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Opinion Dynamics with Topological Gossiping: Asynchronous Updates under Limited Attention

Wilbert Samuel Rossi and Paolo Frasca

Abstract—This paper introduces a general model of opinion dynamics with opinion-dependent connectivity. Agents update their opinions asynchronously: for the updating agent, the new opinion is the average of the k closest opinions within a subset of m agents that are sampled from the population of size n . Depending on k and m with respect to n , the dynamics can have a variety of equilibria, which include consensus and clustered configurations. The model covers as special cases a classical gossip update (if $m = n$) and a deterministic update defined by the k nearest neighbors (if $m = k$). We prove that the dynamics converges to consensus if $n > 2(m - k)$. Before convergence, however, the dynamics can remain for long time in the vicinity of metastable clustered configurations.

Index Terms—Agent-based systems, Large-scale systems, Network analysis and control, Randomized algorithm.

I. INTRODUCTION

THE control community is witnessing an increasing interest in mathematical models of opinion dynamics in social networks [1], [2]. A recurring theme of opinion dynamics is that social influence is countered by some other dynamical feature, thereby preventing the onset of global consensus and leading to complex behaviors. In many popular models, this feature is an opinion-dependent limitation of the connectivity: chief examples are bounded confidence models [3], [4], where social agents influence each other when their opinions are closer than a threshold. This way of defining connectivity, however, seems at odds with reality: it assumes agents having always access to the opinions of all fellow agents and may lead to agents being influenced by a large number of their fellows, possibly the whole population.

Instead, the number of possible interactions is capped by the limited attention capabilities of the individuals. Limitations of attention are well documented in psychology and sociology, for instance by the notion of Dunbar number, and become evermore crucial in today’s age of information bonanza. In online social media, users exchange opinions via the contents they share: out of a randomly generated pool of fresh contents, the online platform selects for each user the best contents in order to maximize engagement, mainly based on similarities between users [5]. Limitations of attention have also been considered in biology by a number of theoretical and experimental studies about flocking in animal groups [6], [7], [8], [9].

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In this paper, we consider for the first time the following multi-agent dynamics. Agents update their opinions asynchronously and the new opinion of the updating agent is the average of the k closest opinions from a subset of m agents, randomly sampled from the population that has size n . This model combines randomness, by sampling from the whole population, and biased selection, by selecting the k closest opinions. Our main result is that the dynamics converges to consensus if $2(m - k) < n$. Even when convergence to consensus is guaranteed, we observe that the convergence time can be very large and the dynamics can develop *metastable* non-consensus states.

Our convergence result is inspired by classical proofs of convergence for randomized consensus dynamics [10, Ch. 3], but its interest and difficulty originate from two distinctive features whose combination distinguishes our model from others in the literature: (1) interactions are not reciprocal; (2) whether two agents interact does not only depend on their two states, but also on the states of all the other agents.

In the literature, models with any of these features are still relatively few. In classical bounded confidence models interactions are reciprocal as long as the interaction thresholds are equal for all agents [3], [11], [12], [13], and any lack of reciprocity makes the analysis much more delicate [14], [15], [16]. In our model, not only interactions are non-reciprocal, but they are also non-metric: whether two agents interact is not solely determined by the distance between their two opinions. For this reason, we follow a consolidated tradition [6] and refer to our connectivity model where agents can interact with their k nearest neighbors as “topological”. Topological interactions are becoming increasingly popular in the applied mathematics community, especially for second order (flocking) models: it has been found that k must be logarithmic in n to ensure connectivity [17] and flocking behavior [18]. Kinetic and continuum models with topological interactions are also actively studied [19], [20], [21]. In the control community, [22] has recently used Petri nets to define a class of models where interactions depend on the opinions of multiple agents: despite some similarity, our model does not belong to this class. Closely related papers are also [23], where the authors define a dynamics in which each agent is influenced by a fixed number of neighbors, and our recent paper [24], which deals the case $m = n$: this special case is simpler to study and does not exhibit metastable states. Instead, the paper [25] introduces random sampling in the metric bounded confidence model. However, the combination of random sampling and topological interactions is unique to the model we present here, which also includes as special case the classical Voter Model when $m = k = 1$.

Outline: Section II introduces the model and studies its equilibria. Section III studies convergence properties: consensus and metastability. Section IV proves the main convergence result before the concluding remarks in Section V.

II. THE DYNAMICAL MODEL

Let n, m and k be three integers such that $1 \leq k \leq m \leq n$, and let $V = \{1, \dots, n\}$ be the set of agents. Each agent is endowed with a scalar opinion $x_i \in \mathbb{R}$, to be updated asynchronously. The update law

$$\mathbf{x}^+ = f(\mathbf{x}, i, S) \quad (1)$$

goes as follows. An agent i and a subset S with m elements are selected from V . The elements of S are ordered by increasing values of $|x_j - x_i|$, where $j \in S$; then, the first k elements of the list (i.e. those with smallest distance from i) form the set N_i^S of current neighbors of i . Should a tie between two or more agents arise, priority is given to agents with lower index. Note that agent i may not belong to N_i^S .¹ Once N_i^S is determined, agent i updates her opinion x_i to

$$x_i^+ = \frac{1}{k} \sum_{j \in N_i^S} x_j,$$

while all the remaining agents do not change their opinions

$$x_j^+ = x_j \quad \text{for every } j \neq i.$$

In the rest of this section, we present some basic properties of (1), as well as some instructive special cases. We begin by some natural definitions. A configuration $\mathbf{x} \in \mathbb{R}^n$ is said to be an *equilibrium* for the asynchronous dynamics (1) if

$$\mathbf{x} = f(\mathbf{x}, i, S) \quad \text{for every choice of } i \text{ and } S.$$

Let $\mathbf{1} \in \mathbb{R}^n$ be the all one vector and let the notation $\mathbf{x}_{N_i^S}$ indicate the sub-vector of \mathbf{x} whose indexes belong to N_i^S . A configuration is said to be *clustered* if

$$\mathbf{x}_{N_i^S} = x_i \mathbf{1}_{N_i^S} \quad \text{for every choice of } i \text{ and } S,$$

that is, if for every node all her neighbors have the same opinion, irrespective of the sampling. Furthermore, a clustered configuration $\mathbf{x} = c\mathbf{1}$, with $c \in \mathbb{R}$, is termed *consensus*.

