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ON THE NORTHCOTT PROPERTY FOR SPECIAL VALUES OF L-FUNCTIONS

FABIEN PAZUKI AND RICCARDO PENGO

ABSTRACT. We propose an investigation on the Northcott, Bogomolov and Lehmer properties for special values of L -functions. We first introduce an axiomatic approach to these three properties. We then focus on the Northcott property for special values of L -functions. We prove that such a property holds for the special value at zero of Dedekind zeta functions of number fields. In the case of L -functions of pure motives, we prove a Northcott property for special values located at the left of the critical strip, assuming the validity of the functional equation.

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1. INTRODUCTION

Traditionally, the expression *Northcott property* is used to talk about the finiteness of the set of algebraic numbers having simultaneously bounded height and degree, proved by Northcott (see [63] and [7, Theorem 1.6.8]). More generally, one can say that a field $F \subseteq \overline{\mathbb{Q}}$ has the **Northcott property** if the sets of elements of F having bounded height are finite. Hence, Northcott's theorem can be reformulated by saying that number fields have the Northcott property. They are not the only fields sharing this property (see for example [15]).

The Northcott property can be relaxed by asking for which fields $F \subseteq \overline{\mathbb{Q}}$ the image of the height $h(F) \subseteq \mathbb{R}_{\geq 0}$ does not admit zero as an accumulation point. If this happens, one says that F has the **Bogomolov property** (see [8]), and in recent years there has been an increasing interest in finding fields having the Bogomolov property. Finally, the Bogomolov property can also be relaxed by looking at fields $F \subseteq \overline{\mathbb{Q}}$ such that the product of the height and the degree of algebraic numbers does not admit zero as an accumulation point. It seems reasonable to us to define this as the **Lehmer property** in view of the famous conjecture of Lehmer which says that $\overline{\mathbb{Q}}$ (and hence each of its subfields) satisfies this property. As of today, there is no known example of a sub-field $F \subseteq \overline{\mathbb{Q}}$ which satisfies the Lehmer property

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but not the Bogomolov property, even if there are many known examples of fields which do not have the Bogomolov property (see [2]).

We are interested in the Northcott, Bogomolov and Lehmer properties for more general notions of heights, related to objects in algebraic geometry. We will define these properties in [Section 2](#) for any set-theoretic function $h: S \rightarrow \Gamma$, where Γ is a partially ordered set, and we will see how some results in Diophantine geometry can be interpreted in this language.

One can view heights as a way of measuring the complexity of arithmetic or geometric objects, such as algebraic numbers, algebraic number fields, abelian varieties, Galois representations. More generally speaking, one considers *mixed motives*, which can be thought of as pieces cut out from the cohomology of smooth and proper algebraic varieties defined over a number field F . Abelian categories of mixed motives have been defined by Jannsen (see [44, § 4]), Huber (see [41, § 22]) and Nori (see [42, Chapter 9]), using linear algebraic data associated to algebraic varieties. In this paper we will only use these notions of mixed motives, without any further reference to the triangulated approach of Voevodsky, studied for example in the book [16]. We will denote by $\mathcal{MM}(F; E)$ the abelian category of mixed motives defined over F which have coefficients in E , following Jannsen's, Huber's or Nori's definition. These definitions are not known to be equivalent (see [42, Remark 6.3.12 and § 10.1]), but our results hold for all three of them.

There are two possible notions of heights which can be defined for mixed motives:

- (a) the ones defined by Kato in [47] (see [Example 2.20](#)), and in particular the height $h_{*, \diamond} := \log H_{*, \diamond}$ defined in (8);
- (b) The functions

$$h_{n, \sigma}(X) := |L^*(X, n)_{\sigma}|$$

which associate to every mixed motive $X \in \mathcal{MM}(F; E)$ defined over a number field F and with coefficients in another number field E , endowed with a chosen embedding $\sigma: E \hookrightarrow \mathbb{C}$, the special value

$$L^*(X, n)_{\sigma} := \lim_{s \rightarrow n} \frac{L(X, s)_{\sigma}}{(s - n)^{\text{ord}_{s=n}(L(X, s)_{\sigma})}}$$

of the L -function $L(X, s)_{\sigma}$ at the point $s = n$, assuming the limit exists.

It is known that the Northcott property for $h_{*, \diamond}$ would have many interesting consequences, and special cases of this Northcott property have been recently proved by Koshikawa and Nguyen (see [Example 2.20](#) for a more detailed discussion). It seems then natural to ask if special values of L -functions, *i.e.* if the heights $h_{n, \sigma}(X)$, satisfy a Northcott property. We note that the integer $n \in \mathbb{Z}$ at which one takes the special value $h_{n, \sigma}(X)$ plays a crucial role in being able to prove whether or not the function $h_{n, \sigma}$ satisfies a Northcott property.

Our motivation for regarding special values of L -functions as possible heights comes from the many (mostly conjectural) relations between these special values and different forms of height pairings, like for example:

- (1) the conjectures of Boyd, which relate certain special values of L -functions (typically, $L^*(H^1(X), 0)$ for a smooth projective curve X) to the Mahler measure of integral polynomials, which has the Northcott property (see [Example 2.14](#));
- (2) the formula of Gross and Zagier, relating special values of L -functions of modular forms to heights of Heegner points, which has been generalised by the far reaching Kudla program (see [Example 2.15](#));
- (3) the conjectures of Colmez and Maillot-Roessler, which relate logarithmic derivatives of Artin L -functions to Faltings's heights and Arakelov Chern classes (see [Example 2.16](#));
- (4) the relation between special values of Dedekind ζ -functions and the volume of hyperbolic manifolds, which can be regarded as a height (see [Example 2.19](#));
- (5) the conjectures of Bloch and Kato, which can be stated as a relation between special values of L -functions and height pairings of algebraic cycles (see [\[6\]](#), as well as the survey [\[23\]](#)).

Thus it seems natural to us to investigate which properties that are typical of heights hold also for special values of L -functions.

Going back to Kato's height $h_{*,\diamond}$, it seems interesting to us to study whether the special values $h_{n,\sigma}(X) := |L^*(X, n)_\sigma|$ associated to a mixed motive $X \in \mathcal{MM}(F; E)$ can be compared to the height $h_{*,\diamond}(X_{(n)})$ of some other motive $X_{(n)}$ associated with X , or even to the height $h_{*,\diamond}(X)$ of X itself. If this is the case, then it would be possible to prove a Northcott property for $h_{*,\diamond}$ by relating it to a Northcott property for the special values $h_{n,\sigma}(X)$. This will be the subject of future research.

Let us outline the contents and main results of this paper. First of all, we devote [Section 2](#) to an axiomatic treatment of properties of heights, such as the Northcott, Bogomolov and Lehmer properties mentioned in this introduction. This provides a new conceptual framework which encompasses most of the known examples of Diophantine properties that have been studied for different height functions. In particular, we consider a height to be any function $h: S \rightarrow \Gamma$ where S is any set and Γ is a partially ordered set. This allows one to talk about Northcott, Bogomolov, and Lehmer properties also for sets of heights, by using the product height [\(3\)](#). For example, one can restate Northcott's theorem either by saying that the logarithmic Weil height $h: \overline{\mathbb{Q}} \rightarrow \mathbb{R}$ has the Northcott property when restricted to a number field $F \subseteq \overline{\mathbb{Q}}$, or by saying that the pair $\{h, \deg\}$ has the Northcott property.

The rest of the paper focuses on the Northcott property for special values of L -functions. As a test case we consider a number field F and the special value of its Dedekind zeta function $\zeta_F^*(0)$, and we obtain the following theorem.

Theorem 1.1. *Let S be the set of isomorphism classes of number fields. Then for every $B \in \mathbb{R}_{>0}$ the set*

$$S_B := \{[F] \in S \mid |\zeta_F^*(0)| \leq B\}$$

is finite.

[Theorem 1.1](#) can be reformulated using the language introduced in [Section 2.1](#). Indeed, let S' denote the set of isomorphism classes of motives with rational coefficients defined over \mathbb{Q} , *i.e.* the set of isomorphism classes of objects in the category $\mathcal{MM}(\mathbb{Q}, \mathbb{Q})$. Observe that we have an embedding $S \hookrightarrow S'$, which sends a number field F to the motive $\underline{H}^0(\mathrm{Spec}(F))$, and that $L(\underline{H}^0(\mathrm{Spec}(F)), s) = \zeta_F(s)$ for every number field F . Thus [Theorem 1.1](#) can be reformulated by saying that the height function

$$\begin{aligned} h_0: S' &\rightarrow \mathbb{R}_{\geq 0} \\ [X] &\mapsto |L^*(X, 0)| \end{aligned}$$

has the Northcott property (see [Definition 2.1](#)), when restricted to the set $S \hookrightarrow S'$.

Let us review the strategy behind the proof of [Theorem 1.1](#), after having observed that we do not impose *a priori* any control on the degree of the number fields. The analytic class number formula and the functional equation for $\zeta_F(s)$ (see [\[61, Corollary VII.5.11\]](#)) imply that

$$(1) \quad \zeta_F^*(0) = -\frac{h_F}{w_F} R_F$$

where h_F, R_F, w_F are respectively the class number, the regulator and the number of roots of unity of F . Using this, we prove that $|\zeta_F^*(0)| \leq B$ implies that the discriminant Δ_F is bounded above (which is not obvious), and conclude by Hermite's discriminant theorem (see [\[61, Theorem III.2.16\]](#)). We give the details in [Section 3](#).

Let us note that it seems very difficult to prove a similar Northcott property for the special value $\zeta_F^*(1)$, as we explain at the end of [Section 3](#). A similar phenomenon holds for the special value $L^*(A, 1)$ taken at the centre of the critical strip of the L -function $L(A, s)$ associated to an abelian variety A , as we explain in [Section 4](#). This shows once again that the position at which we take the special value influences crucially the possibility of proving a Northcott property.

We conclude this paper with a general result, whose proof is the main content of [Section 5](#), valid for motives with an L -function that satisfies a functional equation.

