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# 1 From branchings to flows: a study of an Edmonds' 2 like property to arc-disjoint branching flows

3 **Cláudio Carvalho** ✉

4 Departamento de Computação, Universidade Federal do Ceará, Fortaleza, Brazil

5 **Jonas Costa** ✉

6 Departamento de Computação, Universidade Federal do Ceará, Fortaleza, Brazil

7 **Raul Lopes** ✉ 

8 Université Paris-Dauphine, PSL University, CNRS UMR7243, LAMSADE, Paris, France

9 **Ana Karolinna Maia** ✉

10 Departamento de Computação, Universidade Federal do Ceará, Fortaleza, Brazil

11 **Nicolas Nisse** ✉

12 Université Côte d'Azur, Inria, CNRS, I3S, Sophia-Antipolis, France

13 **Cláudia Linhares Sales** ✉

14 Departamento de Computação, Universidade Federal do Ceará, Fortaleza, Brazil

## 15 — Abstract —

16 An  $s$ -branching flow  $f$  in a network  $\mathcal{N} = (D, u)$ , where  $u$  is the capacity function, is a flow that  
17 reaches every vertex in  $V(D)$  from  $s$  while loosing exactly one unit of flow in each vertex other than  
18  $s$ . Bang-Jensen and Bessy [TCS, 2014] showed that, when every arc has capacity  $n - 1$ , a network  $\mathcal{N}$   
19 admits  $k$  arc-disjoint  $s$ -branching flows if and only if its associated digraph  $D$  contains  $k$  arc-disjoint  
20  $s$ -branchings. Thus a classical result by Edmonds stating that a digraph contains  $k$  arc-disjoint  
21  $s$ -branchings if and only if the indegree of every set  $X \subseteq V(D) \setminus \{s\}$  is at least  $k$  also characterizes  
22 the existence of  $k$  arc-disjoint  $s$ -branching flows in those networks, suggesting that the larger the  
23 capacities are, the closer an  $s$ -branching flow is from simply being an  $s$ -branching. This observation  
24 is further implied by results by Bang-Jensen et al. [DAM, 2016] showing that there is a polynomial  
25 algorithm to find the flows (if they exist) when every arc has capacity  $n - c$ , for every fixed  $c \geq 1$ ,  
26 and that such an algorithm is unlikely to exist for most other choices of the capacities. In this paper,  
27 we investigate how a property that is a natural extension of the characterization by Edmonds' relates  
28 to the existence of  $k$  arc-disjoint  $s$ -branching flows in networks. Although this property is always  
29 necessary for the existence of the flows, we show that it is not always sufficient and that it is hard  
30 to decide if the desired flows exist even if we know beforehand that the network satisfies it. On the  
31 positive side, we show that it guarantees the existence of the desired flows in some particular cases  
32 depending on the choice of the capacity function or on the structure of the underlying graph of  $D$ ,  
33 for example. We remark that, in those positive cases, polynomial time algorithms to find the flows  
34 can be extracted from the constructive proofs.

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Nicolas Nisse;

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46 **1** Introduction

47 Let  $D = (V, A)$  be a digraph. If  $e$  is an arc of  $D$  from a vertex  $v$  to a vertex  $w$ , we may refer  
 48 to  $e$  as  $vw$ . The *in-degree* of a set  $X \subseteq V(D)$ , denoted by  $d_D^-(X)$ , is the number of arcs  
 49 with tail outside of  $X$  and head inside of  $X$ . If  $X = \{v\}$ , we simply write  $d_D^-(v)$  instead of  
 50  $d_D^-(\{v\})$  (omitting braces). For an integer  $n$ , we denote by  $[n]$  the set  $\{1, \dots, n\}$ .

51 A *network*  $\mathcal{N} = (D, u)$  is formed by a digraph  $D = (V, A)$  with a *capacity function*  
 52  $u : A(D) \rightarrow \mathbb{Z}_+$ . If all arcs in  $D$  have capacity  $\lambda$ , we simply write  $u \equiv \lambda$ . A *flow* in  $\mathcal{N}$  is  
 53 a function  $f : A(D) \rightarrow \mathbb{Z}_+$  such that  $f(vw) \leq u(vw)$ ,  $\forall vw \in A(D)$  (note that contrary to  
 54 the classical definition of flows, there is no need to satisfy the conservation constraints). For  
 55 a vertex  $v \in V(D)$ , we define  $f^+(v) = \sum_{vw \in A} f(vw)$  and  $f^-(v) = \sum_{wv \in A} f(wv)$ , that is,  
 56  $f^+(v)$  and  $f^-(v)$  are the amount of flow *leaving* and *entering*  $v$ , respectively. The *balance*  
 57 *vector* of a flow  $f$  in a network  $\mathcal{N} = (D, u)$  is the function  $b_f : V(D) \rightarrow \mathbb{Z}$  associating each  
 58 vertex  $v \in V(D)$  to the value  $f^+(v) - f^-(v)$ . If  $b_f(v) = \ell$ , we may also say that  $v$  has *balance*  
 59  $\ell$ . A vertex  $s \in V(D)$  is a *source* of  $f$  if  $f^+(s) > 0$  and  $f^-(s) = 0$ . An  $(s, t)$ -*cut* on a network  
 60  $\mathcal{N} = (D, c)$  is a bipartition  $(S, \bar{S} = V(D) \setminus S)$  of  $V(D)$  such that  $s \in S$  and  $t \in \bar{S}$ . The  
 61 capacity of an  $(s, t)$ -cut is the sum of the capacities of arcs with tail in  $S$  and head in  $\bar{S}$  and  
 62 it is denoted by  $u(S, \bar{S})$ . For other concepts on graphs and digraphs, we follow standard  
 63 terminology as in [3, 6].

64 Flows are widely studied as they allow, with a certain elegance and simplicity, modeling  
 65 problems in different areas of study such as transportation, logistics and telecommunications.  
 66 In the theoretical field, they are used to solve various problems in graphs and digraphs. A  
 67 long list of results related to flows can be found in [1, 10]. In this work, we deal with one  
 68 variation of the problem where, roughly speaking,  $k$  arc-disjoint flows, having in common the  
 69 same source and with a prescribed balance vector, have to be found.

70 Since a flow in a network can be entirely described by its associated balance vector,  
 71 the goal of a flow problem can be seen as finding a flow  $f$  whose balance vector respects  
 72 a given set of properties. Therefore, the problem that has been considered is: given a  
 73 digraph  $D$  together with a balanced vector  $b : V(D) \rightarrow \mathbb{Z}$ , does there exist a flow  $f$  in  $D$   
 74 such that  $b_f = b$ ? For example, in the decision version of the classical MAXIMUM FLOW  
 75 problem we are given a network  $\mathcal{N} = (D, u)$  together with a positive integer  $k$  and a pair  
 76 of vertices  $s, t \in V(D)$ , and the goal is to decide if  $\mathcal{N}$  admits a flow  $f$  with  $b_f(s) \geq k$  such  
 77 that  $b_f(v) = 0$  for all  $v \in V(D) \setminus \{s, t\}$  and  $b_f(s) = -b_f(t)$ . Such a flow  $f$  is known as an  
 78  $(s, t)$ -*flow* and, in this case, we say that  $f$  has *value*  $b_f(s)$ . This problem can be solved in  
 79 polynomial time [11] (see [16, Chapter 10] for a survey on algorithms for MAXIMUM FLOW).  
 80 Furthermore, by applying a simple reduction to MAXIMUM FLOW (see [3, Lemma 4.2.2]),  
 81 we can solve in polynomial time any flow problem in which the aim is to find a flow  $g$  with  
 82  $\sum_{v \in V(D)} b_g(v) = 0$ .

83 The possibility of considering the coexistence of flows respecting some properties in a  
 84 network gives more power of modeling for this already useful tool. In [9], a more general  
 85 version of the flow problem was investigated. There, the goal was to find a collection of  
 86  $(s_i, t_i)$ -flows that sum to a specific value  $r$  and such that the sum of all the flows in each  
 87 arc respects its capacity. It was shown that this version of the problem is NP-hard since it  
 88 generalizes many hard problems in the literature, including the notoriously hard DIRECTED  
 89 DISJOINT PATHS problem.

