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Abstract

Let $A \subset \mathbb{Z}$ be a finite subset. We denote by $\mathcal{B}(A)$ the set of all integers $n \geq 2$ such that $|nA| > (2n-1)(|A|-1)$, where $nA = A + \cdots + A$ denotes the n-fold sumset of A. The motivation to consider $\mathcal{B}(A)$ stems from Buchweitz's discovery in 1980 that if a numerical semigroup $S \subseteq \mathbb{N}$ is a Weierstrass semigroup, then $\mathcal{B}(\mathbb{N} \setminus S) = \emptyset$. By constructing instances where this condition fails, Buchweitz disproved a longstanding conjecture by Hurwitz (1893). In this paper, we prove that for any numerical semigroup $S \subset \mathbb{N}$ of genus $g \geq 2$, the set $\mathcal{B}(\mathbb{N} \setminus S)$ is finite, of unbounded cardinality as S varies.

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To the memory of our friend and colleague
Fernando Torres (1961-2020)

1 Introduction

Denote $\mathbb{N} = \{0,1,2,3,\dots\}$ and $\mathbb{N}_+ = \mathbb{N} \setminus \{0\} = \{1,2,3,\dots\}$. For $a,b \in \mathbb{Z}$, let $[a,b[= \{z \in \mathbb{Z} \mid a \leq z < b\}$ and $[a,\infty[= \{z \in \mathbb{Z} \mid a \leq z\}$ denote the integer intervals they span. A *numerical semigroup* is a subset $S \subseteq \mathbb{N}$ containing 0, stable under addition and with finite complement in $\mathbb{N}$. Equivalently, it is a subset of $\mathbb{N}$ of the form $S = \langle a_1,\dots,a_n\rangle = \mathbb{N}a_1 + \cdots + \mathbb{N}a_n$ where $\gcd(a_1,\dots,a_n) = 1$. The set $\{a_1,\dots,a_n\}$ is then called a *system of generators* of S, and the smallest such n is called the *embedding dimension* of S.

For a numerical semigroup S, its corresponding *gapset* is the complement $G = \mathbb{N} \setminus S$, its *genus* is $g = |G|$, its *multiplicity* is $m = \min S^*$ where $S^* = S \setminus \{0\}$, its

Frobenius number is $f = \max(\mathbb{Z} \setminus S)$ and its *conductor* is $c = f + 1$. Thus $[c, \infty[\subseteq S$ and c is minimal for this property. Finally, the *depth* of S is $q = \lceil c/m \rceil$.

Given a finite subset $A \subset \mathbb{N}$, we denote by $nA = A + \cdots + A$ the *n-fold sumset of A*. See Section 2 for more details.

Definition 1.1. *Let $A \subset \mathbb{Z}$ be a finite subset. We associate to A the function $\beta = \beta_A \colon \mathbb{N}_+ \to \mathbb{Z}$ defined for all $n \geq 1$ by*

$$\beta_A(n) = |nA| - (2n-1)(|A|-1).$$

Notation 1.2. *We denote by $\mathcal{B}(A)$ the* positive support *in $2 + \mathbb{N}$ of the function β_A, i.e.*

$$\mathcal{B}(A) = \{n \geq 2 \mid \beta_A(n) \geq 1\}.$$

For instance, $2 \in \mathcal{B}(A)$ if and only if $|2A| \geq 3|A| - 2$. Interestingly, the failure of this condition, namely the inequality $|2A| \leq 3|A| - 3$, is the key hypothesis of the famous Freiman's $3k-3$ Theorem in additive combinatorics [5].

Example 1.3. *If $|A| = 0$ or 1, then $\mathcal{B}(A)$ is infinite. Indeed, if $A = \emptyset$, then $|nA| = 0$ and so $\beta_\emptyset(n) = 2n-1$ for all $n \geq 1$. Thus $\mathcal{B}(\emptyset) = 2 + \mathbb{N}$ in that case. Similarly, if $|A| = 1$, then $\beta_A(n) = 1$ for all $n \geq 1$. So here again $\mathcal{B}(A) = 2 + \mathbb{N}$.*

In sharp contrast, Theorem 3.3 below states that *if $S \subset \mathbb{N}$ is a numerical semigroup of genus $g \geq 2$, then $\mathcal{B}(\mathbb{N} \setminus S)$ is finite*.

