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Paths in tournaments, a simple proof of Rosenfeld's Conjecture

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Abstract

Rosenfeld Conjectured [7] in 1972 that there exists an integer $K \geq 8$ such that any tournament of order $n \geq K$ contains any Hamiltonian oriented path. In 2000, Havet and Thomassé [5] proved this conjecture for any tournament with exactly 3 exceptions. We give a simplified proof of this fact.

1 Introduction

A tournament T is an orientation of a complete graph. The set of vertices of T is denoted by $V(T)$ and the set of arcs by $E(T)$. $v(T)$ will denote the order of T , $|V(T)|$. Sometimes, we write $|T|$ instead of $v(T)$. T is said to be an n -tournament if $v(T) = n$.

The out-neighbour (resp. in-neighbour) of a vertex v in T is denoted by $N_T^+(v)$ (resp. $N_T^-(v)$) and its out-degree (resp. in-degree) is denoted by $d_T^+(v)$ (resp. $d_T^-(v)$). We denote by $\delta^+(T)$ (resp. $\delta^-(T)$) the minimal out-degree (resp. in-degree) and by $\Delta^+(T)$ (resp. $\Delta^-(T)$) the maximal out-degree (resp. in-degree). Note that $\delta^-(T) \leq \delta^+(T)$ (resp. $\delta^+(T) \leq \Delta^-(T)$). A tournament T is said to be regular if $d^+(v) = d^-(v) \forall v \in V(T)$. A cyclic triangle is a circuit of length 3. We denote by T_4^+ a tournament composed of a circuit triangle together with a source. A Paley tournament on 7 vertices is a tournament T such that $V(T) = \{v_i, 1 \leq i \leq 7\}$ and $(v_i, v_j) \in E(T)$ if and only if $j - i \equiv 1, 2$ or $4 \pmod{7}$. We write $T' \subseteq T$ whenever T' is a subtournament of T . Let $S \subseteq V(T)$, we denote by $T[S]$ the subtournament of T induced by S . If $T' \subseteq T$ and $S \subseteq V(T)$, we write $T' + S = T[V(T') \cup S]$ and $T' - S = T[V(T') - S]$. Let $v \in V(T)$, $d_S^+(v) = |N_T^+(v) \cap S|$ and $d_S^-(v) = |N_T^-(v) \cap S|$. A subset $\{x_1, \dots, x_r\}$ in T will be denoted by $[x_1, x_r]$.

Let $P = x_1 \dots x_s$ be an oriented path, set $\tilde{P} = x_s \dots x_1$. P is called an s -path, x_1 and x_s are its extremities, x_1 is the origin and x_s is the end. The length of P , $l(P)$, is the number of its arcs. P is said to be directed if all of its arcs are oriented in the same direction. A block of P is a maximal (for \subseteq) directed subpath of P . The path P is said to be of type $P(b_1, \dots, b_m)$ and we write $P = P(b_1, \dots, b_m)$, if P is composed of m successive blocks B_1, \dots, B_m , such that $l(B_i) = b_i$. Moreover, we write $P = P^+(b_1, \dots, b_m)$, if $(x_1, x_2) \in E(P)$. Else, we

write $P = P^-(b_1, \dots, b_m)$. P is said to be antidirected if each block of P is of length 1.

Note that if $P = x_1 \dots x_s = P^+(b_1, \dots, b_m)$ (resp. $P^-(b_1, \dots, b_m)$) and $P' = x'_1 \dots x'_s = P^+(b_1, \dots, b_m)$ (resp. $P^-(b_1, \dots, b_m)$), then P and P' are isomorphic. We write $P \equiv P'$. Furthermore, if we write $x_1 \dots x_s \equiv x'_1 \dots x'_s$, then the mapping:

$$\begin{aligned} f : V(P) &\rightarrow V(P') && \text{is an isomorphism.} \\ x_i &\rightarrow f(x_i) = x'_i \end{aligned}$$

A path P , in a tournament T , is said to be Hamiltonian if $V(P) = V(T)$. Let $P = x_1 \dots x_s$ and $Q = y_1 \dots y_r$ be 2 disjoint paths in T , PQ denotes the path $x_1 \dots x_s y_1 \dots y_r$. In a similar way, we may define the path $P_1 P_2 \dots P_t$ from t pairwise disjoint paths P_1, \dots, P_t of T . Let $P = v_1 \dots v_{i-1} v_i v_{i+1} \dots v_n$ be an oriented path in a tournament T , then $P - v_i$ is the path $v_1 \dots v_{i-1} v_{i+1} \dots v_n$ in T . A strong tournament is such that any two of its vertices can be joined by a directed path. It is known that any tournament T is a transitive union of strong subtournaments (called strong components) I_1, I_2, \dots, I_t that is $(v_i, v_j) \in E(T) \forall i < j, v_i \in I_i$ and $v_j \in I_j$. We write $T = I_1 \dots I_t$. A strong tournament is characterized by the following property due to Camion [2].

Theorem 1. *T is strong if and only if T contains a Hamiltonian circuit.*

As a consequence of the above theorem, if $T = I_1 \dots I_t$, then T contains a directed path ending at x with $V(P) = V(I_1) \cup \dots \cup V(I_i)$ for every $x \in T_i$.

