Hessian of Hausdorff dimension on purely imaginary directions

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Abstract. We extend classical results of Bridgeman-Taylor and McMullen on the Hessian of the Hausdorff dimension on quasi-Fuchsian space to the class of $(1,1,2)$-hyperconvex representations, a class introduced in [41] which includes small complex deformations of Hitchin representations and of Θ-positive representations. We also prove that the Hessian of the Hausdorff dimension of the limit set at the inclusion $\Gamma \to \mathsf{PO}(n,1) \to \mathsf{PU}(n,1)$ is positive definite when Γ is co-compact in $\mathsf{PO}(n,1)$ (unless $n=2$ and the deformation is tangent to $\mathfrak{X}(\Gamma, \mathsf{PO}(2,1))$).

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1. Introduction

One of the most interesting and well studied metrics on the Teichmüller space, the parameter space of hyperbolic structures on a closed surface S of genus $g \geqslant 2$, is the Weil-Petersson metric, a non-complete Riemannian metric. A celebrated result by B.-Taylor [14] and McMullen [36] gives a geometric interpretation of this metric in terms of dynamical invariants of quasi-Fuchsian representations.

Recall that the holonomy representation realizes the Teichmüller space $\mathcal{T}(S)$ as a connected component of the character variety

$$\mathfrak{X}\big(\pi_1 S, \mathsf{PSL}_2(\mathbb{R})\big) := \mathrm{Hom}\left(\pi_1 S, \mathsf{PSL}_2(\mathbb{R})\right) /\!/ \mathsf{PSL}_2(\mathbb{R}),$$

which, in turn, sits as a totally real submanifold of the complex character variety $\mathfrak{X}\big(\pi_1 S, \mathsf{PSL}_2(\mathbb{C})\big)$, endowed with the complex structure J induced by the complex structure of the Lie group $\mathsf{PSL}_2(\mathbb{C})$. A neighborhood of $\mathcal{T}(S)$ in the complex character variety is given by quasi-Fuchsian space $\mathcal{QF}(S)$, the set of conjugacy classes

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of representations $\rho : \pi_1 S \to \mathsf{PSL}_2(\mathbb{C}) = \mathrm{Isom}_0(\mathbb{H}^3)$ preserving a convex subset of $\mathbb{H}^3$ on which they act cocompactly. Any such ρ is thus a quasi-isometric embedding and admits an injective equivariant boundary map $\xi_\rho : \partial\pi_1 S \to \mathbb{CP}^1$ whose image is a Jordan curve. Given $\rho \in \mathfrak{QF}(S)$, we denote by $\mathrm{Hff}(\rho)$ the Hausdorff dimension of this Jordan curve. It is bounded below by 1 and Bowen showed that $\mathrm{Hff}(\rho)$ equals 1 precisely when ρ belongs to the Teichmüller space [8]. The result of B.-Taylor and McMullen realizes the Weil-Petersson metric by looking at the infinitesimal change of the Hausdorff dimension in purely imaginary directions at a representation $\rho \in \mathfrak{T}(S) \subset \mathfrak{QF}(S)$.

Theorem 1.1 (B.-Taylor [14] -McMullen [36])**.** *For each $\rho \in \mathfrak{T}(S)$ and every differentiable curve $(\rho_t)_{t\in(-\varepsilon,\varepsilon)} \subset \mathfrak{T}(S)$ with $\rho_0 = \rho$ it holds*

$$\mathrm{Hess}\,\mathrm{Hff}(J\dot\rho) = \|\dot\rho\|_{WP}.$$

In recent years, convex-cocompactness has been generalized from rank 1 to real-algebraic semisimple Lie groups[1] G of arbitrary rank, via the concept of *Anosov representations* $\rho : \Gamma \to \mathsf{G}_{\mathbb{K}}$, where, for $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, $\mathsf{G}_{\mathbb{K}}$ denotes the group of the $\mathbb{K}$-points of G. Specifying a set Θ of simple roots, let $\mathsf{G}_{\mathbb{K}}/\mathsf{P}_\Theta$ be the space of parabolic subgroups of type Θ. Then Θ-Anosov representations are characterized by admitting a continuous, equivariant, transverse boundary map $\xi_\rho^\Theta : \partial\Gamma \to \mathsf{G}_{\mathbb{K}}/\mathsf{P}_\Theta$ with good dynamical properties [32, 23, 27, 26, 4, 29]. They form open subsets

$$\mathfrak{X}_\Theta(\Gamma, \mathsf{G}_{\mathbb{K}}) = \{\rho \in \mathfrak{X}(\Gamma, \mathsf{G}_{\mathbb{K}}) : \rho \text{ is } \Theta\text{-Anosov}\}$$

of the character variety.

For each $\mathsf{a} \in \Theta$ B.-Canary-Labourie-S. [11] constructed, using the thermodynamic formalism, an analogue of the Weil-Petersson metric on $\mathfrak{X}_\Theta(\Gamma, \mathsf{G}_{\mathbb{K}})$, the *spectral radius pressure form* $\mathbf{P}^{\omega_{\mathsf{a}}}$, where ω_{a} is the fundamental weight associated to a. We will recall this construction on Section 4.

Probably the best studied space of Anosov representations is the $\mathsf{PSL}_d(\mathbb{R})$-Hitchin component. Hitchin introduced a special connected component

$$\mathscr{H}(S, \mathsf{G}_{\mathbb{R}}) \subset \mathfrak{X}(\pi_1 S, \mathsf{G}_{\mathbb{R}}),$$

when $\mathsf{G}_{\mathbb{R}}$ is moreover center-free and simple split, which in the case of $\mathsf{PSL}_d(\mathbb{R})$ can be described as the connected component $\mathscr{H}_d(S) \subset \mathfrak{X}(\pi_1 S, \mathsf{PSL}_d(\mathbb{R}))$ containing a *Fuchsian representation*, i.e. the composition of the holonomy of a hyperbolic structure with the irreducible representation $\pi_1 S \to \mathsf{PSL}_2(\mathbb{R}) \to \mathsf{PSL}_d(\mathbb{R})$. On the $\mathsf{PSL}_d(\mathbb{R})$-Hitchin component B.-Canary-Labourie-S. [12] defined a different pressure form, denoted by $\mathbf{P}^{\mathsf{a}_1}$, to which we will refer here as the *spectral gap pressure form*. They prove that $\mathbf{P}^{\mathsf{a}_1}$ is non-degenerate on $\mathscr{H}_d(S)$ and extends the Weil-Petersson inner product on Teichmüller space, embedded into $\mathscr{H}_d(S)$ as the space of Fuchsian representations.

A corollary of the main result of the paper is a generalization of Theorem 1.1. To state the result, we denote by Π the set of simple (restricted) roots of $\mathsf{G}_{\mathbb{R}}$ and consider the Hitchin component $\mathscr{H}(S, \mathsf{G}_{\mathbb{R}})$ as a subset of $\mathfrak{X}_\Pi(\pi_1 S, \mathsf{G}_{\mathbb{C}})$, the latter equipped the complex structure J induced by the complex structure of $\mathsf{G}_{\mathbb{C}}$. For $\mathsf{a} \in \Pi$ denote by

$$\mathrm{Hff}_{\mathsf{a}}(\rho) = \dim_{\mathrm{Hff}}\left(\xi_\rho^{\mathsf{a}}(\partial\pi_1 S)\right)$$

[1](of non-compact type)

the Hausdorff dimension of the (image of the) limit curve $\xi^{\mathsf{a}}_\rho : \partial\Gamma \to \mathcal{F}_{\mathsf{a}}(\mathsf{G}_{\mathbb{C}})$ for a(ny) Riemannian metric on $\mathcal{F}_{\mathsf{a}}(\mathsf{G}_{\mathbb{C}})$. It follows from Theorem 2.7 that $\mathrm{Hff}_{\mathsf{a}}$ is critical at $\mathscr{H}(S, \mathsf{G}_{\mathbb{R}})$ and thus its Hessian is well defined.

Corollary A. *For every $v \in \mathsf{T}\mathscr{H}(S, \mathsf{G}_{\mathbb{R}})$ and every $\mathsf{a} \in \Pi$ one has*

$$\mathrm{Hess}\,\mathrm{Hff}_{\mathsf{a}}(Jv) = \mathbf{P}^{\mathsf{a}}(v).$$

Moreover, when $\mathsf{G}_{\mathbb{R}} = \mathsf{PSL}_d(\mathbb{R})$ the Hessian of $\mathrm{Hff}_{\mathsf{a}_1} : \mathfrak{X}_\Pi\big(\pi_1 S, \mathsf{PSL}_d(\mathbb{C})\big) \to \mathbb{R}$, at a Hitchin point ρ, is strictly positive on every direction except $\mathsf{T}_\rho\mathscr{H}_d(S)$, where it is degenerate. In particular the Hitchin locus is an isolated minimum for $\mathrm{Hff}_{\mathsf{a}_1}$.

The second statement follows directly from the first, together with the aforementioned non-degeneracy result by B.-Canary-Labourie-S. [12] for the spectral gap pressure form $\mathbf{P}^{\mathsf{a}_1}$.

