# Identifying Young diagrams among residue multisets

Salim Rostam

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**Abstract**

To any Young diagram we can associate the multiset of residues of all its nodes. This paper is concerned with the inverse problem: given a multiset of elements of $\mathbb{Z}/e\mathbb{Z}$, does it comes from a Young diagram? We give a full solution in level one and a partial answer in higher levels for Young multidiagrams, using Fayers's notions of core block and weight of a multipartition. We apply the result in level one to study a shift operation on partitions.

## 1 Introduction

Studying the representation theory of a finite group $G$ over a field $k$ of characteristic zero reduces to find the irreducible representations of $G$. If $G = \mathfrak{S}_n$ is the symmetric group on $n$ letters, the irreducible representations are indexed by the *partitions* of $n$, that is, non-increasing sequences of positive integers $\lambda = (\lambda_1 \geq \cdots \geq \lambda_h > 0)$ with sum $|\lambda| = n$. If the field $k$ is of positive characteristic $p$, some representations may not be written as a direct sum of irreducible ones. Hence, we are also interested in the *blocks* of the group algebra, that is, indecomposable two-sided ideals. These blocks are parametrised by the *$p$-cores* of the partitions of $n$, in particular, any block is uniquely determined by its $p$-core and its *$p$-weight*. Note that the set of irreducible representations can be parametrised by the *$p$-regular* partitions of $n$.

More generally, we can replace the symmetric group $\mathfrak{S}_n$ by a *complex reflection group*. The set of *irreducible* complex reflection groups consists of an infinite family $\{G(r,p,n)\}_{r,p,n}$ where $r, p, n$ are positive integers with $p \mid r$, and also a finite number of exceptions (see [ShTo]). The complex reflection group $G(r,1,n)$ is isomorphic to $(\mathbb{Z}/r\mathbb{Z}) \wr \mathfrak{S}_n \simeq (\mathbb{Z}/r\mathbb{Z})^n \rtimes \mathfrak{S}_n$ and can be seen as the set of $n \times n$ monomial matrices with non-zero entries in the set of complex $r$-roots of unity, while $G(r,p,n)$ is a certain subgroup of $G(r,1,n)$ of index $p$. We can then study the representation theory of a *Hecke algebra* $\mathcal{H}_n^{\mathbf{s}}(q)$ of $G(r,1,n)$, where $\mathbf{s} = (\mathsf{s}_1, \ldots, \mathsf{s}_r) \in \mathbb{Z}^r$ is a multicharge ([ArKo, BrMa, BMR]). The algebra $\mathcal{H}_n^{\mathbf{s}}(q)$ is a particular deformation of the group algebra of $G(r,1,n)$ and $q \in k$ is the deformation parameter. Assume that $q \in k \setminus \{0\}$ and let $e \geq 0$ be its multiplicative order (with $e := 0$ if $q$ is of infinite order). The representation theories of $\mathcal{H}_n^{\mathbf{s}}(q)$ and $G(r,1,n)$ are deeply linked. For instance, if $r = 1$ then $G(1,1,n) \simeq \mathfrak{S}_n$ and the situation is the following: if $\mathcal{H}_n^{\mathbf{s}}(q)$ is semisimple then its irreducible representations are indexed by the partitions of $n$, otherwise they are parametrised by the $e$-regular partitions of $n$ and the blocks are parametrised by the $e$-cores of partitions of $n$. In the general case $r \geq 1$, if $\mathcal{H}_n^{\mathbf{s}}(q)$ is semisimple then its irreducible representations are indexed by the *$r$-partitions* of $n$, that is, $r$-tuples $\boldsymbol{\lambda} = \left(\lambda^{(1)}, \ldots, \lambda^{(r)}\right)$ with $|\boldsymbol{\lambda}| = |\lambda^{(1)}| + \cdots + |\lambda^{(r)}| = n$. On the contrary, if $\mathcal{H}_n^{\mathbf{s}}(q)$ is non-semisimple then the situation is more complex. First, its irreducible representations can be indexed by a non-trivial generalisation of $e$-regular partitions, known as *Kleshchev* $r$-partitions (see [Ar01, ArMa]). Similarly,

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the naive generalisation of $e$-cores to $r$-partitions, the *$e$-multicores*, do not parametrise in general the blocks of $\mathcal{H}_n^{\mathbf{s}}(q)$. Namely, Lyle and Mathas [LyMa] proved that the blocks of $\mathcal{H}_n^{\mathbf{s}}(q)$ are parametrised by the multisets of $\mathbf{s}$-residues modulo $e$ of the $r$-partitions of $n$. To make things more explicit, let $Q = \oplus_{i\in\mathbb{Z}/e\mathbb{Z}}\mathbb{Z}\alpha_i$ be a free abelian group. If $\boldsymbol{\lambda}$ is a multipartition, for any $i \in \mathbb{Z}/e\mathbb{Z}$ we denote by $c_i^{\mathbf{s}}(\boldsymbol{\lambda})$ the number of $i$-nodes of $\boldsymbol{\lambda}$. Then the block *corresponding* to $\boldsymbol{\lambda}$ is

$$\alpha^{\mathbf{s}}(\boldsymbol{\lambda}) \coloneqq \sum_{i\in\mathbb{Z}/e\mathbb{Z}} c_i^{\mathbf{s}}(\boldsymbol{\lambda})\alpha_i \in Q,$$

and we say that $\boldsymbol{\lambda}$ *lies* in $\alpha^{\mathbf{s}}(\boldsymbol{\lambda})$. During their proof, Lyle and Mathas used a generalisation of the $e$-weight to $r$-partitions, introduced by Fayers [Fa06]. If $\boldsymbol{\lambda} = \left(\lambda^{(1)}, \dots, \lambda^{(r)}\right)$ is an $r$-partition, its $\mathbf{s}$-*weight* is given by

$$\mathsf{w}^{\mathbf{s}}(\boldsymbol{\lambda}) \coloneqq \sum_{j=1}^{r} c_{\mathsf{s}_j}^{\mathbf{s}}(\boldsymbol{\lambda}) - \frac{1}{2}\sum_{i\in\mathbb{Z}/e\mathbb{Z}} \left(c_i^{\mathbf{s}}(\boldsymbol{\lambda}) - c_{i+1}^{\mathbf{s}}(\boldsymbol{\lambda})\right)^2.$$

In particular, two $r$-partitions that lie in the same block (*i.e.* $\alpha^{\mathbf{s}}(\boldsymbol{\lambda}) = \alpha^{\mathbf{s}}(\boldsymbol{\mu})$) have the same $\mathbf{s}$-weight. Fayers proved that if $r = 1$ then this definition of $\mathbf{s}$-weight coincides with the usual notion of $e$-weight for partitions. For instance, if the partition $\lambda$ is an $e$-core we have:

$$c_0(\lambda) = \frac{1}{2}\sum_{i\in\mathbb{Z}/e\mathbb{Z}} \left(c_i(\lambda) - c_{i+1}(\lambda)\right)^2$$

(in level 1 we may omit to write the multicharge, assuming, without loss of generality, that $\mathsf{s} = 0$). Moreover, while the fact that the $e$-weight of a partition is non-negative follows from the definition, this is a non-trivial statement for the $\mathbf{s}$-weight in higher levels (see [Fa06]).

In level one, a partition is an $e$-core if and only if its $e$-weight is zero. In higher levels, we still have an implication: if an $r$-partition has $\mathbf{s}$-weight at most $r-1$ then it is an $e$-multicore. However, we can find $e$-multicores of arbitrary large $\mathbf{s}$-weight. Moreover, we can have two $r$-partitions $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ lying in the same block, thus having the same $\mathbf{s}$-weight, but with $\boldsymbol{\lambda}$ (respectively $\boldsymbol{\mu}$) being (resp. not being) an $e$-multicore. In order to obtain a more satisfying notion of core of a multipartition, Fayers [Fa07a] introduced the notion of *core block* and *reduced $e$-multicore* (the latter expression is due to [LyMa]): an $e$-multicore $\boldsymbol{\lambda}$ is $\mathbf{s}$-*reduced* if any multipartition $\boldsymbol{\mu}$ such that $\alpha^{\mathbf{s}}(\boldsymbol{\mu}) = \alpha^{\mathbf{s}}(\boldsymbol{\lambda})$ is an $e$-multicore, in which case the block associated with $\boldsymbol{\lambda}$ is a *core block*. If $r = 1$ then every $e$-core is $\mathsf{s}$-reduced and every block is a core block. Now if $r > 1$, to any multipartition we can still associate a unique core block, however there is no canonical choice for a reduced $e$-multicore inside this core block. Note that Jacon–Lecouvey [JaLe] managed to define what they have called the $(e, \mathbf{s})$-*core* of a multipartition (they also use the notion of *reduced* $(e, \mathbf{s})$-*core*, which is different from the notion of reduced $e$-multicore that we use here). The $(e, \mathbf{s})$-core of an $r$-partition is again an $r$-partition, and the situation is then entirely similar to the level one case, namely, for the combinatorics of blocks. However, the $(e, \mathbf{s})$-core of a multipartition is associated with a possibly different multicharge, which depends on the multipartition.

Now let $p \mid r$. We can use Clifford theory to study the representation theory of $G(r, p, n)$, and this involves the following natural *shift* operation of order $p$ on $r$-partitions:

$$\boldsymbol{\lambda} = \left(\lambda^{(1)}, \dots, \lambda^{(r)}\right) \longmapsto {}^{\sigma}\boldsymbol{\lambda} \coloneqq \left(\lambda^{(r-d+1)}, \dots, \lambda^{(r)}, \lambda^{(1)}, \dots, \lambda^{(r-d)}\right),$$

where $d \coloneqq \frac{r}{p}$. Typically, if $S^{\boldsymbol{\lambda}}$ is an irreducible $G(r, 1, n)$-module then the number of irreducible constituents of $S^{\boldsymbol{\lambda}}$ seen as an $G(r, p, n)$-module will only depend on the cardinality of the orbit of $\boldsymbol{\lambda}$ under the shift operation (see, for instance, [Ar95, ChJa, GeJa, HuMa]). Now take $e \geq 2$ and assume that $p$ also divides $e$. There is a map $\sigma : Q \to Q$ of order $p$ such that if $\mathbf{s} \in \mathbb{Z}^r$ is a multicharge satisfying some compatibility conditions, we have the relation

$$\alpha^{\mathbf{s}}\left({}^{\sigma}\boldsymbol{\lambda}\right) = \sigma \cdot \alpha^{\mathbf{s}}(\boldsymbol{\lambda}). \tag{1.1}$$

In [Ro19a], the author proved that if $r \geq 2$ then a block $\alpha \in Q^{\mathsf{s}}$ that is *stuttering*, that is $\sigma \cdot \alpha = \alpha$, always corresponds to a *stuttering* multipartition, that is, a multipartition $\boldsymbol{\lambda}$ satisfying ${}^{\sigma}\boldsymbol{\lambda} = \boldsymbol{\lambda}$.

The aim of this paper is to study the natural generalisation of Fayers's weight function to $Q = \oplus_{i \in \mathbb{Z}/e\mathbb{Z}} \mathbb{Z}\alpha_i$, and then apply this result in level one to define a shift operation on partitions so that a relation such as (1.1) holds. More precisely, let $\mathsf{s} \in \mathbb{Z}^r$ be a multicharge and for any $\alpha = \sum_{i \in \mathbb{Z}/e\mathbb{Z}} c_i \alpha_i \in Q$ define its $\mathsf{s}$-weight by:

$$\mathsf{w}^{\mathsf{s}}(\alpha) \coloneqq \sum_{j=1}^{r} c_{\mathsf{s}_j} - \frac{1}{2} \sum_{i \in \mathbb{Z}/e\mathbb{Z}} (c_i - c_{i+1})^2 \in \mathbb{Z},$$

so that $\mathsf{w}^{\mathsf{s}}\big(\alpha^{\mathsf{s}}(\boldsymbol{\lambda})\big) = \mathsf{w}^{\mathsf{s}}(\boldsymbol{\lambda})$ for any $r$-partition $\boldsymbol{\lambda}$. We also define the set

$$Q^{\mathsf{s}} \coloneqq \big\{\alpha^{\mathsf{s}}(\boldsymbol{\lambda}) : \boldsymbol{\lambda} \text{ is an } r\text{-partition}\big\} \subseteq Q.$$

Since $\mathsf{w}^{\mathsf{s}}(\boldsymbol{\lambda}) \geq 0$ for any $r$-partition $\boldsymbol{\lambda}$, we have an inclusion $Q^{\mathsf{s}} \subseteq \{\alpha \in Q : \mathsf{w}^{\mathsf{s}}(\alpha) \geq 0\}$. The paper is mainly concerned in studying a reverse inclusion. More precisely, we prove the following results.

**Theorem A** (Corollaries 3.9 and 3.27)**.** *Let $r \geq 2$ and $e > 0$. We have*

$$Q^{\mathsf{s}} \supseteq \big\{\alpha \in Q : \mathsf{w}^{\mathsf{s}}(\alpha) > N'_{r,e} - r\big\},$$

*where* $N'_{r,e} \coloneqq \left\lfloor \frac{r^2}{2e} \left\lfloor \frac{e^2}{4} \right\rfloor \right\rfloor$.

**Theorem B** (Propositions 2.40 and 3.11)**.** *Let $r \geq 1$ and $e \geq 0$. Assume that $r = 1$ or $(r, e) \in \big\{(2,2), (2,3), (3,2)\big\}$. Then*

$$Q^{\mathsf{s}} = \big\{\alpha \in Q : \mathsf{w}^{\mathsf{s}}(\alpha) \geq 0\big\}.$$

*In particular, given a multiset of elements of $\mathbb{Z}/e\mathbb{Z}$, we can easily determine whether it comes from a Young diagram by computing its weight.*

Note that Theorem A does not hold when $e = 0$, whereas Theorem B does (when $r = 1$). If $e > 0$, a set of the form $\big\{\alpha \in Q : \mathsf{w}^{\mathsf{s}}(\alpha) \geq C\big\}$ for $C \geq 0$ is never empty (and even always infinite), so that the inclusion of Theorem A is not trivially true. The main idea of the proof of Theorem A is to prove that the weight function is bounded above by a constant $N_{r,e}$ on the set of *reduced* $e$-multicores. Note that this assertion is wrong if we only consider $e$-multicores, which can have arbitrarily large weight. With some calculations involving binary matrices, we then give a sharp estimation of the bound $N_{r,e}$. Interestingly, as it is mentioned in the statement, Theorem B gives a simple criterion to determine whether an element of $Q$, in particular, a multiset of residues, actually comes from a partition. To the author's knowledge, such a result was not already known and is a non-trivial generalisation of an old result of Robinson–Thrall [RoTh] from $e = 0$ to $e \geq 0$. We will also give a procedure to compute all the corresponding partitions.

Now assume that $r = 1$, take $p$ dividing $e$ and let $\lambda$ be a partition. Shifting the components of the *$e$-quotient* of $\lambda$, we define another partition $\sigma\lambda$. Using Theorem B, we prove the following analogue of (1.1) in level one.

**Theorem C** (Corollary 4.10)**.** *Let $\lambda$ be a partition. With $e' \coloneqq \frac{e}{p}$, we have*

$$\alpha(\sigma\lambda) = \sigma \cdot \alpha(\lambda) \iff |\sigma\lambda| = |\lambda| \iff c_0(\lambda) = c_{e'}(\lambda).$$

In particular, if $\lambda$ is a partition of $n$ then $\sigma\lambda$ is a partition of $m$ with possibly $m \neq n$. Finally, we complete the study of stuttering blocks initiated in [Ro19b], by giving its analogue in level one. It turns out that a similar equivalence holds, now with an additional condition on the weight of the block.

**Theorem D** (Lemma 4.15 and Corollary 4.17)**.** *Assume that $r = 1$ and let $\alpha \in Q^0$. The block $\alpha$ corresponds to a partition $\lambda$ satisfying $\sigma\lambda = \lambda$ if and only if $\sigma \cdot \alpha = \alpha$ and $p \mid \mathsf{w}(\alpha)$.*

We now give a brief overview of the paper. Section 2 introduces the necessary material to define core blocks and give their proprieties as stated in [Fa07a]. In particular, in §2.1 we define the weight of any element of $Q = \oplus_{i\in\mathbb{Z}/e\mathbb{Z}}\mathbb{Z}\alpha_i$ and in §2.2 we define the abaci representations of a partition and recall how to recover the block associated with a partition from its abacus. We use these results to prove that an element $\alpha \in Q$ corresponds to an $e$-core if and only if $\alpha$ has weight 0 (Lemma 2.39) and we deduce the case $r = 1$ of Theorem B (Proposition 2.40). We deduce a simple criterion to determine whether a given collection of residues actually comes from a Young diagram (Corollary 2.41). In §2.4 we recall Fayers's definition of weight for multipartitions, and we reprove that the weight of a multipartition is non-negative using abaci. We also recall an important result from [Fa06] (Proposition 2.55), which expresses the weight of a multipartition as the minimum of the cardinalities of two sets. Finally, in §2.5 we recall from [Fa07a] the notion of core block and reduced multicore, and we prove that any element of $Q$ is canonically associated with a core block (Lemma 2.67). This leads to the definition of the $\mathbf{s}$-*core* of any element of $Q$ (Definition 2.70). We conclude this section by recalling a key result from [Fa07a] (Proposition 2.73), which characterises the abaci of reduced $e$-multicores.

Section 3 is the heart of the paper. In §3.1 we show that the weights of reduced $e$-multicores can be obtained via a simple functional on binary matrices (Lemma 3.3), where the columns are seen as characteristic vectors of subsets of $\{1, \dots, e\}$. In §3.2 we prove that for fixed $r$ and $e$ the weight of a reduced $e$-multicore is bounded above by a constant $N_{r,e}$ (Theorem 3.7). We then deduce the first part of Theorem A, stating that $Q^{\mathbf{s}}$ contains an (infinite) superlevel set for the weight function on $Q$ (Corollary 3.9). We also give the second part of Theorem B, using some results of the next subsection (Proposition 3.11). In §3.3 the aim is to give sharp bounds for the constant $N_{r,e}$. We first compute $N_{r,e}$ for $r = 2$ and $e = 2$ (Propositions 3.13 and 3.14), and we give a lower bound for $N_{r,e}$ using the superadditivity of the sequence $(N_{r,e})_{e\geq 1}$ (Corollary 3.16). The computation of a sharp upper bound is more elaborate. We use the fact that the computation of $N_{r,e}$ reduces to maximising a certain quadratic form with integer coefficients on the $(2^e - 1)$-sphere for the 1-norm (Proposition 3.17). Namely, using a calculation taken from graph theory, we compute the eigenvalues of the matrix $A_{e,k} = (|E \cap F|)_{E,F}$, where $E, F$ run over the subsets of $\{1, \dots, e\}$ of cardinality $k$ (Lemma 3.24). Note that the matrix $A_{e,k}$ often appears in the literature (see, for instance, [Ry81, Ry82]). We then deduce an upper bound for $N_{r,e}$ (Corollary 3.27), and we compute other values of $N_{r,e}$ for small $r$ or $e$, proving that $N_{r,e} = N'_{r,e}$ (where $N'_{r,e}$ is the constant appearing in Theorem A) for (at least) $r \in \{2, 4\}$ and $e \in \{2, \dots, 6\}$ (Proposition 3.47). We conclude the section by a quick study of the asymptotic behaviour of $N_{r,e}$, for $e \to \infty$ (see (3.50)) and $r \to \infty$ (Corollary 3.53).

Finally, Section 4 is devoted to an application of Theorem B in level one. In §4.1 we define a shift operation $\alpha \mapsto \sigma \cdot \alpha$ on $Q$ of order $p$ where $p$ divides $e$, and we characterise the blocks $\alpha \in Q$ such that both $\alpha$ and $\sigma \cdot \alpha$ correspond to a partition (Corollary 4.4). In §4.2 we propose a definition of a shift operation $\lambda \mapsto \sigma\lambda$ on the set of partitions, uniquely defined on the set of $e$-cores by the following requirement: if $\lambda$ is an $e$-core then $\alpha(\sigma\lambda)$ is the core of $\sigma \cdot \alpha(\lambda)$. In particular, if $\sigma \cdot \alpha(\lambda) = \alpha(\mu)$ for some partition $\mu$ then $\sigma\lambda$ is the $e$-core of $\mu$. Moreover, we prove that the two shift operations, on blocks and on partitions, are compatible under some conditions, giving an analogue of (1.1) in level one (Corollary 4.10). Finally, in §4.3 we give some properties of the map $\alpha(\lambda) \mapsto \sigma \cdot \alpha(\lambda)$. We first give some properties involving $e$-cores and $e'$-cores, where $e' := \frac{e}{p}$ (Propositions 4.13 and 4.14). Then, as in [Ro19b] we study the existence of a stuttering partition inside a stuttering block. More precisely, given a partition $\lambda$ such that $\sigma \cdot \alpha(\lambda) = \alpha(\lambda)$, we prove that there exists a partition $\mu$ satisfying $\sigma\mu = \mu$ and $\alpha(\mu) = \alpha(\lambda)$ if and only if $p$ divides the $e$-weight of $\lambda$ (Lemma 4.15 and Corollary 4.17).

## 2 Background on core blocks

We recall here the combinatorics of blocks of Ariki–Koike algebras. We will write $\mathbb{N}$ for $\mathbb{Z}_{\geq 0}$. Let $e \in \mathbb{N}$. If $e > 0$ we identify $\mathbb{Z}/e\mathbb{Z}$ and $\{0, \dots, e-1\}$, and if $e = 0$ then $\mathbb{Z}/e\mathbb{Z} \simeq \mathbb{Z}$.

### 2.1 Partitions

Let $n \in \mathbb{N}$. Let $\lambda$ be a *partition* of $n$, that is, a non-increasing sequence of positive integers $\lambda = (\lambda_1 \geq \cdots \geq \lambda_h > 0)$ with sum $n$. We will write $|\lambda| \coloneqq n$ and $h(\lambda) \coloneqq h$. We denote by $\emptyset$ the empty partition, which satisfies $|\emptyset| = h(\emptyset) = 0$. The Young diagram associated with $\lambda$ is the region of $\mathbb{N}^2$ given by

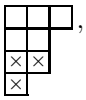
$$\mathcal{Y}(\lambda) \coloneqq \left\{(a,b) \in \mathbb{N}^2 : 1 \leq a \leq h(\lambda) \text{ and } 1 \leq b \leq \lambda_a\right\}.$$

An element of $\mathcal{Y}(\lambda)$ is a *node of* $\lambda$. More generally, we will call *node* any element of $\mathbb{N}^2$. A node $\gamma \notin \mathcal{Y}(\lambda)$ is *addable* (respectively, $\gamma \in \mathcal{Y}(\lambda)$ is *removable*) if $\mathcal{Y}(\lambda) \cup \{\gamma\}$ (resp. $\mathcal{Y} \setminus \{\gamma\}$) is the Young diagram of a partition. A *rim hook* of $\lambda$ is a subset of $\mathcal{Y}(\lambda)$ of the following form:

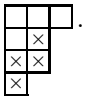
$$r^{\lambda}_{(a,b)} \coloneqq \{(a',b') \in \mathcal{Y}(\lambda) : a' \geq a,\, b' \geq b \text{ and } (a'+1, b'+1) \notin \mathcal{Y}(\lambda)\},$$

where $(a,b) \in \mathcal{Y}(\lambda)$. We say that $r^{\lambda}_{(a,b)}$ is an *$l$-rim hook* if it has cardinality $l$. Note that 1-rim hooks are exactly removable nodes. The set $\mathcal{Y}(\lambda) \setminus r^{\lambda}_{(a,b)}$ is the Young diagram of a certain partition $\mu$, obtained by *unwrapping* or *removing* the rim hook $r^{\lambda}_{(a,b)}$ from $\lambda$. Conversely, we say that $\lambda$ is obtained from $\mu$ by *wrapping on* or *adding* the rim hook $r^{\lambda}_{(a,b)}$. We say that $\lambda$ is an *$e$-core* if $\lambda$ has no $e$-rim hook. Note that if $e = 0$ then every partition is an $e$-core, and if $e = 1$ then the empty partition $\emptyset$ is the only $e$-core. In particular, the combinatorics of 1-cores is very easy, and all the future statements in the paper can easily be proven for $e = 1$.

*Example* 2.1. We consider the partition $\lambda \coloneqq (3,2,2,1)$. An example of a 3-rim hook is

$r^{\lambda}_{(3,1)} =$  ,

and a 4-rim hook is for instance

$r^{\lambda}_{(2,1)} =$  .

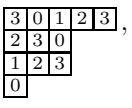
We can check that $\lambda$ has no 5-rim hooks so it is a 5-core. We will see in §2.2 how to use abaci to easily know whether a partition is an $e$-core.