Example 1.4. *Let $S = \langle 3, 7 \rangle$. Then $\mathbb{N} \setminus S = \{1, 2, 4, 5, 8, 11\}$ and $\beta_{\mathbb{N}\setminus S}(n) = 0$ for all $n \geq 2$ as easily seen. In particular, $\mathcal{B}(\mathbb{N} \setminus S) = \emptyset$.*

More generally, it was shown in [8, 10] that $\beta_{\mathbb{N}\setminus S}(n) = 0$ for all *symmetric* numerical semigroups S of multiplicity $m \geq 3$ and all $n \geq 2$. We shall not use this result below, but instead give a short self-contained proof of an immediate consequence, namely that $\mathcal{B}(\mathbb{N} \setminus S)$ is empty in that case.

In fact, $\mathcal{B}(\mathbb{N} \setminus S)$ is empty in most cases. Indeed, Buchweitz discovered in 1980 that the condition $\mathcal{B}(\mathbb{N} \setminus S) = \emptyset$ is necessary for S to be a Weierstrass semigroup. By constructing instances where this condition fails, Buchweitz [1] was able to negate the longstanding conjecture by Hurwitz [6] according to which all numerical semigroups of genus $g \geq 2$ are Weierstrass semigroups. His first counterexample was $S = \langle 13, 14, 15, 16, 17, 18, 20, 22, 23 \rangle$, with corresponding gapset

$$G = \mathbb{N} \setminus S = [1, 12] \cup \{19, 21, 24, 25\}$$

of cardinality 16. Then $2G = [2,50] \setminus \{39,41,47\}$, so that $|2G| = 46$ and $\beta_G(2) = 46 - 3 \cdot 15 = 1$, implying $2 \in \mathcal{B}(G)$ and thus impeding S to be a Weierstrass semigroup. For more information on Buchweitz's condition and Weierstrass semigroups, see e.g. [3, 7].

Here are the contents of this paper. In Section 2, we recall a result of Nathanson in additive combinatorics and we use it to study the asymptotic behavior of the function $\beta_A(n)$. In Section 3, we introduce the Buchweitz set of a numerical semigroup and we prove our main results. Section 4 concludes the paper with open questions on the possible shapes of the sets $\mathcal{B}(A)$.

2 Sumset growth

Given finite subsets A, B of a commutative monoid $(M, +)$, we denote as usual

$$A + B = \{a + b \mid a \in A, b \in B\},$$

the *sumset* of A, B, and $2A = A + A$. More generally, if $n \geq 2$, we denote $nA = A + (n-1)A$, where $1A = A$. The set nA is called the n-fold sumset of A.

A classical question in additive combinatorics is, how does $|nA|$ grow with n? Here we only consider the case $M = \mathbb{Z}$. We shall need the following result of Nathanson [9, Theorem 1.1].

Theorem 2.1. *Let $A_0 \subset \mathbb{N}$ be a finite subset of cardinality $k \geq 2$, containing 0 and such that $\gcd(A_0) = 1$. Let $a_0 = \max(A_0)$. Then there exist integers c, d and subsets $C \subseteq [0, c-2]$, $D \subseteq [0, d-2]$ such that*

$$nA_0 = C \sqcup [c, a_0 n - d] \sqcup (a_0 n - D)$$

for all $n \geq \max\{(|A_0| - 2)(a_0 - 1)a_0, 1\}$.

As pointed out in [9], the hypotheses $0 \in A_0$ and $\gcd(A_0) = 1$ are not really restrictive. Indeed, for any finite set $A \subset \mathbb{Z}$ with $|A| \geq 2$, the simple transformation $A \mapsto A_0 = (A - \alpha)/d$, where $\alpha = \min(A)$ and $d = \gcd(A - \alpha)$, yields a set A_0 satisfying these hypotheses and such that $|nA_0| = |nA|$ for all n. In view of our applications to gapsets, we shall need the following version.

Corollary 2.2. *Let $A \subset \mathbb{N}_+$ be a finite subset containing $\{1, 2\}$. Let $a = \max(A)$. Then there is an integer $b \leq 1$ such that*

$$|nA| = (a-1)n + b$$

for all $n \geq (|A| - 2)(a-2)(a-1)$.

