and (3.3). Write

$$I(g) := \int_0^\infty g(L(s, X_s), X_s) \operatorname{sgn}(B_s) dB_s.$$

Then

$$g_n \cdot M - I(g) = \sum_{k=1}^\infty \Delta_n(k),$$

where, from (3.3) and (3.5),

$$\begin{aligned} \Delta_n(k) &:= \sum_{j=1}^\infty Z_{k,j}^n \int_0^\infty \mathbf{1}_{(a_k^n, a_{k-1}^n)}(X_u^{+,y}) \mathbf{1}_{A_j^n}(L^{+,y}(u, X_u^{+,y})) d\beta_u^{+,y} \\ &\quad - \int_0^\infty g(L^{+,y}(u, X_u^{+,y}), X_u^{+,y}) \mathbf{1}_{(a_k^n, a_{k-1}^n)}(X_u^{+,y}) d\beta_u^{+,y}, \end{aligned}$$

where as before $y = a_k^n$. Since $Z_{k,j}^n$ is $\mathcal{E}_{a_{k-1}^n}^+$ -measurable, hence $\mathcal{E}_{a_k^n}^+$ -measurable, we see that the sum over j in the definition of $\Delta_n(k)$ is $\mathcal{E}_{a_k^n}^+$ -measurable. For the last integral in $\Delta_n(k)$, we use the fact that for $z \leq 0$, $g(\cdot, z)$ is measurable with respect to \mathcal{E}_z^+ . Note that $X_u^{+,y}$ and $L^{+,y}(\cdot, \cdot)$ are measurable with respect to \mathcal{E}_y^+ . Since $X_u^{+,y} \geq y$ and \mathcal{E}_z^+ decreases on z , we deduce that $g(\cdot, X_u^{+,y})$ is \mathcal{E}_y^+ -measurable, and so is $g(L^{+,y}(u, X_u^{+,y}), X_u^{+,y})$. It follows that $\Delta_n(k)$ is measurable with respect to $\mathcal{E}_{a_k^n}^+$.

Now we prove that $(\Delta_n(k))_{k \geq 1}$ is a martingale difference sequence with respect to the filtration $(\mathcal{E}_{a_k^n}^+)_{k \geq 1}$. Indeed, using (3.2) and (3.4) instead of (3.3) and (3.5), one can also write

$$\begin{aligned} \Delta_n(k) &= \sum_{j=1}^\infty Z_{k,j}^n \int_0^\infty \mathbf{1}_{(a_k^n, a_{k-1}^n)}(X_u^{-,x}) \mathbf{1}_{A_j^n}(L^{-,x}(u, X_u^{-,x})) d\beta_u^{-,x} \\ &\quad - \int_0^\infty g(L^{-,x}(u, X_u^{-,x}), X_u^{-,x}) \mathbf{1}_{(a_k^n, a_{k-1}^n)}(X_u^{-,x}) d\beta_u^{-,x}, \end{aligned}$$

with $x = a_{k-1}^n$. Recall from Proposition 2.3 that $\beta^{-,x}$ is $(\mathcal{F}_u^{-,x})$ -Brownian motion, which is independent of \mathcal{E}_x^+ by Corollary 2.4. Then $\beta^{-,x}$ can be seen as $(\mathcal{G}_u^{-,x})$ -Brownian motion.

By Corollary 2.5 (ii), $g(L^{-,x}(u, X_u^{-,x}), X_u^{-,x})$ is progressive with respect to the filtration $(\mathcal{G}_u^{-,x})$, while

$$\mathbf{1}_{(a_k^n, a_{k-1}^n)}(X_u^{-,x}) g_n(L^{-,x}(u, X_u^{-,x}), X_u^{-,x}) = \sum_{j=1}^{\infty} Z_{k,j}^n \mathbf{1}_{(a_k^n, a_{k-1}^n)}(X_u^{-,x}) \mathbf{1}_{A_j^n}(L^{-,x}(u, X_u^{-,x}))$$