More generally, we can successively remove $e$-rim hooks to the partition $\lambda$ until we reach an $e$-core partition. This $e$-core is uniquely defined from $\lambda$, in particular it does not depend on the order we chose to remove the $e$-rim hooks (see Lemma 2.20). We denote by $\overline{\lambda}$ the $e$-core of $\lambda$.

**Definition 2.2.** The *$e$-weight* of $\lambda$, denoted by $\mathsf{w}_e(\lambda)$, is the number of $e$-rim hooks that we remove to $\lambda$ to obtain its $e$-core $\overline{\lambda}$.

Note that $|\lambda| = |\overline{\lambda}| + \mathsf{w}_e(\lambda)e$ if $e > 0$ and $\mathsf{w}_0(\lambda) = 0$. Now let $\mathsf{s} \in \mathbb{Z}$ be a *charge*. Given $i \in \mathbb{Z}/e\mathbb{Z}$, an *$(i,\mathsf{s})$-node* or simply *$i$-node* is a node of $\mathsf{s}$-residue $i$, where the *$\mathsf{s}$-residue* of the node $(a,b) \in \mathbb{N}^2$ is $\mathrm{res}^{\mathsf{s}}(a,b) \coloneqq b - a + \mathsf{s} \pmod e$. More generally, if $i \in \mathbb{Z}$ then an $i$-node will be a node $\gamma \in \mathbb{N}^2$ such that $\mathrm{res}^{\mathsf{s}}(\gamma) = i \pmod e$. We denote by $c^{\mathsf{s}}_i(\lambda)$ the number of $(i,\mathsf{s})$-nodes of $\lambda$.

*Example* 2.3. Take $\mathsf{s} = -1$ and $e = 4$. For the partition $\lambda = (5,3,3,1)$, in each box of $\mathcal{Y}(\lambda)$ we place the value of the corresponding $\mathsf{s}$-residues:

 ,

thus:

$$c_0^{\mathsf{s}}(\lambda) = 3, \quad c_1^{\mathsf{s}}(\lambda) = 2, \quad c_2^{\mathsf{s}}(\lambda) = 3, \quad c_3^{\mathsf{s}}(\lambda) = 4.$$

Note the following simple equality, where $\mathsf{0}$ denotes the charge $0 \in \mathbb{Z}$:

$$c_i^{\mathsf{s}}(\lambda) = c_{i-\mathsf{s}}^{\mathsf{0}}(\lambda). \tag{2.4}$$

*Remark* 2.5. Let $i \in \mathbb{Z}$. The integer $c_i^{\mathsf{s}}(\lambda)$ does not depend on $e$ if $e \geq \max\big(|\lambda|, |i-\mathsf{s}|\big)$, in which case the value of $c_i^{\mathsf{s}}(\lambda)$ is the one for $e = 0$. In particular, we will sometimes deal with the case $e = 0$ by taking $e \gg 0$. Besides, we have $c_i^{\mathsf{s}}(\lambda) = 0$ as soon as $e \geq |i-\mathsf{s}| \geq |\lambda|$ (as soon as $|i-\mathsf{s}| \geq |\lambda|$ if $e = 0$).

A simple consequence of the definition of a rim hook is the following.

**Lemma 2.6.** *Let $\lambda$ be a partition. Any $e$-rim hook of $\lambda$ has exactly one node of residue $i$ for each $i \in \mathbb{Z}/e\mathbb{Z}$.*

Fayers [Fa06] introduced another weight function on partitions: the $\mathsf{s}$-*weight* of $\lambda$, given by

$$\mathsf{w}^{\mathsf{s}}(\lambda) \coloneqq c_{\mathsf{s}}^{\mathsf{s}}(\lambda) - \frac{1}{2} \sum_{i \in \mathbb{Z}/e\mathbb{Z}} \big(c_i^{\mathsf{s}}(\lambda) - c_{i+1}^{\mathsf{s}}(\lambda)\big)^2. \tag{2.7}$$

It is well-defined for $e = 0$ by Remark 2.5. We will see that the $\mathsf{s}$-weight coincide with the $e$-weight, however these weights will have different generalisations to multipartitions (see §2.4).

Let $Q$ be a free $\mathbb{Z}$-module with a basis indexed by $\mathbb{Z}/e\mathbb{Z}$ that we denote by $\{\alpha_i\}_{i \in \mathbb{Z}/e\mathbb{Z}}$. We have $Q = \oplus_{i \in \mathbb{Z}/e\mathbb{Z}} \mathbb{Z}\alpha_i$ and we define $Q_+ \coloneqq \oplus_{i \in \mathbb{Z}/e\mathbb{Z}} \mathbb{N}\alpha_i$. We will also use the element $\delta \in Q_+$ defined by

$$\delta = \begin{cases} \sum_{i \in \mathbb{Z}/e\mathbb{Z}} \alpha_i, & \text{if } e > 0, \\ 0, & \text{if } e = 0. \end{cases}$$

For any $\alpha \in Q$, we denote by $|\alpha| \in \mathbb{Z}$ the sum of its coordinates in the basis $\{\alpha_i\}_{i \in \mathbb{Z}/e\mathbb{Z}}$. For any partition $\lambda$, we define

$$\alpha^{\mathsf{s}}(\lambda) \coloneqq \sum_{\gamma \in \mathcal{Y}(\lambda)} \alpha_{\mathrm{res}^{\mathsf{s}}(\gamma)} = \sum_{i \in \mathbb{Z}/e\mathbb{Z}} c_i^{\mathsf{s}}(\lambda) \alpha_i \in Q_+.$$

Note that $|\alpha^{\mathsf{s}}(\lambda)| = |\lambda|$. We will say that the partition $\lambda$ *lies* in $\alpha \in Q$ if $\alpha^{\mathsf{s}}(\lambda) = \alpha$. Lemma 2.6 can be reformulated as follows.

**Lemma 2.8.** *Let $\lambda$ and $\mu$ be two partitions such that $\mu$ is obtained from $\lambda$ by adding an $e$-rim hook. Then $\alpha^{\mathsf{s}}(\mu) = \alpha^{\mathsf{s}}(\lambda) + \delta$.*

We extend the definition of the $\mathsf{s}$-weight to $Q$ by setting

$$\mathsf{w}^{\mathsf{s}}(\alpha) \coloneqq c_{\mathsf{s}} - \frac{1}{2} \sum_{i \in \mathbb{Z}/e\mathbb{Z}} (c_i - c_{i+1})^2 \in \mathbb{Z},$$

for any $\alpha = \sum_{i \in \mathbb{Z}/e\mathbb{Z}} c_i \alpha_i \in Q$. Note that $\mathsf{w}^{\mathsf{s}}(\alpha) \in \mathbb{Z}$ indeed, since $\sum_{i \in \mathbb{Z}/e\mathbb{Z}} \big(c_i - c_{i+1}\big) = 0$ and thus $\sum_{i \in \mathbb{Z}/e\mathbb{Z}} \big(c_i - c_{i+1}\big)^2$ is even. Moreover, we have

$$\mathsf{w}^{\mathsf{s}}(\lambda) = \mathsf{w}^{\mathsf{s}}\big(\alpha^{\mathsf{s}}(\lambda)\big), \tag{2.9}$$

and if $e > 0$ then

$$\mathsf{w}^{\mathsf{s}}(\alpha + h\delta) = \mathsf{w}^{\mathsf{s}}(\alpha) + h, \tag{2.10}$$

for any $h \in \mathbb{Z}$. Finally, we define

$$Q^{\mathsf{s}} \coloneqq \{\alpha^{\mathsf{s}}(\lambda) : \lambda \text{ partition of } n \text{ for some } n \in \mathbb{N}\}.$$

In §2.2 we will give a parametrisation of $Q^{\mathsf{s}}$ using abaci, while in §2.3 we will give an implicit description of $Q^{\mathsf{s}}$, using a superlevel set for the $\mathsf{s}$-weight function.

## 2.2 Abaci

Let $\lambda$ be a partition. The *charged beta-number* associated with the partition $\lambda$ and the charge $\mathsf{s}$ is the sequence $\beta^{\mathsf{s}}(\lambda) = \big(\beta^{\mathsf{s}}(\lambda)_a\big)_{a\geq 1}$ defined by

$$\beta^{\mathsf{s}}(\lambda)_a := \mathsf{s} + \lambda_a - a \in \mathbb{Z},$$

for any $a \geq 1$. It is a (strictly) decreasing sequence of integers. For $a > h(\lambda)$ we have $\beta^{\mathsf{s}}(\lambda)_a = \mathsf{s} - a$, and conversely if $\beta = (\beta_a)_{a\geq 1}$ is a decreasing sequence of integers such that $\beta_a = \mathsf{s} - a$ for $a \gg 0$ then $\beta = \beta^{\mathsf{s}}(\lambda)$ for some partition $\lambda$. Representing $\beta^{\mathsf{s}}(\lambda)$ by a copy of $\mathbb{Z}$ where we put a bead on position $i \in \mathbb{Z}$ if $i \in \beta^{\mathsf{s}}(\lambda)$ and a gap otherwise gives the $\mathsf{s}$-*abacus* associated with $\lambda$.

*Example* 2.11. The $\mathsf{0}$-beta-number associated with the empty partition is $(-1, -2, -3, \dots)$ and the associated charged abacus is

*Example* 2.12. The $\mathsf{1}$-beta-number associated with the partition $(4, 3, 3, 1)$ is $(4, 2, 1, -2, -4, -5, \dots)$ and the associated charged abacus is

The next result is well-known, see for instance [Ol, Proposition (1.8)].

**Lemma 2.13.** *Let $l \in \mathbb{N}$. The partition $\lambda$ has an $l$-rim hook if and only if there is an element $b \in \beta(\lambda)$ such that $b - l \notin \beta(\lambda)$. In that case, if $\mu$ is the partition that we obtain by removing this $l$-rim hook, then $\beta(\mu)$ is obtained by replacing $b$ by $b - l$ in $\beta(\lambda)$ and then sorting in decreasing order.*

Unless mentioned otherwise, we now assume that $e > 0$. For each $i \in \{0, \dots, e-1\}$ we can define $\mathfrak{b}_i^{\mathsf{s}}(\lambda) \in \mathbb{Z}$ to be the largest element of $\beta^{\mathsf{s}}(\lambda)$ congruent to $i$ modulo $e$. By [Fa07a, Lemma 3.2], if $\lambda$ is an $e$-core we have the following relation:

$$\mathsf{s} = \frac{e+1}{2} + \frac{1}{e} \sum_{i \in \mathbb{Z}/e\mathbb{Z}} \mathfrak{b}_i^{\mathsf{s}}(\lambda). \tag{2.14}$$

Setting $y_i^{\mathsf{s}}(\lambda) := \frac{1}{e}\big(\mathfrak{b}_i^{\mathsf{s}}(\lambda) - i\big) + 1 \in \mathbb{Z}$ for each $i \in \{0, \dots, e-1\}$, the relation (2.14) becomes

$$\mathsf{s} = \sum_{i=0}^{e-1} y_i^{\mathsf{s}}(\lambda). \tag{2.15}$$

**Proposition 2.16.** *Let $y_0, \dots, y_{e-1} \in \mathbb{Z}$. There exists an $e$-core $\lambda$ such that $y_i = y_i^{\mathsf{s}}(\lambda)$ for all $i \in \{0, \dots, e-1\}$ if and only if $y_0 + \dots + y_{e-1} = \mathsf{s}$.*

*Proof.* The direct implication follows from (2.15). Now assume that $y_0 + \dots + y_{e-1} = \mathsf{s}$ and define $\mathfrak{b}_i := e(y_i - 1) + i$ for all $i \in \{0, \dots, e-1\}$. The (unique) decreasing sequence of integers $\beta = (\beta_a)_{a\geq 1}$ defined by:

- for all $i \in \{1, \dots, e-1\}$, the integer $\mathfrak{b}_i$ is the largest element of $\beta$ congruent to $i$ modulo $e$;
- for all $h \in \beta$ we have $h - e \in \beta$;

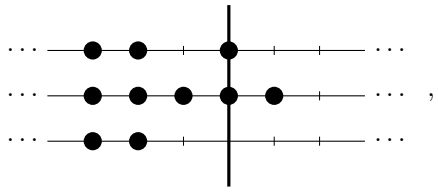
is the charged beta-number associated with some partition $\lambda$ and charge $\mathsf{s}' \in \mathbb{Z}$. By Lemma 2.13 the partition $\lambda$ is an $e$-core and by (2.15) we have $\mathsf{s}' = \mathsf{s}$. This concludes the proof. □

The abacus representation of a partition that we now use has been first introduced by James [Ja]; we use here the setting of [LyMa]. We dispose the elements of $\beta^{\mathsf{s}}(\lambda)$ on an abacus with now $e$ runners, the *charged $e$-abacus of $\lambda$*, each runner being a horizontal copy of $\mathbb{Z}$ and displayed in the following way: the 0-th runner is on the bottom and the origins of

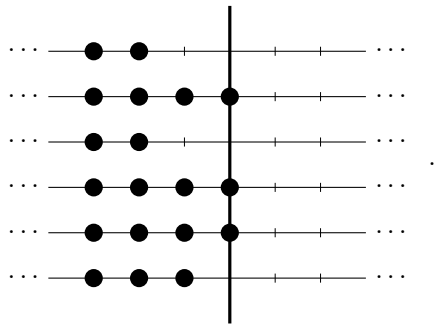
each copy of $\mathbb{Z}$ are aligned with respect to a vertical line. We record the elements of $\beta^{\mathsf{s}}(\lambda)$ on this abacus according to the following rule: there is a bead at position $j \in \mathbb{Z}$ on the runner $i \in \{0, \dots, e-1\}$ if and only if there exists $a \geq 1$ such that $\beta^{\mathsf{s}}(\lambda)_a = i + je$. For each $i \in \{0, \dots, e-1\}$, the $i$-th runner corresponds to the charged abacus (as defined at the beginning of §2.2) of a certain partition $\lambda^{[i]}$. The $e$-tuple $\left(\lambda^{[0]}, \dots, \lambda^{[e-1]}\right)$ is the *$e$-quotient* of $\lambda$. In particular, any partition is uniquely determined by its $e$-core and its $e$-quotient.

*Remark* 2.17. The $e$-quotient of $\lambda$ depends on the charge $\mathsf{s}$ only up to a shift: if $\left(\lambda^{[0]}, \dots, \lambda^{[e-1]}\right)$ is the $e$-quotient of $\lambda$ computed on the $\mathsf{s}$-charged $e$-abacus then $\left(\lambda^{[e-1]}, \lambda^{[0]}, \dots, \lambda^{[e-2]}\right)$ is the $e$-quotient computed on the $(\mathsf{s}+1)$-charged $e$-abacus.

*Example* 2.18. We take the charge $\mathsf{s} = 1$ and we consider the partition $\lambda := (4, 3, 3, 1)$ as in Example 2.12. The associated charged 3-abacus is



and the associated charged 6-abacus is



Counting the number of gaps down each bead (continuing counting on the left starting from the top runner when reaching the bottom one) recovers the underlying partition. The 3-quotient of $\lambda$ is $\left(\emptyset, \emptyset, (1)\right)$ and its 6-quotient has only empty partitions.

The $e$-abacus representation is particularly adapted to the addition and deletion of rim hooks: we give two particular cases, as corollaries of Lemma 2.13.

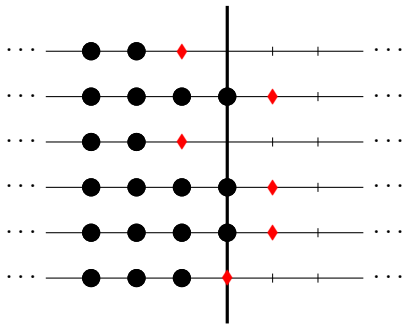
**Lemma 2.19.** *Let $\lambda$ be a partition and $i \in \{0, \dots, e-1\}$.*

- *The partition $\lambda$ has a removable $i$-node if and only if we can move a bead from runner $i$ to runner $i-1$ (to runner $e-1$ if $i = 0$), keeping the same position $j \in \mathbb{Z}$ (from position $j$ to $j-1$ if $i = 0$).*
- *The partition $\lambda$ has an addable $i$-node if and only if we can move a bead from runner $i$ to runner $i+1$ (to runner $0$ if $i = e-1$), keeping the same position $j \in \mathbb{Z}$ (from position $j$ to $j+1$ if $i = e-1$).*

**Lemma 2.20.** *Let $\lambda$ be a partition.*

- *The partition $\lambda$ has an $e$-rim hook if and only if on some runner we can slide a bead at some position $j \in \mathbb{Z}$ to the previously free position $j-1$. Hence, the partition $\lambda$ is an $e$-core if and only if its associated $e$-abacus has no gaps, that is, there are no gaps between two beads on a same runner.*
- *The partition $\lambda$ has an addable $e$-rim hook if and only if on some runner we can slide a bead at some position $j \in \mathbb{Z}$ to the previously free position $j+1$. In particular, to any partition we can add at least $e$ different $e$-rim hooks.*

*Remark* 2.21. Let $\lambda$ be an $e$-core. For any $i \in \{0, \dots, e-1\}$, the integer $y_i^{\mathsf{s}}(\lambda)$ corresponds to the position of the first gap on runner $i$ (pictured by ♦ in the abacus below). For instance, by Lemma 2.20 the partition $\lambda = (4, 3, 3, 1)$ of Example 2.18 is a 6-core and we have $y^{\mathsf{s}}(\lambda) = (0, 1, 1, -1, 1, -1)$. In particular, note that (2.15) is satisfied indeed.



Note that Lemma 2.20 implies

$$\mathsf{w}_e(\lambda) = \sum_{i=0}^{e-1} \left|\lambda^{[i]}\right|. \tag{2.22}$$

The next result follows from Lemma 2.19 (see, for instance, [Ro19b, Lemma 2.11]).

**Proposition 2.23.** *Assume that $\lambda$ is an $e$-core. For any $i \in \{0, \dots, e-1\}$ we have:*

$$y_i^{\mathsf{s}}(\lambda) = c_i^{\mathsf{s}}(\lambda) - c_{i+1}^{\mathsf{s}}(\lambda) + y_i^{\mathsf{s}}(\emptyset).$$

Note that we can compute the value of $y_i^{\mathsf{s}}(\emptyset)$.

**Lemma 2.24.** *Write $\mathsf{s} = be + \mathsf{s}'$, where $b \in \mathbb{Z}$ and $\mathsf{s}' \in \{0, \dots, e-1\}$. For any $i \in \{0, \dots, e-1\}$ we have:*

$$y_i^{\mathsf{s}}(\emptyset) = \begin{cases} b+1, & \text{if } i \in \{0, \dots, \mathsf{s}'-1\}, \\ b, & \text{otherwise.} \end{cases}$$

*Proof.* We have $\beta^{\mathsf{s}}(\emptyset)_a = \mathsf{s} - a$ for all $a \geq 1$. Thus, for any $a \in \{1, \dots, e\}$, if $i \in \{0, \dots, e-1\}$ is the residue modulo $e$ of $\mathsf{s} - a$ then $\mathfrak{b}_i^{\mathsf{s}}(\emptyset) = \mathsf{s} - a$. Thus, if $a$ describes $\{1, \dots, \mathsf{s}'\}$ then $i := \mathsf{s}' - a$ describes $\{0, \dots, \mathsf{s}'-1\}$, moreover $i = \mathsf{s} - a \pmod e$ thus $\mathfrak{b}_i^{\mathsf{s}}(\emptyset) = \mathsf{s} - a = be + i$ and $y_i^{\mathsf{s}}(\emptyset) = \frac{1}{e}\big(\mathfrak{b}_i^{\mathsf{s}}(\emptyset) - i\big) + 1 = b+1$. Similarly, if $a \in \{\mathsf{s}'+1, \dots, e\}$ then with $i := \mathsf{s}' - a + e \in \{\mathsf{s}', \dots, e-1\}$ we have $i = \mathsf{s} - a \pmod e$ thus $\mathfrak{b}_i^{\mathsf{s}}(\emptyset) = be + i - e$ and $y_i^{\mathsf{s}}(\emptyset) = b$. □

For any $i \in \mathbb{Z}/e\mathbb{Z}$, define

$$x_i^{\mathsf{s}}(\lambda) := y_i^{\mathsf{s}}(\lambda) - y_i^{\mathsf{s}}(\emptyset).$$

By Proposition 2.23 (and (2.15)), for an $e$-core $\lambda$ we have

$$x_i^{\mathsf{s}}(\lambda) = c_i^{\mathsf{s}}(\lambda) - c_{i+1}^{\mathsf{s}}(\lambda), \qquad \text{for all } i \in \mathbb{Z}/e\mathbb{Z}, \tag{2.25}$$

$$\sum_{i \in \mathbb{Z}/e\mathbb{Z}} x_i^{\mathsf{s}}(\lambda) = 0. \tag{2.26}$$

As for the $e$-quotient, the family $x^{\mathsf{s}}(\lambda) = \big(x_i^{\mathsf{s}}(\lambda)\big)_{i \in \mathbb{Z}/e\mathbb{Z}}$ depends on the charge $\mathsf{s}$ only up to a shift (by (2.4) and (2.25)). The following proposition is a direct consequence of (2.15) and Proposition 2.16.

**Proposition 2.27.** *Let $\mathsf{s} \in \mathbb{Z}$ and $x_0, \dots, x_{e-1} \in \mathbb{Z}$. There exists an $e$-core $\lambda$ such that $x_i = x_i^{\mathsf{s}}(\lambda)$ for all $i \in \{0, \dots, e-1\}$ if and only if $x_0 + \cdots + x_{e-1} = 0$.*

For completeness, we also give the version of Proposition 2.27 in the case $e = 0$. We recall that every partition $\lambda$ is a 0-core, and we define the integers $x_i^{\mathsf{s}}(\lambda)$ for any $i \in \mathbb{Z}$ using (2.25). By Remark 2.5 the family $\big(x_i^{\mathsf{s}}(\lambda)\big)_{i \in \mathbb{Z}}$ has finite support, equality (2.26) is still satisfied and by (2.25) we have

$$x_i^{\mathsf{s}}(\lambda) \in \begin{cases} \{0, 1\}, & \text{if } i \geq \mathsf{s}, \\ \{0, -1\}, & \text{if } i < \mathsf{s}. \end{cases} \tag{2.28}$$

**Proposition 2.29** ([RoTh, (3.9)]). *Let $(x_i)_{i\in\mathbb{Z}} \in \mathbb{Z}^{\mathbb{Z}}$ with finite support. There exists a partition $\lambda$ such that $x_i = x_i^{\mathsf{s}}(\lambda)$ for all $i \in \mathbb{Z}$ if and only if*

$$x_i \in \begin{cases} \{0,1\}, & \text{if } i \geq \mathsf{s}, \\ \{0,-1\}, & \text{if } i < \mathsf{s}, \end{cases} \tag{2.30}$$

*for all $i \in \mathbb{Z}$, and*

$$\sum_{i\in\mathbb{Z}} x_i = 0.$$

*Proof.* We have just seen that these conditions are necessary. Now let $(x_i)_{i\in\mathbb{Z}} \in \mathbb{Z}^{\mathbb{Z}}$ with finite support satisfy the conditions of Proposition 2.29 and let $N > 0$ such that $x_i = 0$ if $|i| \geq N$. We define a family $(c_i)_{i\in\mathbb{Z}}$ by setting $c_N \coloneqq 0$ and

$$c_i \coloneqq \begin{cases} c_{i+1} + x_i, & \text{if } i < N, \\ c_{i-1} - x_{i-1}, & \text{if } i > N. \end{cases}$$

By assumption we have $c_i = 0$ if $|i| \geq N$, moreover $c_i - c_{i+1} = x_i$ for all $i \in \mathbb{Z}$. By (2.30), it is clear that we can construct a partition $\lambda$ such that $c_i = c_i^{\mathsf{s}}(\lambda)$ for all $i \in \mathbb{Z}$. □

The following result shows that the $e$-weight $\mathsf{w}_e$ (defined in Definition 2.2) and the $\mathsf{s}$-weight $\mathsf{w}^{\mathsf{s}}$ (defined in (2.7)) of a partition coincide.

**Proposition 2.31** ([Fa06, Proposition 2.1]). *Let $e \in \mathbb{N}$. We have $\mathsf{w}_e(\lambda) = \mathsf{w}^{\mathsf{s}}(\lambda)$.*

We are thus able to recover the integers $c_i^{\mathsf{s}}(\lambda)$ from $x^{\mathsf{s}}(\lambda)$.