Theorem 1.2 (Northcott property at the left of the critical strip). *Let F and E be two number fields. Fix an integer $w \in \mathbb{Z}$ and any norm $|\cdot|: E \otimes \mathbb{C} \rightarrow \mathbb{R}_{\geq 0}$. Let S be the set of isomorphism classes of elements in $\mathcal{MM}(F; E)$. Then for every $B_1, B_2 \in \mathbb{R}_{\geq 0}$ and every $n \in \mathbb{Z}$ such that $n < w/2$, the set*

$$(2) \quad S_{B_1, B_2} := \{[X] \in S \mid X \cong \mathrm{gr}_w^{\mathcal{W}}(X), |L^*(X, n)| \leq B_1, \dim(X) \leq B_2\}$$

is finite, under the assumption that the motivic L -functions $L(X, s)$ are well defined, can be meromorphically continued to the whole complex plane, and satisfy the functional equation [\(18\)](#). In other words, the height $h_n: S \rightarrow \mathbb{R}_{\geq 0}$ defined as $h_n(X) := |L^(X, n)|$ has the Northcott property, when restricted to the subset of isomorphism classes of pure motives of weight $w > 2n$.*

To obtain this result, the basic idea is to use the functional equation to bound the conductor of X from above, and deduce finiteness. Once again, this shows the different behaviour of the various functions $h_{n,\sigma}(X) := |L^*(X, n)_\sigma|$ with respect to the choice of n .

2. PROPERTIES OF HEIGHTS

2.1. General definitions. In complete generality we may say that a *height* (or *height function*) on a set S is a function $h: S \rightarrow \Gamma$ with values in a partially ordered set Γ . The aim of this section is to describe various properties of height functions in this generality. Let us start with the Northcott property.

Definition 2.1. Let $h: S \rightarrow \Gamma$ be a height function, and let \mathbb{S} be a collection of subsets of S . Then the height h has:

- (i) the *fibre-wise \mathbb{S} -Northcott property* if and only if the fibres of h lie in \mathbb{S} ;
- (ii) the *\mathbb{S} -Northcott property* if and only if $\{s \in S \mid h(s) \leq \gamma\} \in \mathbb{S}$ for every $\gamma \in \Gamma$.

When \mathbb{S} is the collection of finite subsets of S it will usually be omitted from the notation.

The previous definition readily generalises to sets of height functions as follows.

Definition 2.2. If $\mathbf{h} = \{h_i: S \rightarrow \Gamma_i\}_{i \in I}$ is a set of height functions we say that \mathbf{h} has one of the properties described in [Definition 2.1](#) if and only if the “product height”

$$(3) \quad \begin{aligned} \tilde{\mathbf{h}}: S &\rightarrow \prod_{i \in I} \Gamma_i \\ s &\mapsto (h_i(s))_{i \in I} \end{aligned}$$

has these properties, where the set $\prod_{i \in I} \Gamma_i$ is endowed with the product order.

Before moving on, let us observe that

$$(4) \quad \boxed{h \text{ has } \mathbb{S}\text{-Northcott}} + \boxed{\mathbb{S} \text{ is lower-closed}} \Rightarrow \boxed{h \text{ has fibre-wise } \mathbb{S}\text{-Northcott}}$$

where \mathbb{S} is called *lower-closed* if for all $Y \subseteq X \subseteq S$ then $X \in \mathbb{S} \Rightarrow Y \in \mathbb{S}$. Moreover, if \mathbb{S} is the collection of finite subsets of S then

$$\boxed{h \text{ has fibre-wise Northcott}} + \boxed{h(S) \text{ is upper-finite}} \Rightarrow \boxed{h \text{ has Northcott}}$$

where we say that $X \subseteq \Gamma$ is *upper-finite* if $X_{\leq \gamma} := \{x \in X \mid x \leq \gamma\}$ is finite for all $\gamma \in \Gamma$.

Let us now shift to the definition of the Bogomolov property. This uses the concepts of *essential infimum* and *successive infima*, that we now review.

Definition 2.3. Let Γ be a partially ordered set, let $X \subseteq \Gamma$ and let \mathbb{X} be a collection of subsets of X . Write $X_{\leq \gamma} := \{x \in X \mid x \leq \gamma\}$ for every $\gamma \in \Gamma$. Then X has an \mathbb{X} -*essential infimum* (resp. \mathbb{X} -*essential minimum*) if the set

$$\Xi(X, \mathbb{X}) := \{\gamma \in \Gamma \mid X_{\leq \gamma} \notin \mathbb{X}\} \subseteq \Gamma$$

has an infimum (resp. a minimum). In this case we denote the set of infima (resp. minima) of $\Xi(X, \mathbb{X})$ by $\mu_{\text{ess}}(X, \mathbb{X}) \subseteq \bar{\Gamma}$. Here $\bar{\Gamma} := \Gamma \sqcup \{+\infty\}$ is the partially ordered set obtained by adjoining to Γ a global maximum $+\infty$. In particular, $\mu_{\text{ess}}(X, \mathbb{X}) = \{+\infty\}$ if and only if $\Xi(X, \mathbb{X}) = \emptyset$, *i.e.* if and only if $X_{\leq \gamma} \in \mathbb{X}$ for every $\gamma \in \Gamma$.

Definition 2.4. Let Γ be a partially ordered set and let $k \in \mathbb{N}$. Then a subset $X \subseteq \Gamma$ has *at least k successive sets of infima* (respectively *at least k successive sets of minima*) if:

- (i) X is bounded from below;
- (ii) whenever $k \geq 1$, X has at least $k - 1$ successive sets of infima (resp. sets of minima) and the set $X \setminus X_{k-1}$ has an infimum (resp. minimum). In this case, we denote by $\mu_k(X) \subseteq \Gamma$ the set of infima (resp. minima) of $X \setminus X_{k-1}$. Moreover, the set X_{k-1} is defined by induction as $X_0 := \emptyset$ and

$$X_{k-1} := X_{k-2} \cup U_{k-1}$$

for any $k \geq 2$, where $U_{k-1} \subseteq \Gamma$ denotes the union of connected components of the set $X \cup \{\mu_{k-1}(X)\}$ that contain an element of $\mu_{k-1}(X)$. These connected components are taken with respect to the subspace topology induced on $X \cup \{\mu_{k-1}(X)\}$ by the order topology on Γ .

It is easy to see that for every $j \in \mathbb{Z}_{\geq 1}$ and every $x_j \in \mu_j(X)$ and $x_{j+1} \in \mu_{j+1}(X)$ the inequality $x_j \leq x_{j+1}$ holds. Moreover, if $\mu_{j+1}(X) = \mu_j(X)$ for some $j \in \mathbb{Z}_{\geq 1}$ then X has at least k successive infima for every $k \in \mathbb{N}$ and $\mu_k(X) = \mu_j(X)$ for every $k \geq j$. This leads to the following definition.

Definition 2.5. Let Γ be a partially ordered set. Then any subset $X \subseteq \Gamma$ has *exactly k successive sets of infima* (respectively *exactly k successive sets of minima*) for some $k \in \mathbb{N}$ if it has at least k successive sets of infima (resp. sets of minima) and at least one of the following holds:

- (i) X does not have at least $k + 1$ successive sets of infima (resp. sets of minima);
- (ii) $\mu_{k+1}(X) = \mu_k(X)$.

We are now ready to give the definition of Bogomolov property.

Definition 2.6. Let $h: S \rightarrow \Gamma$ be a height function. Then h has *Bogomolov number* $\mathcal{B}(h) \in \mathbb{N}$ if the set $h(S) \subseteq \Gamma$ has exactly $\mathcal{B}(h)$ successive sets of infima, denoted by $\mu_j(h)$ for every $j \in \{1, \dots, \mathcal{B}(h)\}$.

Definition 2.7. Let $h: S \rightarrow \Gamma$ be a height function. Then h has:

- (i) the *very weak Bogomolov property* if and only if $\mathcal{B}(h) \geq 0$, *i.e.* if and only if the set $h(S) \subseteq \Gamma$ is bounded from below;
- (ii) the *weak Bogomolov property* if and only if $\mathcal{B}(h) \geq 1$ and $\mu_1(h) \in h(S)$, *i.e.* if and only if $h(S)$ has at least one minimum;

- (iii) the *Bogomolov property* if and only if either $|h(S)| = 1$ or $\mathcal{B}(h) \geq 2$ and $\mu_1(h) \in h(S)$, i.e. if and only if the minima of $h(S)$ are isolated.

Moreover, for any collection \mathbb{S} of subsets of S the height h has:

- (iv) the \mathbb{S} -*essential Bogomolov property* if the set $h(S) \subseteq \Gamma$ has an $h(\mathbb{S})$ -essential infimum.

The previous definition readily generalises to sets of height functions.

Definition 2.8. If $\mathbf{h} = \{h_i: S \rightarrow \Gamma_i\}_{i \in I}$ is a set of height functions, we write $\mathcal{B}(\mathbf{h})$ and $\mu_{\text{ess}}(\mathbf{h}, \mathbb{S})$ for the Bogomolov number and the essential infimum of the product height (3), and we say that \mathbf{h} has one of the various Bogomolov properties if and only if $\tilde{\mathbf{h}}$ does.

Clearly one has the chains of implications

$$\boxed{h \text{ has Bogomolov}} \Rightarrow \boxed{h \text{ has weak Bogomolov}} \Rightarrow \boxed{h \text{ has very weak Bogomolov}} \\ \boxed{h \text{ has Northcott}} \Rightarrow \boxed{h \text{ has Bogomolov}}.$$

We can define the Lehmer property by generalising slightly the above definition.

Definition 2.9. Let $\mathbf{h} = \{h_i: S \rightarrow \Gamma_i\}_{i \in I}$ be a set of heights, and let $\alpha: \prod_{i \in I} \Gamma_i \rightarrow \Gamma$ be any map of sets, where Γ is a partially ordered set. Then the *Lehmer number* $\mathcal{L}(\mathbf{h}, \alpha) \in \mathbb{N}$ is defined to be the Bogomolov number of the height

$$S \xrightarrow{\tilde{\mathbf{h}}} \prod_{i \in I} \Gamma_i \xrightarrow{\alpha} \Gamma$$

and the successive infima of $\alpha(\tilde{\mathbf{h}}(S))$ are denoted by $\mu_j(\mathbf{h}, \alpha)$ for $j \in \{1, \dots, \mathcal{L}(\mathbf{h}, \alpha)\}$. Moreover, the pair (\mathbf{h}, α) has:

- (i) the *very weak Lehmer property* if and only if $\alpha \circ \tilde{\mathbf{h}}$ has the very weak Bogomolov property;
- (ii) the *weak Lehmer property* if and only if $\alpha \circ \tilde{\mathbf{h}}$ has the weak Bogomolov property;
- (iii) the *Lehmer property* if and only if $\alpha \circ \tilde{\mathbf{h}}$ has the Bogomolov property.