is $(\mathcal{G}_u^{-,x})$ -progressive as well. Therefore, one can write

$$(3.6) \quad \Delta_n(k) = \int_0^{\infty} \mathbf{1}_{(a_k^n, a_{k-1}^n)}(X_u^{-,x}) (g_n - g)(L^{-,x}(u, X_u^{-,x}), X_u^{-,x}) d\beta_u^{-,x},$$

with $x = a_{k-1}^n$. It follows that (since $\mathcal{G}_0^{-,x} = \mathcal{E}_x^+$)

$$\mathbb{E}[\Delta_n(k) \mid \mathcal{E}_{a_{k-1}^n}^+] = 0.$$

In other words, the process $j \rightarrow \sum_{k=1}^j \Delta_n(k)$ is a martingale and we have

$$\mathbb{E}[(g_n \cdot M - I(g))^2] = \sum_{k=1}^{\infty} \mathbb{E}[\Delta_n(k)^2].$$

From (3.6), we get

$$\begin{aligned} \mathbb{E}[\Delta_n(k)^2] &= \mathbb{E} \left[\int_0^{\infty} \mathbf{1}_{(a_k^n, a_{k-1}^n)}(X_u^{-,x}) ((g_n - g)(L^{-,x}(u, X_u^{-,x}), X_u^{-,x}))^2 du \right] \\ &= \mathbb{E} \left[\int_{a_k^n}^{a_{k-1}^n} (g_n(\ell, z) - g(\ell, z))^2 d\ell dz \right], \end{aligned}$$

where in the second equality we have used the occupation time formula. It follows that

$$\mathbb{E}[(g_n \cdot M - I(g))^2] = \|g_n - g\|^2$$

which goes to 0 as $n \rightarrow \infty$. Thus we get that $g \cdot M = I(g)$. \square

Remark 3.3. Fix $b < 0$. Similarly to (3.1) we may define a martingale measure \widehat{M} by

$$\widehat{M}_r(A) := W(\mathbf{1}_{A \times [b, b+r]}),$$

for any Borel set $A \subset \mathbb{R}_+$ of finite Lebesgue measure and $r \geq 0$. The analogs of Propositions 3.1 and 3.2 hold for \widehat{M} . Specifically, $(\widehat{M}_r, r \geq 0)$ is a martingale measure with respect to the filtration $(\mathcal{E}_{b+r}^-, r \geq 0)$. Moreover, we may define in a similar way the stochastic integral $f \cdot \widehat{M}$ for any $f \in \widehat{\mathcal{L}}^2$, where $\widehat{\mathcal{L}}^2$ denotes the space of $(\mathcal{E}_{b+r}^-)_{r \geq 0}$ -predictable functions f such that $\mathbb{E}[\int_{\mathbb{R}_+} d\ell \int_{[b, \infty)} f^2(\ell, x) dx] < \infty$. Then for any $g \in \widehat{\mathcal{L}}^2$

such that $s \mapsto g(L(s, X_s), X_s)$ admits a version which is progressive with respect to the Brownian filtration (\mathcal{B}_s) , we have

$$(3.7) \quad g \cdot \widehat{M} = \int_0^\infty g(L(s, X_s), X_s) dX_s - \frac{1-\mu}{\mu} \int_{-\infty}^0 g(0, x) dx, \quad \text{a.s.}$$

With a slight abuse of notation, we shall write $g \cdot \widehat{M} \equiv \int_{\mathbb{R}_+ \times [b, \infty)} g(\ell, x) W(d\ell, dx)$ and $(g \mathbf{1}_{\mathbb{R}_+ \times [b, t)}) \cdot \widehat{M} \equiv \int_{\mathbb{R}_+ \times [b, t)} g(\ell, x) W(d\ell, dx)$ for any $t \geq b$. Then $\int_{\mathbb{R}_+ \times [b, b+r)} g(\ell, x) W(d\ell, dx)$, is an (\mathcal{E}_{b+r}^-) -continuous martingale with quadratic variation process $\int_{\mathbb{R}_+ \times [b, b+r)} g^2(\ell, x) d\ell dx$ for $r \geq 0$.