The complement of a digraph D , denoted by \overline{D} , is the digraph obtained from D by reversing all its arcs.

In 1971, Grünbaum [4] proved the following theorem:

Theorem 2. *Any tournament contains any Hamiltonian antidirected path with exactly 3 exceptions: a cyclic triangle (T_3), a regular tournament on 5 vertices (T_5) and a Paley tournament on 7 vertices (T_7).*

Set $\mathcal{T}_{3,5,7} = \{T_3, T_5, T_7\}$. Note that if $T \in \mathcal{T}_{3,5,7}$ contains a copy of a path P , then any vertex in T is an origin of a copy of P . Rosenfeld [7], in 1972, inspired by the work of Grünbaum, conjectured that there exists $K \geq 8$ such that any tournament of order $n \geq K$ contains any Hamiltonian oriented path. The case of directed path being Rédei's theorem [6]. Alspach, Rosenfeld [1] and Straight [9] proved Rosenfeld's conjecture on paths of 2 blocks. In 1973, Forcade [3] proved Rosenfeld's conjecture for any tournament of order 2^n . Thomason [10] was the first one to give a general answer. He proved, in 1986, that there exists $n_0 < 2^{128}$ such that for all $n \geq n_0$ any tournament of order n contains any Hamiltonian oriented path. Havet and Thomassé settled the problem by proving that the three exceptions of Grünbaum are the only tournaments not satisfying Rosenfeld's conjecture. Havet and Thomassé's proof consists of giving a refinement of a key idea introduced by Thomason saying that any set of $b_1 + 1$ vertices in an n -tournament contains an origin of any $(n - 1)$ -path whose first block is of length b_1 . They proved that if $s^+(x, y) \geq b_1 + 1$, then x or y is an origin of a copy of such path where $s^+(x, y) = |\{z \in T \text{ such that } z \text{ can be reached from } x \text{ or } y \text{ by a directed path}\}|$. This new performance allowed them to remark that proving the existence of an $(n - 1)$ -oriented path in any n -tournament T is equivalent to the existence of any Hamiltonian path P in

this tournament unless (T, P) is one of the 69 exceptions that were verified one by one. In this paper, we give a simple proof of the result without treating all these exceptions.

2 The main result

In our proof, we are going to use the following theorem due to El-sahili and Ghazo-Hanna [8].

Theorem 3. *A path lies in a tournament T if and only if it lies in \overline{T} .*

The following Lemmas will be useful in the sequel.

Lemma 1. *Let T be a tournament then $|\{v \in T / T - v \in \mathcal{T}_{3,5,7}\}| \leq 2$.*

Proof. Suppose that T contains 3 distinct vertices, v_1, v_2, v_3 such that $T - v_i \in \mathcal{T}_{3,5,7}$, $1 \leq i \leq 3$. We may suppose without loss of generality that $(v_1, v_2), (v_2, v_3) \in E(T)$. Since $T - v_1 \in \mathcal{T}_{3,5,7}$ and $T - v_3 \in \mathcal{T}_{3,5,7}$, then $N_{T-v_3}^-(v_1) = N_{T-v_1}^-(v_3)$ and $N_{T-v_1}^+(v_3) = N_{T-v_3}^+(v_1)$, but $v_2 \in N_{T-v_3}^+(v_1)$ and $v_2 \notin N_{T-v_1}^+(v_3)$, which is a contradiction. \square

We may deduce from this Lemma more precise conclusions. Indeed, if $T \in \mathcal{T}_{3,5,7}$, then $\forall x \in T, \exists y \in N^+(x)$ such that $T - \{x, y\} \notin \mathcal{T}_{3,5,7}$.

Lemma 2. *(Simple Lemma) Let $n \geq 4$. Suppose that any s -tournament ($s < n$) contains any s -path unless if the tournament is in $\mathcal{T}_{3,5,7}$ and the path is antidirected. Let T be an n -tournament with a vertex v such that $d^-(v) = 0$ and let $P = x_1 \dots x_n$ be a non directed path. Unless $P = P^+(1, 2)$ and $T - v$ is a cyclic triangle, we have:*

1. T contains a copy of P with origin $x \neq v$.
2. Any of the vertices of $T - v$ is an origin of a copy of P if $T - v \in \mathcal{T}_{3,5,7}$.

Proof.

1. Let $j \in [2, n]$ be the minimal integer such that $d_P^-(x_j) = 0$. If $j = n$, then $T - v$ contains a path P' such that $P'v \equiv P$ since otherwise $P - x_j$ is antidirected and so $P = P^+(1, 2)$ and $T = T_4^+$ which is a contradiction. Otherwise, $1 < j < n$. In this case, $P - x_j + (x_{j-1}, x_{j+1})$ or $P - x_j + (x_{j+1}, x_{j-1})$ is not antidirected. Suppose that $P' = P - x_j + (x_{j-1}, x_{j+1})$ is not antidirected, then $T - v$ contains a path $v_1 v_2 \dots v_{j-1} v_{j+1} \dots v_n \equiv P'$ and so $v_1 \dots v_{j-1} v v_{j+1} \dots v_n \equiv P$ and $v_1 \neq v$.
2. If $T - v \in \mathcal{T}_{3,5,7}$, then we may suppose that a copy of P' may be found in $T - v$ starting at any one of the vertices of $T - v$.