90 Two flows  $f$  and  $g$  in a network  $\mathcal{N} = (D, u)$  are arc-disjoint if  $f(vw) \cdot g(vw) = 0$ ,  
 91  $\forall vw \in A(D)$ . In [2], the problem of finding arc-disjoint flows was introduced and studied.  
 92 The authors considered many constraints showing that some generalize important problems,

93 known to be hard, and that there are cases where polynomial-time algorithms are possible.  
 94 Amongst other results, they showed that the following are NP-complete: the problem of  
 95 deciding if there are two arc-disjoint flows in a network where all arcs have capacity one, the  
 96 problem of deciding if there are two arc-disjoint flows with the same balance vectors in a  
 97 network where all arcs have capacity at most 2, and the problem of deciding if there are two  
 98 arc-disjoint  $(s, t)$ -flows in a network where all arcs have capacity at most two.

99 We say that a digraph  $D$  is an  $s$ -branching if there is a directed path from  $s$  to every  
 100 other vertex in  $D$  and the underlying<sup>1</sup> graph of  $D$  is a tree. We also say that  $D$  has root  $s$ .  
 101 There is an extensive literature concerning the study of branchings (see for example chapter  
 102 9 of [3]), given their relevance both from practical and theoretical point of view due to its  
 103 numerous applications, some of which we enumerate below. A classical result by Edmonds [8]  
 104 characterizes the digraphs containing  $k$  arc-disjoint  $s$ -branchings and later proofs of the same  
 105 theorem done in [14, 18] show polynomial-time algorithms to find such branchings if they  
 106 exist.

107 ► **Theorem 1** (Edmonds' branching theorem [8]). *A digraph  $D = (V, A)$  has  $k$  arc-disjoint*  
 108  *$s$ -branchings if and only if  $d_D^-(X) \geq k$  for all  $X \subseteq V(D) \setminus \{s\}$  with  $X \neq \emptyset$ .*

109 A stronger version of this result, allowing each branching to have its own set of roots,  
 110 can be seen in [16, Chapter 53]. There are many applications for Edmonds' Theorem on  
 111 arc-disjoint branchings: it can be used, for example, to prove Menger's Theorem [15], to  
 112 characterize arc-connectivity [17], and to characterize branching cover [12]. The problem of  
 113 finding disjoint branchings was recently studied in the setting of temporal graphs in [19]. In  
 114 this paper, we consider branchings in terms of flow.

115 An  $s$ -branching flow in a network  $\mathcal{N} = (D, u)$  is a flow  $f$  such that  $b_f(s) = n - 1$  and  
 116  $b_f(v) = -1$  for every  $v \in V(D) \setminus \{s\}$ . In other words,  $f$  reaches all vertices of  $D$  and each  
 117 vertex other than  $s$  retains one unit of flow. Here, we study the problem that, given a  
 118 digraph  $D$  and a source  $s$  and an integer  $k$ , aims at deciding whether there exist  $k$  arc-disjoint  
 119  $s$ -branching flows.

120 **Related work:** Finding one  $s$ -branching flow in a given network is easy: since  $\sum_{v \in V(D)} b_f(v) =$   
 121  $0$ , we can reduce this problem to the problem of finding one  $(s, t)$ -flow, as discussed above.  
 122 We can also find  $k$  arc-disjoint  $s$ -branching flows in polynomial time when  $u \equiv n - 1$ . In [2]  
 123 it was shown that, in this case,  $\mathcal{N}$  admits an  $s$ -branching flow if and only if  $D$  contains an  
 124  $s$ -branching. Thus, applying Theorem 1, the authors provided a characterization of networks  
 125 admitting  $k$  arc-disjoint  $s$ -branching flows, which are exactly the networks constructed on  
 126 digraphs containing  $k$  arc-disjoint  $s$ -branchings. They also provided a polynomial time  
 127 algorithm that finds such flows if they exist. We now discuss how the tractability of this  
 128 problem, in general, depends on the choice of the capacity function.

129 In [2], the authors showed that the problem of deciding if a network has  $k$  arc-disjoint  
 130  $s$ -branching flows is NP-complete if every arc has capacity at most two, and in [4] this result  
 131 was extended to networks with capacity at most  $\ell$ , for every fixed  $\ell \geq 2$ . For most choices of  
 132 larger capacities, the problem remains hard. In [4] it was shown that, unless the *Exponential*  
 133 *Time Hypothesis* [13] fails, there is no polynomial time algorithm for the problem of finding  
 134  $k$  arc-disjoint  $s$ -branching flows in a network  $\mathcal{N} = (D, u)$  with  $u \equiv \lambda$  for any choice of  $\lambda$   
 135 such that  $n/2 \leq \lambda \leq n - (\log n)^{1+\varepsilon}$ ,  $\varepsilon > 0$ , even if  $D$  is acyclic. In [7] the authors adapted  
 136 this last proof to show that the same holds if  $(\log n)^{1+\varepsilon} \leq \lambda \leq n/2$ . The last two results

<sup>1</sup> The underlying graph of a digraph  $D$  is the undirected graph obtained by ignoring the orientation of the arcs of  $D$ .

137 have an intersection point, which is when  $u \equiv n/2$ . The two constructions result in the same  
 138 network in this case, and it is worth noticing that it gives a polynomial time reduction. So,  
 139 for  $u \equiv n/2$ , the problem is NP-hard. On the positive side, in [4] it was also shown that  
 140 the problem is solvable in polynomial time when  $\lambda = n - c$ , for fixed  $k$  and  $c \geq 1$ . More  
 141 precisely, the authors showed that the problem is XP for fixed  $k$  and  $c$  when the goal is to  
 142 find  $k$  arc-disjoint  $s$ -branching flows. This was later improved to an FPT algorithm with the  
 143 same parameter in [5].

144 In short, the results presented in the previous paragraph roughly say that the problem  
 145 of deciding if a given network with  $u \equiv n - c$  contains  $k$  arc-disjoint  $s$ -branching flows  
 146 admits a polynomial time algorithm for every fixed  $c \geq 1$ , and is unlikely to admit such an  
 147 algorithm for most other choices of  $u$  under reasonable complexity hypothesis. Thus any  
 148 *general* characterization of networks admitting such flows must also be hard to verify. In the  
 149 following, we extend to  $s$ -branching flows the property characterizing networks containing  $k$   
 150 arc-disjoint  $s$ -branchings, which was originally proposed in [7], .

151 For a vertex  $s \in V(D)$  and two non-negative integers  $k, \lambda$ , we say that  $D$  is  $(k, \lambda, s)$ -  
 152 *sufficient* if, for every  $X \subseteq V(D) \setminus \{s\}$  with  $X \neq \emptyset$ , we have

$$153 \quad d_D^-(X) \geq k \left\lceil \frac{|X|}{\lambda} \right\rceil. \quad (1)$$

154  
 155 If  $\mathcal{N} = (D, u)$  with  $u \equiv \lambda$  is a network constructed on such a digraph, we may also say that  
 156  $\mathcal{N}$  is  $(k, \lambda, s)$ -sufficient.

157 It is not hard to visualize the relationship between this property and the one in the  
 158 statement of Theorem 1: if  $\lambda = n - 1$ , then every arc of  $D$  has enough capacity to send  
 159 as many units of flow as needed to reach any set  $X \subseteq V(D)$ . Thus, capacities are not an  
 160 issue and Inequality 1 states that  $d_D^-(X) \geq k$ , as in Theorem 1, since  $|X| \leq n - 1$ . For other  
 161 choices of  $\lambda$ , note that every  $s$ -branching flow on  $\mathcal{N}$  must reach a set of vertices  $X$  with at  
 162 least  $|X|$  units of flow to cover it. Thus each  $s$ -branching flow in  $\mathcal{N}$  uses at least  $\lceil |X|/\lambda \rceil$   
 163 arcs to cover  $X$ . More formally, the following was proved.

164 ► **Proposition 2.** [7] *Let  $\mathcal{N} = (D, u)$  be a network with  $u \equiv \lambda$  and  $s \in V(D)$ . If  $\mathcal{N}$  admits  
 165  $k$  arc-disjoint  $s$ -branching flows, then  $D$  is  $(k, \lambda, s)$ -sufficient.*

166 **Our contributions:** In Section 2, we show how to construct a family of a  $(k, \lambda, s)$ -sufficient  
 167 networks not containing  $k$  arc-disjoint  $s$ -branching flows. On the positive side, we show that  
 168 some positive cases of sufficiency do occur depending on the structure of  $D$  and on the choice  
 169 of  $k$  and  $\lambda$ .