Proof. Set $A_0 = A - 1$ and $a_0 = a - 1$. Then A_0 contains $\{0,1\}$, hence it satisfies the hypotheses of Theorem 2.1. Using the same notation, its conclusion implies

$$|nA_0| = a_0 n + b \tag{1}$$

for all $n \geq \max\{(|A_0|-2)(a_0-1)a_0, 1\}$, where

$$b = |C| + |D| - c - d + 1.$$

Note that $b \leq 1$ since $|C| \leq \max(0, c-1)$, $|D| \leq \max(0, d-1)$. The desired statement follows from (1) since $|nA| = |nA_0|$ for all $n \geq 0$. □

2.1 Asymptotic behavior of $\beta_A(n)$

We now study the evolution of $\beta_A(n)$ as n grows.

Theorem 2.3. *Let $A \subset \mathbb{N}_+$ be a finite set containing $\{1,2\}$. Let $f = \max(A)$ and $g = |A|$. Then*

$$\lim_{n\to\infty} \beta_A(n) = \begin{cases} -\infty & \text{if} \quad f \leq 2g-2, \\ +\infty & \text{if} \quad f \geq 2g. \end{cases}$$

Finally if $f = 2g-1$, then $\beta_A(n)$ is constant and nonpositive for n large enough.

Proof. By Corollary 2.2, we have $|nA| = (f-1)n + b$ for some integer $b \leq 1$ and for n large enough. Hence

$$\begin{aligned} \beta_A(n) &= (f-1)n + b - (2n+1)(g-1) \\ &= (f-2g+1)n + b + 1 - g \end{aligned}$$

for n large enough. The claims for $f \leq 2g-2$ and $f \geq 2g$ follow. If $f = 2g-1$, then $\beta_A(n) = b + 1 - g \leq 0$ for n large enough, since $b \leq 1$ and $g \geq 2$. □

Corollary 2.4. *Let $A \subseteq \mathbb{N}_+$ be a finite set containing $\{1,2\}$. Let $f = \max(A)$ and $g = |A|$. Then $\mathcal{B}(A)$ is finite if and only if $f \leq 2g-1$.*

Proof. If $f \geq 2g$, then $\lim_{n\to\infty} \beta_A(n) = \infty$ by the theorem, whence $\beta_A(n) \geq 1$ for all large enough n. Thus $\mathcal{B}(A)$ is infinite in this case. If $f \leq 2g-1$, the theorem implies $\beta_A(n) \leq 0$ for n large enough, whence $\mathcal{B}(A)$ is finite in that case. □

3 Application to numerical semigroups

Definition 3.1. *Let $S \subseteq \mathbb{N}$ be a numerical semigroup. We define the* Buchweitz set of S *as* $\mathrm{Buch}(S) = \mathcal{B}(\mathbb{N} \setminus S)$. *Explicitly, setting $G = \mathbb{N} \setminus S$, we have*

$$\begin{aligned} \mathrm{Buch}(S) &= \{n \geq 2 \mid |nG| > (2n-1)(|G|-1)\} \\ &= \{n \geq 2 \mid \beta_G(n) \geq 1\}. \end{aligned}$$

In this section, we first prove that $\mathrm{Buch}(S)$ is *finite* for all numerical semigroups S of genus $g \geq 2$. We then show, by explicit construction, that the cardinality of $\mathrm{Buch}(S)$ may be arbitrarily large.

3.1 Finiteness of $\mathrm{Buch}(S)$

We start with a well known inequality linking the Frobenius number and the genus of a numerical semigroup.

Proposition 3.2. *Let $S \subset \mathbb{N}$ be a numerical semigroup with Frobenius number f and genus $g \geq 1$. Then $f \leq 2g-1$.*

Proof. Let $x \in S \cap [0,f]$. Then $f - x \notin S$ since S is stable under addition and $x + (f-x) = f \notin S$. Hence, the map $x \mapsto f - x$ induces an injection

$$S \cap [0,f] \hookrightarrow \mathbb{N} \setminus S.$$

Since $|S \cap [0,f]| = (f+1) - g$, it follows that $f \leq 2g-1$, as claimed. □

Recall that S is said to be *symmetric* if $|S \cap [0,f]| = |\mathbb{N} \setminus S|$, i.e. if $f = 2g-1$. A classical result of Sylvester states that any numerical semigroup of the form $S = \langle a,b \rangle$ with $\gcd(a,b) = 1$ is symmetric.