**Corollary 2.32.** *Let $e \in \mathbb{N}$.*

- *Assume that $e > 0$. If $\lambda$ is an $e$-core, then*

$$c_{\mathsf{s}}^{\mathsf{s}}(\lambda) = \frac{1}{2}\|x^{\mathsf{s}}(\lambda)\|^2 = \frac{1}{2}\sum_{i=0}^{e-1} x_i^{\mathsf{s}}(\lambda)^2,$$
$$c_{\mathsf{s}+i}^{\mathsf{s}}(\lambda) = \frac{1}{2}\|x^{\mathsf{s}}(\lambda)\|^2 - x_{\mathsf{s}}^{\mathsf{s}}(\lambda) - \cdots - x_{\mathsf{s}+i-1}^{\mathsf{s}}(\lambda),$$

  *for all $i \in \{1,\dots,e-1\}$.*
- *Assume that $e = 0$. We have*

$$c_{\mathsf{s}}^{\mathsf{s}}(\lambda) = \frac{1}{2}\|x^{\mathsf{s}}(\lambda)\|^2 = \frac{1}{2}\sum_{i\in\mathbb{Z}} x_i^{\mathsf{s}}(\lambda)^2,$$
$$c_{\mathsf{s}+i}^{\mathsf{s}}(\lambda) = \frac{1}{2}\|x^{\mathsf{s}}(\lambda)\|^2 - x_{\mathsf{s}}^{\mathsf{s}}(\lambda) - \cdots - x_{\mathsf{s}+i-1}^{\mathsf{s}}(\lambda),$$
$$c_{\mathsf{s}-i}^{\mathsf{s}}(\lambda) = \frac{1}{2}\|x^{\mathsf{s}}(\lambda)\|^2 + x_{\mathsf{s}-1}^{\mathsf{s}}(\lambda) + \cdots + x_{\mathsf{s}-i+1}^{\mathsf{s}}(\lambda),$$

  *for all $i \in \mathbb{Z}_{>0}$.*

## 2.3 Blocks for partitions

Let $e \in \mathbb{N}$. We first recall some standard facts concerning blocks associated with partitions.

**Lemma 2.33.** *If $\lambda$ and $\mu$ are two $e$-cores with $\alpha^{\mathsf{s}}(\lambda) = \alpha^{\mathsf{s}}(\mu)$ then $\lambda = \mu$.*

*Proof.* The statement is clear if $e = 0$. If $e > 0$, by (2.25) we have $x^{\mathsf{s}}(\lambda) = x^{\mathsf{s}}(\mu)$. We thus conclude that $y_i^{\mathsf{s}}(\lambda) = y_i^{\mathsf{s}}(\mu)$ and thus $\mathfrak{b}_i^{\mathsf{s}}(\lambda) = \mathfrak{b}_i^{\mathsf{s}}(\mu)$ for all $i \in \{0,\dots,e-1\}$ whence $\lambda = \mu$. □

Recall from the introduction that if $\lambda$ and $\mu$ are two partitions, we say that $\lambda$ and $\mu$ *lie in the same block* if $\alpha^{\mathsf{s}}(\lambda) = \alpha^{\mathsf{s}}(\mu)$. For instance, Lemma 2.33 says that two different $e$-cores cannot lie in the same block. Together with Lemma 2.8, (2.9) and Proposition 2.31, we deduce the next lemma, which can be seen as a combinatorial version of the so-called "Nakayama's Conjecture".

**Lemma 2.34** ([JaKe, Theorem 2.7.41])**.** *Two partitions that lie in the same block share the same $e$-core.*

In particular, the elements $\alpha^{\mathsf{s}}(\lambda)$ encode the blocks of the associated *Iwahori–Hecke algebra*, see for instance [Ma, 5.38 Corollary]. Finally, by (2.9), Proposition 2.31 and Lemma 2.34, we have the following result.

**Lemma 2.35.** *Let $\alpha \in Q^{\mathsf{s}}$ with $\mathsf{w}^{\mathsf{s}}(\alpha) = 0$. Then there is a unique partition lying in $\alpha$, and this partition is an $e$-core.*

*Remark* 2.36. We will see in Lemma 2.39 that the statement of Lemma 2.35 remains true if we replace $\alpha \in Q^{\mathsf{s}}$ with $\alpha \in Q$.

We can now give a 1:1-parametrisation of the set $Q^{\mathsf{s}} = \{\alpha^{\mathsf{s}}(\lambda) : \lambda \text{ partition of } n \text{ for some } n \in \mathbb{N}\}$ by $\mathbb{Z}^{e-1} \times \mathbb{N}$, if $e > 0$. By Lemmas 2.33 and 2.35, for any $\alpha \in Q^{\mathsf{s}}$ we can denote by $\lambda_\alpha$ the (unique) common $e$-core of the partitions lying in $\alpha$.

**Proposition 2.37.** *Assume that $e > 0$. The map*

$$Q^{\mathsf{s}} \longrightarrow \big\{x = (x_0, \dots, x_{e-1}) \in \mathbb{Z}^e : x_0 + \cdots + x_{e-1} = 0\big\} \times \mathbb{N},$$

*given by $\alpha \mapsto \big(x^{\mathsf{s}}(\lambda_\alpha), \mathsf{w}^{\mathsf{s}}(\alpha)\big)$, is a bijection. Its inverse is given by*

$$(x, w) \longmapsto \left(\frac{1}{2}\|x\|^2 + w\right) \sum_{i=0}^{e-1} \alpha_i - \sum_{i=1}^{e-1} (x_{\mathsf{s}} + \cdots + x_{\mathsf{s}+i-1})\alpha_{\mathsf{s}+i},$$

*where the indices of $x = (x_0, \dots, x_{e-1})$ are taken modulo $e$.*

*Remark* 2.38. Using Proposition 2.29 and Corollary 2.32, we can also give a version of Proposition 2.37 in the case $e = 0$, using the map $\alpha \mapsto x^{\mathsf{s}}(\lambda_\alpha)$. Note that in this case, an element $\alpha = \sum_{i \in \mathbb{Z}} c_i \alpha_i \in Q$ is in $Q^{\mathsf{s}}$ if and only if

$$c_i - c_{i+1} \in \begin{cases} \{0, 1\}, & \text{if } i \geq \mathsf{s}, \\ \{0, -1\}, & \text{if } i < \mathsf{s}. \end{cases}$$

We now want to give an implicit description of $Q^{\mathsf{s}}$. Note that in the following results we consider the $\mathsf{s}$-weight inside $Q$ and not $Q_+$.

**Lemma 2.39.** *We have $\big\{\alpha^{\mathsf{s}}(\lambda) : \lambda \text{ is an } e\text{-core}\big\} = \{\alpha \in Q : \mathsf{w}^{\mathsf{s}}(\alpha) = 0\}$.*

*Proof.* We first assume that $e > 0$. The inclusion $\subseteq$ follows from (2.9) and Proposition 2.31. Now let $\alpha = \sum_{i \in \mathbb{Z}/e\mathbb{Z}} c_i \alpha_i \in Q$ such that $\mathsf{w}^{\mathsf{s}}(\alpha) = 0$ and define $x_i := c_i - c_{i+1}$ for all $i \in \mathbb{Z}/e\mathbb{Z}$. By Proposition 2.27 we know that $x_i = x_i^{\mathsf{s}}(\lambda)$ for some $e$-core $\lambda$. By Corollary 2.32 and since $\mathsf{w}^{\mathsf{s}}(\alpha) = 0$ we have

$$\begin{aligned} c_{\mathsf{s}}^{\mathsf{s}}(\lambda) &= \frac{1}{2} \sum_{i \in \mathbb{Z}/e\mathbb{Z}} x_i^{\mathsf{s}}(\lambda)^2 \\ &= \frac{1}{2} \sum_{i \in \mathbb{Z}/e\mathbb{Z}} x_i^2 \\ &= c_{\mathsf{s}}, \end{aligned}$$

and

$$\begin{aligned} c_{\mathsf{s}+i}^{\mathsf{s}}(\lambda) &= c_{\mathsf{s}}^{\mathsf{s}}(\lambda) - x_{\mathsf{s}}^{\mathsf{s}}(\lambda) - \cdots - x_{\mathsf{s}+i-1}^{\mathsf{s}}(\lambda) \\ &= c_{\mathsf{s}}^{\mathsf{s}} - x_{\mathsf{s}}^{\mathsf{s}} - \cdots - x_{\mathsf{s}+i-1}^{\mathsf{s}} \\ &= c_{\mathsf{s}+i}, \end{aligned}$$

thus $\alpha^{\mathsf{s}}(\lambda) = \alpha$.

We now assume that $e = 0$. Again $\subseteq$ follows from (2.9) and Proposition 2.31, thus let $\alpha = \sum_{i\in\mathbb{Z}} c_i\alpha_i \in Q$ such that $\mathsf{w}^{\mathsf{s}}(\alpha) = 0$. We can repeat the proof of the case $e > 0$, using Proposition 2.29 instead of Proposition 2.27. However, we have to ensure that $c_i - c_{i+1} \in \begin{cases}\{0,1\}, & \text{if } i \geq \mathsf{s},\\ \{0,-1\}, & \text{if } i < \mathsf{s}.\end{cases}$ Since $\mathsf{w}^{\mathsf{s}}(\alpha) = 0$ we have

$$\begin{aligned} 2c_{\mathsf{s}} &= \sum_{i\in\mathbb{Z}}(c_i - c_{i+1})^2\\ &= \sum_{i\geq\mathsf{s}}(c_i - c_{i+1})^2 + \sum_{i<\mathsf{s}}(c_i - c_{i+1})^2\\ &\geq \sum_{i\geq\mathsf{s}}(c_i - c_{i+1}) + \sum_{i<\mathsf{s}}(c_{i+1} - c_i)\\ &= c_{\mathsf{s}} + c_{\mathsf{s}}, \end{aligned}$$

thus we deduce that

$$(c_i - c_{i+1})^2 = \begin{cases} c_i - c_{i+1}, & \text{if } i \geq \mathsf{s},\\ c_{i+1} - c_i, & \text{if } i < \mathsf{s},\end{cases}$$

for all $i \in \mathbb{Z}$, thus we obtain the desired result. Another way to see the result for $e = 0$ is to consider $e > 0$ large enough so that we can apply the previous case (using Remark 2.5). □

**Proposition 2.40.** *We have* $Q^{\mathsf{s}} = \{\alpha \in Q : \mathsf{w}^{\mathsf{s}}(\alpha) \geq 0\}$.

*Proof.* Assume first that $e > 0$. Again $\subseteq$ follows from (2.9) and Proposition 2.31. Let $\alpha = \sum_{i\in\mathbb{Z}/e\mathbb{Z}} c_i\alpha_i \in Q$ such that $\mathsf{w}^{\mathsf{s}}(\alpha) \geq 0$. By (2.10), setting $\hat{\alpha} := \alpha - \mathsf{w}^{\mathsf{s}}(\alpha)\delta \in Q$ we have $\mathsf{w}^{\mathsf{s}}(\hat{\alpha}) = 0$ thus by Lemma 2.39 we have $\hat{\alpha} = \alpha^{\mathsf{s}}(\lambda)$ for some $e$-core $\lambda$. Now if $\mu$ is any partition obtained from $\lambda$ by adding $\mathsf{w}^{\mathsf{s}}(\alpha)$ times an $e$-rim hook we have $\alpha^{\mathsf{s}}(\mu) = \alpha^{\mathsf{s}}(\lambda) + \mathsf{w}^{\mathsf{s}}(\alpha)\delta = \hat{\alpha} + \mathsf{w}^{\mathsf{s}}(\alpha)\delta = \alpha$ which concludes the proof in the case $e > 0$.

Now if $e = 0$, if $\alpha = \sum_{i\in\mathbb{Z}} c_i\alpha_i \in Q$ is such that $\mathsf{w}^{\mathsf{s}}(\alpha) \geq 0$ then as in the proof of Lemma 2.39 we have

$$2c_{\mathsf{s}} \geq \sum_{i\in\mathbb{Z}}(c_i - c_{i+1})^2 \geq \sum_{i\geq\mathsf{s}}(c_i - c_{i+1}) + \sum_{i<\mathsf{s}}(c_{i+1} - c_i) = 2c_{\mathsf{s}},$$

thus we have in fact $\mathsf{w}^{\mathsf{s}}(\alpha) = 0$ and we apply Lemma 2.39. □

An interesting consequence of Proposition 2.40 is the following result, which was, up to the author's knowledge, not already known. It allows to determine whether a given multiset of residues actually comes from a Young diagram and non-trivially extends a result of Robinson–Thrall [RoTh] for $e = 0$ to $e \geq 0$.

**Corollary 2.41.** *Let $e \in \mathbb{N}$ and let $(c_i)_{i\in\mathbb{Z}/e\mathbb{Z}}$ be a sequence of integers indexed by $\mathbb{Z}/e\mathbb{Z}$ (with finite support if $e = 0$). There exists a partition $\lambda$ satisfying $c_i^{\mathsf{s}}(\lambda) = c_i$ for all $i \in \mathbb{Z}/e\mathbb{Z}$ if and only if $\sum_{i\in\mathbb{Z}/e\mathbb{Z}} (c_i - c_{i+1})^2 \leq 2c_{\mathsf{s}}$.*

*Remark* 2.42. If a partition as in Corollary 2.41 exists, Lemma 2.39 and the proof of Proposition 2.40 show how to compute the associated core, and thus all the partitions with the same multiset of residues (that is, lying in the same block).

The aim of Section 3 is to study an analogue of Proposition 2.40 in higher levels. By (2.10) we obtain the following corollary of Proposition 2.40.

**Corollary 2.43.** *Assume $e > 0$. For any $\alpha \in Q$ and any $h \in \mathbb{Z}$ we have:*

$$\alpha + h\delta \in Q^{\mathsf{s}} \iff h \geq -\mathsf{w}^{\mathsf{s}}(\alpha).$$

*Example* 2.44. Assume $e \geq 2$ and take $\mathsf{s} = \mathsf{0}$. Let $n \in \mathbb{N}$ and define $\alpha \coloneqq n\alpha_0 \in Q_+$. We have $\mathsf{w}^0(\alpha) = n - n^2 \leq 0$. If $n \in \{0, 1\}$ then $\mathsf{w}^0(\alpha) = 0$ thus $\alpha \in Q^{\mathsf{s}}$, and indeed if $n = 0$ then $\alpha = 0 = \alpha^0(\emptyset)$ and if $n = 1$ then $\alpha = \alpha_0 = \alpha^0\big((1)\big)$.

Thus, we now assume $n \geq 2$, in particular $\mathsf{w}^0(\alpha) = n - n^2 < 0$. We have

$$\mathsf{w}^0\big(\alpha + (n^2 - n)\delta\big) = 0.$$

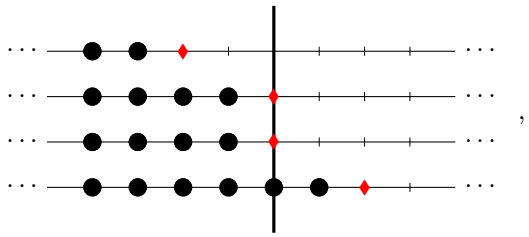
Thus, by Lemma 2.39 there is an $e$-core $\lambda$ such that

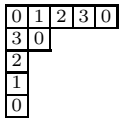
$$\alpha^0(\lambda) = \alpha + (n^2 - n)\delta = n^2\alpha_0 + (n^2 - n)\sum_{i=1}^{e-1} \alpha_i,$$

in other words $c_0^0(\lambda) = n^2$ and $c_i^0(\lambda) = n^2 - n$ for all $i \in \{1, \dots, e-1\}$. In particular, we have:

$$|\lambda| = n^2 + (e-1)(n^2 - n) = en^2 - (e-1)n.$$

By (2.25), the $e$-core $\lambda$ satisfies $x_0^0(\lambda) = -x_{e-1}^0(\lambda) = n$, with the other integers $x_i^0(\lambda)$ being 0. For instance, with $e = 4$ and $n = 2$ then by Remark 2.21 the charged 4-abacus of $\lambda$ is:



thus $\lambda = (5, 2, 1, 1, 1)$, with Young diagram (with residues)  thus $c_0^0(\lambda) = 4$ and $c_i^0(\lambda) = 2$ for all $i \in \{1, 2, 3\}$ indeed.

*Example* 2.45. Assume $e \geq 2$ and take $\mathsf{s} = \mathsf{0}$. Let $n \in \mathbb{N}$, fix $i_0 \in \{1, \dots, e-1\}$ and define $\alpha \coloneqq n\alpha_{i_0} \in Q_+$. We have $\mathsf{w}^0(\alpha) = -n^2 \leq 0$. If $n = 0$ then $\alpha = 0 = \alpha^0(\emptyset)$, thus we now assume that $n \geq 1$. We have $\mathsf{w}^0(\alpha) = -n^2 < 0$ and

$$\mathsf{w}^0\big(\alpha + n^2\delta\big) = 0,$$

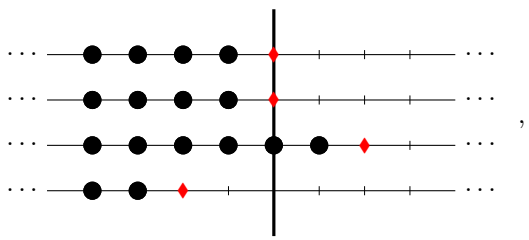
thus there is an $e$-core $\lambda$ such that

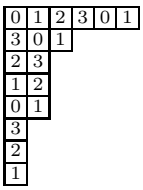
$$\alpha^0(\lambda) = \big(n^2 + n\big)\alpha_{i_0} + n^2 \sum_{\substack{i \in \mathbb{Z}/e\mathbb{Z} \\ i \neq i_0}} \alpha_i,$$

in other words $c_{i_0}^0(\lambda) = n^2 + n$ and $c_i^0(\lambda) = n^2$ for all $i \neq i_0$. In particular, we have:

$$|\lambda| = n^2 + n + (e-1)n^2 = en^2 + n.$$

The $e$-core $\lambda$ satisfies $x_{i_0}^0(\lambda) = -x_{i_0-1}^0(\lambda) = n$, with the other integers $x_i^0(\lambda)$ being 0. Note that for (the excluded case) $i_0 = 0$ we obtain the same $e$-tuple $x^0(\lambda)$ as in Example 2.44, although the value of $|\lambda|$ is different. For $e = 4$ and $n = 2$ and $i_0 = 1$, the charged 4-abacus of $\lambda$ is:



thus $\lambda = (6,3,2,2,2,1,1,1)$, with Young diagram (with residues)  thus $c_1^0(\lambda) = 6$

and $c_i^0(\lambda) = 4$ for all $i \neq 1$ indeed.

## 2.4 Multipartitions

Let $r \geq 1$ and $e \in \mathbb{N}$. Let $\boldsymbol{\lambda}$ be an *$r$-partition* (or *multipartition*) of $n$, that is, an $r$-tuple $\boldsymbol{\lambda} = \big(\lambda^{(1)}, \dots, \lambda^{(r)}\big)$ of partitions such that $\big|\lambda^{(1)}\big| + \cdots + \big|\lambda^{(r)}\big| = n$. The Young diagram of the $r$-partition $\boldsymbol{\lambda}$ is the subset of $\mathbb{N}^2 \times \{1, \dots, r\}$ defined by

$$\mathcal{Y}(\boldsymbol{\lambda}) \coloneqq \bigcup_{j=1}^{r} \mathcal{Y}\big(\lambda^{(j)}\big) \times \{j\}.$$

An element of $\mathcal{Y}(\boldsymbol{\lambda})$ is a node of $\boldsymbol{\lambda}$, more generally any element of $\mathbb{N}^2 \times \{1, \dots, r\}$ is a node. An $e$-rim hook of $\boldsymbol{\lambda}$ is an $e$-rim hook of $\lambda^{(j)}$ for some $j \in \{1, \dots, r\}$. We say that $\boldsymbol{\lambda}$ is an *$e$-multicore* if $\lambda^{(j)}$ is an $e$-core for all $j \in \{1, \dots, r\}$, and the *$e$-multicore* of $\boldsymbol{\lambda}$ is the $r$-partition $\overline{\boldsymbol{\lambda}}$ given by $\overline{\boldsymbol{\lambda}} \coloneqq \big(\overline{\lambda^{(1)}}, \dots, \overline{\lambda^{(r)}}\big)$.

Now let $\mathbf{s} = (\mathsf{s}_1, \dots, \mathsf{s}_r) \in \mathbb{Z}^r$ be a *multicharge*. The $\mathbf{s}$-residue $\operatorname{res}^{\mathbf{s}}(\gamma)$ of a node $\gamma = (a, b, j)$ is the $\mathsf{s}_j$-residue of the node $(a, b)$, in other words

$$\operatorname{res}^{\mathbf{s}}(a, b, j) = \operatorname{res}^{\mathsf{s}_j}(a, b) = b - a + \mathsf{s}_j \pmod{e}.$$

Again, given $i \in \mathbb{Z}/e\mathbb{Z}$ an $(i, \mathbf{s})$-node (or simply $i$-node) is a node of $\mathbf{s}$-residue $i$. We denote by $c_i^{\mathbf{s}}(\boldsymbol{\lambda})$ the number of $(i, \mathbf{s})$-nodes of $\boldsymbol{\lambda}$. We define

$$\alpha^{\mathbf{s}}(\boldsymbol{\lambda}) \coloneqq \sum_{\gamma \in \mathcal{Y}(\boldsymbol{\lambda})} \alpha_{\operatorname{res}^{\mathbf{s}}(\gamma)} = \sum_{i \in \mathbb{Z}/e\mathbb{Z}} c_i^{\mathbf{s}}(\boldsymbol{\lambda}) \alpha_i \in Q_+.$$

Following [Fa06], the *$\mathbf{s}$-weight* (or simply *weight*) of an $r$-partition $\boldsymbol{\lambda}$ is

$$\mathsf{w}^{\mathbf{s}}(\boldsymbol{\lambda}) = \sum_{j=1}^{r} c_{\mathsf{s}_j}^{\mathbf{s}}(\boldsymbol{\lambda}) - \frac{1}{2} \sum_{i \in \mathbb{Z}/e\mathbb{Z}} \big(c_i^{\mathbf{s}}(\boldsymbol{\lambda}) - c_{i+1}^{\mathbf{s}}(\boldsymbol{\lambda})\big)^2.$$

Note that we recover the corresponding definition given at §2.1 when $r = 1$.

*Remark* 2.46. (See, for instance, [JaLe, §3.1][1]) This definition of $\mathbf{s}$-weight, which can be directly computed from the Young diagram, can be interpreted in Lie-theoretic terms as follows. The Fock space $\mathcal{F}_{\mathbf{s}} = \oplus_{\boldsymbol{\lambda}} \mathbb{Q}(v)\boldsymbol{\lambda}$, where $v$ is an indeterminate and $\boldsymbol{\lambda}$ runs over all the $r$-partitions, has a structure of an integrable $\mathcal{U}_v(\widehat{\mathfrak{sl}}_e)$-module. For the corresponding weight space decomposition, each $r$-partition $\boldsymbol{\lambda}$ is a weight vector and applying Kac's bilinear form on the corresponding weight gives a scalar $\|\boldsymbol{\lambda}\|^{\mathbf{s}}$. We then have the relation $\mathsf{w}^{\mathbf{s}}(\boldsymbol{\lambda}) = \|\boldsymbol{\emptyset}\|^{\mathbf{s}} - \|\boldsymbol{\lambda}\|^{\mathbf{s}}$ (see, for instance, [JaLe, Proposition 3.5]).

We extend the definition of the $\mathbf{s}$-weight to $Q$ by setting

$$\mathsf{w}^{\mathbf{s}}(\alpha) \coloneqq \sum_{j=1}^{r} c_{\mathsf{s}_j} - \frac{1}{2} \sum_{i \in \mathbb{Z}/e\mathbb{Z}} \big(c_i - c_{i+1}\big)^2 \in \mathbb{Z},$$

for any $\alpha = \sum_{i \in \mathbb{Z}/e\mathbb{Z}} c_i \alpha_i \in Q$. Note that $\mathsf{w}^{\mathbf{s}}(\alpha) \in \mathbb{Z}$ for the same reason as in level one, and

$$\mathsf{w}^{\mathbf{s}}(\boldsymbol{\lambda}) = \mathsf{w}^{\mathbf{s}}\big(\alpha^{\mathbf{s}}(\boldsymbol{\lambda})\big).$$

We now recall some results from [Fa06]. The first one follows from Lemma 2.6.

[1] In this remark, the reference [JaLe] refers to the `arXiv` version, since a part of the exposition has been removed from the published version.