Clustered configurations are equilibria and are formed by *clusters* with a sufficiently large number of agents that share the same opinion. In order to make the latter claim more precise, let $V_i = \{j : x_j = x_i\}$ denote the cluster of nodes that share i 's opinion.

Lemma 1 (Clustered configurations): A configuration is clustered if and only if $|V_i| \geq n - m + k$ for every i .

Proof: Assume $|V_i| \geq n - m + k$ for every i . Fix i and note that at least $n - m + k$ agents j (including i) have $x_j = x_i$: such nodes have zero distance from i . The choice of S excludes at most $n - m$ agents out of those, so $|S \cap V_i| \geq k$ and, hence, $N_i^S \subseteq V_i$. Since this fact holds for every i , the configuration is clustered. For the converse, assume that there exists i with $|V_i| \leq n - m + k - 1$. Then, there is a choice of S for which

¹Agent i not belonging to N_i^S arises in two ways: if $i \notin S$ or if $i \in S$ but so do other k elements with $x_j = x_i$ and $j < i$. However, in the latter case the opinion of i remains well represented by its neighbors N_i^S .

N_i^S contains a node j with $x_j \neq x_i$ and not in V_i , violating the definition of clustered configuration. ■

This lemma implies that clustered configurations allow up to

$$\left\lfloor \frac{n}{n - m + k} \right\rfloor$$

distinct clusters V_i , since each cluster must contain at least $n - m + k$ agents. For the special case of consensus we therefore have the following immediate corollary.

Corollary 2 (Consensus): Consensus is the only feasible clustered configuration if and only if

$$n < 2(n - m + k). \quad (2)$$

We shall see in the next section that actually condition (2) is also sufficient to guarantee convergence to consensus. The consensus condition (2) can be rewritten as

$$n > 2(m - k) \quad \text{or as} \quad k > m - n/2,$$

thus making it more readable: the condition requires m to be small and k to be large. Large values of k imply more interactions, while small values of m imply that interactions are more random than determined by the distance between opinions. Both randomness and volume of the interactions favor the mixing of opinions, that leads to consensus.

The reader can build a more concrete understanding of dynamics (1) (and of our last comment) by considering the extreme cases when m equals either k or n . If $m = k$, then $S = N_i$ so the k neighbors of i are precisely given by set S . In other words, the reordering does not play a role and the consensus is the only possible equilibrium. For example, if $k = m = 1$, then the dynamics becomes the classical Voter Model, in which the chosen agent i selects a second agents j and copies her opinion; the process continues until only one of the original opinions survives.

If $m = n$, then $S = V$ and hence the sets N_i are constructed purely on topological terms: the sampling step that involves S does not play any role. The sets N_i contain the k agents whose opinions are closest to x_i (possibly, equal to x_i) with ties sorted as discussed before. If $k = 1$, then agents keep their own opinion and every configuration is an equilibrium. If $k = 2$, then each agent averages her opinion with the closest agent apart himself. For general k , this dynamics was studied in [24]. In this case, there exist non-clustered configurations that are equilibria, see [24] for two examples (one using the tie break rule and one not). Instead, for $m < n$, the following result shows that all equilibria are clustered configurations.

Proposition 3 (Equilibria are clustered): Let $m < n$. If configuration \mathbf{x} is an equilibrium, then it is clustered.

Proof: We prove the contrapositive, i.e. if configuration \mathbf{x} is not clustered, then it is not an equilibrium. If \mathbf{x} is not clustered, then there exist i and S such that $x_j \neq x_i$ for some $j \in N_i^S$: we fix such i and S . To prove the claim, we use $m < n$ to choose an alternative S' such that $\frac{1}{k} \sum_{\ell \in N_i^{S'}} x_\ell \neq \frac{1}{k} \sum_{\ell \in N_i^S} x_\ell$. There are three cases.

Case 1: $x_j \neq x_i$ for all $j \in N_i^S$. It is sufficient to choose S' by replacing any $j \in N_i^S$ by i itself. Agent i is not in $N_i^{S'}$ but is certainly in $N_i^{S'}$: since $x_i \neq x_j$ the above follows.

Case 2a: $x_\ell = x_i$ for some $\ell \in N_i^S$, and there exists $e \notin S$ such that $x_e \neq x_i$. Choose S' by replacing any $\ell \in N_i^S$ such that $x_\ell = x_i$ by e . Hence, set $N_i^{S'}$ contains a new element s (not necessarily e) such that $x_s \neq x_\ell$, and the above follows.

Case 2b: $x_\ell = x_i$ for some $\ell \in N_i^S$, and there exists $e \notin S$ such that $x_e = x_i$. In this case, choose S' by replacing any $j \in N_i^S$ such that $x_j \neq x_i$ by agent e . Then, set $N_i^{S'}$ contains e with $x_e = x_i$, but $x_e \neq x_j$ and the above follows.