Theorem 3.3. *Let $S \subseteq \mathbb{N}$ be a numerical semigroup of genus $g \geq 2$. Then* $\mathrm{Buch}(S)$ *is finite.*

Proof. Let $G = \mathbb{N} \setminus S$. Then $\mathrm{Buch}(S) = \mathcal{B}(G)$ by definition. We have $g = |G| \geq 2$. Let $f = \max(G)$ be the Frobenius number of S. Let $m = \min(S \setminus \{0\})$ be the multiplicity of S. Then $m \geq 2$ since $g \geq 2$, and $[1, m-1] \subseteq G$.

Assume first $m \geq 3$. Then $\{1,2\} \subseteq G$. Hence Corollary 2.4 applies, and since $f \leq 2g-1$ by Proposition 3.2, it yields that $\mathcal{B}(G)$ is finite, as desired.

Assume now $m = 2$. Then $S = \langle 2, b \rangle$ with b odd and $b \geq 5$ since $|G| \geq 2$. At this point, we might conclude the proof right away using what is known in

the symmetric case [8, 10]. However, for the convenience of the reader, let us give a short self-contained argument. We have $G = \{1,3,\dots,b-2\}$, i.e. all odd numbers from 1 to $b-2$. Hence $G-1 = \{0,2,\dots,b-3\}$ and $\gcd(G-1) = 2$. Set $A = (G-1)/2 = [0,k]$, where $k = (b-3)/2$. For all $n \geq 1$, we have

$$|nG| = |nA| = |nk+1| = n(|G|-1)+1.$$

Therefore $\beta_G(n) = -(n-1)|G|+n$, whence $\beta_G(n) \leq 0$ for all $n \geq 2$. It follows that $\mathcal{B}(G) = \emptyset$ and we are done. □

3.2 Unboundedness of $|\mathrm{Buch}(S)|$

We show here, by explicit construction, that $|\mathrm{Buch}(S)|$ may be arbitrarily large.

Proposition 3.4. *For any integer $b \geq 3$, there exists a numerical semigroup S such that* $\mathrm{Buch}(S) = [2,b]$.

Proof. Let $k = b-2$, and let S be the numerical semigroup of multiplicity $m = 6k+15$ and depth $q = 2$ whose corresponding gapset $G = \mathbb{N} \setminus S$ is given by

$$G = [1,m-1] \sqcup \{2m-7, 2m-5, 2m-2, 2m-1\}. \tag{2}$$

We claim that $\mathrm{Buch}(S) = [2,k+2]$. Indeed, we will show a more precise statement, namely

$$\beta_G(n) = \begin{cases} 1 & \text{if } n = 2, \\ 2 & \text{if } 3 \leq n \leq k+2, \\ -6(n-k-3) & \text{if } n \geq k+3. \end{cases}$$

Let $A = (2m-1) - G$. Then $\beta_G(n) = \beta_A(n)$ since $|nG| = |nA|$ for all $n \geq 1$. We have

$$A = [0,1] \sqcup \{4,6\} \sqcup [m, 2(m-1)].$$

Let us compute $2A$ and $3A$. We obtain

$$\begin{aligned} 2A &= [0,2] \sqcup [4,8] \sqcup \{10,12\} \sqcup [m, 4(m-1)], \\ 3A &= [0,14] \cup \{16,18\} \cup [m, 6(m-1)]. \end{aligned}$$

In general, we have

$$nA = ([0,6n-4] \sqcup \{6n-2, 6n\}) \cup [m, 2n(m-1)] \tag{3}$$

for all $n \geq 3$, as easily verified by induction on n.

