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Alexey Glutsyuk

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Density of thin film billiard reflection pseudogroup in Hamiltonian symplectomorphism pseudogroup

Alexey Glutsyuk*†‡§

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Abstract

Reflections from hypersurfaces act by symplectomorphisms on the space of oriented lines with respect to the canonical symplectic form. We consider an arbitrary C^∞-smooth hypersurface $\gamma \subset \mathbb{R}^{n+1}$ that is either a global strictly convex closed hypersurface, or a germ of hypersurface. We deal with the pseudogroup generated by compositional ratios of reflections from γ and of reflections from its small deformations. In the case, when γ is a global convex hypersurface, we show that the latter pseudogroup is dense in the pseudogroup of Hamiltonian diffeomorphisms between subdomains of the phase cylinder: the space of oriented lines intersecting γ transversally. We prove an analogous local result in the case, when γ is a germ. The derivatives of the above compositional differences in the deformation parameter are Hamiltonian vector fields calculated by Ron Perline. To prove the main results, we find the Lie algebra generated by them and prove its C^∞-density in the Lie algebra of Hamiltonian vector fields. We also prove analogues of the above results for hypersurfaces in Riemannian manifolds.

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*CNRS, France (UMR 5669 (UMPA, ENS de Lyon), UMI 2615 (ISC J.-V.Poncelet)). E-mail: aglutsyu@ens-lyon.fr

†HSE University, Moscow, Russia

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1 Main results: density of thin film planar billiard reflection pseudogroup

It is well-known that billiard reflections acting on the space of oriented geodesics preserve the canonical symplectic form [1, 2, 6, 7, 10, 16]. However

only a tiny part of symplectomorphisms are realized by reflections. There is an important open question stated in [14]: which symplectomorphisms can be realized by compositions of reflections?

In the present paper we consider the case of billiards in Euclidean spaces. Namely, we deal with a hypersurface $\gamma \subset \mathbb{R}^{n+1}$: either a strictly convex closed hypersurface, or a germ of hypersurface. We investigate compositional ratios of reflections from γ and reflections from its small deformations. They were introduced and studied by Ron Perline [13]. He had shown that their derivatives in the parameter are Hamiltonian vector fields and calculated their Hamiltonian functions. See Subsection 1.4 below.

We show that the Lie algebra generated by the above Hamiltonian vector fields is dense in the Lie algebra of all the Hamiltonian vector fields (symplectic vector fields in the case, when γ is a germ of curve). We apply this result to the pseudogroup generated by the above compositional ratios of reflections. In the case of a strictly convex closed hypersurface we show that the latter pseudogroup is C^∞-dense in the pseudogroup of Hamiltonian symplectomorphisms between open subdomains of the phase cylinder: the space of oriented lines intersecting γ transversally. We prove analogous statement for germs of hypersurfaces. The corresponding results are stated in Subsections 1.1 and 1.2 respectively.

In the case of a germ of planar curve, when $n = 1$, we show that the above pseudogroup coming from reflections is C^∞-dense in the local pseudogroup of symplectomorphisms between simply connected subdomains of a small region in the space of oriented lines. See the statements of the corresponding results in Subsection 1.3.

Density of the above pseudogroups is proved in Section 4.

For the proof of main results we find the Lie algebra generated by the above-mentioned Hamiltonian functions from [13] and prove its C^∞-density in the space of C^∞-smooth functions. The corresponding results are presented in Subsection 1.4 and proved in Sections 2 and 3 in the cases, when $n = 1$ and $n \geq 2$ respectively.

We show that Perline's formula for the above vector fields, which are derivatives of the above compositional ratios of reflections, remains valid for hypersurfaces in Riemannian manifolds. Then we deduce the above pseudogroup density results in this case. These results are stated in Subsection 1.5 and proved in Section 5.

In Subsection 1.6 we state some open questions.

1.1 Main density results: case of global hypersurface

Here we deal with billiards in $\mathbb{R}^{n+1}$, $n \geq 1$: reflections from hypersurfaces, which act on the space of oriented lines in $\mathbb{R}^{n+1}$, see the next paragraph. It is well-known that the space of all oriented lines is canonically diffeomorphic to the tangent bundle of the unit sphere S^n, and it carries a canonical symplectic form induced from the standard symplectic form on TS^n. All the reflections from hypersurfaces act symplectomorphically [1, 2, 6, 7, 10, 16].

Let $\gamma \subset \mathbb{R}^{n+1}$ be a closed convex hypersurface. Recall that the reflection $\mathcal{T}_\gamma$ from γ acts on the space of oriented lines in $\mathbb{R}^{n+1}$ as follows. If a line L is either disjoint from γ, or tangent to it, then it is fixed by $\mathcal{T}_\gamma$. If L intersects γ transversally, then we take its last point A of intersection with γ (in the sense of orientation of the line L). We set $\mathcal{T}_\gamma(L)$ to be its image under reflection from the tangent hyperplane $T_A\gamma$ and orient it by a vector in $T_A\mathbb{R}^{n+1}$ directed inside the compact domain bounded by γ.

Let now $\vec{N}$ be the exterior normal vector field on the hypersurface γ. Let $f : \gamma \to \mathbb{R}$ be a C^∞-smooth function. Consider the following deformation of the surface γ:

$$\gamma_\varepsilon = \gamma_{\varepsilon,f} := \{x + \varepsilon \vec{N}(x) \mid x \in \gamma\}. \tag{1.1}$$

We deal with reflections $\mathcal{T}_\gamma$ and $\mathcal{T}_{\gamma_\varepsilon}$ acting on oriented lines. Recall that the *phase cylinder* is the domain

$$\Pi = \Pi_\gamma := \{\text{the oriented lines intersecting } \gamma \text{ transversally}\},$$

$$\Delta\mathcal{T}_\varepsilon = \Delta\mathcal{T}_{\varepsilon,f} := \mathcal{T}_{\gamma_\varepsilon}^{-1} \circ \mathcal{T}_\gamma, \quad v_f := \frac{d\Delta\mathcal{T}_\varepsilon}{d\varepsilon}|_{\varepsilon=0}. \tag{1.2}$$

For every compact subset $K \Subset \Pi$ the symplectic mapping $\Delta\mathcal{T}_\varepsilon$ is well-defined on K, whenever ε is small enough dependently on K. Hence, v_f are symplectic vector fields on Π. They are Hamiltonian with Hamiltonian functions given in [13, p.23], see also (1.5). We prove the following theorem.

Theorem 1.1 *The Lie algebra generated by the vector fields v_f, $f \in C^\infty(\gamma)$, is C^∞-dense in the Lie algebra of Hamiltonian vector fields on Π_γ.*

Using Theorem 1.1, we prove density results for pseudogroups generated by the compositional differences $\Delta\mathcal{T}_\varepsilon$. To state them, let us recall the following well-known definitions.

Consider a collection of C^∞-smooth diffeomorphisms between domains on some manifold (e.g., the space of oriented lines in $\mathbb{R}^{n+1}$). We consider each diffeomorphism together with its domain of definition and all its restrictions to smaller domains. We will deal with those compositions of diffeomorphisms and of their inverses that are well-defined on some domains. Recall

that the collection of all the above compositions is called a *pseudogroup*. (In the case when we take only compositions of mappings and not of their inverses, the above collection of compositions is called a *pseudo-semigroup*.)

Definition 1.2 For a given sequence of domains V_n we say that the intersections $V_n \cap V$ converge to V, if each compact subset $K \Subset V$ is contained in V_n whenever n is large enough (depending on K). Let g be a C^∞-smooth diffeomorphism defined on a domain V: we deal with it as with the pair (g, V). Recall that a sequence g_n of elements in the pseudogroup converges to a mapping g on V in the C^∞-topology, if g is well-defined on V, g_n are defined on a sequence of domains V_n such that $V_n \cap V \to V$, and $g_n \to g$ uniformly on compact subsets in V with all the derivatives. The C^∞*-closure* of a given pseudogroup consists of the mappings forming the pseudogroup and the limits of the above converging sequences.

For every positive-valued mapping $\delta : C^\infty(\gamma) \to \mathbb{R}_+$ set

$$\mathcal{G}(\delta) := \text{ the pseudogroup generated by the collection of mappings} \quad (1.3)$$

$$\{\Delta\mathcal{T}_{\varepsilon,h} \mid h \in C^\infty(\gamma), 0 \le \varepsilon \le \delta(h)\}.$$

Let us recall the following well-known definition.

Definition 1.3 Let M be a symplectic manifold, and let $V \subset M$ be an open domain. A symplectomorphism $F : V \to V_1 \subset M$ is (M-) *Hamiltonian,* if there exists a smooth family $F_t : V \to V_t \subset M$ of symplectomorphisms parametrized by $t \in [0, 1]$, $V_0 = V$, $F_0 = Id$, $F_1 = F$, such that for every $t \in [0, 1]$ the derivative $\frac{dF_t}{dt}$ is a Hamiltonian vector field on V_t. (In the case when $V = V_t = M$, this definition coincides with the usual definition of Hamiltonian diffeomorphism of a manifold [3, definition 4.2.4].)

Remark 1.4 Let M be a two-dimensional topological cylinder equipped with a symplectic form, and let $V \subset M$ be a subcylinder with compact closure and smooth boundary. Further assume that V is a deformation retract of the ambient cylinder M. Then not every area-preserving map $F : V \to U \subset M$ is Hamiltonian. A necessary condition for being Hamiltonian is that for every boundary component L of the subcylinder V the self-intersecting domain bounded by L and by its image $F(L)$ has zero signed area. This follows from results presented in [3, chapters 3, 4]. The above-defined Hamiltonian symplectomorphisms between domains in a symplectic manifold form a pseudogroup: a composition of two M-Hamiltonian symplectomorphisms $U \to V$ and $V \to W$ is M-Hamiltonian.

Theorem 1.5 *For every mapping $\delta : C^\infty(\gamma) \to \mathbb{R}_+$ the C^∞-closure of the pseudogroup $\mathcal{G}(\delta)$ contains the whole pseudogroup of Π-Hamiltonian diffeomorphisms between domains in Π. In other words, for every domain $V \subset \Pi$ and every Π-Hamiltonian symplectomorphism $F : V \to W \subset \Pi$ there exists a sequence F_n of finite compositions of mappings $\Delta\mathcal{T}_{\varepsilon_j,h_j}$, $0 \le \varepsilon_j \le \delta(h_j)$, F_n being defined on domains V_n with $V_n \cap V \to V$, such that $F_n \to F$ on V in the C^∞-topology.*

Definition 1.6 We say that two hypersurfaces are (α, k)-close if the distance between them in the C^k-topology is no greater than α. Here closeness means that the hypersurfaces are diffeomorphically parametrized by the same manifold, and their parametrizations can be chosen so that the C^k-distance between them is no greater than α.

Corollary 1.7 *For every $\alpha > 0$ (arbitrarily small) and $k \in \mathbb{N}$ the C^∞-closure of the pseudogroup generated by reflections from hypersurfaces (α, k)-close to γ contains the whole pseudogroup of Π-Hamiltonian symplectomorphisms between domains in Π.*

Theorems 1.1 will be proved in Subsection 2.5 (for $n = 1$) and 3.3 (for $n \ge 2$). Theorem 1.5 and Corollary 1.7 will be proved in Section 4.

1.2 Main results in the local case

Here we consider the case, when γ is a germ of (not necessarily convex) C^∞-smooth hypersurface in $\mathbb{R}^{n+1}$ at some point x_0 and state local versions of the above results.

Each function $f : \gamma \to \mathbb{R}$ defines a deformation $\gamma_\varepsilon = \gamma_{\varepsilon,f}$ of the germ γ given by (1.1). Let $\Delta\mathcal{T}_\varepsilon$, v_f be the same, as in (1.2). Fix an arbitrary small contractible neighborhood of the point x_0 in γ, and now denote by γ the latter neighborhood. It is a local hypersurface parametrized by a contractible open domain in $\mathbb{R}^n$. Fix an arbitrary domain $\Pi = \Pi_\gamma$ in the space of oriented lines in $\mathbb{R}^{n+1}$ such that each point in Π represents a line L intersecting γ transversally at one point $x = x(L)$. Let $w = w(L) \in T_x\gamma$ denote the orthogonal projection to $T_x\gamma$ of the directing vector of L at x. We identify all the tangent spaces $T_x\gamma$ by projecting them orthogonally to an appropriate coordinate subspace $\mathbb{R}^n \subset \mathbb{R}^{n+1}$. For simplicity in the local case under consideration we will choose Π to be diffeomorphic to a contractible domain in $\mathbb{R}^{2n}$ via the correspondence sending L to (x, w).

Theorem 1.8 *The Lie algebra of vector fields on* Π_γ *generated by the fields* v_f, $f \in C^\infty(\gamma)$, *see (1.2), is* C^∞*-dense in the Lie algebra of Hamiltonian vector fields on* Π_γ.

The definitions of pseudogroup $\mathcal{G}(\delta)$ and (α, k)-close hypersurfaces for local hypersurfaces (germs) are the same, as in the previous subsection.

Theorem 1.9 *The statements of Theorem 1.5 and Corollary 1.7 hold in the case, when* γ *is a local hypersurface: a* C^∞*-smooth germ of hypersurface in* $\mathbb{R}^{n+1}$.

Theorems 1.8 will be proved in Section 2.5 (for $n = 1$) and 3.3 (for $n \geq 2$). Theorem 1.9 will be proved in Section 4.

1.3 Case of germs of planar curves: density in symplectic vector fields and maps

Let now γ be a C^∞-smooth germ of planar curve in $\mathbb{R}^2$ at a point O. Let Π be the same, as in the previous subsection.

Theorem 1.10 *In the case, when* γ *is a germ of planar curve, the Lie algebra generated by the fields* v_f *is dense in the Lie algebra of symplectic vector fields on* Π.

Theorem 1.11 *If* γ *is a germ of planar curve, then the statements of Theorems 1.5 and Corollary 1.7 hold with density in the pseudogroup of symplectomorphisms between simply connected domains in* Π.

Theorem 1.10 is proved in Subsection 2.5. Theorem 1.11 is proved in Subsection 4.2.

1.4 Hamiltonian functions for vector fields v_f and the Lie algebra generated by them

Let us recall a well-known presentation of the symplectic structure on Π coming from the identification of the outwards directed unit tangent bundle on γ with the unit ball bundle $T_{<1}\gamma \subset T\gamma$ [1, 2, 6, 7, 10, 16]. To each oriented line λ intersecting γ transversally we put into correspondence its last intersection point $x(\lambda)$ with γ in the sense of orientation of the line λ and the unit vector $u \in T_{x(\lambda)}\mathbb{R}^n$ directing λ. To a unit vector $u \in T_x\mathbb{R}^{n+1}$, $x \in \gamma$, we put into correspondence its orthogonal projection $w := \pi_\perp(u) \in T_x\gamma$ to the tangent hyperplane of γ at x; one has $||w|| \leq 1$. In the case, when

γ is a strictly convex closed hypersurface, the composition of the above correspondences yields a diffeomorphism

$$J : \Pi \mapsto T_{<1}\gamma, \quad \lambda \mapsto (x(\lambda), w(\lambda)) \in T_{<1}\gamma := \{(x, w) \in T\gamma \mid ||w|| < 1\}.$$

In the case, when γ is a germ, J is a diffeomorphism of the corresponding domain Π in the space of oriented lines onto an open subset $J(\Pi) \subset T_{<1}\gamma$. The tangent bundle $T\gamma$ consists of pairs (x, w), $x \in \gamma$, $w \in T_x\gamma$, and carries the Liouvillian 1-form $\alpha \in T^*(T\gamma)$ defined as follows: for every $x \in \gamma$, $w \in T_x\gamma$ and $v \in T_{(x,w)}(T\gamma)$ one has

$$\alpha(v) =< w, \pi_*(v) >, \quad \pi_* = d\pi, \quad \pi \text{ is the projection } T\gamma \to \gamma. \tag{1.4}$$

The *standard symplectic form* on $T\gamma$ is given by the differential

$$\omega := d\alpha.$$

The above Liouville form α and symplectic form ω are well-defined on every Riemannian manifold γ.