It is easy to observe that we have the following implications

$$\boxed{h' \text{ has very weak Bogomolov}} + \boxed{\alpha \circ \tilde{\mathbf{h}} \geq h'} \Rightarrow \boxed{(\mathbf{h}, \alpha) \text{ has very weak Lehmer}} \\ \boxed{h' \text{ has weak Bogomolov}} + \boxed{\alpha \circ \tilde{\mathbf{h}} \geq h'} \Rightarrow \boxed{(\mathbf{h}, \alpha) \text{ has weak Lehmer}} \\ \boxed{h' \text{ has Bogomolov}} + \boxed{\alpha \circ \tilde{\mathbf{h}} \geq h'} \Rightarrow \boxed{(\mathbf{h}, \alpha) \text{ has Lehmer}}$$

where $h': S \rightarrow \Gamma$ is any height and $\alpha \circ \tilde{\mathbf{h}} \geq h'$ means that $\alpha(\tilde{\mathbf{h}}(s)) \geq h'(s)$ for every $s \in S$.

2.2. Examples of successive infima. We devote this subsection to the study of examples of successive infima and minima. In particular, we will see that our definitions [Definition 2.4](#) and [Definition 2.5](#) recover the notions of successive infima and minima present in Arakelov geometry, due to Minkowski (for lattices) and Zhang (for heights associated to hermitian line bundles).

Example 2.10. Let $\Gamma = \mathbb{R}$. In this case the order topology coincides with the Euclidean topology. Then every set which has at least zero successive infima (*i.e.* is bounded from below) has also has at least n successive infima for every $n \in \mathbb{N}$. Moreover, if $X \subseteq \mathbb{R}$ is a finite union of open intervals $X = \bigcup_{i=1}^k (a_i, b_i)$ with $a_1 < b_1 < a_2 < b_2 < \dots$, then it is easy to see that X has exactly k successive infima, with $\mu_i(X) = a_i$ for every $i \in \{1, \dots, k\}$. Finally, if $X \subseteq \mathbb{R}$ is countable then X has exactly $k \in \mathbb{Z}_{\geq 1}$ successive minima if and only if there exists a Cauchy sequence $\{x_n\}_{n \in \mathbb{N}} \subseteq X$ such that $|\{x \in X \mid x \leq x_n, \forall n \in \mathbb{N}\}| = k$.

Example 2.11 (Minkowski). Let $\Lambda \subseteq \mathbb{R}^n$ be a lattice, and let $g: \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ be any distance function (see [\[13, Chapter IV\]](#)), *i.e.* any continuous function such that $g(t\mathbf{x}) = |t|g(\mathbf{x})$ for all $t \in \mathbb{R}$. Then the image of the map

$$\begin{aligned} \Lambda &\rightarrow \mathbb{R}_{\geq 0} \times \mathbb{N} \\ \lambda &\mapsto (g(\lambda), \dim_{\mathbb{R}}(V_{g,\lambda})) \end{aligned} \quad \text{where} \quad V_{g,\lambda} := \langle \{x \in \Lambda \mid g(x) \leq g(\lambda)\} \rangle_{\mathbb{R}}$$

has exactly n successive infima, which are given by the pairs $(\mu_j(\Lambda, g), j)$ for some sequence

$$0 < \mu_1(\Lambda, g) \leq \mu_2(\Lambda, g) \leq \dots \leq \mu_n(\Lambda, g) < +\infty$$

with $\mu_j(\Lambda, g) \in \mathbb{R}_{>0}$ for every $j \in \{1, \dots, n\}$. The numbers $\{\mu_j(\Lambda, g)\}$ are usually called *successive minima* of the function g on the lattice Λ (see [\[13, Chapter VIII\]](#)). However, these numbers are really infima and not minima in general.

Example 2.12 (Zhang). Let $\mathcal{X} \rightarrow \text{Spec}(\mathbb{Z})$ be an arithmetic variety of dimension d , as defined in [\[79\]](#), and let $\text{Cl}(X)$ be the set of closed sub-schemes of the generic fibre $X := \mathcal{X}_{\mathbb{Q}}$. Fix $\overline{\mathcal{L}}$ to be a relatively semi-ample hermitian line bundle on \mathcal{X} with ample generic fibre, and let $h_{\overline{\mathcal{L}}}: X(\overline{\mathbb{Q}}) \rightarrow \mathbb{R}$ be the associated height. Then the image of the map

$$\begin{aligned} \text{Cl}(X) &\rightarrow \mathbb{R} \times \mathbb{N} \\ Y &\mapsto (\inf\{h_{\overline{\mathcal{L}}}(x) \mid x \in X(\overline{\mathbb{Q}}) \setminus Y(\overline{\mathbb{Q}})\}, \dim(Y)) \end{aligned}$$

has exactly $d + 1$ successive infima, which are given by pairs $(\mu_j(\mathcal{X}, \overline{\mathcal{L}}), j)$ for some sequence

$$\mu_0(\mathcal{X}, \overline{\mathcal{L}}) \leq \mu_1(\mathcal{X}, \overline{\mathcal{L}}) \leq \dots \leq \mu_d(\mathcal{X}, \overline{\mathcal{L}}) \leq +\infty$$

with $\mu_j(\mathcal{X}, \overline{\mathcal{L}}) \in \mathbb{R}$ for every $j \in \{0, \dots, d-1\}$ and $\mu_d(\mathcal{X}, \overline{\mathcal{L}}) \in \mathbb{R} \sqcup \{+\infty\}$. It is easy to see that $\mu_d(\mathcal{X}, \overline{\mathcal{L}}) = +\infty$ if and only if X is irreducible, and that for every $j \in \{0, \dots, d-1\}$ we have $\mu_j(\mathcal{X}, \overline{\mathcal{L}}) = e_{d-j}(\overline{\mathcal{L}})$, where $e_1(\overline{\mathcal{L}}) \geq \dots \geq e_d(\overline{\mathcal{L}})$ is the sequence defined in [\[79, § 5\]](#).

2.3. Heights and their relations to special values of L -functions. We devote the rest of this section to list examples of heights and their properties, and to relate these examples to special values of L -functions. Let us start with the logarithmic Weil height, which was the main inspiration to give the general definitions in [Section 2.1](#).

Example 2.13 (logarithmic Weil height). Let $h: \overline{\mathbb{Q}} \rightarrow \mathbb{R}$ be the absolute logarithmic Weil height (see [\[7, Definition 1.5.4\]](#)), and let $\deg: \overline{\mathbb{Q}} \rightarrow \mathbb{Z}_{\geq 1}$ denote the degree $\deg(\alpha) := [\mathbb{Q}(\alpha): \mathbb{Q}]$. It is immediate to see that h does not have the fibre-wise Northcott property (with respect to the collection of finite subsets of $\overline{\mathbb{Q}}$), for example because $h(\zeta) = 0$ for any root of unity $\zeta \in \overline{\mathbb{Q}}$. Hence h does not have the Northcott property. It is also immediate to see that the same holds for \deg . However, Northcott's theorem (see [\[7, Theorem 1.6.8\]](#)) shows that the set $\mathbf{h} = \{h, \deg\}$ has the Northcott property. Moreover, it is immediate to see that h has the weak Bogomolov property, because $0 \in \mathbb{R}$ is a minimum for $h(\overline{\mathbb{Q}})$, attained exactly at the roots of unity (see [\[7, Theorem 1.5.9\]](#)). However, it is easy to see that this minimum is not isolated, because for example $\lim_{n \rightarrow +\infty} h(\sqrt[n]{2}) = 0$. Hence $\mathcal{B}(h) = 1$, and h does not have the Bogomolov property. Finally, asking whether the set $\mathbf{h} = \{h, \deg\}$ has the Lehmer property with respect to the function

$$\begin{aligned} \pi: \mathbb{R} \times \mathbb{Z}_{\geq 1} &\rightarrow \mathbb{R} \\ (x, d) &\mapsto xd \end{aligned}$$

is equivalent to Lehmer's celebrated problem (see [\[7, § 1.6.15\]](#)).

Let us mention some of the recent work concerning Northcott, Bogomolov and Lehmer properties relative to the logarithmic Weil height. First of all, it is known that h has the Northcott property, or the Bogomolov property, when restricted to suitable infinite sub-extensions of $\overline{\mathbb{Q}}$ (see the introduction of [\[14\]](#) for a survey of known results). Moreover, Smyth's theorem [\[7, Theorem 4.4.15\]](#) says that (\mathbf{h}, π) has the Lehmer property when restricted to the set $S \subseteq \overline{\mathbb{Q}}$ of algebraic numbers which are not Galois-conjugate to their inverse. Finally, Dobrowolski's theorem [\[7, Theorem 4.4.1\]](#) says that, if we let

$$\begin{aligned} \alpha: \mathbb{R} \times \mathbb{Z}_{\geq 1} &\rightarrow \mathbb{R} \\ (x, d) &\mapsto xd \left(\frac{\log(3d)}{\log \log(3d)} \right)^3 \end{aligned}$$

then the pair (\mathbf{h}, α) has Lehmer's property.

Let us conclude by observing that the results of [\[1\]](#), combined with the class number formula [\(1\)](#), show that for every number field F with unit rank $r_F := \dim_{\mathbb{Q}}(\mathcal{O}_F^{\times} \otimes_{\mathbb{Z}} \mathbb{Q})$ there exists a basis $\{\gamma_1, \dots, \gamma_{r_F}\} \subseteq \mathcal{O}_F^{\times} \otimes_{\mathbb{Z}} \mathbb{Q}$ such that $\{\gamma_1, \dots, \gamma_{r_F}\} \subseteq \mathcal{O}_F^{\times}$ and

$$(5) \quad \frac{h_F d_F^{r_F} (2r_F)!}{2w_F (r_F!)^4} \prod_{i=1}^{r_F} h(\gamma_i) \leq |\zeta_F^*(0)| \leq \frac{h_F d_F^{r_F}}{w_F} \prod_{i=1}^{r_F} h(\gamma_i)$$

which shows that the special value $\zeta_F^*(0)$ of the Dedekind ζ -function associated to a number field F is commensurable to a product of Weil heights. The terms appearing in (5) are given by $d_F := [F : \mathbb{Q}]$, $h_F := |\text{Pic}(\mathcal{O}_F)|$ and $w_F := |(\mathcal{O}_F^\times)_{\text{tors}}|$.