**Lemma 2.47** ([Fa06, Corollary 3.4]). *Assume that $e > 0$ and let $\boldsymbol{\lambda}$ be an $r$-partition obtained from $\boldsymbol{\mu}$ by adding an $e$-rim hook. We have $\mathsf{w}^{\mathsf{s}}(\boldsymbol{\lambda}) = \mathsf{w}^{\mathsf{s}}(\boldsymbol{\mu}) + r$.*

More generally, if $e > 0$ then for any $\alpha \in Q$ and any $h \in \mathbb{Z}$ we have

$$\mathsf{w}^{\mathsf{s}}(\alpha + h\delta) = \mathsf{w}^{\mathsf{s}}(\alpha) + rh. \tag{2.48}$$

If $\boldsymbol{\lambda}$ is an $e$-multicore, for any $i \in \mathbb{Z}/e\mathbb{Z}$ and $j \in \{1, \dots, r\}$ we define

$$\begin{aligned}
\mathfrak{b}^{\mathsf{s}}_{ij}(\boldsymbol{\lambda}) &\coloneqq \mathfrak{b}^{\mathsf{s}_j}_i\big(\lambda^{(j)}\big),\\
y^{\mathsf{s}}_{ij}(\boldsymbol{\lambda}) &\coloneqq y^{\mathsf{s}_j}_i\big(\lambda^{(j)}\big),\\
x^{\mathsf{s}}_{ij}(\boldsymbol{\lambda}) &\coloneqq x^{\mathsf{s}_j}_i\big(\lambda^{(j)}\big).
\end{aligned}$$

Moreover, if $e > 0$ then for any $j, k \in \{1, \dots, r\}$ we define

$$u_{j,k}(\boldsymbol{\lambda}) \coloneqq -x^{\mathsf{s}}_{\mathsf{s}_j,j}(\boldsymbol{\lambda}) - \cdots - x^{\mathsf{s}}_{\mathsf{s}_j+i-1,j}(\boldsymbol{\lambda}),$$

if $\mathsf{s}_j \neq \mathsf{s}_k$, where $i \geq 1$ is any integer such that $i = \mathsf{s}_k - \mathsf{s}_j \pmod e$ (recall (2.26)). We define $u_{j,k}(\boldsymbol{\lambda}) \coloneqq 0$ if $\mathsf{s}_j = \mathsf{s}_k$. The point of this definition is that

$$c^{\mathsf{s}_j}_{\mathsf{s}_k}(\lambda^{(j)}) = \frac{1}{2}\big\|x^{\mathsf{s}_j}\big(\lambda^{(j)}\big)\big\|^2 + u_{j,k}(\boldsymbol{\lambda}), \tag{2.49}$$

for all $j, k \in \{1, \dots, r\}$, by Corollary 2.32

**Lemma 2.50** ([Fa06, Proposition 3.5]). *Write $\boldsymbol{\lambda} = \big(\lambda^{(1)}, \dots, \lambda^{(r)}\big)$. If $\boldsymbol{\lambda}$ is a multicore then*

$$\mathsf{w}^{\mathsf{s}}(\boldsymbol{\lambda}) = \sum_{1 \leq j < k \leq r} \mathsf{w}^{(\mathsf{s}_j, \mathsf{s}_k)}\big(\lambda^{(j)}, \lambda^{(k)}\big).$$

Note that Lemma 2.50 can be proved by a direct calculation using the elements $x^{\mathsf{s}}_{ij}(\boldsymbol{\lambda})$ together with (2.25) and (2.49). Using the same idea, we will give a proof of the next proposition.

**Proposition 2.51** ([Fa06, Corollary 3.9]). *For any $r$-partition $\boldsymbol{\lambda}$ we have $\mathsf{w}^{\mathsf{s}}(\boldsymbol{\lambda}) \geq 0$.*

*Proof.* By Lemma 2.47 it suffices to consider the case where $\boldsymbol{\lambda}$ is a multicore, moreover by Lemma 2.50 it suffices to consider the case $r = 2$. Besides, by Remark 2.5 we can assume $e > 0$. By (2.4) and since $\mathsf{w}^{(\mathsf{s}_1,\mathsf{s}_2)}(\lambda^{(1)}, \lambda^{(2)}) = \mathsf{w}^{(\mathsf{s}_2,\mathsf{s}_1)}(\lambda^{(2)}, \lambda^{(1)})$, we can further assume that $\mathsf{s} = (0, \mathsf{s})$ with $\mathsf{s} \in \{0, \dots, e-1\}$. We have, using (2.25) and (2.49),

$$\begin{aligned}
\mathsf{w}^{\mathsf{s}}(\boldsymbol{\lambda}) &= c^{\mathsf{s}}_0(\boldsymbol{\lambda}) + c^{\mathsf{s}}_{\mathsf{s}}(\boldsymbol{\lambda}) - \frac{1}{2}\sum_{i \in \mathbb{Z}/e\mathbb{Z}} \big(c^{\mathsf{s}}_i(\boldsymbol{\lambda}) - c^{\mathsf{s}}_{i+1}(\boldsymbol{\lambda})\big)^2\\
&= \sum_{j=1}^{2}\Big(c^{\mathsf{s}_j}_0(\lambda^{(j)}) + c^{\mathsf{s}_j}_{\mathsf{s}}(\lambda^{(j)})\Big) - \frac{1}{2}\sum_{i \in \mathbb{Z}/e\mathbb{Z}}\left(\sum_{j=1}^{2}\Big(c^{\mathsf{s}_j}_i(\lambda^{(j)}) - c^{\mathsf{s}_j}_{i+1}(\lambda^{(j)})\Big)\right)^2\\
&= \Big(\big\|x^0(\lambda^{(1)})\big\|^2 + u_{1,2}(\boldsymbol{\lambda})\Big) + \Big(\big\|x^{\mathsf{s}}(\lambda^{(2)})\big\|^2 + u_{2,1}(\boldsymbol{\lambda})\Big) - \frac{1}{2}\sum_{i \in \mathbb{Z}/e\mathbb{Z}} \big(x^{\mathsf{s}}_{i1}(\boldsymbol{\lambda}) + x^{\mathsf{s}}_{i2}(\boldsymbol{\lambda})\big)^2\\
&= \big\|x^0(\lambda^{(1)})\big\|^2 + \big\|x^{\mathsf{s}}(\lambda^{(2)})\big\|^2 + u_{1,2}(\boldsymbol{\lambda}) + u_{2,1}(\boldsymbol{\lambda}) - \frac{1}{2}\sum_{i \in \mathbb{Z}/e\mathbb{Z}} \big(x^{\mathsf{s}}_{i1}(\boldsymbol{\lambda})^2 + x^{\mathsf{s}}_{i2}(\boldsymbol{\lambda})^2 + 2x^{\mathsf{s}}_{i1}(\boldsymbol{\lambda})x^{\mathsf{s}}_{i2}(\boldsymbol{\lambda})\big)\\
&= \frac{1}{2}\Big(\|x^0(\lambda^{(1)})\|^2 + \|x^{\mathsf{s}}(\lambda^{(2)})\|^2\Big) + u_{1,2}(\boldsymbol{\lambda}) + u_{2,1}(\boldsymbol{\lambda}) - \sum_{i \in \mathbb{Z}/e\mathbb{Z}} x^{\mathsf{s}}_{i1}(\boldsymbol{\lambda})x^{\mathsf{s}}_{i2}(\boldsymbol{\lambda})\\
&= \frac{1}{2}\Big(\|x^0(\lambda^{(1)})\|^2 + \|x^{\mathsf{s}}(\lambda^{(2)})\|^2\Big) + u_{1,2}(\boldsymbol{\lambda}) + u_{2,1}(\boldsymbol{\lambda}) - \sum_{i \in \mathbb{Z}/e\mathbb{Z}} x^0_i(\lambda^{(1)})x^{\mathsf{s}}_i(\lambda^{(2)})\\
&= \frac{1}{2}\|x^0(\lambda^{(1)}) - x^{\mathsf{s}}(\lambda^{(2)})\|^2 + u_{1,2}(\boldsymbol{\lambda}) + u_{2,1}(\boldsymbol{\lambda}).
\end{aligned}$$

If $\mathsf{s} = 0$ we have $u_{1,2}(\boldsymbol{\lambda}) = u_{2,1}(\boldsymbol{\lambda}) = 0$ so that $\mathsf{w}^{\mathbf{s}}(\boldsymbol{\lambda}) \geq 0$ as desired, otherwise we have $\mathsf{s} \in \{1, \dots, e-1\}$ so that

$$u_{1,2}(\boldsymbol{\lambda}) = -x^{\mathbf{s}}_{0,1}(\boldsymbol{\lambda}) - \cdots - x^{\mathbf{s}}_{\mathsf{s}-1,1}(\boldsymbol{\lambda}),$$
$$u_{2,1}(\boldsymbol{\lambda}) = -x^{\mathbf{s}}_{\mathsf{s},2}(\boldsymbol{\lambda}) - \cdots - x^{\mathbf{s}}_{e-1,2}(\boldsymbol{\lambda}).$$

Now by (2.26) we have

$$u_{2,1}(\boldsymbol{\lambda}) = x^{\mathbf{s}}_{0,2}(\boldsymbol{\lambda}) + \cdots + x^{\mathbf{s}}_{\mathsf{s}-1,2}(\boldsymbol{\lambda}).$$

Hence, setting $z \coloneqq x^{\mathsf{0}}(\lambda^{(1)}) - x^{\mathsf{s}}(\lambda^{(2)})$ we obtain

$$\mathsf{w}^{\mathbf{s}}(\boldsymbol{\lambda}) = \frac{1}{2}\|z\|^2 - z_0 - \cdots - z_{\mathsf{s}-1},$$

thus we conclude by the below Lemma 2.52, recalling (2.26). □

**Lemma 2.52.** *Assume that $e > 0$. Let $z = (z_1, \dots, z_e) \in \mathbb{Z}^e$ such that $z_1 + \cdots + z_e = 0$. Then $\|z\|^2 \geq 2(z_1 + \cdots + z_{\mathsf{s}})$ for any $\mathsf{s} \in \{1, \dots, e\}$.*

*Proof.* First, note that for any integer $t \in \mathbb{Z}$ we have $t^2 \geq 2t$ unless if $t = 1$. We define the following two sets:

$$I \coloneqq \{i \in \{1, \dots, \mathsf{s}\} : z_i = 1\},$$
$$J \coloneqq \{i \in \{1, \dots, e\} : z_i \leq 0\},$$

in particular $I \cap J = \emptyset$. We have

$$0 = z_1 + \cdots + z_e \geq \sum_{i \in J} z_i + \sum_{i \in I} z_i,$$

so that $|I| + \sum_{i \in J} z_i \leq 0$. For any $i \in J$ we have $z_i \in \mathbb{Z}_{\leq 0}$ thus

$$z_i^2 - 2z_i \geq z_i^2 \geq -z_i,$$

hence we obtain

$$\begin{aligned}
\|z\|^2 - 2(z_1 + \cdots + z_{\mathsf{s}}) &\geq \sum_{i \in I}(z_i^2 - 2z_i) + \sum_{i \in J}(-z_i) \\
&= \sum_{i \in I}(-1) - \sum_{i \in J} z_i \\
&= -|I| - \sum_{i \in J} z_i \\
&\geq 0,
\end{aligned}$$

as desired. □

*Remark* 2.53. Fayers' proof of Proposition 2.51 uses Proposition 2.55 (from which the result is immediate). The proof we give here has the advantage to be more direct from the definition of $\mathsf{w}^{\mathbf{s}}$ and $c_i^{\mathbf{s}}$ (a similar remark holds for the proof of Lemma 2.50).

Note that $\mathsf{w}^{\mathbf{s}}(\boldsymbol{\lambda}) \geq r \sum_{j=1}^{r} \mathsf{w}^{\mathsf{s}_j}(\lambda^{(j)})$. Indeed, by Propositions 2.31 and 2.51 the inequality holds when $\boldsymbol{\lambda}$ is an $e$-multicore, and we conclude by Lemma 2.47.

Finally, if $\boldsymbol{\lambda}$ is an $e$-multicore, for any $i \in \mathbb{Z}/e\mathbb{Z}$ and $j, k \in \{1, \dots, r\}$, as in [Fa06, Fa07a] we define

$$\gamma^{\mathbf{s}}_{i,jk}(\boldsymbol{\lambda}) \coloneqq \frac{\mathfrak{b}^{\mathbf{s}}_{ij}(\boldsymbol{\lambda}) - \mathfrak{b}^{\mathbf{s}}_{ik}(\boldsymbol{\lambda})}{e} = y^{\mathbf{s}}_{ij}(\boldsymbol{\lambda}) - y^{\mathbf{s}}_{ik}(\boldsymbol{\lambda}) \in \mathbb{Z}.$$

These integers depend on the multicharge $\mathbf{s}$, however for any $i, l \in \mathbb{Z}/e\mathbb{Z}$ and $j, k \in \{1, \dots, r\}$ the set defined by the integers

$$\gamma_{il,jk}(\boldsymbol{\lambda}) \coloneqq \gamma^{\mathbf{s}}_{i,jk}(\boldsymbol{\lambda}) - \gamma^{\mathbf{s}}_{l,jk}(\boldsymbol{\lambda}), \tag{2.54}$$

does not.

**Proposition 2.55** ([Fa06, Proposition 3.8]). *Assume that $r = 2$ and let $\boldsymbol{\lambda}$ be a bicore. Assume that $\gamma_{il,12}(\boldsymbol{\lambda}) \leq 2$ for all $i, l \in \mathbb{Z}/e\mathbb{Z}$. Then $\mathsf{w}^{\mathbf{s}}(\boldsymbol{\lambda})$ is the smaller of the two integers*

$$\#\big\{i \in \mathbb{Z}/e\mathbb{Z} : \gamma_{il,12}(\boldsymbol{\lambda}) = 2 \text{ for some } l \in \mathbb{Z}/e\mathbb{Z}\big\},$$

*and*

$$\#\big\{l \in \mathbb{Z}/e\mathbb{Z} : \gamma_{il,12}(\boldsymbol{\lambda}) = 2 \text{ for some } i \in \mathbb{Z}/e\mathbb{Z}\big\}.$$

Proposition 2.55 immediately implies that if $\boldsymbol{\lambda}$ is a bicore satisfying $\gamma_{il,12}(\boldsymbol{\lambda}) \leq 2$ for all $i, l \in \mathbb{Z}/e\mathbb{Z}$ then $\mathsf{w}^{\mathbf{s}}(\boldsymbol{\lambda}) \leq e$. We will see that we can strengthen this inequality for a certain class of multipartitions. One on the aims of this paper is to study the consequences of such an inequality, namely, on the blocks of multipartitions.

*Remark* 2.56. By [Fa06, Lemma 3.7], the $\mathbf{s}$-weight function on $e$-multicores is not bounded. For instance, if $e > 0$ assume that $r = 2$ and $\mathbf{s} = (0, 0)$, and let $\lambda$ and $\mu$ be two bicores. By (2.25) and Corollary 2.32 we have

$$\begin{aligned}
\mathsf{w}^{\mathbf{s}}(\lambda, \mu) &= 2c_0^0(\lambda, \mu) - \frac{1}{2} \sum_{i \in \mathbb{Z}/e\mathbb{Z}} \big(c_i^0(\lambda, \mu) - c_{i+1}^0(\lambda, \mu)\big)^2 \\
&= 2c_0^0(\lambda) + 2c_0^0(\mu) - \frac{1}{2} \sum_{i \in \mathbb{Z}/e\mathbb{Z}} \big(c_i^0(\lambda) - c_{i+1}^0(\lambda) + c_i^0(\mu) - c_{i+1}^0(\mu)\big)^2 \\
&= \big\|x^0(\lambda)\big\|^2 + \big\|x^0(\mu)\big\|^2 - \frac{1}{2} \sum_{i \in \mathbb{Z}/e\mathbb{Z}} \big(x_i^0(\lambda) + x_i^0(\mu)\big)^2 \\
&= \frac{1}{2}\big\|x^0(\lambda) - x^0(\mu)\big\|^2.
\end{aligned}$$

Thus, if we chose the $e$-core $\mu$ so that $x^0(\mu) = -x^0(\lambda)$ (which satisfies the conditions of Proposition 2.27 indeed) we obtain

$$\mathsf{w}^{\mathbf{s}}(\lambda, \mu) = 2\big\|x^0(\lambda)\big\|^2 = c_0^0(\lambda),$$

which is not bounded since we can find $e$-cores with arbitrary large number of 0-nodes (see, for instance, Example 2.44). If $e = 0$ one can simply consider the partition $\lambda^{[h]} := (h, \dots, h)$ where $h \geq 1$ is repeated $h$ times, and see that $\mathsf{w}^{\mathbf{s}}(\lambda^{[h]}, \emptyset) = \mathsf{w}^0(\lambda^{[h]}) + c_0^0(\lambda^{[h]}) = 0 + h = h$.

## 2.5 Blocks for multipartitions

The combinatorics between an $r$-partition $\boldsymbol{\lambda}$ and $\alpha^{\mathbf{s}}(\boldsymbol{\lambda})$ is more intricate than in the case $r = 1$. For instance, the natural notion of $e$-multicore no more suffices to distinguish the blocks between two $r$-partitions. However, as in the level 1 case the quantities $\alpha^{\mathbf{s}}(\boldsymbol{\lambda})$ determine the blocks of the associated *Ariki–Koike algebra* (see [LyMa]).

*Example* 2.57. Let $r = 2$, $e = 2$ and $\mathbf{s} = (0, 1)$. The bipartitions $\boldsymbol{\lambda} = \big((1), (1)\big)$ and $\boldsymbol{\mu} = \big((2), \emptyset\big)$ lie in the same block $\alpha_0 + \alpha_1$, however $\boldsymbol{\lambda}$ is a bicore whereas $\boldsymbol{\mu}$ is not.

Again, we define

$$Q^{\mathbf{s}} := \{\alpha^{\mathbf{s}}(\boldsymbol{\lambda}) : \boldsymbol{\lambda} \text{ is an } r\text{-partition of } n \text{ for some } n \in \mathbb{N}\}.$$

*Remark* 2.58. As we mentioned in §2.1, the case $e = 1$ is particularly easy to deal with, and indeed we have here $Q^{\mathbf{s}} = \mathbb{N}\alpha_0 = Q_+$.

Since $\alpha^{\mathbf{s}}(\boldsymbol{\lambda}) = \sum_{j=1}^r \alpha^{\mathbf{s}_j}(\lambda^{(j)})$ we have $Q^{\mathbf{s}} = \sum_{j=1}^r Q^{\mathbf{s}_j}$, however the sum is not direct. In particular, using the 1:1-parametrisation of §2.3 for each $Q^{\mathbf{s}_j}$, we can give a naive parametrisation of $Q^{\mathbf{s}}$ but this parametrisation will not be 1:1. We will rather aim at a generalisation of Lemma 2.39 and Proposition 2.40, looking for an implicit description of $Q^{\mathbf{s}}$. It turns out that the following simple result, which comes from Lemmas 2.8 and 2.20, is the key of what follows.

**Proposition 2.59.** *Let $\alpha \in Q^{\mathbf{s}}$ and $h \in \mathbb{N}$. Then $\alpha + h\delta \in Q^{\mathbf{s}}$.*

We have the following (weak) generalisation of Corollary 2.43 (see also [Fa07b, Theorem 4.7]).

**Proposition 2.60.** *Assume $e > 0$. For any $\alpha \in Q$ and any $h \geq -\max\{\mathsf{w}^{\mathsf{s}_j}(\alpha) : j \in \{1, \dots, r\}\}$ we have*

$$\alpha + h\delta \in Q^{\mathsf{s}}.$$

*Proof.* Let $j \in \{1, \dots, r\}$. If $h \geq -\mathsf{w}^{\mathsf{s}_j}(\alpha)$ then by Corollary 2.43 we have $\alpha + h\delta \in Q^{\mathsf{s}_j}$. We conclude the proof since $Q^{\mathsf{s}_j} \subseteq Q^{\mathsf{s}} = \sum_{k=1}^{r} Q^{\mathsf{s}_k}$. □

As we saw in Example 2.57, the raw notion of $e$-multicores is not adapted to the study of blocks.

**Definition 2.61** ([Fa07a])**.** Let $\alpha \in Q$. We say that $\alpha$ is a *core block* if $\alpha \in Q^{\mathsf{s}}$ and either:

- $e = 0$;
- $e > 0$ and $\alpha - \delta \notin Q^{\mathsf{s}}$.

By Lemmas 2.8 and 2.20, a block $\alpha \in Q^{\mathsf{s}}$ is a core block if and only if every $r$-partition $\boldsymbol{\lambda}$ such that $\alpha^{\mathsf{s}}(\boldsymbol{\lambda}) = \alpha$ is an $e$-multicore.

*Remark* 2.62. In particular, if $r = 1$ then $\alpha^{\mathsf{s}}(\lambda)$ is a core block if and only if $\lambda$ is an $e$-core, in which case $\lambda$ is the only $e$-core (and partition) that lies in $\alpha^{\mathsf{s}}(\lambda)$ (see Lemma 2.35).

*Remark* 2.63. If $e = 1$ then the only $e$-multicore is the empty multipartition, thus there is only one core block, which is $0 \in Q^{\mathsf{s}}$.

*Remark* 2.64. The notion of core block is related to the notion of *maximal weight* for (irreducible) modules on Kac–Moody algebras: see, for instance, [JaLe, §3.1] for the connection with Kac–Moody algebras and [Ca, §20.3] for the notion of maximal weights.

**Definition 2.65** ([LyMa])**.** A multipartition $\boldsymbol{\lambda}$ is an $\mathsf{s}$-*reduced $e$-multicore* if $\alpha^{\mathsf{s}}(\boldsymbol{\lambda})$ is a core block.

Note that the term *core multipartition* has a different meaning, see [Fa19, JaLe]. Any $\mathsf{s}$-reduced $e$-multicore is an $e$-multicore, and the converse holds when $r = 1$ by Remark 2.62.

**Lemma 2.66.** *Let $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(r)})$ be an $\mathsf{s}$-reduced multicore. Then for each $1 \leq j < k \leq r$, the bipartition $(\lambda^{(j)}, \lambda^{(k)})$ is an $(\mathsf{s}_j, \mathsf{s}_k)$-reduced bicore.*

*Proof.* We prove the contraposition. Assume that $(\lambda^{(j)}, \lambda^{(k)})$ is not an $(\mathsf{s}_j, \mathsf{s}_k)$-reduced bicore. Then we can find a bipartition $(\mu^{(j)}, \mu^{(k)})$ in its block which is not a bicore, in particular we can remove an $e$-rim hook to $\mu^{(j)}$ or $\mu^{(k)}$. But then the multipartition

$$(\lambda^{(1)}, \dots, \lambda^{(j-1)}, \mu^{(j)}, \lambda^{(j+1)}, \dots, \lambda^{(k-1)}, \mu^{(k)}, \lambda^{(k+1)}, \dots, \lambda^{(r)})$$

lie in the same block as $\boldsymbol{\lambda}$ and is not an $e$-multicore, thus $\boldsymbol{\lambda}$ is not a reduced $\mathsf{s}$-multicore. □

**Lemma 2.67.** *Assume that $e > 0$ and let $\alpha \in Q$. There is a unique integer $h \in \mathbb{Z}$ such that $\alpha - h\delta$ is a core block and we have $h = \max\{k \in \mathbb{Z} : \alpha - k\delta \in Q^{\mathsf{s}}\}$. Moreover, we have $\alpha \in Q^{\mathsf{s}} \iff h \geq 0$.*

*Proof.* Write $H \coloneqq \{k \in \mathbb{Z} : \alpha - k\delta \in Q^{\mathsf{s}}\}$. The set $H$ is non-empty by Proposition 2.60 and bounded above by (2.48) and Proposition 2.51. If $h \coloneqq \max H$, then by definition we have $\alpha - h\delta \in Q^{\mathsf{s}}$ and $\alpha - (h+1)\delta \notin Q^{\mathsf{s}}$ thus $\alpha - h\delta$ is a core block. We now prove the unicity. If $h' \in \mathbb{Z}$ is such that $\alpha - h'\delta$ is a core block then in particular $\alpha - h'\delta \in Q^{\mathsf{s}}$ thus $h' \leq h$ by maximality of $h$. Now if $h' < h$ then

$$\alpha - (h'+1)\delta = (\alpha - h\delta) + (h - h' - 1)\delta,$$

but the left-hand side is not in $Q^{\mathsf{s}}$ since $\alpha - h'\delta$ is a core block while the right-hand side is in $Q^{\mathsf{s}}$ by Proposition 2.59, thus $h' = h$. Finally, we have $\alpha \in Q^{\mathsf{s}} \iff 0 \in H \iff h \geq 0$, which concludes the proof. □

*Remark* 2.68. If $e > 0$, Lemma 2.67 ensures that $Q/\langle\delta\rangle \simeq \mathbb{Z}^{e-1}$ is in 1:1-correspondence with the set of core blocks. In particular, if $e \geq 2$ then given a multicharge there is an infinite number of core blocks.