4 Proofs of Theorems 1.2 and 1.1

By Tanaka's formula, for any $r \geq 0$ and $x \in \mathbb{R}$,

$$(4.1) \quad (X_r - x)^- = (-x)^- - \int_0^r \mathbf{1}_{\{X_s \leq x\}} dX_s + \frac{1}{2} L(r, x).$$

Part (i) of Theorems 1.2 and 1.1: Applying (4.1) to $r = \tau_a^0$ gives that $L(\tau_a^0, x) = 2 \int_0^{\tau_a^0} \mathbf{1}_{\{X_s \leq x\}} dX_s$ for all $x \in \mathbb{R}$. Let $h \geq 0$. Taking $x = -h$ and $x = 0$, and using the fact that $L(\tau_a^0, 0) = a$, we obtain that

$$(4.2) \quad \begin{aligned} L(\tau_a^0, -h) &= a - 2 \int_0^{\tau_a^0} \mathbf{1}_{\{-h < X_s \leq 0\}} dX_s \\ &= a - 2 \int_0^{\tau_a^0} \mathbf{1}_{\{-h < X_s \leq 0\}} \text{sgn}(B_s) dB_s + \left(2 - \frac{2}{\mu}\right) \min(h, |I_{\tau_a^0}|), \end{aligned}$$

where the second equality follows from (2.4) and the fact that $\int_0^{\tau_a^0} \mathbf{1}_{\{-h < X_s \leq 0\}} dI_s = \int_0^{\tau_a^0} \mathbf{1}_{\{I_s > -h\}} dI_s = I_{\min(T_{-h}, I_{\tau_a^0})} = -\min(h, |I_{\tau_a^0}|)$.

To deal with the stochastic integral with respect to (B_s) in (4.2), we shall use Proposition 3.2. First we remark that for $x \leq 0$, $L(\tau_a^0, x)$ is measurable with respect to \mathcal{E}_x^+ . In fact, let $u := A_{\tau_a^0}^{+, x}$. Then $\alpha_u^{+, x} = \tau_a^0$ as τ_a^0 is an increasing time for $A^{+, x}$. Therefore $L(\tau_a^0, x) = L^{+, x}(A_{\tau_a^0}^{+, x}, x)$. Since $\{A_{\tau_a^0}^{+, x} \geq t\} = \{L(\alpha_t^{+, x}, 0) \leq a\} = \{L^{+, x}(t, 0) \leq a\}$, we obtain that $L(\tau_a^0, x)$ is measurable with respect to \mathcal{E}_x^+ .

Let $g(\ell, x) := \mathbf{1}_{\{0 \leq \ell \leq L(\tau_a^0, x)\} \cap \{-h < x \leq 0\}}$. Using the continuity of local times $L(\tau_a^0, x)$ on x and the fact that $\mathbb{E}[\int_{\mathbb{R}_+ \times \mathbb{R}} g^2(\ell, x) d\ell dx] < \infty$, we get that $g \in \mathcal{L}^2$.

Observe that a.s., $g(L(s, X_s), X_s) = \mathbf{1}_{\{0 \leq s \leq \tau_a^0, -h < X_s \leq 0\}}$, ds-a.e. This follows from the fact that $\int_0^\infty \mathbf{1}_{\{L(s, X_s) \leq L(\tau_a^0, X_s), s > \tau_a^0\}} ds = \int_{-\infty}^\infty dx \int_{\tau_a^0}^\infty \mathbf{1}_{\{L(s, x) \leq L(\tau_a^0, x)\}} ds L(s, x) = 0$, by the occupation time formula. Then $s \mapsto g(L(s, X_s), X_s)$ admits a version which is (\mathcal{B}_s) -progressive and we are entitled to apply Proposition 3.2 to see that

$$\int_0^{\tau_a^0} \mathbf{1}_{\{-h < X_s \leq 0\}} \text{sgn}(B_s) dB_s = \int_{\mathbb{R}_+ \times (-h, 0]} \mathbf{1}_{\{0 \leq \ell \leq L(\tau_a^0, x)\}} W(d\ell, dx) = \int_{-h}^0 W([0, L(\tau_a^0, x)], dx).$$