Corollary A brings further evidence that the spectral gap pressure form is more geometric than the spectral radius pressure form, and shares more similarities to the classical Weil-Petersson metric. A key ingredient in its proof is the notion of $(1,1,2)$-hyperconvex representations, studied in P.-S.-W. [41] (see Theorem 2.7). These are representations $\rho : \Gamma \to \mathsf{PSL}_d(\mathbb{C})$ that are Anosov with respect to the first two simple roots and whose boundary maps satisfy an additional transversality condition (see Section 2.2). The main result of [41] then yields that, on the open set

$$\mathfrak{X}^{\pitchfork}_{\{\mathsf{a}_1,\mathsf{a}_2\}}\big(\Gamma, \mathsf{PGL}_d(\mathbb{C})\big) = \big\{\rho \in \mathfrak{X}\big(\Gamma, \mathsf{PGL}_d(\mathbb{C})\big) : \ (1,1,2)\text{-hyperconvex}\big\}$$

the Hausdorff dimension of the limit set $\xi^1_\rho(\partial\Gamma)$ equals the *critical exponent* $h^{\mathsf{a}_1}_\rho$ for the first root (see Section 2.2 for the definition of the critical exponent) and is thus analytic.

Theorem A. *Let Γ be a word hyperbolic group with $\partial\Gamma$ homeomorphic to a circle and let $\rho \in \mathfrak{X}^{\pitchfork}_{\{\mathsf{a}_1,\mathsf{a}_2\}}\big(\Gamma, \mathsf{PSL}_d(\mathbb{R})\big)$ be a regular point of the character variety $\mathfrak{X}\big(\Gamma, \mathsf{PGL}_d(\mathbb{C})\big)$. Then for every differentiable curve $(\rho_t)_{t\in(-\varepsilon,\varepsilon)} \subset \mathfrak{X}\big(\Gamma, \mathsf{PSL}_d(\mathbb{R})\big)$ with $\rho_0 = \rho$ one has*

$$\mathrm{Hess}\,\mathrm{Hff}_{\mathsf{a}_1}(J\dot\rho) = \mathbf{P}^{\mathsf{a}_1}(\dot\rho).$$

Thanks to the work of [41, 40], Theorem A applies not only to Hitchin components but to Θ-positive Anosov representations into indefinite orthogonal groups [24]. However, in all these cases, we do not know for which roots a the associated pressure form is non-degenerate.

Corollary B. *Let $\mathcal{P}\big(\Gamma, \mathsf{SO}_0(p,q)\big)$ denote the set of Θ-positive Anosov representations. Then for every $v \in \mathsf{T}\mathcal{P}\big(\Gamma, \mathsf{SO}_0(p,q)\big)$ and every $\mathsf{a} \in \{\mathsf{a}_1, \ldots, \mathsf{a}_{p-2}\}$ one has*

$$\mathrm{Hess}\,\mathrm{Hff}_{\mathsf{a}}(Jv) = \mathbf{P}^{\mathsf{a}}(v).$$

The second main result of the paper is a generalization of B. [10] to Anosov representations of word hyperbolic groups that are not necessarily virtual surface groups.

Theorem B. *Let Γ be a word hyperbolic group and let $\rho \in \mathfrak{X}^{\pitchfork}_{\{\mathsf{a}_1,\mathsf{a}_2\}}\big(\Gamma, \mathsf{PSL}_d(\mathbb{C})\big)$ be a smooth point. Assume moreover that*

$$\mathrm{Hff}_{\mathsf{a}_1} : \mathfrak{X}^{\pitchfork}_{\{\mathsf{a}_1,\mathsf{a}_2\}}\big(\Gamma, \mathsf{PSL}_d(\mathbb{C})\big) \to \mathbb{R}_+$$

is critical at ρ. *Then* $\operatorname{Hess}_\rho \operatorname{Hff}_{\mathsf{a}_1}$ *is positive semidefinite on a subspace of dimension at least half the real dimension of* $\mathfrak{X}^{\pitchfork}_{\{\mathsf{a}_1,\mathsf{a}_2\}}\big(\Gamma, \mathsf{PSL}_d(\mathbb{C})\big)$. *If, furthermore, the pressure form* $\mathbf{P}^{\mathsf{a}_1}$ *is non-degenerate, then there are no local maxima.*

This theorem applies in particular to all convex cocompact Kleinian groups in $\mathsf{PSL}_2(\mathbb{C})$, a result not covered by [10]. In general, admitting $(1,1,2)$-hyperconvex representations is a relatively restrictive assumption on the group Γ, as, for example, it implies that its boundary has dimension smaller than 2. However there are many classes of subgroups of $\mathsf{PSL}_2(\mathbb{C})$ admitting $(1,1,2)$-hyperconvex representations with Zariski dense image in $\mathsf{PSL}_d(\mathbb{C})$. It is not yet known if the spectral gap pressure form is non-degenerate for such representations, and we hope that Theorem B will encourage research in this direction.

The proof of both Theorems A and B relies on pluriharmonicity of length functions. The mechanism behind both proofs is relatively versatile, and can be applied in many situations. As an example we use it in Corollary 5.4 to prove that, at a Fuchsian representation, the critical exponent $h^{\omega_1}(\rho)$ relative to the first fundamental weight necessarily increases along purely imaginary deformations (see Section 5.5 for details).

Our last result is another application of the previous techniques in a rank one situation. Recall that a representation $\rho : \Gamma \to \mathsf{PU}(n,1)$ is convex-cocompact if and only if it is projective Anosov when $\mathsf{PU}(n,1)$ is considered as a subgroup of $\mathsf{PGL}_{n+1}(\mathbb{C})$ through the standard inclusion. Moreover, for $\gamma \in \Gamma$ the real length of the associated closed geodesic is the spectral radius $\omega_1\Big(\lambda\big(\rho(\gamma)\big)\Big)$.

If $\rho : \Gamma \to \mathsf{PU}(n,1)$ is convex co-compact then we denote by $\operatorname{Hff}_{\partial\mathbb{CH}^n}(\rho)$ the Hausdorff dimension the limit set of Γ in the visual boundary of the complex-hyperbolic space, with respect to a visual metric. A celebrated result by Sullivan [45] in real-hyperbolic space, and further extended to arbitrary negatively curved manifolds by Yue [48], asserts that if $\rho : \Gamma \to \mathsf{PU}(n,1)$ is convex-co-compact then

$$\operatorname{Hff}_{\partial\mathbb{CH}^n}(\Gamma) = h^{\omega_1}(\rho).$$

Assume now that Γ is a co-compact lattice of $\mathsf{PSO}(n,1)$ and denote by ι the inclusion

$$\iota : \Gamma \to \mathsf{PSU}(n,1),$$

where the inclusion of $\mathsf{PSO}(n,1) \to \mathsf{PSU}(n,1)$ is given by extending the coefficients. Bourdon [6] proved that ι is a global rigid minima for $\operatorname{Hff}_{\partial\mathbb{CH}^n}$ among convex cocompact representations of Γ in $\mathsf{PU}(n,1)$. In section 5.4 we prove the following strengthening.

Theorem C. *Assume* ι *is a regular point of the character variety* $\mathfrak{X}\big(\Gamma, \mathsf{PSU}(n,1)\big)$, *then* $\operatorname{Hess}_\iota \operatorname{Hff}_{\partial\mathbb{CH}^n}$ *is positive definite in any direction not tangent to* $\mathfrak{X}\big(\Gamma, \mathsf{PSO}(n,1)\big)$.

If $n > 2$ then Mostow's classical rigidity result states that ι is an isolated point of $\mathfrak{X}\big(\Gamma, \mathsf{PSO}(n,1)\big)$ so Theorem C implies that the Hessian of the Hausdorff dimension at ι is positive definite.

Remark 1.2. If ρ is a small deformation of ι in $\mathsf{PSU}(n,1)$ then it is a convex cocompact subgroup of $\mathsf{PSU}(n,1)$ and moreover, by P.-S.-W. [41, Corollary 8.5], the Hausdorff dimension of the limit set for a *Riemannian metric* on $\partial\mathbb{CH}^n$ also coincides with $h^{\omega_1}(\rho)$, so in Theorem C we can consider $\operatorname{Hff}_{\partial\mathbb{CH}^n}$ either as the Hausdorff dimension for a visual metric or for a Riemannian metric.

Outline of the paper. In Section 2 we discuss the background on Anosov representations needed in the paper: after reviewing the basic definitions we discuss, in Section 2.2, the results of [41] which singled out $(1,1,2)$-hyperconvex representations. In Section 2.3 we recall the basic facts about higher rank Teichmüller theories needed to deduce Corollaries A and B from Theorem A; finally in Section 2.4 we discuss an important dynamical viewpoint on Anosov representations: these can be thought of as reparametrizations of the geodesic flow of Γ, and are thus amenable to the thermodynamic formalism. In Section 3 we discuss the thermodynamic formalism needed to define, in Section 4, the pressure forms. We conclude the paper in Section 5 introducing the main technical tool of the paper, pluriharmonicity of dynamical intersection, which is directly used to prove Theorems A, B and C.

2. Anosov representations

In this section we introduce the necessary background on Anosov representations, and recall how they give rise to reparametrizations of the geodesic flow.

2.1. Basic notions.

We recall the Cartan and the Jordan-Lyapunov projections and the characterization of Anosov representations we are going to use.

Let G be a semisimple real-algebraic Lie group of non-compact type with finite center, for $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$ denote by $\mathsf{G}_{\mathbb{K}}$ the group of the $\mathbb{K}$-points of G. Fix a maximal compact subgroup $\mathsf{K}<\mathsf{G}_{\mathbb{K}}$ with Lie algebra $\mathfrak{k}$. We denote by $\mathsf{E}<\mathfrak{k}^{\perp}$ a Cartan subalgebra, and by $\Delta\subset\mathsf{E}^*$ a choice of simple roots. This corresponds to the choice of a Weyl chamber in E, which we will denote by E^+. In the case $\mathsf{G}=\mathsf{PSL}_d$ we identify E^+ with

$$\Big\{(x_1,\dots,x_d)\in\mathbb{R}^d|x_1\geqslant\ldots\geqslant x_d,\sum x_i=0\Big\}.$$

Every element $g\in\mathsf{G}_{\mathbb{K}}$ can be written as a product

$$g=k_1\exp\big(\sigma(g)\big)k_2$$

for $k_1,k_2\in\mathsf{K}$ and a unique element $\sigma(g)\in\mathsf{E}^+$, the *Cartan projection* of g. If $\mathsf{G}_{\mathbb{R}}=\mathsf{PSL}_d(\mathbb{R})$, the numbers $\sigma_i(g)$ are the logarithms of the square roots of the eigenvalues of the symmetric matrix g^tg. If $\mathsf{a}\in\Delta$ then we denote by ω_{a} its associated fundamental weight.