\square

By analogy, the above lemma is valid if $d^+(v) = 0$ with the only exception where $P = P^-(1, 2)$ and $T - v$ is a cyclic triangle. Using the same reasoning, we may deduce that if P contains a vertex x_j with $1 < j < n$ such that $d_P^-(x_j) = 0$, then T contains a copy of P such that v is not an extremity of P .

We are going now to present the proof of the main result.

Theorem 4. *Any n -tournament contains any n -path unless the three exceptions of Grünbaum.*

Proof. Let T be an n -tournament and let $P = x_1 \dots x_n$ be an n -path. If $n \leq 4$, then any n -path is either directed, antidirected or of two blocks. By theorem 3, the problem will be solved if we prove the existence of \hat{P} in \hat{T} , where $\hat{P} \in \{P, \overline{P}\}$ and $\hat{T} \in \{T, \overline{T}\}$. We may suppose, without loss of generality, that $\Delta^+(T) \geq \Delta^+(\overline{T})$. We argue by induction on $|T| \geq 5$. The case $\delta^-(T) = 0$ can be deduced from simple lemma and Rédei's theorem. Set $\delta^-(T) = i \geq 1$ and suppose, without loss of generality, that $(x_i, x_{i+1}) \in E(P)$, since otherwise, we use \overline{P} . Then $x_i \dots x_n$ is of type $P^+(b_1, b_2, \dots, b_r)$ for some $b_1, \dots, b_r \geq 1$. Let $v \in T$ such that $d^-(v) = \delta^-(T)$, $T_1 = T[N^-(v)]$ and $T_2 = T[N^+(v)]$. We will treat the two cases according to the existence of a copy of $x_1 \dots x_i$ in T_1 .

Case 1. $T_1 \not\supseteq v_1 \dots v_i \equiv x_1 \dots x_i$.

In this case, $x_1 \dots x_i$ is antidirected and $T_1 \in \mathcal{T}_{3,5,7}$. If $b_1 \geq 3$, let $a \in T_2$ such that $T_2 - a \supseteq v_{i+3} \dots v_n \equiv x_{i+3} \dots x_n$. Then $T_1 + a \supseteq v_1 \dots v_{i+1} \equiv x_1 \dots x_{i+1}$ such that $v_{i+1} \neq a$. So $v_1 \dots v_{i+1} v v_{i+3} \dots v_n \equiv P$.

In the case $b_1 \leq 2$, for $(\mathbf{x}_i, \mathbf{x}_{i-1}) \in \mathbf{E}(\mathbf{P})$, suppose that $b_1 = 1$, we discuss if $T_2 \supseteq v_{i+2} \dots v_n \equiv x_{i+2} \dots x_n$ or not. In the first case, suppose that there exists $x \in T_1$ such that $(v_{i+2}, x) \in E(T)$. By simple lemma, $T_1 + v \supseteq v_1 \dots v_{i+1} \equiv x_1 \dots x_{i+1}$ with $v_{i+1} = x$, so $v_1 \dots v_n \equiv P$. If $(x, v_{i+2}) \in E(T) \forall x \in T_1$, let t be the minimal integer ($t \geq i+2$) such that $d_P^-(x_t) = 0$. If there exists $y \in T_1$ such that $T_2 + y \supseteq w_{i+1} \dots w_n \equiv x_{i+1} \dots x_n$ with $w_{i+1} = v_{i+2}$, then $(T_1 - y) + v \supseteq v_1 \dots v_i \equiv x_1 \dots x_i$ and so $v_1 \dots v_i w_{i+1} \dots w_n \equiv P$. Otherwise, $N^-(x) \cap [v_{i+2}, v_t] = \emptyset$ and $(v_{t+1}, x) \in E(T) \forall x \in T_1$. In this case, let $y \in T_1$. $Q = (v_{t+1} y v_t v_{t-1} \dots v_{i+2} - v_t) - v_{i+2}$ is a directed path. If $t \geq i+3$, then $l(Q) = b_2 - 1$. $T_2[v_{t+2}, v_n] + \{v_t, v\} \supseteq w_t \dots w_n \equiv x_t \dots x_n$ with $w_t = v$ or, by simple lemma, $w_t = v_t$. In the other hand, by simple lemma, $T_1 + v_{i+2} \supseteq v_1 \dots v_i v_{i+1} \equiv x_1 \dots x_{i+1}$ with $v_{i+1} = y$ and $v_i \neq v_{i+2}$, so $v_1 \dots v_i \tilde{Q} w_t \dots w_n \equiv P$. If $t = i+2$, let $x \in T_1$, $T_1 - x \supseteq v_2 \dots v_i \equiv x_2 \dots x_i$. If $(v_{t+1}, v_{t+2}) \in E(T)$, $v_{t+1} v_2 \dots v_i v_t x v v_{t+2} \dots v_n \equiv P$. If $(v_{t+2}, v_{t+1}) \in E(T)$ and there exists $y \in T_1$ such that $(v_{t+2}, y) \in E(T)$, consider $(a, b) \in E(T_1 - y)$, $(T_1 - \{a, b, y\}) + v \supseteq v_1 \dots v_{i-2} \equiv x_1 \dots x_{i-2}$ and so $v_1 \dots v_{i-2} v_t a b v_{t+1} y v_{t+2} \dots v_n \equiv P$. Else, let $z \in T_1$. $(T_1 - z) + v_{t+1} \supseteq v_1 \dots v_i \equiv x_1 \dots x_i$ with $v_i \neq v_{t+1}$ and $(T[v_{t+2}, v_n]) + v \supseteq w_{t+1} \dots w_n \equiv x_{t+1} \dots x_n$ with $w_{t+1} = v_{t+2}$ then $v_1 \dots v_i v_t z w_{t+1} \dots w_n \equiv P$.