By exhausting the cases, we have shown that the non clustered configuration \mathbf{x} does not satisfy the equilibrium condition for some S' . ■

III. CONVERGENCE TO CONSENSUS & METASTABILITY

The dynamical process based on (1) converges to a consensus from any initial configuration provided that $n > 2(m-k)$ and that the sequence of agents i and subsets S used in the updates are i.i.d. uniform random variables over V .

Theorem 4 (Convergence to consensus): Let n, m, k with $1 \leq k \leq m \leq n$ be given. Let $t \in \mathbb{Z}_{\geq 0}$ and let $\{\mathcal{J}(t), t \geq 0\}$ be a sequence of independent and uniformly distributed random variables over V . Let $\{\mathcal{S}(t), t \geq 0\}$ be a sequence of independent and uniformly distributed random elements of $\mathcal{P}_m(V)$, where $\mathcal{P}_m(V)$ is the set of subsets of V with cardinality m . Consider the discrete time dynamics

$$\mathbf{x}(t+1) = f(\mathbf{x}(t), \mathcal{J}(t), \mathcal{S}(t)) \quad \text{for every } t \geq 0,$$

with initial state $\mathbf{x}(0) = \mathbf{x}^0 \in \mathbb{R}^n$. If $n > 2(m-k)$, then

$$\lim_{t \rightarrow \infty} \mathbf{x}(t) = \mathbf{1}c \quad \text{almost surely}$$

for any \mathbf{x}^0 , with $c \in [\min_i(x_i^0), \max_i(x_i^0)]$.

The result continues to hold for choices of $\mathcal{J}(t)$ and $\mathcal{S}(t)$ not uniformly distributed over V and $\mathcal{P}_m(V)$, respectively, as long as the probability to sample each agent and subset is constant and positive.

The proof of Theorem 4, available in Sect. IV, is based on exhibiting a suitable finite “shrinking sequence”, which appears infinitely often with probability one, thus guaranteeing almost sure convergence. However, the convergence time can be very large. For this reason, even if the assumptions of Theorem 4 are satisfied and therefore the dynamics is guaranteed to converge to consensus, one can fail to actually observe consensus in simulations. Instead, simulations often show the formation of two (or more) distinct groups of agents, with the agents in the same group that have same opinion while the opinions of the two groups are clearly apart. Such configurations, although bound to be transient by Theorem 4, can remain qualitatively unchanged for many iterations: agents seldom leave their group and, even then, the same agent often returns to her original group the following time she updates opinion. Fig. 1 shows two significant examples. We refer to this kind of configurations, which are not equilibria but around which the dynamics remains for a long time, as to *metastable* configurations.

Metastable configurations can last very long because the probability that, at any time step, a node leaves her group is small. In order to clarify this observation, we can estimate this “exit probability” in a special but instructive case.

A. Exponential exit time

Consider $n > 2(m-k)$ agents split into two disjoint subsets A and B of cardinalities a and b , respectively. Assume that all agents in the same subset share the same opinion, α and β , respectively, with $\alpha \neq \beta$. Moreover, let $a \geq n - m + k$, implying that A is a cluster (its agents cannot leave), while B is not since $b \leq m - k < n - m + k$.

The probability to exit the above configuration, i.e. that an agent leaves her group, is

$$\mathbb{P}(\text{exit}) = \mathbb{P}(\text{exit} | i \in B) \frac{b}{n},$$

since only the agents in B can change opinion. The conditional exit event occurs if N_i^S includes at least one element of A , i.e. if S includes at least $m - k + 1$ elements of A . Let $X \sim \text{Hypergeometric}(n, a, m)$ be the number of elements of A in S , drawn in m draws without replacement from a population of size n that contains exactly a elements of A . Theorems 1 and 4 in [26] imply

$$\mathbb{P}(X - ma/n \geq m\epsilon) \leq \exp(-2m\epsilon^2)$$

where $0 < \epsilon < b/n$. We have

$$\begin{aligned} \mathbb{P}(\text{exit} | i \in B) &= \mathbb{P}(X \geq m - k + 1) \\ &= \mathbb{P}(X - ma/n \geq m - k + 1 - m(1 - b/n)) \\ &= \mathbb{P}\left(X - ma/n \geq m\left(\frac{b}{n} - \frac{k-1}{m}\right)\right) \\ &\leq \exp\left(-2m\left(\frac{b}{n} - \frac{k-1}{m}\right)^2\right) \end{aligned}$$

by taking $\epsilon = \frac{b}{n} - \frac{k-1}{m}$ (under the assumptions $b > \frac{(k-1)n}{m}$ and $k > 1$). The expected exit time (which we denote by τ) is geometrically distributed with parameter $\mathbb{P}(\text{exit})$. Hence,

$$\tau = \frac{1}{\mathbb{P}(\text{exit})} \geq \frac{n}{b} \exp\left(2m\left(\frac{b}{n} - \frac{k-1}{m}\right)^2\right).$$

As long as b (and therefore m) is proportional to n , this bound implies that τ is (at least) exponential in n .