170 In Section 3, we show that  $(k, \lambda, s)$ -sufficiency guarantees the existence of  $k$  arc-disjoint  
 171  $s$ -branching flows in a network  $\mathcal{N} = (D, u)$  when  $D$  is an  $s$ -branching where parallel arcs are  
 172 allowed, and when  $D$  is a collection of internally disjoint multi-paths starting at  $s$  and all  
 173 ending on the same vertex  $t \in V(D)$ . In other words, the flows are guaranteed to exist when  
 174 the underlying simple graph of  $D$  is a tree or a spindle, and thus the considered property  
 175 characterizes the existence of the flows in those classes of networks. Additionally, we show  
 176 that we also have this guarantee in general digraphs when  $k = 1$ , when  $\lambda = 1$ , and when  
 177  $\lambda = |V(D)| - 2$ . Finally, we show that, for a network  $\mathcal{N} = (D, u)$  with  $u \equiv \lambda$  and an integer  
 178  $p$  satisfying  $p\lambda \geq n - 1$ , if  $d_D^-(X) \geq pk$  for all non-empty  $X \subseteq V(D) \setminus \{s\}$ , then  $\mathcal{N}$  admits  $k$   
 179 arc-disjoint  $s$ -branching flows.

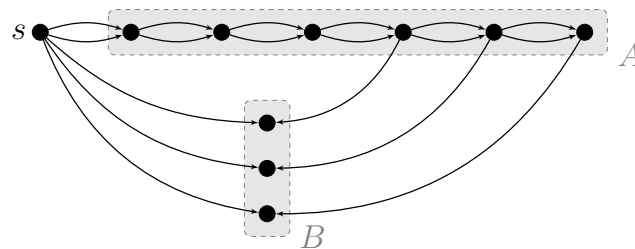
180 In Section 4, we show that the analyzed property is not enough to improve the tractability  
 181 of the problem of finding the desired flows. Namely, we show that it is NP-complete to decide

182 if a network  $\mathcal{N}$  admits  $k$  arc-disjoint  $s$ -branching flows even if its underlying simple graph is  
 183 constructed by appending a path to the end of a spindle and we know beforehand that  $\mathcal{N}$   
 184 is  $(k, \lambda, s)$ -sufficient. Since this property can be easily verified in this class and it is always  
 185 necessary for the existence of the flows, it is no surprise that it is not enough to guarantee  
 186 the existence of the  $k$  arc-disjoint  $s$ -branching flows.

187 In Section 5 we close the paper with some open questions. We remark that polynomial  
 188 time algorithms to find the flows can be extracted from the provided constructive proofs of  
 189 the aforementioned results.

190 **2 Networks on  $(k, \lambda, s)$ -sufficient digraphs not containing the flows**

191 In Figure 1 we give an example of a  $(2, n - 3, s)$ -sufficient network that does not contains 2  
 192 arc-disjoint  $s$ -branching flows, and we show that it can be generalised to  $(2, n - c, s)$ -sufficient  
 193 networks, for every fixed  $c \geq 3$  (Theorem 3). Thus the best that we can expect is that every  
 194  $(2, n - 2, s)$ -sufficient network contains the desired  $s$ -branching flows. In Section 3 we show  
 that this is indeed true for general networks (c.f. Theorem 13).



195 **Figure 1** Example of a network on a  $(2, n - 3, s)$ -sufficient digraph that does not contain 2  
 196 arc-disjoint  $s$ -branching flows. Here every arc has capacity  $n - 3$ .

195

196 **Theorem 3.** *For every  $c \geq 3$ , there exists a  $(2, n - c, s)$ -sufficient network  $\mathcal{N}$  which does*  
 197 *not admit 2 arc-disjoint  $s$ -branching flows.*

198 **Proof.** Let  $\mathcal{N}$  be a network as in Figure 1, considering that  $B$  has exactly  $c$  vertices and  
 199  $|A| = n - c - 1$ , with each of the last  $c$  vertices of  $A$  connected to a different vertex of  $B$ .  
 200 First, we show that  $\mathcal{N}$  is  $(2, n - c, s)$ -sufficient. For any set  $X \subseteq V(D) \setminus \{s\}$ , let  $a = |X \cap A|$   
 201 and  $b = |X \cap B|$ . Thus,  $|X| = a + b$ .

202 If  $|X| \leq n - c$ , then we must show that  $d^-(X) \geq 2 \lceil \frac{|X|}{n-c} \rceil = 2$ . If  $a > 0$ , one can see that  
 203 there are two arcs entering  $X \cap A$ , then  $d^-(X) \geq 2 + b \geq 2$ . Otherwise, we have that  $b > 0$   
 204 and  $d^-(X) = 2b \geq 2$ .

205 If  $|X| > n - c$ , then we must show that  $d^-(X) \geq 2 \lceil \frac{|X|}{n-c} \rceil = 4$ . Note that in this case  
 206  $a > 0$  and  $b \geq 2$ . So, we have  $d^-(X) \geq 2 + b \geq 4$ .

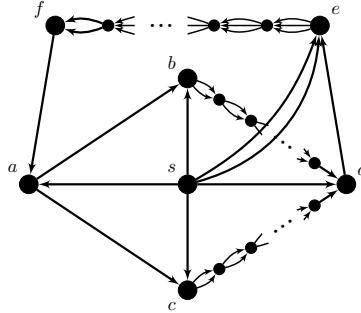
207 Now we show that  $\mathcal{N}$  does not admit 2 arc-disjoint  $s$ -branching flows. Observe that each  
 208 flow can use at most one arc from  $A$  to  $B$ , by the size of  $A$  and the capacity of the arcs  
 209 (specially the arcs leaving  $s$  to  $A$ ). So, there is a vertex  $v$  in  $B$  that must be reached by each  
 210 flow through an arc coming from  $s$ , but there is only one arc from  $s$  to  $v$ . ◀

211 Next, we show how to construct a family of  $(k, \lambda, s)$ -sufficient networks that do not contain  
 212  $k$  arc-disjoint  $s$ -branching flows.

213 **Theorem 4.** *For all  $\lambda \geq 2$  and for all even  $k \geq 2$ , there exists a  $(k, \lambda, s)$ -sufficient network*  
 214  *$\mathcal{N} = (D, u)$ , with  $u \equiv \lambda$ , that does not admit  $k$  arc-disjoint  $s$ -branching flows.*

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215 **Proof.** Given an even  $k \geq 2$  and  $\lambda \geq 2$ , start the construction of  $\mathcal{N}$  by adding the vertices  
 216  $s, a, b, c$  and  $d$  then add an arc from  $s$  to every other vertex along with the arcs  $ab, ac, bd,$  and  
 217  $cd$ . Subdivide the arc  $bd$  ( $cd$ )  $\lambda - 2$  times and let  $B$  ( $C$ ) be the union of  $\{b\}$  ( $\{c\}$ ) and the set  
 218 of vertices obtained by the subdivisions of  $bd$  ( $cd$ ) (note that if  $\lambda = 2$  nothing changes at this  
 219 step, which means that  $B = \{b\}$  and  $C = \{c\}$ ), and then double the arcs between vertices of  
 220  $B$  ( $C$ ). Let  $P$  be a path with  $\lambda$  vertices and denote by  $e$  and  $f$ , respectively, the first and  
 221 last vertex of  $P$ . Triple the first  $\lambda - 2$  arcs of  $P$  and double the last one (the one that enters  
 222  $f$ ). After that, add  $P$  to  $D$  along with the arcs  $de, fa$  and two parallel arcs  $se$ . To complete  
 223 the construction, replace each arc by  $k/2$  copies of itself and set  $u \equiv \lambda$ . See Figure 2.