Now, let us study the higher-dimensional generalisation of the function

$$\begin{aligned} \pi \circ \tilde{\mathbf{h}} : \overline{\mathbb{Q}} &\rightarrow \mathbb{R} \\ \alpha &\mapsto h(\alpha) \deg(\alpha) \end{aligned}$$

appearing in [Example 2.13](#).

Example 2.14 (Mahler's measure). Let $\mathbb{G}_{m,\mathbb{C}}^\infty := \varprojlim_{n \in \mathbb{N}} \mathbb{G}_{m,\mathbb{C}}^n$ denote the inverse limit of the complex algebraic tori $\mathbb{G}_{m,\mathbb{C}}^n$ with respect to the projections on the last coordinate. Then the global sections of the structure sheaf $\mathcal{O}_{\mathbb{G}_{m,\mathbb{C}}^\infty}$ are given by the ring of Laurent polynomials in any number of variables. Moreover, the *logarithmic Mahler measure* is defined as

$$\begin{aligned} m : \Gamma(\mathbb{G}_{m,\mathbb{C}}^\infty, \mathcal{O}_{\mathbb{G}_{m,\mathbb{C}}^\infty}) &\rightarrow \mathbb{R} \\ P &\mapsto \int_{\mathbb{T}^\infty} \log |P| d\mu_{\mathbb{T}^\infty}. \end{aligned}$$

where $\mathbb{T}^\infty := \varprojlim_{n \in \mathbb{N}} \mathbb{T}^n$ denotes the inverse limit of the real analytic tori $\mathbb{T}^n := (S^1)^n$ with respect to the projections on the last coordinates, and $\mu_{\mathbb{T}^\infty}$ denotes the unique Haar probability measure on \mathbb{T}^∞ .

The height m has the weak Bogomolov property if one restricts it to the ring

$$\Gamma(\mathbb{G}_{m,\mathbb{Z}}^\infty, \mathcal{O}_{\mathbb{G}_{m,\mathbb{Z}}^\infty}) = \mathbb{Z}[x_1^{\pm 1}, x_2^{\pm 1}, \dots]$$

of Laurent polynomials with integral coefficients, because for every $P \in \mathbb{Z}[x_1^{\pm 1}, x_2^{\pm 1}, \dots]$ one has that $m(P) \geq 0$ and $m(P) = 0$ if and only if P is a product of cyclotomic polynomials evaluated at monomials (see [9]). In particular, $m(P) = m(\tilde{P})$ for every $P \in \mathbb{Z}[x_1^{\pm 1}, \dots]$, where $\tilde{P} \in \mathbb{Z}[x_1, \dots]$ denotes the polynomial obtained by cleaning the denominators of P . Finally, if we let

$$\begin{aligned} \delta : \Gamma(\mathbb{G}_{m,\mathbb{C}}^\infty, \mathcal{O}_{\mathbb{G}_{m,\mathbb{C}}^\infty}) &\rightarrow \mathbb{Z}_{\geq 1} \\ P &\mapsto \sum_{i=1}^{+\infty} i \deg_{x_i}(\tilde{P}) \end{aligned}$$

then the pair (m, δ) has the Northcott property, when restricted to $\Gamma(\mathbb{G}_{m,\mathbb{Z}}^\infty, \mathcal{O}_{\mathbb{G}_{m,\mathbb{Z}}^\infty})$. Indeed, this follows from [57], which gives the inequality

$$\exp(m(P)) = \exp(m(\tilde{P})) \geq 2^{-\sum_{i=1}^{+\infty} \deg_{x_i}(\tilde{P})} \sum_{\mathbf{j}} |a_{\mathbf{j}}|$$

where $\{a_{\mathbf{j}}\}_{\mathbf{j}} \subseteq \mathbb{Z}$ are the coefficients of $\tilde{P} = \sum_{\mathbf{j}} a_{\mathbf{j}} x^{a_{\mathbf{j}}}$ written in multi-index notation.

Conjectural relations of Mahler's measure with special values of L -functions come from the work of Boyd [10]. A celebrated example of these relations is that, for every $k \in \mathbb{Z} \setminus \{0, \pm 4\}$, there should exist a rational number $\alpha_k \in \mathbb{Q}^\times$ such that

$$(6) \quad L'(E_k, 0) = \alpha_k m \left(x + \frac{1}{x} + y + \frac{1}{y} + k \right)$$

where $E_k: y^2 = x^3 + (k^2 - 8)x^2 + 16x$ is an elliptic curve. Moreover, the computational evidence gathered in [10] shows that it is reasonable to expect that $\alpha_k \in \mathbb{Z}$ for all but finitely many k . If this is true, then the relation (6) and the Northcott property of the Mahler measure would entail a Northcott property for the special values $L'(E_k, 0)$.

We have seen that the Mahler measure of a polynomial $P \in \mathbb{Z}[x_1, \dots, x_n]$ can be seen as a way of measuring the complexity of the zero locus of P , which is a sub-variety of $\mathbb{G}_{m, \mathbb{Z}}^n$. Let us briefly outline another height defined for the closed sub-varieties of a given arithmetic variety, which depends more canonically on the choice of a model.

Example 2.15 (Canonical height). Let X be an algebraic variety defined over a number field F . The term canonical height (usually denoted by \hat{h}_X^{can}) is often used for a height function which measures the arithmetic complexity of closed sub-varieties of X .

The simplest case is of course the Néron-Tate height $\hat{h}_{X, \phi, D}^{\text{can}}: X(\bar{F}) \rightarrow \mathbb{R}$ associated to an endomorphism $\phi: X \rightarrow X$ and a divisor $D \in \text{Div}(X)$ such that $\phi^*(D) \sim \alpha D$ for some $\alpha \in \mathbb{R}_{>1}$. We refer the interested reader to [40, § B.4] for an introduction to this notion of canonical height, with a special focus on the case of abelian varieties.

Canonical heights for higher dimensional subvarieties have been defined by Zhang (see [79, 80]), Philippon (see [66, 67, 68]) and Faltings (see [22] and [71, Chapter III, § 6]). We refer also the interested reader to [37, § 3] for an introduction to canonical heights in toric varieties, and to [37, Corollary 6.3] for their relation with the Mahler measure.

The canonical height is related to special values of L -functions by a far reaching program initiated by the seminal work of Gross and Zagier (see [35, 33], as well as the modern reviews [18, 78]). This was later continued by the groundbreaking works of Kudla and collaborators (see [34, 51]). We refer the interested reader to the survey article [52], as well as to the monograph [53]. Finally, we mention the recent work of Li and Zhang [55], which settles the local Kudla-Rapoport conjecture in the unitary case.

In the following, example we focus on the case of abelian varieties, for which a more intrinsic definition of height has been given by Faltings in [21].

Example 2.16 (Faltings height). Let $\mathcal{A}(\bar{\mathbb{Q}})$ be the set of isomorphism classes of abelian varieties defined over $\bar{\mathbb{Q}}$, and let $h: \mathcal{A}(\bar{\mathbb{Q}}) \rightarrow \mathbb{R}$ be the stable Faltings height (see [21, Section 3] and [19, Page 27], which use two different normalizations). Then the set $\{h, \dim\}$ has the very weak Bogomolov property, since one has the lower bound $h(A) \geq -\log(\sqrt{2\pi}) \dim(A)$ (see [27, Corollary 8.4]). Then [19, Page 29] shows that h has the weak Bogomolov property if we

restrict to the set $\mathcal{A}_1(\overline{\mathbb{Q}})$ of $\overline{\mathbb{Q}}$ -isomorphism classes of elliptic curves defined over $\overline{\mathbb{Q}}$. Moreover, [56] and [12] show that $h: \mathcal{A}_1(\overline{\mathbb{Q}}) \rightarrow \mathbb{R}$ has the Bogomolov property *tout court*. It seems reasonable to ask whether the set $\{h, \dim\}$ has the Bogomolov property.

Finally, Faltings's celebrated theorem [21, Theorem 1], combined with Zarhin's "trick" [59, Remark 16.12], shows that the set $\{h, \dim, \deg\}$ has the Northcott property. The degree function is defined by

$$\begin{aligned} \deg: \mathcal{A}(\overline{\mathbb{Q}}) &\rightarrow \mathbb{N} \\ A &\mapsto \min\{[F: \mathbb{Q}] \mid A \text{ is defined over } F\} \end{aligned}$$

where we say that an abelian variety A defined over a field \mathbb{L} is defined over a sub-field \mathbb{K} if there exists an abelian variety A' defined over \mathbb{K} and such that $A \cong A' \times_{\text{Spec}(\mathbb{K})} \text{Spec}(\mathbb{L})$. Then \deg is well defined, because every abelian variety defined over $\overline{\mathbb{Q}}$ can be defined over a number field (see [36, Théorème 8.8.2]). It has also been recently proved that (if one assumes Artin's and Colmez's conjectures) the function h satisfies Northcott's property, if we restrict to the subset of isomorphism classes of abelian varieties with complex multiplication (see [60, Theorem 1.4]).

For abelian varieties with complex multiplication, the stable Faltings height $h: \mathcal{A}(\overline{\mathbb{Q}}) \rightarrow \mathbb{R}$ is expected to be related to L -functions by Colmez's conjecture [17, Conjecture 0.4] which predicts the relation

$$(7) \quad -h(A) \stackrel{?}{=} \sum_{\chi} m_{(E, \Phi)}(\chi) \left(\frac{L'(\chi, 0)}{L(\chi, 0)} + \log(f_{\chi}) \right)$$

where (E, Φ) is the CM-type of A and the sum runs over all the Artin characters

$$\chi: G_{\mathbb{Q}} \rightarrow \mathbb{C}^{\times}$$

whose value on complex conjugation $c \in G_{\mathbb{Q}} := \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ equals $\chi(c) = -1$. This implies in particular that $L(\chi, 0) \in \mathbb{C}^{\times}$. Moreover, $f_{\chi} \in \mathbb{N}$ denotes the Artin conductor of χ and the family of rational numbers $\{m_{(E, \Phi)}(\chi)\}_{\chi} \subseteq \mathbb{Q}$ is defined by the equality

$$\frac{1}{[G_{\mathbb{Q}}: \text{Stab}(\Phi)]} \sum_{\sigma \in G_{\mathbb{Q}}/\text{Stab}(\Phi)} |\Phi \cap \sigma \circ \Phi| = \sum_{\chi} m_{(E, \Phi)}(\chi) \chi(\sigma)$$

which holds for every $\sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. In particular, $m_{(E, \Phi)}(\chi) = 0$ for all but finitely many Artin characters.