IV. CONVERGENCE PROOF

We begin by introducing some useful notation. Given a subset S of V , we denote its complement by S^c . The power set of V is denoted by $\mathcal{P}(V)$ and the set of subsets of V with cardinality m by $\mathcal{P}_m(V)$. We now introduce some convenient functions of the configuration \mathbf{x} , as illustrated in Fig. 2. The function $\mu : \mathbb{R}^n \rightarrow V$ returns the index of the smallest component of vector \mathbf{x} , with ties sorted

$$\mu(\mathbf{x}) := \min(\arg \min_i x_i),$$

where the outer min sorts possible ties. The function $\Omega : \mathbb{R}^n \rightarrow \mathcal{P}_m(V)$ returns the indices of the m largest components of the vector \mathbf{x} , with ties sorted, that is $\Omega(\mathbf{x}) := \{\Omega_1(\mathbf{x}), \Omega_2(\mathbf{x}), \dots, \Omega_m(\mathbf{x})\}$, where

$$\Omega_1(\mathbf{x}) := \min\left(\arg \max_{i \in V} x_i\right)$$

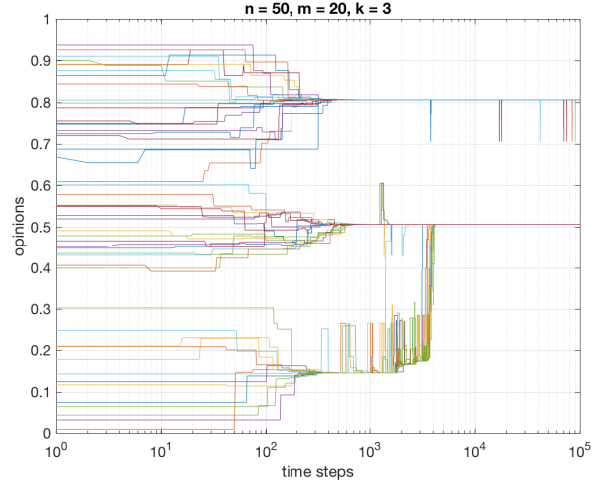
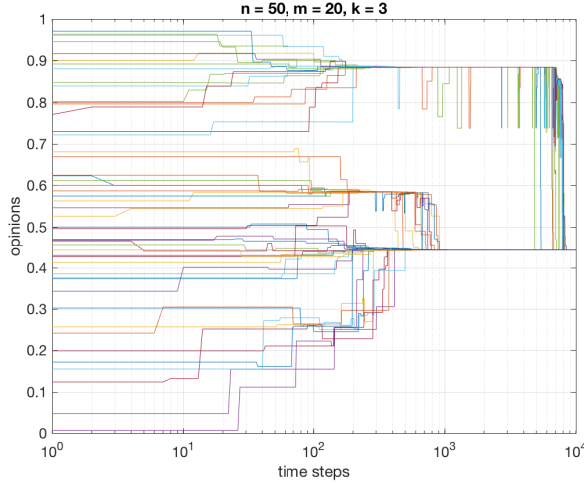


Fig. 1. Simulations of model (1) under the assumptions of Corollary 2 and Theorem 4. The initial configuration \mathbf{x}^0 is such that $x_i^0 \sim \mathcal{U}(0, 1)$ for all i . The simulation in the left plot shows a metastable configuration that satisfies the assumptions of Subsection III-A between (roughly) time step 1000 and 8000: from time 1000, one can identify a cluster of 34 agents with approximately the same opinion and a smaller group with larger opinions.

$$\begin{aligned} \Omega_2(\mathbf{x}) &:= \min \left(\arg \max_{i \notin \{\Omega_1(\mathbf{x})\}} x_i \right) \\ &\vdots \\ \Omega_m(\mathbf{x}) &:= \min \left(\arg \max_{i \notin \{\Omega_1(\mathbf{x}), \dots, \Omega_{m-1}(\mathbf{x})\}} x_i \right) \end{aligned}$$

The function $Y : \mathbb{R}^n \rightarrow \mathcal{P}_{n-m+k}(V)$ returns the $n - m + k$ smallest elements of the vector \mathbf{x} (with ties not sorted):

$$Y(\mathbf{x}) := (\Omega(\mathbf{x}))^c \cup N_{\mu(\mathbf{x})}^{\Omega(\mathbf{x})}.$$

The function $y : \mathbb{R}^n \rightarrow \mathbb{R}$ returns the opinion of the largest element among those in $Y(\mathbf{x})$,

$$y(\mathbf{x}) := \max_{i \in Y(\mathbf{x})} x_i = \max_{i \in N_{\mu(\mathbf{x})}^{\Omega(\mathbf{x})}} x_i.$$

Finally, we define the analogous functions for the largest index $M(\mathbf{x}) := \mu(-\mathbf{x})$, the subset of the m smallest indices $\omega(\mathbf{x}) := \Omega(-\mathbf{x})$, the subset of the $n - m + k$ largest components $Z(\mathbf{x}) := Y(-\mathbf{x})$, and the lowest opinion in $Z(\mathbf{x})$, i.e. $z(\mathbf{x}) := y(-\mathbf{x})$. The next equivalence will prove crucial.

Lemma 5: Given n, m, k with $1 \leq k \leq m \leq n$, consider $\mathbf{x} \in \mathbb{R}^n$ and the quantities $y(\mathbf{x})$ and $z(\mathbf{x})$. Then, $z(\mathbf{x}) \leq y(\mathbf{x})$ for every $\mathbf{x} \in \mathbb{R}^n$ if and only if $n > 2(m - k)$.

Proof: We prove the equivalent claim that there exists $\mathbf{x} \in \mathbb{R}^n$ with $z(\mathbf{x}) > y(\mathbf{x})$ if and only if $n \leq 2(m - k)$. Assume $n \leq 2(m - k)$ and consider $\mathbf{x} \in \mathbb{R}^n$ such that

$$x_1 < \dots < x_{n-m+k} < \dots < x_{m-k+1} < \dots < x_n$$

where $m - k + 1 > n - m + k$. The above vector has $\mu(\mathbf{x}) = 1$, $\Omega(\mathbf{x}) = \{n - m + 1, \dots, n\}$, $Y(\mathbf{x}) = \{1, \dots, n - m + k\}$ and hence, $y(\mathbf{x}) = x_{n-m+k}$. Symmetrically, $z(\mathbf{x}) = x_{m-k+1}$ but that means $z(\mathbf{x}) > y(\mathbf{x})$.