■ **Figure 2** Counter-example for  $k = 2$  and  $\lambda \geq 2$

224 Here, we say that a set  $X \subseteq V(D)$  is *satisfied* if  $d_D^-(X) \geq k \lceil |X|/\lambda \rceil$ .  
 225 To prove that  $D$  is  $(k, \lambda, s)$ -sufficient, we consider the case where  $k = 2$  because for a  
 226 larger  $k$  we multiply the in-degree of each subset of  $V(D) - s$  by  $k/2$ . Observe that, except  
 227 for  $s$ , every vertex has in-degree at least 2 in  $D$  and that for every  $X \subseteq V(D) - s$  each vertex  
 228 from  $\{a, b, c, d\}$  which belongs to  $X$  sums at least one unit to its in-degree and the vertex  $e$   
 229 sums to two. Consider a set  $X \subseteq V(D) - s$ . If  $D[X]$  has a cycle, then  $d_D^-(X) \geq 5$ , once  $X$   
 230 necessarily contains  $a, d, P$  and either  $B$  or  $C$  (or both), but its also true that  $|X| \geq 2\lambda + 1$ ,  
 231 which means that should be at least one extra arc entering in  $X$  for it to be satisfied. To  
 232 verify the existence of this extra arc assume without loss of generality that  $B \subset X$  (the  
 233 case where  $C \subset X$  is symmetric) and take the longest  $(z, d)$ -path in  $D[X]$  such that  $z \in C$ .  
 234 If no such path exists, there is at least one arc going from  $C$  to  $d$  which was not counted  
 235 before and if the path exists, then either  $z = c$  or the two arcs  $yz$  enters  $X$ , where  $y$  is the  
 236 in-neighbor of  $z$  which, by the choice of  $z$ , cannot be in  $X$ .  
 237 Now consider that  $D[X]$  is acyclic. Note that each source in  $D[X]$  contributes with at  
 238 least two units in the in-degree of  $X$ . Since each weak component of  $D[X]$  has at least  
 239 one source, then every  $X$  with  $|X| \leq \lambda$  is satisfied. Similarly, the same also holds when  
 240  $\lambda < |X| \leq 2\lambda$  if  $D[X]$  has more than one weak component or only one component with many  
 241 sources. Otherwise  $D[X]$  has only one weak component  $W$  with a single source  $w$ , and we  
 242 have 5 cases:  
 243 ■  $w \in B$ . Observe that in this case  $\{d, e\} \subset X$ , once that  $|X| \geq \lambda + 1$  and  $|B| = \lambda - 1$ ,  
 244 which means that  $d_D^-(X) \geq 5$ .  
 245 ■  $w \in C$ . It's analogous to the previous one.  
 246 ■  $w = a$ . In this case, there are at least two vertices from  $\{b, c, d\}$  in  $X$ . That is,  $d^-(X) \geq 4$ .  
 247 ■  $w = d$ . If  $d$  is the source, then  $e \in X$ , and thus  $d_D^-(X) \geq 5$ .  
 248 ■  $w \in V(P)$ . Except for  $f$ , every vertex of  $P$  has in-degree three and since  $X$  must contains  
 249  $a$ , then  $d_D^-(X) \geq 4$ . When  $w = f$ , besides  $a$ ,  $X$  must also contains at least one vertex  
 250 from  $\{b, c\}$  which guaranties  $d_D^-(X) \geq 4$ .

251 When  $2\lambda + 1 \leq |X| \leq 3\lambda$ , the in-degree of  $X$  must be at least 6. We start with the case  
 252 that  $D[X]$  has just one weak component. If  $w$  is a source in  $D[X]$ , again we have 5 subcases:  
 253 ■  $w \in B$ . In this case, since  $|B \cup \{d\}| = \lambda$ , either there is a vertex in  $C$  which is also a source  
 254 in  $D[X]$  or  $a \in X$ . In both situations,  $d, e \in X$  because  $|B \cup C| < 2\lambda$ . Thus,  $d_D^-(X) \geq 6$ .  
 255 ■  $w \in C$ . It's analogous to the previous one.  
 256 ■  $w = a$ . Once  $|\{a, d\} \cup B \cup C| = 2\lambda$ , if  $a$  is a source, there is a path in  $D[X]$  that starts in  
 257  $a$ , goes through  $B$  or  $C$ ,  $d$  and ends in some vertex of  $P$ .  
 258 ■  $w = d$ . When  $d$  is a source, for sure  $\{a, b, c, d, e\} \subset X$ .  
 259 ■  $w \in V(P)$ . Here, in order to avoid cycles, we know that if  $w = f$ , necessarily  $a, b, c$  and  $d$   
 260 are also included in  $X$  and if  $w \neq f$ , at least  $a, b$  and  $c$  are included in  $X$  but, in this  
 261 case,  $w$  already contributes with 3 for the in-degree of  $X$ . Thus  $d_D^-(X) \geq 6$ .

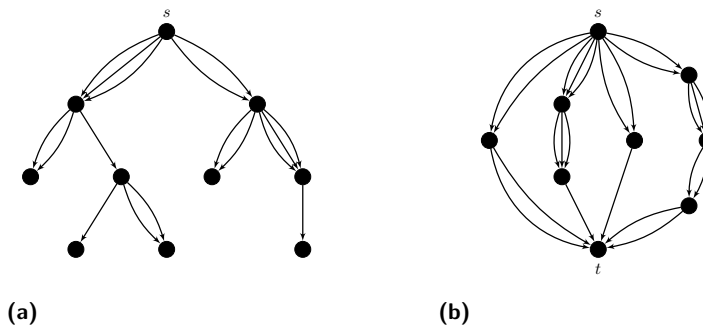
262 If  $D[X]$  has more than 2 weak components we already know that  $d_D^-(X) \geq 6$ . If  $D[X]$   
 263 has two weak components  $Y, Z$ , one of them, say  $Y$ , has at least  $\lambda + 1$  vertices and hence, as  
 264 in the previous cases,  $d_D^-(Y) \geq 4$ . Thus,  $d^-(X) \geq 6$ .

265 Notice that if there are  $k$  arc-disjoint  $s$ -branching flows  $x_1, x_2, \dots, x_k$  in  $\mathcal{N}$ , as  $d_D^-(B) =$   
 266  $d_D^-(C) = k$ , each one of these  $k$  arcs has to be used by a different flow, and carry at least  
 267  $\lambda - 1$  units of flow, that is, one unit for each vertex of  $B$  and  $C$ . Now observe that  $k/2$  of the  
 268 arcs which enter in  $B$  come from  $a$  and the same for the ones that enter  $C$ , and then  $a$  must  
 269 receive  $\lambda$  units of flow on each of its  $k$  incoming arcs in order to send the proper amount  
 270 of flow to  $B$  and  $C$ . Assume without loss of generality that the  $k/2$  copies of  $sa$  are going  
 271 to be used by  $x_1, x_2, \dots, x_{k/2}$  and the  $k/2$  copies of  $fa$  are left for  $x_{k/2+1}, x_{k/2+2}, \dots, x_k$ .  
 272 Since the in-degree of  $f$  is also equal to  $k$ ,  $f$  can receive at most  $\lambda$  units of flow of each  
 273  $x_1, x_2, \dots, x_k$  and hence it can send at most  $\lambda - 1$  units of any of these flows to  $a$ . Therefore,  
 274  $\mathcal{N}$  does not admits  $k$  arc-disjoint  $s$ -branching flows. ◀

275 **3 Networks on  $(k, \lambda, s)$ -sufficient digraphs containing the flows**

276 In this section, we show some positive results regarding  $(k, \lambda, s)$ -sufficiency. We remark  
 277 that the constructive proofs of those results can be naturally adapted to polynomial-time  
 278 algorithms to find the desired flows in those networks.

279 We say that a digraph  $D = (V, A)$  is a *multi-branching* if  $D$  is an out-branching when  
 280 we ignore its parallel arcs and, for  $\ell \geq 1$ , we denote by  $\mathcal{D}_\ell$  the class of *multi-spindles*. That  
 281 is, the class of digraphs formed by a source vertex  $s$ , a sink vertex  $t$ ,  $\ell$  pairwise internally  
 282 vertex-disjoint multi-paths  $P_1, \dots, P_\ell$  from  $s$  to  $t$ , each with  $p_i \geq 1$  internal vertices,  $1 \leq i \leq \ell$ ,  
 283 respectively. See Figures 3a and 3b for an example of a multi-branching and an example of a  
 284 multi-spindle, respectively.



■ **Figure 3** Examples of multi-branching (a) and multi-spindle (b)



285 ► **Theorem 5.** *Let  $D$  be a  $(k, \lambda, s)$ -sufficient multi-branching. Then the network  $\mathcal{N} = (D, u)$ ,*  
 286 *with  $u \equiv \lambda$ , admits  $k$  arc-disjoint  $s$ -branching flows.*

287 The following result is useful for proving Theorem 5.

288 ► **Lemma 6.** *Let  $D$  be a  $(k, \lambda, s)$ -sufficient digraph and let  $W$  be a nonempty subset of*  
 289  *$V(D) \setminus \{s\}$  such that  $N_D^+(V(D) \setminus W) = \{w\}$ ,  $w \in W$ . Then  $D[W]$  is  $(k, \lambda, w)$ -sufficient.*

290 **Proof.** Since  $w$  is the unique vertex of  $W$  which has in-neighbours in  $V(D) \setminus W$ , then we  
 291 have  $d_{D[W]}^-(v) = d_D^-(v)$  for every  $v \in W \setminus \{w\}$ . Thus, for every  $X \subseteq W \setminus \{w\}$  we have  
 292  $d_{D[W]}^-(X) = d_D^-(X)$ . ◀

293 **Proof of Theorem 5.** We are going to use induction on the height  $h$  of the multi-branching  
 294  $D$  with root  $s$  to construct  $k$  arc-disjoint  $s$ -branching flows  $f_1, f_2, \dots, f_k$  in  $\mathcal{N}$ .