The above diffeomorphism J is known to be a symplectomorphism [1, 2, 6, 7, 10, 16]. In what follows we switch from Π to $T_{<1}\gamma$; the images of the vector fields v_f under the diffeomorphism J are symplectic vector fields on $T_{<1}\gamma$, which will be also denoted by v_f.

Remark 1.12 For every function f on γ the corresponding vector field v_f given by (1.2) is a well-defined vector field on the space of all pairs (x, u), where $x \in \gamma$ and $u \in T_x\mathbb{R}^{n+1}$ is a vector transversal to γ. Therefore, the corresponding vector field v_f on $J(\Pi)$ is well-defined on all of $T_{<1}\gamma$. *In what follows, whenever the contrary is not specified, we will deal with v_f as a vector field on $T_{<1}\gamma$.*

Theorem 1.13 *[13, p.23] For every C^∞-smooth function $f : \gamma \to \mathbb{R}$ the corresponding vector field v_f is Hamiltonian with the Hamiltonian function*

$$H_f(x, w) := -2\sqrt{1 - ||w||^2} f(x). \tag{1.5}$$

Remark 1.14 The formula from [13, p.23] was given in terms of unit tangent bundle: the space of pairs (x, u), $x \in \gamma$, $u \in T_x\mathbb{R}^{n+1}$ being a unit vector directed outside γ. The Hamiltonian function from loc. cit. is $H_f(x, u) = -2 < u, \vec{N}(x) > f(x)$. Taking into account that the latter scalar product is equal to $\sqrt{1 - ||w||^2}$, $w = \pi_\perp(u)$, this yields (1.5).

Proposition 1.15 *The vector space over $\mathbb{R}$ generated by the Poisson brackets of the functions $H_f(x,w)$ given by (1.5) for all f is a Lie algebra, where each element can be represented as a sum of at most $2n+1$ Poisson brackets. It consists of the functions on $T_{<1}\gamma$ of type $\eta(w)$, where η is an arbitrary C^∞-smooth 1-form on γ. Thus, it is identified with the space of 1-forms. The Lie algebra structure thus obtained on the space of 1-forms is isomorphic to the standard Lie algebra structure on the space of vector fields via the duality isomorphism $T^*\gamma \to T\gamma$ given by the metric.*

Proposition 1.15 is proved in Subsection 3.1.

Theorem 1.16 *Let γ be an arbitrary Riemannian manifold (neither necessarily embedded, nor necessarily compact). Let $\mathfrak{H}$ denote the Lie algebra generated by the Hamiltonian functions (1.5) on $T_{<1}\gamma$ constructed from all the C^∞-functions f on γ under the Poisson bracket. The Lie algebra $\mathfrak{H}$ is C^∞-dense in the Lie algebra of all the C^∞-functions on the unit tangent bundle $T_{<1}\gamma$: dense in the topology of uniform C^∞-convergence on compact subsets.*

Below we formulate more precise versions of Theorem 1.16, which give explicitly the Lie algebra generated by functions (1.5) in different cases. To obtain its simpler description, we deal with the following renormalization isomorphism

$$Y : T_{<1}\gamma \to T\gamma,\ (x,w) \mapsto (x,y),\ y = y(w) := \frac{w}{\sqrt{1-||w||^2}} \in T_x\gamma. \quad (1.6)$$

This equips $T\gamma$ with the pushforward symplectic structure $Y_*\omega$.

Convention 1.17 *Sometimes we rewrite functions in $(x,w) \in T_{<1}\gamma$ as functions of $(x,y) \in T\gamma$. By definition, the Poisson bracket of two functions in (x,y) is calculated with respect to the pushforward symplectic form $Y_*\omega$. It coincides with the pushforward by Y of their Poisson bracket as functions of (x,w), with respect to the canonical symplectic form ω on $T_{<1}\gamma$.*

Example 1.18 For every function f on γ the corresponding Hamiltonian function $H_f(x,w)$ of the vector field v_f written in the new coordinate y is equal to

$$H_f(x,w) = H_{0,f}(x,y) := -\frac{2f(x)}{\sqrt{1+||y||^2}}. \quad (1.7)$$

This follows from (1.5) and (1.6).

Let $S^k(T^*\gamma)$ denote the space of those C^∞-smooth functions on $T\gamma$ whose restrictions to the fibers $T_x\gamma$ are homogeneous polynomials of degree k. For every $\phi \in S^k(T^*\gamma)$ we write $\phi = \phi(y)$ as a polynomial in $y \in T_x\gamma$ with coefficients depending C^∞-smoothly on $x \in \gamma$. Set

$$H_{k,\phi} := \frac{\phi(y)}{\sqrt{1+||y||^2}}, \quad H_{k,\phi} \in C^\infty(T\gamma), \tag{1.8}$$

$$\Lambda_k := \{H_{k,\phi} \mid \phi \in S^k(T^*\gamma)\}.$$

Example 1.19 Let $n = 1$ and let, for simplicity, γ be connected. Then γ is either an interval, or a circle, equipped with a Riemannian metric. Let s be the length element on γ. Then y is just one variable, each element in $S^k(T^*\gamma)$ is a product $\phi = h(s)y^k$, $h \in C^\infty(\gamma)$. In this case, when $n = 1$, we will use a simplified notations replacing ϕ by h and writing $H_{k,\phi}$ as $H_{k,h}$:

$$H_{k,\phi} = H_{k,h} := \frac{h(s)y^k}{\sqrt{1+y^2}}; \qquad \Lambda_k = \{H_{k,h} \mid h \in C^\infty(\gamma)\}. \tag{1.9}$$

In the case, when γ is a closed planar curve with induced metric, the phase cylinder Π is indeed a cylinder: it is diffeomorphic to the product of a circle and an interval via the following correspondence. To each oriented line L intersecting γ transversally we put into correspondence the intersection point with γ (identified with its natural parameter s), running all of $\gamma \simeq S^1$, and the intersection angle $\theta \in (0, \pi)$. The above-defined symplectic form on Π is equal to $\sin\theta ds \wedge d\theta$. The natural parameter yields a canonical trivialization of the tangent bundle to γ. After this trivialization the corresponding vectors w and y become just real numbers that are equal to

$$w = \cos\theta, \qquad y = \cot\theta. \tag{1.10}$$

In the renormalized coordinates y the Lie algebra $\mathfrak{H}$ admits the following description.

Theorem 1.20 *Let γ be the same, as in Theorem 1.16. If $n = \dim\gamma \geq 2$, then one has*

$$\mathfrak{H} = \oplus_{k\geq 0}\Lambda_k. \tag{1.11}$$

Theorem 1.21 *Statement (1.11) remains valid in the case, when in $n = 1$ and γ is an interval equipped with a Riemannian metric.*

It appears, that in the case, when $n = 1$ and γ is a topological circle, statement (1.11) is not true. To state its version in this special case, let us introduce the following notations. Let $d \in \mathbb{Z}_{\geq 0}$, $H_{d,h}$ be functions on the phase cylinder Π given by formula (1.9), and let Λ_d be their space, see (1.9). Let us consider that the length of the curve γ is equal to 2π, rescaling the metric by constant factor. Set

$$\Lambda_{d,0} := \{H_{d,h} \in \Lambda_d \mid \int_0^{2\pi} h(s)ds = 0\}.$$

For every odd polynomial vanishing at zero with derivative,

$$P(y) = \sum_{j=1}^{k} a_j y^{2j+1}, \tag{1.12}$$

set

$$\widetilde{P}(x) := x^{-\frac{1}{2}} P(x^{\frac{1}{2}}) = \sum_{j=1}^{k} a_j x^j. \tag{1.13}$$

Theorem 1.22 *Let $n = 1$ and γ be a topological circle equipped with a Riemannian metric. The Lie algebra $\mathfrak{H}$ generated by the functions $H_{0,f}$, $f \in C^\infty(\gamma)$, see (1.9), is*

$$\mathfrak{G}_{glob} := \Lambda_1 \oplus (\oplus_{d \in 2\mathbb{Z}_{\geq 0}} \Lambda_d) \oplus (\oplus_{d \in 2\mathbb{Z}_{\geq 1}+1} \Lambda_{d,0}) \oplus \Psi, \tag{1.14}$$

$$\Psi := \{\frac{P(y)}{\sqrt{1+y^2}} \mid P(y) \textit{ is a polynomial as in (1.12) with } \widetilde{P}'(-1) = 0\}. \tag{1.15}$$

Theorems 1.16, 1.20, 1.21, 1.22 will be proved in Sections 2 (for $n = 1$) and 3 (for $n \geq 2$).

1.5 Case of hypersurfaces in Riemannian manifolds

Let M be a complete Riemannian manifold. Let $\gamma \subset M$ be a closed strictly convex hypersurface bounding a domain $\Omega \subset M$ homeomorphic to a ball. Let for every geodesic Γ lying in small neighborhood $U = U(\overline{\Omega}) \subset M$ the intersection $\Gamma \cap \Omega$ be either empty, or an interval bounded by two points in $\partial\Omega$. The space of geodesics intersecting Ω will be called the *phase cylinder* and denoted by Π. The billiard ball map $\mathcal{T}_\gamma$ of reflection from γ acts on the space of oriented geodesics in the same way, as in Subsection 1.1. This

action is symplectic with respect to the standard symplectic form on the space of oriented geodesics that is given by symplectic reduction (Melrose construction, see [1, 2, 6, 7, 10, 16]). The phase cylinder Π is symplectomorphic to the unit ball bundle $T_{<1}\gamma$ equipped with the standard symplectic form, as in Subsection 1.4.

Let us repeat Perline's thin film billiard construction. For every C^∞-smooth function $f : \gamma \to \mathbb{R}$ consider the family of hypersurfaces γ_ε consisting of the points $\gamma_\varepsilon(x)$ defined as follows. For every $x \in \gamma$ consider the geodesic $\Gamma_N(x)$ through the point x that is orthogonal to γ and directed out of Ω. The point $\gamma_\varepsilon(x)$ is obtained from the point x by shift of (signed) distance $\varepsilon f(x)$ along the geodesic $\Gamma_N(x)$; one has $\gamma_\varepsilon(x) \notin \overline{\Omega}$, if $f(x) > 0$, and $\gamma_\varepsilon(x) \in \Omega$, if $f(x) < 0$.

Theorem 1.23 *Consider the following compositional ratio and its derivative*

$$\Delta\mathcal{T}_\varepsilon = \Delta\mathcal{T}_{\varepsilon,f} := \mathcal{T}_{\gamma_\varepsilon}^{-1} \circ \mathcal{T}_\gamma, \quad v_f := \frac{d\Delta\mathcal{T}_{\varepsilon,f}}{d\varepsilon}|_{\varepsilon=0}.$$

The derivative v_f is a Hamiltonian vector field on the phase cylinder $\Pi \simeq T_{<1}\gamma$ with the same Hamiltonian function $H_f(x,w) = -2\sqrt{1-||w||^2}f(x)$, as in (1.5).

Theeorem 1.23 follows easily from its Euclidean version due to R.Perline (Theorem 1.13 in Subsection 1.4).

We will deal not only with the global case, when γ is a closed hypersurface, as above, but also with the case, when γ is a germ of hypersurface.

Theorem 1.24 *The statements of Theorem 1.5 and Corollary 1.7 hold for any hypersurface γ as above and for any germ of hypersurface in any Riemannian manifold.*

Proofs of Theorems 1.23 and 1.24 will be given in Section 5.

1.6 Historical remarks and an open problem

In 1999 R.Peirone studied dynamics of billiard in thin film formed by a hypersurface γ and its given deformation γ_ε with small ε. He proved the following transitivity result for small ε: *for every two points $p_1, p_2 \in \gamma$ there exists a billiard orbit starting at p_0 that lands at p_1 after sufficiently many reflections* [12]. A series of results on dynamics in thin film billiard, including results mentioned in Subsection 1.1 (calculation of the vector field v_f and its Hamiltonian function), together with relation to geodesic flow were obtained

in [13]. For other results and open problems, e.g., on relations to integrable PDE's, see [13, sections 8, 9] and references to [13].

Corollary 1.7 states that the C^∞-closure of the *pseudogroup* generated by reflections $\mathcal{T}_{\gamma_\varepsilon} = \mathcal{T}_{\varepsilon,f}$, $0 \le \varepsilon < \delta(f)$, and their inverses contains the whole pseudogroup of Π-Hamiltonian diffeomorphisms between domains in the phase cylinder $\Pi = \Pi_\gamma$.

Open Problem. Is it true that for every closed strictly convex hypersurface $\gamma \subset \mathbb{R}^{n+1}$ the C^∞-closure of the *pseudo-semigroup* generated by reflections from the hypersurface γ and from its small deformations *(without including their inverses)* contains the whole pseudogroup of Hamiltonian diffeomorphisms between domains in the phase cylinder Π?

2 The Lie algebra in the case or curves. Proof of Theorems 1.21, 1.22, 1.16, 1.1, 1.8

In the present section we consider the case, when γ is a connected curve equipped with a Riemannian metric. We prove Theorems 1.21, 1.22 and Theorem 1.16 for curves. To do this, first in Subsection 2.1 we calculate Poisson brackets of functions of type $H_{d,h}$ from (1.9). Then we treat separately two cases, when γ is respectively either an interval (Subsection 2.2), or a circle (Subsections 2.3, 2.4), and prove Theorems 1.21, 1.22, 1.16. Then we deduce Theorems 1.1, 1.8, 1.10 in Subsection 2.5.

2.1 Calculation of Poisson brackets

We work in the space $T_{<1}\gamma = \gamma \times (-1,1)$ equipped with coordinates (s,w). Here s is the natural parameter of the curve γ, and $w \in (-1,1)$ is the coordinate of tangent vectors to γ with respect to the basic vector $\frac{\partial}{\partial s}$. We identify a point of the curve γ with the corresponding parameter s. Recall that the canonical symplectic structure of $T_{<1}\gamma$ is the standard symplectic structure $dw \wedge ds$. Therefore the Poisson bracket of two functions F and G is equal to

$$\{F,G\} = \frac{\partial F}{\partial w}\frac{\partial G}{\partial s} - \frac{\partial F}{\partial s}\frac{\partial G}{\partial w}. \tag{2.1}$$

We write formulas for Poisson brackets of functions from (1.9) in the coordinates

$$(s,y), \quad y = \frac{w}{\sqrt{1-w^2}},$$

in which they take simpler forms. Recall that for every $d \in \mathbb{Z}_{\geq 0}$ and every function $h(s)$ we set

$$H_{d,h}(s,y) := \frac{y^d}{\sqrt{1+y^2}} h(s), \quad H_{-1,h}(s,y) := 0,$$

see (1.9), and for every function $h(s)$ on γ the vector field v_h on $T_{<1}\gamma$ is Hamiltonian with the Hamiltonian function $-2H_{0,h} = H_{0,-2h}$, see (1.7).