*Remark* 2.69. Let $\alpha \in Q$ and let $h \in \mathbb{Z}$ such that $\hat{\alpha} \coloneqq \alpha - h\delta$ is a core block. By (2.10) we have $h = \frac{\mathsf{w}^{\mathbf{s}}(\alpha)-\mathsf{w}^{\mathbf{s}}(\hat{\alpha})}{r}$. Note that the quantity $\mathsf{w}^{\mathbf{s}}(\hat{\alpha})$ can be computed from $\alpha$, see [Fa07b]. However, the formula given in [Fa07b] does not seem suitable for the purpose of this paper.

**Definition 2.70.** Assume that $e > 0$ and let $\alpha \in Q$. We say that the unique core block of the form $\alpha - h\delta$ for $h \in \mathbb{Z}$ as in Lemma 2.67 is the $\mathbf{s}$-*core* of $\alpha$.

We will simply use *core* instead of $\mathbf{s}$-core when the multicharge $\mathbf{s}$ is understood from the context. Given a multipartition $\boldsymbol{\lambda}$, by Lemma 2.8 to compute the core of $\alpha^{\mathbf{s}}(\boldsymbol{\lambda})$ we can proceed as follows. We first look at the $e$-multicore $\overline{\boldsymbol{\lambda}}$ of $\boldsymbol{\lambda}$. If every $r$-partition lying in the same block as $\overline{\boldsymbol{\lambda}}$ is an $e$-multicore then we are done (in which case $\alpha^{\mathbf{s}}(\overline{\boldsymbol{\lambda}})$ is the core of $\alpha^{\mathbf{s}}(\boldsymbol{\lambda})$), otherwise we repeat the procedure with an $r$-partition in the block of $\overline{\boldsymbol{\lambda}}$ that is not an $e$-multicore.

*Remark* 2.71. The terminology fits with the usual notion of $e$-core when $r = 1$: if $\lambda$ and $\mu$ are two partitions then $\alpha^{\mathbf{s}}(\lambda)$ is the $\mathbf{s}$-core of $\alpha^{\mathbf{s}}(\mu)$ if and only if $\lambda$ is the $e$-core of $\mu$ (by Remark 2.62 together with Lemmas 2.8 and 2.33).

By Lemma 2.47 and Proposition 2.51, we have the following particular case of core block. The aim of this paper is to give a (weak) converse.

**Proposition 2.72.** *Let $\alpha \in Q^{\mathbf{s}}$ with $\mathsf{w}^{\mathbf{s}}(\alpha) \leq r - 1$. Then $\alpha$ is a core block.*

We say that a multicharge $\mathbf{s}' \in \mathbb{Z}^e$ is *compatible* with $\mathbf{s}$ (or simply *compatible* if $\mathbf{s}$ is clear from the context) if $\mathbf{s}' - \mathbf{s} \in e\mathbb{Z}$. In particular, note that if $\mathbf{s}'$ is a compatible multicharge then a multicore $\boldsymbol{\lambda}$ is $\mathbf{s}$-reduced if and only if it is $\mathbf{s}'$-reduced. We have the following characterisation of core blocks in terms of abaci.

**Proposition 2.73** ([Fa07a, Theorem 3.1])**.** *Assume that $e > 0$ and let $\boldsymbol{\lambda}$ be an $e$-multicore. The following assertions are equivalent.*

(*i*) *The $e$-multicore $\boldsymbol{\lambda}$ is $\mathbf{s}$-reduced.*

(*ii*) *There exist a compatible multicharge $\mathbf{s}'$ and integers $\mathfrak{b}_1, \dots, \mathfrak{b}_e \in \mathbb{Z}$ such that*

$$\mathfrak{b}^{\mathbf{s}'}_{ij}(\boldsymbol{\lambda}) \in \{\mathfrak{b}_i, \mathfrak{b}_i + e\},$$

*for all $i \in \mathbb{Z}/e\mathbb{Z}$ and $j \in \{1, \dots, r\}$.*

(*iii*) *There exist a compatible multicharge $\mathbf{s}'$ and integers $\sigma_1, \dots, \sigma_r \in \mathbb{Z}$ such that*

$$\gamma^{\mathbf{s}'}_{i,jk}(\boldsymbol{\lambda}) \leq \sigma_j - \sigma_k + 1,$$

*for all $i \in \mathbb{Z}/e\mathbb{Z}$ and $j, k \in \{1, \dots, r\}$.*

(*iv*) *For any compatible multicharge $\mathbf{s}'$, there exist integers $\sigma_1, \dots, \sigma_r \in \mathbb{Z}$ such that*

$$\gamma^{\mathbf{s}'}_{i,jk}(\boldsymbol{\lambda}) \leq \sigma_j - \sigma_k + 1,$$

*for all $i \in \mathbb{Z}/e\mathbb{Z}$ and $j, k \in \{1, \dots, r\}$.*

**Corollary 2.74.** *Assume that $e > 0$. An $e$-multicore $\boldsymbol{\lambda}$ is $\mathbf{s}$-reduced if and only if there exists a compatible multicharge $\mathbf{s}'$ such that*

$$\gamma^{\mathbf{s}'}_{i,jk}(\boldsymbol{\lambda}) \leq 1,$$

*for all $i \in \mathbb{Z}/e\mathbb{Z}$ and $j, k \in \{1, \dots, r\}$.*

*Proof.* If the $e$-multicore $\boldsymbol{\lambda}$ is $\mathbf{s}$-reduced then by Proposition 2.73(*ii*) we can write $\mathfrak{b}^{\mathbf{s}'}_{ij}(\boldsymbol{\lambda}) = \mathfrak{b}_i + \epsilon_{ij}e$ for all $i \in \mathbb{Z}/e\mathbb{Z}$ and $j \in \{1, \dots, r\}$, where $\epsilon_{ij} \in \{0, 1\}$ and $\mathbf{s}'$ is a compatible multicharge. Thus, we obtain

$$\gamma^{\mathbf{s}'}_{i,jk}(\boldsymbol{\lambda}) = \frac{\mathfrak{b}^{\mathbf{s}'}_{ij}(\boldsymbol{\lambda}) - \mathfrak{b}^{\mathbf{s}'}_{ik}(\boldsymbol{\lambda})}{e} = \epsilon_{ij} - \epsilon_{ik} \leq 1,$$

for all $i \in \mathbb{Z}/e\mathbb{Z}$ and $j, k \in \{1, \dots, r\}$. Conversely, if there exists a compatible multicharge $\mathbf{s}'$ such that $\gamma^{\mathbf{s}'}_{i,jk}(\boldsymbol{\lambda}) \leq 1$ for all $i \in \mathbb{Z}/e\mathbb{Z}$ and $j, k \in \{1, \dots, r\}$ then item *(iii)* of Proposition 2.73 is satisfied with $\sigma_1 = \cdots = \sigma_r$ thus the $e$-multicore $\boldsymbol{\lambda}$ is $\mathbf{s}$-reduced. □

# 3 Implicit description of the blocks

We will give in this section the central results of the paper. We will first give a formula to compute the weight as a map from binary matrices. Let $r \geq 2$ and $e \geq 1$.

## 3.1 Weight of a tuple of subsets

We first begin with the following standard fact. To an $r$-tuple of subsets $(E_1, \dots, E_r)$ of $\{1, \dots, e\}$ we can associate the $e \times r$ binary matrix $\mathcal{E}$ such that the characteristic vector $\mathbf{1}_{E_j}$ of $E_j$ is the $j$-th column of $\mathcal{E}$ for any $j \in \{1, \dots, r\}$. Conversely, to an $e \times r$ binary matrix $\mathcal{E}$ we can associate the $r$-tuple $(E_1, \dots, E_r)$ of subsets of $\{1, \dots, e\}$ such that for any $j \in \{1, \dots, r\}$ the $j$-th column of $\mathcal{E}$ is exactly $\mathbf{1}_{E_j}$.

Now recall from Proposition 2.73*(ii)* that to any $\mathbf{s}$-reduced $e$-multicore we can associate an $e \times r$ binary matrix. Conversely, given an $e \times r$ binary matrix $\mathcal{E} = (\epsilon_{ij})$ and integers $\mathfrak{b}_1, \dots, \mathfrak{b}_e \in \mathbb{Z}$ such that $\mathfrak{b}_i = i \pmod e$ for each $i \in \{1, \dots, e\}$, we can construct an $\mathbf{s}$-reduced $e$-multicore $\boldsymbol{\lambda} = \boldsymbol{\lambda}(\mathcal{E}, (\mathfrak{b}_i))$, where the multicharge $\mathbf{s}$ is determined by (2.14). Note that the integers

$$\gamma^{\mathbf{s}}_{i,jk}(\boldsymbol{\lambda}) = \epsilon_{ij} - \epsilon_{ik}, \tag{3.1}$$

do not depend on the choice of $(\mathfrak{b}_i)$ so that, recalling Proposition 2.55, the weight $\mathsf{w}^{\mathbf{s}}(\boldsymbol{\lambda})$ only depends on $\mathcal{E}$. Now if $(E_1, \dots, E_r)$ is the $r$-tuple of subsets of $\{1, \dots, e\}$ corresponding to $\mathcal{E}$, we can thus write:

$$\mathsf{w}(E_1, \dots, E_r) \coloneqq \mathsf{w}^{\mathbf{s}}(\boldsymbol{\lambda}).$$

*Example* 3.2. Take $e = 3$ and $r = 2$. We consider the following subsets of $\{1, 2, 3\}$:

$$E_1 = \{1, 2, 3\}, \quad E_2 = \{3\},$$

which correspond to the $3 \times 2$ binary matrix $\mathcal{E} = (\epsilon_{ij}) = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 1 \end{pmatrix}$. We also consider the following integers:

$$\mathfrak{b}_1 = 1, \quad \mathfrak{b}_2 = 5, \quad \mathfrak{b}_3 = -3,$$

which satisfy the condition $\mathfrak{b}_i = i \pmod e$. For $i \in \{1, 2, 3\}$ and $j \in \{1, 2\}$ we define $\mathfrak{b}_{ij} = \mathfrak{b}_i + \epsilon_{ij} e$. We obtain:

$$\begin{aligned} \mathfrak{b}_{11} &= 1 + 3 = 4, & \mathfrak{b}_{12} &= 1 + 0 = 1, \\ \mathfrak{b}_{21} &= 5 + 3 = 8, & \mathfrak{b}_{22} &= 5 + 0 = 5, \\ \mathfrak{b}_{31} &= -3 + 3 = 0, & \mathfrak{b}_{32} &= -3 + 3 = 0. \end{aligned}$$

Recall that $\mathfrak{b}_{ij}$ is the largest element of $\beta^{\mathbf{s}_j}(\lambda^{(j)})$ congruent to $i$ modulo $e$. We obtain that the charged beta-number associated with $\lambda^{(1)}$ is:

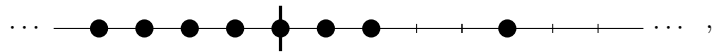
$$\begin{aligned} \beta^{\mathbf{s}_1}(\lambda^{(1)}) &= \{4, 1, -2, -5, -8, \dots\} \sqcup \{8, 5, 2, -1, -4, -7, \dots\} \sqcup \{0, -3, -6, -9, \dots, \} \\ &= \{8, 5, 4, 2, 1, 0, -1, -2, -3, -4, \dots\}, \end{aligned}$$

thus the associated charged abacus is:

thus $\lambda^{(1)} = (3, 1, 1)$. Similarly, we have:

$$\begin{aligned} \beta^{\mathbf{s}_2}(\lambda^{(2)}) &= \{1, -2, -5, -8, \dots\} \sqcup \{5, 2, -1, -4, -7, \dots, \} \sqcup \{0, -3, -6, -9, \dots, \} \\ &= \{5, 2, 1, 0, -1, -2, -3, -4, \dots\}, \end{aligned}$$

thus the associated charged abacus is:



thus $\lambda^{(2)} = (2)$. We can deduce the associated multicharge $\mathbf{s} = (\mathsf{s}_1, \mathsf{s}_2)$, but we can also use the formula

$$\mathsf{s}_j = \frac{3+1}{2} + \frac{1}{3}\sum_{i=1}^{3} \mathfrak{b}_{ij},$$

for $j \in \{1,2\}$, thus obtaining $\mathbf{s} = (6,4)$. Finally, to the pair $(E_1, E_2) = \big(\{1,2,3\}, \{3\}\big)$ of subsets of $\{1,2,3\}$ we have associated the $(6,4)$-reduced 3-multicore $\boldsymbol{\lambda} = \big((3,1,1),(2)\big)$, and the calculation of $\mathsf{w}(E_1,E_2)$ follows.

**Lemma 3.3.** *Assume that $r = 2$ and let $E_1, E_2 \subseteq \{1,\dots,e\}$. We have:*

$$\mathsf{w}(E_1,E_2) = \min(|E_1|,|E_2|) - |E_1 \cap E_2|.$$

*Proof.* Let $\mathfrak{b}_1,\dots,\mathfrak{b}_e \in \mathbb{Z}$ with $\mathfrak{b}_i = i \pmod e$, let $\mathbf{s}$ be the multicharge associated with $(\mathfrak{b}_i)$ and let $\boldsymbol{\lambda}$ be the $\mathbf{s}$-reduced $e$-multicore associated with $(\mathfrak{b}_i)$ and pair $(E_1,E_2)$ of subsets of $\{1,\dots,e\}$. By (3.1), for any $i \in \mathbb{Z}/e\mathbb{Z}$ we have

$$\gamma^{\mathbf{s}}_{i,12}(\boldsymbol{\lambda}) = \mathbf{1}_1(i) - \mathbf{1}_2(i),$$

so that $\gamma^{\mathbf{s}}_{i,12}(\boldsymbol{\lambda}) = 1$ if and only if $i \in E_1 \setminus E_2$. Hence, recalling (2.54),

$$\begin{aligned}\gamma_{il,12}(\boldsymbol{\lambda}) = 2 &\iff \gamma^{\mathbf{s}}_{i,12}(\boldsymbol{\lambda}) = 1 \text{ and } \gamma^{\mathbf{s}}_{l,12}(\boldsymbol{\lambda}) = -1\\ &\iff i \in E_1 \setminus E_2 \text{ and } l \in E_2 \setminus E_1,\end{aligned}$$

so that

$$A \coloneqq \#\{i \in \mathbb{Z}/e\mathbb{Z} : \gamma_{il,12}(\boldsymbol{\lambda}) = 2 \text{ for some } l \in \mathbb{Z}/e\mathbb{Z}\} = \begin{cases} |E_1 \setminus E_2|, & \text{if } |E_2 \setminus E_1| \neq 0,\\ 0, & \text{otherwise,}\end{cases}$$

and similarly

$$B \coloneqq \#\{l \in \mathbb{Z}/e\mathbb{Z} : \gamma_{il,12}(\boldsymbol{\lambda}) = 2 \text{ for some } i \in \mathbb{Z}/e\mathbb{Z}\} = \begin{cases} |E_2 \setminus E_1|, & \text{if } |E_1 \setminus E_2| \neq 0,\\ 0, & \text{otherwise.}\end{cases}$$

Looking whether 0 is in $\big\{|E_1 \setminus E_2|, |E_2 \setminus E_1|\big\}$ or not, we obtain:

$$\min(A,B) = \min\big(|E_1 \setminus E_2|, |E_2 \setminus E_1|\big).$$

Recalling from Proposition 2.55 that $\mathsf{w}(E_1,E_2) = \min(A,B)$, we thus obtain:

$$\begin{aligned}\mathsf{w}(E_1,E_2) &= \min\Big(|E_1| - |E_1 \cap E_2|,\ |E_2| - |E_1 \cap E_2|\Big)\\ &= \min(|E_1|,|E_2|) - |E_1 \cap E_2|,\end{aligned}$$

as announced. □

Note that if $E_1,\dots,E_r \subseteq \{1,\dots,e\}$ then by Lemma 2.50 we have:

$$\mathsf{w}(E_1,\dots,E_r) = \sum_{1 \le j < k \le r} \Big[\min(|E_j|,|E_k|) - |E_j \cap E_k|\Big]. \tag{3.4}$$

We conclude this subsection by the following particular case of Lemma 3.3.

**Lemma 3.5.** *For any $E \subseteq \{1,\dots,e\}$ we have:*

$$\mathsf{w}(E,E) = \mathsf{w}(E,\emptyset) = \mathsf{w}(E,\{1,\dots,e\}) = 0.$$

*Remark* 3.6. Recalling Remark 2.63, if $e = 1$ then we recover the fact that $\mathsf{w}(E) = 0$ for any subset $E \subseteq \{1,\dots,e\}$.

## 3.2 Bound for the weight of core blocks

Recall from Remark 2.68 that there is an infinite number of reduced $e$-multicores.

**Theorem 3.7.** *Let $r \geq 2$ and $e > 0$. There exists a constant $C \geq 0$ such that for all multicharge $\mathbf{s}$ and each $r$-partition $\boldsymbol{\lambda}$ that is an $\mathbf{s}$-reduced $e$-multicore we have*

$$\mathsf{w}^{\mathbf{s}}(\boldsymbol{\lambda}) \leq C.$$

Note that the statement is wrong if $e = 0$ (see Remark 2.56 or the coming Remark 3.10).

*Proof.* We first prove the case $r = 2$, which is clear since by Proposition 2.55 and Corollary 2.74 we have $\mathsf{w}^{\mathbf{s}}(\boldsymbol{\lambda}) \leq e$ (we could also have use Lemma 3.3). We now assume that $r \geq 3$. Since $\boldsymbol{\lambda}$ is a multicore, by Lemma 2.50 we have

$$\mathsf{w}^{\mathbf{s}}(\boldsymbol{\lambda}) = \sum_{1 \leq j < k \leq r} \mathsf{w}^{(\mathsf{s}_j, \mathsf{s}_k)}\big(\lambda^{(j)}, \lambda^{(k)}\big).$$

By Lemma 2.66 we know that each $\big(\lambda^{(j)}, \lambda^{(k)}\big)$ is a reduced $(\mathsf{s}_j, \mathsf{s}_k)$-bicore, thus using the case $r = 2$ we find

$$\mathsf{w}^{\mathbf{s}}(\boldsymbol{\lambda}) \leq e\binom{r}{2},$$

which concludes the proof. □

**Definition 3.8.** We denote by $N_{r,e}$ the smallest possible constant $C$ as in Theorem 3.7, that is:

$$N_{r,e} = \sup\big\{\mathsf{w}^{\mathbf{s}}(\boldsymbol{\lambda}) : \mathbf{s} \in \mathbb{Z}^r \text{ a multicharge and } \boldsymbol{\lambda} \text{ an } \mathbf{s}\text{-reduced } e\text{-multicore}\big\} < \infty.$$

**Corollary 3.9.** *Assume that $e > 0$. We have*

$$Q^{\mathbf{s}} \supseteq \big\{\alpha \in Q : \mathsf{w}^{\mathbf{s}}(\alpha) > N_{r,e} - r\big\}.$$

*Proof.* Let $\alpha \in Q$ and assume that $\alpha \notin Q^{\mathbf{s}}$. We will prove that $\mathsf{w}^{\mathbf{s}}(\alpha) \leq N_{r,e} - r$. By Lemma 2.67, we can find $h \in \mathbb{Z}$ such that $\alpha - h\delta$ is a core block, and since $\alpha \notin Q^{\mathbf{s}}$ we have $h < 0$. Now by Theorem 3.7 we have $\mathsf{w}^{\mathbf{s}}(\alpha - h\delta) \leq N_{r,e}$ thus by (2.48) we have $\mathsf{w}^{\mathbf{s}}(\alpha) \leq N_{r,e} + rh \leq N_{r,e} - r$. This concludes the proof. □

Note that the inclusion of Corollary 3.9 is not trivial, that is, the set $\big\{\alpha \in Q : \mathsf{w}^{\mathbf{s}}(\alpha) > N_{r,e} - r\big\}$ is not empty, and in fact it is even infinite (by (2.48)).

*Remark* 3.10. The result of Corollary 3.9 does not hold if $e = 0$. For instance, assume that $r = 2$ and take $\mathbf{s} = (0, \mathsf{s})$ for $\mathsf{s} \geq 0$. For any $h \geq 1$, as in Remark 2.56 we consider the partition $\lambda^{[h]} = (h, \dots, h)$ where $h$ is repeated $h$ times and we define

$$\alpha^{[h]} := \alpha^0(\lambda^{[h]}) + \alpha_{h+\mathsf{s}+1} = \sum_{|i|<h} (h - |i|)\alpha_i + \alpha_{h+\mathsf{s}+1}.$$

Let $(c_i)_{i\in\mathbb{Z}} \in \mathbb{Z}^{\mathbb{Z}}$ so that $\alpha^{[h]} = \sum_{i\in\mathbb{Z}} c_i\alpha_i$. Since $c_{h+\mathsf{s}} = 0 < c_{h+\mathsf{s}+1}$ and $\mathbf{s} = (0, \mathsf{s})$, we have $\alpha^{[h]} \notin Q^{\mathbf{s}}$ (by (2.25) and (2.28)), however

$$\begin{aligned}
\mathsf{w}^{\mathbf{s}}(\alpha^{[h]}) &= 2c_0 - \frac{1}{2}\sum_{i\in\mathbb{Z}}(c_i - c_{i+1})^2 \\
&= 2c_0 - \frac{1}{2}\left(\sum_{i=-h}^{h-1}(c_i - c_{i+1})^2 + (c_{h+\mathsf{s}} - c_{h+\mathsf{s}+1})^2 + (c_{h+\mathsf{s}+1} - c_{h+\mathsf{s}+2})^2\right) \\
&= 2h - \frac{1}{2}\left(\sum_{i=-h}^{h-1} 1^2 + 1^2 + 1^2\right) \\
&= 2h - \frac{1}{2}(2h + 2) \\
&= h - 1.
\end{aligned}$$

Note that this calculation remains valid for $e \geq 2(h+1)+\mathsf{s}$. This proves that we can find elements in $Q \setminus Q^{\mathsf{s}}$ of arbitrarily large weight.

We will give in §3.3 more information about the constant $N_{r,e}$. In particular, in the following particular cases the situation is the same as in level one (cf. Proposition 2.40).

**Proposition 3.11.** *Assume that* $(r,e) \in \{(2,2),(2,3),(3,2)\}$. *Then* $Q^{\mathsf{s}} = \{\alpha \in Q : \mathsf{w}^{\mathsf{s}}(\alpha) \geq 0\}$.

*Proof.* We will see in Proposition 3.13 (respectively, Proposition 3.33) that $N_{r,e} = 1$ (resp. $N_{r,e} = 2$) if $r = 2$ and $e \in \{2,3\}$ (resp. $r = 3$ and $e = 2$). In these three cases we have $N_{r,e} \leq r-1$, thus

$$\{\alpha \in Q : \mathsf{w}^{\mathsf{s}}(\alpha) \geq 0\} \subseteq \{\alpha \in Q : \mathsf{w}^{\mathsf{s}}(\alpha) \geq N_{r,e} - r + 1\}.$$

We conclude the proof by Proposition 2.51 and Corollary 3.9 since we then find

$$Q^{\mathsf{s}} \subseteq \{\alpha \in Q : \mathsf{w}^{\mathsf{s}}(\alpha) \geq 0\} \subseteq \{\alpha \in Q : \mathsf{w}^{\mathsf{s}}(\alpha) > N_{r,e} - r\} \subseteq Q^{\mathsf{s}}.$$

□

## 3.3 Optimality

We now assume that $e \geq 2$. The aim of this subsection is to give sharp bounds for $N_{r,e}$. Note that we have seen during the proof of Theorem 3.7 that $N_{r,e} \leq \frac{er(r-1)}{2}$. In fact, we will see that this first bound is roughly the value of $N_{r,e}$ multiplied by a factor of 4 (see Corollary 3.29). We will also compute $N_{r,e}$ for small values of $e$ or $r$ (see §3.3.1 and §3.3.3). We first begin by gathering some elementary results.

**Lemma 3.12.** *Let* $k \in \{1,\dots,e-1\}$.

(*i*) *We have:*

$$k\binom{e-1}{k} = (e-1)\binom{e-2}{k-1},$$

*and:*

$$(e-k)\binom{e}{k} = e\binom{e-1}{k}.$$

(*ii*) *The quantity* $k(e-k)$ *is maximal for* $k = \lfloor \frac{e}{2} \rfloor$, *the corresponding maximal value being* $\lfloor \frac{e^2}{4} \rfloor$.

*Proof.* (*i*) The first identity is standard. For the second one, using the identity $\binom{n}{m} = \binom{n}{n-m}$ twice and the first one we have:

$$(e-k)\binom{e}{k} = (e-k)\binom{e}{e-k} = e\binom{e-1}{e-k-1} = e\binom{e-1}{k}.$$

(*ii*) The first assertion is clear. For the second one, the result is clear if $e$ is even, and if $e$ is odd then $\lfloor \frac{e}{2} \rfloor = \frac{e-1}{2}$ and:

$$\left\lfloor \frac{e}{2} \right\rfloor \left(e - \left\lfloor \frac{e}{2} \right\rfloor\right) = \frac{e-1}{2}\frac{e+1}{2} = \frac{e^2-1}{4} = \left\lfloor \frac{e^2}{4} \right\rfloor.$$

□

### 3.3.1 Values for small parameters

We will here compute $N_{r,e}$ when $r = 2$ or $e = 2$.