In the other case, $T_2 \not\supseteq v_{i+2} \dots v_n \equiv x_{i+2} \dots x_n$, so $T_2 \in \mathcal{T}_{3,5,7}$ and $x_{i+2} \dots x_n$ is antidirected. If $b_2 = 1$, then P is antidirected treated in Theorem 2. Otherwise, $b_2 = 2$. Let $a \in T_2$, then $T_1 + a \supseteq v_1 \dots v_{i+1} \equiv x_1 \dots x_{i+1}$ such that $v_{i+1} \neq a$. Since $v_{i+1} \in T_1 \in \mathcal{T}_{3,5,7}$, then $d_{T_2}^-(v_{i+1}) \geq 2$, there exists $b \in T_2 - a$ such that $(b, v_{i+1}) \in E(T)$. If $T_2 - \{a, b\} \supseteq Q \equiv x_{i+4} \dots x_n$, then $v_1 \dots v_{i+1} b v Q \equiv P$. Otherwise, there exists $c \in T_2 - \{a, b\}$ such that $(c, b) \in E(T)$, then, by simple lemma, $(T_2 - \{a, b\}) + v \supseteq Q \equiv x_{i+3} \dots x_n$ starting by c , then $v_1 \dots v_{i+1} b Q \equiv P$.

For $b_1 = 2$, here also we study if $T_2 \supseteq v_{i+2} \dots v_n \equiv x_{i+2} \dots x_n$ or not. In the first case, let $x \in N^-(v_{i+2}) \cap T_1$, if any, by simple lemma, $T_1 + v \supseteq v_1 \dots v_{i+1} \equiv x_1 \dots x_{i+1}$ with $v_{i+1} = x$ and so $v_1 \dots v_n \equiv x_1 \dots x_n$. If $N^-(v_{i+2}) \cap T_1 = \emptyset$, v can be inserted inside $v_{i+3} \dots v_n$ to obtain a path $Q \equiv x_{i+2} \dots x_n$ starting at v_{i+3} in $(T_2 - v_{i+2}) + v$. Similarly, by simple lemma, if $N^-(v_{i+3}) \cap T_1 = \emptyset$, we may find a copy of P in T , so, we may suppose in the sequel that $d_{T_1}^-(v_{i+2}) = d_{T_1}^-(v_{i+3}) = 0$.