Proposition 2.1 *For every $d, k \in \mathbb{Z}_{\geq 0}$ and any two functions $f(s)$, $g(s)$ one has*

$$\{H_{d,f}, H_{k,g}\} = H_{d+k-1, dfg'-kf'g} + H_{d+k+1,(d-1)fg'-(k-1)f'g}. \tag{2.2}$$

Proof For every $m \in \mathbb{Z}_{\geq 0}$ one has

$$\frac{y^m}{\sqrt{1+y^2}} = \frac{w^m}{(\sqrt{1-w^2})^{m-1}},$$

$$\frac{\partial}{\partial w}\left(\frac{y^m}{\sqrt{1+y^2}}\right) = \frac{mw^{m-1}}{(\sqrt{1-w^2})^{m-1}} + \frac{(m-1)w^{m+1}}{(\sqrt{1-w^2})^{m+1}} = my^{m-1} + (m-1)y^{m+1}.$$

Substituting the latter expression to (2.1) yields

$$\{H_{d,f}, H_{k,g}\} = \frac{(dy^{d-1} + (d-1)y^{d+1})y^k fg' - (ky^{k-1} + (k-1)y^{k+1})y^d gf'}{\sqrt{1+y^2}}.$$

This implies (2.2). □

2.2 Case, when γ is an interval. Proof of Theorems 1.21, 1.16, 1.10

Recall that for every $d \in \mathbb{Z}_{\geq 0}$ Λ_d denotes the vector space of functions of the type $H_{d,f}$, see (1.9), where $f(s)$ runs through all the C^∞-smooth functions in one variable. Let

$$\pi_k : \oplus_{d=0}^{+\infty} \Lambda_d \to \Lambda_k$$

denote the projection to the k-th component.

Proposition 2.2 *One has*

$$\{\Lambda_0, \Lambda_0\} = \Lambda_1, \quad \{\Lambda_1, \Lambda_1\} \subset \Lambda_1, \tag{2.3}$$

$$\{\Lambda_d, \Lambda_k\} \subset \Lambda_{d+k-1} \oplus \Lambda_{d+k+1} \text{ whenever } (d,k) \neq (1,1), (0,0), \tag{2.4}$$

$$\pi_{k+1}(\{\Lambda_0, \Lambda_k\}) = \Lambda_{k+1} \text{ for every } k \geq 1. \tag{2.5}$$

Proof Inclusion (2.4) and the right inclusion in (2.3) follow immediately from (2.2). Let us prove the left formula in (2.3). One has

$$\{H_{0,f}, H_{0,g}\} = H_{1,f'g-g'f}, \tag{2.6}$$

by (2.2). It is clear that each function $\eta(s)$ can be represented by an expression $f'g - g'f$, since the functions in question are defined on an interval. For example, one can take $f = \int_{s_0}^{s} \eta(\tau)d\tau$ and $g \equiv 1$. This proves the left formula in (2.3). The proof of statement (2.5) is analogous. □

Proof of Theorem 1.21. The Hamiltonians of the vector fields v_f are the functions $-2H_{0,f}$. The Lie algebra $\mathfrak{H}$ generated by them coincides with $\oplus_{k=0}^{\infty}\Lambda_k$, by Proposition 2.2. This proves Theorem 1.21. □

Proof of Theorem 1.16 for γ being an interval. Each function from the direct sum $\mathfrak{H} = \oplus_{k=0}^{\infty}\Lambda_k$ is $\frac{1}{\sqrt{1+y^2}}$ times a polynomial in y with coefficients depending on s. The latter polynomials include all the polynomials in (s, y), which are C^∞-dense in the space of C^∞ functions in $(s, y) \in \gamma \times \mathbb{R}$ (Weierstrass Theorem). Therefore, $\mathfrak{H}$ is also dense. Theorem 1.16 is proved in the case, when γ is an interval. □

2.3 Case of closed curve. Proof of Theorem 1.22

Let $H_{d,h}$ be functions given by formula (1.9); now $h(s)$ being C^∞-smooth functions on the circle equipped with the natural length parameter s. Recall that we consider that its length is equal to 2π, rescaling the metric by constant factor. Let Λ_d be the vector space of all the functions $H_{d,h}$. Set

$$\Lambda_{d,0} := \{H_{d,h} \in \Lambda_d \mid \int_0^{2\pi} h(s)ds = 0\}.$$

For every C^∞-smooth function $h : S^1 = \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{R}$ set

$$\widehat{h} := \int_0^{2\pi} h(s)ds.$$

In the proof of Theorem 1.22 we use the following four propositions.

Proposition 2.3 *For every $d, k \in \mathbb{Z}_{\geq 0}$ and every pair of smooth functions $f(s)$ and $g(s)$ on the circle one has*

$$\{H_{d,f}, H_{k,g}\} = H_{d+k-1,h_{d+k-1}(s)} + H_{d+k+1,h_{d+k+1}(s)},$$

$$(d + k - 2)\widehat{h}_{d+k-1} = (d + k)\widehat{h}_{d+k+1}. \tag{2.7}$$

Proof The first formula in (2.7) holds with

$$h_{d+k-1} = (d+k)fg' - k(fg)',\ h_{d+k+1} = (d+k-2)fg' - (k-1)(fg)', \quad (2.8)$$

by (2.2). This together with the fact that the derivative $(fg)'$ has zero average implies that the ratio of averages of the functions $h_{d+k\mp 1}$ is equal to $\frac{d+k}{d+k-2}$ and proves (2.7). □

It is clear that the Lie algebra $\mathfrak{H}$ is contained in the direct sum of the subspaces Λ_j, by (2.2). Recall that for every $j \in \mathbb{Z}_{\geq 0}$ by π_j we denote the projection of the latter direct sum to the j-th component Λ_j.

Proposition 2.4 *For every $d, k \in \mathbb{Z}_{\geq 0}$ one has*

$$\{\Lambda_d, \Lambda_k\} \subset \Lambda_{d+k-1} \oplus \Lambda_{d+k+1}, \quad (2.9)$$

$$\pi_{d+k+1}(\{\Lambda_d, \Lambda_k\}) = \pi_{d+k+1}(\{\Lambda_{d,0}, \Lambda_{k,0}\}) = \begin{cases} 0 \ \textit{if}\ d = k = 1, \\ \Lambda_{d+k+1},\ \textit{if}\ d + k \neq 2, \\ \Lambda_{3,0},\ \textit{if}\ \{d, k\} = \{0, 2\}. \end{cases} \quad (2.10)$$

In particular,

$$\{\Lambda_0, \Lambda_0\} = \Lambda_1. \quad (2.11)$$

Proof For $d = k = 1$ formula (2.10) follows from (2.2). For $d = 0$, $k = 2$ one has $\pi_3(\{\Lambda_0, \Lambda_2\}) \subset \Lambda_{3,0}$, by (2.7): the left-hand side in (2.7) vanishes, hence, $\widehat{h}_{d+k+1} = \widehat{h}_3 = 0$. Let us prove that in fact, the latter inclusion is equality and moreover,

$$\pi_3(\{\Lambda_{0,0}, \Lambda_{2,0}\}) = \Lambda_{3,0}. \quad (2.12)$$

Indeed, for every two functions f and g on the circle one has

$$\{H_{0,f}, H_{2,g}\} = H_{1,-2f'g} + H_{3,-(fg)'},$$

by (2.2). It is clear that every function h on the circle with zero average is a derivative of some function $-f$ on the circle (we choose f with zero average). Hence, $h = -f' = (fg)'$ with $g \equiv 1$. This already implies the formula $\pi_3(\{\Lambda_0, \Lambda_2\}) = \Lambda_{3,0}$, but not the above formula (2.12): the function $g \equiv 1$ does not have zero average. To prove (2.12), let us show that every function f with zero average can be represented as a sum $\sum_{j=1}^4 f_j g_j$ with f_j and g_j being smooth functions of zero average. Indeed, f is the sum of a linear combination $f_{=1}(s) = ae^{is} + \bar{a}e^{-is}$, $a \in \mathbb{C}$, and a Fourier series $f_{\geq 2}$

containing only e^{ins} with $|n| \geq 2$. It is clear that $f_{=1}(s) = e^{3is}(e^{-3is} f_{=1}(s))$ and $f_{\geq 2} = e^{is}(e^{-is} f_{\geq 2})$, and the latter are products of two complex functions with zero average. Their real parts are obviously sums of pairs of such products. Therefore, f can be represented as a sum of four such products. This together with the above discussion implies that for every function h with zero average the function $H_{3,h}$ is the π_3-projection of a sum of four Poisson brackets $\{H_{0,f_j}, H_{2,g_j}\}$ with f_j, g_j being of zero average. This proves (2.12) and the third formula in (2.10).

Let us now treat the remaining middle case: $d + k \neq 2$. To do this, it suffices to show that every smooth function $h(s)$ on the circle can be represented as a finite sum

$$h = \sum_{l=1}^{N} h_{d+k+1,l}, \ h_{d+k+1,l} := (d-1) f_l g_l' - (k-1) f_l' g_l, \tag{2.13}$$

see (2.7) and (2.2), where $f_l(s)$ and $g_l(s)$ are smooth functions on the circle with zero average. Moreover, it suffices to prove the same statement for complex-valued functions. Indeed, if (2.13) holds for a complex function $h(s)$ and a finite collection of pairs $(f_l(s), g_l(s))$ of complex functions, $l = 1, \ldots, N$, then the similar equality holds for the function $\operatorname{Re} h(s)$ and the collection of pairs $(\operatorname{Re} f_l, \operatorname{Re} g_l)$, $(-\operatorname{Im} f_l, \operatorname{Im} g_l)$ taken for all l. Let, say, $d \neq 1$. Let us write a complex function $h(s)$ as a Fourier series

$$h(s) = \sum_{n \in \mathbb{Z}} a_n e^{ins}.$$

Set

$$f_1(s) = e^{is}, \ g_1(s) = \sum_{n \in \mathbb{Z}_{\neq 1}} b_n e^{i(n-1)s}.$$

Set $h_{d+k+1,1} := (d-1) f_1 g_1' - (k-1) f_1' g_1$. One has

$$h(s) - h_{d+k+1,1}(s) = a_1 e^{is} + \sum_{n \neq 1} (a_n - i b_n((d-1)(n-1) - (k-1))) e^{ins}. \tag{2.14}$$

We would like to make the above difference zero. For each individual $n \neq 1$ one can solve the equation

$$a_n - i b_n((d-1)(n-1) - (k-1)) = 0,$$

provided that $(d-1)(n-1) \neq k-1$, i.e., $n \neq n(d,k) := \frac{k-1}{d-1} + 1$. Take b_n found from the above equation for all $n \neq 1, n(d,k)$. They yield a converging

and C^∞-smooth Fourier series

$$g_1(s) = \sum_{n\neq 1, n(d,k)} b_n e^{i(n-1)s},$$

since so is $h(s) = \sum_{n\in\mathbb{Z}} a_n e^{ins}$ and $b_n = o(a_n)$, as $n \to \infty$. The corresponding function $h_{d+k+1,1}$, see (2.13), satisfies the equality

$$h(s) - h_{d+k+1,1}(s) = a_1 e^{is} + a_{n(d,k)} e^{isn(d,k)}. \tag{2.15}$$

Now we set $f_2(s) = e^{(p+1)is}$ with some $p \in \mathbb{Z} \setminus \{0, -1, n(d,k) - 1\}$, and we would like to find a function

$$g_2(s) = c_1 e^{-pis} + c_2 e^{i(n(d,k)-(p+1))s}$$

such that $h = h_{d+k+1,1} + h_{d+k+1,2}$, see (2.13). The latter equation is equivalent to the equation

$$a_1 e^{is} + a_{n(d,k)} e^{isn(d,k)} = (d-1) f_2(s) g_2'(s) - (k-1) f_2'(s) g_2(s), \tag{2.16}$$

by (2.15). Its right-hand side divided by i equals $c_1(-p(d-1) - (p+1)(k-1))e^{is} + c_2((d-1)(n(d,k) - (p+1)) - (k-1)(p+1))e^{isn(d,k)}$. Therefore, one can find constant coefficients c_1, c_2 in the definition of the function g_2 such that equation (2.16) holds, if

$$(n(d,k)-(p+1))(d-1)-(p+1)(k-1) \neq 0,\ p(d-1)+(p+1)(k-1) \neq 0. \tag{2.17}$$

The left-hand sides of inequalities (2.17) are linear non-homogeneous functions in p with coefficients at p being equal to $\mp(d+k-2) \neq 0$. Hence, choosing appropriate $p \in \mathbb{Z} \setminus \{0, -1, n(d,k) - 1\}$ one can achieve that inequalities (2.17) hold and hence, equation (2.16) can be solved in c_1, c_2. Finally, we have solved equation (2.13) with an arbitrary complex function h and $N = 2$, in complex functions f_l, g_l, $l = 1, 2$ with zero averages. This together with the above discussion finishes the proof of statement (2.10). Statement (2.11) follows from the second statement in (2.10). □

Proposition 2.5 *The Lie algebra $\mathfrak{H}$ contains the direct sum*

$$\mathfrak{G}_{glob,0} := \Lambda_1 \oplus (\oplus_{d\in 2\mathbb{Z}_{\geq 0}} \Lambda_d) \oplus (\oplus_{d\in 2\mathbb{Z}_{\geq 1}+1} \Lambda_{d,0}).$$

Proof The algebra $\mathfrak{H}$ contains Λ_0 and $\Lambda_1 = \{\Lambda_0, \Lambda_0\}$, see (2.11). It also contains Λ_2, by the latter statement and since $\{\Lambda_0, \Lambda_1\} \subset \Lambda_0 \oplus \Lambda_2$, see (2.9),

and $\pi_2(\{\Lambda_0, \Lambda_1\}) = \Lambda_2$, by (2.10). Hence, $\mathfrak{H} \supset \oplus_{j=0}^2 \Lambda_j$. Analogously one has $\mathfrak{H} \supset \Lambda_{3,0}$, by the latter statement, and (2.9), (2.10) applied to $(d,k) = (0,2)$. Hence, $\mathfrak{H} \supset (\oplus_{j=0}^2 \Lambda_j) \oplus \Lambda_{3,0}$. One has $\mathfrak{H} \supset \Lambda_4$, by the latter statement and (2.9), (2.10) applied to $(d,k) = (0,3)$. Hence, $\mathfrak{H} \supset (\oplus_{0 \le j \le 4, j \ne 3} \Lambda_j) \oplus \Lambda_{3,0}$. Let us show that $\mathfrak{H} \supset \Lambda_5$. Indeed, the bracket $\{\Lambda_0, \Lambda_4\}$ is contained in $\Lambda_3 \oplus \Lambda_5$, see (2.9). For every functions f and g on the circle one has $\{H_{0,f}, H_{4,g}\} = H_{3,h_3} + H_{5,h_5}$, where the ratio of averages of the functions h_3 and h_5 is equal to 2, see (2.7). Therefore, if the average of the function h_5 vanishes, then so does the average of the function h_3. This implies that the subspace of those elements in $\{\Lambda_0, \Lambda_4\}$ whose projections to Λ_5 have zero averages coincides with the subspace with the analogous property for the projection π_3. This together with the above statement implies that $\mathfrak{H}$ contains $\Lambda_{5,0}$. Applying the above argument successively to Poisson brackets $\{\Lambda_0, \Lambda_n\}$, $n \ge 5$, we get the statement of Proposition 2.5. $\square$

Proposition 2.6 *The Lie algebra $\mathfrak{H}$ is the direct sum of the subspace $\mathfrak{G}_{glob,0}$ and a vector subspace Ψ in*

$$\mathcal{P} = \{\frac{P(y)}{\sqrt{1+y^2}} \mid P \text{ is an odd polynomial }, \ P'(0) = 0\}.$$

The corresponding subspace $\sqrt{1+y^2}\Psi$ in the space of polynomials is generated by the polynomials

$$R_j(y) := jy^{2j-1} + (j-1)y^{2j+1}, \ j \in \mathbb{N}, \ j \ge 2. \tag{2.18}$$