**Proposition 3.13** (Case $r = 2$)**.** *We have* $N_{2,e} = \lfloor \frac{e}{2} \rfloor$.

*Proof.* By Lemma 3.3, it suffices to prove that for any $E, F \subseteq \{1, \dots, e\}$ we have $\min(|E|, |F|) - |E \cap F| \leq \lfloor \frac{e}{2} \rfloor$ and that equality happen for some $E, F$. Writing $E = (E \cap F) \sqcup E'$ and $F = (E \cap F) \sqcup F'$, we have

$$\min(|E|, |F|) - |E \cap F| = \min(|E'|, |F'|) \leq \left\lfloor \frac{e}{2} \right\rfloor,$$

since $E' \cap F' = \emptyset$. We conclude the proof since equality holds when $|E| = \lfloor \frac{e}{2} \rfloor$ and $F = E^c$. □

**Proposition 3.14** (Case $e = 2$)**.** *We have* $N_{r,2} = \left\lfloor \frac{r^2}{4} \right\rfloor$.

*Proof.* Let $E_1, \dots, E_r \subseteq \{1, 2\}$. By Lemma 3.5, if $E_j = \emptyset$ or $E_j = \{1, 2\}$ for some $j \in \{1, \dots, r\}$ then $\mathsf{w}(E_1, \dots, E_r)$ does not decrease if we replace $E_j$ by $\{1\}$. Hence, we can assume that for all $j \in \{1, \dots, r\}$ we have $E_j = \{1\}$ or $E_j = \{2\}$. Let $s \in \{0, \dots, r\}$ so that, after reordering, we have $E_1 = \cdots = E_s = \{1\}$ and $E_{s+1} = \cdots = E_r = \{2\}$. Then by Lemmas 2.50 and 3.5 we have

$$\mathsf{w}(E_1, \dots, E_r) = s(r - s)\, \mathsf{w}(\{1\}, \{2\}) = s(r - s),$$

and this concludes the proof by Lemma 3.12*(ii)*. □

### 3.3.2 Bounds

Recall that we always assume that $e \geq 2$ (recall from Remark 3.6 that $N_{r,1} = 0$), with the exception of Lemma 3.15. We will first give a lower bound for $N_{r,e}$.

**Lemma 3.15.** *Let $r \geq 2$. The sequence $(N_{r,e})_{e \geq 1}$ is superadditive, in other words:*

$$N_{r,e+e'} \geq N_{r,e} + N_{r,e'},$$

*for any $e, e' \geq 1$.*

*Proof.* Let $E_1, \dots, E_r \subseteq \{1, \dots, e\}$ and $F_1, \dots, F_r \subseteq \{1, \dots, e'\}$. For any $j \in \{1, \dots, r\}$ we define $F'_j \coloneqq \{i + e : i \in F_j\} \subseteq \{e+1, \dots, e+e'\}$. Note that $E_j \cap F'_k = \emptyset$ for all $1 \leq j < k \leq r$. For any $j \in \{1, \dots, r\}$, we define $G_j \coloneqq E_j \sqcup F'_j \subseteq \{1, \dots, e + e'\}$. For any $1 \leq j < k \leq r$ we have

$$\begin{aligned} \min(|G_j|, |G_k|) &= \min(|E_j| + |F_j|, |E_k| + |F_k|) \\ &\geq \min(|E_j|, |E_k|) + \min(|F_j|, |F_k|), \end{aligned}$$

and, recalling that $E_j \cap F'_k = \emptyset$,

$$\begin{aligned} G_j \cap G_k &= (E_j \sqcup F'_j) \cap (E_k \sqcup F'_k) \\ &= (E_j \cap E_k) \sqcup (F'_j \cap F'_k), \end{aligned}$$

so that $\mathsf{w}(G_j, G_k) \geq \mathsf{w}(E_j, E_k) + \mathsf{w}(F_j, F_k)$. By Lemma 2.50 we obtain $\mathsf{w}(G_1, \dots, G_r) \geq \mathsf{w}(E_1, \dots, E_r) + \mathsf{w}(F_1, \dots, F_r)$ and this conclude the proof since $N_{r,e+e'} \geq \mathsf{w}(G_1, \dots, G_r)$ and by taking the maximum on $E_1, \dots, E_r$ and $F_1, \dots, F_r$. □

**Corollary 3.16.** *For any $e \geq 2$ we have* $N_{r,e} \geq \lfloor \frac{e}{2} \rfloor \lfloor \frac{r^2}{4} \rfloor$.

*Proof.* Since $N_{r,1} = 0$ we have $N_{r,e} \geq N_{r,2\lfloor \frac{e}{2} \rfloor} \geq \lfloor \frac{e}{2} \rfloor N_{r,2}$ by Lemma 3.15 and we conclude by Proposition 3.14. □

We will now give an upper bound for $N_{r,e}$. The next proposition is the key of the next results. We write $\mathcal{P}_e \coloneqq \mathcal{P}(\{1,\dots,e\})$ for the powerset of $\{1,\dots,e\}$, the set of subsets of $\{1,\dots,e\}$. We denote by $\|\cdot\|_1$ the 1-norm on $\mathbb{R}^{\mathcal{P}_e}$, given by $\|x\|_1 = \sum_{E\in\mathcal{P}_e}|x_E|$ for all $x = (x_E)_{E\in\mathcal{P}_e} \in \mathbb{R}^{\mathcal{P}_e}$. For any $x, y \in \mathbb{R}^{\mathcal{P}_e}$ we write $\langle x, y\rangle \coloneqq x^\intercal y$ for the canonical scalar product. We have $\langle x, x\rangle = \|x\|^2$, where $\|\cdot\|$ is the Euclidean norm as in Section 2. Let $A_e = (a_{E,F})_{E,F\in\mathcal{P}_e}$ be the matrix given by $a_{E,F} \coloneqq \mathsf{w}(E, F)$ for all $E, F \in \mathcal{P}_e$. The matrix $A_e$ is symmetric of size $2^e$ with all its diagonal entries being 0 and only non-negative entries. We denote by $q_e$ the quadratic form given by $q_e(x) \coloneqq \frac{1}{2}\langle x, A_e x\rangle$ for all $x \in \mathbb{R}^{\mathcal{P}_e}$.

**Proposition 3.17.**

(*i*) *If $x \in \mathbb{N}^{\mathcal{P}_e}$ then $q_e(x) \in \mathbb{N}$.*

(*ii*) *We have:*

$$N_{r,e} = \max_{\substack{x\in\mathbb{N}^{\mathcal{P}_e}\\ \|x\|_1=r}} q_e(x).$$

*Proof.* We have a one-to-one correspondence between unordered $r$-tuples $(E_1,\dots,E_r)$ of subsets of $\{1,\dots,e\}$ and elements $x = (x_E)_{E\in\mathcal{P}_e} \in \mathbb{N}^{\mathcal{P}_e}$ with $\|x\|_1 = r$ given as follows:

$$x_E = \#\big\{j \in \{1,\dots,r\} : E_j = E\big\}.$$

For such elements, by Lemmas 2.50 and 3.5 we have

$$\begin{aligned}
\mathsf{w}(E_1,\dots,E_r) &= \sum_{1\le j<k\le r} \mathsf{w}(E_j, E_k)\\
&= \sum_{1\le j<k\le r} a_{E_j,E_k}\\
&= \frac{1}{2}\sum_{1\le j,k\le r} a_{E_j,E_k}\\
&= \frac{1}{2}\sum_{E,F\in\mathcal{P}_e} a_{E,F}x_E x_F\\
&= q_e(x),
\end{aligned}$$

which concludes the proof. □

We will now study the quadratic form $q_e$. Our first aim is to prove that it suffices to study the restrictions of $q_e$ to the subspaces of $\mathbb{R}^{\mathcal{P}_e}$ corresponding to subsets of $\{1,\dots,e\}$ of same cardinalities.

**Lemma 3.18.** *Let $E, F \subseteq \{1,\dots,e\}$. We have $\mathsf{w}(E, F) = \mathsf{w}(E^c, F^c)$.*

*Proof.* A simple calculation gives

$$\begin{aligned}
\mathsf{w}(E^c, F^c) &= \min(|E^c|, |F^c|) - |E^c \cap F^c|\\
&= \min(e - |E|, e - |F|) - |(E \cup F)^c|\\
&= e - \max(|E|, |F|) - e + |E \cup F|\\
&= -\max(|E|, |F|) + |E| + |F| - |E \cap F|\\
&= \min(|E|, |F|) - |E \cap F|\\
&= \mathsf{w}(E, F).
\end{aligned}$$

□

**Lemma 3.19.** *Let $E_1,\dots,E_r \subseteq \{1,\dots,e\}$. Let $m \coloneqq \min_{1\le j\le r}|E_j|$ and assume that $m < e$. Let $j_0$ such that $|E_{j_0}| = m$ and let $x \in \{1,\dots,e\} \setminus E_{j_0}$. For $j \in \{1,\dots,r\}$, we define*

$$\widetilde{E}_j \coloneqq \begin{cases} E_j, & \text{if } j \ne j_0,\\ E_{j_0} \cup \{x\}, & \text{if } j = j_0. \end{cases}$$

*Then*

$$\mathsf{w}(\widetilde{E}_1,\dots,\widetilde{E}_r) = \mathsf{w}(E_1,\dots,E_r) + \#\big\{j\in\{1,\dots,r\} : |E_j| > m \text{ and } x\notin E_j\big\} \\ - \#\big\{j\in\{1,\dots,r\} : |E_j| = m \text{ and } x\in E_j\big\}.$$

*Proof.* Let $j\neq k\in\{1,\dots,e\}$. If $j,k\neq j_0$ then $\mathsf{w}(\widetilde{E}_j,\widetilde{E}_k) = \mathsf{w}(E_j,E_k)$. If $k=j_0$ then, recalling that $\mathbf{1}_j$ denotes the characteristic vector of $E_j$,

$$\begin{aligned}\mathsf{w}(\widetilde{E}_j,\widetilde{E}_{j_0}) &= \mathsf{w}(E_j, E_{j_0}\sqcup\{x\}) \\ &= \min(|E_j|, m+1) - |E_j\cap E_{j_0}| - \mathbf{1}_j(x) \\ &= \begin{cases}\mathsf{w}(E_j,E_{j_0}) - \mathbf{1}_j(x), & \text{if } |E_j| = m,\\ \mathsf{w}(E_j,E_{j_0}) + 1 - \mathbf{1}_j(x), & \text{if } |E_j| > m.\end{cases}\end{aligned}$$

Thus, by Lemma 2.50 we have

$$\begin{aligned}\mathsf{w}(\widetilde{E}_1,\dots,\widetilde{E}_r) - \mathsf{w}(E_1,\dots,E_r) &= \sum_{\substack{1\le j\le r\\ j\neq j_0}} \big[\mathsf{w}(\widetilde{E}_j,\widetilde{E}_{j_0}) - \mathsf{w}(E_j,E_{j_0})\big] \\ &= \#\big\{j\in\{1,\dots,r\} : |E_j| > m\big\} - \#\big\{j\in\{1,\dots,r\} : x\in E_j\big\}.\end{aligned}$$

Writing

$$\big\{j\in\{1,\dots,r\} : |E_j| > m\big\} = \big\{j\in\{1,\dots,r\} : |E_j| > m \text{ and } x\in E_j\big\} \\ \sqcup \big\{j\in\{1,\dots,r\} : |E_j| > m \text{ and } x\notin E_j\big\},$$

and

$$\big\{j\in\{1,\dots,r\} : x\in E_j\big\} = \big\{j\in\{1,\dots,r\} : x\in E_j \text{ and } |E_j| > m\big\} \\ \sqcup \big\{j\in\{1,\dots,r\} : x\in E_j \text{ and } |E_j| = m\big\},$$

gives

$$\mathsf{w}(\widetilde{E}_1,\dots,\widetilde{E}_r) - \mathsf{w}(E_1,\dots,E_r) = \#\big\{j\in\{1,\dots,r\} : |E_j| > m \text{ and } x\notin E_j\big\} \\ - \#\big\{j\in\{1,\dots,r\} : x\in E_j \text{ and } |E_j| = m\big\},$$

as announced. □

**Proposition 3.20.** *There exist $E_1,\dots,E_r\subseteq\{1,\dots,e\}$ with $|E_1| = \cdots = |E_r|$ such that $\mathsf{w}(E_1,\dots,E_r) = N_{r,e}$.*

*Proof.* Let $E_1,\dots,E_r\subseteq\{1,\dots,e\}$. It suffices to prove that we can find $\widetilde{E}_1,\dots,\widetilde{E}_r\subseteq\{1,\dots,e\}$ satisfying $|\widetilde{E}_1| = \cdots = |\widetilde{E}_r|$ such that $\mathsf{w}(\widetilde{E}_1,\dots,\widetilde{E}_r)\ge\mathsf{w}(E_1,\dots,E_r)$.

Let $m := \min_{1\le j\le r}|E_j|$ and $M := \max_{1\le j\le r}|E_j|$. If $m = M$ then we are done, thus we now assume that $m < M$. Let $j_m, j_M\in\{1,\dots,e\}$ such that $|E_{j_m}| = m$ and $|E_{j_M}| = M$. We have $e - m + M > e$ by assumption, thus $E_{j_m}^c\cap E_{j_M}\neq\emptyset$, in particular we can pick $x\in E_{j_m}^c\cap E_{j_M}$. We first assume that:

$$\#\big\{j\in\{1,\dots,r\} : |E_j| > m \text{ and } x\notin E_j\big\} \ge \#\big\{j\in\{1,\dots,r\} : |E_j| = m \text{ and } x\in E_j\big\}. \tag{3.21}$$

We define:

$$N_m := \#\big\{j\in\{1,\dots,r\} : |E_j| = m\big\},$$
$$N_M := \#\big\{j\in\{1,\dots,r\} : |E_j| = M\big\}.$$

By Lemma 3.19 applied with the family $E_1, \dots, E_r$ and $x \notin E_{j_m}$, we can construct a family $\widetilde{E}_1, \dots, \widetilde{E}_r \subseteq \{1, \dots, e\}$ satisfying $\mathsf{w}(\widetilde{E}_1, \dots, \widetilde{E}_r) \geq \mathsf{w}(E_1, \dots, E_r)$ such that either

$$\max_{1 \leq j \leq r} |\widetilde{E}_j| = M,$$
$$\min_{1 \leq j \leq r} |\widetilde{E}_j| = m + 1,$$

or

$$\max_{1 \leq j \leq r} |\widetilde{E}_j| = M,$$
$$\min_{1 \leq j \leq r} |\widetilde{E}_j| = m,$$
$$\#\big\{j \in \{1, \dots, r\} : |\widetilde{E}_j| = m\big\} = N_m - 1,$$
$$\#\big\{j \in \{1, \dots, r\} : |\widetilde{E}_j| = M\big\} = N_M,$$

thus in both cases we conclude by induction on $(M - m, N_m + N_M) \in \mathbb{N}^2$.

Thus, we now assume that (3.21) fails, that is,

$$\#\big\{j \in \{1, \dots, r\} : |E_j| > m \text{ and } x \notin E_j\big\} < \#\big\{j \in \{1, \dots, r\} : |E_j| = m \text{ and } x \in E_j\big\}. \tag{3.22}$$

Defining $F_j \coloneqq E_j^c$ for all $j \in \{1, \dots, e\}$, we have

$$\begin{aligned}
\#\big\{j \in \{1, \dots, r\} : |F_j| > e - M \text{ and } x \notin F_j\big\} &= \#\big\{j \in \{1, \dots, r\} : |E_j| < M \text{ and } x \in E_j\big\} \\
&\geq \#\big\{j \in \{1, \dots, r\} : |E_j| = m \text{ and } x \in E_j\big\} \\
&> \#\big\{j \in \{1, \dots, r\} : |E_j| > m \text{ and } x \notin E_j\big\} \text{ (by (3.22))} \\
&\geq \#\big\{j \in \{1, \dots, r\} : |E_j| = M \text{ and } x \notin E_j\big\} \\
&= \#\big\{j \in \{1, \dots, r\} : |F_j| = e - M \text{ and } x \in F_j\big\}.
\end{aligned}$$

Since we have chosen $x$ such that $x \notin F_{j_M}$ with $|F_{j_M}| = e - M = \min_{1 \leq j \leq r} |F_j|$, we can apply Lemma 3.19 to the family $F_1, \dots, F_r$ and $x \notin F_{j_M}$. We thus find a family $\widetilde{F}_1, \dots, \widetilde{F}_r \subseteq \{1, \dots, e\}$ satisfying $\mathsf{w}(\widetilde{F}_1, \dots, \widetilde{F}_r) > \mathsf{w}(F_1, \dots, F_r)$, and this concludes the proof since $\mathsf{w}(F_1, \dots, F_r) = \mathsf{w}(E_1, \dots, E_r)$ by Lemmas 2.50 and 3.18. □

For any $k \in \{0, \dots, e\}$, let $q_{e,k}$ be the restriction of $q_e$ to the subspace $\mathbb{R}^{\mathcal{P}_{e,k}}$, where $\mathcal{P}_{e,k}$ is the subset of $\mathcal{P}_e$ given by the subsets $E \subseteq \{1, \dots, e\}$ of size $k$. We have $q_{e,k}(x) = \frac{1}{2}\langle x, A_{e,k} x \rangle$ for all $x \in \mathbb{R}^{\mathcal{P}_{e,k}}$, where $A_{e,k} = (a_{E,F})_{E,F \in \mathcal{P}_{e,k}}$ is the real symmetric matrix of size $\binom{e}{k}$ given by

$$a_{E,F} = \mathsf{w}(E, F) = k - |E \cap F|,$$

for all $E, F \in \mathcal{P}_{e,k}$. In particular, as for $A_e$, the matrix $A_{e,k}$ has only 0 entries in the diagonal. By Lemmas 3.5 and 3.18 and Proposition 3.20, we have

$$N_{r,e} = \max_{1 \leq k \leq \lfloor \frac{e}{2} \rfloor} \max_{\substack{x \in \mathbb{N}^{\mathcal{P}_{e,k}} \\ \|x\|_1 = r}} q_{e,k}(x). \tag{3.23}$$

The matrix $A_{e,k}$ appears at some places in the literature (see, for instance, [Ry81, Ry82]). The result of Lemma 3.24 is maybe well-known; we provide a proof for convenience.

**Lemma 3.24.** *Recall that $e \geq 2$ and let $k \in \{1, \dots, e-1\}$. The eigenvalues of $A_{e,k}$ are the following:*

$$k\binom{e-1}{k}, \qquad \text{with multiplicity } 1,$$
$$0, \qquad \text{with multiplicity } \binom{e}{k} - e,$$
$$-\binom{e-2}{k-1}, \qquad \text{with multiplicity } e - 1.$$

*Moreover, the constant vector $\mathbf{1} \in \mathbb{R}^{\mathcal{P}_{e,k}}$ is an eigenvector for the eigenvalue $k\binom{e-1}{k}$.*

*Proof.* We follow the computation of the eigenvalues for adjacency matrices of strongly regular graphs (see, for instance, [GoRo]). For any $m, n \geq 1$, we denote by $J_{m,n}$ the $m \times n$ matrix filled with ones, and we define $J_n := J_{n,n}$. We also write $I_n$ for the $n \times n$ identity matrix.

We first note that for any $E \in \mathcal{P}_{e,k}$ we have,

$$\begin{aligned}\sum_{F \in \mathcal{P}_{e,k}} a_{E,F} &= \sum_{F \in \mathcal{P}_{e,k}} (k - |E \cap F|) \\ &= \sum_{\ell=0}^{k-1} (k-\ell) \binom{k}{\ell} \binom{e-k}{k-\ell} \\ &= \sum_{\ell=1}^{k} \ell \binom{k}{\ell} \binom{e-k}{\ell} \\ &= \sum_{\ell=1}^{k} k \binom{k-1}{\ell-1} \binom{e-k}{e-k-\ell},\end{aligned}$$

thus, using Vandermonde's identity we obtain,

$$\begin{aligned}\sum_{F \in \mathcal{P}_{e,k}} a_{E,F} &= k \binom{e-1}{e-k-1} \\ &= k \binom{e-1}{k},\end{aligned}$$

so that the constant vector $\mathbf{1}$ is an eigenvector of $A_{e,k}$ with eigenvalue $k\binom{e-1}{k}$.

Now let $B = (b_{iE})_{\substack{1 \leq i \leq e \\ E \in \mathcal{P}_{e,k}}}$ be the $e \times \binom{e}{k}$ incidence matrix associated with the elements of $\{1, \ldots, e\}$ and $\mathcal{P}_{e,k}$. In other words, for any $i \in \{1, \ldots, e\}$ and $E \in \mathcal{P}_{e,k}$ we have $b_{iE} = \mathbf{1}_E(i)$, where $\mathbf{1}_E$ is the characteristic vector of $E \subseteq \{1, \ldots, e\}$. The matrix $B^\intercal B$ is square with rows and columns indexed by $\mathcal{P}_{e,k}$ and for any $E, F \in \mathcal{P}_{e,k}$ we have

$$(B^\intercal B)_{EF} = \sum_{i=1}^{e} \mathbf{1}_E(i) \mathbf{1}_F(i) = |E \cap F|,$$

so that $A_{e,k} = kJ_{\binom{e}{k}} - B^\intercal B$. We will now compute $A_{e,k}^2$. First, note that the matrix $BB^\intercal$ is square of size $e$ and for any $i, j \in \{1, \ldots, e\}$ we have

$$(BB^\intercal)_{ij} = \sum_{E \in \mathcal{P}_{e,k}} \mathbf{1}_E(i) \mathbf{1}_E(j) = \begin{cases} \binom{e-1}{k-1}, & \text{if } i = j, \\ \binom{e-2}{k-2}, & \text{if } i \neq j \end{cases}$$

(with the convention $\binom{e-2}{k-2} = 0$ if $k = 1$), so that

$$BB^\intercal = \binom{e-2}{k-2} J_e + \left( \binom{e-1}{k-1} - \binom{e-2}{k-2} \right) I_e = \binom{e-2}{k-2} J_e + \binom{e-2}{k-1} I_e.$$

It is clear that $B^\intercal J_e$ is a scalar multiple of $J_{\binom{e}{k},e}$, similarly

$$J_{\binom{e}{k}}^2, \quad J_{\binom{e}{k}} B^\intercal B \text{ and } J_{\binom{e}{k},e} B,$$

are scalar multiples of $J_{\binom{e}{k}}$. We deduce that

$$\begin{aligned} A_{e,k}^2 &= \left(kJ_{\binom{e}{k}} - B^\intercal B\right)^2, \\ &= \alpha J_{\binom{e}{k}} + B^\intercal(BB^\intercal)B \\ &= \beta J_{\binom{e}{k}} + \binom{e-2}{k-1} B^\intercal B \\ &= \gamma J_{\binom{e}{k}} - \binom{e-2}{k-1} A_{e,k}, \end{aligned} \tag{3.25}$$

for some scalars $\alpha, \beta, \gamma$. Now let $x$ be any eigenvector of $A_{e,k}$ orthogonal to $\mathbf{1}$ and let $\lambda$ be its associated eigenvalue. Since $\langle x, \mathbf{1}\rangle = 0$ we have $J_{\binom{e}{k}} x = 0$ and thus (3.25) gives

$$A_{e,k}^2 x = -\binom{e-2}{k-1} A_{e,k} x,$$

thus $\lambda^2 = -\binom{e-2}{k-1}\lambda$ thus $\lambda = 0$ or $\lambda = -\binom{e-2}{k-1}$. Now if $m$ (respectively $m_0$) denotes the multiplicity of the eigenvalue $-\binom{e-2}{k-1}$ (resp. 0), what precedes proves that $k\binom{e-1}{k}$ is of multiplicity 1 and we have

$$\begin{aligned} 1 + m_0 + m &= \binom{e}{k}, \\ k\binom{e-1}{k} - m\binom{e-2}{k-1} &= \operatorname{tr}(A_{e,k}) = 0, \end{aligned}$$

recalling that the diagonal entries of $A_{e,k}$ are 0. By Lemma 3.12 we find that $m = k\binom{e-1}{k}/\binom{e-2}{k-1} = e-1$ and $m_0 = \binom{e}{k} - e$, which concludes the proof. □

**Proposition 3.26.** *Let $k \in \{1, \dots, e-1\}$. For any $x \in \mathbb{R}^{\mathcal{P}_{e,k}}$ we have*

$$q_{e,k}(x) \leq \frac{\|x\|_1^2}{2e} k(e-k).$$

*Proof.* For any $x \in \mathbb{R}^{\mathcal{P}_{e,k}}$ we have

$$x = \frac{\langle x, \mathbf{1}\rangle}{\|\mathbf{1}\|^2}\mathbf{1} + x_\perp,$$

where $x_\perp \in \mathbb{R}^{\mathcal{P}_{e,k}}$ is orthogonal to $\mathbf{1}$. By Lemma 3.24, the vector $x_\perp$ is a sum of eigenvectors for $A_{e,k}$ with non-positive eigenvalues, thus $q_{e,k}(x_\perp) \leq 0$ and:

$$q_{e,k}(x) = \frac{\langle x, \mathbf{1}\rangle^2}{\|\mathbf{1}\|^4} q_{e,k}(\mathbf{1}) + q_{e,k}(x_\perp) \leq \frac{\langle x, \mathbf{1}\rangle^2}{\|\mathbf{1}\|^4} q_{e,k}(\mathbf{1}).$$

Now $|\langle x, \mathbf{1}\rangle| \leq \|x\|_1$ by the triangle inequality and, again by Lemma 3.24,

$$q_{e,k}(\mathbf{1}) = \frac{1}{2}\langle \mathbf{1}, A_{e,k}\mathbf{1}\rangle = \frac{k}{2}\binom{e-1}{k}\|\mathbf{1}\|^2.$$

Hence, we have

$$q_{e,k}(x) \leq \frac{\|x\|_1^2}{\|\mathbf{1}\|^2}\frac{k}{2}\binom{e-1}{k} = \frac{\|x\|_1^2}{\binom{e}{k}}\frac{k}{2}\binom{e-1}{k}.$$

We obtain the announced result since $\frac{\binom{e-1}{k}}{\binom{e}{k}} = \frac{e-k}{e}$ by Lemma 3.12. □

**Corollary 3.27.** *We have*

$$N_{r,e} \leq \frac{r^2}{2e}\left\lfloor \frac{e^2}{4} \right\rfloor.$$

*Proof.* For any $k \in \{1, \dots, e-1\}$, by Proposition 3.26 we know that $q_{e,k}(x) \leq \frac{r^2}{2e} k(e-k)$ if $\|x\|_1 = r$. By (3.23) this concludes the proof since $k(e-k) \leq \lfloor \frac{e^2}{4} \rfloor$ by Lemma 3.12(*ii*). □

**Definition 3.28.** For any $r \geq 0$ and $e \geq 1$ we define $N'_{r,e} \coloneqq \left\lfloor \frac{r^2}{2e} \lfloor \frac{e^2}{4} \rfloor \right\rfloor$.