Let $\Theta\subset\Delta$ be a subset of simple roots. We denote by $\mathsf{P}_\Theta<\mathsf{G}$ the associated parabolic subgroup, by $\check{\mathsf{P}}_\Theta$ the opposite associated parabolic group and by

$$\mathsf{E}_\Theta:=\bigcap_{\mathsf{a}\in\Delta\setminus\Theta}\ker(\mathsf{a})$$

the Lie algebra of the center of the Levi group $\mathsf{P}_\Theta\cap\check{\mathsf{P}}_\Theta$. It comes equipped with the natural projection

$$p_\Theta:\mathsf{E}\to\mathsf{E}_\Theta \qquad (1)$$

parallel to $\bigcap_{\mathsf{a}\in\Theta}\ker\mathsf{a}$. Finally let $\mathsf{E}^*_\Theta<\mathsf{E}^*$ be the subspace generated by the fundamental weights associated to elements in Θ

$$\mathsf{E}^*_\Theta:=\langle\omega_{\mathsf{a}}|\ \mathsf{a}\in\Theta\rangle=\{\varphi\in\mathsf{E}^*|\ \varphi\circ p_\Theta=\varphi\}.$$

One has that $\mathsf{E}_\Delta=\mathsf{E}$ and P_Δ is a minimal parabolic subgroup.

Let Γ be a finitely generated discrete group and denote by $|\,|$ the word length for a fixed finite symmetric generating set.

Definition 2.1. Let $\Theta \subset \Delta$. A representation $\rho : \Gamma \to \mathsf{G}_{\mathbb{K}}$ is Θ-*Anosov* if there exist positive constants c, μ such that for all $\gamma \in \Gamma$ and $\mathsf{a} \in \Theta$ one has

$$\mathsf{a}\Big(\sigma\big(\rho(\gamma)\big)\Big) \geqslant \mu|\gamma| - c. \tag{2}$$

A $\{\mathsf{a}_1\}$-Anosov representation $\rho : \Gamma \to \mathsf{PGL}_d(\mathbb{K})$ will be called *projective Anosov.*

Note that this is not the original definition given in Labourie and Guichard-W. [32, 23], but a characterization due to Kapovich-Leeb-Porti and Bochi-Potrie-S. [27, 4]. Note also that there is a recent characterization by Kassel-Potrie [29] only in terms of the Jordan-Lyapunov projection (see below for the definition) rather than the Cartan projection.

Anosov representations are quasi-isometric embeddings, thus in particular they are injective and have discrete image. It was proven in [27] (see also [4]) that only word hyperbolic groups admit Anosov representations; we will denote by $\partial\Gamma$ the Gromov boundary of the group Γ.

A key property of Anosov representations is the existence of equivariant boundary maps with good dynamical properties [32, 23, 22, 27, 4]. With our definition, the existence of boundary maps for such representations is a Theorem of [27] and [4]. From now on we will restrict ourselves, without loss of generality, to self-opposite subsets $\Theta \subset \Delta$

Theorem 2.2 (Kapovich-Leeb-Porti [27])**.** *Let $\rho : \Gamma \to \mathsf{G}_{\mathbb{K}}$ be Θ-Anosov. Then there exist a unique dynamics preserving, continuous, transverse equivariant boundary map*

$$\xi_\rho^\Theta : \partial\Gamma \to \mathsf{G}_{\mathbb{K}}/\mathsf{P}_\Theta.$$

If $\mathsf{G} = \mathsf{PGL}_d$ and $\Theta = \{\mathsf{a}_r\}$, then $\mathsf{G}_{\mathbb{K}}/\mathsf{P}_\Theta = \mathcal{G}_r(\mathbb{K}^d)$ and we write $\xi_\rho^r = \xi_\rho^{\{\mathsf{a}_r\}}$.

It was proven in [23] that it is possible to reduce the study of general $\{\mathsf{a}\}$-Anosov representations to projective Anosov representations. Indeed one can use the following result by Tits, since for the representations Λ_{a} below one has

$$\mathsf{a}_1\Big(\sigma\big(\Lambda_{\mathsf{a}}(\rho(\gamma))\big)\Big) = \mathsf{a}\Big(\sigma\big(\rho(\gamma)\big)\Big).$$

Proposition 2.3 (Tits [46])**.** *For every $\mathsf{a} \in \Delta$ there exists an irreducible proximal representation $\Lambda_{\mathsf{a}} : \mathsf{G}_{\mathbb{R}} \to \mathsf{PGL}_d(\mathbb{R})$ whose highest restricted weight is $l\omega_{\mathsf{a}}$ for some $l \in \mathbb{N}$.*

Recall that the Jordan decomposition states that every $g \in \mathsf{G}_{\mathbb{K}}$ can be written uniquely as a commuting product $g = g_e g_h g_n$, where g_e is elliptic, g_h is $\mathbb{R}$-split and g_n is unipotent. The *Jordan-Lyapunov* projection $\lambda : \mathsf{G}_{\mathbb{K}} \to \mathsf{E}^+$ is defined by the logarithm of the eigenvalues of g_h with multiplicities and in decreasing order. If $\mathsf{G} = \mathsf{PGL}_d$, this corresponds to the logarithm of the modulus of the roots of the characteristic polynomial of g with multiplicities and in decreasing order, and we denote by

$$\lambda(g) = \big(\lambda_1(g), \ldots, \lambda_d(g)\big) \in \big\{(x_1, \ldots, x_d) \in \mathbb{R}^d |\ x_1 \geqslant \ldots \geqslant x_d, \sum x_i = 0\big\}$$

its coordinates.

We will denote by $\Lambda_\rho \subset \mathsf{E}^+$ the *limit cone* of the subgroup $\rho(\Gamma) < \mathsf{G}_{\mathbb{K}}$. This is the cone given by

$$\Lambda_\rho := \overline{\{\mathbb{R}_+ \cdot \lambda(\rho(\gamma)) |\ \gamma \in \Gamma\}}.$$

It was proven by Benoist [3] that, provided $\rho(\Gamma)$ is Zariski dense, Λ_ρ is convex and has non-empty interior.

For every functional $\varphi \in \mathsf{E}^*$ that is positive on the limit cone Λ_ρ we denote by $h^\varphi(\rho)$ the *critical exponent* of the Dirichlet series

$$s \mapsto \sum_{\gamma \in \Gamma} e^{-s\varphi\left(\sigma\left(\rho(\gamma)\right)\right)},$$

it can be computed as

$$h^\varphi(\rho) = \inf\left\{s : \sum_{\gamma\in\Gamma} e^{-s\varphi\left(\sigma\left(\rho(\gamma)\right)\right)} < \infty\right\} = \sup\left\{s : \sum_{\gamma\in\Gamma} e^{-s\varphi\left(\sigma\left(\rho(\gamma)\right)\right)} = \infty\right\}.$$

2.2. **Hyperconvex representations.** We begin with the following definition from [41]:

Definition. A $\{\mathsf{a}_1, \mathsf{a}_2\}$-Anosov representation $\rho : \Gamma \to \mathsf{PGL}_d(\mathbb{K})$ is called $(1,1,2)$-*hyperconvex* if for every triple of pairwise distinct points $x, y, z \in \partial\Gamma$ one has

$$\left(\xi^1_\rho(x) \oplus \xi^1_\rho(y)\right) \cap \xi^{d-2}_\rho(z) = \{0\}.$$

The following is a direct consequence of the uniqueness of boundary maps:

Lemma 2.4. *The complexification of a real hyperconvex representation is hyperconvex (over $\mathbb{C}$).*

An important property of (1,1,2)-hyperconvex representations, established in [41] is that, for these representations, the Hausdorff dimension of the limit curve for a Riemannian metric on $\mathbb{P}(\mathbb{K}^d)$ computes the critical exponent for the first simple root. If ρ is $\{\mathsf{a}_1\}$-Anosov, the root a_1 is positive on the limit cone (recall Equation (2)) and thus its critical exponent is well defined. We then have the following.

Theorem 2.5 (P.-S.-W. [41])**.** *Let $\rho : \Gamma \to \mathsf{PGL}_d(\mathbb{K})$ be $(1,1,2)$-hyperconvex, then*

$$\dim_{\mathrm{Hff}}\left(\xi^1(\partial\Gamma)\right) = h^{\mathsf{a}_1}_\rho.$$

A second important property of $(1,1,2)$-hyperconvex representations into $\mathsf{PSL}_d(\mathbb{R})$ was established in P.-S.-W. [41] (and independently in Zimmer-Zhang [49]) is the following: if Γ is such that $\partial\Gamma$ is homeomorphic to a circle, then the image of the boundary map ξ^1_ρ is a C^1-curve. As a result we get

Theorem 2.6 (P.-S.-W. [41])**.** *Let $\rho : \Gamma \to \mathsf{PSL}_d(\mathbb{R})$ be $(1,1,2)$-hyperconvex. If $\partial\Gamma$ is homeomorphic to a circle then* $\dim_{\mathrm{Hff}}\left(\xi^1(\partial\Gamma)\right) = 1$.