Proof The direct sum $\oplus_{j \ge 0} \Lambda_j \supset \mathfrak{H}$ is the direct sum of the spaces $\mathfrak{G}_{glob,0}$ and $\mathcal{P}$, by definition. This together with Proposition 2.5 implies that $\mathfrak{H}$ is the direct sum of the subspace $\mathfrak{G}_{glob,0}$ and a subspace $\Psi \subset \mathcal{P}$. Let us describe the subspace Ψ. To do thus, consider the projection

$$\pi_{odd>1} : \oplus_{j \ge 0} \Lambda_j \to \oplus_{j \in 2\mathbb{Z}_{\ge 1}+1} \Lambda_j.$$

Claim 1. *The projection $\pi_{odd>1}\mathfrak{H}$ lies in $\mathfrak{H}$. It is spanned as a vector space over $\mathbb{R}$ by the subspace $\Lambda_{3,0}$ and some Poisson brackets $\{H_{d,a(s)}, H_{k,b(s)}\}$ with $d+k \ge 4$ being even. All the above brackets with all the functions $a(s)$, $b(s)$ with zero average lie in $\pi_{odd>1}\mathfrak{H}$.*

Proof The inclusion $\pi_{odd>1}\mathfrak{H} \subset \mathfrak{H}$ follows from the fact that $\pi_{odd>1}$ is the projection along the vector subspace $\Lambda_1 \oplus (\oplus_{j \in 2\mathbb{Z}_{\ge 0}} \Lambda_j) \subset \mathfrak{G}_{glob,0} \subset \mathfrak{H}$. Each element of the Lie algebra $\mathfrak{H}$ is represented as a sum of a vector in Λ_0 and a

linear combination of Poisson brackets $\{H_{d,a}, H_{k,b}\}$, by definition and (2.2). The latter Poisson brackets lie in $\Lambda_{d+k-1} \oplus \Lambda_{d+k+1}$, by (2.9), and thus, have components of the same parity $d+k+1 (\mathrm{mod}\, 2)$. Note that if $d+k-1=1$, then the above bracket lies in $\Lambda_1 \oplus \Lambda_{3,0} \subset \mathfrak{G}_{glob,0} \subset \mathfrak{H}$, by (2.10), and its $\pi_{odd>1}$-projection lies in $\Lambda_{3,0}$. The two last statements together imply the second statement of the claim. If $a(s)$ and $b(s)$ have zero average, then $H_{d,a}, H_{k,b} \in \mathfrak{G}_{glob,0} \subset \mathfrak{H}$, thus, $\{H_{d,a}, H_{k,b}\} \in \mathfrak{H}$. Hence, the latter bracket lies in $\pi_{odd>1}\mathfrak{H}$, if $d+k$ is even and no less than 4. This proves the claim. $\square$

Taking projection π_Ψ of a vector $w \in \mathfrak{H}$ to Ψ consists of first taking its projection

$$\pi_{odd>1} w = \sum_{j=1}^{k} H_{2j+1, f_j(s)} = \frac{1}{\sqrt{1+y^2}} \sum_{j=1}^{k} f_j(s) y^{2j+1}$$

and then replacing each $f_j(s)$ in the above right-hand side by its average $\widehat{f_j}$:

$$\pi_\Psi w = \frac{1}{\sqrt{1+y^2}} P_w(y), \;\; P_w(y) = \sum_{j=1}^{k} \widehat{f_j} y^{2j+1}.$$

If $w = \{H_{d,a}, H_{k,b}\}$ with $d+k = 2j \geq 4$, then $P_w(y) = cR_j(y)$, see (2.18), $c \in \mathbb{R}$, which follows from (2.7). This together with Claim 1 implies that the vector space $\sqrt{1+y^2}\Psi$ is contained in the vector space spanned by the polynomials R_j. Now it remains to prove the converse: each R_j is contained in Ψ. To this end, we have to find the above functions a and b with zero average so that $P_w(y) \neq 0$, i.e., so that the above constant factor c be non-zero. Let $b(s)$ be an arbitrary smooth non-constant function on the circle with zero average. Set $a(s) = b'(s)$. Let $d+k = 2j \geq 4$. Then

$$w = \{H_{d,a}, H_{k,b}\} = \frac{1}{\sqrt{1+y^2}} (f_{j-1}(s) y^{2j-1} + f_j(s) y^{2j+1}),$$

$$f_{j-1} = 2j(b')^2 - k(b'b)',$$

by (2.8). The function $f_{j-1}(s)$ has positive average, since so does its first term, while its second term has zero average. Therefore, $P_w = cR_j$, $c > 0$. Proposition 2.6 is proved. $\square$

Lemma 2.7 *The vector subspace generated by the polynomials R_j from (2.18) coincides with the space of odd polynomials $P(y)$ with $P'(0) = 0$ such that the corresponding polynomial $\widetilde{P}(x) = x^{-\frac{1}{2}} P(x^{\frac{1}{2}})$ has vanishing derivative at -1.*

As it is shown below, the lemma is implied by the following proposition.

Proposition 2.8 *An odd polynomial*

$$P_k(y) = \sum_{j=1}^{k} a_j y^{2j+1} \tag{2.19}$$

is a linear combination of polynomials $R_\ell(y)$, see (2.18), if and only if

$$\sum_{j=1}^{k} (-1)^j j a_j = 0. \tag{2.20}$$

Proof The polynomials (2.18) obviously satisfy (2.20). In the space of odd polynomials $P(y)$ with $P'(0) = 0$ of any given degree $d \geq 5$ equation (2.20) defines a hyperplane. The polynomials (2.18) of degree no greater than d also generate a hyperplane there. Hence, these two hyperplanes coincide. The proposition is proved. □

Proof of Lemma 2.7. For every odd polynomial $P(y)$ as in (2.19) one has

$$\widetilde{P}(x) = \sum_{j=1}^{k} a_j x^j.$$

Hence, equation (2.20) is equivalent to the equation $\widetilde{P}'(-1) = 0$. Lemma 2.7 is proved. □

Proof of Theorem 1.22. Theorem 1.22 follows from Proposition 2.6 and Lemma 2.7. □

2.4 Proof of Theorem 1.16 for closed curve

Theorem 1.16 is implied by Theorem 1.22 proved above and the following lemma.

Lemma 2.9 *The Lie algebra $\mathfrak{G}_{glob}$, see (1.14), is C^∞-dense in the space of smooth functions on $T_{<1}\gamma \simeq \gamma_s \times \mathbb{R}_y$.*

Proof Let us multiply each function from $\mathfrak{G}_{glob}$ by $\sqrt{1+y^2}$; then each function becomes a polynomial in y with coefficients being smooth functions on a circle. All the polynomials in y with coefficients as above are C^∞-dense in the space of functions in $(s, y) \in \gamma \times \mathbb{R}$, by Weierstrass Theorem. The

polynomials realized by functions from $\mathfrak{G}_{glob}$ in the above way are exactly the polynomials that are represented in unique way as sums of at most four polynomials of the following types:

1) any polynomial of degree at most 2;

2) any even polynomial containing only monomials of degree at least 4;

3) any odd polynomial of type $P(y;s) = \sum_{j=1}^{k} a_j(s)y^{2j+1}$ with coefficients $a_j(s)$ being of zero average;

4) any odd polynomial of type $P(y) = \sum_{j=1}^{k} b_j y^{2j+1}$ with constant coefficients b_j and $\widetilde{P}'(-1) = 0$.

For the proof of Lemma 2.9 it suffices to show that the odd polynomials of type 4) are C^∞-dense in the space of odd polynomials in y with constant coefficients and vanishing derivative at 0.

Take an arbitrary odd polynomial of type $Q(y) = \sum_{j=1}^{k} b_j y^{2j+1}$. The polynomial $\widetilde{Q}(x) := x^{-\frac{1}{2}}Q(x^{\frac{1}{2}}) = \sum_{j=1}^{k} b_j x^k$ can be approximated in the topology of uniform convergence with derivatives on segments $[0, A]$ with A arbitrarily large by polynomials $\widetilde{R}(x)$ with $\widetilde{R}(0) = 0$ and $\widetilde{R}'(-1) = 0$. Indeed, let us extend the restriction $\widetilde{Q}|_{\{x\geq 0\}}$ to a C^∞-smooth function on the semi-interval $[-1, +\infty)$ with vanishing derivative at -1. Thus extended function can be approximated by polynomials $\widetilde{H}_n(x)$. One can normalize the above polynomials $\widetilde{H}_n$ to vanish at 0 and to have zero derivative at -1 by adding a small linear non-homogeneous function $a_n x + b_n$. Then the corresponding polynomials $H_n(y) := y\widetilde{H}_n(y^2)$ are of type 4) and approximate $Q(y)$. This together with the above discussion proves Lemma 2.9. This finishes the proof of Theorem 1.16 for curves. □

2.5 Proof of Theorems 1.1, 1.8 and 1.10 for $n = 1$

Theorems 1.1 and 1.8 follow from Theorem 1.16.

In the case, when γ is a germ of curve (a local curve parametrized by an interval), the bundle $T_{<1}\gamma$ is a contractible space identified with a rectangle in the coordinates (s, w). Therefore, each symplectic vector field on $T_{<1}\gamma$ is Hamiltonian. This together with Theorem 1.8 implies density of the Lie algebra generated by the fields v_f in the space of symplectic vector fields. This proves Theorem 1.10.

3 The Lie algebra in higher dimensions. Proof of Proposition 1.15 and Theorems 1.20, 1.16, 1.1

Here we consider the case, when $n \geq 2$ (whenever the contrary is not specified). In Subsection 3.1 we give a formula for Poisson brackets of functions $H_{k,\phi}$ with $\phi \in S^k(T^*\gamma)$ and prove Proposition 1.15. Then in Subsection 3.2 we find the Lie algebra generated by the functions $H_f(x, w)$ and prove Theorem 1.20. In Subsection 3.3 we prove Theorems 1.16 and 1.1.

3.1 Poisson brackets and their calculations for $n \geq 1$. Proof of Proposition 1.15

The results of the present subsection are valid for all $n \in \mathbb{N}$.

Let γ be a n-dimensional Riemannian manifold, $x \in \gamma$. Fix some orthonormal coordinates $z = (z_1, \dots, z_n)$ on $T_x\gamma$. Recall that the *normal coordinates* centered at x on a neighborhood $U = U(x) \subset \gamma$ is its parametrization by an open subset in $\mathbb{R}^n_z \simeq T_x\gamma$ that is given by the exponential mapping: $\exp : T_x\gamma \to \gamma$. It is well-known that in thus constructed normal coordinates z on U the Riemannian metric has the same first jet, as the standard Euclidean metric $dz_1^2 + \dots + dz_n^2$.

Proposition 3.1 *Fix arbitrary normal coordinates $(z_1, \dots, z_n)$ centered at $x_0 \in \gamma$ on a neighborhood $U = U(x_0) \subset \gamma$. For every $x \in U$ and every vector $w \in T_x\gamma$ let $(w_1, \dots, w_n)$ denote its components in the basis $\frac{\partial}{\partial z_j}$ in $T_x\gamma$. This yields global coordinates $(z_1, \dots, z_n; w_1, \dots, w_n)$ on $T\gamma|_U$. At the points of the fiber $T_{x_0}\gamma \subset T\gamma$ the canonical symplectic form $\omega = d\alpha$ coincides with the standard symplectic form $\omega_e := \sum_{j=1}^n dw_j \wedge dz_j$.*

Proof Let α and $\alpha_e := \sum_{j=1}^n w_j dz_j$ denote the Liouville forms (1.4) on $T\gamma$ defined respectively by the metric under question and the standard Euclidean metric $dz_1^2 + \dots + dz_n^2$. Their 1-jets at each point (x_0, w) of the fiber $T_{x_0}\gamma$ coincide, since both metrics have the same 1-jet at x_0. Therefore, $\omega = d\alpha = d\alpha_e = \omega_e$ on $T_{x_0}\gamma$. □

Corollary 3.2 *In the conditions of Proposition 3.1 for every two smooth functions F and G on an open subset in $\pi^{-1}(U) \subset T\gamma$ their Poisson bracket at points of the fiber $T_{x_0}\gamma$ is equal to the standard Poisson bracket in the coordinates (z, w):*

$$\{F, G\} = \frac{dF}{dw}\frac{dG}{dz} - \frac{dF}{dz}\frac{dG}{dw} \quad \text{on } T_{x_0}\gamma. \tag{3.1}$$

Proof of Proposition 1.15. Fix a point $x_0 \in \gamma$ and a system of normal coordinates $(z_1, \dots, z_n)$ centered at x_0. Let $(w_1, \dots, w_n)$ be the corresponding coordinates on tangent spaces to γ introduced above. For every $f, g \in C^\infty(\gamma)$ one has

$$\begin{aligned} &\{f(z)\sqrt{1-||w||^2}, g(z)\sqrt{1-||w||^2}\}|_{z=0} \\ &= -\sqrt{1-||w||^2}\sum_j \left(f\frac{\partial g}{\partial z_j} - g\frac{\partial f}{\partial z_j}\right)\frac{w_j}{\sqrt{1-||w||^2}} \qquad (3.2) \\ &= (gdf - fdg)(w) = (2gdf - d(gf))(w) : \end{aligned}$$

the derivative in z of the function $\sqrt{1-||w||^2}$ vanishes at the points of the fiber $\{z = 0\}$, since the metric in question has the same first jet at 0, as the Euclidean metric (normality of coordinates). Thus, the latter Poisson bracket is the function on $T_{<1}\gamma$ given by a 1-form.

Claim 1. *The Poisson bracket (3.1) of any two 1-forms $\eta_1(w)$, $\eta_2(w)$ considered as functions on $T\gamma$ is also a 1-form. The dual Poisson bracket on vector fields on γ induced via the duality isomorphism $T^*\gamma \to T\gamma$ given by the metric is the usual Lie bracket.*

Proof The result of taking Poisson bracket (3.1) of two 1-forms in normal chart at the fiber $T_{x_0}\gamma$ is obviously a linear functional of w. This is true for every $x_0 \in \gamma$ and the corresponding normal chart. Thus, the bracket is linear on fibers, and hence, is given by a 1-form. The fact that the duality given by the metric transforms the Poisson bracket on 1-forms to the Lie bracket on the vector fields follows from (3.2). □

Claim 1 already implies that the space of functions on $T_{<1}\gamma$ given by 1-forms is a Lie algebra dual to the usual Lie algebra of vector fields.

Now it remains to show that each 1-form is a sum of at most $2n+1$ Poisson brackets (3.2). To do this, we use the next proposition.