As above, let $t \geq i+3$ be the minimal integer such that $d_P^-(x_t) = 0$. If $t = i+3$, since $T_1 \in \mathcal{T}_{3,5,7}$, then $d_{T_2}^+(x) \geq 1 \forall x \in T_1$, since otherwise, $\Delta^+(\overline{T}) > \Delta^+(T)$. If there exists $x \in T_1$ and $a \in T_2$ such that $(x, a) \in E(T)$ and $d_{T_2-v_{i+2}}^-(a) \geq 1$, let $b \in N_{T_2-v_{i+2}}^-(a)$. By simple lemma, $T_1 + v_{i+2} \supseteq v_1 \dots v_{i+1} \equiv x_1 \dots x_{i+1}$ with $v_{i+1} = x$. $(T_2 - \{a, v_{i+2}\}) + v \supseteq v_{i+3} \dots v_n \equiv x_{i+3} \dots x_n$ with $v_{i+3} = v$ or, by simple lemma, $v_{i+3} = b$, so $v_1 \dots v_{i+1} a v_{i+3} \dots v_n \equiv P$. Otherwise, if $(v_{i+2}, a) \in E(T)$, then the problem is solved by considering $v_1 \dots v_{i+1} \equiv x_1 \dots x_{i+1}$ in $T_1 + v_{i+3}$. Otherwise, there exists $a \in T_2$ such that $\forall x \in T_1, \forall y \in T_2 - a$, we have $(x, a), (y, x), (a, y) \in E(T)$. Let $y \in T_2 - a$ such that $T_2 - \{a, y\} \notin \mathcal{T}_{3,5,7}$. $T_1 + y \supseteq v_1 \dots v_{i+1} \equiv x_1 \dots x_{i+1}$ with $v_{i+1} \neq y$ and $T_2 - \{a, y\} \supseteq v_{i+4} \dots v_n \equiv x_{i+4} \dots x_n$ then $v_1 \dots v_{i+1} a v_{i+4} \dots v_n \equiv P$. If $t > i+3$, let $x \in T_1$, $T_1 - x \supseteq v_2 \dots v_i \equiv x_2 \dots x_i$. If $(T_2 - v_{i+2}) + \{v, x, v_i\} \supseteq w_i \dots w_n \equiv x_i \dots x_n$ with $w_i \in \{v_{i+3}, v_i\}$, then $v_{i+2} v_2 \dots v_{i-1} w_i \dots w_n \equiv P$. Otherwise, $\forall x \in T_1 N^-(x) \cap [v_{i+4}, v_t] = \emptyset$ and $(v_{t+1}, x) \in E(T)$. Let $x, y \in T_1$ such that $(x, y) \in E(T)$ and $T_1 - \{x, y\} \supseteq v_2 \dots v_{i-1} \equiv x_2 \dots x_{i-1}$. As above, $(T_2 - \{v_{i+2}, v_{i+3}\}) + \{y, v\} \supseteq w_{i+2} \dots w_n \equiv x_{i+2} \dots x_n$ with $w_{i+2} \in \{y, v_{i+4}\}$, so $v_{i+2} v_2 \dots v_{i-1} v_{i+3} x w_{i+2} \dots w_n \equiv P$. If $T_2 \not\supseteq v_{i+2} \dots v_n \equiv x_{i+2} \dots x_n$ then, $x_{i+2} \dots x_n$ is antirected and $T_2 \in \mathcal{T}_{3,5,7}$. If $|T_2| > |T_1|$, the problem is solved by considering $\overline{P}(b_1 = 1)$. Otherwise, $|T_1| = |T_2|$. Let $x \in T_1$. $T_2 + x \supseteq v_{i+1} \dots v_n \equiv x_{i+1} \dots x_n$ such that $v_{i+1} \neq x$. If there exists $y \in T_1 - x$ such that $(y, v_{i+1}) \in E(T)$, then $(T_1 + v) - x \supseteq w_1 \dots w_i \equiv x_1 \dots x_i$ with $w_i = y$ and so $w_1 \dots w_i v_{i+1} \dots v_n \equiv P$, unless $T_1 - \{x, y\} \in \mathcal{T}_{3,5,7}$, then there exists $z \in T_1 - \{x, y\}$ such that $(y, z) \in E(T)$. By simple lemma, $(T_1 - \{x, y\}) + v \supseteq w_1 \dots w_{i-1}$ with $w_{i-1} = z$, and so $w_1 \dots w_{i-1} y v_{i+1} \dots v_n \equiv P$. Otherwise T_1 and T_2 are cyclic triangles. The problem is solved unless if for $u \in T, T[N^+(u)], T[N^-(u)] \in \mathcal{T}_3$. So, if $u \in T_1$, then $d_{T_2}^+(u) = 1$. Else, $d_{T_1}^+(u) = 2$. Set $V(T_1) = \{x, y, z\}$, then $T_2 + x \supseteq v_4 \dots v_7 \equiv x_4 \dots x_7$. Since $x_4 \dots x_7$ is antirected, then $v_5 = x$. Suppose, without loss of generality that, $(y, v_4) \in E(T)$, then $z v y v_4 \dots v_7 \equiv P$.

If $(x_{i-1}, x_i) \in \mathbf{E}(\mathbf{P})$, let $x \in T_1$. $T_1 - x \supseteq Q_1 \equiv x_1 \dots x_{i-1}$. If $T_2 + x \supseteq v_{i+1} \dots v_n \equiv x_{i+1} \dots x_n$ and $v_{i+1} \neq x$, $Q_1 v v_{i+1} \dots v_n \equiv P$. If $v_{i+1} = x$, then, by simple lemma, $T_1 + v \supseteq v_1 \dots v_{i+1} \equiv x_1 \dots x_{i+1}$ such that $v_{i+1} = x$. Then $v_1 \dots v_n \equiv P$. If $T_2 + x$ contains no copy of $x_{i+1} \dots x_n$, then this path is antirected, so we consider $\tilde{P} = y_1 \dots y_n$ to remark that P or \overline{P} contains the arcs (y_i, y_{i-1}) and (y_i, y_{i+1}) , we recover a previous case.

Case 2. $T_1 \supseteq v_1 \dots v_i \equiv x_1 \dots x_i$.

If $b_1 \geq 2$, then either $T_2 \supseteq v_{i+2} \dots v_n \equiv x_{i+2} \dots x_n$ or not. In the first case, $v_1 \dots v_i v v_{i+2} \dots v_n \equiv P$ and in the last case, $T_2 \in \mathcal{T}_{3,5,7}$ and $x_{i+2} \dots x_n$ is antirected. If $\exists x \in T_2$ such that $(v_i, x) \in E(T)$, then, by simple lemma, $T_2 + v \supseteq v_{i+1} \dots v_n \equiv x_{i+1} \dots x_n$ with $v_{i+1} = x$. Thus $v_1 \dots v_i v_{i+1} \dots v_n \equiv P$. Otherwise, $(x, v_i) \in E(T) \forall x \in T_2$, we have $(v_i, v_j) \in E(T)$ whenever $j < i$, since otherwise $\Delta^+(\overline{T}) \geq d_T^+(v_i) = d_T^-(v_i) > |T_2| = d^+(v) = \Delta^+(T)$, a contradiction. By simple lemma, $T_2 + v_i$ contains a path $v_{i+1} \dots v_n \equiv x_{i+1} \dots x_n$ with $v_{i+1} \in T_2$. If $i = 1$, then $v v_{i+1} \dots v_n \equiv P$. Otherwise, let $j < i$ be the maximal integer such that $d_P^+(x_j) = 0$, then $v_1 \dots v_{j-1} v v_j \dots v_{i-1} \equiv x_1 \dots x_i$. Since $(v_i, v_{i-1}) \in E(T)$, then $\exists x \in T_2$ such that $(v_{i-1}, x) \in E(T)$. By simple lemma, we suppose that $v_{i+1} = x$ and so $v_1 \dots v_{j-1} v v_j \dots v_{i-1} v_{i+1} \dots v_n \equiv P$.