Thus, for fundamental groups of surfaces, the Hausdorff dimension is constant and minimal on the real (1,1,2)-hyperconvex locus.

2.3. **Higher rank Teichmüller Theory.** A higher Teichmüller space is a union of connected components of a character variety $\mathfrak{X}(\pi_1 S, \mathsf{G}_\mathbb{R})$ only consisting of Anosov representations.

Historically, the first family of higher Teichmüller spaces are Hitchin components. They arise whenever $\mathsf{G}_\mathbb{R}$ is a center free real-split simple Lie group. In this case there is a unique *principal* subalgebra $\mathfrak{sl}_2(\mathbb{R}) < \mathfrak{g}_\mathbb{R}$ characterized by the property that the centralizer of $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ has minimal dimension (Kostant [31]). The *Hitchin component* $\mathscr{H}(S, \mathsf{G}_\mathbb{R}) \subset \mathfrak{X}(\pi_1 S, \mathsf{G}_\mathbb{R})$ is the connected component containing *Fuchsian representations*: the composition of the holonomy of a hyperbolization

$\pi_1 S \to \mathsf{PSL}_2(\mathbb{R})$ and the morphism $\mathsf{PSL}_2(\mathbb{R}) \to \mathsf{G}_\mathbb{R}$ induced by the inclusion of the principal subalgebra. Representations in the Hitchin component are Anosov with respect to the minimal parabolic subgroup [32].[2] Furthermore, representations in the Hitchin component are hyperconvex:

Theorem 2.7 ([32, 41, 44])**.** *Let $\mathsf{G}_\mathbb{R}$ be a simple split center-free real group. For every $\rho \in \mathscr{H}(S, \mathsf{G}_\mathbb{R})$ and $\mathsf{a} \in \Pi$ the representation $\Lambda_\mathsf{a}\rho : \pi_1 S \to \mathsf{PSL}(V, \mathbb{R})$ is $(1,1,2)$-hyperconvex.*

Proof. This was established, for the groups $\mathsf{G}_\mathbb{R} = \mathsf{PSL}_d(\mathbb{R})$, $\mathsf{PSp}(2n, \mathbb{R})$, $\mathsf{PSO}(n, n+1)$ or the split form of the exceptional complex Lie group G_2, by Labourie [32], for $\mathsf{G} = \mathsf{SO}(n, n)$ by P.-S.-W. [41, Theorem 9.9]. The general case follows from S. [44, Remark 5.14]. □

The second family of higher Teichmüller spaces are spaces of maximal representations in Hermitian Lie groups $\mathsf{G}_\mathbb{R}$ [16]. Our results here do not apply in this setting. Maximal representations are, in general, only Anosov with respect to one root a, which therefore doesn't belong to the Levi-Anosov subspace. Even though we know that for maximal representations the critical exponent h^a_ρ is constant and equal to one ([40, Theorem 1.2]), it is not clear if a spectral gap pressure metric $\mathbf{P}^\mathsf{a}$ can be constructed in this case. Moreover, since maximal representations are, in general, not (1,1,2)-hyperconvex, it is not known if, for complex deformations in $\rho : \Gamma \to \mathsf{G}_\mathbb{C}$, the critical exponent h^a_ρ equals the Hausdorff dimension of the limit set.

Conjecturally there are two further families of higher Teichmüller spaces, given by Θ-positive representations as introduced in [24, 47]. Θ-positive representations exist when $\mathsf{G}_\mathbb{R}$ is locally isomorphic to $\mathsf{SO}(p, q)$, $p < q$, or when $\mathsf{G}_\mathbb{R}$ belongs to a special family of exceptional Lie groups. In a forthcoming article, Guichard, Labourie and W. [25] prove that Θ-positive representations are Θ-Anosov; in particular, in the case of $\mathsf{SO}(p, q)$, $p < q$, Θ-positive representations are Anosov with respect to the first $p - 1$ roots. Since this article is not yet available, we will here consider Θ-positive Anosov representations, and use the following result from [40].

Theorem 2.8 ([40, Theorem 10.1])**.** *Let $\rho : \Gamma \to \mathsf{SO}(p, q)$ be a Θ-positive Θ-Anosov representation. For every $\mathsf{a} \in \{\mathsf{a}_1, \ldots, \mathsf{a}_{p-2}\}$ the representation $\Lambda_\mathsf{a}\rho : \pi_1 S \to \mathsf{PSL}(V, \mathbb{R})$ is $(1,1,2)$-hyperconvex.*

Note that when $\mathsf{G}_\mathbb{R}$ admits a Θ-positive structure, Guichard and W. conjectured several years ago, see also [24, 47], that then there exist additional connected components (namely the conjectured components of Θ-positive representations) in the representation variety, which are not distinguished by characteristic numbers. This conjecture has been proven by Collier [17] in the case of $\mathsf{G}_\mathbb{R} = \mathsf{SO}(n, n+1)$ and by Aparicio-Arroyo, Bradlow, Collier, García-Prada, Gothen, and Oliveira [1] in the case of $\mathsf{G}_\mathbb{R} = \mathsf{SO}(p, q)$ using methods from the theory of Higgs bundles.

[2] In fact, Labourie proved this for $\mathsf{G}_\mathbb{R} = \mathsf{PSL}_d(\mathbb{R})$, which implies the result also for symplectic groups and odd-dimensional orthogonal groups. Fock and Goncharov gave a characterization of representations in the Hitchin component as positive representations, from which the Anosov property can be deduced with a little work.

2.4. **Reparametrizations of geodesic flows.** In this section we describe a very useful dynamical viewpoint on Anosov representations from S. [43] and B.-Canary-Labourie-S. [11], which makes them amenable to the thermodynamic formalism: Any Anosov representation gives rise to a reparametrization of the geodesic flow.

Given a hyperbolic group $\mathsf{\Gamma}$ we denote by $\mathsf{U}\mathsf{\Gamma}$ the Gromov geodesic flow; this is a metric space endowed with a topologically transitive flow ϕ whose periodic orbits correspond to conjugacy classes in $\mathsf{\Gamma}$. If $\mathsf{\Gamma}$ admits an Anosov representation then ϕ is moreover metric Anosov [11]. Note that, if $\mathsf{\Gamma}$ is the fundamental group of a compact negatively curved manifold M, we can choose $\mathsf{U}\mathsf{\Gamma} = \mathsf{U}M$; more generally, whenever $\mathsf{\Gamma}$ admits an Anosov representation, its geodesic flow can be explicitly constructed with the aid of the associated boundary maps [11, Theorem 1.10].

If $\alpha > 0$, we denote by $\mathrm{Hol}_\alpha(\mathsf{U}\mathsf{\Gamma}, \mathbb{R})$ the space of α-Hölder continuous functions on $\mathsf{U}\mathsf{\Gamma}$ and by $\mathrm{Hol}(\mathsf{U}\mathsf{\Gamma}, \mathbb{R})$ is the space of all Hölder continuous functions. If $f \in \mathrm{Hol}(\mathsf{U}\mathsf{\Gamma}, \mathbb{R})$ and $a \in [\mathsf{\Gamma}]$ is a conjugacy class, then we define the f-period of a by

$$\ell_f(a) = \int_0^{\ell(a)} f(\phi_t(x))dt$$

where $x \in a$. If $f \in \mathrm{Hol}_\alpha(\mathsf{U}\mathsf{\Gamma}, \mathbb{R}_+)$, we obtain a new flow ϕ^f on $\mathsf{U}\mathsf{\Gamma}$ called the *reparametrization* of ϕ by f. The flow ϕ^f is given by the formula

$$\phi_t(x) = \phi^f_{k_f(x,t)}(x) \tag{3}$$

where $k_f(x,t) = \int_0^t f(\phi_s x)ds$ for all $x \in X$ and $t \in \mathbb{R}$. The flow ϕ^f is Hölder orbit equivalent to ϕ and if $a \in [\mathsf{\Gamma}]$, then $\ell_f(a)$ is the period of a in the flow ϕ^f.

In [11, Section 4] B.-Canary-Labourie-S. associate to any projective Anosov representation a reparametrization of the geodesic flow $\mathsf{U}\mathsf{\Gamma}$. They prove the following statement, the second part is proved in [11, Section 6].

Proposition 2.9 ([11])**.** *Let $\rho : \mathsf{\Gamma} \to \mathsf{PGL}_d(\mathbb{R})$ be a projective Anosov representation. Then there exists a positive Hölder-continuous function $f^{\lambda_1}_\rho : \mathsf{U}\mathsf{\Gamma} \to \mathbb{R}_{>0}$ such that for every conjugacy class $[\gamma] \in [\mathsf{\Gamma}]$ one has*

$$\ell_\gamma(f^{\lambda_1}_\rho) = \lambda_1\big(\rho(\gamma)\big).$$

Moreover, if $\{\rho_u\}_{u\in D}$ is an analytic family of such representations, then one can choose $f^{\omega_1}_{\rho_u}$ so that the function $u \mapsto f^{\lambda_1}_{\rho_u}$ is analytic.

Proposition 2.9 together with Tits Proposition 2.3 directly give the following from Potrie-S. [39], where $\mathbb{K} = \mathbb{R}$ case is treated. When $\mathbb{K} = \mathbb{C}$ the result follows at once by considering $\mathsf{G}_\mathbb{C}$ as a real group. Recall equation (1) for the definition of $p_\Theta : \mathsf{E} \to \mathsf{E}_\Theta$.