Let us determine $|nA|$ for all $n \geq 1$. Note first that *the union in* (3) *is disjoint if and only if* $6n+1 \leq m$. Moreover, as $m = 6k+15$, we have

$$6n+1 \leq m \Longleftrightarrow n \leq k+2.$$

In contrast, if $n \geq k+3$, i.e. if $6n-3 \geq m$, then the union in (3) collapses to a single interval and we get

$$nA = [0, 2n(m-1)].$$

Summarizing, we have

$$|nA| = \begin{cases} m+3 & \text{if} \quad n = 1, \\ 3m+7 & \text{if} \quad n = 2, \\ (2n-1)(m-1)+6n-1 & \text{if} \quad 3 \leq n \leq k+2, \\ 2n(m-1)+1 & \text{if} \quad n \geq k+4. \end{cases}$$

The stated formula for $\beta_G(n) = \beta_A(n) = |nA| - (2n-1)(|A|-1)$ follows. Hence $\mathrm{Buch}(S) = [2, k+2]$, as claimed. □

This family of numerical semigroups was inspired by the *PF-semigroups* introduced in [4].

3.3 More intervals

What are the possible shapes of $\mathrm{Buch}(S)$ when S varies? We do not know in general. By Proposition 3.4, any finite integer interval I with $|I| \geq 2$ and $\min(I) = 2$ may be realized as $I = \mathrm{Buch}(S)$ for some numerical semigroup S. Here we present families of numerical semigroups S realizing as $\mathrm{Buch}(S)$ all finite integer intervals I with $|I| \geq 2$ and $\min(I) \in \{3,4,5,6\}$.

Proposition 3.5. *Let $k \geq 1$. Let S be the numerical semigroup of multiplicity $m = 6k+19$ and depth $q = 2$ whose corresponding gapset $G = \mathbb{N} \setminus S$ is given by*

$$G = [1, m-1] \sqcup \{2m-7, 2m-6, 2m-2, 2m-1\}. \tag{4}$$

Then $\mathrm{Buch}(S) = [3, k+3]$.

Proof. Let again $A = (2m-1) - G = [0,1] \sqcup \{5,6\} \sqcup [m, 2(m-1)]$. We then have

$$\begin{aligned} 2A &= [0,2] \sqcup [5,7] \sqcup [10,12] \sqcup [m, 4(m-1)], \\ 3A &= [0,3] \sqcup [5,8] \sqcup [10,13] \sqcup [15,18] \sqcup [m, 6(m-1)], \\ 4A &= [0,24] \cup [m, 8(m-1)]. \end{aligned}$$

It follows that $nA = [0,6n] \cup [m, 2n(m-1)]$ for all $n \geq 4$. In particular, if $6n \geq m$ then $nA = [0, 2n(m-1)]$. Therefore,

$$\beta_G(n) = \beta_A(n) = \begin{cases} 0 & \text{if } n = 2, \\ 1 & \text{if } n = 3, \\ 4 & \text{if } 4 \leq n \leq k+3, \\ 6k - 6n + 22 & \text{if } n \geq k+4. \end{cases}$$

Hence $\mathrm{Buch}(S) = [3, k+3]$, as claimed. □

Proposition 3.6. *For $k \geq 1$ and $i \in \{1,2,3\}$, let S_i be the numerical semigroup with $G_i = \mathbb{N} \setminus S_i$ given by*

$$\begin{aligned} G_1 &= [1, m_1 - 1] \sqcup \{2m_1 - 6, 2m_1 - 2, 2m_1 - 1\}, \\ G_2 &= [1, m_2 - 1] \sqcup \{2m_2 - 10, 2m_2 - 4, 2m_2 - 3, 2m_2 - 2\}, \\ G_3 &= [1, m_3 - 1] \sqcup \{2m_3 - 10, 2m_3 - 9, 2m_3 - 2\}, \end{aligned}$$

where $m_1 = 4k + 22$, $m_2 = 7k + 44$ and $m_3 = 5k + 55$, respectively. Then

$$\mathrm{Buch}(S_1) = [4, k+4], \ \mathrm{Buch}(S_2) = [5, k+5], \ \mathrm{Buch}(S_3) = [6, k+6].$$

Proof. Similar to the proofs of Propositions 3.4 and 3.6. We omit it here. □

Having realized all finite integer intervals I with $|I| \geq 2$ and $\min(I) \in [2,6]$ as $I = \mathrm{Buch}(S)$ for a suitable numerical semigroups S, is it possible to do the same for all finite integer intervals I with $\min(I) \geq 7$? We do not know in general. But here is a particular case where $\min(I)$ can be arbitrarily large. It is based on a family of numerical semigroups found in [8].