We have seen in the previous example that Colmez's conjectural formula (7) involves the *Artin conductor* $f_{\chi} \in \mathbb{N}$ associated to an Artin character $\chi: G_{\mathbb{Q}} \rightarrow \mathbb{C}^{\times}$. By definition $f_{\chi} := N(\mathfrak{f}_{\rho})$ coincides with the norm of the conductor associated to any complex representation $\rho: G_{\mathbb{Q}} \rightarrow \text{GL}_n(\mathbb{C})$ such that $\chi = \text{tr} \circ \rho$, where $\text{tr}: \text{GL}_n(\mathbb{C}) \rightarrow \mathbb{C}$ denotes the trace map. The aim of the next example is to show that the association $\rho \mapsto \mathfrak{f}_{\rho}$ behaves almost like a height. In particular it satisfies a Northcott property, at least if we include the Archimedean places in the definition of the conductor.

Example 2.17 (Conductors of complex representations). Let F be a number field, fix $n \in \mathbb{N}$ and let $\mathcal{A}_n(F)$ be the set of cuspidal automorphic representations of $\mathrm{GL}_n(\mathbb{A}_F)$ (see [43, § 1]). Then [11, Corollary 9] shows that the *analytic conductor* $\mathcal{C}: \mathcal{A}_n(F) \rightarrow \mathbb{R}_{\geq 1}$, which is defined in [43, Equation (31)], satisfies the Northcott property. In particular, the $n = 1$ case shows that the set of Hecke characters $\psi: \mathbb{A}_F^\times \rightarrow \mathbb{C}^\times$ with bounded analytic conductor is finite.

Let now $\mathcal{W}_{\mathbb{C}}(F)$ be the set of isomorphism classes of pairs (V, ρ) where V is a finite dimensional complex vector space and $\rho: W_F \rightarrow \mathrm{GL}(V)$ is a continuous semi-simple representation of the Weil group W_F (see [73, § 1]). Then there is a function $\mathfrak{f}: \mathcal{W}_{\mathbb{C}}(F) \rightarrow \mathcal{O}_F$ sending each (V, ρ) to its global Artin conductor ideal $\mathfrak{f}_\rho \subseteq \mathcal{O}_F$ (see [61, Chapter VII, § 11]). Moreover, the Archimedean local Langlands correspondence, explained for example in [48], allows one to associate to each $(V, \rho) \in \mathcal{W}_{\mathbb{C}}(F)$ an Archimedean conductor $\mathcal{C}_\infty((V, \rho)) \in \mathbb{R}$, defined in exactly the same way as the Archimedean part of the analytic conductor of a cuspidal automorphic form. Then [3, Theorem 3.3] can be combined with our previous discussion to show that the function $\mathcal{C}: \mathcal{W}_{\mathbb{C}}(F) \rightarrow \mathbb{R}$ defined as $\mathcal{C}((V, \rho)) := N_{F/\mathbb{Q}}(\mathfrak{f}_\rho) \mathcal{C}_\infty((V, \rho))$ satisfies the Northcott property. Let us observe that:

- (i) one can consider all the number fields at once as follows: if $\mathcal{W}_{\mathbb{C}}$ denotes the set of isomorphism classes of triples (F, V, ρ) , where F is a number field and $(V, \rho) \in \mathcal{W}_{\mathbb{C}}(F)$, then [69, Property (a2)] shows that the composite map $\mathcal{C} \circ \mathrm{Ind}: \mathcal{W}_{\mathbb{C}} \rightarrow \mathcal{W}_{\mathbb{C}}(\mathbb{Q}) \rightarrow \mathbb{R}$ satisfies the Northcott property, where $\mathrm{Ind}: \mathcal{W}_{\mathbb{C}} \rightarrow \mathcal{W}_{\mathbb{C}}(\mathbb{Q})$ sends (F, V, ρ) to the induced representation on $W_{\mathbb{Q}} \supseteq W_F$;
- (ii) the conductor \mathfrak{f}_ρ is related to L -functions by means of the functional equation (see [73, Theorem 3.5.3]), which will also be exploited in [Section 5](#)

The following example is the analogue of the previous one for representations valued in vector spaces defined over \mathbb{Q}_ℓ .

Example 2.18 (Conductors of ℓ -adic representations). Let $\ell \in \mathbb{N}$ be a prime number and let F be a number field. We denote by M_F^0 the set of non-Archimedean places of F , and for every $v \in M_F^0$ we write $\mathrm{Frob}_v \subseteq \mathrm{Gal}(\overline{F}/F)$ for the conjugacy class of geometric Frobenius elements relative to v . We define $\mathcal{G}_\ell(F)$ to be the set of isomorphism classes of pairs (V, ρ) where V is a finite dimensional vector space over \mathbb{Q}_ℓ and $\rho: \mathrm{Gal}(\overline{F}/F) \rightarrow \mathrm{GL}(V)$ is a continuous semi-simple representation satisfying the following properties:

- the set $S_\rho^{(\mathrm{ram})} \subseteq M_F^0$ of non-Archimedean places at which ρ is ramified is finite;
- the set $S_\rho^{(\mathrm{int})} \subseteq M_F^0$ of non-Archimedean places $v \in M_F^0$ such that $\mathrm{tr}(\rho(\mathrm{Frob}_v)) \in \mathbb{Z}$ has finite complement. Here $\mathrm{tr}: \mathrm{GL}(V) \rightarrow \mathbb{C}$ denotes the trace.

Let now $(V, \rho) \in \mathcal{G}_\ell(F)$. We set $S_\rho := S_\rho^{(\mathrm{ram})} \cup (M_F^0 \setminus S_\rho^{(\mathrm{int})})$ and we denote by \mathbf{T}_ρ the family of finite sets $T \subseteq M_F^0$ such that $T \cap S_\rho = \emptyset$ and the restriction map

$$\bigcup_{v \in T} \mathrm{Frob}_v \rightarrow \mathrm{Gal}(K/F)$$

is surjective for every extension $F \subseteq K$ of number fields which is unramified outside S_ρ and such that $[K : F] \leq \ell^{2 \dim(V)^2}$. We define two functions $\pi : \mathcal{G}_\ell(F) \rightarrow \mathbb{N}$ and $\tau : \mathcal{G}_\ell(F) \rightarrow \mathbb{N}$ as

$$\begin{aligned}\pi(V, \rho) &:= \max\{\text{char}(\kappa_v) : v \in S_\rho^{(\text{ram})}\} \\ \tau(V, \rho) &:= \min_{T \in \mathbf{T}_\rho} (\max\{|\text{tr}(\rho(\text{Frob}_v))| : v \in T\})\end{aligned}$$

where κ_v denotes the residue field of F at v . Note in particular that $\mathbf{T}_\rho \neq \emptyset$, as follows from a combination of Chebotarev's density theorem and Hermite's theorem.

Then [20, Théorème 1] shows that the set $\mathbf{h} = \{\dim, \pi, \tau\}$ has the Northcott property. Moreover, the functions π and τ are related to more classical invariants as follows:

- (i) $\pi(V, \rho) \leq \mathcal{C}_0(V, \rho)$, where $\mathcal{C}_0(V, \rho) := N_{F/\mathbb{Q}}(\mathfrak{f}_\rho)$ denotes the norm of the conductor ideal $\mathfrak{f}_\rho \subseteq \mathcal{O}_F$ associated to ρ (see for example [75]). Hence the set $\mathbf{h} = \{\dim, \mathcal{C}_0, \tau\}$ has the Northcott property;
- (ii) $\tau(V, \rho) \leq \dim(V) \tilde{\tau}(V, \rho)$, where $\tilde{\tau} : \mathcal{G}_\ell(F) \rightarrow \mathbb{R}$ is the function defined by

$$\tilde{\tau}(V, \rho) := \min_{T \in \mathbf{T}_\rho} (\max\{|\sigma| : \sigma \in \text{Sp}(\rho(\text{Frob}_v))\})$$

where, for any $f \in \text{End}(V)$, we denote by $\text{Sp}(f)$ the set of its eigenvalues. In particular, if we restrict to the subset $\mathcal{M}_\ell(F) \subseteq \mathcal{G}_\ell(F)$ consisting of those Galois representations that admit a weight filtration with finitely many non-zero graded pieces (see [45, § 2]), then the sets $\{\dim, \pi, w_{\max}\}$ and $\{\dim, \mathcal{C}_0, w_{\max}\}$ have the Northcott property, where $w_{\max} : \mathcal{M}_\ell(F) \rightarrow \mathbb{N}$ sends a representation to the greatest of its weights.

Let us conclude by making the following observations:

- (a) the semi-simplifications of the ℓ -adic étale cohomology groups $H_{\text{ét}}^i(X_{\overline{F}}; \mathbb{Q}_\ell(j))$ associated to a smooth and proper variety X defined over F which has good reduction at all the primes of F lying above ℓ give rise to elements of $\mathcal{M}_\ell(F)$ which are pure of weight $i - 2j$. For these Galois representations the set S_ρ equals the set of primes of F which either lie above ℓ or are primes of bad reduction for X . This follows from the smooth and proper base change theorem for étale cohomology, combined with Deligne's proof of the Weil conjectures (see [44, Appendix C]).
- (b) we can consider all the number fields at once, as we did in Example 2.17, by defining \mathcal{G}_ℓ as the set of isomorphism classes of triples (F, V, ρ) where F is a number field and $(V, \rho) \in \mathcal{G}_\ell(F)$. Then [69, Property (a2)] implies that the sets $\{\dim \circ \text{Ind}, \pi \circ \text{Ind}, \tau\}$ and $\{\dim, \mathcal{C}_0 \circ \text{Ind}, \tau\}$ have the Northcott property. Here $\text{Ind} : \mathcal{G}_\ell \rightarrow \mathcal{G}_\ell(\mathbb{Q})$ is again the map sending (F, V, ρ) to the representation induced on $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \supseteq \text{Gal}(\overline{F}/F)$;
- (c) the conductor \mathfrak{f}_ρ is supposed to be related to the L -function $L(\rho, s)$ by means of the conjectural functional equation (compare with [73, § 4.5]).