Combining Corollaries 3.16 and 3.27 yields the following final bounds.

**Corollary 3.29.** *For any $e, r \geq 2$ we have*

$$\left\lfloor \frac{e}{2} \right\rfloor \left\lfloor \frac{r^2}{4} \right\rfloor \leq N_{r,e} \leq N'_{r,e} \leq \frac{er^2}{8}$$

We have the following particular case.

**Proposition 3.30.** *Assume that $e, r \geq 2$ are both even. Then*

$$N_{r,e} = N'_{r,e} = \frac{er^2}{8}.$$

We end this part by the following result, which will be useful later.

**Lemma 3.31.** *For any $r \geq 0$ and $e \geq 1$ we have:*

$$N'_{r,e} = \left\lfloor \frac{r^2}{2\binom{e}{k}} \lambda \right\rfloor,$$

*where $k \coloneqq \lfloor \frac{e}{2} \rfloor$ and $\lambda \coloneqq k\binom{e-1}{k}$ (the largest eigenvalue of $A_{e,k}$).*

*Proof.* By Lemma 3.12(*i*) we have $\frac{\binom{e-1}{k}}{\binom{e}{k}} = \frac{e-k}{e}$. Thus, we have:

$$\begin{aligned} \frac{r^2}{2\binom{e}{k}} \lambda &= \frac{r^2}{2\binom{e}{k}} k \binom{e-1}{k}. \\ &= \frac{r^2}{2e} k(e-k) \\ &= \frac{r^2}{2e} \left\lfloor \frac{e^2}{4} \right\rfloor, \end{aligned}$$

where the last equality follows from Lemma 3.12(*ii*), and this concludes the proof. □

### 3.3.3 More values for small parameters

Let $r, e \geq 2$. We now aim to compute the value of $N_{r,e}$ for $r \in \{3, 4\}$ or $e \in \{3, \dots, 6\}$. For these values, we will see that, except for $r = 3$, we have $N_{r,e} = N'_{r,e}$ (see Proposition 3.47).

**With small $r$** We begin by the following result, which will be in fact interesting only in the case $r = 3$.

**Lemma 3.32.** *If $r \geq 3$ then $N_{r,e} \leq \lfloor \frac{(r-1)re}{6} \rfloor$.*

*Proof.* Let $E_1, \dots, E_r \subseteq \{1, \dots, e\}$ and define $a_{jk} \coloneqq \mathsf{w}(E_j, E_k) = \min(|E_j|, |E_k|) - |E_j \cap E_k|$ for all $1 \leq j < k \leq r$. By (3.4) and since $a_{jk} \in \mathbb{N}$, it suffices to show that

$$\sum_{1 \leq j < k \leq r} a_{jk} \leq \frac{(r-1)re}{6}.$$

Let $1 \leq j < k < l \leq r$; note that this is possible since $r \geq 3$. We have

$$\begin{aligned} a_{jk} + a_{kl} + a_{jl} = {} & \min(|E_j|, |E_k|) + \min(|E_k|, |E_l|) + \min(|E_j|, |E_l|) \\ & - |E_j \cap E_k| - |E_k \cap E_l| - |E_j \cap E_l| \\ \leq {} & |E_j| + |E_k| + |E_l| - |E_j \cap E_k| - |E_k \cap E_l| - |E_j \cap E_l| \\ \leq {} & |E_j \cup E_k \cup E_l| - |E_j \cap E_k \cap E_l| \\ \leq {} & |E_j \cup E_k \cup E_l| \\ \leq {} & e. \end{aligned}$$

Now, in the sum $\sum_{1 \leq j < k < l \leq r} (a_{jk} + a_{kl} + a_{jl})$ each couple $(j, k)$ for $1 \leq j < k \leq r$ appears exactly $r - 2$ times, thus

$$\sum_{1 \leq j < k < l \leq r} (a_{jk} + a_{kl} + a_{jl}) = (r-2) \sum_{1 \leq j < k \leq r} a_{jk}.$$

Thus, we find that

$$(r-2) \sum_{1 \leq j < k \leq r} a_{jk} \leq e \binom{r}{3},$$

and thus

$$\sum_{1 \leq j < k \leq r} a_{jk} \leq e \frac{r(r-1)}{6},$$

whence the result. □

**Proposition 3.33** (Case $r = 3$)**.** *We have $N_{3,e} = e$.*

*Proof.* By Corollary 3.16 and Lemma 3.32 we have

$$2 \left\lfloor \frac{e}{2} \right\rfloor \leq N_{3,e} \leq e, \tag{3.34}$$

in particular $N_{3,e} = e$ if $e$ is even. (Note that if we use Corollary 3.16 instead of Lemma 3.32 we only obtain $N_{r,e} \leq \frac{9e}{8}$.) Thus, we can now assume that $e$ is odd and let us write $e = 2e' + 3$ with $e' \geq 0$. We first assume that $e' \geq 1$. By Lemma 3.15 we have

$$N_{3,2e'} + N_{3,3} \leq N_{3,e}.$$

Since $2e'$ is even we have $N_{3,2e'} = 2e'$, thus we obtain, using (3.34) for the right inequality,

$$e - 3 + N_{3,3} \leq N_{3,e} \leq e.$$

Thus, to conclude it suffices to prove that $N_{3,3} = 3$. We already know by (3.34) that $N_{3,3} \leq 3$, thus it suffices to find $E_1, E_2, E_3 \subseteq \{1, 2, 3\}$ such that $\mathsf{w}(E_1, E_2, E_3) = 3$. Using (3.4), this equality is satisfied with $E_i := \{i\}$ for any $i \in \{1, 2, 3\}$ and this concludes the proof. □

**Proposition 3.35** (Case $r = 4$)**.** *We have:*

$$N_{4,e} = \begin{cases} 2e, & \text{if } e \text{ is even,} \\ 2e - 1, & \text{if } e \text{ is odd.} \end{cases}$$

*Proof.* If $e$ is even then we know by Proposition 3.30 that $N_{4,e} = \frac{e4^2}{8} = 2e$. We thus now assume that $e$ is odd. By Corollary 3.27 we obtain

$$N_{4,e} \leq \frac{8}{e} \left\lfloor \frac{e^2}{4} \right\rfloor < \frac{8}{e} \frac{e^2}{4} = 2e,$$

thus $N_{4,e} \leq 2e - 1$. If $e > 3$, as in the proof of Proposition 3.33, using Lemma 3.15 we have

$$N_{4,e-3} + N_{4,3} \leq N_{4,e},$$

thus, since $e-3$ is even,

$$2e-6+N_{4,3} \leq N_{4,e}.$$

Hence, to conclude it suffices to prove that $N_{4,3}=5$. We have seen that $N_{4,3} \leq 2\cdot 3-1=5$, thus it suffices to find $E_1,\dots,E_4 \subseteq \{1,\dots,e\}$ with $\mathsf{w}(E_1,\dots,E_4)=5$. Setting $E_i \coloneqq \{i\}$ for all $i \in \{1,2,3\}$ and $E_4 \coloneqq E_3$, we have

$$\mathsf{w}(E_j,E_k)=\begin{cases}1, & \text{if } j \in \{1,2\},\\ 0, & \text{otherwise,}\end{cases}$$

for any $1 \leq j < k \leq 4$, thus $\mathsf{w}(E_1,\dots,E_4)=\sum_{1\leq j<k\leq 4}\mathsf{w}(E_j,E_k)=3+2=5$. This concludes the proof. □

**With small $e$** The next results will mainly consist in using the fact that $\mathbf{1}$ is an eigenvector of $A_{e,k}$.

**Proposition 3.36** (Case $e=3$)**.** *We have* $N_{r,3}=\left\lfloor \frac{r^2}{3} \right\rfloor$.

*Proof.* By Corollary 3.27 we have $N_{r,3} \leq \frac{r^2}{6}\left\lfloor \frac{9}{4} \right\rfloor = \frac{r^2}{3}$, hence $N_{r,3} \leq \left\lfloor \frac{r^2}{3} \right\rfloor$. Thus, by (3.23) it suffices to find $x \in \mathbb{N}^{\mathcal{P}_{e,1}}$ with $\|x\|_1=r$ such that $q_{e,1}(x)=\left\lfloor \frac{r^2}{3} \right\rfloor$. To that extent, write $r=3a+h$ with $a \geq 1$ and $h \in \{-1,0,1\}$ and consider $x \coloneqq a\mathbf{1}+h\mathbf{1}_E \in \mathbb{Z}^{\mathcal{P}_{e,1}}$, where $\mathbf{1}_E$ is any characteristic vector of an element of $\mathcal{P}_{e,1}$. Note that $x \in \mathbb{N}^{\mathcal{P}_{e,1}}$ and $\|x\|_1 = a\binom{3}{1}+h = 3a+h = r$ indeed. We have, recalling that $a_{E,E}=0$ and from Lemma 3.24 that $A_{e,1}\mathbf{1}=\binom{2}{1}\mathbf{1}=2\,\mathbf{1}$,

$$\begin{aligned}
q_{e,1}(x) &= \frac{1}{2}\langle x, A_{e,1}x\rangle \\
&= \frac{1}{2}\left(a^2\langle \mathbf{1}, A_{e,1}\mathbf{1}\rangle + h^2\langle \mathbf{1}_E, A_{e,1}\mathbf{1}_E\rangle + 2ah\langle \mathbf{1}, A_{e,1}\mathbf{1}_E\rangle\right) \\
&= \frac{1}{2}\left(2a^2\|\mathbf{1}\|^2+4ah\right) \\
&= 3a^2+2ah \\
&= \frac{r^2-h^2}{3} \\
&= \begin{cases}\dfrac{r^2}{3}, & \text{if } r=0 \pmod 3,\\ \dfrac{r^2-1}{3}, & \text{if } r=\pm 1 \pmod 3,\end{cases} \\
&= \left\lfloor \frac{r^2}{3} \right\rfloor.
\end{aligned}$$

This concludes the proof. □

**Proposition 3.37** (Case $e \in \{4,6\}$)**.** *We have* $N_{r,4}=\left\lfloor \frac{r^2}{2} \right\rfloor$ *and* $N_{r,6}=\left\lfloor \frac{3r^2}{4} \right\rfloor = 3\left\lfloor \frac{r^2}{4} \right\rfloor$.

*Proof.* If $r$ is even then the result is true by Proposition 3.30. Now if $r$ is odd, by Corollary 3.29 we have

$$\begin{aligned}
2\left\lfloor \frac{r^2}{4} \right\rfloor \leq N_{r,4} \leq \left\lfloor \frac{r^2}{2} \right\rfloor, \\
3\left\lfloor \frac{r^2}{4} \right\rfloor \leq N_{r,6} \leq \left\lfloor \frac{3r^2}{4} \right\rfloor.
\end{aligned}$$

Now we have

$$\left\lfloor \frac{r^2}{2} \right\rfloor = \frac{r^2-1}{2}, \qquad \left\lfloor \frac{r^2}{4} \right\rfloor = \frac{r^2-1}{4},$$

which concludes the proof for $e = 4$. For $e = 6$ we note that

$$3\left\lfloor \frac{r^2}{4} \right\rfloor = 3\frac{r^2-1}{4} = \frac{3r^2}{4} - \frac{3}{4} = \left\lfloor \frac{3r^2}{4} \right\rfloor,$$

which concludes the proof. □

We will now deal with the case $e = 5$.

**Definition 3.38.** For any $r \geq 0$ and $e \geq 2$, we define

$$Q_{r,e} \coloneqq \max_{\substack{x \in \mathbb{N}^{\mathcal{P}_{e,k}} \\ \|x\|_1 = r}} q_{e,k}(x),$$

where $k \coloneqq \lfloor \frac{e}{2} \rfloor$.

Recalling (3.23) and Corollary 3.29, for any $r \geq 2$ and $e \geq 2$ we have

$$Q_{r,e} \leq N_{r,e} \leq N'_{r,e}. \tag{3.39}$$

*Remark* 3.40. We will see that when $e = 5$ then $Q_{r,e} = N_{r,e}$ (Lemma 3.44). It is unclear whether this equality holds in full generality, in particular we state the next results with the quantity $Q_{r,e}$.

Recall from Definition 3.28 that $N'_{r,e} = \lfloor \frac{r^2}{2e} \lfloor \frac{e^2}{4} \rfloor \rfloor$.

**Lemma 3.41.** *Let $r \geq 0$ and $e \geq 2$. Assume that $Q_{r,e} = N'_{r,e}$. Then $Q_{r+\binom{e}{k},e} = N'_{r+\binom{e}{k},e}$, where $k \coloneqq \lfloor \frac{e}{2} \rfloor$.*

*Proof.* By assumption, we can find $x \in \mathbb{N}^{\mathcal{P}_{e,k}}$ with $\|x\|_1 = r$ such that $q_{e,k}(x) = N'_{r,e}$ (with $k = \lfloor \frac{e}{2} \rfloor$). We have, where $\lambda \coloneqq k\binom{e-1}{k}$ is the highest eigenvalue of $A_{e,k}$ (see Lemma 3.24),

$$\begin{aligned} q_{e,k}(x + \mathbf{1}) &= q_{e,k}(x) + q_{e,k}(\mathbf{1}) + \langle x, A_{e,k}\mathbf{1} \rangle \\ &= Q_{r,e} + \frac{\lambda}{2}\binom{e}{k} + \lambda r, \end{aligned}$$

thus $Q_{r+\binom{e}{k},e} \geq Q_{r,e} + \frac{\lambda}{2}\binom{e}{k} + \lambda r$. Note that $\frac{\lambda}{2}\binom{e}{k} = q_{e,k}(\mathbf{1})$ is an integer by Proposition 3.17(*i*). Now, recalling the equality $N'_{r,e} = \left\lfloor \frac{r^2}{2\binom{e}{k}}\lambda \right\rfloor$ from Lemma 3.31 we find:

$$\begin{aligned} N'_{r+\binom{e}{k},e} &= \left\lfloor \frac{r^2 + \binom{e}{k}^2 + 2\binom{e}{k}r}{2\binom{e}{k}}\lambda \right\rfloor \\ &= \left\lfloor \frac{r^2}{2\binom{e}{k}}\lambda + \frac{\lambda}{2}\binom{e}{k} + \lambda r \right\rfloor \\ &= \left\lfloor \frac{r^2}{2\binom{e}{k}}\lambda \right\rfloor + \frac{\lambda}{2}\binom{e}{k} + \lambda r. \end{aligned}$$

Finally, we have

$$\begin{aligned} N'_{r+\binom{e}{k},e} &\geq Q_{r+\binom{e}{k},e} \\ &= Q_{r,e} + \left(Q_{r+\binom{e}{k},e} - Q_{r,e}\right) \\ &\geq N'_{r,e} + \left(N'_{r+\binom{e}{k},e} - N'_{r,e}\right) \\ &= N'_{r+\binom{e}{k},e}, \end{aligned}$$

which concludes the proof. □

**Lemma 3.42.** *Let $r \geq 0$ and $e \geq 2$. Let $k \coloneqq \lfloor \frac{e}{2} \rfloor$ and assume that there exists $x \in \{0,1\}^{\mathcal{P}_{e,k}}$ with $\|x\|_1 = r$ such that $q_{e,k}(x) = N'_{r,e}$ (in particular, we have $Q_{r,e} = N'_{r,e}$). Then $Q_{\binom{e}{k}-r,e} = N'_{\binom{e}{k}-r,e}$.*

*Proof.* Note that $\mathbf{1} - x \in \mathbb{N}^{\mathcal{P}_{e,k}}$ since the entries of $x$ are at most 1. With similar calculations as in Lemma 3.41, we thus have:

$$\begin{aligned} Q_{\binom{e}{k}-r,e} &\geq q_{e,k}(\mathbf{1} - x) \\ &= q_{e,k}(x) + q_{e,k}(\mathbf{1}) - \langle x, A_{e,k}\mathbf{1} \rangle \\ &= Q_{r,e} + \frac{\lambda}{2}\binom{e}{k} - \lambda r \\ &= N'_{r,e} + \frac{\lambda}{2}\binom{e}{k} - \lambda r \\ &= N'_{\binom{e}{k}-r,e}, \end{aligned}$$

thus we conclude by (3.39). □

*Remark* 3.43. Assume that $r = 1$. We have $N'_{1,e} \leq \frac{e}{8}$ by Corollary 3.29, thus if $e \leq 7$ we have $N'_{1,e} = 0$. In particular, the assumption $Q_{r,e} = N'_{r,e}$ of Lemma 3.41 is satisfied. Moreover, since any $x \in \mathbb{N}^{\mathcal{P}_{e,k}}$ with $\|x\|_1 = 1$ has entries in $\{0,1\}$, the assumption of Lemma 3.42 is also satisfied (using (3.39)).

We give a last preliminary result, which is a refinement of (3.23) (see Remark 3.40).

**Lemma 3.44.** *Assume that $e = 5$. For any $r \geq 2$ we have $Q_{r,e} = N_{r,e}$.*

*Proof.* By (3.23), Proposition 3.26 and Corollary 3.16, it suffices to prove that

$$\left\lfloor \frac{r^2}{2e} k'(e - k') \right\rfloor < \left\lfloor \frac{e}{2} \right\rfloor \left\lfloor \frac{r^2}{4} \right\rfloor,$$

for all $k' \in \left\{1, \dots, \lfloor \frac{e}{2} \rfloor - 1\right\}$. In our setting, this reduces to proving

$$\left\lfloor \frac{2r^2}{5} \right\rfloor < 2 \left\lfloor \frac{r^2}{4} \right\rfloor. \tag{3.45}$$

If $r$ is even, we have $\lfloor \frac{2r^2}{5} \rfloor \leq \frac{2r^2}{5} < \frac{r^2}{2} = 2\lfloor \frac{r^2}{4} \rfloor$ thus (3.45) holds. If $r$ is odd we have $\lfloor \frac{r^2}{4} \rfloor = \frac{r^2-1}{4}$ and

$$\frac{2r^2}{5} < \frac{r^2 - 1}{2} \iff 4r^2 < 5r^2 - 5,$$

thus (3.45) holds since $r \geq 2$ is odd. □

**Proposition 3.46** (Case $e = 5$)**.** *For any $r \geq 2$ we have $N_{r,5} = N'_{r,5} = \lfloor \frac{3r^2}{5} \rfloor$.*

*Proof.* By Lemma 3.41 and Lemma 3.44, it suffices to prove the equality for $2 \leq r < \binom{5}{2} = 10$, together with $Q_{r,5} = N'_{r,5}$ for $r \in \{0,1\}$. For $r = 0$ we have $Q_{r,5} = N'_{r,5} = 0$, and the equality also holds for $r = 1$ by Remark 3.43. By Lemma 3.42 and again Remark 3.43, we thus have $N_{r,5} = N'_{r,5}$ for $r = 9$.

It remains to prove $N_{r,5} = N'_{r,5}$ for $r \in \{2, \dots, 8\}$. It follows from Propositions 3.13 and 3.33 that the equality $N_{r,5} = N'_{r,5}$ holds for $r \in \{2,3\}$. Moreover, the proofs of these propositions guarantee that Lemma 3.42 can be applied for $r \in \{2,3\}$, so that the equality $N_{r,5} = N'_{r,5}$ also holds for $r \in \{7,8\}$ (using Lemma 3.44). The same can be said for $r = 4$ (Proposition 3.35) and thus $r = 6$, however we need to carefully check that the conditions of Lemma 3.42 can be satisfied when $r = 4$. Following the proofs of Propositions 3.14 and 3.35, taking

$$\begin{aligned} E_1 &\coloneqq \{1,3\}, \quad E_2 \coloneqq \{2,4\}, \\ E_3 &\coloneqq \{1,5\}, \quad E_4 \coloneqq \{2,5\}, \end{aligned}$$

by (3.4) we have $\mathsf{w}(E_1,\dots,E_4) = \sum_{1\le j<k\le 4}\big(2-|E_j\cap E_k|\big) = 9 = N_{4,5}$ indeed and the corresponding $x\in\mathbb{N}^{\mathcal{P}_{5,2}}$ has entries in $\{0,1\}$. We could have also noticed that $N'_{6,5} = \lfloor\frac{3\cdot 36}{5}\rfloor = \lfloor\frac{3\cdot 35}{5}+\frac{3}{5}\rfloor = 21$, and by Lemma 3.15 we have

$$N_{6,5}\ge N_{6,3}+N_{6,2},$$

thus by Propositions 3.14 and 3.36 we have

$$N_{6,5}\ge\left\lfloor\frac{6^2}{3}\right\rfloor+\left\lfloor\frac{6^2}{4}\right\rfloor = 2\cdot 6+3^2 = 21,$$

thus $N_{6,5} = N'_{6,5}$ by Corollary 3.29 (note that this alternative proof does not work for $r\in\{7,8\}$).

It thus remains to treat the case $r=5$. We define:

$$E_1 \coloneqq \{1,2\},\quad E_2\coloneqq\{1,3\},$$

$$E_5\coloneqq\{2,4\},\qquad E_3\coloneqq\{3,5\},$$

$$E_4\coloneqq\{4,5\},$$

so that $|E_j\cap E_{j+1}|=1$ for all $j\in\{1,\dots,5\}$ (with $E_6\coloneqq E_1$), every other pairwise intersection being empty. By (3.4), we deduce that

$$\mathsf{w}(E_1,\dots,E_5) = 5\cdot 1+5\cdot 2 = 15,$$

thus $N_{5,5}\ge 15$. Since $N'_{5,5} = \lfloor\frac{3\cdot 5^2}{5}\rfloor = 15$, by Corollary 3.29 we obtain that $N_{5,5} = N'_{5,5}$ and this concludes the proof. □

**All in one** We now gather all the results we have proven in these small cases. Recall that $N_{r,e}$ (resp. $N'_{r,e}$) is defined in Definition 3.8 (resp. Definition 3.28).

**Proposition 3.47.** *Let $e,r\ge 2$. We have*

$$N_{r,e} = N'_{r,e},$$

*in (at least) the following cases:*

$$r\in\{2,4\}\quad\text{or}\quad e\in\{2,\dots,6\}.$$

Note that if $r=3$ then $N_{3,e}=e$ by Proposition 3.33, however if for instance $e$ is even we have $N'_{3,e} = \lfloor\frac{9e}{8}\rfloor = e+\lfloor\frac{e}{8}\rfloor > N_{3,e}$ as soon as $e\ge 8$.