In view of (4.2), this yields Theorem 1.2 (i) for each fixed $a > 0$ and $h \in [0, |I_{\tau_a^0}|]$. Since the processes are continuous in h and càdlàg in a , they coincide except on a null set.

Theorem 1.1 (i) follows quite simply from Theorem 1.2 (i): for given $a > 0$, since $h \mapsto \int_{-h}^0 W([0, L(\tau_a^0, x)], dx)$ is an $(\mathcal{E}_{-h}^+)_{h \geq 0}$ -continuous martingale with quadratic variation process $\int_{-h}^0 L(\tau_a^0, x) dx$, it follows from the Dambis–Dubins–Schwarz theorem that there exists (\mathcal{E}_{-h}^+) -Brownian motion γ such that $\int_{-h}^0 W([0, L(\tau_a^0, x)], dx) = \int_0^h \sqrt{L(\tau_a^0, s)} d\gamma_s$. Going back to (4.2), we see that for all $0 \leq h \leq \inf\{s \geq 0 : L(\tau_a^0, -s) = 0\} = |I_{\tau_a^0}|$,

$$L(\tau_a^0, -h) = a - 2 \int_0^h \sqrt{L(\tau_a^0, s)} d\gamma_s + \left(2 - \frac{2}{\mu}\right)h,$$

proving Theorem 1.1 (i).

Part (ii) of Theorems 1.2 and 1.1: Let $b < 0$ and $T_b := \inf\{t \geq 0 : X_t = b\}$. For $h \in [0, |b|]$, we get from (4.1) that

$$\begin{aligned} L(T_b, b+h) &= 2h + 2 \int_0^{T_b} \mathbf{1}_{\{X_s \leq b+h\}} dX_s \\ (4.3) \qquad &= 2 \int_0^{T_b} \mathbf{1}_{\{X_s \leq b+h\}} \operatorname{sgn}(B_s) dB_s + \frac{2}{\mu}h, \end{aligned}$$

where the second equality follows from (2.4) again and the fact that $\int_0^{T_b} \mathbf{1}_{\{X_s \leq b+h\}} dI_s = \int_0^{T_b} \mathbf{1}_{\{I_s \leq b+h\}} dI_s = -h$.

The main difference with Part (i) is the measurability. As a matter of fact, for any $x \geq b$, $L(T_b, x)$ is \mathcal{E}_x^- -measurable: observe that $L(T_b, x) = L^{-,x}(A_{T_b}^{-,x}, x)$ and for any $t \geq 0$, $\{A_{T_b}^{-,x} > t\} = \{T_b > \alpha_t^{-,x}\} = \{\inf_{0 \leq s \leq \alpha_t^{-,x}} X_s > b\} = \{\inf_{0 \leq s \leq t} X_s^{-,x} > b\}$ is \mathcal{E}_x^- -measurable.

Let $g(\ell, x) := \mathbf{1}_{\{0 < \ell \leq L(T_b, x)\} \times \{b \leq x \leq b+h\}}$. We can check as in (i) that we may apply (3.7) to get that

$$\int_0^{T_b} \mathbf{1}_{\{X_s \leq b+h\}} \operatorname{sgn}(B_s) dB_s = \int_{\mathbb{R}_+ \times [b, b+h]} \mathbf{1}_{\{0 \leq \ell \leq L(T_b, b+h)\}} W(d\ell, dx) = \int_b^{b+h} W([0, L(T_b, x)], dx),$$

proving, in view of (4.3), Theorem 1.2 (ii). Furthermore by Remark 3.3, the process $h \rightarrow \int_b^{b+h} W([0, L(T_b, x)], dx)$ is an (\mathcal{E}_{b+h}^-) -continuous martingale with quadratic variation process $\int_b^{b+h} L(T_b, x) dx$. This easily yields Theorem 1.1 (ii). \square