We conclude this roundup of examples by talking about two more geometric examples of height: the volume of hyperbolic manifolds and the heights of mixed motives defined by Kato.

Example 2.19 (Volumes of hyperbolic manifolds). Let \mathcal{H} be the set of isomorphism classes of hyperbolic manifolds of finite volume. Then it is conjectured that the volume $\text{vol}: \mathcal{H} \rightarrow \mathbb{R}_{\geq 0}$ has the Bogomolov property, and that the minimum is attained at an arithmetic hyperbolic manifold $M \cong \mathfrak{h}_n/\Gamma$, where Γ is an arithmetic subgroup of the isometry group of the hyperbolic space \mathfrak{h}_n (see [5]). Then, if we restrict to the set \mathcal{H}^{ar} of isomorphism classes of arithmetic hyperbolic manifolds, it is conjectured that the set $\mathbf{h} = \{\text{vol}, \dim, \deg\}$ has the Northcott property, where the degree is defined by $\deg(M) := [\mathbb{Q}(\text{tr}(\pi_1(M)^{(2)})) : \mathbb{Q}]$. Here we denote by $\pi_1(M)^{(2)}$ the sub-group generated by the squares, and by $\text{tr}: \pi_1(M) \rightarrow \mathbb{C}$ the trace map induced from the embedding of $\pi_1(M)$ into the isomorphism group of \mathfrak{h}_n . This Northcott property has been proved for three dimensional arithmetic hyperbolic manifolds (see [46]).

The relations of hyperbolic volumes with special values of L -functions comes from the formula

$$\zeta_F^*(-1) \sim_{\mathbb{Q}^\times} \text{vol} \left(\frac{\mathfrak{h}_3^{r_2(F)}}{\Gamma} \right)$$

which holds for any number field F . Here Γ is a finite-index and torsion-free subgroup of the group $\mathcal{O}^{(1)} \subseteq \mathcal{O}$ of units having norm one in some order $\mathcal{O} \subseteq \mathcal{B}$ in a totally definite quaternion algebra $\mathcal{B} \neq \text{Mat}_{2 \times 2}(K)$ defined over K (see [76, Example IV.1.5]).

Example 2.20 (Heights of motives). Let F and E be two number fields, such that F is the base field and E is the field of coefficients of the abelian category of mixed motives $\mathcal{MM}(F; E)$, defined following one of the constructions provided by Jannsen (see [44, § 4]), Huber (see [41, § 22]) or Nori (see [42, Chapter 9]).

Then Kato constructs in [47] a series of height functions which measure the complexity of an object $X \in \mathcal{MM}(F; E)$, using the v -adic Hodge theory corresponding to any place $v \in M_F$. One of the richest examples of such a height is given by the function

$$(8) \quad \begin{aligned} h_{*, \diamond}: \mathcal{MM}(F; E) &\rightarrow \mathbb{R} \\ X &\mapsto h_{\diamond}(X) + \sum_{w \in \mathbb{Z}} h_*(\text{gr}_w^{\mathcal{W}}(X)) \end{aligned}$$

which is the logarithmic version of the height $H_{*, \diamond}$ defined in [47, § 1.7.1]. Here

$$(9) \quad \text{gr}_w^{\mathcal{W}}(X) := \frac{\mathcal{W}_w(X)}{\mathcal{W}_{w-1}(X)}$$

denotes the graded piece of X with respect to the ascending weight filtration \mathcal{W} , and the various heights $h_*(\text{gr}_w^{\mathcal{W}}(X))$ appearing in (8) are a generalisation of Faltings's height (see [Example 2.16](#)) to pure motives. On the other hand, the height $h_{\diamond}(X)$ measures the distance between X and the semi-simplification

$$X^{\text{ss}} := \bigoplus_{w \in \mathbb{Z}} \text{gr}_w^{\mathcal{W}}(X)$$

and thus can be seen as a measure of the mixed nature of X .

It is extremely interesting to study the Northcott property for the height $h_{*,\diamond}$, in view of the many consequences that this would have, which are investigated in [47, § 2]. In particular, [47, Proposition 2.1.17] shows that the Northcott property for $h_{*,\diamond}$ implies the finite generation of motivic cohomology, which would be a motivic analogue of the Mordell-Weil theorem. Special instances of the Northcott property for the height $h_{*,\diamond}$ have been recently proved to hold by Koshikawa (see [49, 50]) and Nguyen (see [62]). In particular, Koshikawa shows that $h_{*,\diamond}$ has the Northcott property when restricted to the set of pure motives X which are isogenous to a fixed pure motive X_0 . We note that in this case $h_{*,\diamond}(X) = h_*(X)$ because $h_\diamond(X) = 0$ for pure motives X . Koshikawa's result is reminiscent of the similar Northcott property for the Faltings height (see [21, § 4]), which allowed Faltings himself to show the Tate conjecture for abelian varieties.

3. SPECIAL VALUES INSIDE THE CRITICAL STRIP: NUMBER FIELDS

We give now a proof of [Theorem 1.1](#).

Proof (of [Theorem 1.1](#)). Let $B > 0$ be a real number, and recall that S denotes the set of isomorphism classes of number fields. We will prove that $|\zeta_F^*(0)| \leq B$ implies that $|\Delta_F|$ is bounded above, and conclude by Hermite's discriminant theorem (see [61, Theorem III.2.16]).

By the class number formula (1), if $|\zeta_F^*(0)| \leq B$, then we have

$$(10) \quad \frac{h_F R_F}{w_F} \leq B.$$

The proof proceeds with two steps: first we prove that inequality (10) implies an upper bound on R_F . This will lead to finiteness, except possibly for CM fields. The second step is proving finiteness of CM fields with $|\zeta_F^*(0)|$ bounded from above.

Step 1: Observe first of all that for every number field F of degree $d_F := [F : \mathbb{Q}]$ and number of roots of unity $w_F := |(\mathcal{O}_F^\times)_{\text{tors}}|$, the inequality $\varphi(w_F) \leq d_F$ holds true, where φ is Euler's totient function. Indeed $\mathbb{Q}(\mu_{w_F}) \subseteq F$, where $\mu_n \subseteq \overline{\mathbb{Q}}$ denotes the group of n -th roots of unity. Then one can use the easy estimate $2\varphi(n) \geq \sqrt{n}$ to get that $w_F \leq 4d_F^2$. Now, for every number field F one has that $R_F \geq c_1 c_2^{d_F}$, where $c_1 = (11.5)^{-39}$ and $c_2 = 1.15$, as it was proved by Zimmert in [81, Satz 3] (see also [70] for a simpler proof, and [26] for a more general statement). This surely implies the weaker inequality $w_F \leq c_3 \sqrt{R_F}$ for some absolute constant $c_3 \in \mathbb{R}_{>0}$. Going back to (10) and using $h_F \geq 1$, we see that $|\zeta_F^*(0)| \leq B$ implies that $R_F \leq (c_3 B)^2$. By [64, Theorem 1.1] we obtain that the set

$$\{[F] \in S \setminus S_{\text{CM}} \mid |\zeta_F^*(0)| \leq B\}$$

is finite, where S_{CM} is the set of isomorphism classes of CM number fields. We recall briefly the argument here for completeness. Observe that Zimmert's inequality $R_F \geq c_1 c_2^{d_F}$ implies that number fields with regulator bounded from above have degree bounded from above (recall

$c_2 > 1$). Then we can use [25, Theorem C] providing us with the inequality

$$(11) \quad R_F \geq \frac{c_4}{d_F^{2d_F}} \left(\log \left(\frac{|\Delta_F|}{d_F^{d_F}} \right) \right)^{r_1(F)+r_2(F)-1-r_0(F)}$$

where $c_4 \in \mathbb{R}_{>0}$ is an absolute constant, $r_1(F)$ is the number of real embeddings of F , $r_2(F)$ is the number of pairs of complex embeddings of F , and

$$r_0(F) := \max \{r_1(L) + r_2(L) - 1 \mid L \subsetneq F\}$$

is the biggest unit rank of proper sub-fields of F . This gives a useful upper bound on the discriminant if and only if F is not a CM field. Indeed, we always have

$$r_0(F) \leq r_1(F) + r_2(F) - 1$$

and the equality $r_0(F) = r_1(F) + r_2(F) - 1$ is satisfied if and only if F is a CM field (see [64, Proposition 3.7]). The final step is Hermite's discriminant theorem, which shows that the discriminant has the Northcott property.