295 The base case is  $h = 1$ . In this case  $D$  is a star and, since it is  $(k, \lambda, s)$ -sufficient, there  
 296 are at least  $k$  arcs from  $s$  to every other vertex and we can use one arc for each flow  $f_j$ , for  
 297  $j \in [k]$ .

298 Induction step: Suppose that the statement holds for multi-branchings of height  $h < q$ .  
 299 Assume that  $D$  has height  $q$  and let  $r_1, r_2, \dots, r_p$  be the out-neighbours of  $s$  in  $D$ .

300 Let  $B_{r_i}$  be the subgraph of  $B_s$  that is a multi-branching with root  $r_i$ , for each  $i \in [p]$ .  
 301 Observe that, by Lemma 6,  $B_{r_i}$  is  $(k, \lambda, r_i)$ -sufficient and has height  $h_i \leq q - 1$ , thus, by  
 302 induction hypothesis,  $B_{r_i}$  has  $k$  arc-disjoint  $r_i$ -branching flows  $f_1^i, f_2^i, \dots, f_k^i$ . We know that  
 303  $d^-(r_i) \geq k \lceil \frac{|B_{r_i}|}{\lambda} \rceil$  and then we can use  $\lceil \frac{|B_{r_i}|}{\lambda} \rceil$  of these arcs to send the proper amount of  
 304 flow from  $s$  to  $r_i$  on each  $x_j$  to complete the  $k$   $s$ -branching flows. ◀

305 Since a multi-path is also a multi-branching, Theorem 5 generalizes the result of [7] for  
 306 multi-paths.

307 ► **Theorem 7.** *If  $D$  is a  $(k, \lambda, s)$ -sufficient digraph in  $\mathcal{D}_\ell$ , then the network  $\mathcal{N} = (D, u)$  with*  
 308  *$u \equiv \lambda$  admits  $k$  arc-disjoint  $s$ -branching flows.*

309 **Proof.** Let  $D$  be a digraph in  $\mathcal{D}_\ell$  that is  $(k, \lambda, s)$ -sufficient. By definition,  $D$  is composed  
 310 by the multi-paths  $P_1, P_2, \dots, P_\ell$  and  $V(P_i) \cap V(P_j) = \{s, t\}$  for every  $1 \leq i < j \leq \ell$ . We  
 311 denote the  $p_i$  internal vertices of  $P_i$  by  $v_{p_i}^i, v_{p_i-1}^i, \dots, v_2^i, v_1^i$  in this order, for every  $i \in [\ell]$ . For  
 312 every  $j \in [p_i]$ , we define  $r_j^i \geq 0$  so that  $d_D^-(v_j^i) = k \lceil j/\lambda \rceil + r_j^i$ . Remark that  $d_D^-(v_j^i) \geq k \lceil j/\lambda \rceil$ ,  
 313 otherwise the set  $\{v_j^i, v_{j-1}^i, \dots, v_1^i\}$  would contradict the fact that  $D$  is  $(k, \lambda, s)$ -sufficient,  
 314 since the only arcs entering in it are those arriving at  $v_j^i$ . Considering only the vertices with  
 315 an index multiple of  $\lambda$ , that is, the vertices  $v_{j\lambda}^i$  for every  $1 \leq j \leq \lfloor p_i/\lambda \rfloor$ , let

$$316 \quad r(P_i) = \begin{cases} k, & \text{if } p_i < \lambda, \\ \min\{r_{j\lambda}^i \mid 1 \leq j \leq \lfloor p_i/\lambda \rfloor\}, & \text{otherwise.} \end{cases}$$

317 For all  $i \in [\ell]$ , we define  $e_i$  as the number of arcs from  $v_1^i$  to  $t$  and  $m_i = \min\{r(P_i), e_i\}$ .  
 318 Let  $k' = \sum_{i=1}^{\ell} m_i$ .

319 Assume without loss of generality that  $P_1, P_2, \dots, P_q$  are the paths on which  $r(P_i) \leq e_i$ , for  
 320  $i \in [q]$ . Then  $k' = \sum_{i=1}^q r(P_i) + \sum_{i=q+1}^{\ell} e_i$ . Now, for every  $i \in [q]$ , we choose a  $j_i \in [\lfloor p_i/\lambda \rfloor]$   
 321 which satisfies  $d_D^-(v_{j_i\lambda}^i) = k j_i + r(P_i)$  and we define  $X_i = \{v_{j_i\lambda}^i, v_{(j_i\lambda)-1}^i, \dots, v_1^i\}$ . Consider  
 322 the set  $X = \bigcup_{i=1}^q X_i \cup \{t\}$ . We have that

$$323 \quad d_D^-(X) = \sum_{i=1}^q d_D^-(X_i) + \sum_{i=q+1}^{\ell} e_i = k \sum_{i=1}^q j_i + \sum_{i=1}^q r(P_i) + \sum_{i=q+1}^{\ell} e_i = k \sum_{i=1}^q j_i + k'. \quad (2)$$

324 Since  $D$  is  $(k, \lambda, s)$ -sufficient and  $|X_i| = j_i \lambda$

$$325 \quad d_D^-(X) \geq k \left\lceil \frac{|X|}{\lambda} \right\rceil = k \left\lceil \frac{1}{\lambda} + \frac{\lambda \sum_{i=1}^q j_i}{\lambda} \right\rceil = k \sum_{i=1}^q j_i + k. \quad (3)$$

326 Combining (2) and (3) we conclude that  $k' \geq k$ . To finally construct the  $k$  arc disjoint  
327 branching flows, we need the following claim.

328  $\triangleright$  **Claim 8.** There are  $k$  arc-disjoint flows  $x_1^i, \dots, x_k^i$  on the network  $\mathcal{N}_i = (P_i, u)$ , with  
329  $u \equiv \lambda$ , such that  $x_1^i, \dots, x_{m_i}^i$  are branching flows in  $\mathcal{N}_i$  and  $x_{m_i+1}^i, \dots, x_k^i$  are branching  
330 flows in  $\mathcal{N}_i - t$ .

331 *Proof.* We show how to construct the flows  $x_1^i, x_2^i, \dots, x_k^i$ . For every  $j \in [p_i]$ , the vertex  $v_j^i$   
332 must receive  $j + 1$  units of flow on the first  $m_i$  flows and  $j$  units on remaining  $k - m_i$  flows.  
333 Since  $d_D^-(v_j^i) = d_{P_i}^-(v_j^i) \geq k \lceil j/\lambda \rceil$ , we can use a distinct group of  $\lceil j/\lambda \rceil$  arcs to send  $j$  units  
334 of flow on each one of the  $k$  flows and we only have to argue how to send the extra unit  
335 for the flows  $x_1^i, \dots, x_{m_i}^i$ . If  $j$  is a multiple of  $\lambda$ , then there are at least another  $r(P_i)$  extra  
336 arcs entering  $v_j^i$  and since  $m_i \leq r(P_i)$  they can be used to send the extra units (note that, if  
337  $r(P_i) = 0$  then  $m_i = 0$  and no extra unit is needed). Otherwise,  $j$  is not multiple of  $\lambda$  and  
338 when we send  $j$  units of flow through  $\lceil j/\lambda \rceil$  arcs, there is an arc that will not be used in  
339 the maximum capacity and so we can use it to send the extra unit. Applying this method  
340 iteratively from  $v_{p_i}^i$  to  $v_1^i$ , we arrive at  $v_1^i$  with 2 units of flow on the flows  $x_1^i, \dots, x_{m_i}^i$  and we  
341 can send the extra unit of each flow to  $t$  because  $m_i \leq e_i$ .  $\triangleleft$

342 For each  $i \in [\ell]$ , we compute the flows  $x_1^i, x_2^i, \dots, x_k^i$  as in the Claim 8. Observe that  $k'$  is  
343 the number of flows that reach  $t$ . If  $k' = k$ , then we can rename these  $k$  flows in such a way  
344 that each  $x_j = \bigcup_{i=1}^{\ell} x_j^i$  is an  $s$ -branching flow on  $\mathcal{N}$ . Finally, if  $k' > k$ , we take a flow that  
345 reaches  $t$  and we modify it so that it doesn't reach  $t$  anymore, and we repeat this process  
346 until there are only  $k$  flows that reach  $t$ .  $\blacktriangleleft$

347 It is worth to notice that the above proofs for multi-branchings and multi-spindles together  
348 with Proposition 2, besides giving a complete characterization of the digraphs in these classes  
349 having  $k$  arc-disjoint branching flows, they lead to polynomial-time algorithms to find such  
350 flows, once testing the  $(k, \lambda, s)$ -sufficiency for multi-branchings and multi-spindles can be  
351 done in polynomial time. The algorithms work for every value of  $\lambda$ , even those for which the  
352 problem of finding  $k$  arc-disjoint branching flows is known to be hard in general.