5 Extension to the two-sided μ -process

In this Section, we shall explore the strong Markov property at the hitting times of a μ -process defined on \mathbb{R} and present an analogue of Theorem 1.2. This result, apart from its own interest, will be useful in a forthcoming work on the duality of Jacobi stochastic flows.

Let $(B_t)_{t \in \mathbb{R}}$ be a two-sided Brownian motion, which means that for $t \leq 0$, $B_t = B'_{-t}$, where B' is a standard Brownian motion independent of $(B_t)_{t \geq 0}$. Denote by $(\mathfrak{L}'_t)_{t \geq 0}$ the local time process at position zero of B' .

Recall that $X_t = |B_t| - \mu \mathfrak{L}_t$, for $t \geq 0$. For $t \leq 0$, we let $X_t := |B'_{-t}| + \mu \mathfrak{L}'_{-t}$. We call $(X_t)_{t \in \mathbb{R}}$ a two-sided μ -process. Fix $\mu > 0$. Notice that $X_t \rightarrow \infty$ as $t \rightarrow -\infty$, and $T_r < 0$ when $r > 0$.

We naturally extend the notation $T_r := \inf\{t \in \mathbb{R} : X_t = r\}$ for $r \in \mathbb{R}$,

$$L(t, x) := \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_{-\infty}^t \mathbf{1}_{\{x \leq X_s \leq x + \varepsilon\}} ds, \quad x \in \mathbb{R},$$

the local time accumulated by $(X_t, t \in \mathbb{R})$ at position x up to time t , and

$$\tau_a^x := \inf\{t \in \mathbb{R} : L(t, x) > a\}, \quad a \geq 0,$$

the inverse local time at position x . We define now for bounded Borel functions g with compact support,

$$(5.1) \quad W(g) := \int_{-\infty}^{\infty} g(L(t, X_t), X_t) \operatorname{sgn}(B_t) dB_t.$$

The stochastic integral has to be understood as an integral with respect to the Brownian motion $B^{(r)} := (B_{t+T_r}, t \geq 0)$ where r is any positive real such that $g(\ell, x) = 0$ for all $x \geq r$ (that $B^{(r)}$ is a standard Brownian motion comes from the fact that $(B'_{T_r+t}, t \in [0, |T_r|])$ is distributed as $(B_t, t \in [0, T_{-r}])$). We will see in the following theorem that W defines a white noise.

Similarly to Notation 2.1, for $x \in \mathbb{R}$, we can consider the process $(X_u^{-,x}, u \geq 0)$ obtained by gluing the excursions of X below x : that is we set for $t \in \mathbb{R}$, $A_t^{-,x} := \int_{-\infty}^t \mathbf{1}_{\{X_s \leq x\}} ds$, $\alpha_u^{-,x} := \inf\{t \in \mathbb{R} : A_t^{-,x} > u\}$, and $X_u^{-,x} := X_{\alpha_u^{-,x}}$ for $u \geq 0$. The excursion filtration \mathcal{E}^- is defined as $\mathcal{E}_x^- := \sigma(X_u^{-,x}, u \geq 0)$, for all $x \in \mathbb{R}$.