Corollary 2.10 ([39, Cor. 4.5])**.** *Let $\rho : \mathsf{\Gamma} \to \mathsf{G}_\mathbb{K}$ be Θ-Anosov, then there exists a positive Hölder-continuous function $f^\Theta_\rho : \mathsf{U}\mathsf{\Gamma} \to \mathsf{E}_\Theta$ such that for every conjugacy class $[\gamma] \in [\mathsf{\Gamma}]$ one has*

$$\ell_\gamma(f^\Theta_\rho) = p_\Theta\Big(\lambda\big(\rho(\gamma)\big)\Big).$$

Moreover, if $\{\rho_u\}_{u\in D}$ is an analytic family of such representations, then one can choose $f^\Theta_{\rho_u}$ so that the function $u \mapsto f^\Theta_{\rho_u}$ is analytic.

Thus, Corollary 2.10 readily implies that if ρ is Θ-Anosov then for every $\varphi \in (\mathsf{E}_\Theta)^*$ that is strictly positive on $\Lambda_\rho - \{0\}$ there exists a reparametrization of the geodesic flow of Γ whose periods are given by[3]

$$\varphi\big(\lambda(\rho(\gamma))\big).$$

namely, if we denote by $f_\rho^\varphi = \varphi(f_\rho^\Theta)$ then one considers the flow $\phi^{f_\rho^\varphi}$. We will need in the following that, in this situation, the critical exponent $h^\varphi(\rho)$ is also the entropy of the flow $\phi^{f_\rho^\varphi}$. This can be found for example in Ledrappier [33], S. [43] and on Glorieux-Monclair-Tholozan [21] for the general version.

Proposition 2.11. *Let $\rho : \Gamma \to \mathsf{G}_\mathbb{K}$ be Θ-Anosov. For each $\varphi \in \mathsf{E}_\Theta^*$ strictly positive on $\Lambda_\rho - \{0\}$ it holds that*

$$h^\varphi(\rho) = \lim_{T\to\infty} \frac{\log \#\Big\{\gamma \in [\Gamma] |\ \varphi\Big(\lambda\big(\rho(\gamma)\big)\Big) < T\Big\}}{T}$$

This applies, in particular, to the root a_1 if a representation ρ is (1,1,2)-hyperconvex.

3. Thermodynamic formalism

We now briefly describe the thermodynamic formalism introduced by Bowen, Ruelle, Parry, Pollicott (among others), and in particular the pressure function on the space of Hölder observables on a metric space with a Hölder flow (see [42]). This will then be used, in Section 4, to define various pressure forms $\mathbf{P}^\varphi$ on subsets of the representation variety $\mathfrak{X}(\Gamma, \mathsf{PSL}_d(\mathbb{R}))$ by assigning to each representation ρ the Hölder function f_ρ^φ on the geodesic flow space $\mathsf{U}\Gamma$ of the group.

For a moment we forget about representations and let X be a compact metric space with a Hölder continuous flow $\phi = \{\phi_t : X \to X\}_{t\in\mathbb{R}}$ without fixed points. We denote by O the collection of periodic orbits of the flow ϕ. For $a \in O$, we let $\ell(a)$ be the length of the periodic orbit a.

As in Section 2.4 we denote by $\mathrm{Hol}_\alpha(X, \mathbb{R})$ the space of α-Hölder continuous functions on X for some $\alpha > 0$, and we set the f-period of $a \in O$ to be

$$\ell_f(a) = \int_0^{\ell(a)} f(\phi_t(x))dt.$$

Two maps $f, g \in \mathrm{Hol}_\alpha(X, \mathbb{R})$ are called *Livšic cohomologuous* if there exists $U : X \to \mathbb{R}$ such that, for all $x \in X$, then

$$f(x) - g(x) = \left.\frac{\partial}{\partial t}\right|_{t=0} U(\phi_t x).$$

It follows that if f and g are Livšic cohomologous then $\ell_f(a) = \ell_g(a)$ for all $a \in O$. If $f \in \mathrm{Hol}_\alpha(X, \mathbb{R}_+)$, we denote by ϕ^f the *reparametrization* of ϕ by f, which is the flow on X defined by (3).

We let $\mathcal{M}_\phi$ be the set of ϕ-invariant probability measures on X. In particular if δ_a is the Lebesgue measure on the periodic orbit a, then $\hat{\delta}_a = \delta_a/\ell(a) \in \mathcal{M}_\phi$. For $\mu \in \mathcal{M}_\phi$ we denote by $h(\phi, \mu)$ its metric entropy. Then, for $f \in \mathrm{Hol}_\alpha(X, \mathbb{R})$, the *topological pressure* is

$$P(f) = \sup_{m\in\mathcal{M}_\phi} \left\{ h(\phi, m) + \int_X f dm \right\}.$$

[3] Recall that for every $\varphi \in (\mathsf{E}_\Theta)^*$ one has $\varphi \circ p_\Theta = \varphi$.

Note that the topological pressure P depends on the flow ϕ, but we will omit this in the notation. The *topological entropy* of a flow is given by $h_{\text{top}}(\phi) = P_\phi(0)$. A measure m_f that attains this supremum is called an *equilibrium state* for f and an equilibrium state for the zero function is called a *measure of maximal entropy.*

We note that $P(f)$ only depends on the Livšic cohomology class of f.

Lemma 3.1 (S. [43, Lemma 2.4])**.** *Let ϕ be a Hölder continuous flow on a compact metric space X and $f \in \text{Hol}_\alpha(X, \mathbb{R}_+)$. Then*

$$P(-hf) = 0$$

if and only if $h = h_{\text{top}}(\phi^f)$. Moreover, if m is an equilibrium state of $-h_{\text{top}}(\phi)f$, then fm is a positive multiple of a measure of maximal entropy for the flow ϕ^f.

We now restrict to *transitive metric Anosov flows.* In the manifold setting a metric Anosov flow ϕ corresponds to a standard Anosov flow where the unit tangent bundle of X has a ϕ-invariant decomposition $T_1(X) = E_- \oplus E_0 \oplus E_+$ where E_- is contracting under the flow, E_0 is the direction of the flow and E_+ is contracting under the flow reverse flow of ϕ (see [38] for details). We have the following theorem of Livšic.

Theorem 3.2 (Livšic's Theorem, [35])**.** *Let ϕ be a transitive metric Anosov flow. If $f \in \text{Hol}_\alpha(X, \mathbb{R})$ then $\ell_f(a) = 0$ for all $a \in O$ if and only if f is Livšic cohomologous to* 0.

It follows that for metric Anosov flows, the Livšic cohomology class of f is determined by its periods.

Given $f \in \text{Hol}_\alpha(X, \mathbb{R})$ we let

$$R_T(f) = \{a \in O \mid \ell_f(a) \leqslant T\}.$$

Then we have the following;

Theorem 3.3 (Bowen [7], Bowen-Ruelle [9], Pollicott [38])**.** *Let ϕ be a transitive metric Anosov flow and $f \in \text{Hol}_\alpha(X, \mathbb{R}_+)$ nowhere vanishing. Then*

$$h(f) = \lim_{T\to\infty} \frac{\log \# R_T(f)}{T} = h_{\text{top}}(\phi^f)$$

is finite and positive. Moreover for all $g \in \text{Hol}_\alpha(X, \mathbb{R})$ there exists a unique equilibrium state m_g for g. The measure of maximal entropy μ_ϕ for the flow ϕ is

$$\mu_\phi = \lim_{T\to\infty} \frac{1}{\# R_T(1)} \sum_{a \in R_T(1)} \frac{\delta_a}{\ell(a)}.$$

Furthermore for Anosov flows the derivatives of the Pressure function satisfy the following.

Proposition 3.4 (Parry-Pollicott [37])**.** *Let ϕ be a transitive metric Anosov flow and $f, g \in \text{Hol}_\alpha(X, \mathbb{R})$. Then*

(i) *The function $t \to P(f + tg)$ is analytic*
(ii) *The first derivative satisfies*

$$\left.\frac{\partial P(f + tg)}{\partial t}\right|_{t=0} = \int g dm_f,$$

where m_f is the equilibrium state for f.

(iii) *If* $\int g dm_f = 0$ *(mean-zero) then*

$$\left.\frac{\partial^2 P(f+tg)}{\partial t^2}\right|_{t=0} = \lim_{T\to\infty}\int\left(\int_0^T g(\phi_s(x))ds\right)^2 dm_f(x) = \mathrm{Var}(g, m_f).$$

(iv) *If* $\mathrm{Var}(g, m_f) = 0$ *then* g *is Livšic cohomologous to zero.*

Using the above, in [36] McMullen defined the Pressure semi-norm as follows. We let $\mathcal{P}(X)$ be the space of pressure zero functions, i.e.

$$\mathcal{P}(X) = \{F \in \mathrm{Hol}(X, \mathbb{R}) \mid P(F) = 0\}.$$

Then by Proposition 3.4(ii), the tangent space to $\mathcal{P}(X)$ at F can be identified with

$$\mathsf{T}_F(\mathcal{P}(X)) = \left\{g \in \mathrm{Hol}(X, \mathbb{R}) \mid \int g dm_F = 0\right\},$$

where m_F is the equilibrium state for F. Then the *pressure semi-norm* of $g \in \mathsf{T}_F(\mathcal{P}(X))$ is

$$\mathbf{P}(g) = -\frac{\mathrm{Var}(g, m_F)}{\int F dm_F}.$$

By Proposition 3.4 it follows that $\mathbf{P}(g)$ only depends on the Livšic-cohomology class $[g]$ and is positive definite in the sense that it is zero if and only if $[g] = 0$. Therefore it can be considered as a (positive-definite) metric on the space of Livšic cohomology classes.