Proposition 3.3 *On every C^∞-smooth manifold γ each smooth 1-form η can be represented as a finite sum $\sum_{\ell=1}^{2n} g_\ell df_\ell$, where f_ℓ, g_ℓ are smooth functions on γ.*

Proof Consider γ as an embedded submanifold in $\mathbb{R}^{2n}_{s_1,\dots,s_{2n}}$ (Whitney Theorem). We would like to show that the form η can written as the restriction to γ of some differential 1-form $\widetilde{\eta}$ on $\mathbb{R}^{2n}$. To do this, fix a tubular neighborhood $\Gamma^\delta \subset \mathbb{R}^{2n}$ given by the Tubular Neighborhood Theorem. Here $\delta = \delta(x) > 0$ is a smooth function,

$$\Gamma^\delta := \{p \in \mathbb{R}^{2n} \mid \operatorname{dist}(p, x) < \delta(x) \text{ for some } x \in \gamma\}. \qquad (3.3)$$

Each point $p \in \Gamma^\delta$ has the unique closest point $x = \pi_\delta(p) \in \gamma$, and the projection $\pi_\delta : \Gamma^\delta \to \gamma$ is a submersion. The projection π_δ allows to extend the form η to the pullback form $\hat{\eta} := \pi_\delta^* \eta$ on Γ^δ, which coincides with η on γ. Take now an arbitrary smooth bump function β on Γ^δ that is identically equal to 1 on γ and vanishes on a neighborhood of the boundary $\partial\Gamma^\delta$. The 1-form $\widetilde{\eta} := \beta\hat{\eta}$ extended by zero outside the tubular neighborhood Γ^δ is a global smooth 1-form on $\mathbb{R}^{2d}$ whose restriction to γ coincides with η. Taking its coordinate representation $\widetilde{\eta} = \sum_{\ell=1}^{2n} \widetilde{\eta}_\ell ds_\ell$ and putting $g_\ell = \widetilde{\eta}_\ell$, $f_\ell = s_\ell$ yields the statement of the proposition. $\square$

Let now η be an arbitrary smooth 1-form on γ. Let f_ℓ, g_ℓ be the same, as in Proposition 3.3. Set $g_{2n+1} = 1$, $f_{2n+1} = \sum_{\ell=1}^{2n} g_\ell f_\ell$. Then

$$2\eta = 2\sum_{\ell=1}^{2n} g_\ell df_\ell = \sum_{\ell=1}^{2n}(2g_\ell df_\ell - d(g_\ell f_\ell)) + (2g_{2n+1}df_{2n+1} - d(g_{2n+1}f_{2n+1})).$$

Therefore, 2η, and hence η can be presented as a sum of at most $2n+1$ Poisson brackets (3.2). This finished the proof of Proposition 1.15. $\square$

For every $x \in \gamma$ let $\mathrm{Sym}^k(T_x^*\gamma)$ denote the space of symmetric k-linear forms on $T_x\gamma$: each of them sends a collection of k vectors in $T_x\gamma$ to a real number and is symmetric under permutations of vectors. The union of the latter spaces through all $x \in \gamma$ is a vector bundle of tensors. The space of its sections will be denoted by $\mathrm{Sym}^k(T^*\gamma)$. Similarly, by $S^k(T_x^*\gamma)$ we will denote the space of degree k homogeneous polynomials as functions on the vector space $T_x\gamma$. The space $S^k(T^*\gamma)$ introduced before is the space of sections of the vector bundle over γ with fibers $S^k(T_x^*\gamma)$. It is well-known that the mapping $\chi_{sym} : \mathrm{Sym}^k(T_x^*\gamma) \to S^k(T_x^*\gamma)$ sending a k-linear form $\Phi(y_1, \ldots, y_k)$ to the polynomial $\phi(y) := \Phi(y, \ldots, y)$ is an isomorphism, which induces a section space isomorphism $\mathrm{Sym}^k(T^*\gamma) \to S^k(T^*\gamma)$. The notion of covariant derivative of a tensor bundle section $\Phi \in \mathrm{Sym}^k(T^*\gamma)$ is well-known. For every $x \in \gamma$, $\nu \in T_x\gamma$, $\phi \in S^k(T^*\gamma)$ the *covariant derivative* $\nabla_\nu\phi \in S^k(T_x^*\gamma)$ along a vector $\nu \in T_x\gamma$ is

$$\nabla_\nu\phi := \chi_{sym}(\nabla_\nu(\chi_{sym}^{-1}\phi)). \tag{3.4}$$

Remark 3.4 For every $\phi \in S^k(T^*\gamma)$ one has

$$V_k(\phi)(y) := (\nabla_y\phi)(y) \in S^{k+1}(T^*\gamma). \tag{3.5}$$

Let $z = (z_1, \ldots, z_n)$ be normal coordinates on γ centered at x, and let $(w_1, \ldots, w_n)$ be the corresponding coordinates on the fibers of the bundle

$T\gamma$ in the basis $\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_n}$. For every $\nu \in T_x\gamma$ and every $\phi \in S^k(T^*\gamma)$ considered as a polynomial in w with coefficients depending on z the covariant derivative (3.4) is obtained from the polynomial ϕ by replacing its coefficients by their derivatives along the vector ν. This follows from definition, since in normal coordinates the Christoffel symbols vanish.

Convention 3.5 *Everywhere below for every $\phi \in S^k(T^*\gamma)$, $x \in \gamma$ and $\nu \in T_x\gamma$ by $\frac{d_y\phi(y)}{d\nu}$ we denote* **the derivative of the polynomial** *$\phi(y)$ considered as a function of $y \in T_x\gamma$ along the vector ν (treating ν as a tangent vector to $T_x\gamma$ at 0, after identifying the vector space $T_x\gamma$ with its tangent space at 0 by translations). For every function $f \in C^\infty(\gamma)$ by $\frac{d_y\phi(y)}{d\nabla f}$ we denote the above derivative calculated for $\nu = \nabla f(x)$ at each point $x \in \gamma$. Note that the latter derivative lies in $S^{k-1}(T^*\gamma)$, by definition.*

Proposition 3.6 *For every $k, m \in \mathbb{Z}_{\geq 0}$ and every $\phi \in S^k(T^*\gamma)$, $f \in S^m(T^*\gamma)$ the Poisson bracket of the corresponding functions $H_{k,\phi}$ and $H_{m,f}$ from (1.8) (see Convention 1.17) lies in $\Lambda_{k+m-1} \oplus \Lambda_{k+m+1}$. In the case, when $m = 0$, i.e., $f = f(x)$ is a function of x, one has*

$$\{H_{k,\phi}, H_{0,f}\} = \frac{\frac{d_y\phi(y)}{d\nabla f} + (\nabla_y(f\phi))(y) + (k-2)\phi(y) < y, \nabla f >}{\sqrt{1 + ||y||^2}}. \tag{3.6}$$

Here ∇f is the gradient of the function $f(x)$ with respect to the metric of γ, and $\nabla_y(f\phi)$ is the above-defined covariant derivative of the form $f\phi \in S^k(T^*\gamma)$ along the vector y.

Proof Let us prove the first statement of the proposition. Fix an arbitrary $x_0 \in \gamma$. It suffices to show that the restriction to $T_{x_0}\gamma$ of the Poisson bracket is $(\sqrt{1 + ||y||^2})^{-1}$ times a sum of two homogeneous polynomials in y of degrees $k + m - 1$ and $k + m + 1$. Let us pass back to the initial unit ball bundle $T_{<1}\gamma$. Fix some normal coordinates $(z_1, \dots, z_n)$ centered at z_0 and the corresponding coordinates $w = (w_1, \dots, w_n)$ on the fibers. One has

$$H_{k,\phi} = \frac{\phi(w)}{(\sqrt{1 - ||w||^2})^{k-1}}; \tag{3.7}$$

here the squared norm $||w||^2$ is given by the metric and depends on the point $z \in \gamma$. This follows by definition and since

$$||y||^2 = \frac{||w||^2}{1 - ||w||^2}, \ ||y||^2 + 1 = (1 - ||w||^2)^{-1}. \tag{3.8}$$

Let us calculate the Poisson bracket $\{H_{k,\phi}, H_{m,f}\}$ at points in $T_{z_0}\gamma$ by formula (3.1). The partial derivative $\frac{\partial H_{k,\phi}}{dz_j}$ is $(\sqrt{1-||w||^2})^{-(k-1)}$ times a degree k homogeneous polynomial in w obtained from the polynomial $\phi(w)$ by replacing its coefficients by their partial derivatives in z_j. This follows from definition and the fact that $\sqrt{1-||w||^2}$ has zero derivatives in z_j at points of the fiber $T_{x_0}\gamma$. The latter fact follows from normality of the coordinates z_j. The partial derivative $\frac{\partial H_{k,\phi}}{\partial w_j}$ is a sum of a degree $k-1$ homogeneous polynomial in w divided by $(\sqrt{1-||w||^2})^{k-1}$ and a degree $k+1$ homogeneous polynomial in w divided by $(\sqrt{1-||w||^2})^{k+1}$ (Leibniz rule). Finally, the expression (3.1) for the above Poisson bracket is a sum of terms of the two following types: 1) a degree $k+m-1$ homogeneous polynomial in w divided by $(\sqrt{1-||w||^2})^{k+m-2}$; 2) a degree $k+m+1$ homogeneous polynomial in w divided by $(\sqrt{1-||w||^2})^{k+m}$. The terms of types 1) and 2) are homogeneous polynomials in y of degrees respectively $k+m\pm 1$ divided by $\sqrt{1+||y||^2}$, by (3.8). This implies the first statement of the proposition.

Let now $f = f(x)$ be a function of $x \in \gamma$. Let us prove formula (3.6). For $z = 0$ (i.e., $x = x_0$) one has

$$\frac{\partial H_{k,\phi}}{\partial w_j} = (\sqrt{1-||w||^2})^{1-k}\frac{\partial}{\partial w_j}\phi(w) + \frac{(k-1)\phi(w)w_j}{(\sqrt{1-||w||^2})^{k+1}},$$

$$\frac{\partial H_{0,f}}{\partial w_j} = -\frac{w_j}{\sqrt{1-||w||^2}}f, \quad \frac{\partial H_{0,f}}{\partial z_j} = \sqrt{1-||w||^2}\frac{\partial f}{\partial z_j},$$

$$\sum_{j=1}^{n}\frac{\partial H_{k,\phi}}{\partial w_j}\frac{\partial H_{0,f}}{\partial z_j} = (\sqrt{1-||w||^2})^{2-k}\frac{d_w\phi(w)}{d\nabla f} + \frac{(k-1)\phi(w) < \nabla f, w >}{(\sqrt{1-||w||^2})^k}, \tag{3.9}$$

$$\sum_{j=1}^{n}\frac{\partial H_{k,\phi}}{\partial z_j}\frac{\partial H_{0,f}}{\partial w_j} = -\frac{f}{(\sqrt{1-||w||^2})^k}\sum_{j=1}^{n} w_j\frac{\partial \phi(w)}{\partial z_j} = -\frac{(\nabla_w\phi)(w)f}{(\sqrt{1-||w||^2})^k}, \tag{3.10}$$

which follows from Remark 3.4. Substituting formulas (3.9), (3.10) and the formula $< \nabla f, w > \phi(w) + (\nabla_w\phi)(w)f = (\nabla_w(f\phi))(w)$ to (3.1) yields

$$\{H_{k,\phi}, H_{0,f}\} = \frac{\frac{d_w\phi(w)}{d\nabla f}}{(\sqrt{1-||w||^2})^{k-2}}$$

$$+\frac{(k-2)\phi(w) < \nabla f, w >}{(\sqrt{1-||w||^2})^k} + \frac{1}{(\sqrt{1-||w||^2})^k}(\nabla_w(f\phi))(w),$$

which is equivalent to (3.6). Proposition 3.6 is proved. □

Remark 3.7 Formula (3.6) remains valid for $n = 1$ and yields formula (2.2) from Subsection 2.1. Indeed, in this case $\phi = h(s)y^k$, $f = f(s)$. The first term in the numerator in (3.6) is equal to $khf'y^{k-1}$. The second term equals $(hf)'y^{k+1}$. The third term equals $(k-2)hf'y^{k+1}$. Thus, the numerator is equal to $khf'y^{k-1} + ((k-1)hf' + fh')y^{k+1}$. This together with (3.6) yields (2.2).

3.2 The Lie algebra. Proof of Theorem 1.20

Proposition 3.8 *The Lie algebra $\mathfrak{H}$ generated by functions $H_{k,\phi}$ for all $k \in \mathbb{Z}_{\geq 0}$ and $\phi \in S^k(T^*\gamma)$ with respect to the Poisson bracket is contained in $\Lambda := \oplus_{k\geq 0}\Lambda_k$, and the latter direct sum is a Lie algebra.*

Proposition 3.8 follows from Proposition 3.6.

In what follows by π_k we denote the projection $\pi_k : \Lambda \to \Lambda_k$.

We will deal with the linear operators V_k given by (3.5):

$$V_k : S^k(T^*\gamma) \mapsto S^{k+1}(T^*\gamma), \quad \phi(y) \mapsto (\nabla_y \phi)(y).$$

Proposition 3.9 *One has*

$$\pi_{k+1}\{\Lambda_k, \Lambda_0\} = \begin{cases} \Lambda_{k+1} \text{ for } k \neq 2, \\ \frac{1}{\sqrt{1+||y||^2}} V_2(S^2(T^*\gamma)) \text{ for } k = 2. \end{cases} \tag{3.11}$$

Proof The higher order part of a Poisson bracket $\{H_{k,\phi}, H_{0,f}\}$ is equal to

$$\{H_{k,\phi}, H_{0,f}\}_{k+1} = \frac{1}{\sqrt{1+||y||^2}}((k-2)\phi(y) < \nabla f, y > +(\nabla_y(\phi f))(y)), \tag{3.12}$$

by (3.6). If either $f \equiv 1$, or $k = 2$, then the above expression in the brackets is reduced to $(\nabla_y(\phi f))(y) = V_k(\phi f)(y)$. Therefore, each element in $\frac{1}{\sqrt{1+||y||^2}} V_k(S^k(T\gamma))$ is realized by a Poisson bracket, and hence,

$$\frac{1}{\sqrt{1+||y||^2}} V_k(S^k(T\gamma)) \subset \{\Lambda_k, \Lambda_0\}. \tag{3.13}$$

In particular, this implies the statement of the proposition for $k = 2$. To treate the case, when $k \neq 2$, we use the following proposition.

Proposition 3.10 *For every $k \in \mathbb{Z}_{\geq 1}$ the vector subspace in $S^{k+1}(T^*\gamma)$ generated by all the products $\phi(y) < \nabla f, y >= \phi(y)\frac{df}{dy}$ with $\phi \in S^k(T^*\gamma)$ and $f \in C^\infty(\gamma)$ coincides with all of $S^{k+1}(T^*\gamma)$. Moreover, each element in $S^{k+1}(T^*\gamma)$ can be represented as a sum of at most $2n$ products as above.*

Proof Consider γ as an embedded submanifold in $\mathbb{R}^{2n}_{s_1,\dots,s_{2n}}$ (Whitney Theorem); the embedding needs not be isometric.

Claim 3. *Each $\phi \in S^{k+1}(T^*\gamma)$ is the restriction to $T\gamma$ of some $h \in S^{k+1}(T^*\mathbb{R}^{2n})$.*

Proof Let $\delta(x) > 0$ be a smooth function on γ with $\delta_{sup} := \sup \delta \ll 1$ that defines a tubular neighborhood $\Gamma^\delta \subset \mathbb{R}^{2n}$ of the submanifold γ, see (3.3). Let $\pi_\delta : \Gamma^\delta \to \gamma$ be the projection, which is a submersion. Let us extend ϕ to a form $\widetilde{\phi} := \pi_\delta^* \phi \in S^{k+1}(\Gamma^\delta)$ as the projection pullback. Let $\psi : \Gamma^\delta \to \mathbb{R}$ be a bump function that is identically equal to 1 on a neighborhood of the submanifold γ and vanishes on a neighborhood of the boundary $\partial\Gamma^\delta$. Then the form $h := \psi\widetilde{\phi}$ extended as zero outside the tubular neighborhood Γ^δ becomes a well-defined form $h \in S^{k+1}(T^*\mathbb{R}^{2n})$ whose restriction to $T\gamma$ coincides with ϕ. □

Let us consider the standard trivialization of the tangent bundle $T\mathbb{R}^{2n}$ given by translations to $T_0\mathbb{R}^{2n}$. Let $y = (y_1, \dots, y_{2n})$ be the corresponding coordinates on the fibers. Every $h \in S^{k+1}(\mathbb{R}^{2n})$ is a homogeneous polynomial in y with coefficients being C^∞-smooth functions on $\mathbb{R}^{2n}$. Therefore, h can be decomposed as

$$h(y) = \sum_{j=1}^{2n} \phi_j(y) y_j = \sum_{j=1}^{2n} \phi_j(y) ds_j(y), \ \phi_j \in S^k(T^*\mathbb{R}^{2n}).$$

The restriction of the latter decomposition to $T\gamma$ together with Claim 1 imply the second (and hence, the first) statement of Proposition 3.10. □

The statement of Proposition 3.9 for $k \neq 2$ follows from formula (3.12), statement (3.13) and Proposition 3.10. □

Corollary 3.11 *The sum $\Lambda_0 \oplus \Lambda_1 \oplus \Lambda_2$ and the subspace $\frac{1}{\sqrt{1+||y||^2}} V_2(S^2(T^*\gamma))$ in Λ_3 are contained in the Lie algebra $\mathfrak{H}$. If $\Lambda_3 \subset \mathfrak{H}$, then $\mathfrak{H} = \oplus_{k \geq 0} \Lambda_k$.*

Proof The first statement of the corollary follows from definition and the statement of Proposition 3.9 for $k = 0, 1, 2$. Its second statement follows from the statement of Proposition 3.9 for $k \geq 3$. □

Now for the proof of Theorems 1.20 it suffices to show that $\Lambda_3 \subset \mathfrak{H}$. As it is shown below, this is implied by formula (3.6) and the following lemma.