Proposition 3.7. *For any integer $k \geq 1$, there is a numerical semigroup S such that* $\mathrm{Buch}(S) = [7 + 2k, 7 + 4k]$.

Proof. For $k \geq 1$, let S be the numerical semigroup minimally generated by the set $T_1 \cup T_2 \cup T_3$, where

$$\begin{aligned} T_1 &= [44 + 27k + 4k^2, 79 + 51k + 8k^2], \\ T_2 &= [81 + 51k + 8k^2, 84 + 53k + 8k^2], \\ T_3 &= [87 + 53k + 8k^2, 87 + 54k + 8k^2]. \end{aligned}$$

The corresponding gapset $G = \mathbb{N} \setminus S$ is then given by

$$G = [1, 43 + 27k + 4k^2] \cup \{80 + 51k + 8k^2, 85 + 53k + 8k^2, 86 + 53k + 8k^2\}.$$

Let $A = (86 + 53k + 8k^2) - G$. Then

$$A = [0,1] \sqcup \{6 + 2k\} \sqcup [43 + 26k + 4k^2, 85 + 53k + 8k^2],$$

of cardinality $|A| = 46 + 27k + 4k^2$. The n-fold sumsets of A are then given by

$$nA = [0,n] \cup \left(\bigcup_{i=1}^{n} [i(6+2k), i(6+2k) + n - i] \right) \cup \left[43 + 26k + 4k^2, (85 + 53k + 8k^2)n \right]. \tag{5}$$

• Assume first $2 \leq n < 6 + 2k$. In this case, we have

$$\begin{aligned} &0 < n < 6 + 2k < 6 + 2k + n - 1 < \cdots < (n-1)(6+2k) \\ &< (n-1)(6+2k) + n - 1 < n(6+2k) < (7+2k)(6+2k) + 1 \\ &= 43 + 26k + 4k^2 < (85 + 53k + 8k^2)n. \end{aligned}$$

Thus, all the sets appearing in (5) are disjoint and the cardinality of nA is equal to

$$(n+1) + \sum_{i=1}^{n} i + \left(\left(8k^2 + 53k + 85 \right) n - (4k^2 + 26k + 43) + 1 \right) = \\ -41 - 26k - 4k^2 + (173n)/2 + 53kn + 8k^2 n + n^2/2.$$

Thus,

$$\beta_G(n) = (-41 - 26k - 4k^2 + (173n)/2 + 53kn + 8k^2 n + n^2/2) \\ - \left(4k^2 + 27k + 46 - 1 \right) (2n - 1) = \\ (4+k) - (\frac{7}{2} + k)n + \frac{1}{2}n^2 \tag{6}$$

for every $n \in [2, 5 + 2k]$.

The only difference between the case $n = 6 + 2k$ and the previous one is that the sets $[0,n]$ and $[6 + 2k, 6 + 2k + n - 1]$ have a nonempty intersection, equal to $\{6 + 2k\}$. Replacing n by $6 + 2k$ and subtracting one, we obtain $\beta_G(6 + 2k) = 0$.

• Assume now $6+2k < n \leq 11+4k$. The sequence of sets $[0,n]$ and

$$[6+2k, 6+2k+n-1], \ldots, [(n-5-2k)(6+2k), (n-5-2k)(6+2k)+(5+2k)]$$

verifies that the intersection of any two consecutive terms is nonempty. Moreover, their union is the interval $[0,(n-5-2k)(6+2k)+(5+2k)]$ whose cardinality is equal to $(n-5-2k)(6+2k)+(5+2k)+1$. For $i = n-4-2k, \ldots, 6+2k$ the intervals are disjoint with all the others sets appearing in the expression (5); the cardinality of the union of these sets is equal to $\sum_{i=n-2k-5}^{2k+5} i$. For every $i = 7+2k, \ldots, n$ the intersection

$$[i(6+2k), i(6+2k)+n-i] \cap [43+26k+4k^2, (8k^2+53k+85)n]$$

is nonempty, except for $n = 7+2k$. Since $(7+2k)(6+2k) = 42+26k+4k^2$, the set

$$\left(\bigcup_{i=7+2k}^{n} [i(6+2k), i(6+2k)+n-i] \right) \cup \left[43+26k+4k^2, (85+53k+8k^2)n\right]$$