353 We now consider the cases of  $k$  or  $\lambda$  with value 1 and for that we need the three following  
354 results.

355  $\blacktriangleright$  **Theorem 9 (Max-flow min-cut).** *In any network  $\mathcal{N} = (D, u)$  with source  $s$  and sink  $t$ , the  
356 value of a maximum flow is equal to the capacity of a minimum cut.*

357  $\blacktriangleright$  **Theorem 10.** [2] *Let  $k$  be an integer and  $\mathcal{N} = (D, u)$  be a network with  $u \equiv 1$  and a  
358 prescribed balance vector  $b$  such that  $b \not\equiv 0$ . There exist  $k$  arc-disjoint flows in  $\mathcal{N}$ , all with  
359 balance vector  $b$ , if and only if  $\mathcal{N}$  has a flow  $f$  with balance vector  $b_f \equiv kb$ . Hence one can  
360 decide the existence of these flows in polynomial time.*

361  $\blacktriangleright$  **Lemma 11 (Adapted from [3]).** *Given a network  $\mathcal{N} = (D, u)$  and a prescribed balance  
362 vector  $b$ . Let  $M = \sum_{v: b(v) > 0} b(v)$  and let  $\mathcal{N}' = (D', u')$  be a network defined as follows:*

363  $\blacksquare$   $V(D') = V(D) \cup \{s', t'\};$

364  $\blacksquare$   $A(D') = A(D) \cup \{s'u : u \in V(D), b(u) > 0\} \cup \{vt' : v \in V(D), b(v) < 0\};$

## 23:10 A study of an Edmonds' like property to arc-disjoint branching flows

365 ■  $u'(a) = u(a), \forall a \in A(D), u'(s'u) = b(u)$  if  $b(u) > 0$  and  $u'(vt') = -b(v)$  if  $b(v) < 0$ .

366 Then,  $\mathcal{N}$  admits a flow  $f$  with balance  $b$  if and only if  $\mathcal{N}'$  admits a  $(s', t')$ -flow  $f'$  with  
367 value  $M$ .

368 ► **Theorem 12.** Let  $D$  be a  $(k, \lambda, s)$ -sufficient digraph. Then, if  $\lambda = 1$  or  $k = 1$ , the network  
369  $\mathcal{N} = (D, u)$  with  $u \equiv \lambda$  admits  $k$  arc-disjoint  $s$ -branching flows.

370 **Proof.** We start by showing a result that is common to both cases. Let  $b_f$  be the following  
371 balance vector:  $b_f(s) = k(n - 1)$  and  $b_f(v) = -k$ , for all  $v \in V(D) - s$ , where  $n = |V(D)|$ .  
372 Let  $\mathcal{N}'$  be the network obtained from  $\mathcal{N}$  and  $b_f$ , as described in Lemma 11, and  $(S, \bar{S})$  be  
373 a minimum  $(s', t')$ -cut in  $\mathcal{N}'$ . We show that  $u(S, \bar{S}) = k(n - 1)$  which, by the Theorem 9,  
374 implies that there is an  $(s', t')$ -flow with value  $k(n - 1)$  in  $\mathcal{N}'$ . Remark that, by construction  
375 of  $\mathcal{N}'$ ,  $u'(s's) = k(n - 1)$ ,  $u'(vt') = k$  for every  $v \in D \setminus s$  and  $u'(a) = \lambda$  for every  $a \in A(D)$ .  
376 It is easy to see that  $u(S, \bar{S}) \leq k(n - 1)$  because  $u(s', V(D') - s') = k(n - 1)$ . Then, we  
377 might assume that  $s \in S$  and we define  $X = \bar{S} \setminus \{t'\}$ . It follows that

$$378 \quad u(S, \bar{S}) = u(S, X) + u(S, \{t'\}) = \lambda d_D^-(X) + k(n - |X| - 1). \quad (4)$$

379 Case  $\lambda = 1$ . Since  $D$  is  $(k, 1, s)$ -sufficient,  $d_D^-(X) \geq k|X|$  and resuming (4), we have  
380  $u(S, \bar{S}) \geq k|X| + k(n - |X| - 1) = k(n - 1)$ . Thus,  $u(S, \bar{S}) = k(n - 1)$  and, by the Lemma 11,  
381  $\mathcal{N}$  admits a flow with balance  $b_f$ . The result follows from Theorem 10 which states that  $\mathcal{N}$   
382 admits  $k$  arc-disjoint  $s$ -branching flows if and only if it admits a single flow  $f$  with balance  
383  $b_f$ .

384 Case  $k = 1$ . Here,  $D$  is  $(1, \lambda, s)$ -sufficient, which means that  $d_D^-(X) \geq (|X|/\lambda)$ , and so  
385  $\lambda d_D^-(X) \geq |X|$ . Replacing this in (4), we obtain  $u(S, \bar{S}) \geq |X| + n - |X| - 1 = n - 1$ . Thus,  
386  $u(S, \bar{S}) = k(n - 1)$ . Similarly to the previous case, we conclude, by the Lemma 11, that  $\mathcal{N}$   
387 admits a flow  $f$  with balance  $b_f$ . As  $k = 1$ , we have that  $f$  is a branching flow and the result  
388 follows. ◀

389 We also show that networks on  $(2, n - 2, s)$ -sufficient digraphs contain 2 arc-disjoint  
390  $s$ -branching flows. This is a slightly improvement on the result by [2] when  $k = 2$ , which  
391 states that Edmonds' characterizaion of digraphs containing 2 arc-disjoint  $s$ -branchings also  
392 characterizes the existence of 2 arc-disjoint  $s$ -branching flows.

393 ► **Theorem 13.** Let  $\mathcal{N} = (D, u)$  be a  $(2, n - 2, s)$ -sufficient network with  $u \equiv n - 2$ . Then  
394  $\mathcal{N}$  admits 2 arc-disjoint  $s$ -branching flows.

395 **Proof.** Since  $\mathcal{N} = (D, u)$  is a  $(2, n - 2, s)$ -sufficient network, then for every  $X \subset V(D) \setminus \{s\}$   
396 we have  $d_D^-(X) \geq 2$ , and for  $X = V(D) \setminus \{s\}$  we have  $d_D^-(X) \geq 4$ . By Edmonds' Theorem,  
397  $D$  has two arc-disjoint  $s$ -branchings. Let us call these  $s$ -branchings  $B^b$  and  $B^g$  and let us  
398 colour blue the arcs used by  $B^b$ , green the arcs used by  $B^g$ , and black the arcs of  $D$  not used  
399 by any of those branchings (observe that every vertex in  $V(D) \setminus \{s\}$  is the head of exactly  
400 one blue and exactly one green arc). The idea is to construct the desired  $s$ -branching flows  
401 using the arcs of  $B^b$  and  $B^g$ . This will be possible unless we are forced to send  $n - 1$  units  
402 of flow through an arc, which can only happen from an arc that leaves  $s$ . In this particular  
403 case, we show how to modify the branchings to avoid the problem.

404 Let  $r_1^b, \dots, r_\ell^b$  be the out-neighbours of  $s$  in  $B^b$  and let  $B_1^b, \dots, B_\ell^b$  be the maximal sub-  
405 branchings of  $B^b \setminus \{s\}$  with roots  $r_1^b, \dots, r_\ell^b$ , respectively. If  $\ell > 1$ , then we can construct a  
406 branching flow  $f_b$  from  $B^b$  by sending  $|B_i^b|$  units of flow through  $sr_i^b$ , for  $1 \leq i \leq \ell$ , which  
407 is possible because  $u \equiv n - 2$ . The same applies to  $B^g$  and we will exchange  $b$  to  $g$  in the  
408 above notation when it concerns  $B^g$  in the rest of the proof.

409 If  $s$  has only one out-neighbour  $r^b$  in  $B^b$ , but there is a black arc  $sv$ , we can construct  
 410  $f_b$  from  $B^b \cup sv$ . We have to consider two cases: if  $v \neq r^b$ , let  $uv \in A(B^b)$  and let  $B_v^b$  be  
 411 the maximal connected subgraph of  $B^b \setminus \{u\}$  with root  $v$ . Then we send  $|B_v^b|$  units of flow  
 412 through  $sv$  and  $|B^b \setminus \{s\}| - |B_v^b|$  through  $sr^b$ . Otherwise,  $v = r^b$  and we have two copies of  
 413 the arc  $sv$ . So we can send 1 unit of flow through one and  $n - 2$  through the other. In both  
 414 cases, for the remaining blue arcs, we just need to adjust the proper amount of flow. The  
 415 same applies to  $B^g$ .