Lemma 3.12 *Let $n \geq 2$. For every $k \in \mathbb{N}$ consider two $\mathbb{R}$-linear mappings*

$$G_k^+ : S^k(T^*\gamma) \otimes_{\mathbb{R}} C^\infty(\gamma) \to S^{k+1}(T^*\gamma), \ \phi \otimes_{\mathbb{R}} f \mapsto \phi(y) \frac{df}{dy}, \tag{3.14}$$

$$G_k^- : S^k(T^*\gamma) \otimes_{\mathbb{R}} C^\infty(\gamma) \to S^{k-1}(T^*\gamma),\ \phi \otimes_{\mathbb{R}} f \mapsto \frac{d_y\phi(y)}{d\nabla f}. \tag{3.15}$$

For every even k, e.g., $k = 4$, one has $G_k^-(\operatorname{Ker} G_k^+) = S^{k-1}(T^*\gamma)$: more precisely, there exists a $d_{k,n} \in \mathbb{N}$, $d_{k,n} < (2N)^{k+1}$, such that for every $h \in S^{k-1}(T^*\gamma)$ there exists a collection of $d_{k,n}$ pairs

$$(\phi_j, f_j),\ \phi_j \in S^k(T^*\gamma),\ f_j \in C^\infty(\gamma),$$

such that

$$\sum_j \phi_j(y)\frac{df}{dy} \equiv 0, \quad \sum_j \frac{d_y\phi_j(y)}{d\nabla f_j} = h. \tag{3.16}$$

Lemma 3.12 is proved below. In its proof we deal with γ as an embedded submanifold in $\mathbb{R}^N_{(s_1,\dots,s_N)}$ (Whitney Theorem, in which one can take $N = 2n$), equipped with its intrinsic Riemannian metric (not coinciding with the restriction to $T\gamma$ of the standard Euclidean metric). Let us trivialize the tangent bundle $T\mathbb{R}^N$ by translation to the origin and denote $y = (y_1, \dots, y_N)$ the corresponding coordinates on the tangent spaces. Let $\mathcal{P}^k = \mathcal{P}^k(T^*\mathbb{R}^N) \subset S^k(T^*\mathbb{R}^N)$ be the subspace of degree k homogeneous polynomials in y with constant coefficients. Let $\mathbb{L} \subset C^\infty(\mathbb{R}^N)$ denote the N-dimensional vector subspace over $\mathbb{R}$ generated by the coordinate functions $s_1, \dots, s_N$.

Remark 3.13 The operator G_k^+ given by formula (3.14) extends as a well-defined linear operator $S^k(T^*\mathbb{R}^N) \to S^{k+1}(T^*\mathbb{R}^N)$ by the same formula, which will be also denoted by G_k^+. For every $h \in S^k(T^*\mathbb{R}^N)$ the restriction $(G_k^+h)|_{T\gamma}$ coinsides with the image of the restriction $h|_{T\gamma} \in S^k(T^*\gamma)$ under the operator G_k^+ acting on $S^k(T^*\gamma)$. In particular, the restriction to $T\gamma$ of the kernel of the operator G_k^+ in the space $S^k(T^*\mathbb{R}^N)$ is contained in the kernel of the operator G_k^+ acting on $S^k(T^*\gamma)$.

Step 1: finding basis of the kernel of G_k^+ in $\mathcal{P}^k \otimes \mathbb{L}$.

Proposition 3.14 *For every $k \in \mathbb{N}$ the kernel $\operatorname{Ker}^k_{pol}$ of the restriction to $\mathcal{P}^k \otimes \mathbb{L}$ of the linear operator G_k^+ is the vector space with the basis*

$$Q_{m,i,j} := y^m(y_i \otimes s_j - y_j \otimes s_i),\ m = (m_1, \dots, m_N) \in \mathbb{Z}^N_{\geq 0}, \tag{3.17}$$

$$|m| := \sum_\ell m_\ell = k - 1, \quad i, j = 1, \dots, N, \quad j > i,$$

$$j \geq \max \operatorname{ind}(m) := \max\{\ell \mid m_\ell > 0\}.$$

Proof It is obvious that $G_k^+ Q_{m,i,j} = 0$. Let us show that the elements $Q_{m,i,j}$ are linearly independent. Indeed, if there were a linear dependence, then there would be a linear dependence between some $Q_{m,i,j}$, for which the corresponding monomial $P := y^m y_i y_j$ is the same: a fixed monomial. All the $Q_{m,\ell,\mu}$ corresponding to P have $\mu = j$, by definition, while ℓ runs through those indices for which P contains y_ℓ. It is clear that these $Q_{m,\ell,j}$ are linearly independent, – a contradiction.

Fix an arbitrary $Q \in \mathrm{Ker}^k_{pol}$:

$$Q = \sum_{j,m} c_{j,m} y^m \otimes s_j; \ |m| = k. \tag{3.18}$$

Let us show that Q is a linear combination of elements $Q_{m,i,j}$. In the proof we use the formula

$$G_k^+(y^m \otimes s_j) = y_j y^m \text{ for every } m \in \mathbb{Z}^N_{\geq 0}, \ |m| = k, \tag{3.19}$$

which follows from definition. Without loss of generality we can and will consider that $i \geq \ell_0 := \max \mathrm{ind}(m)$ for every (i,m) with $c_{i,m} \neq 0$. Indeed, if the latter inequality does not hold for some (i,m), one can achieve it by adding $c_{i,m}(y^m y_{\ell_0}^{-1}(y_i \otimes s_{\ell_0} - y_{\ell_0} \otimes s_i) = c_{i,m} Q_{m',i,\ell_0}$ to Q. This operation kills the (i,m)-th terms and replaces it by an (ℓ_0, m')-th terms. Here m' is obtained from m by replacing m_{ℓ_0} by $m_{\ell_0} - 1$. Let us show that $Q = 0$. Indeed, $G_k^+(c_{i,m} y^m \otimes s_i) = c_{i,m} y_i y^m$ should cancel out with similar monomials coming from other $(j, \widetilde{m}) \neq (i,m)$, since $G_k^+ Q = 0$. Therefore, there exist $j \neq i$ and $\widetilde{m} \neq m$ for which $c_{j,\widetilde{m}} \neq 0$ and $y_j y^{\widetilde{m}} = y_i y^m$. Hence, y^m is divisible by y_j and $j \leq \ell_0$. On the other hand, $j \geq \max \mathrm{ind}(\widetilde{m})$, by the above assumption on all the monomials in Q, and $y^{\widetilde{m}}$ is divisible by y_i. Hence, $\ell_0 \geq j > i \geq \ell_0$, since $j \neq i$, – a contradiction. Therefore, $Q = 0$. The proposition is proved. □

Step 2: proof of an Euclidean version of Lemma 3.12. In what follows by $\mathcal{P}_0^k \subset \mathcal{P}^k$ we denote the subspace of polynomials with zero average along the unit sphere.

Lemma 3.15 *Let $N \geq 2$. Consider the mapping $g_k^- : \mathcal{P}^k \otimes \mathbb{L} \to \mathcal{P}^{k-1}$ acting by the formula*

$$g_k^- : P(y) \otimes f \mapsto \frac{dP(y)}{d\nabla f}; \ f = \sum_{j=1}^N c_j s_j, \ c_j = const, \ \nabla f = (c_1, \ldots, c_N), \tag{3.20}$$

i.e., the above gradient is taken with respect to the standard Euclidean metric. One has $g_k^-(\mathrm{Ker}_{pol}^k) = \mathcal{P}_0^{k-1}$. *The latter image coincides with all of* $\mathcal{P}^{k-1}$, *if and only if* k *is even.*

In the proof of Lemma 3.15 we will use the following equivariance and invariance properties of the operators G_k^+, g_k^- and the kernel Ker_{pol}^k under the actions of $\mathrm{GL}_N(\mathbb{R})$ and $\mathrm{O}(N)$ on $\mathcal{P}^k$ and $\mathcal{P}^k \otimes \mathbb{L}$:

$$H(P) := P \circ H, \quad H(P \otimes s) := (P \circ H) \otimes (s \circ H)$$

$$\text{for every } H \in \mathrm{GL}_N(\mathbb{R}),\ P \in \mathcal{P}^k,\ s \in \mathbb{L}.$$

Proposition 3.16 *1) The restriction of the mapping* G_k^+ *to* $\mathcal{P}^k \otimes \mathbb{L}$ *is equivariant under the action of the linear group* $\mathrm{GL}_N(\mathbb{R})$ *on the image and the preimage:* $G_k^+(H(P \otimes s))(y) = H(Pds)(y) = P(Hy)ds(HY)$ *for every* $H \in \mathrm{GL}_N(\mathbb{R})$. *In particular, the kernel* Ker_{pol}^k *is* $\mathrm{GL}_N(\mathbb{R})$*-invariant.*

2) The mapping g_k^- *is equivariant under action of the orthogonal group* $\mathrm{O}(N)$*:* $g_k^-(H(P \otimes s)) = H((g_k^-)(P \otimes s))$ *for every* $H \in \mathrm{O}(N)$. *In particular, the image* $g_k^-(\mathrm{Ker}_{pol}^k)$ *is* $\mathrm{O}(N)$*-invariant.*

3) The latter image is generated by derivatives of monomials y^m, $|m| = k-1$, *along the vector fields* $v_{ij} := y_i \frac{\partial}{\partial y_j} - y_j \frac{\partial}{\partial y_i}$, $i \neq j$; *each* v_{ij} *generates the one-dimensional Lie algebra of the standard* $\mathrm{SO}(2)$*-action on* $\mathbb{R}^N$ *in the variables* (y_i, y_j).

Proof Statements 1) and 2) of the proposition follow from definition. The image $g_k^-(\mathrm{Ker}_{pol}^k)$ is generated by the polynomials

$$g_k^-(Q_{m,i,j}) = g_k^-(y^m(y_i \otimes s_j - y_j \otimes s_i)) = y_i \frac{\partial y^m}{\partial y_j} - y_j \frac{\partial y^m}{\partial y_i}, \ \ |m| = k-1, \quad (3.21)$$

by Proposition 3.14. The latter right-hand side is the derivative of the polynomial y^m along the generator v_{ij} of the SO(2)-action on the variables (y_i, y_j). This proves Proposition 3.16. □

Proposition 3.17 *The derivatives from Proposition 3.16, Statement 3), lie in* $\mathcal{P}_0^{k-1}$.

Proof Averaging the derivative $\frac{dy^m}{dv_{ij}}$ along the SO(2)-action in the variables (y_i, y_j) yields zero, analogously to the well-known fact that the derivative of a function on a circle has zero average. Every function on the unit sphere in $\mathbb{R}^N$ having zero average along the above SO(2)-action has zero average

on the whole unit sphere as well, since the volume form of the sphere is the product of the family of length elements of the SO(2)-orbits and a measure transversal to the foliation of the sphere by SO(2)-orbits. This proves Proposition 3.17. □

Proof of Lemma 3.15. One has $g_k^-(\mathrm{Ker}^k_{pol}) \subset \mathcal{P}_0^{k-1}$ (Proposition 3.17 and Proposition 3.16, Statement 3)). For even k one has $\mathcal{P}_0^{k-1} = \mathcal{P}^{k-1}$, since each odd degree homogeneous polynomial has zero average along the unit sphere: the antipodal map changes the sign of such a polynomial. Therefore, it suffices to prove that each polynomial in $\mathcal{P}_0^{k-1}$ lies in $g_k^-(\mathrm{Ker}^k_{pol})$. We prove this statement by induction in the number N of variables.

Induction base: $N = 2$. A homogeneous polynomial in (y_1, y_2) has zero average along the unit circle, if and only if it is the derivative of another homogeneous polynomial by the vector field v_{12} generating the SO(2)-action. Indeed, writing the restriction to the unit circle of a homogeneous polynomial of degree $k-1$ as a trigonometric polynomial in $\phi = \arctan(\frac{y_2}{y_1})$ of the same degree reduces the above statement to the following well-known one: a trigonometric polynomial of a given degree has zero average, if and only if it is the derivative of another trigonometric polynomial of the same degree. Therefore, the space $\mathcal{P}_0^{k-1}$ of polynomials in (y_1, y_2) coincides with $g_k^-(\mathrm{Ker}^k_{pol})$, by Proposition 3.16 (Statement 3)) and Proposition 3.17.

Induction step. Let the statement $\mathcal{P}_0^{k-1} = g_k^-(\mathrm{Ker}^k_{pol})$ be proved for all $N \le d$, $N \ge 2$. Let us prove it for $N = d+1$. Fix an arbitrary $P \in \mathcal{P}_0^{k-1}$. It can be represented as the sum

$$P = Q + R,\ Q \text{ has zero average along the } \mathrm{SO}(2) \text{ action in } (y_1, y_2);$$

$$\text{the polynomial } \quad R \quad \text{is } \mathrm{SO}(2) - \text{invariant.}$$

Namely, R is the average of the polynomial P under the above SO(2)-action. Let us show that Q, R, and hence, P lie in $g_k^-(\mathrm{Ker}^k_{pol})$. The polynomials Q and R are homogeneous of the same degree $k-1$. The polynomial Q is a derivative, as above, see the proof of the induction base. Therefore, $Q \in g_k^-(\mathrm{Ker}^k_{pol})$. One has

$$R(y) = \sum_{j=0}^{[\frac{N}{2}]} (y_1^2 + y_2^2)^j R_j(y_3, \dots, y_N),$$

$$R_j \text{ are homogeneous polynomials, } \deg R_j = N - 2j,$$

by SO(2)-invariance. For every $j \in \mathbb{N}$ the SO(2)-average of the monomial y_2^{2j} is equal to $c_j(y_1^2 + y_2^2)^j$, $c_j > 0$. Set

$$\widetilde{R}(y) := \sum_{j=0}^{[\frac{N}{2}]} c_j^{-1} y_2^{2j} R_j(y_3, \dots, y_N).$$

The difference $\widetilde{R} - R$ has zero average along the SO(2)-action, by construction. Therefore, it lies in $g_k^-(\mathrm{Ker}_{pol}^k)$, see the above argument. Now it remains to show that $\widetilde{R} \in g_k^-(\mathrm{Ker}_{pol}^k)$. The polynomial $\widetilde{R}$ has zero average along the unit sphere in $\mathbb{R}^N$, by assumption. On the other hand, it is a polynomial of $N-1$ variables $y_2, \dots, y_N$. Therefore, it has zero average along the $(N-2)$-dimensional unit sphere in $\mathbb{R}^{N-1}_{y_2,\dots,y_N}$. Hence, it lies in the g_k^--image of the kernel Ker_{pol}^k from the space $\mathcal{P}^k \otimes \mathbb{L}$ in variables $(y_2, \dots, y_N)$, by the induction hypothesis. The induction step is over. Lemma 3.15 is proved. □

Step 3: proof of Lemma 3.12 in the general case. Recall that $S^k(T^*\gamma)$ is the space of sections of a smooth vector bundle E_k whose fiber over each point $x \in \gamma$ is the space of degree k homogeneous polynomials on the tangent space $T_x\gamma$. The bundle E_k is isomorphic to the k-th symmetric power of the cotangent bundle $T^*\gamma$, see the discussion before Remark 3.4. To prove Lemma 3.12, we show that for every even k each section of the bundle E_{k-1} is a finite linear combinations of the restrictions to TS of elements of the image $G_k^-(\mathrm{Ker}_{pol}^k)$ with C^∞-smooth coefficients. To this end, we will show that the latter image spans the space of degree $k-1$ homogeneous polynomials at each point $x \in \gamma$: see the following definition.