*Proof.* First, note that the result holds if both $r$ and $e$ are even, by Proposition 3.30. We have

$$N'_{r,e} = \begin{cases}\left\lfloor\dfrac{r^2e}{8}\right\rfloor, & \text{if } e \text{ is even,}\\[2ex] \left\lfloor\dfrac{r^2(e^2-1)}{8e}\right\rfloor, & \text{if } e \text{ is odd.}\end{cases}$$

For $r=2$, we deduce that (with $e\ge 2$ odd)

$$N'_{2,e} = \left\lfloor\frac{e^2-1}{2e}\right\rfloor = \left\lfloor\frac{e}{2}-\frac{1}{2e}\right\rfloor = \left\lfloor\frac{e}{2}\right\rfloor,$$

thus $N'_{2,e}=N_{2,e}$ by Proposition 3.13. For $r=4$, we have (with $e\ge 2$ odd)

$$N'_{4,e} = \left\lfloor\frac{2(e^2-1)}{e}\right\rfloor = 2e-\left\lfloor\frac{2}{e}\right\rfloor = 2e-1,$$

thus $N'_{4,e}=N_{4,e}$ by Proposition 3.35. The equalities $N'_{r,e}=N_{r,e}$ for $e\in\{2,\dots,6\}$ are straightforward from Propositions 3.14, 3.36, 3.37 and 3.46. □

### 3.3.4 Asymptotics

In this section, we will describe the asymptotic behaviour of the two sequences $(N_{r,e})_{e\geq 2}$ and $(N_{r,e})_{r\geq 2}$ for $r, e \geq 2$. We first recall the following basic fact, which will be used without any further reference.

**Lemma 3.48.** *Let $(u_n)_{n\geq 1}$ be a sequence of real numbers that converges to $+\infty$ and let $x \in \mathbb{R}$. Then the sequence $\left(\frac{\lfloor xu_n \rfloor}{u_n}\right)_{n\geq 1}$ converges to $x$.*

We also recall the following standard result.

**Lemma 3.49** (Fekete's superadditive lemma)**.** *Let $(u_n)_{n\geq 1}$ be a superadditive sequence. Then $\left(\frac{u_n}{n}\right)_{n\geq 1}$ has a limit in $[-\infty, +\infty]$ as $n \to +\infty$.*

By Lemmas 3.15 and 3.49, we know that the sequence $\left(\frac{N_{r,e}}{e}\right)_{e\geq 1}$ has a limit $N_{r,\infty} \in [0, +\infty]$. By Corollary 3.29 we have

$$\frac{1}{2}\left\lfloor \frac{r^2}{4} \right\rfloor \leq N_{r,\infty} \leq \frac{r^2}{8}, \tag{3.50}$$

in particular if $r$ is even then $N_{r,\infty} = \frac{r^2}{8}$.

*Remark* 3.51. By Propositions 3.13, 3.33 and 3.35, we have $N_{2,\infty} = \frac{1}{2}$, $N_{3,\infty} = 1$ and $N_{4,\infty} = 2$.

Our aim is now to give a similar result when $r$ grows to infinity.

**Proposition 3.52.** *The sequence $\left(\frac{N_{r,e}}{r^2}\right)_{r\geq 2}$ converges to $N_{\infty,e} \coloneqq \max_{\substack{y\in\mathbb{R}^{\mathcal{P}_e} \\ \|y\|_1=1}} q_e(y)$.*

*Proof.* By Corollary 3.29, we know that the sequence $\left(N_{r,e}/r^2\right)_{r\geq 2}$ is bounded. Hence, to prove that it converges to $N_{\infty,e}$ it suffices to prove that any converging subsequence converges to $N_{\infty,e}$. We thus consider a converging subsequence of $\left(N_{r,e}/r^2\right)_{r\geq 2}$ and let $\ell \in \mathbb{R}$ be its limit. To avoid the notation being overloaded, we still denote this subsequence by $\left(N_{r,e}/r^2\right)_{r\geq 2}$. By Proposition 3.17, we have $N_{r,e}/r^2 \leq N_{\infty,e}$ for any $r \geq 2$ thus $\ell \leq N_{\infty,e}$. We will now prove the reverse inequality.

Let $x = (x_E) \in \mathbb{R}^{\mathcal{P}_e}$. Defining $|x| \coloneqq (|x_E|)_{E\in\mathcal{P}_e}$ we have $\||x|\|_1 = \|x\|_1$. Since the entries of $A_e$ are non-negative we have $q_e(x) \leq q_e(|x|)$. Hence, if $x \in \mathbb{R}^{\mathcal{P}_e}$ is such that $q_e(x) = \max_{\|y\|_1=1} q_e(y)$ then we can assume that $x$ has only non-negative coordinates. In particular, we can take a sequence of rational vectors with non-negative entries $(x_n)_{n\geq 1}$ with limit $x$ and satisfying $\|x_n\|_1 = 1$ for all $n \geq 1$, where the entries of $x_n$ have the same denominator $b_n$. Noting that $b_n$ is not necessarily coprime with its associated numerator, we can always assume that $b_n \to \infty$. Then by Proposition 3.17 we have $N_{b_n,e} \geq q_e(b_n x_n) = b_n^2 q_e(x_n)$, thus since $q_e$ is continuous we obtain $\ell \geq q_e(x) = N_{\infty,e}$. This concludes the proof. □

By Corollary 3.29 we have

$$\frac{1}{4}\left\lfloor \frac{e}{2} \right\rfloor \leq N_{\infty,e} \leq \frac{1}{2e}\left\lfloor \frac{e^2}{4} \right\rfloor.$$

We will be able here to determine the exact value of the limit $N_{\infty,e}$.

**Corollary 3.53.** *For any $e \geq 2$ we have*

$$N_{\infty,e} = \frac{1}{2e}\left\lfloor \frac{e^2}{4} \right\rfloor.$$

*Proof.* Since $Q_{0,e} = N'_{0,e} = 0$, we deduce from Lemma 3.41 and (3.39) that $Q_{r,e} = N_{r,e} = N'_{r,e} = \left\lfloor \frac{r^2}{2e}\left\lfloor \frac{e^2}{4} \right\rfloor \right\rfloor$ as soon as $r$ is of the form $r = \alpha\left(\frac{e}{\lfloor \frac{e}{2} \rfloor}\right)$ for some $\alpha \in \mathbb{Z}_{\geq 1}$. This proves that $(N_{r,e}/r^2)_r$ has a subsequence that converges to $\frac{1}{2e}\left\lfloor \frac{e^2}{4} \right\rfloor$ and this concludes the proof. □

# 4 Application in level one

Let $e \geq 2$. We will here apply the implicit description of the set $Q^{\mathsf{s}}$ given at Proposition 2.40 to study a shift operation on partitions, and relate it to a shift operation on blocks that is naturally defined in higher levels.

In this whole section we assume that we are in level one, that is, we have $r = 1$. In particular, without loss of generality we can assume that the charge is zero and therefore we will often omit to write the corresponding superscripts. For instance, the weight of $\alpha = \sum_{i \in \mathbb{Z}/e\mathbb{Z}} c_i \alpha_i \in Q$ is given by

$$\mathsf{w}(\alpha) = c_0 - \frac{1}{2} \sum_{i \in \mathbb{Z}/e\mathbb{Z}} (c_i - c_{i+1})^2,$$

and, recalling Lemma 2.24, if $\lambda$ is an $e$-core and $i \in \{0, \dots, e-1\}$ then $x_i(\lambda) = y_i(\lambda)$. The only case where we keep the superscript is for $Q^0 = \{\alpha(\lambda) : \lambda \text{ is a partition}\}$, to avoid confusion with the ambient abelian group $Q \supseteq Q^0$. Finally, we define

$$Q^0_* \coloneqq \{\alpha(\lambda) : \lambda \text{ is an } e\text{-core}\}.$$

We recall from Lemma 2.39 that $Q^0_* = \big\{\alpha \in Q : \mathsf{w}(\alpha) = 0\big\}$. Let $e' \in \{1, \dots, e-1\}$ and let $p \in \{2, \dots, e\}$ be the order of $e'$ in $\mathbb{Z}/e\mathbb{Z}$.

## 4.1 Shifting blocks

**Definition 4.1.** We define the $\mathbb{Z}$-linear map $\sigma : Q \to Q$ by $\sigma \cdot \alpha_i \coloneqq \alpha_{i-e'}$ for all $i \in \mathbb{Z}/e\mathbb{Z}$.

The map $\sigma$ is an automorphism of $Q$ of order $p$. As we saw in the introduction, the automorphism $\sigma$ appears together with a shift operation $\boldsymbol{\lambda} \mapsto {}^{\sigma}\boldsymbol{\lambda}$ on $r$-partitions when $r \geq 2$. This shift operation on $r$-partitions is defined by a (power of a) cyclic permutation of the components of $\boldsymbol{\lambda}$, and we have (under suitable conditions on the multicharge $\mathbf{s}$):

$$\alpha^{\mathbf{s}}\big({}^{\sigma}\boldsymbol{\lambda}\big) = \sigma \cdot \alpha^{\mathbf{s}}(\boldsymbol{\lambda})$$

(see (1.1)). In this Section 4 we are interested in the case $r = 1$. As we saw in Definition 4.1, the operation $\alpha \mapsto \sigma \cdot \alpha$ on $Q$ is still defined, but it is unclear what the operation $\lambda \mapsto {}^{\sigma}\lambda$ should be. Our aim is to study some properties of such an operation.

Note that :

$$\sigma \cdot \delta = \delta, \tag{4.2}$$

recalling that $\delta = \sum_{i \in \mathbb{Z}/e\mathbb{Z}} \alpha_i \in Q$.

**Proposition 4.3.** *Let $\alpha = \sum_{i \in \mathbb{Z}/e\mathbb{Z}} c_i \alpha_i \in Q$. We have $\mathsf{w}(\sigma \cdot \alpha) = \mathsf{w}(\alpha) + c_{e'} - c_0$.*

*Proof.* By definition, we have

$$\sigma \cdot \alpha = \sum_{i \in \mathbb{Z}/e\mathbb{Z}} c_{i+e'} \alpha_i,$$

thus

$$\begin{aligned}
\mathsf{w}(\sigma \cdot \alpha) &= c_{e'} - \frac{1}{2} \sum_{i \in \mathbb{Z}/e\mathbb{Z}} \big(c_{i+e'} - c_{i+e'+1}\big)^2 \\
&= c_{e'} - \frac{1}{2} \sum_{i \in \mathbb{Z}/e\mathbb{Z}} (c_i - c_{i+1})^2 \\
&= \mathsf{w}(\alpha) - c_0 + c_{e'}.
\end{aligned}$$

□

Using Lemma 2.39 and Proposition 2.40, we deduce the following corollary.

**Corollary 4.4.** *Let* $\alpha = \sum_{i\in\mathbb{Z}/e\mathbb{Z}} c_i\alpha_i \in Q^0$. *Then*

$$\sigma\cdot\alpha \in Q^0 \iff c_0 \leq c_{e'} + \mathsf{w}(\alpha),$$

*the right-hand side being an equality if and only if* $\sigma\cdot\alpha \in Q^0_*$. *In particular, if* $\alpha \in Q^0_*$ *then*

$$\sigma\cdot\alpha \in Q^0 \iff c_0 \leq c_{e'},$$

*the right-hand side being an equality if and only if* $\sigma\cdot\alpha \in Q^0_*$.

## 4.2 Shifting partitions

In the continuation of Corollary 4.4, given a partition $\lambda$ such that $\sigma\cdot\alpha(\lambda) \in Q^0$ we show how to obtain the partitions that lie in $\sigma\cdot\alpha(\lambda)$. The constructions and results that we present in this subsection can be obtained using *Scopes isometries* (see [Sc] and for instance [Fa06]), a Scopes isometry consisting in swapping two (adjacent) runners in the $e$-abacus.

**Definition 4.5.** Let $\lambda$ be an $e$-core. We denote by $\sigma\lambda$ the unique $e$-core such that

$$x(\sigma\lambda) = \big(x_{e'}(\lambda),\dots,x_{e-1}(\lambda),x_0(\lambda),\dots,x_{e'-1}(\lambda)\big).$$

Note that $\sigma\lambda$ is well-defined by Proposition 2.27 since $x_{e'}(\lambda)+\cdots+x_{e-1}(\lambda)+x_0(\lambda)+\cdots+x_{e'-1}(\lambda) = 0$.

**Proposition 4.6.** *Let* $\lambda$ *be an* $e$*-core and let* $z \coloneqq c_0(\lambda) - c_{e'}(\lambda)$. *We have*

$$\alpha(\sigma\lambda) = \sigma\cdot\alpha(\lambda) + z\delta.$$

*Proof.* By (2.25) we have $x_0(\lambda)+\cdots+x_{e'-1}(\lambda) = z$, moreover by Corollary 2.32 we have $c_0(\sigma\lambda) = c_0(\lambda)$. Hence, for any $i \in \mathbb{Z}/e\mathbb{Z}$ we have by (2.25)

$$\begin{aligned} c_i(\sigma\lambda) &= c_0(\sigma\lambda) - x_0(\sigma\lambda) - \cdots - x_{i-1}(\sigma\lambda)\\ &= c_0(\lambda) - x_{e'}(\lambda) - \cdots - x_{e'+i-1}(\lambda)\\ &= c_0(\lambda) + z - x_0(\lambda) - \cdots - x_{e'+i-1}(\lambda)\\ &= c_{e'+i}(\lambda) + z. \end{aligned}$$

We conclude since

$$\alpha(\sigma\lambda) = \sum_{i\in\mathbb{Z}/e\mathbb{Z}} c_i(\sigma\lambda)\alpha_i = \sum_{i\in\mathbb{Z}/e\mathbb{Z}} \big(c_{e'+i}(\lambda)+z\big)\alpha_i = \sum_{i\in\mathbb{Z}/e\mathbb{Z}} c_i(\lambda)\alpha_{i-e'} + z\delta = \sigma\cdot\alpha(\lambda) + z\delta.$$

□

The next result shows that Definition 4.5 fits with what we can expect between $\sigma\cdot\alpha(\lambda)$ and $\alpha(\sigma\lambda)$ when $\lambda$ is an $e$-core. Recall from Definition 2.70 that any $\alpha \in Q$ has an associated core.

**Corollary 4.7.** *Let* $\lambda$ *be a partition and let* $\overline{\lambda}$ *be its* $e$*-core. Then* $\alpha(\sigma\overline{\lambda})$ *is the core of* $\sigma\cdot\alpha(\lambda)$.

*Proof.* By definition, the partition $\sigma\overline{\lambda}$ is an $e$-core thus the block $\alpha(\sigma\overline{\lambda})$ is a core block (recalling Remark 2.62). The assertion then follows from Lemma 2.67 and Proposition 4.6. □

In particular, it follows from Remark 2.71 that if $\lambda$ is a partition such that $\sigma\cdot\alpha(\lambda) \in Q^0$ then $\sigma\overline{\lambda}$ is the common $e$-core of the partitions that lie in $\sigma\cdot\alpha(\lambda)$. We now propose a generalisation of Definition 4.5 when $\lambda$ is not necessarily an $e$-core.

**Definition 4.8.** Let $\lambda$ be a partition and for any $i \in \{0,\dots,e-1\}$, let $R_i$ be the $i$-th runner of the $e$-abacus of $\lambda$. We define $\sigma\lambda$ to be the partition whose $e$-abacus is obtained as follows: for any $i \in \{0,\dots,e-1\}$, the $i$-th runner is $R_{i+e'}$ (where addition is modulo $e$).

*Remark* 4.9. The partition $\sigma\lambda$ is the (unique) partition with $e$-core $\sigma\overline{\lambda}$ and $e$-quotient $\big(\lambda^{[e']},\dots,\lambda^{[e-1]},\lambda^{[0]},\dots,\lambda^{[e'-1]}\big)$. In particular by (2.22) we have $\mathsf{w}_e(\lambda)=\mathsf{w}_e\big(\sigma\lambda\big)$.

**Corollary 4.10.** *Let $\lambda$ be a partition. We have*

$$\alpha\left(\sigma\lambda\right)=\sigma\cdot\alpha(\lambda)\iff|\sigma\lambda|=|\lambda|\iff c_0(\lambda)=c_{e'}(\lambda).$$

*Proof.* By Lemma 2.6 we have $c_0(\lambda)-c_{e'}(\lambda)=c_0(\overline{\lambda})-c_{e'}(\overline{\lambda})$, thus from (4.2), Proposition 4.6 and Remark 4.9 we obtain

$$\alpha(\sigma\lambda)=\sigma\cdot\alpha(\lambda)+\left(c_0(\lambda)-c_{e'}(\lambda)\right)\mathbf{1}.$$

Since $|\sigma\cdot\alpha(\lambda)|=|\alpha(\lambda)|=|\lambda|$, we deduce that $|\sigma\lambda|=|\lambda|+c_0(\lambda)-c_{e'}(\lambda)$ and this concludes the proof. □

*Remark* 4.11. Is is easy to construct an $e$-core (and thus, a partition) that satisfies the equivalent conditions of Corollary 4.10. Indeed, if $\lambda$ is an $e$-core then by (2.25) we have $c_0(\lambda)=c_{e'}(\lambda)$ if and only if $x_0(\lambda)+\cdots+x_{e'-1}(\lambda)=0$.

## 4.3 Some properties

In this subsection, we always assume that $e'$ divides $e$. In particular, the order $p$ of $e'$ in $\mathbb{Z}/e\mathbb{Z}$ is $p=\frac{e}{e'}$. We will count both modulo $e$ and $e'$, thus to avoid ambiguities we will add a prime when we compute modulo $e'$. For instance, for any partition $\lambda$ and for any $i'\in\{0,\dots,e'-1\}$ we will denote by $c'_{i'}(\lambda)$ the number of nodes $\gamma$ of $\lambda$ such that $\operatorname{res}(\gamma)=i'$ (mod $e'$). The next lemma is immediate.

**Lemma 4.12.** *Let $i'\in\{0,\dots,e'-1\}$. For any partition $\lambda$ we have*

$$c'_{i'}(\lambda)=\sum_{k=0}^{p-1}c_{i'+ke'}(\lambda).$$

Let $Q'=\oplus_{j\in\mathbb{Z}/e'\mathbb{Z}}\mathbb{Z}\alpha'_j$ be a free abelian group with basis $\{\alpha'_j\}_{j\in\mathbb{Z}/e'\mathbb{Z}}$. We consider the $\mathbb{Z}$-linear map $\pi:Q\to Q'$ determined by

$$\pi(\alpha_i):=\alpha'_{i\bmod e'},$$

for all $i\in\mathbb{Z}/e\mathbb{Z}$. Note that $i$ mod $e'$ is well-defined since we have assumed that $e'$ divides $e$. It follows from the definition of the automorphism $\sigma$ that

$$\pi(\sigma\cdot\alpha)=\pi(\alpha)\in Q',$$

for any $\alpha\in Q$. We then deduce from Lemma 2.34 the following result.

**Proposition 4.13.** *Let $\alpha\in Q^0$ so that $\sigma\cdot\alpha\in Q^0$. If a partition $\lambda$ (respectively, $\mu$) lies in $\alpha$ (resp. $\sigma\cdot\alpha$) then $\lambda$ and $\mu$ share the same $e'$-core.*

Recall the classical result that any $e'$-core is an $e$-core (cf. Lemma 2.13).

**Proposition 4.14.** *Let $\lambda$ be an $e$-core and assume that $\sigma\cdot\alpha(\lambda)=\alpha(\lambda)$. Then $\lambda$ is an $e'$-core.*

*Proof.* Since $\sigma\cdot\alpha(\lambda)=\alpha(\lambda)$ we have $c_i(\lambda)=c_{i+e'}(\lambda)$ for any $i\in\{0,\dots,e-1\}$. We deduce that

$$\sum_{i=0}^{e-1}\left(c_i(\lambda)-c_{i+1}(\lambda)\right)^2=p\sum_{i'=0}^{e'-1}\left(c_{i'}(\lambda)-c_{i'+1}(\lambda)\right)^2,$$

and, by Lemma 4.12,

$$c'_{i'}(\lambda)=pc_{i'}(\lambda),$$

for any $i' \in \{0, \dots, e'-1\}$. By Proposition 2.31, we obtain

$$\begin{aligned}
\mathsf{w}_{e'}(\lambda) &= c'_0(\lambda) - \frac{1}{2}\sum_{i'=0}^{e'-1}\left(c'_{i'}(\lambda) - c'_{i'+1}(\lambda)\right)^2 \\
&= pc_0(\lambda) - \frac{p^2}{2}\sum_{i'=0}^{e'-1}\left(c_{i'}(\lambda) - c_{i'+1}(\lambda)\right)^2 \\
&= pc_0(\lambda) - \frac{p}{2}\sum_{i=0}^{e-1}\left(c_i(\lambda) - c_{i+1}(\lambda)\right)^2 \\
&= p\mathsf{w}_e(\lambda) \\
&= 0,
\end{aligned}$$

thus $\lambda$ is an $e'$-core. □

The previous proposition involves a block $\alpha \in Q$ satisfying $\alpha = \sigma \cdot \alpha$. We say that such a block is *stuttering*. In a similar spirit as in [Ro19b], we will now study the relationship between stuttering blocks and stuttering partitions, that is, partitions $\lambda$ satisfying $\sigma\lambda = \lambda$.

**Lemma 4.15.** *Let $\lambda$ be a partition that satisfies $\sigma\lambda = \lambda$. Then $\sigma \cdot \alpha(\lambda) = \alpha(\lambda)$ and $p \mid \mathsf{w}_e(\lambda)$.*

*Proof.* The first assertion is clear by Corollary 4.10 since $|\sigma\lambda| = |\lambda|$. For the second assertion, since $\sigma\lambda = \lambda$ then we know by Remark 4.9 that the $e$-quotient of $\lambda$ is

$$\left(\lambda^{[0]}, \dots, \lambda^{[e'-1]}, \lambda^{[0]}, \dots, \lambda^{[e'-1]}, \dots, \lambda^{[0]}, \dots, \lambda^{[e'-1]}\right),$$

where the sequence $\lambda^{[0]}, \dots, \lambda^{[e'-1]}$ is repeated $p$ times. Thus by (2.22) we have

$$\mathsf{w}_e(\lambda) = p\sum_{i=0}^{e'-1}\left|\lambda^{[i]}\right|,$$

which concludes the proof. □

The aim is now to give a converse statement for Lemma 4.15, proving Theorem D from the introduction.

**Proposition 4.16.** *Let $\lambda$ be an $e$-core. If $\sigma \cdot \alpha(\lambda) = \alpha(\lambda)$ then $\sigma\lambda = \lambda$.*

*Proof.* The assumption implies that $c_0(\lambda) = c_{e'}(\lambda)$. Hence, by Corollary 4.10 we have $\alpha(\sigma\lambda) = \sigma \cdot \alpha(\lambda) = \alpha(\lambda)$, thus $\lambda$ and $\sigma\lambda$ lie in the same block. This concludes since $\lambda$ is an $e$-core (recalling Lemma 2.34). □

**Corollary 4.17.** *Let $\alpha \in Q^0$. If $\alpha = \sigma \cdot \alpha$ and $p \mid \mathsf{w}(\alpha)$ then there exists $\lambda$ with $\alpha(\lambda) = \alpha$ satisfying $\sigma\lambda = \lambda$.*

*Proof.* Let $\mu$ be a partition with $\alpha(\mu) = \alpha$ and let $\overline{\mu}$ be the $e$-core of $\mu$. By Lemma 2.8 and (4.2), since $\sigma \cdot \alpha(\mu) = \alpha(\mu)$ we have $\sigma \cdot \alpha(\overline{\mu}) = \alpha(\overline{\mu})$. Thus, by Proposition 4.16 we have $\sigma\overline{\mu} = \overline{\mu}$. By assumption and (2.9), we can write $\mathsf{w}_e(\mu) = wp$ with $w \in \mathbb{N}$. Now let $\lambda$ be the partition whose $e$-abacus is obtained as follows: for each $i \in \{0, \dots, p-1\}$, we slide $w$ times the rightmost bead on runner $ie'$ of the $e$-abacus corresponding to $\overline{\mu}$. By Lemmas 2.8 and 2.20, the partition $\lambda$ satisfies $\alpha(\lambda) = \alpha(\overline{\mu}) + pw\mathbf{1} = \alpha(\overline{\mu}) + \mathsf{w}_e(\mu)\mathbf{1} = \alpha(\mu)$, and by construction $\sigma\lambda = \lambda$. □

Note that for any $h \in \mathbb{N}$, by (4.2) the block $\alpha = h\mathbf{1} \in Q^0$ satisfies $\alpha = \sigma \cdot \alpha$, however, we have $p \mid \mathsf{w}(\alpha)$ if and only if $p \mid h$.

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