Step 2: We now want to prove that the set

$$(12) \quad \{[F] \in S_{\text{CM}} \mid |\zeta_F^*(0)| \leq B\}$$

is finite, where S_{CM} is the set of isomorphism classes of CM fields. To do so observe that for a CM field F of degree $d_F := [F: \mathbb{Q}]$ and with maximal real sub-field denoted F^+ , the inequality $R_F \geq 2^{\frac{d_F}{2}-1} R_{F^+}$ holds true (see [64, Proposition 3.7]), and thus any upper bound on $|\zeta_F^*(0)|$ entails an upper bound on R_{F^+} . This implies by [64, Theorem 1.1] that if F is an element of the set given in (12), then F^+ belongs to a finite set of isomorphism classes of totally real fields. Hence to conclude we can assume that F^+ is fixed. Then any upper bound on $|\zeta_F^*(0)|$ implies an upper bound on h_F , which in turn implies the finiteness of the set given in (12) by results of Siegel and Stark. To be more precise, when $F^+ = \mathbb{Q}$ (hence F is an imaginary quadratic field), Siegel proved the following: for any fixed $\varepsilon > 0$ there exists a constant $c_5(\varepsilon) > 0$, such that for any imaginary quadratic field F

$$h_F \geq c_5(\varepsilon) |\Delta_F|^{\frac{1}{2}-\varepsilon}.$$

This implies that the set of isomorphism classes of imaginary quadratic fields of class number bounded from above is finite by Hermite's theorem (see [28] for a short and elegant proof of Siegel's result). If $F^+ \neq \mathbb{Q}$ we can use a result of Stark, who proved the following: for any fixed $\varepsilon > 0$ there exists a constant $c_6(\varepsilon) > 0$, such that for any CM field F of degree $d_F \geq 4$,

$$(13) \quad h_F \geq \frac{c_6(\varepsilon)^{d_F}}{d_F g(F^+)} \left(\frac{|\Delta_F|}{|\Delta_{F^+}|^2} \right)^{\frac{1}{2}-\frac{1}{d_F}} |\Delta_F|^{\frac{1}{2}-\frac{2}{d_F}-\varepsilon}$$

(see [72, Theorem 2]) where for every number field κ we set $g(\kappa) = 1$ if there is a tower $\mathbb{Q} = \kappa_0 \subseteq \dots \subseteq \kappa_n = \kappa$ such that $\kappa_i \subseteq \kappa_{i+1}$ is Galois, and $g(\kappa) = [\kappa: \mathbb{Q}]!$ otherwise. Since we fixed F^+ then (13) gives us immediately an upper bound for the absolute discriminant Δ_F , which depends on h_F (recall $d_F \geq 4$). This in turn implies the finiteness of isomorphism

classes of CM fields with $F^+ \neq \mathbb{Q}$ fixed and h_F bounded, again by Hermite's theorem. Putting everything together, we have proved that the set given in (12) is finite and thus we completed the proof. \square

Remark 3.1. If we look at the special value of $\zeta_F(s)$ at $s = 1$, the class number formula reads

$$\zeta_F^*(1) = \frac{2^{r_1}(2\pi)^{r_2}}{w_F} \frac{h_F R_F}{\sqrt{|\Delta_F|}}.$$

By the Brauer-Siegel Theorem [54, Chapter XVI], when the degree of F is bounded in a family, we have $|\Delta_F|^{\frac{1}{2}-\varepsilon} \ll h_F R_F \ll |\Delta_F|^{\frac{1}{2}+\varepsilon}$. It seems thus difficult to derive any Northcott property for $|\zeta_F^*(1)|$. This phenomenon will reappear in the case of abelian varieties, which we will study in the next section.

4. SPECIAL VALUES INSIDE THE CRITICAL STRIP: ABELIAN VARIETIES

In this section, we investigate the possible Northcott property of the special value at the integer $s = 1$ of the L -functions $L(A, s) := L(\underline{H}^1(A), s)$ associated to abelian varieties A defined over a number field F . The main outcome of the discussion is that even assuming the Birch and Swinnerton-Dyer conjecture, it is not possible, so far, to prove a Northcott property in this case. According to the heuristics of Watkins about elliptic curves E over \mathbb{Q} (see [77]), one is in fact lead to the conclusion that the Northcott property for $L^*(E, 1)$ could be unlikely to hold in general.

We note that it is not clear whether we can expect a similar Northcott property as in the case of the special values $\zeta_F^*(0)$ which we considered in the previous section. First of all, if we want to follow the strategy that we used in the previous section, we should relate the special value $L^*(A, 1)$ to some regulator determinant. This relation was given by the class number formula (1) in the case of the special value $\zeta_F^*(0)$ studied in the previous section, and was thus unconditional. On the other hand, $L^*(A, 1)$ is related to a regulator determinant by conjectural equality

$$(14) \quad L^*(A, 1) \stackrel{?}{=} \left(\frac{\prod_{v \in M_F^0} c_v(A)}{|A(F)_{\text{tors}}| |A^\vee(F)_{\text{tors}}|} \right) \frac{|\text{III}(A/F)| R_{A/F}}{\Omega_A^{-1}}$$

which is the subject of the celebrated conjecture by Birch and Swinnerton-Dyer (see [74]).

Now, the first step in the proof of Theorem 1.1 was observing that the quantity $|(\mathcal{O}_F^\times)_{\text{tors}}|$ appearing in the class number formula (1) is clearly bounded from above by a polynomial in the degree $[F : \mathbb{Q}]$ of the number field F . An analogous statement for abelian varieties is the content of the following conjecture.

Conjecture 4.1 (Torsion conjecture). *For every $d \in \mathbb{N}_{\geq 1}$ and every $g \in \mathbb{N}_{\geq 1}$ there exists a natural number $c(g, d) \in \mathbb{N}$ such that for all number fields F of degree $d = [F : \mathbb{Q}]$, for all g -dimensional abelian varieties A defined over F , we have $|A(F)_{\text{tors}}| \leq c(g, d)$.*

We recall that, in the case of elliptic curves, [Conjecture 4.1](#) is proved to be true, thanks to work of Merel (see [\[58\]](#)). Moreover, the prime number theorem shows easily that

$$|A(F)_{\text{tors}}| |A^\vee(F)_{\text{tors}}| \ll (\log |N_{F/\mathbb{Q}}(\mathfrak{f}_A)|)^{4 \dim(A)}$$

as explained in [\[38, Lemma 3.6\]](#).

Now, observing that the Tamagawa numbers $c_v(A)$ are positive integers, we see that any upper bound for the quantity $|L^*(A, 1)|$ entails an upper bound for the quantity

$$(15) \quad \frac{|\text{III}(A/F)| R_{A/F}}{\Omega_A^{-1}}$$

if one assumes the validity of the formula [\(14\)](#), and of [Conjecture 4.1](#). Since our goal is to study the Northcott property for the quantity $|L^*(A, 1)|$, it would be useful to compare the quantity [\(15\)](#) to other quantities for which a Northcott property is already known to hold. The best candidates for this are the stable Faltings height $h_{\text{st}}(A)$ and the norm of the conductor ideal \mathfrak{f}_A of the abelian variety A .

This is exactly the same strategy which was achieved in the proof of [Theorem 1.1](#), where the quantity $h_F R_F$ was compared to the quantity $|\Delta_F|$, which satisfies the Northcott property thanks to Hermite's theorem. However, there is one fundamental difference between the proof of [Theorem 1.1](#) and the current discussion: both the numerator and the denominator of the ratio [\(15\)](#) are comparable to something satisfying a Northcott property, at least conjecturally. In that respect, that case of $|L^*(A, 1)|$ is closer to the case of $|\zeta_F^*(1)|$ described at the end of the previous section.

Let us be more precise. First of all, one has that

$$H(A) \ll \Omega_A^{-1} \ll H(A)(\log(H(A)))^{\dim(A)/2}$$

as shown in [\[38, Lemma 3.7\]](#). Recent works of Hindry [\[38\]](#), Hindry-Pacheco [\[39\]](#) and Griffon [\[29, 30, 31, 32\]](#) on the analogue of the Brauer-Siegel estimate for abelian varieties show that the numerator of [\(15\)](#) is also expected to be comparable (in some cases) to $H(A)$. Hence it is necessary to gain further evidence in order to be able to decide if a Northcott property for the special value $L^*(A, 1)$ associated to abelian varieties holds in some cases. In particular, both the numerator and denominator of the ratio [\(15\)](#) appear to be comparable (in some cases) to $H(A)$. This makes it extremely difficult to prove, or even expect, a Northcott property for the special value $L^*(E, 1)$. In fact, the heuristics proposed by Watkins in [\[77\]](#) provide some evidence to expect that $L^*(E, 1)$ does not satisfy a Northcott property. Indeed, Watkins's work predicts the existence of infinitely many elliptic curves E defined over \mathbb{Q} for which $|\text{III}(E/F)| R_{E/F}$ is bounded (see in particular [\[77, § 4.5\]](#)).

In the case of elliptic curves, one knows from [\[4\]](#) that the following inequality holds

$$(16) \quad \frac{R_{E/F}}{|E(F)_{\text{tors}}| |E^\vee(F)_{\text{tors}}|} \gg h^{\frac{r_{E/F}-4}{3}} (\log(3h))^{\frac{2r_{E/F}+2}{3}}$$

where $h := \max\{1, h(j(E))\}$ is a quantity comparable with the stable Faltings height (see for instance [65, Lemma 3.2]), and $r_{E/F} := \text{rk}(E(F))$. The inequality (16) shows that a part of the right hand side of (14) can indeed be related to some height, even if this relation is too weak to conclude (even assuming the validity of the Birch and Swinnerton-Dyer conjecture) that the special value $L^*(E, 1)$ satisfies a Northcott property.

5. SPECIAL VALUES AT THE LEFT OF THE CRITICAL STRIP AND FUNCTIONAL EQUATIONS

The aim of this section is to show how to get a Northcott property for special values of motivic L -functions at the left of the critical strip using the conjectural functional equation. To do so, we need to introduce a certain amount of notation relative to the theory of motives.

First of all, we fix a number field F , over which our motives will be defined, and a number field E , which is the field of coefficients for our motives. Then the works of Jannsen (see [44, § 4]), Huber (see [41, § 22]) and Nori (see [42, Chapter 9]) show that we can define (in three different ways) an abelian category $\mathcal{MM}(F; E)$ of *mixed motives* defined over F with coefficients in E , which is constructed from suitable categories of vector spaces with extra structure, defined over F . The three different constructions of Jannsen, Huber and Nori are not known to be equivalent (see [42, Remark 6.3.12 and § 10.1]), but our result holds for each of them.

Let us now recall that each object $X \in \mathcal{MM}(F; E)$ is endowed with an increasing *weight filtration* \mathcal{W} , and for every $w \in \mathbb{Z}$ one says that X is *pure of weight* w if $X \cong \text{gr}_w^{\mathcal{W}}(X)$. Moreover, if one denotes by M_F the set of places of the number field F , then one can associate to every motive $X \in \mathcal{MM}(F; E)$ and every place $v \in M_F$ the realisations $R_v(X)$. These are v -adic Galois representations for every non-Archimedean place $v \in M_F^0$, and they are mixed Hodge structures for every Archimedean place $v \in M_F^\infty$.