For the converse, assume that $\mathbf{x} \in \mathbb{R}^n$ with $z(\mathbf{x}) > y(\mathbf{x})$ exists, meaning $\max_{i \in Y(\mathbf{x})} x_i < \min_{i \in Z(\mathbf{x})} x_i$. Both sets $Y(\mathbf{x})$ and $Z(\mathbf{x})$ contain $n - m + k$ elements, so sets $\{j : x_j \leq \max_{i \in N_{\mu(\mathbf{x})}} x_i\}$ and $\{j : x_j \geq \min_{i \in N_{M(\mathbf{x})}} x_i\}$ contain at least $n - m + k$ elements each. Since these two sets are disjoint, vector \mathbf{x} has at least $n \geq 2(n - m + k)$ components, which means $n \leq 2(m - k)$. ■

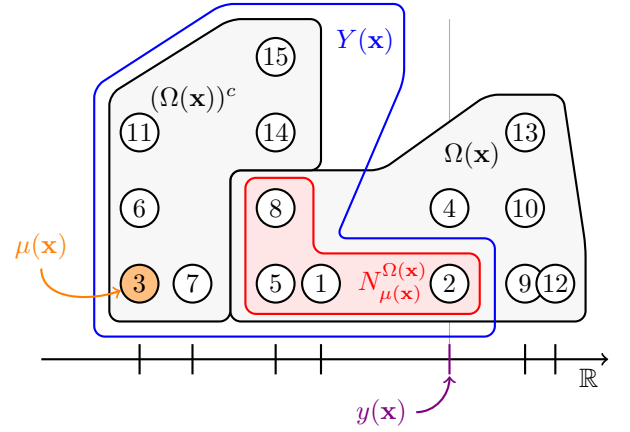


Fig. 2. Illustration of the notation defined in Sect. IV when $n = 15$, $m = 9$, $k = 4$. The horizontal axis reports the values of x_i for $i \in \{1, \dots, 15\}$.

A. Intermediate lemmas regarding the dynamics

For $t \in \mathbb{Z}_{\geq 0}$, let $\mathbf{x}(t) \in \mathbb{R}^n$ be the sequence of opinion vectors, $i(t) \in V$ a sequence of agents and $S(t) \subset V$ a sequence of subsets of agents. Given an initial configuration $\mathbf{x}(0) = \mathbf{x}^0$, we consider the general dynamics

$$\mathbf{x}(t+1) = f(\mathbf{x}(t), i(t), S(t)) \quad \text{for every } t \geq 0, \quad (3)$$

where $i(t)$ is the agent that updates her opinion at time t .

As a stepping stone to our convergence result, the next two lemmas study the following special case: the agent with smallest opinion is the one that updates her opinion, while the subset S is always chosen to contain the agents with largest opinion. The first result regards monotonicity properties of the trajectories.

Lemma 6: Given n, m, k with $1 \leq k \leq m \leq n$, consider

$$\mathbf{x}(t+1) = f(\mathbf{x}(t), \mu(\mathbf{x}(t)), \Omega(\mathbf{x}(t))) \quad \text{for every } t \geq 0,$$

with $\mathbf{x}(0) = \mathbf{x}^0 \in \mathbb{R}^n$. Then:

- the sequences $Y(\mathbf{x}(t))$ and $y(\mathbf{x}(t))$ are constant;

- for every $i \in Y(\mathbf{x}(0))$, the sequence $x_i(t)$ is non-decreasing and satisfies $x_i(t) \leq y(\mathbf{x}(0))$;
- for every $i \notin Y(\mathbf{x}(0))$, the sequence $x_i(t)$ is constant.

Proof: The proof goes by induction. First, consider the trivial case $x_{\mu(\mathbf{x}(t))}(t) = y(t)$, which implies $x_i(t) = y(t)$ for every $i \in Y(\mathbf{x}(t))$ and hence, for every $i \in N_{\mu(\mathbf{x}(t))}^{\Omega(\mathbf{x}(t))}$. Then, $x_{\mu(\mathbf{x}(t))}(t+1) = y(t)$ so nothing changes. Next, consider the case $x_{\mu(\mathbf{x}(t))}(t) < y(t)$. We have

$$x_{\mu(\mathbf{x}(t))}(t+1) = \frac{1}{k} \sum_{j \in N_{\mu(\mathbf{x}(t))}^{\Omega(\mathbf{x}(t))}} x_j(t) \in (x_{\mu(\mathbf{x}(t))}(t), y(t)).$$

Therefore,

$$\{i : x_i(t) < y(t)\} = \{i : x_i(t+1) < y(t)\} \neq \emptyset,$$

$$\{i : x_i(t) = y(t)\} = \{i : x_i(t+1) = y(t)\} \neq \emptyset$$

and $\{i : x_i(t) > y(t)\} = \{i : x_i(t+1) > y(t)\}$. Moreover, the cardinality of the set $\{i : x_i(t) < y(t)\}$ is strictly smaller than $n - m + k$. This implies that, while $\Omega(\mathbf{x}(t+1))$ might be different than $\Omega(\mathbf{x}(t))$, $Y(\mathbf{x}(t+1)) = Y(\mathbf{x}(t))$ and $y(t+1) = y(t)$. The claims follow by induction and by observing that only the agents $i \in Y(\mathbf{x}(0))$ can update their opinions at some time $t \geq 0$ and $x_i(t+1) \in [x_i(t), y(t)]$. ■

The second result ensures the desired “shrinking” effect.