Definition 3.18 Let γ be a C^∞-smooth manifold (not necessarily compact). Let $\pi : E \to \gamma$ be a C^∞-smooth finite-dimensional vector bundle. A finite collection of its sections $f_1, \dots, f_\ell$ is called *generating,* if for every $x \in \gamma$ the vectors $f_j(x) \in E_x$ span the fiber E_x over x.

Proposition 3.19 *Let k be an even number. Let $Q_{m,i,j} \in \mathrm{Ker}_{pol}^k \subset \mathcal{P}^k \otimes \mathbb{L}$ be the same, as in (3.17). The G_k^--images of their restrictions to $T\gamma$ are generating sections of the vector bundle E_{k-1}.*

Proof Fix an arbitrary point $x \in \gamma$. Let us choose a Euclidean scalar product on $\mathbb{R}^N$ and orthonormal coordinates $(s_1, \dots, s_N)$ centered at 0 so that

- the vector subspace parallel to $T_x\gamma$ be the coordinate subspace $\mathbb{R}^n_{s_1,\dots,s_n}$;

- the translation pushforward of the Riemannian metric on $T_x\gamma$ to $\mathbb{R}^n_{s_1,\dots,s_n}$ be the standard Euclidean metric given by $ds_1^2+\dots+ds_n^2$;

- the radius vector of the point x be orthogonal to $T_x\gamma$.

One can achieve this by applying a linear transformation and choosing appropriate scalar product. These operations change neither the space $\mathcal{P}^k\otimes\mathbb{L}$, nor Ker^k_{pol} (Proposition 3.16, Statement 1)). Let us equip the tangent spaces to $\mathbb{R}^N$ with the coordinates $y_1,\dots,y_N$ obtained from $s_1,\dots,s_N$ by translations. This identifies the tangent subspace $T_x\gamma$ with $\mathbb{R}^n_{y_1,\dots,y_n}$. For every $Q\in\mathcal{P}^k\otimes\mathbb{L}$ containing only s_j and y_j with $j\le n$ the image $G_k^-(Q|_{T\gamma})$ coincides on the fiber $T_x\gamma$ with the restriction to it of the form $g_k^-(Q)$, by construction. The images $g_k^-(Q_{m,i,j})$ generate the space of all the degree $k-1$ homogeneous polynomials in $(y_1,\dots,y_n)$, since $Q_{m,i,j}$ generate Ker^k_{pol} (Proposition 3.14) and by Lemma 3.15. This proves Proposition 3.19. □

Theorem 3.20 *Let γ and $\pi:E\to\gamma$, be as in the above definition. Let $f_1,\dots,f_\ell$ be a generating collection of sections of the bundle E. Then every C^∞-smooth section of the bundle E is a linear combination of the sections $f_1,\dots,f_\ell$ with coefficients being C^∞-smooth functions on γ.*

Proof Set $d=\dim E$. Each point of the manifold γ has a neighborhood U such that there exists a collection of distinct indices $j_1,\dots,j_d$ for which the values $f_{j_1}(x),\dots,f_{j_d}(x)$ are linearly independent at every $x\in U$. Each section F on such a neighborhood is a linear combination $F=\sum_{i=1}^d\eta_i f_{j_i}$, where η_i are C^∞-smooth functions on U. If F has compact support in U, then so do η_i.

Now fix a locally finite at most countable covering of the manifold γ by open subsets U_α as above and the corresponding partition of unity consisting of functions ρ_α compactly supported in U_α. Let F be an arbitrary C^∞-smooth section of the bundle E. Each function $\rho_\alpha F$ is compactly supported in U_α and hence $\rho_\alpha F=\sum_{i=1}^d\eta_{i,\alpha}f_{j_{i,\alpha}}$, where $\eta_{i,\alpha}$ are compactly supported in U_α. Writing $F=\sum_\alpha(\rho_\alpha F)$ and replacing $\rho_\alpha F$ by the latter linear combinations yields that F is a linear combitation of the sections f_j with C^∞-smooth coefficients. The theorem is proved. □

Proof of Lemma 3.12. Each element $h\in S^{k-1}(T^*\gamma)$ is a linear combination of the images $G_k^-(Q_{m,i,j}|_{T\gamma})$ with coefficients $\eta_{m,i,j}\in C^\infty(\gamma)$, by Proposition 3.19 and Theorem 3.20; $Q_{m,i,j}=y^m y_i\otimes s_j-y^m y_j\otimes s_i$. The elements $h_{m,i,j}:=(\eta_{m,i,j}y^m y_i)\otimes s_j-(\eta_{m,i,j}y^m y_j)\otimes s_i\in S^k(T^*\gamma)$ lie in $\mathrm{Ker}\,G_k^+$, by construction. One has $G_k^-h_{m,i,j}=\eta_{m,i,j}G_k^-(Q_{m,i,j}|_{T\gamma})$, since the operator G_k^- is C^∞-linear in the first tensor factor: $G_k^-(\eta\phi\otimes f)=\eta G_k^-(\phi\otimes f)$ for ev-

ery $\eta \in C^\infty(\gamma)$, by definition, see (3.15). Therefore, $h = G_k^-(\sum_{m,i,j} h_{m,i,j})$. This proves Lemma 3.12 for the number $d_{k,n}$ being equal to the number of elements $Q_{m,i,j}$ with $|m| = k-1$, $N = 2n$: $d_{k,n} < (2N)^{k+1}$. □

Proposition 3.21 *The space Λ_3 is contained in $\{\Lambda_4, \Lambda_0\}$.*

Proof The Poisson bracket space $\{\Lambda_0, \Lambda_4\}$ is contained in $\Lambda_3 \oplus \Lambda_5$. Fix an arbitrary $A = \frac{h(y)}{\sqrt{1+||y||^2}} \in \Lambda_3$; $h \in S^3(T^*\gamma)$. Let us show that $A \in \{\Lambda_4, \Lambda_0\}$. To do this, fix an element

$$X := \sum_{i=1}^d \phi_i \otimes f_i \ \in \ S^4(T^*\gamma) \otimes_{\mathbb{R}} C^\infty(\gamma), \quad X \in \operatorname{Ker} G_4^+, \ G_4^-(X) = h.$$

It exists by Lemma 3.12. Set

$$H_i := \frac{\phi_i(y)}{\sqrt{1+||y||^2}} \in \Lambda_4, \ F_i := \frac{f_i(x)}{\sqrt{1+||y||^2}} \in \Lambda_0, \ B := \sum_{i=1}^d \{H_i, F_i\} \in \Lambda_3 \oplus \Lambda_5.$$

One has

$$\pi_3(B) = A, \quad \pi_5(B) = \frac{(\nabla_y \phi)(y)}{\sqrt{1+||y||^2}}, \ \phi := \sum_{i=1}^d f_i \phi_i \in S^4(T^*\gamma), \tag{3.22}$$

by construction and (3.6). Now replacing B by

$$\widetilde{B} := B - \{\frac{\phi(y)}{\sqrt{1+||y||^2}}, \frac{1}{\sqrt{1+||y||^2}}\}$$

we cancel the remainder $\pi_5(B)$ in π_5 without changing π_3 and get that $\pi_5(\widetilde{B}) = 0$, $\pi_3(\widetilde{B}) = A$. The proposition is proved. □

Proof of Theorems 1.20. Theorems 1.20 follows from Corollary 3.11 and Proposition 3.21. □

3.3 Proofs of Theorems 1.16, 1.1 and 1.8 for $n \geq 2$

Proposition 3.22 *Let $U \subset \gamma$ be a local chart identified with a contractible domain in $\mathbb{R}^n$. The restrictions to $(T\gamma)|_U$ of functions from the space $\Lambda := \oplus_{k=0}^\infty \Lambda_k$ are C^∞-dense in the space of C^∞-functions on $(T\gamma)|_U$.*

Proof Let $z = (z_1, \dots, z_n)$ be the coordinates of the chart U. The restricted tangent bundle $(T\gamma)|_U$ is the direct product $\mathbb{R}^n_y \times U_z$. The restrictions to $(T\gamma)_U$ of functions from the space $\sqrt{1+||y||^2}\Lambda$ are polynomials in y with coefficients C^∞-smoothly depending on z. The polynomials in variables (y, z) are C^∞-dense in the space of C^∞-smooth functions on $(T\gamma)|_U$. Therefore, $\sqrt{1+||y||^2}\Lambda$ is dense there, and hence, so is Λ. $\square$

Proof of Theorem 1.16. Fix a locally finite covering of the manifold γ by local contractible charts U_ℓ. Let $(\rho_\ell)_{\ell=1,2,\dots}$ be a partition of unity, $\operatorname{supp}\rho_\ell \Subset U_\ell$. Fix an arbitrary $h \in C^\infty(T\gamma)$. For every ℓ the function $h\rho_\ell$ is supported in U_ℓ. Hence, $h\rho_\ell$ is a C^∞-limit of the restrictions to $(T\gamma)|_{U_\ell}$ of functions $h_{k,\ell} \in \Lambda$, by Proposition 3.22. For every ℓ fix an arbitrary function $\alpha_\ell(z)$ compactly supported in U_ℓ that is identically equal to 1 on a neighborhood of the support $\operatorname{supp}(h\rho_\ell)$. The functions $\widetilde{h}_{k,\ell} := \alpha_\ell(z)h_{k,\ell}$ lie in Λ and converge to $h\rho_\ell$, as $k \to \infty$, in the C^∞-topology of the space of functions on $T\gamma$, by construction. The sums $\sum_{\ell=1}^m \widetilde{h}_{k,\ell}$ lie in Λ for every k and m and C^∞-converge to h, as k and m tend to infinity, by construction and local finiteness of covering. Finally, Λ is C^∞-dense in $C^\infty(T\gamma)$.

Recall that the Lie algebra $\mathfrak{H}$ consists of functions on $T_{<1}\gamma$. It is transformed to Λ by the diffeomorphism $T_{<1}\gamma \to T\gamma$, $(x, w) \mapsto (x, y := \frac{w}{\sqrt{1-||w||^2}})$, by Theorem 1.20. This together with density of the algebra Λ implies density of the algebra $\mathfrak{H}$ in the space $C^\infty(T_{<1}\gamma)$ and proves Theorem 1.16. $\square$

Theorems 1.1 and 1.8 follow immediately from Theorems 1.16 and 1.13.

4 Density of pseudo-groups. Proofs of Theorems 1.5, 1.9, 1.11 and Corollary 1.7

Let γ be either a global strictly convex closed hypersurface in $\mathbb{R}^{n+1}$, or a germ of hypersurface in $\mathbb{R}^{n+1}$. It is supposed to be C^∞-smooth. In what follows Π will denote an open subset in the space of oriented lines in $\mathbb{R}^{n+1}$ where the reflection from the curve γ is well-defined: it is either the phase cylinder, as in Subsection 1.1, or diffeomorphic to a contractible domain in $\mathbb{R}^{2n}$. First we prove Theorems 1.5, 1.9 and Corollary 1.5. Then we prove Theorem 1.11.

4.1 Density in Hamiltonian symplectomorphism pseudogroup. Proofs of Theorems 1.5, 1.9 and Corollary 1.5

Definition 4.1 Let M be a manifold, $V \subset M$ be an open subset, v be a smooth vector field on M, and $t \in \mathbb{R}$, $t \neq 0$. We say that the time t flow map g_v^t is *well-defined on* V, if all the flow maps g_v^τ with $\tau \in (0,t]$ ($\tau \in [t,0)$, if $t<0$) are well-defined on V, that is, the corresponding differential equation has a well-defined solution for every initial condition in V for all the time values $\tau \in (0,t]$ (respectively, $\tau \in [t,0)$).

Recall that $\mathfrak{H}$ denotes the Lie algebra of functions on Π generated by the space Λ_0 of functions of type $H_f(x,w) = -2f(x)\sqrt{1-||w||^2}$ with respect to the Poisson bracket for all $f \in C^\infty(\gamma)$.

Proposition 4.2 *Let $\mathcal{F}$ denote the pseudogroup generated by flow maps (well-defined on domains in the sense of the above definition) of the Hamiltonian vector fields with Hamiltonian functions from the Lie algebra $\mathfrak{H}$. For every mapping $\delta : C^\infty(\gamma) \to \mathbb{R}_+$ the C^∞-closure $\overline{\mathcal{G}(\delta)}$ of the corresponding pseudogroup $\mathcal{G}(\delta)$, see (1.3), contains $\mathcal{F}$.*

Proof Let L_0 denote the space of Hamiltonian vector fields with Hamiltonian functions from the space Λ_0. The Lie algebra of Hamiltonian vector fields with Hamiltonians in $\mathfrak{H}$ consists of finite linear combinations of successive commutators of vector fields from the space L_0.

Claim 1. *The well-defined flow maps (on domains) of each iterated commutator of a collection of vector fields in L_0 are contained in $\overline{\mathcal{G}(\delta)}$.*

Proof Induction in the number of Lie brackets in the iterated commutator.

Basis of induction. The space L_0 consists of the derivatives $v_h := \frac{d}{d\varepsilon}\Delta\mathcal{T}_{\varepsilon,h}$, $\Delta\mathcal{T}_{\varepsilon,h} = \mathcal{T}_{\gamma_{\varepsilon,h}}^{-1} \circ \mathcal{T}_\gamma$, for all C^∞-smooth functions h on γ. Fix an arbitrary function $h \in C^\infty(\gamma)$, a domain $W \subset \Pi$ and a $t \in \mathbb{R}\setminus\{0\}$, say $t>0$, such that the flow map $g_{v_h}^t$ is well-defined on W. Note that the mapping $F_\varepsilon := (\Delta\mathcal{T}_{\varepsilon,h})^{[\frac{t}{\varepsilon}]}$ converges to $g_{v_h}^t$, as $\varepsilon \to 0$, whenever it is defined. Since $g_{v_h}^t$ is well-defined on W, we therefore see that F_ε is well-defined on a domain $W_\varepsilon \subset \Pi$ such that $W_\varepsilon \cap W \to W$, as $\varepsilon \to 0$, and F_ε converges to $g_{v_h}^t$ on W in the C^∞ topology. Note that $F_\varepsilon \in \mathcal{G}(\delta)$. Therefore, $g_{v_h}^t|_W \in \overline{\mathcal{G}(\delta)}$. Thus, the flow maps of the vector fields in L_0 are contained in $\overline{\mathcal{G}(\delta)}$.