The *dynamical intersection* is defined in [11] as follows; if $f, g \in \mathrm{Hol}_\alpha(X, \mathbb{R})$ are positive, then their dynamical intersection is

$$\mathbf{I}(f, g) = \lim_{T\to\infty}\frac{1}{\#R_T(f)}\sum_{a\in R_T(f)}\frac{\ell_g(a)}{\ell_f(a)} = \frac{\int g\mathbf{d}m_{-h_f f}}{\int f\mathbf{d}m_{-h_f f}}. \tag{4}$$

The last equality follows from [11, Sec. 3.4]. Similar definitions have been studied in different situations, for example by Bonahon [5], Burger [15] and Knieper [30].

The *renormalized dynamical intersection* is

$$\mathbf{J}(f, g) := \frac{h(g)}{h(f)}\mathbf{I}(f, g).$$

Proposition 3.5 (B.-Canary-Labourie-S. [11, Proposition 3.8])**.** *For every pair of positive Hölder-continuous functions* f *and* g *one has* $\mathbf{J}(f, g) \geqslant 1$. *In particular* $\mathbf{J}(f, \cdot)$ *is critical at* f *which gives*

$$\left.\frac{\partial}{\partial t}\right|_{t=0}\log h(f_t) = \left.\frac{\partial}{\partial t}\right|_{t=0}\mathbf{I}(f, f_t), \tag{5}$$

where $(f_t)_{t\in(-\varepsilon,\varepsilon)}$ *is a* C^1 *curve of positive Hölder-continuous functions with* $f_0 = f$.

Then we have:

Theorem 3.6 (B.-Canary-Labourie-S. [11])**.** *Let* ϕ *be a transitive metric Anosov flow on a compact metric space* X. *If* $f_t \in \mathrm{Hol}(X, \mathbb{R}_+), t \in (-1, 1)$ *is a* 1*-parameter family and* $F_t = -h_{f_t} f_t$ *then*

$$\left.\frac{\partial^2}{\partial t^2}\right|_{t=0}\mathbf{J}(f_0, f_t) = \mathbf{P}\big(\dot{F}_0\big)$$

The following proposition characterizes degenerate vectors for the second derivative of $\mathbf{J}$.

Proposition 3.7 (B.-Canary-Labourie-S. [11, Lemma 9.3]). *Let $(f_t)_{t\in(-\varepsilon,\varepsilon)}$ be a C^1 curve of positive Hölder-continuous functions. Then $(\partial^2/\partial t^2)|_{t=0}\mathbf{J}(f_0,f_t)=0$ if and only if for every periodic orbit τ one has*

$$\left.\frac{\partial}{\partial t}\right|_{t=0} h(f_t)\ell_\tau(f_t)=0.$$

4. Pressure forms

Now we will apply the thermodynamic formalism to representations. For this we make use of the interpretation of a Θ-Anosov representation as a reparametrization of the geodesic flow as explained in Section 2.4.

Given any functional $\varphi\in\mathsf{E}^*_\Theta$ that is positive on the limit cone, one can associate a reparametrization f^φ_ρ of the geodesic flow on Γ. Here we describe in detail two special cases of this construction which play an important role in the paper

4.1. **Spectral radius pressure form.** Let ρ,η be two projective Anosov representations (with possibly different target groups). They both give rise to reparametrizations of the geodesic flow $f^{\omega_1}_\rho$ and $f^{\omega_1}_\eta$, where ω_1 is the first fundamental weight.

We define the *spectral radius dynamical intersection* of the two projective-Anosov representations ρ,η to be the dynamical intersection between $f^{\omega_1}_\rho$ and $f^{\omega_1}_\eta$:

$$\mathbf{I}^{\omega_1}(\rho,\eta)=\mathbf{I}(f^{\omega_1}_\rho,f^{\omega_1}_\eta).$$

Analogously we define $\mathbf{J}^{\omega_1}(\rho,\eta)$. Moreover, given a C^1 curve $(\rho_t)_{t\in(-\varepsilon,\varepsilon)}$ of projective Anosov representations the *spectral radius pressure norm* of $\dot\rho_0$ is defined by

$$\mathbf{P}^{\omega_1}_\rho(\dot\rho_0)=\left.\frac{\partial^2}{\partial t^2}\right|_{t=0}\mathbf{J}^{\omega_1}(\rho_0,\rho_t)\geqslant 0.$$

The spectral radius pressure norm induces a positive semidefinite symmetric bilinear two form at the smooth points of $\{\mathsf{a}_1\}$-Anosov representations. However positive semi-definiteness is as far as thermodynamics goes, and one needs geometric arguments to establish non-degeneracy. In [11] B.-Canary-Labourie-S. prove non-degeneracy under some mild assumptions, giving

Theorem 4.1 (B.-Canary-Labourie-S. [11, Theorem 1.4]). *Let Γ be word hyperbolic, and $\mathsf{G}_\mathbb{R}<\mathsf{PGL}_d(\mathbb{R})$ be reductive. The spectral radius pressure form is an analytic Riemannian metric on the space $\mathcal{C}_g(\Gamma,\mathsf{G}_\mathbb{R})$ of conjugacy classes of $\mathsf{G}_\mathbb{R}$-generic, regular, irreducible, projective Anosov representations.*

Recall that a representation $\rho:\Gamma\to\mathsf{G}_\mathbb{R}$ is $\mathsf{G}_\mathbb{R}$-generic if its Zariski closure contains elements whose centralizer is a maximal torus in $\mathsf{G}_\mathbb{R}$, and it is regular if it is a smooth point of the algebraic variety $\mathrm{Hom}(\Gamma,\mathsf{G}_\mathbb{R})$.

4.2. **Spectral gap pressure form.** We now consider two $\{\mathsf{a}_1,\mathsf{a}_2\}$-Anosov representations ρ,η (with possibly different target groups). As explained in Section 2.4 they define reparametrizations $f^{\mathsf{a}_1}_\rho$ and $f^{\mathsf{a}_1}_\eta$ of the geodesic flow.

We define the *spectral gap dynamical intersection* of ρ and η to be the dynamical intersection between $f^{\mathsf{a}_1}_\rho$ and $f^{\mathsf{a}_1}_\eta$:

$$\mathbf{I}^{\mathsf{a}_1}(\rho,\eta)=\mathbf{I}(f^{\mathsf{a}_1}_\rho,f^{\mathsf{a}_1}_\eta),$$

and analogously for $\mathbf{J}^{\mathsf{a}_1}(\rho,\eta)$. Given a C^1 curve $(\rho_t)_{t\in(-\varepsilon,\varepsilon)}$ of such $\{\mathsf{a}_1,\mathsf{a}_2\}$-representations the *spectral gap pressure norm* of $\dot\rho_0$ is defined by

$$\mathbf{P}^{\mathsf{a}_1}_\rho(\dot\rho_0) = \left.\frac{\partial^2}{\partial t^2}\right|_{t=0} \mathbf{J}^{\mathsf{a}_1}(\rho_0,\rho_t) \geqslant 0.$$

The spectral gap pressure norm induces a semidefinite symmetric bilinear two form on smooth points of $\{\mathsf{a}_1,\mathsf{a}_2\}$-Anosov representations. This looks very similar to the spectral radius pressure norm. It is, however, in general harder to check when the spectral gap pressure form is non-degenerate. As far as the authors know this has, so far, only been established for the Hitchin component in $\mathsf{PSL}_d(\mathbb{R})$:

Theorem 4.2 (B.-Canary-Labourie-S. [12, Theorem 1.6])**.** *Let $\mathsf{G}_\mathbb{R}$ denote either $\mathsf{PSL}_d(\mathbb{R})$, $\mathsf{PSp}(2n,\mathbb{R})$, $\mathsf{PSO}(n,n+1)$ or the split form of the exceptional complex Lie group G_2. Then the spectral gap pressure form is positive definite on the Hitchin component $\mathscr{H}(S,\mathsf{G}_\mathbb{R})$.*

4.3. **Vanishing directions.** Complex conjugation of matrices is an external automorphism of $\mathsf{PSL}_d(\mathbb{C})$ and thus induces an involution

$$\tau : \mathfrak{X}\big(\Gamma,\mathsf{PSL}_d(\mathbb{C})\big) \to \mathfrak{X}\big(\Gamma,\mathsf{PSL}_d(\mathbb{C})\big)$$

whose fixed point set contains $\mathfrak{X}\big(\Gamma,\mathsf{PSL}_d(\mathbb{R})\big)$. If $\rho\in\mathfrak{X}\big(\Gamma,\mathsf{PSL}_d(\mathbb{R})\big)$ is a regular point, then the differential $d_\rho\tau$ splits the tangent space as a sum of *purely imaginary vectors* and the tangent space to the real characters:

$$\mathsf{T}_\rho\mathfrak{X}\big(\Gamma,\mathsf{PSL}_d(\mathbb{C})\big) = \ker(d_\rho\tau+\mathrm{id})\oplus\mathsf{T}_\rho\mathfrak{X}\big(\Gamma,\mathsf{PSL}_d(\mathbb{R})\big);$$

the almost complex structure J of $\mathfrak{X}\big(\Gamma,\mathsf{PSL}_d(\mathbb{C})\big)$ interchanges this splitting.