For \mathcal{E}_x^+ , we let $A_t^{+,x} := t$ if $t \leq T_x$, and for $t > T_x$, $A_t^{+,x} := T_x + \int_{T_x}^t \mathbf{1}_{\{X_s > x\}} ds$. For $s \in \mathbb{R}$, we set $\alpha_s^{+,x} := \inf\{t \in \mathbb{R}, A_t^{+,x} > s\}$, and $X_s^{+,x} := X_{\alpha_s^{+,x}}$. The excursion filtration \mathcal{E}^+ is defined as $\mathcal{E}_x^+ := \sigma(X_s^{+,x}, s \in \mathbb{R})$, for all $x \in \mathbb{R}$. Note that \mathcal{E}_x^+ is decreasing in x whereas \mathcal{E}_x^- is increasing.

Theorem 5.1. Fix $\mu > 0$. Equation (5.1) defines a white noise on $\mathbb{R}_+ \times \mathbb{R}$.

(i) Almost surely for all $a > 0$, $r \in \mathbb{R}$ and $h \in [0, |I_{\tau_a^r} - r|]$,

$$(5.2) \quad L(\tau_a^r, r - h) = a - 2 \int_{r-h}^r W([0, L(\tau_a^r, x)], dx) + \left(2 - \frac{2}{\mu}\right) h,$$

where $I_t := \inf_{-\infty < s \leq t} X_s, t \in \mathbb{R}$, denotes the infimum process of X .

(ii) Almost surely for all $a > 0$, $r \in \mathbb{R}$ and $h \geq 0$,

$$(5.3) \quad L(\tau_a^r, r + h) = a + 2 \int_r^{r+h} W([0, L(\tau_a^r, x)], dx) + \frac{2}{\mu} h.$$

The stochastic integral $\int_{r-h}^r W([0, L(\tau_a^r, x)], dx)$ is the stochastic integral with respect to the martingale measure $M_h^{(r)}(\bullet) := W(\mathbf{1}_{\bullet \times [r-h, r]})$, $h \geq 0$, associated to the filtration $(\mathcal{E}_{r-h}^+)_{h \geq 0}$. The stochastic integral $\int_r^{r+h} W([0, L(\tau_a^r, x)], dx)$ is the stochastic integral with respect to the martingale measure $\widehat{M}_h^{(r)}(\bullet) := W(\mathbf{1}_{\bullet \times [r, r+h]})$, $h \geq 0$, associated to the filtration $(\mathcal{E}_{r+h}^-)_{h \geq 0}$.

Proof. Notice that for any $r \in \mathbb{R}$, $X^{(r)} := (X_{T_r+t} - r, t \geq 0)$ is distributed as $(X_t, t \geq 0)$. Therefore we can apply Theorem 1.2 to $X^{(r)}$. As in (1.3), we define $W^{(r)}$ the white noise associated to $X^{(r)}$: for any Borel function $g : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$ such that $\int_{\mathbb{R}_+} dl \int_{\mathbb{R}} g^2(\ell, x) dx < \infty$,

$$W^{(r)}(g) := \int_0^\infty g(L^{(r)}(t, X_t^{(r)}), X_t^{(r)}) \operatorname{sgn}(B_t^{(r)}) dB_t^{(r)},$$

where $L^{(r)}(\cdot, \cdot)$ denote the local times of $X^{(r)}$. Let $\tau_a^{x, (r)}$ be the associated inverse local times. By Theorem 1.2 (i) applied to $X^{(r)}$, for $a > 0$,

$$(5.4) \quad L^{(r)}(\tau_a^{0, (r)}, -h) = a - 2 \int_{-h}^0 W^{(r)}([0, L^{(r)}(\tau_a^{0, (r)}, x)], dx) + \left(2 - \frac{2}{\mu}\right) h, \quad h \in [0, |I_{\tau_a^{0, (r)}}^{(r)}|],$$

where $I_t^{(r)} := \inf_{0 \leq s \leq t} X_s^{(r)}$, $t \geq 0$, is the infimum process of $X^{(r)}$. Notice that $\tau_a^{0, (r)} = \tau_a^r - T_r$, $L^{(r)}(t, x) = L(t + T_r, r + x)$ for $x \leq 0$, and for $t \geq 0$, $I_t^{(r)} = I_{t+T_r} - r$ where $I_t := \inf_{-\infty < s \leq t} X_s$. Moreover, for any $x \leq y$,