Lemma 7: Given n, m, k with $1 \leq k \leq m \leq n$, consider

$$\mathbf{x}(t+1) = f(\mathbf{x}(t), \mu(\mathbf{x}(t)), \Omega(\mathbf{x}(t))) \quad \text{for every } t \geq 0,$$

with $\mathbf{x}(0) = \mathbf{x}^0 \in \mathbb{R}^n$. If $T := n - m + k - 1$, then

$$y(\mathbf{x}(T)) - \min_i x_i(T) \leq (1 - \frac{1}{k}) (y(\mathbf{x}(0)) - \min_i x_i(0)).$$

Proof: For a generic $t \geq 0$, consider the quantity

$$x_{\mu(\mathbf{x}(t))}(t+1) = \frac{1}{k} \sum_{j \in N_{\mu(\mathbf{x}(t))}^{\Omega(\mathbf{x}(t))}} x_j(t).$$

Recall that there exists $j^* \in N_{\mu(\mathbf{x}(t))}^{\Omega(\mathbf{x}(t))}$ such that

$$x_{j^*}(t) = \max_{j \in N_{\mu(\mathbf{x}(t))}^{\Omega(\mathbf{x}(t))}} x_j(t) = y(\mathbf{x}(t)),$$

while in general $x_j(t) \geq x_{\mu(\mathbf{x}(t))}(t)$. Therefore,

$$x_{\mu(\mathbf{x}(t))}(t+1) \geq \frac{k-1}{k} x_{\mu(\mathbf{x}(t))}(t) + \frac{1}{k} y(\mathbf{x}(t)).$$

Using Lemma 6, we have $y(\mathbf{x}(t)) = y(\mathbf{x}(0))$ and

$$x_{\mu(\mathbf{x}(t))}(t) \geq x_{\mu(\mathbf{x}(t))}(0) \geq x_{\mu(\mathbf{x}(0))}(0).$$

Hence,

$$\begin{aligned} x_{\mu(\mathbf{x}(t))}(t+1) &\geq \frac{k-1}{k} x_{\mu(\mathbf{x}(0))}(0) + \frac{1}{k} y(\mathbf{x}(0)) \\ &= x_{\mu(\mathbf{x}(0))}(0) + \frac{1}{k} (y(\mathbf{x}(0)) - x_{\mu(\mathbf{x}(0))}(0)) \end{aligned}$$

Next, consider the set

$$S(t) = \{i : x_i(t) < x_{\mu(\mathbf{x}(0))}(0) + \frac{1}{k} (y(\mathbf{x}(0)) - x_{\mu(\mathbf{x}(0))}(0))\},$$

and observe that either $S(t) = \emptyset$ or $|S(t+1)| = |S(t)| - 1$ because $\mu(\mathbf{x}(t)) \notin S(t+1)$. Since set $S(0)$ contains at most $T := n - m + k - 1$ elements, set $S(T)$ is empty. Hence,

$$x_i(T) \geq x_{\mu(\mathbf{x}(0))}(0) + \frac{1}{k} (y(0) - x_{\mu(\mathbf{x}(0))}(0))$$

for every i , a fact that implies

$$x_{\mu(\mathbf{x}(T))}(T) \geq x_{\mu(\mathbf{x}(0))}(0) + \frac{1}{k} (y(0) - x_{\mu(\mathbf{x}(0))}(0)).$$

Using Lemma 6 we know that $Y(\mathbf{x}(t)) = Y(\mathbf{x}(0))$ for every $t \geq 0$ and that for every i therein, $x_i(t) \leq y(\mathbf{x}(t)) = y(\mathbf{x}(0))$. Therefore,

$$\begin{aligned} y(\mathbf{x}(T)) - x_{\mu(\mathbf{x}(T))}(T) &\leq y(\mathbf{x}(0)) - x_{\mu(\mathbf{x}(0))}(0) \\ &\quad - \frac{1}{k} (y(\mathbf{x}(0)) - x_{\mu(\mathbf{x}(0))}(0)) \end{aligned}$$

and the thesis follows because $x_{\mu(\mathbf{x}(t))} = \min_i x_i(t)$. ■

The following corollary follows from Lemma 6 and 7 using the properties $M(\mathbf{x}) = \mu(-\mathbf{x})$ and $\Omega(\mathbf{x}) = \omega(-\mathbf{x})$.

Corollary 8: Given n, m, k with $1 \leq k \leq m \leq n$ and an initial configuration $\mathbf{x}^0 \in \mathbb{R}^n$ consider the dynamics

$$\mathbf{x}(t+1) = f(\mathbf{x}(t), M(\mathbf{x}(t)), \omega(\mathbf{x}(t))) \quad \text{for every } t \geq 0,$$

with $\mathbf{x}(0) = \mathbf{x}^0$. Then:

- $Z(\mathbf{x}(t))$ and $z(\mathbf{x}(t))$ are constant;
- for every $i \in Z(\mathbf{x}(0))$ the sequence $x_i(t)$ is non-increasing and satisfies $x_i(t) \geq z(\mathbf{x}(0))$;
- for every $i \notin Z(\mathbf{x}(0))$ the sequence $x_i(t)$ is constant.

Moreover, provided $T = n - m + k - 1$,

$$\max_i x_i(T) - z(\mathbf{x}(T)) \leq (1 - \frac{1}{k}) (\max_i x_i(0) - z(\mathbf{x}(0))),$$

The next lemma describes a vital “shrinking sequence”.