416 Thus we can assume w.l.o.g. that every out-neighbour of  $s$  in  $D$  is either blue or green  
 417 and, from such arcs, exactly one is blue ( $sr^b$ ) and the remaining are green. Notice that, since  
 418  $d_D^-(V(D) \setminus \{s\}) \geq 4$ , there must be at least three green arcs from  $s$  to  $D \setminus \{s\}$ .

419 Since  $B^g$  is a  $s$ -branching,  $r^b$  appears in only one  $B_i^g$ , and let us consider a  $r_j^g$  for which  
 420  $B_j^g \neq B_i^g$ . Except for  $sr_j^g$ , there are no green arcs entering or leaving  $B_j^g$ . Then, let  $E$  be the  
 421 set of blue arcs entering  $V(B_j^g)$  together with all blue arcs between vertices in  $V(B_j^g)$ . We  
 422 now recolour blue the arcs in  $A(B_j^g) \cup sr_j^g$  and yellow the arcs in  $E$ . After this recolouring,  
 423 a vertex  $w \neq s$  is the head of exactly one blue arc and one yellow arc if it is in  $V(B_j^g)$ ,  
 424 otherwise it is still the head of exactly one blue and one green arc. This means that the  
 425 vertices of  $V(D)$  along with the blue arcs will form a  $s$ -branching  $B^{b'}$ . In particular, there  
 426 are two blue arcs leaving  $s$  in  $B^{b'}$  and we can construct the branching flow  $f_{b'}$  as before.

427 We now recolour the arcs of  $E$  green. Again, we have the guarantee that every vertex in  
 428  $B_j^g$  has exactly one green arc arriving at it (the former yellow arcs), and, since the green arcs  
 429 outside  $A(B_j^g) \cup sr_j^g$  remain unaltered, the new set of all green arcs also form a  $s$ -branching  
 430  $B^{g'}$ . Once more, we have at least two arcs leaving  $s$  in  $B^{g'}$ , and we can construct  $f_{g'}$  as  
 431 before. ◀

432 We remark that, by a result of [4], deciding if a network with  $u \equiv n - 2$  admits arc-disjoint  
 433 flows can be done in polynomial time. Since Theorem 13 completes the proof that a network  
 434 is  $(2, n - 2, s)$ -sufficient if and only if it admits 2 arc-disjoint  $s$ -branching flows (together  
 435 with Proposition 2), we get that it can be decided in polynomial time whether a digraph is  
 436  $(2, n - 2, s)$ -sufficient.

437 We end this section showing that, for particular choices of  $d_D^-(X)$  and  $\lambda$ , a simple stronger  
 438 condition than the one presented in the statement of Proposition 2 guarantees the existence  
 439 of  $k$  arc-disjoint  $s$ -branching flows in a given network.

440 ▶ **Theorem 14.** *Let  $\mathcal{N} = (D, u)$  be a network with  $u \equiv \lambda$  and  $\lambda \geq 2$ , and  $p$  be an integer  
 441 such that  $p\lambda \geq n - 1$ . If  $d_D^-(X) \geq pk$  for all  $X \subseteq V(D) \setminus \{s\}$  with  $X \neq \emptyset$ , then  $\mathcal{N}$  admits  $k$   
 442 arc-disjoint  $s$ -branching flows.*

443 **Proof.** By Theorem 1, there are  $pk$  arc-disjoint  $s$ -branchings  $B_1, \dots, B_{pk}$  in  $D$ . For  $i \in [k]$ ,  
 444 let  $D_i$  be the digraph with  $V(D_i) = V(D)$ , with  $A(D_i) = A(B_{(i-1)p+1}) \cup \dots \cup A(B_{ip})$ , and  
 445  $\mathcal{N}_i = (D_i, u)$ .

446 Now, for every non-empty  $X \subseteq V(D) \setminus \{s\}$ , we conclude that  $d_{D_i}^-(X) \geq p$  since each  $B_i$   
 447 with  $i \in [pk]$  must reach  $X$  at least once. Moreover, by our choice of  $p$ , we have that every  
 448  $\mathcal{N}_i$  is  $(1, \lambda, s)$ -sufficient. Thus, by Theorem 12, we have that each  $\mathcal{N}_i$  admits an  $s$ -branching  
 449 flow  $x_i$ . Finally, as every arc of  $D$  appears in at most one digraph  $D_i$ , we conclude that the  
 450  $s$ -branching flows  $x_1, \dots, x_k$  are pairwise arc-disjoint. The result follows since each of these  
 451 flows can be promptly used to construct an  $s$ -branching flow in  $\mathcal{N}$ , by copying arcs and flow  
 452 functions used by each flow  $x_j$  with  $j \in [k]$ . ◀

453 **4 Hardness of finding the flows in  $(k, \lambda, s)$ -sufficient networks**

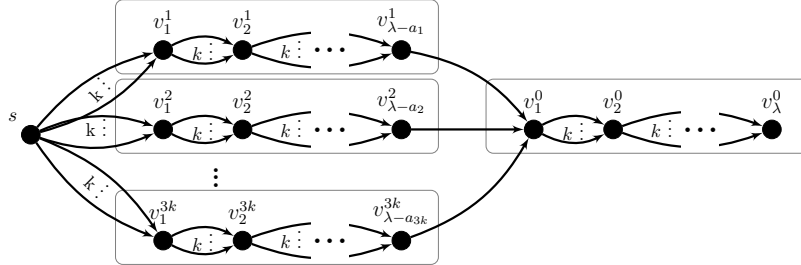
454 We showed in Sections 2 and 3 that there are networks  $\mathcal{N} = (D, u)$ , with  $D$  being  $(k, \lambda, s)$ -  
 455 sufficient, which have  $k$  arc-disjoint  $s$ -branching flows and others that do not have such flows.  
 456 We believe it is not always computationally easy to check the  $(k, \lambda, s)$ -sufficiency of a digraph,  
 457 but, even if we know that  $D$  has such property, it is hard to decide if the desired flows exist,  
 458 even if the network is constructed on a digraph formed by appending a path to the sink  
 459 vertex of a multi-spindle.

460 **► Theorem 15.** *Given two integers  $k$  and  $\lambda$  it is NP-complete to decide whether a  $(k, \lambda, s)$ -*  
 461 *sufficient network  $N = (D, u)$  with  $u \equiv \lambda$  admits  $k$  arc-disjoint  $s$ -branching flows.*

462 **Proof.** We are going to reduce the 3-PARTITION problem to our problem. Given a set of  
 463 natural numbers  $S = \{a_1, a_2, \dots, a_{3k}\}$  and a positive integer  $\lambda$  such that  $\lambda/4 < a_i < \lambda/2$ , for  
 464 each  $a_i \in S$ , and such that  $\sum_{i=1}^{3k} a_i = k\lambda$ , the 3-PARTITION problem consists in deciding the  
 465 following: can  $S$  be partitioned in  $k$  subsets  $S_1, S_2, \dots, S_k$  so that  $\sum_{a_j \in S_i} a_j = \lambda$ ,  $1 \leq i \leq k$ ?  
 466 Observe that each  $S_i$  should be composed by exactly 3 elements of  $S$ .

467 From  $S$ , we construct a network  $\mathcal{N} = (D, u)$  such that  $D$  is a  $(k, \lambda, s)$ -sufficient network  
 468 that admits  $k$  arc-disjoint branching flows  $x_1, x_2, \dots, x_k$  if and only if  $S$  is a *yes* instance of  
 469 the 3-PARTITION problem.

470 We begin the construction by adding a vertex  $s$ , a multi-path  $P_0 = (v_1^0, v_2^0, \dots, v_\lambda^0)$  and,  
 471 for every  $i \in [3k]$ , multi-paths  $P_i = (v_1^i, v_2^i, \dots, v_{\lambda-a_i}^i)$ , each one with  $k$  parallel arcs between  
 472 consecutive vertices. Also, for every  $i \in [3k]$ , we add one arc  $v_{\lambda-a_i}^i v_1^0$  and  $k$  parallel arcs  $sv_1^i$ .  
 473 See Figure 4. Finally, we set  $u \equiv \lambda$  to conclude the construction.