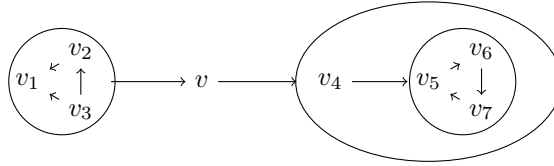
Now, we will study the case $b_1 = 1$. Suppose that $N^+(v_i) \cap T_2 \neq \emptyset$. Let I_1, I_2, \dots, I_t be the strong connected components of T_2 such that $T_2 = I_1 \dots I_t$ and

let $l = \max\{j, N^+(v_i) \cap I_j \neq \emptyset\}$, $I = I_1 I_2 \dots I_l$ and $s = |I|$. We discuss according to the value of s :

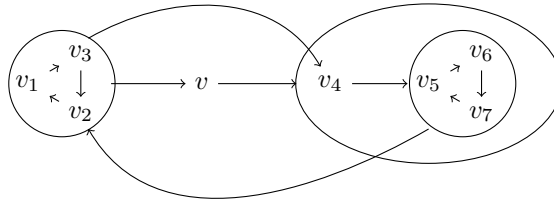
For the case $s > \mathbf{b}_2$, let $Q = u_1 \dots u_{b_2+1}$ be a directed path in I such that $u_{b_2+1} \in N^+(v_i)$ and let $j = i + b_2 + 2$. So either $T_2 - [u_2, u_{b_2+1}]$ contains a path $v_j \dots v_n \equiv x_j \dots x_n$, and in this case $v_1 \dots v_i u_{b_2+1} \dots u_2 v v_j \dots v_n \equiv P$, or $T_2 - [u_2, u_{b_2+1}] \in \mathcal{T}_{3,5,7}$ and $x_j \dots x_n$ is antidirected. By simple lemma, $(T_2 - [u_2, u_{b_2+1}]) + v$ contains a path $v_{j-1} \dots v_n \equiv x_{j-1} \dots x_n$ with $v_{j-1} = u_1$. In this case, $v_1 \dots v_i u_{b_2+1} \dots u_2 v_{j-1} \dots v_n \equiv P$.

Now we will study the case $s = \mathbf{b}_2$. Let $Q = u_1 \dots u_{b_2}$ be a directed path in I such that $u_{b_2} \in N^+(v_i)$ and let $j = i + b_2 + 2$. If $T_2 - I \notin \mathcal{T}_{3,5,7}$ or $x_j \dots x_n$ is not antidirected, then $v_1 \dots v_i u_{b_2} \dots u_1 v v_j \dots v_n \equiv P$ where $T_2 - I \supseteq v_j \dots v_n \equiv x_j \dots x_n$. Otherwise, we will continue the proof depending on the orientation of $x_i x_{i-1}$ and on the value of i . If $(x_{i-1}, x_i) \in E(P)$ or $i = 1$, let $a \in T_2 - I$. By simple lemma, $(T_2 - (I \cup a)) + v_i$ contains a path $v_j \dots v_n \equiv x_j \dots x_n$ with $v_j \neq v_i$, then $v_1 \dots v_{i-1} v a u_{b_2} \dots u_1 v v_j \dots v_n \equiv P$. Otherwise, $(x_i, x_{i-1}) \in E(P)$ and $i \geq 2$. If $N^+(v_{i-1}) \cap (T_2 - I) \neq \emptyset$, let $a \in N^+(v_{i-1}) \cap (T_2 - I)$. As above $\exists r < i$ such that $v_1 \dots v_{r-1} v v_r \dots v_{i-1} \equiv x_1 \dots x_i$ and by simple lemma $(T_2 - (I \cup a)) + v_i \supseteq v_j \dots v_n \equiv x_j \dots x_n$ with $v_j \neq v_i$, so $v_1 \dots v_{r-1} v v_r \dots v_{i-1} a u_{b_2} \dots u_1 v v_j \dots v_n \equiv P$. Otherwise, let abc be a cyclic triangle in $T_2 - I$. Then, if $b_2 \geq 2$, $v_1 \dots v_{i-1} a v_i b u_{b_2} \dots u_3 v v_j \dots v_n \equiv P$ with $v_j \dots v_n \subseteq (T_2 - (I \cup \{a, b\})) + \{u_1, u_2\}$ such that $v_j \dots v_n \equiv x_j \dots x_n$. If $b_2 = b_3 = 1$, $v_1 \dots v_{i-1} a v_i b c v v_{j+2} \dots v_n \equiv P$ where $v_{j+2} \dots v_n \subseteq T_2 - \{a, b, c\}$. If $b_2 = 1, b_3 = 2$ and $T_2 - I \in \{T_5, T_7\}$, choose $d \in T_2 - \{u_1, a, b, c\}$ such that $T_2 - \{u_1, a, b, c, d\} \notin \mathcal{T}_{3,5,7}$ (d exists by Lemma 1), then $v_1 \dots v_{i-1} d v_i a b c v v_{j+3} \dots v_n \equiv P$ where $v_{j+3} \dots v_n \subseteq T_2 - \{a, b, c, d\}$.