With a standard symmetry argument (see for example B.-Canary-S. [13, Section 5.8]), we get:

Lemma 4.3. *Let $\rho:\Gamma\to\mathsf{PSL}_d(\mathbb{R})$ be $\{\mathsf{a}_1\}$-Anosov and let v be a purely imaginary direction at ρ. Then $\mathbf{P}^{\omega_1}(v)=0$. If ρ is moreover $\{\mathsf{a}_2\}$-Anosov, then $\mathbf{P}^{\mathsf{a}_1}(v)=0$.*

Proof. Let us prove on the second statement, the first one being analogous. Consider a differentiable curve $(\rho_t)_{t\in(-\varepsilon,\varepsilon)}\subset\mathfrak{X}_{\{\mathsf{a}_1,\mathsf{a}_2\}}(\Gamma,\mathsf{PSL}_d(\mathbb{C}))$ such that $\rho_0=\rho$, $\dot\rho_0=v$ and $\tau\rho_t=\rho_{-t}$. For every conjugacy class $[\gamma]\in[\Gamma]$, the functions

$$t\mapsto \ell_\gamma(f^{\mathsf{a}_1}_{\rho_t}) = (\lambda_1-\lambda_2)\big(\rho_t(\gamma)\big) \quad\text{ and }\quad t\mapsto h(f^{\mathsf{a}_1}_{\rho_t})$$

are invariant under $t\mapsto -t$ and are thus critical at 0. Consequently, for every conjugacy class, the function $t\mapsto h(f^{\mathsf{a}_1}_{\rho_t})\ell_\gamma(f^{\mathsf{a}_1}_{\rho_t})$ is critical at 0 and hence Proposition 3.7 implies that $\mathbf{P}^{\mathsf{a}_1}(v)=0$. ☐

5. Pluriharmonicity of length functions and its consequences

In this section we prove the main results stated in the Introduction.

5.1. **Pluriharmonic length functions.** If $\rho,\eta\in\mathfrak{X}_\Theta(\Gamma,\mathsf{G}_\mathbb{C})$ and $\varphi\in(\mathsf{E}_\Theta)^*$ is strictly positive on $\big(\Lambda_\rho\cup\Lambda_\eta\big)-\{0\}$, then one can define their *φ-dynamical intersection* by

$$\mathbf{I}^\varphi(\rho,\eta) = \mathbf{I}(f^\varphi_\rho,f^\varphi_\eta) = \lim_{T\to\infty}\frac{1}{\#R_T(f^\varphi_\rho)}\sum_{[\gamma]\in R_T(f^\varphi_\rho)}\frac{\varphi\Big(\lambda\big(\eta(\gamma)\big)\Big)}{\varphi\Big(\lambda\big(\rho(\gamma)\big)\Big)}, \tag{6}$$

where $f^\varphi_\rho=\varphi(f^\Theta_\rho)$ is given by Corollary 2.10.

Recall that a function is *pluriharmonic* if it is locally the real part of a holomorphic function. The argument from B.-Taylor [14, Section 5] applies directly and one has the following result.

Proposition 5.1. *Consider $\rho \in \mathfrak{X}_\Theta(\Gamma, \mathsf{G}_\mathbb{C})$ and $\varphi \in (\mathsf{E}_\Theta)^*$ that is strictly positive in $\Lambda_\rho - \{0\}$. Then the function*

$$\mathbf{I}^\varphi_\rho = \mathbf{I}^\varphi(\rho, \cdot) : \mathfrak{X}_\Theta(\Gamma, \mathsf{G}_\mathbb{C}) \to \mathbb{R}$$

is pluriharmonic (when defined).

Recall from Potrie-S. [39, Corollary 4.9] that the map $\eta \mapsto \mathbb{P}\big(p_\Theta(\Lambda_\eta)\big)$ is continuous on $\mathfrak{X}_\Theta(\Gamma, \mathsf{G}_\mathbb{K})$, when considering the Hausdorff topology on compact subsets of $\mathbb{P}\big((\mathsf{E}_\Theta)^*\big)$. Thus the domain of definition of $\mathbf{I}^\varphi_\rho$ is an open subset of $\mathfrak{X}_\Theta(\Gamma, \mathsf{G}_\mathbb{C})$ that contains, in particular, ρ. The proposition implies then that $\mathbf{I}^\varphi_\rho$ is (defined and) pluriharmonic on a neighborhood of ρ.

Proof. Consider $\eta \in \mathfrak{X}_\Theta(\Gamma, \mathsf{G}_\mathbb{C})$ such that $\varphi|\Lambda_\eta - \{0\}$ is striclty positive. It follows then from Bochi-Potrie-S. [4, Proposition 5.11] that there exists a neighborhood $\mathcal{U}$ of η such that the constants in Definition 2.1 hold for every $\psi \in \mathcal{U}$. This implies that the convergence in the definition of $\mathbf{I}^\varphi(\rho, \cdot)$ is uniform on compact subsets of its domain of definition. For each $T > 0$, the truncated sum in equation (6) is the real part of a holomorphic function and thus Theorem 1.23 from Axler-Bourdon-Ramey [2] yields the result. □

5.2. **Proof of Theorem A.** Let $\rho \in \mathfrak{X}\big(\pi_1 S, \mathsf{PSL}_d(\mathbb{R})\big)$ be $(1,1,2)$-hyperconvex and assume that it is a regular point of the character variety $\mathfrak{X}\big(\pi_1 S, \mathsf{PSL}_d(\mathbb{R})\big)$. Consider a tangent vector $v \in \mathsf{T}_\rho\mathfrak{X}\big(\pi_1 S, \mathsf{PSL}_d(\mathbb{R})\big)$. Note that then Jv is a purely imaginary tangent direction in $\mathsf{T}_\rho\mathfrak{X}\big(\pi_1 S, \mathsf{PSL}_d(\mathbb{C})\big)$. Thus, Lemma 4.3 implies that for any C^1 curve $(\rho_t)_{t\in(-\varepsilon,\varepsilon)}$ with $\rho_0 = \rho$, $\dot\rho_0 = Jv$ and $\tau\rho_t = \rho_{-t}$ we have

$$0 = \mathbf{P}^{\mathsf{a}_1}(Jv) = \left.\frac{\partial^2}{\partial t^2}\right|_{t=0} \mathbf{J}^{\mathsf{a}_1}(\rho_0, \rho_t). \tag{7}$$

Recall that $\mathbf{I}(f, f) = 1$ and that if ρ is $(1,1,2)$-hyperconvex then Theorem 2.6 states that $h^{\mathsf{a}_1}_\rho = 1$. Moreover, as observed in the proof of Lemma 4.3, $\dot h^{\mathsf{a}_1}(\dot\rho_0) = 0$, so developing the last term of equation (7) one obtains

$$0 = \mathrm{Hess}_\rho(h^{\mathsf{a}_1})(Jv) + \mathrm{Hess}_\rho \mathbf{I}^{\mathsf{a}_1}_\rho(Jv).$$

Proposition 5.1 states that $\mathbf{I}^{\mathsf{a}_1}_\rho$ is pluriharmonic, so $\mathrm{Hess}_\rho \mathbf{I}^{\mathsf{a}_1}_\rho(Jv) = -\mathrm{Hess}_\rho \mathbf{I}^{\mathsf{a}_1}_\rho(v)$ and thus

$$\mathrm{Hess}_\rho h^{\mathsf{a}_1}(Jv) = \mathrm{Hess}_\rho \mathbf{I}^{\mathsf{a}_1}_\rho(v).$$

Lemma 2.4 implies that, at least for small t, ρ_t is $(1,1,2)$-hyperconvex (over $\mathbb{C}$) and thus Theorem 2.5 yields $h^{\mathsf{a}_1}(\rho_t) = \mathrm{Hff}_{\mathsf{a}_1}(\rho_t)$. Finally, since $h^{\mathsf{a}_1} \equiv 1$ in a neighborhood of ρ in $\mathfrak{X}\big(\pi_1 S, \mathsf{PSL}_d(\mathbb{R})\big)$ one has

$$\mathrm{Hess}_\rho \mathbf{I}^{\mathsf{a}_1}_\rho(v) = \mathbf{P}^{\mathsf{a}_1}(v).$$

The result follows.

5.3. **Proof of Theorem B.** By Theorem 2.5 $\mathrm{Hff}_1 = h^{\mathsf{a}_1}$ in a neighborhood of ρ, and thus by assumption, the latter is critical at ρ. Since $\mathbf{J}^{\mathsf{a}_1}(\rho,\cdot)$ is also critical at ρ (Proposition 3.5) one concludes that $\mathbf{I}^{\mathsf{a}_1}_\rho$ is critical at ρ and thus its Hessian is well defined.

By Proposition 5.1 $\mathbf{I}^{\mathsf{a}_1}_\rho$ is pluriharmonic and thus one has (as before) that for every $v \in \mathsf{T}_\rho\mathfrak{X}\big(\Gamma, \mathsf{PSL}_d(\mathbb{C})\big)$

$$\mathrm{Hess}_\rho \mathbf{I}^{\mathsf{a}_1}_\rho(Jv) = -\mathrm{Hess}_\rho \mathbf{I}^{\mathsf{a}_1}_\rho(v).$$

One concludes that the $(+,0,-)$ signature of $\mathrm{Hess}_\rho \mathbf{I}^{\mathsf{a}_1}_\rho$ is of the form $(p,2k,p)$ for some $p \leqslant$ half $\dim_{\mathbb{R}} \mathfrak{X}\big(\Gamma, \mathsf{PSL}_d(\mathbb{C})\big)$. Moreover, by Theorem 3.6 one has

$$0 \leqslant \mathbf{P}^{\mathsf{a}_1}(Jv) = \mathrm{Hess}_\rho h^{\mathsf{a}_1}(Jv) - h^{\mathsf{a}_1}_\rho \mathrm{Hess}_\rho \mathbf{I}^{\mathsf{a}_1}_\rho(v),$$

so that $\mathrm{Hess}_\rho \mathbf{I}^{\mathsf{a}_1}_\rho(v) \geqslant 0$ implies $\mathrm{Hess}_\rho h^{\mathsf{a}_1}(Jv) \geqslant 0$. In particular $\mathrm{Hess}_\rho h^{\mathsf{a}_1}$ is positive semidefinite on a subspace of dimension at least

$$\dim_{\mathbb{R}} \mathfrak{X}\big(\Gamma, \mathsf{PSL}_d(\mathbb{C})\big) - p \geqslant \frac{1}{2}\dim_{\mathbb{R}} \mathfrak{X}\big(\Gamma, \mathsf{PSL}_d(\mathbb{C})\big)$$

and the theorem is proven.