is equal to $[42+26k+4k^2, (85+53k+8k^2)n]$, and the cardinality of this set is $\left(\left(8k^2+53k+85\right)n - 4k^2 - 26k - 42\right) + 1$. Putting all the above together, we have that if $6+2k < n \leq 11+4k$, the set nA has cardinality equal to

$$\begin{aligned}((n-5-2k)(6+2k)+(5+2k)+1)+ & \\ \sum_{i=n-2k-5}^{2k+5} i + \left(\left(8k^2+53k+85\right)n - 4k^2 - 26k - 42\right) + 1 = & \\ -65-46k-8k^2+(193n)/2+57kn+8k^2n-n^2/2, &\end{aligned}$$

and therefore

$$\begin{aligned}(7)\quad \beta_G(n) = -65-46k-8k^2+(193n)/2+57kn+8k^2n-n^2/2 & \\ -(2n-1)(46+27k+4k^2-1) = & \\ (-20-19k-4k^2)+(3k+\frac{13}{2})n-\frac{n^2}{2} &\end{aligned}$$

for every $n \in [6+2k, 11+4k]$.

• Finally, assume $11+4k<n$. The set

$$[0,n]\cup(\bigcup_{i=1}^{6+2k}[i(6+2k),i(6+2k)+n-i])\cup[43+26k+4k^2,(85+53k+8k^2)n]$$

is equal to $[0,(85+53k+8k^2)n]$ and the remaining intervals are contained in this union. So we have $nA=[0,(85+53k+8k^2)n]$ and therefore

$$\beta_G(n)=-\left(4k^2+27k+46-1\right)(2n-1)+\left(8k^2+53k+85\right)n+1= \\ (4k^2+27k+46)-(k+5)n \tag{8}$$

for every $n>11+4k$.

Combining $\beta_G(6+2k)=0$ with the formulation of (6), (7) and (8) for $\beta_G(n)$, we get the following formulas:

$$\beta_G(n)=\begin{cases}(4+k)-(\frac{7}{2}+k)n+\frac{1}{2}n^2 & \text{if } 2\le n<6+2k,\\ 0 & \text{if } n=6+2k,\\ (-20-19k-4k^2)+(3k+\frac{13}{2})n-\frac{n^2}{2} & \text{if } 6+2k<n\le 11+4k,\\ (4k^2+27k+46)-(k+5)n & \text{if } 11+4k<n.\end{cases}$$

Let $k\ge 1$ be fixed. For $2\le n<6+2k$, the formula of $\beta_G(n)$ is a degree two polynomial in n with positive leading coefficient such that $\beta_G(2)=-1-k<0$ and $\beta_G(5+2k)=-1-k<0$. We have therefore $\beta_G(n)<0$ for every $n=2,\dots,5+2k$.

If $n>11+4k$, we now have that $\beta_G(n)$ is a degree one polynomial with negative leading coefficient and such that $\beta_G(12+4k)=-14-5k<0$. So $\beta_G(n)<0$ for every $n>11+4k$.

Finally, if $6+2k<n\le 11+4k$ the function $\beta_G(n)$ is a degree two polynomial in n with negative leading coefficient. As in addition $\beta_G(7+2k)=1+k$, $\beta_G(7+4k)=1$ and $\beta_G(8+4k)=-k$, the only positive values that we have in this part are for $n\in[7+2k,7+4k]$.

Since $\beta_G(n)\le 0$ except for $n\in[7+2k,7+4k]$, the set $\mathrm{Buch}(S)$ is equal to $[7+2k,7+4k]$. □

4 Concluding remarks

The current knowledge on the structure of $\mathcal{B}(A)$ for finite subsets $A\subset\mathbb{Z}$ is very scarce, even for gapsets. Do they have some special shape or property? We end this paper with three questions based on the few currently available observations.

Question 4.1. *Let $A \subset \mathbb{Z}$ be a finite subset, or more specifically a gapset. Is the set $\mathcal{B}(A)$ always an interval of integers?*

Question 4.2. *Even more so, is the function $\beta_A(n)$ unimodal?*

Question 4.3. *In sharp contrast with the above questions, let $T \subset 2+\mathbb{N}$ be any finite subset. Does there exist a finite subset $A \subset \mathbb{N}$, or more specifically a gapset, such that $\mathcal{B}(A) = T$?*

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