Lemma 9: Given n, m, k with $1 \leq k \leq m \leq n$, let $T = n - m + k - 1$ and consider the dynamics (3) with

$$i(t) = \begin{cases} \mu(\mathbf{x}(t)) & \text{for } t \in \{0, \dots, T-1\} \\ M(\mathbf{x}(t)) & \text{for } t \in \{T, \dots, 2T-1\} \end{cases}$$

$$S(t) = \begin{cases} \Omega(\mathbf{x}(t)) & \text{for } t \in \{0, \dots, T-1\} \\ \omega(\mathbf{x}(t)) & \text{for } t \in \{T, \dots, 2T-1\}. \end{cases}$$

Let $\gamma := 1 - \frac{1}{k}$. If $n > 2(m - k)$, then

$$\max_i x_i(2T) - \min_i x_i(2T) \leq \gamma (\max_i x_i(0) - \min_i x_i(0)).$$

Proof: To keep the notation compact we additionally introduce $\alpha(t) := \min_i x_i(t)$, $\beta(t) := \max_i x_i(t)$ and write $y(t) := y(\mathbf{x}(t))$ and $z(t) := z(\mathbf{x}(t))$. We have

$$\begin{aligned} \beta(2T) - \alpha(2T) &= \beta(2T) - z(2T) + z(2T) - \alpha(2T) \\ &\leq \gamma (\beta(T) - z(T)) + z(T) - \alpha(T) \end{aligned}$$

using Corollary 8 with initial configuration $\mathbf{x}(T)$. Then,

$$\begin{aligned} &= \gamma (\beta(T) - y(T)) + \gamma (y(T) - z(T)) + z(T) - \alpha(T) \\ &\leq \gamma (\beta(T) - y(T)) + y(T) - z(T) + z(T) - \alpha(T) \end{aligned}$$

since $\gamma < 1$ while $y(T) - z(T) \geq 0$ by Lemma 5. Then,

$$\begin{aligned} &= \gamma (\beta(T) - y(T)) + y(T) - \alpha(T) \\ &\leq \gamma (\beta(0) - y(0)) + \gamma (y(0) - \alpha(0)) \\ &= \gamma (\beta(0) - \alpha(0)) \end{aligned}$$

using Lemma 6 and 7 with initial configuration $\mathbf{x}(0)$. ■

B. Proof of Theorem 4

Let $\delta(t) = \max_i x_i(t) - \min_i x_i(t)$ and observe that for any $\mathbf{x}^0, i(t)$ and $S(t)$ it holds $0 \leq \delta(t+1) \leq \delta(t)$ for every $t \geq 0$, because in dynamics (3) the agent with highest (lowest) opinion cannot increase (decrease) it. We introduce the sequence of events $\{A_t, t \geq 2T-1\}$ where the event A_t is the occurrence in the time window $\{t-(2T-1), \dots, t\}$ of the “shrinking sequence” introduced in Lemma 9. In the same lemma we proved that, given event A_t , we have $\delta(t+1) \leq (1 - \frac{1}{k}) \delta(t-2T+1)$. Observe that

$$0 \leq \lim_{t \rightarrow \infty} \delta(t) \leq \lim_{t \rightarrow \infty} (1 - \frac{1}{k})^{n_t} \delta(0)$$

where n_t is the number of times event A_s occurred up to time t . If $\mathbb{P}(A_t \text{ infinitely often}) = 1$ then $n_t \rightarrow \infty$ for $t \rightarrow \infty$ and the rightmost limit above is zero almost surely. Hence, $\lim_{t \rightarrow \infty} \delta(t)$ almost surely, which implies the convergence to consensus. Moreover, $c \in [\min_i(x_i^0), \max_i(x_i^0)]$ because every update (3) is a convex combination of current opinions.

It remains to prove $\mathbb{P}(A_t \text{ infinitely often}) = 1$. The events of the sequence $\{A_t, t \geq 2T-1\}$ are not independent but those in the subsequence $\{A_{t_h}, h \geq 1\}$ where $t_h = 2Th-1$ are. Each of those events has constant non-zero probability

$$\mathbb{P}(A_{t_h}) = (n \binom{n}{m})^{-2T},$$

thus $\sum_{h=1}^{\infty} \mathbb{P}(A_{t_h}) = \infty$. Hence, $\{A_t \text{ i.o.}\} \supset \{A_{t_h} \text{ i.o.}\}$. From the second Borel-Cantelli lemma [27, Ch. 2, Thm 18.2] $\mathbb{P}(A_t \text{ infinitely often}) \geq \mathbb{P}(A_{t_h} \text{ infinitely often}) = 1$.

V. CONCLUSION

In this paper we have introduced a new model of opinion dynamics with opinion-dependent connectivity, which captures the intuitive idea of interactions taking place between the closest individuals. Even though simulations suggest that the dynamics converge for any k, m, n , proving a general convergence result is an open problem. As a contribution in this direction, we have given a necessary and sufficient condition for all equilibria to be consensus states and we have proved that, under the same condition, the dynamics asymptotically converges to consensus. However, the transient features *metastable* non-consensus states, which dominate the qualitative behavior of the dynamics, because solutions can stay close to them for times that are exponential in the number of agents. A detailed analysis of these metastable states might benefit from suitable mean-field approximations: we leave this approach to future research. Another valuable research direction is extending the model to include a social network, whose graph would encode possible biases or restrictions in the sampling process.