5.4. **Proof of Theorem C.** Let Γ be a co-compact lattice in $\mathsf{PSO}(n,1)$ such that the inclusion $\iota : \Gamma \to \mathsf{PSO}(n,1)$ defines, after extending coefficients, a regular point of the character variety $\mathfrak{X}\big(\Gamma, \mathsf{PSU}(n,1)\big)$. Theorem 2.2 (and the Remark following it) in Cooper-Long-Thistlethwaite [18] assert that ι is then a regular point of the $\mathsf{PSL}_{n+1}(\mathbb{R})$ character variety $\mathfrak{X}\big(\Gamma, \mathsf{PSL}_{n+1}(\mathbb{R})\big)$.

Moreover, since $\mathfrak{so}(n,1)$ is the fixed point set of an involution in $\mathfrak{sl}_{n+1}(\mathbb{R})$, one has the decomposition $\mathfrak{sl}_{n+1}(\mathbb{R}) = \mathfrak{so}(n,1) \oplus \mathfrak{s}$ with $[\mathfrak{s},\mathfrak{s}] \subset \mathfrak{so}(n,1)$. One readily sees that

$$\mathfrak{su}(n,1) = \mathfrak{so}(n,1) \oplus i\mathfrak{s} \subset \mathfrak{sl}_{n+1}(\mathbb{C}). \tag{8}$$

The twisted cohomology $H^1_\iota\big(\Gamma, \mathfrak{sl}_{n+1}(\mathbb{R})\big)$ splits as

$$H^1_\iota\big(\Gamma, \mathfrak{sl}_{n+1}(\mathbb{R})\big) = H^1_\iota\big(\Gamma, \mathfrak{so}(n,1)\big) \oplus H^1_\iota(\Gamma, \mathfrak{s}).$$

Consequently, by equation (8) the subspace $H^1_\iota(\Gamma,\mathfrak{s}) \subset H^1_\iota(\Gamma, \mathfrak{sl}_{n+1}(\mathbb{C}))$ is sent bijectively to $H^1_\iota(\Gamma, i\mathfrak{s})$ when multiplied by the complex structure J, i.e.

$$J \cdot H^1_\iota(\Gamma,\mathfrak{s}) = H^1_\iota(\Gamma, i\mathfrak{s}). \tag{9}$$

We will need the following generalization of Crampon [19].

Theorem 5.2 (Potrie-S. [39, Theorem 7.2])**.** *Assume $\rho \in \mathfrak{X}\big(\Gamma, \mathsf{PSL}_{n+1}(\mathbb{R})\big)$ has finite kernel and divides a proper open convex set of $\mathbb{P}(\mathbb{R}^{n+1})$. Then the entropy*

$$h^{\omega_1}(\rho) \leqslant n-1$$

and equality holds only if ρ has values in $\mathsf{PSO}(n,1)$.

This has the following useful consequence.

Corollary 5.3. *The spectral radius pressure form $\mathbf{P}^{\omega_1}$ on $\mathfrak{X}\big(\Gamma, \mathsf{PGL}_{n+1}(\mathbb{R})\big)$ is non-degenerate at ι.*

Proof. When $n = 2$ this follows directly from Theorem 4.1, but if $n > 2$, the embedding $\mathfrak{so}(n,1) \subset \mathfrak{sl}_{n+1}(\mathbb{R})$ is not $\mathsf{PGL}_{n+1}(\mathbb{R})$-generic so, even though $\iota(\Gamma)$ is irreducible, we need additional arguments. Nevertheless, by Theorem 5.2, the entropy function $\rho \mapsto h^{\omega_1}(\rho)$ is critical at ι, so by Proposition 3.7 one only needs to verify that the set

$$\left\{d_\iota \omega_1^\gamma : [\gamma] \in [\Gamma]\right\}$$

spans the cotangent space $\mathsf{T}_\iota^* \mathfrak{X}\big(\Gamma, \mathsf{PSL}_{n+1}(\mathbb{R})\big)$, where $\omega_1^\gamma : \mathfrak{X}\big(\Gamma, \mathsf{PGL}_{n+1}(\mathbb{R})\big) \to \mathbb{R}$ is the function

$$\rho \mapsto \omega_1\Big(\lambda\big(\rho(\gamma)\big)\Big).$$

As ι is irreducible and projective Anosov, this is the content of B.-Canary-Labourie-S. [11, Proposition 10.1]. $\square$

Consider then $v \in H^1_\iota(\Gamma, \mathfrak{s}) \subset \mathsf{T}_\iota \mathfrak{X}\big(\Gamma, \mathsf{PSL}_{n+1}(\mathbb{R})\big)$, by equation (9) the purely imaginary vector $J \cdot v \in \mathsf{T}_\iota \mathfrak{X}\big(\Gamma, \mathsf{PSL}_{n+1}(\mathbb{C})\big)$ belongs to $H^1_\iota\big(\Gamma, \mathsf{PSU}(n,1)\big)$ and represents thus a non-trivial infinitesimal deformation of ι inside $\mathsf{PSU}(n,1)$. As in Lemma 4.3 we choose a differentiable curve $(\rho_t)_{t\in(-\varepsilon,\varepsilon)} \subset \mathfrak{X}\big(\Gamma, \mathsf{PSU}(n,1)\big)$ with $\rho_0 = \iota$ and $\dot\rho_0 = Jv$ and $\tau\rho_t = \rho_{-t}$.

By Lemma 4.3 we have that

$$0 = \mathbf{P}^{\omega_1}_\iota(Jv) = \left.\frac{\partial^2}{\partial t^2}\right|_{t=0} \mathbf{J}^{\omega_1}(\iota, \rho_t). \tag{10}$$

Expanding the second term, and using that both $h^{\omega_1}(\rho_t)$ and $\mathbf{I}^{\omega_1}_\iota(\rho_t)$ are critical at $t = 0$ (as in the proof of Lemma 4.3) and that $\mathbf{I}^{\omega_1}_\iota$ is pluriharmonic, we get

$$0 = \operatorname{Hess}_\iota(h^{\omega_1})(Jv) - (n-1)\operatorname{Hess}_\iota(\mathbf{I}^{\omega_1}_\iota)(v).$$

On the other hand

$$\mathbf{P}^{\omega_1}_\iota(v) = \operatorname{Hess}_\iota(h^{\omega_1})(v) + (n-1)\operatorname{Hess}_\iota(\mathbf{I}^{\omega_1}_\iota)(v).$$

Which in turn gives

$$\operatorname{Hess}_\iota(h^{\omega_1})(Jv) = \mathbf{P}^{\omega_1}_\iota(v) - \operatorname{Hess}_\iota(h^{\omega_1})(v) > 0,$$

since $\mathbf{P}^{\omega_1}_\iota(v) > 0$ by Corollary 5.3, and $-\operatorname{Hess}_\iota(h^{\omega_1})(v) \geqslant 0$ since by Theorem 5.2 ι is a global maxima of h^{ω_1} among deformations in $\mathsf{PSL}_{n+1}(\mathbb{R})$. The result then follows.

5.5. **The Hessian of the entropy at the Fuchsian locus of the Hitchin component.** Applying the same techniques as in the last section we can also show the following result on the Hitchin component.

Corollary 5.4. *Let $\iota \in \mathscr{H}_d(S)$ be a representation $\pi_1 S \to \mathsf{PSL}_2(\mathbb{R}) \to \mathsf{PSL}_d(\mathbb{R})$ in the embedded Teichmüller space. Then* $\operatorname{Hess}(h^{\omega_1}_\iota)$ *is positive definite on purely imaginary directions of* $\mathsf{T}_\iota \mathfrak{X}\big(\pi_1 S, \mathsf{PSL}_d(\mathbb{C})\big)$.

Proof. We mimic the last paragraph. In this case the pressure form $\mathbf{P}^{\omega_1}$ is positive definite on $\mathsf{T}_\iota \mathscr{H}\big(S, \mathsf{PSL}_d(\mathbb{R})\big)$ directly by Theorem 4.1. One gets, through the same arguments, that

$$\operatorname{Hess}_\rho(h^{\omega_1})(Jv) = \mathbf{P}^{\omega_1}(v) - \operatorname{Hess}(h^{\omega_1})(v).$$

As we already observed, the first term on the right hand side is positive by Theorem 4.1, while $\operatorname{Hess}(h^{\omega_1})(v) \leqslant 0$ since, by Potrie-S. [39, Theorem A], Fuchsian representations are maxima for the entropy within the Hitchin locus. The corollary follows. $\square$

We refer the reader to Dey-Kapovich [28] (see also Ledrappier [33] and Link [34]) for an interpretation of the critical exponent $h^{\omega_1}(\rho)$ as the Hausdorff dimension of the limit set with respect to a visual metric, i.e. a metric with respect to which the group action is conformal.

Finally, it would be interesting to relate Corollary 5.4, or an analog of it, to the recent work by Dai-Li [20] studying the translation lengths on the symmetric space of $\mathsf{PSL}_d(\mathbb{C})$, when one deforms a Fuchsian representation along its Hitchin fiber.

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