Induction step. Let v and w be vector fields on Π such that their well-defined flow maps lie in $\overline{\mathcal{G}(\delta)}$. Let us prove the same statement for their commutator $[v,w]$. Note that for small τ the flow maps g_v^τ, g_w^τ are well-defined on domains in Π converging to Π, as $\tau \to 0$. Let $t \in \mathbb{R}\setminus\{0\}$,

say $t > 0$, and a domain $D \subset \Pi$ be such that the flow map $g^t_{[v,w]}$ be well-defined on D. Then $g^t_{[v,w]}|_D \in \overline{\mathcal{G}(\delta)}$, by a classical argument: the composition $(g_v^{\frac{t}{N}} \circ g_w^{\frac{t}{N}} \circ g_v^{-\frac{t}{N}} \circ g_w^{-\frac{t}{N}})^{N^2}$ is well-defined on a domain W_N with $W_N \cap D \to D$, as $N \to \infty$; it belongs to $\overline{\mathcal{G}(\delta)}$ and converges to $g^t_{[v,w]}$ on D in the C^∞-topology. The induction step is done. The claim is proved. □

Claim 2. *Let well-defined flow maps of vector fields v and w lie in $\overline{\mathcal{G}(\delta)}$. Then well-defined flow maps of all their linear combinations also lie in $\overline{\mathcal{G}(\delta)}$.*
Proof It suffices to prove the claim for the sum $v + w$, since $g^t_{cv} = g^{ct}_v$. The composition $(g_v^{\frac{t}{N}} g_w^{\frac{t}{N}})^N$ obviously converges to g^t_{v+w} on every domain where the latter flow map is well-defined. This proves the claim. □

Claims 1 and 2 together imply the statement of Proposition 4.2. □

Recall the following well-known notion.

Definition 4.3 We say that a symplectomorphism F of a symplectic manifold M has *compact support* if it is equal to the identity outside some compact subset. We say that it is *Hamiltonian with compact support,* if it can be connected to the identity by a smooth path F_t in the group of symplectomorphisms $M \to M$, $F_0 = Id$, $F_1 = F$, such that the vector fields $\frac{dF_t}{dt}$ are Hamiltonian with compact supports contained in one and the same compact subset.

Proposition 4.4 *The group of Hamiltonian symplectomorphisms with compact support is dense in the pseudogroup of M-Hamiltonian symplectomorphisms between domains in the ambient manifold M.*

Proof Consider the Hamiltonian vector fields $\frac{dH_t}{dt}$ on domains V_t from the definition of a M-Hamiltonian symplectomorphism $V \to U \subset M$ (see Definition 1.3); let $g_t : V_t \to \mathbb{R}$ denote the corresponding Hamiltonian functions. We may identify V_t with V by the mappings H_t and consider g_t as one function g on $V \times [0,1]_t$. We can approximate it by functions g_n with compact supports; $g_n \to g$ in the C^∞-topology, i.e., the convergence is uniform with all derivatives on compact subsets. The approximating functions yield a family of globally defined functions $g_{t,n} : M \to \mathbb{R}$ with compact supports contained in V_t. Let $v_{n,t}$ denote the Hamiltonian vector fields with Hamiltonians $g_{t,n}$. Then the time 1 map of the non-autonomous differential equation defined by $v_{t,n}$ converges to F on V in the C^∞-topology, as $n \to \infty$. □

Proposition 4.5 *The whole group of Hamiltonian symplectomorphisms* $\Pi \to \Pi$ *with compact support lies in the* C^∞*-closure of the pseudogroup generated by well-defined flow maps of Hamiltonian vector fields with Hamiltonian functions from the Lie algebra* $\mathfrak{H}$*.*

Proof Consider a Hamiltonian symplectomorphism $F : \Pi \to \Pi$ with compact support, let $v(x,t) = \frac{dF_t}{dt}$ be the corresponding family of Hamiltonian vector fields with compact supports. The map F is the C^∞-limit of compositions of time $\frac{1}{n}$ flow maps of the autonomous Hamiltonian vector fields $v(x, \frac{k}{n})$, $k = 0, \dots, n-1$, as $n \to \infty$. Each above Hamiltonian vector field $v(x, \frac{k}{n})$ can be approximated by Hamiltonian vector fields with Hamiltonians in $\mathfrak{H}$, by Theorems 1.1 and 1.8. Then F becomes approximated by products of their flows. This implies the statement of Proposition 4.5. $\square$

Proof of Theorems 1.5: global and local cases. Each Π-Hamiltonian symplectomorphism between domains in Π is a limit of a converging sequence of Hamiltonian symplectomorphisms $\Pi \to \Pi$ with compact supports in the C^∞-topology, by Proposition 4.4. The latter symplectomorphisms are, in their turn, limits of compositions of well-defined flow maps of vector fields with Hamiltonians from the algebra $\mathfrak{H}$, by Proposition 4.5. For every mapping $\delta : C^\infty(\gamma) \to \mathbb{R}_+$ the flow maps under question lie in $\overline{\mathcal{G}(\delta)}$, by Proposition 4.2. This proves Theorem 1.5 in global and local cases. $\square$

Proof of Corollary 1.7 and Theorem 1.9: global and local cases. Fix some $\alpha > 0$ and $k \in \mathbb{N}$. We can choose a mapping $\delta : C^\infty(\gamma) \to \mathbb{R}_+$ so that for every $h \in C^\infty(\gamma)$ and every $\varepsilon \in [0, \delta(h))$ the curve $\gamma_{\varepsilon,h}(s) = \gamma(s) + \varepsilon \vec{N}(s) h(s)$ is (α, k)-close to γ. Then the statement of Corollary 1.7 follows immediately from Theorem 1.5. This proves Theorem 1.9. $\square$

4.2 Special case of germ of planar curve. Density in symplectomorphisms: proof of Theorem 1.11

To prove Theorem 1.11, which states similar approximability of (a priori non-Hamiltonian) symplectomorphisms in the case, when γ is a germ of planar curve, we use the following three well-known propositions communicated to the author by Felix Schlenk.

Proposition 4.6 *Let* M *be an oriented manifold. Let* $V \subset M$ *be an open subset with compact closure and smooth boundary that is* C^∞*-smoothly diffeomorphic to a closed ball. Then every orientation-preserving diffeomorphism* $F : \overline{V} \to F(\overline{V}) \Subset M$ *extends to a* C^∞*-smooth diffeomorphism*

$F : M \to M$ isotopic to identity with compact support. The isotopy can be chosen C^∞-smooth and so that its diffeomorphisms coincide with the identity outside some compact subset.

Proof (Felix Schlenk). **Step 1.** It suffices to show that

Lemma 4.7 *The space of smooth orientation-preserving embeddings $\varphi\colon \overline{B}^n \to M$ is connected.*

Indeed, given V as in the proposition, choose a diffeomorphism $\varphi\colon \overline{B}^n \to V$. By the lemma, we find a smooth family $\varphi_t\colon \overline{B}^n \to M$ of smooth embeddings, $t \in [0,1]$, with $\varphi_0 = \varphi$ and $\varphi_1 = F \circ \varphi$. Consider the vector field $v(x,t) = \frac{d}{dt}\varphi_t(x)$ that generates this isotopy. The vector field v is defined on the compact subset $\big\{(x,t) \mid x \in \varphi_t(\overline{B}^n)\big\}$ of $M \times [0,1]$. Choose any smooth extension $\tilde{v}$ of v to $M \times [0,1]$ that vanishes outside a compact set. Then the flow maps of $\tilde{v}$, $t \in [0,1]$, form the desired isotopy.

Step 2. Proof of the lemma.

Let $\varphi_0, \varphi_1\colon \overline{B}^n \to M$ be two smooth orientation-preserving embeddings. Take a smooth isotopy g_t, $t \in [0,1]$, of M that moves $\varphi_0(0)$ to $\varphi_1(0)$. Choose $\varepsilon > 0$ such that

$$g_1\big(\varphi_0(\overline{B}^n_\varepsilon)\big) \subset \varphi_1(\overline{B}^n).$$

Consider the smooth family of embeddings $\overline{B}^n_1 \to M$ defined by

$$\varphi_0^s(x) = \varphi_0(sx), \quad s \in [\varepsilon, 1]. \tag{4.1}$$

By definition, φ_1 is the restriction of a smooth embedding $\tilde{\varphi}_1$ of the open ball $B^n_{1+\delta}$. We now have two smooth orientation-preserving embeddings $g_1 \circ \varphi_0^\varepsilon, \varphi_1\colon \overline{B}^n_1 \to \tilde{\varphi}_1(B^n_{1+\delta})$. Since $\tilde{\varphi}_1(B^n_{1+\delta})$ is diffeomorphic to $\mathbb{R}^n$, the classical Alexander trick shows that the embeddings $g_1 \circ \varphi_0^\varepsilon, \varphi_1$ can be connected, see e.g. [15, Appendix A, formula (A.1)]. Since $g_1 \circ \varphi_0^\varepsilon$ is connected to φ_0^ε by $g_t \circ \varphi_0^\varepsilon$, and since φ_0^ε is connected to φ_0 by (4.1), the lemma and the proposition are proven. □

Proposition 4.8 *Let M be a two-dimensional symplectic manifold, and let $V \subset M$ be an open subset with compact closure and smooth boundary. Let $\overline{V}$ be C^∞-smoothly diffeomorphic to a disk. Then every symplectomorphism $F : \overline{V} \to F(\overline{V}) \Subset M$ can be extended to a symplectomorphism $F : M \to M$ with compact support. Moreover, F can be chosen isotopic to the identity via symplectomorphisms with compact supports contained in one and the same compact subset in M.*

Proof (Felix Schlenk). Consider an arbitrary extension G of F to a diffeomorphism $G : M \to M$ with compact support that is isotopic to the identity through diffeomorphisms $G_s : M \to M$ with compact supports contained in a common compact subset $K \subset M$, $K \supset (\overline{V} \cup F(\overline{V}))$; $s \in [0,1]$, $G_0 = Id$, $G_1 = G$. It exists by Proposition 4.6. Set $W := M \setminus K$. Let ω be the symplectic form of the manifold M. Consider its pullback $\alpha := G^*(\omega)$, which is a symplectic form on M. The forms $\alpha_t := t\alpha + (1-t)\omega$, $t \in [0,1]$, are also symplectic: they are obviously closed, and they are positive area forms, since M is two-dimensional. This is the place we use two-dimensionality. The forms α_t are cohomologous to ω, the corresponding areas of the manifold M are equal, and $\alpha_t \equiv \omega$ on $\overline{V} \cup W$. One has $\alpha_0 = \omega$, $\alpha_1 = \alpha$. There exists a family of diffeomorphisms $S_t : M \to M$, $t \in [0,1]$, $S_0 = Id$, such that $S_t^*(\alpha_t) = \omega$ and S_t coincide with the identity on the set $V \cup W$, where $\omega = \alpha_t$. This follows from the relative version of Moser's deformation argument [8, exercise 3.18]; see also [11], [5, p.11]. The composition $\Phi := G \circ S_1 : M \to M$ preserves the symplectic form ω and coincides with G on $V \cup W$. Hence, $\Phi|_V = F$, $\Phi|_W = Id$.

Applying the above construction of the symplectomorphism Φ to the diffeomorphisms G_s instead of G yields a smooth isotopy Φ_s of the symplectomorphism $\Phi = \Phi_1$ to the identity via symplectomorphisms with supports in K. (But now G_s is not necessarily symplectic on V, and $(\Phi_s)|_V$ is not necessarily the identity.) The proposition is proved. □

Proposition 4.9 *Every symplectomorphism with compact support of an open topological disk equipped with a symplectic structure is a Hamiltonian symplectomorphism with compact support.*

Proof Every symplectomorphism in question is isotopic to the identity via a smooth family F_t of symplectomorphisms with compact support, by Proposition 4.8. The derivatives $\frac{dF_t}{dt}$ form a t-dependent family of divergence-free vector fields. Each divergence-free vector field X on a topological disk equipped with a symplectic form ω is symplectic. Hence, $L_X\omega = d(i_X\omega) = 0$, by Cartan's formula. Thus, $i_X\omega$ is a closed 1-form and hence, exact, $i_X\omega = dH$, since the underlying manifold is a topological disk. This proves the proposition. □

Remark 4.10 Proposition 4.9 is a part of the following well-known fact: *the group of compactly-supported symplectomorphisms of an open symplectic topological disk is contractible and path-connected by smooth paths.* The same statement holds for symplectomorphisms of $\mathbb{R}^4$ (Gromov's theorem

[9, p.345, theorem 9.5.2]). In higher dimensions it is not known whether a similar statement is true. For every $k \in \mathbb{N}$ there exists an exotic symplectic structure on $\mathbb{R}^{4k}$ admitting a symplectomorphism $\mathbb{R}^{4k} \to \mathbb{R}^{4k}$ with compact support that is not smoothly isotopic to the identity in the class of (not necessarily symplectic) diffeomorphisms [4, theorem 1.1].

Proof of Theorem 1.11. Let $V \subset \Pi$ be an arbitrary simply connected domain. Let us show that each symplectomorphism $F : V \to F(V) \subset \Pi$ can be approximated by elements of the pseudogroup $\mathcal{G}(\delta)$ for every δ. Consider an exhaustion $V_1 \Subset V_2 \Subset \dots$ of the domain $V = V_\infty$ by simply connected domains with compact closures in V and smooth boundaries. Each restriction $F|_{V_n}$ extends to a symplectomorphism with compact support, by Proposition 4.8. The latter extension is Hamiltonian with compact support, by Proposition 4.9. The elements of the pseudogroup $\mathcal{G}(\delta)$ accumulate to the restrictions to V of all the Hamiltonian symplectomorphisms $\Pi \to \Pi$ with compact support, see the above proof of Theorems 1.5, 1.9. Therefore, some sequence of elements in $\mathcal{G}(\delta)$ converges to F on V in the C^∞-topology. This proves the analogue of Theorem 1.5 with density in the pseudogroup of symplectomorphisms between simply connected domains in Π. The proof of similar analogue of Corollary 1.7 repeats the proof of Corollary 1.7 with obvious changes. Theorem 1.11 is proved. □

5 Proof for hypersurfaces in Riemannian manifolds

Here we prove Theorems 1.23 and 1.24.

Proof of Theorem 1.23. Fix a point $x \in \gamma$. Let $V(x)$ be its neighborhood in M equipped with normal coordinates for the metric of the ambient manifold M. We deal with two metrics on V: the metric of the manifold M (which will be denoted by g) and the Euclidean metric in the normal coordinates (which will be denoted by g_E). Consider the reflections $\mathcal{T}_\varepsilon$ with $\varepsilon \in [0, \delta]$ as mappings acting on compact subsets in the unit ball bundle $T_{<1}\gamma$ (after the identifications of geodesics and unit tangent vectors, see Subsection 1.4). The mapping $\mathcal{T}$ of reflection from γ in the metric g has the same 1-jet, as the reflection from γ in the metric g_E, at each point of the fiber over x: at each point (x, w), $w \in T_x\gamma$, with $||w|| := ||w||_g = ||w||_{g_E} < 1$. This holds, since the metrics in question have the same 1-jet at x. Similarly, the mappings $\mathcal{T}_\varepsilon$ constructed for both metrics have the same 1-jets at points $(x, w, 0)$ as functions on $(T_{<1}\gamma) \times [0, \delta]$. Therefore, the derivatives

$v_f = \frac{d\Delta\mathcal{T}_\varepsilon}{d\varepsilon}|_{\varepsilon=0}$ calculated for both metrics coincide at all points (x, w) with $w \in T_x\gamma$, $||w|| < 1$. This together with Theorem 1.23 applied to the Euclidean metric g_E implies that v_f coincides with the Hamiltonian vector field of the Hamiltonian function $H_f(x, w) = -2\sqrt{1 - ||w||^2}f(x)$ at each point (x, w) as above. Hence, the field v_f calculated for the given metric g coincides with the latter Hamiltonian vector field everywhere, since the choice of the point x was arbitrary. Theorem 1.23 is proved. □

Proof of Theorem 1.24. The Lie algebra generated by the Hamiltonian functions $H_f(x, w)$ of the above vector fields v_f on $T_{<1}\gamma$ is dense in the space of all the C^∞-smooth functions on $T_{<1}\gamma$ (Theorem 1.16). Therefore, the Lie algebra generated by the vector fields v_f is dense in the Lie algebra of Hamiltonian vector fields on Π (in both cases, when γ is either a closed hypersurface, as above, or a germ). Afterwards the proof of Theorem 1.24 repeats the arguments from Section 4. □

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