$$W^{(r)}(\mathbf{1}_{A \times [x, y]}) = \int_0^\infty \mathbf{1}_{A \times [x, y]}(L^{(r)}(t, X_t^{(r)}), X_t^{(r)}) \operatorname{sgn}(B_t^{(r)}) dB_t^{(r)}$$

where we recall that $B_t^{(r)} = B_{t+T_r}$, $t \geq 0$. We deduce that if $x \leq y \leq 0$ and A is a bounded Borel set of \mathbb{R}_+ ,

$$(5.5) \quad \begin{aligned} W^{(r)}(\mathbf{1}_{A \times [x, y]}) &= \int_0^\infty \mathbf{1}_{A \times [x, y]}(L(t + T_r, X_{t+T_r}), X_{t+T_r} - r) \operatorname{sgn}(B_t^{(r)}) dB_t^{(r)} \\ &= \int_{T_r}^\infty \mathbf{1}_{A \times [r+x, r+y]}(L(t, X_t), X_t) \operatorname{sgn}(B_t) dB_t \\ &= W(\mathbf{1}_{A \times [r+x, r+y]}). \end{aligned}$$

We deduce that (5.1) defines a white noise on $\mathbb{R}_+ \times (-\infty, r)$, hence on $\mathbb{R}_+ \times \mathbb{R}$ since r can be made arbitrary large. Equation (5.4) becomes

$$L(\tau_a^r, r - h) = a - 2 \int_{r-h}^r W([0, L(\tau_a^r, x)], dx) + \left(2 - \frac{2}{\mu}\right) h, \quad h \in [0, |I_{\tau_a^r} - r|].$$

It is (5.2). We prove now (5.3). Let $r' > r + h$ arbitrary. Using Tanaka's formula applied to $(X_{T_{r'}+t}, t \geq 0)$, we have for any $x \in \mathbb{R}$ and $t \geq T_{r'}$,

$$(5.6) \quad (X_t - x)^- = (r' - x)^- - \int_{T_{r'}}^t \mathbf{1}_{\{X_s \leq x\}} dX_s + \frac{1}{2}(L(t, x) - L(T_{r'}, x)).$$

Taking $t = \tau_a^r$, and $x = r$ then $x = r + h$, we get

$$L(\tau_a^r, r + h) = 2h + a + 2 \int_{T_{r'}}^{\tau_a^r} \mathbf{1}_{\{r < X_s \leq r+h\}} dX_s = a + 2 \int_{T_{r'}}^{\tau_a^r} \mathbf{1}_{\{r < X_s \leq r+h\}} \operatorname{sgn}(B_s) dB_s + \frac{2}{\mu} h$$

where the second equality follows from (2.4) and the fact that $\int_{T_{r'}}^{\tau_a^r} \mathbf{1}_{\{r < X_s \leq r+h\}} dI_s = \int_{T_{r'}}^{\tau_a^r} \mathbf{1}_{\{I_s \leq r+h\}} dI_s = -h$. By Proposition 3.2 applied to $(X^{(r')}, B^{(r')})$,

$$\begin{aligned} \int_{T_{r'}}^{\tau_a^r} \mathbf{1}_{\{r < X_s \leq r+h\}} \operatorname{sgn}(B_s) dB_s &= \int_0^{\tau_a^{r-r', (r')}} \mathbf{1}_{\{r-r' < X_s^{(r')} \leq r+h-r'\}} \operatorname{sgn}(B_s^{(r')}) dB_s^{(r')} \\ &= \int_{r-r'}^{r+h-r'} W^{(r')}([0, L(\tau_a^r, x)], dx) \\ &= \int_r^{r+h} W([0, L(\tau_a^r, x)], dx) \end{aligned}$$

by (5.5) applied to $r = r'$. This proves (5.3) and completes the proof of the theorem. \square

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