For the remaining cases, $T_2 = T_4^+$. The possible situations are the following. If $|T_1| = 2$, then a copy of P is found by considering \widetilde{P} . If $|T_1| = 3$, we have the following two cases:



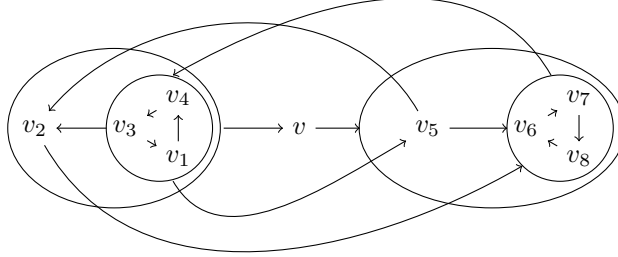
$$v_5 v_4 v v_7 v_6 v_3 v_1 v_2 \equiv P.$$



$$v_7 v_6 v_5 v_2 v_3 v v_4 v_1 \equiv P.$$

If $|T_1| = 4$, the problem is solved unless $T_1 = \overline{T_2}$ and $x_1 \dots x_4 \equiv x_9 \dots x_6$.

In this case, we have:



$$v_3 v v_2 v_1 v_4 v_6 v_7 v_8 v_5 \equiv P.$$

Finally, $s \leq b_2 - 1$. Let $u_1 \dots u_s$ be a directed path in T_2 such that $(v_i, u_s) \in E(T)$. If $s \geq 2$, then $b_2 \geq 3$. Let Q be a directed path of length $b_2 - s - 1$ in $T_2 - [u_1, u_s]$ such that $(T_2 - Q) - [u_1, u_s] \supseteq Q_1 \equiv x_{i+b_2+2} \dots x_n$, and let Q_2 be a Hamiltonian directed path in $T[\{u_1, \dots, u_{s-2}, u_s\}]$. If $(x_{i-1}, x_i) \in E(P)$ or $|T_1| = 1$ then $v_1 \dots v_{i-1} v u_s v_i Q u_{s-1} \dots u_1 Q_1 \equiv P$. Otherwise, $(x_i, x_{i-1}) \in E(P)$, then $(T_1 - v_i) + v \supseteq v'_1 \dots v'_i \equiv x_1 \dots x_i$ such that $v'_i = v_{i-1}$. We will insert Q, v, v_i, u_s into $v_1 \dots v_{i-1} u_{s-1} \dots u_1 Q_1$ to obtain a copy of P according to the arcs between v_{i-1}, v_i, u_{s-1} and u_s . $(v_{i-1}, u_s) \in E(T) \Rightarrow v'_1 \dots v'_i u_s v_i Q u_{s-1} \dots u_1 Q_1 \equiv P$. $(u_s, v_{i-1}), (u_{s-1}, v_i) \in E(T) \Rightarrow v_1 \dots v_{i-1} u_s Q v v_i u_{s-1} \dots u_1 Q_1 \equiv P$. $(u_s, v_{i-1}), (v_i, u_{s-1}), (u_{s-1}, v_{i-1}) \in E(T) \Rightarrow v_1 \dots v_{i-1} u_{s-1} u_s v_i Q u_{s-2} \dots u_1 v Q_1 \equiv P$. The only remainder case is $(u_s, v_{i-1}), (v_i, u_{s-1}), (v_{i-1}, u_{s-1}) \in E(T)$. In this case, $v'_1 \dots v'_i u_{s-1} v_i Q Q_2 Q_1 \equiv P$. Now, we will treat the case $s = 1$ and $b_2 \geq 2$. If $(x_i, x_{i-1}) \in E(P)$, then $|T_2| \geq 3$. If $\exists a \in (T_2 - u_1) \cap N^-(v_{i-1})$ and $b_2 = \Delta^+(T)$, let Q be a Hamiltonian directed path in $T_2 - \{u_1, a\}$, then $v_1 \dots v_{i-1} a v_i Q u_1 v \equiv P$. If $b_2 < \Delta^+(T)$, let Q_1 be a directed path of length $b_2 - 2$ in $T_2 - \{a, u_1\}$ then $T_2 - (V(Q_1) \cup \{a\}) \supseteq Q_2$ such that $v_1 \dots v_{i-1} a v_i Q_1 v Q_2 \equiv P$. If $(T_2 - u_1) \cap N^-(v_{i-1}) = \emptyset$, then $(T_1 - v_{i-1}) \subseteq N^-(v_{i-1}) \cap N^-(u_1)$ and $(u_1, v_{i-1}) \in E(T)$. Since $b_2 \geq 2$, then $x_{i+2} \dots x_n$ is neither of type $P^+(1, 2)$ nor directed starting from x_{i+2} . By simple lemma, $(T_2 - u_1) + v \supseteq v_{i+2} \dots v_n \equiv x_{i+2} \dots x_n$ such that $v_{i+2} \neq v$. For $|T_1| > 2$, if $\exists j < i - 1$ such that $(v_j, v_i) \in E(T)$, then since $(T_1 - \{v_{i-1}, v_i, v_j\}) + u_1 \supseteq Q \equiv x_1 \dots x_{i-2}$, so $Q v_{i-1} v_j v_i v_{i+2} \dots v_n \equiv P$. If $(v_i, v_j) \in E(T)$ whenever $j < i$, then $T_1 - \{v_1, v_{i-1}\} \supseteq Q_1 \equiv x_1 \dots x_{i-2}$ and $Q_1 v_{i-1} v_1 \equiv x_1 \dots x_i$. Then the problem is solved since $(v_i, v_1) \in E(T)$. For $|T_1| = 2$, if $\exists a \in T_2 - u_1$ such that $(T_2 - a) \supseteq v_{i+3} \dots v_n \equiv x_{i+3} \dots x_n$ with $v_{i+3} \neq u_1$, then $a v_1 v v_2 v_{i+3} v_n \equiv P$. Otherwise, by simple lemma, $x_5 \dots x_n$ is a directed path. Let $a, b \in T_2 - u_1$ such that $(a, b) \in E(T)$. $|T_2| > 3$, since otherwise, $N_T^+(b) = \{v_2\}$ then $d_T^+(b) = 1$ and so $\Delta^+(\overline{T}) > \Delta^+(T)$, a contradiction. Let Q be a Hamiltonian directed path in $T_2 - \{a, b, u_1\}$, then $v v_1 b a u_1 Q v_2 \equiv P$. For $(x_{i-1}, x_i) \in E(P)$ or $i = 1$, if $T_2 - u_1 \supseteq Q \equiv x_{i+3} \dots x_n$, then $v_1 \dots v_{i-1} v u_1 v_i Q \equiv P$. Otherwise, $T_2 - u_1 \in \mathcal{T}_{3,5,7}$ and $x_{i+3} \dots x_n$ is antidircted. If $b_2 = 3$, let $a \in T_2 - u_1$, then $T_2 - \{a, u_1\} \supseteq Q_1$ such that $v_1 \dots v_{i-1} v a u_1 v_i Q_1 \equiv P$. Otherwise, $b_2 = 2$. If $|T_1| < |T_2|$, then, by considering \tilde{P} or \overline{P} , we recover one of the previous cases. If $|T_1| = |T_2|$, then the problem is solved unless T_1 is isomorphic to \overline{T}_2