These realisations are used to define the local L -factors $L(R_v(X), s)_\sigma$, associated to every place $v \in M_F$ and every embedding $\sigma: E \hookrightarrow \mathbb{C}$. In particular, the Archimedean L -factors are defined as

$$(17) \quad L(\underline{H}/K, s)_\sigma := \begin{cases} \prod_{j \in \mathbb{Z}} \Gamma_{\mathbb{R}}(s - j + \varepsilon_j)^{n_{j,\sigma}^+(\underline{H}/\mathbb{R})} \Gamma_{\mathbb{R}}(s - j + (1 - \varepsilon_j))^{n_{j,\sigma}^-(\underline{H}/\mathbb{C})}, & \text{if } K \cong \mathbb{R}, \\ \prod_{j \in \mathbb{Z}} \Gamma_{\mathbb{C}}(s - j)^{n_{j,\sigma}(\underline{H}/\mathbb{C})}, & \text{if } K \cong \mathbb{C}, \end{cases}$$

for every mixed Hodge structure \underline{H}/K defined over a complete Archimedean field K . We recall that a mixed Hodge structure \underline{H}/\mathbb{C} consists of a triple $(H, W_\bullet(H), F^\bullet(H_{\mathbb{C}}))$, where H is a vector space defined over \mathbb{Q} , endowed with an increasing filtration $W_\bullet(H)$ (called *weight filtration*), and F^\bullet is a decreasing filtration (called *Hodge filtration*) on the vector space $H_{\mathbb{C}} := H \otimes_{\mathbb{Q}} \mathbb{C}$. Moreover, a mixed Hodge structure \underline{H}/\mathbb{R} consists simply of a mixed Hodge structure defined over \mathbb{C} together with an action of $\text{Gal}(\mathbb{C}/\mathbb{R})$, which amounts to a direct sum decomposition $H = H^+ \oplus H^-$ (at the level of \mathbb{Q} -vector spaces), which is compatible with the weight filtration and such that the Hodge filtration is induced from a filtration defined on the real vector

space $(H_{\mathbb{C}})^{\text{Gal}(\mathbb{C}/\mathbb{R})} := H^+ \oplus iH^-$. Let us also recall that the functions $\Gamma_{\mathbb{C}}(s) := (2\pi)^{-s}\Gamma(s)$ and $\Gamma_{\mathbb{R}}(s) := (\pi^{-s/2}/\sqrt{2})\Gamma(s/2)$ appearing in (17) follow Deninger's normalisation (see [23, Remarque 12.2]). Moreover, for every mixed Hodge structure \underline{H} defined over \mathbb{C} , every $j \in \mathbb{Z}$ and every embedding $\sigma: E \hookrightarrow \mathbb{C}$ we set

$$n_{j,\sigma}(\underline{H}_{/\mathbb{C}}) := \dim_{\mathbb{C}}(\text{gr}_{\gamma}^j(H_{\mathbb{C}}) \otimes_{E \otimes \mathbb{C}, \sigma} \mathbb{C})$$

where γ^{\bullet} denotes the decreasing filtration on $H_{\mathbb{C}}$ defined as

$$\gamma^n(H_{\mathbb{C}}) := F^n(H_{\mathbb{C}}) \cap \overline{F}^n(H_{\mathbb{C}}) = (F^n(H_{\mathbb{C}}) \cap H) \otimes \mathbb{C}.$$

Finally, if \underline{H} is a mixed Hodge structure defined over \mathbb{R} we define

$$n_{j,\sigma}^{\varepsilon}(\underline{H}_{/\mathbb{C}}) := \dim_{\mathbb{C}}(\text{gr}_{\gamma}^j(H_{\mathbb{C}}) \otimes_{E \otimes \mathbb{C}, \sigma} \mathbb{C})^{\varepsilon}$$

for every integer $j \in \mathbb{Z}$, every embedding $\sigma: E \hookrightarrow \mathbb{C}$ and every sign $\varepsilon \in \{+, -\}$. Here, the complex vector space $(\text{gr}_{\gamma}^j(H_{\mathbb{C}}) \otimes_{E \otimes \mathbb{C}, \sigma} \mathbb{C})^{\pm}$ is defined as the ± 1 -eigenspace of the involution induced on $\text{gr}_{\gamma}^j(H_{\mathbb{C}}) \otimes_{E \otimes \mathbb{C}, \sigma} \mathbb{C}$ by the action of $\text{Gal}(\mathbb{C}/\mathbb{R})$ on H and on \mathbb{C} .

To conclude the introductory part of this section, let us recall that the local L -factors $L(R_v(X), s)_{\sigma}$ are conjectured to be defined over E (see [23, § 3.3]). If this is the case, one can put them together in the formal Euler products

$$L(X, s)_{\sigma} := \prod_{v \in M_F \setminus S} L(R_v(X), s)_{\sigma}$$

where $S \subseteq M_F$ denotes any finite set of places. These formal products are known to converge for $\Re(s) \geq 1 + w_{\max}(X)/2$, where $w_{\max}(X)$ is the maximum $w \in \mathbb{Z}$ such that $\text{gr}_w^{\mathcal{W}}(X) \neq 0$. Moreover, they are conjectured to have a meromorphic continuation to the whole complex plane, and the completed L -function

$$\widehat{L}(X, s) := (L_{\emptyset}(X, s)_{\sigma})_{\sigma \in \text{Hom}(E, \mathbb{C})}: \mathbb{C} \dashrightarrow E \otimes \mathbb{C} \cong \prod_{\sigma \in \text{Hom}(E, \mathbb{C})} \mathbb{C}$$

is conjectured to satisfy the functional equation

$$(18) \quad \widehat{L}(X, s) = \varepsilon(X, s) \widehat{L}(X^{\vee}, 1 - s)$$

where $X^{\vee} \in \mathcal{MM}(F; E)$ is the dual (with respect to the tensor product) of X , and the ε -factor is defined as $\varepsilon(X, s) := a(X) e^{b(X)s}$, for two numbers $a(X), b(X) \in E \otimes \mathbb{C}$.

We are finally ready to prove the Northcott property for special values of motivic L -functions, taken at the left of the critical strip, that we announced in Theorem 1.2.

Proof (of Theorem 1.2). Our aim is to prove that the set S_{B_1, B_2} defined in (2) is finite. Recall that S_{B_1, B_2} is defined using a fixed norm $|\cdot|$ on the finite dimensional complex vector space $E \otimes \mathbb{C}$. Applying (18) we see that the bound $|L^*(X, n)| \leq B_1$ is equivalent to

$$(19) \quad |\varepsilon(X, n)| \leq B_1 |L^*(X^{\vee}, 1 - n)|^{-1} \frac{|L_{\infty}^*(X, n)|}{|L_{\infty}^*(X^{\vee}, 1 - n)|}$$

where $L_\infty := \prod_{v \in M_F^\infty} L(R_v(X), s)_\mathbb{C}$ denotes the Archimedean part of the completed L -function $\widehat{L}(X, s)$. Since $\dim(X)$ is bounded from above, we see from the definition of the Archimedean component of the L -function that there exists $B_3 \in \mathbb{R}_{\geq 0}$ (depending on B_2) such that

$$(20) \quad X \in S_{B_1, B_2} \Rightarrow \frac{|L_\infty^*(X, n)|}{|L_\infty^*(X^\vee, 1 - n)|} \leq B_3$$

which can be combined with (19) to get that

$$(21) \quad |\varepsilon(X, n)| \leq (B_1 B_3) |L^*(X^\vee, 1 - n)|^{-1}$$

for every $X \in S_{B_1, B_2}$. We note that, in order for the implication (20) to hold, it would have been sufficient, in the definition of the set S_{B_1, B_2} given in (2), to bound the quantity

$$\mathfrak{h}_\infty(X) := \max_{\substack{j \in \mathbb{Z} \\ \sigma \in \text{Hom}(E, \mathbb{C})}} \left(\{n_{j, \sigma}(R_v(X)) \mid v \in M_F^\mathbb{C}\} \cup \{n_{j, \sigma}^\varepsilon(R_v(X)) \mid \varepsilon \in \{+, -\}, v \in M_F^\mathbb{R}\} \right)$$

instead of the dimension $\dim(X)$. Clearly, bounding $\dim(X)$ implies a bound of $\dim(R_v(X))$ for every $v \in M_F$, and this in turn implies a bound for $\mathfrak{h}_\infty(X)$.

Now, the assumption that $X \cong \text{gr}_w^\mathcal{W}(X)$, *i.e.* that X is pure of weight w , implies that for every non-Archimedean place $v \in M_F^0$ the absolute values of the roots of the polynomial $f_{R_v(X)}$ attached to the restriction of the Galois representation $R_v(X)$ to $\text{Gal}(\overline{F_v}/F_v)$ (see [24, § 3.3]) are bounded by a function depending only on w , which is equal to $|\mathbb{N}_{K/\mathbb{Q}}(\mathfrak{p}_v)|^{w/2}$ for almost all places $v \in M_F^0$. Moreover, since X is supposed to be pure of weight w , the dual X^\vee is isomorphic to the Tate twist $X(w)$, which shows that

$$|L^*(X^\vee, 1 - n)| = |L^*(X, w + 1 - n)|.$$

Combining this with the previous observation we see that

$$(22) \quad X \in S_{B_1, B_2} \Rightarrow |L^*(X^\vee, 1 - n)| \geq B_4$$

for some $B_4 \in \mathbb{R}$, depending only on w and n . Hence, putting together (21) and (22) we see that $|\varepsilon(X, n)| \leq B_5$ for every $X \in S_{B_1, B_2}$, where $B_5 := B_1 B_3 / B_4$.

To conclude, it is sufficient to recall that for every $X \in S_{B_1, B_2}$ we have

$$|\varepsilon(X, n)| = |\Delta_F|^{\frac{w+1}{2} \dim(R_v(X))} |\mathbb{N}_{F/\mathbb{Q}}(\mathfrak{f}_{R_v(X)})|^{\frac{w+1}{2} - n} \leq B_5$$

where $v \in M_F^0$ is any non-Archimedean place at which X has good reduction (see [69, § 12, Proposition]). Thus we see that both the dimension $\dim(X)$ and the norm of the conductor of the Galois representations $R_v(X)$ are bounded from above. Hence we can apply the Northcott property for the conductor that we have seen in Example 2.18 to see that there are only finitely many $R_\ell(X)$, up to isomorphism. Since $\mathcal{MM}(F; E)$ is one of the categories of mixed motives defined by Jannsen, Huber or Nori, every motive is determined by its realisations, and therefore we have also finitely many elements in S_{B_1, B_2} . \square

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