474 **■ Figure 4** The network  $\mathcal{N} = (D, c)$  constructed from an instance of the 3-PARTITION problem

474 First, let us check that  $D$  is  $(k, \lambda, s)$ -sufficient. By contradiction, assume that there  
 475 exists a set  $X \subseteq V(D) - s$  such that  $d_D^-(X) < k \lceil |X|/\lambda \rceil$ . Note that  $X \neq V(D) - s$  since  
 476  $|V - s| = (2k + 1)\lambda$  and  $d_D^-(V - s) = k(3k)$ . Recall that the in-degree of every vertex of  $P_i$   
 477 is  $k$  and thus the in-degree of  $X$  is at least  $k$ . Therefore  $X$  cannot be entirely contained  
 478 in a unique  $P_i$ ,  $i \in [3k]$ , once, in this case,  $|X| \leq \lambda$  and then  $\lceil |X|/\lambda \rceil = 1$ . Similarly,  
 479  $X$  cannot be entirely contained in  $P_0$ , since there are  $3k$  arcs arriving in  $v_1^0$ . Now, let  
 480  $I = \{i : 1 \leq i \leq 3k, V(P_i) \cap X \neq \emptyset\}$ . Observe that  $d_D^-(X) \geq k|I|$ . We are going now to  
 481 consider the following cases:

- 482 (i)  $V(P_0) \cap X = \emptyset$ . Then  $|X| \leq \lambda|I|$  and  $\lceil |X|/\lambda \rceil \leq |I|$ , so  $d_D^-(X) \geq k|I| \geq k \lceil |X|/\lambda \rceil$ , a  
 483 contradiction to the choice of  $X$ .
- 484 (ii)  $X \cap V(P_0) \neq \emptyset$ . Then  $|X| \leq \lambda|I| + \lambda = \lambda(|I| + 1)$ . Take the smallest  $j$  such that  $v_j^0 \in X$ .  
 485 We analyze two possibilities:

- 486 (ii.1)  $j > 1$ . Then there are  $k$  arcs  $v_{j-1}^0 v_j^0$  that enter  $X$ , that is,  $d^-(X) \geq k|I| + k =$   
 487  $k(|I| + 1) \geq k \lceil |X|/\lambda \rceil$  since  $|I|$  is a positive integer and  $|X| \leq \lambda(|I| + 1)$ , a  
 488 contradiction.
- 489 (ii.2)  $j = 1$ . Therefore  $v_1^0 \in X$  and  $d^-(X) \geq k|I|$ . Again, we consider some cases:
- 490 (ii.2.1) If there are at least  $k$  different values of  $i$  for which  $v_{\lambda-a_i}^i \notin X$ , then  
 491 such vertices contribute with  $k$  to the in-degree of  $X$ , and again we have  
 492  $d^-(X) \geq k|I| + k$ .
- 493 (ii.2.2) If there is only  $1 \leq l < k$  different values of  $i$  for which  $v_{\lambda-a_i}^i \notin X$ ,  
 494 then, in order to complete the argument, we need to estimate more  
 495 precisely the size of  $X$ . Recall that  $\lambda/4 < a_i < \lambda/2$ . So,  $|X| \leq \lambda|I| -$   
 496  $\frac{\lambda|I|}{4} + \lambda = \lambda(\frac{3|I|}{4} + 1)$ . Observe that the amount of arcs entering  $X$  is  
 497  $k|I| + l = \frac{3k|I|}{4} + \frac{k|I|}{4} + l$ . Since, in this case,  $|I| > 2k$ , we have that  
 498  $\frac{k|I|}{4} \geq \frac{k^2}{2} > k$  if  $k > 2$ , which we can consider true by the definition of  
 499 the 3-PARTITION problem ( $k < 3$  would be directly a negative instance).  
 500 Then,  $d^-(X) \geq \frac{3k|I|}{4} + k = k(\lambda(\frac{3|I|}{4} + 1)/\lambda)$ , a contradiction.
- 501 (ii.2.3) If all the last vertices are in  $X$ , then  $d^-(X) \geq d^-(V - s)$ , and we have  
 502 another contradiction, since  $|X| < |V(D) - s|$ .

503 As we got a contradiction on each case, it follows that  $D$  is  $(k, \lambda, s)$ -sufficient.

504 Now we assume that there exists  $k$  subsets  $S_1, S_2, \dots, S_k$  so that  $\sum_{a_j \in S_i} a_j = \lambda$ ,  $1 \leq i \leq k$ .  
 505 In order to construct the arc-disjoint  $s$ -branching flows  $x_1, x_2, \dots, x_k$  in  $\mathcal{N}$ , we establish that  
 506 each of the  $k$  copies of a multiple arc belongs to a different flow.

507 For every  $1 \leq i \leq k$ , we construct  $x_i$  in the following way: If  $a_j \notin S_i$ , the arc  $sv_1^j$  of  $x_i$   
 508 will carry  $\lambda - a_j$  units of flow. If  $a_j \in S_i$ , the arc  $sv_1^j$  of  $x_i$  will carry  $\lambda$  units of flow. Since  
 509  $\sum_{a_j \in S_i} a_j = \lambda$ , the three arcs  $v_{\lambda-a_j}^j$  for  $a_j \in S_i$  will send the amount of flow needed to  
 510 complete the branching flow in  $P_0$ .

511 Finally, we assume that there exist  $k$  arc-disjoint  $s$ -branching flows  $x_1, x_2, \dots, x_k$  in  
 512  $\mathcal{N}$ . We first claim that  $x_r(v_{\lambda-a_i}^i v_1^0) \leq a_i$ , for all  $r \in [k], i \in [q]$ . Since  $x_1, x_2, \dots, x_k$  are  
 513 arc-disjoint, there is only one flow  $x_t$  which is positive in  $v_{\lambda-a_i}^i$ . If  $x_t(v_{\lambda-a_i}^i v_1^0) > a_i$ , then, as  
 514  $|V(P_i)| = \lambda - a_i$ ,  $v_1^i$  must receive more than  $\lambda$  units of flow in  $x_t$ , that is,  $x_t$  uses at least  
 515 two copies of the arc  $sv_1^i$ . This is a contradiction because there are only  $k$  copies of  $sv_1^i$  and  
 516 each flow must use exactly one copy. Thus, the claim follows. Each  $x_r$  must reach  $v_1^0$  with  $\lambda$   
 517 units of flow. Since  $\lambda/4 < a_i < \lambda/2$ , each  $x_r$  should use three arcs of the type  $v_{\lambda-a_j}^j v_1^0$ , and  
 518 the correspondent  $a_j$  forms a  $S_r$ . Then  $S_1, S_2, \dots, S_k$  is an *yes* instance of the 3-PARTITION  
 519 problem. ◀

520 Observe that, by the proof of Theorem 15, the networks constructed from negative  
 521 instances of the 3-PARTITION problem are also examples that  $(k, \lambda, s)$ -sufficiency does not  
 522 guarantee the existence of  $k$  arc-disjoint  $s$ -branching flows.

## 523 5 Concluding remarks

524 In this work, we studied the characterization of networks admitting  $k$  arc-disjoint  $s$ -branching  
 525 flows. We showed that, in some cases (Theorems 5 and 7), an “Edmonds like” condition,  
 526 which we call  $(k, \lambda, s)$ -sufficiency, is enough to guarantee the existence of the  $k$  arc-disjoint  
 527  $s$ -branching flows but, in other cases, a stronger statement is needed (Section 2). This  
 528 proves that  $(k, \lambda, s)$ -sufficiency in general is not enough to guarantee the existence of the  
 529 desired  $s$ -branching flows in networks, although the cases for which it does are still interesting

530 because, among other reasons, they have been resulting in polynomial-time algorithms to find  
 531 the flows. Some of the positive results we prove consider restrictions on the structure of the  
 532 digraphs on which the networks are built: we show that the analyzed property guarantees the  
 533 existence of the flows on multi-spindles and on multi-branchings. In other words, we prove  
 534 that the flows always exist in  $(k, \lambda, s)$ -sufficient networks built on digraphs whose underlying  
 535 simple graphs are trees or spindles.

536 Another direction of this research would be to work on the characterization through a  
 537 stronger condition, as cited before. Some interesting complexity questions are still open,  
 538 such as a possible dichotomy in DAG's between the easy and hard cases. In other words, it  
 539 would be very interesting to know if there a class of digraphs  $\mathcal{H}$ , all of which are DAGs, such  
 540 that every  $(k, \lambda, s)$ -sufficient network constructed on a digraph  $D \in \mathcal{H}$  contains  $k$  arc-disjoint  
 541  $s$ -branching flows, but that the same is not true for every  $D \notin \mathcal{H}$ .

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