REFERENCES

- [1] A. V. Proskurnikov and R. Tempo, “A tutorial on modeling and analysis of dynamic social networks. Part I,” *Annual Reviews in Control*, vol. 43, pp. 65–79, Mar. 2017.
- [2] A. Proskurnikov and R. Tempo, “A tutorial on modeling and analysis of dynamic social networks. Part II,” *Annual Reviews in Control*, vol. 45, pp. 166–190, 2018.
- [3] U. Krause, “A discrete nonlinear and non-autonomous model of consensus formation,” *Communications in Difference Equations*, pp. 227–236, 2000.
- [4] G. Deffuant, D. Neau, F. Amblard, and G. Weisbuch, “Mixing beliefs among interacting agents,” *Advances in Complex Systems*, vol. 03, no. 01n04, pp. 87–98, 2000.
- [5] D. Lazer, “The rise of the social algorithm,” *Science*, vol. 348, no. 6239, pp. 1090–1091, 2015.
- [6] M. Ballerini, N. Cabibbo, R. Candelier, A. Cavagna, E. Cisbani, I. Giardina, V. Lecomte, A. Orlandi, G. Parisi, A. Procaccini, M. Viale, and V. Zdravkovic, “Interaction ruling animal collective behavior depends on topological rather than metric distance: Evidence from a field study,” *Proceedings of the National Academy of Sciences*, vol. 105, no. 4, pp. 1232–1237, 2008.
- [7] E. Cristiani, P. Frasca, and B. Piccoli, “Effects of anisotropic interactions on the structure of animal groups,” *Journal of Mathematical Biology*, vol. 62, no. 4, pp. 569–588, 2011.
- [8] S. Martin, “Multi-agent flocking under topological interactions,” *Systems & Control Letters*, vol. 69, pp. 53–61, 2014.
- [9] A. Aydoğdu, P. Frasca, C. D’Apice, R. Manzo, J. Thornton, B. Gachomo, T. Wilson, B. Cheung, U. Tariq, W. Saidel, and B. Piccoli, “Modeling birds on wires,” *Journal of Theoretical Biology*, vol. 415, pp. 102–112, 2017.
- [10] F. Fagnani and P. Frasca, *Introduction to Averaging Dynamics over Networks*, ser. Lecture Notes in Control and Information Sciences. Springer Nature, 2017.
- [11] V. D. Blondel, J. M. Hendrickx, and J. N. Tsitsiklis, “On Krause’s multi-agent consensus model with state-dependent connectivity,” *IEEE Transactions on Automatic Control*, vol. 54, no. 11, pp. 2586–2597, 2009.
- [12] F. Ceragioli and P. Frasca, “Continuous and discontinuous opinion dynamics with bounded confidence,” *NonLinear Analysis and its Applications B*, vol. 13, no. 3, pp. 1239–1251, 2012.
- [13] C. Canuto, F. Fagnani, and P. Tilli, “An Eulerian approach to the analysis of Krause’s consensus models,” *SIAM Journal on Control and Optimization*, vol. 50, no. 1, pp. 243–265, 2012.
- [14] A. Mirtabatabaei and F. Bullo, “Opinion dynamics in heterogeneous networks: Convergence conjectures and theorems,” *SIAM Journal on Control and Optimization*, vol. 50, no. 5, pp. 2763–2785, 2012.
- [15] B. Chazelle and C. Wang, “Inertial Hegselmann-Krause systems,” *IEEE Transactions on Automatic Control*, vol. 62, no. 8, pp. 3905–3913, Aug 2017.
- [16] G. Chen, W. Su, W. Mei, and F. Bullo, “Convergence properties of the heterogeneous Deffuant-Weisbuch model,” *arXiv preprint arXiv:1901.02092*, 2019.
- [17] P. Balister, B. Bollobás, A. Sarkar, and M. Walters, “Connectivity of random k -nearest-neighbour graphs,” *Advances in Applied Probability*, vol. 37, no. 1, pp. 1–24, 2005.
- [18] C. Chen, G. Chen, and L. Guo, “On the minimum number of neighbors needed for consensus of flocks,” *Control Theory and Technology*, vol. 15, no. 4, pp. 327–339, 2017.
- [19] A. Blanchet and P. Degond, “Topological interactions in a Boltzmann-type framework,” *Journal of Statistical Physics*, vol. 163, no. 1, pp. 41–60, Apr 2016.
- [20] R. Shvydkoy and E. Tadmor, “Topological models for emergent dynamics with short-range interactions,” *arXiv preprint*, 2018.
- [21] P. Degond and M. Pulvirenti, “Propagation of chaos for topological interactions,” *The Annals of Applied Probability*, vol. 29, no. 4, pp. 2594–2612, 2019.
- [22] D. Angeli and S. Manfredi, “A Petri net approach to consensus in networks with joint-agent interactions,” *Automatica*, vol. 110, p. 108466, 2019.
- [23] A. Aydoğdu, M. Caponigro, S. McQuade, B. Piccoli, N. Pouradier Duteil, F. Rossi, and E. Trélat, “Interaction network, state space, and control in social dynamics,” in *Active Particles, Volume 1: Advances in Theory, Models, and Applications*, N. Bellomo, P. Degond, and E. Tadmor, Eds. Springer, 2017, pp. 99–140.
- [24] W. S. Rossi and P. Frasca, “Asynchronous opinion dynamics on the k -nearest-neighbors graph,” in *2018 IEEE Conference on Decision and Control (CDC)*, Dec. 2018, pp. 3648–3653.
- [25] D. Fotakis, D. Palyvos-Giannas, and S. Skoulakis, “Opinion dynamics with local interactions,” in *25th International Joint Conference on Artificial Intelligence (IJCAI)*, July 2016, pp. 279–285.
- [26] W. Hoeffding, “Probability inequalities for sums of bounded random variables,” *Journal of the American Statistical Association*, vol. 58, no. 301, pp. 13–30, 1963.
- [27] A. Gut, *Probability: A Graduate Course*. Springer, 2013.