and $x_1 \dots x_i \equiv x_n \dots x_{n-i+1}$. Then $\exists a \in T_1$ such that $d_{T_1}^+(a) = 0$, $T_1 - a \in \mathcal{T}_{3,5,7}$ and $N^-(a) \cap T_2 = \{u_1\}$. Let bcd be a directed path in $T_1 - a$ and let $b' \in T_2 - u_1$. $(T_1 - \{a, b, c, d\}) + v \supseteq Q'$ and $T_2 - \{b', u_1\} \supseteq Q''$ such that $Q' u_1 b c d b' a Q'' \equiv P$. From now on, we may suppose that any copy $y_1 \dots y_i$ of $x_1 \dots x_i$ in T_1 satisfies the condition that $N^+(y_i) \cap T_2 = \emptyset$. Thus, by simple lemma, we can deduce that $x_i \dots x_1$ is directed or is of type $P^+(1, 2)$ with $T_1 - y_i$ is a cyclic triangle. For $i = 1$, if $x_{i+2} \dots x_n$ is not directed, then choose $a \in T_2$ such that $(T_2 - a) + v \supseteq Q$ with $a v_1 Q \equiv P$. Otherwise, by considering \overline{P} , the problem is solved since $b_1 \geq 2$. For $i \geq 2$, let $a \in T_2 \cap N^-(v_{i-1})$, then $(T_2 - a) + v \supseteq Q$ such that $v_1 \dots v_{i-1} a v_i Q \equiv P$ unless $x_{i+2} \dots x_n$ is directed or $\in P^+(1, 2)$ with $T_2 - a$ is a cyclic triangle. If $\exists b \in (T_2 - a) \cap N^+(a)$, then, if $T_2 - a = bcd$ and $x_{i+2} \dots x_n = P^+(1, 2)$, $v_1 \dots v_{i-1} a b d v_i c v \equiv P$ or $x_{i+2} \dots x_n$ is directed and $v_1 \dots v_{i-1} a b v Q v_i \equiv P$, where Q is a Hamiltonian directed path in $T_2 - \{a, b\}$. If $(T_2 - a) \cap N^+(a) = \emptyset$, since $d_{T_2}^-(v_{i-1}) = 1$, then $d_{T_1}^+(v_{i-1}) = 0$ and $i = 2$. Thus, by considering $\tilde{P} = y_1 \dots y_n$ or $\overline{\tilde{P}} = y_1 \dots y_n$, the problem is solved since \tilde{P} or $\overline{\tilde{P}}$ contains the arcs (y_1, y_2) and (y_2, y_3) . □

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