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Density of thin film planar billiard reflection pseudogroup in Hamiltonian symplectomorphism pseudogroup

Alexey Glutsyuk^{*†‡§}

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Abstract

Reflections from planar curves act by symplectomorphisms on the space of oriented lines with respect to the canonical symplectic form. We consider an arbitrary planar curve γ that is either a germ, or a strictly convex closed curve. In the case of a germ we show that reflections from its small deformations and their inverse transformations generate a pseudogroup that is dense in the pseudogroup of symplectomorphisms between simply connected subdomains of an appropriate domain in the space of oriented lines. In the case of a global strictly convex closed curve we prove a similar density statement in the pseudogroup of Hamiltonian diffeomorphisms between subdomains of the phase cylinder: the space of oriented lines intersecting the given curve transversally.

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1 Main results: density of thin film planar billiard reflection pseudogroup

It is well-known that billiard reflections acting on the space of oriented geodesics preserve the canonical symplectic form [1, 2, 6, 7, 10, 15]. However only a tiny part of symplectomorphisms are realized by reflections. There is an important open question stated in [13]: which symplectomorphisms can be realized by compositions of reflections?

In the present paper we consider the case of planar billiards. Namely we deal with a planar curve γ : either a germ, or a strictly convex closed curve. We consider the pseudogroup of compositions of reflections from its deformations γ_ε and of their inverse transformations. In the case of a germ we show that this pseudogroup is C^∞ -dense in the local pseudogroup of symplectomorphisms between simply connected subdomains of a small region in the space of oriented lines. In the case of a strictly convex closed curve we show that the corresponding pseudogroup is C^∞ -dense in the pseudogroup of Hamiltonian symplectomorphisms between open subdomains of the phase cylinder: the space of oriented lines intersecting γ transversally.

In Subsection 1.1 we state the local version of main result: for germs. In Subsection 1.2 we state the global version: for closed curves.

1.1 Local case: germs of curves

Let $\gamma \subset \mathbb{R}^2$ be a germ of a planar C^∞ -smooth curve at a point O equipped with natural length parameter s , $s(O) = 0$. The parameter s induces an orientation of the curve γ , which will be called the positive orientation. Consider the (germ of) space Π of oriented lines intersecting γ transversally at one point and directed outside the domain U bounded by γ for which γ is a positively oriented boundary. It is identified with the space of pairs (s, ϕ) , where s is the point of intersection of an oriented line L with γ and ϕ is the oriented angle between the oriented lines L and $T_s\gamma$. As $L \in \Pi$ varies in the

pencil of lines passing through a given point $s \in \gamma$, the corresponding angle ϕ varies in an interval $(\psi_1(s), \psi_2(s))$, $0 < \psi_1(s) < \phi < \psi_2(s) < \pi$. Thus, the domain Π is a topological disk.

The billiard reflection from the curve γ , which will be denoted by \mathcal{T}_γ , is defined on Π as follows. Take an oriented line L as above, let A denote its intersection point with γ . The image $\mathcal{T}_\gamma(L)$ is the line symmetric to L with respect to the tangent line $T_A\gamma$ and directed at A inside the above domain U bounded by γ .

It is well-known that billiard reflections with respect to all curves preserve the canonical symplectic (area) form on the space of oriented lines [1, 2, 6, 7, 10, 15]. In the above coordinates (s, ϕ) the canonical invariant symplectic form is

$$\omega = \sin \phi ds \wedge d\phi.$$

We consider the following deformations γ_ε of the curve γ . Let $\vec{n}(s)$ be the unit normal field on γ directed outside the domain U , and let $h(s)$ be a C^∞ -smooth function on γ . Set

$$\gamma_\varepsilon(s) = \gamma_{\varepsilon,h}(s) := \gamma(s) + \varepsilon h(s) \vec{n}(s).$$

We study the compositional differences of the corresponding reflections:

$$\Delta\mathcal{T}_\varepsilon = \Delta\mathcal{T}_{\varepsilon,h} := \mathcal{T}_{\gamma_\varepsilon}^{-1} \circ \mathcal{T}_\gamma.$$

Remark 1.1 The domain Π is not invariant under the reflection from the curve γ . But for every C^∞ -smooth function h and every ε small enough (depending on h) the composition $\Delta\mathcal{T}_\varepsilon$ is well-defined on a subdomain in Π converging to Π , as $\varepsilon \rightarrow 0$, and $\Delta\mathcal{T}_\varepsilon \rightarrow Id$ in the C^∞ -topology, that is, uniformly with all the derivatives on compact subsets in Π . In general, for every $k \in \mathbb{N}$ and every sequence of parametrized curves γ_n converging to γ in the C^k -distance the reflections from the curves γ_n are well-defined on a sequence of subdomains in Π converging to Π , and they converge to \mathcal{T}_γ in the C^{k-1} -topology: uniformly with derivatives up to order $k-1$ on compact subsets in Π .

The derivative

$$v_h := \left. \frac{d\Delta\mathcal{T}_{\varepsilon,h}}{d\varepsilon} \right|_{\varepsilon=0}$$

is a symplectic vector field on Π with respect to the area form ω .

The above compositions were studied by Ron Perline [12], who calculated their derivatives v_h in any dimension.

We study the Lie algebra generated by the vector fields v_h coming from all the deformations $\gamma_{\varepsilon,h}$. We prove that the latter Lie algebra is C^∞ -dense in the Lie algebra of all the divergence free vector fields on Π . As a corollary, this implies that the pseudogroup of compositions of reflections from curves and of their inverses is dense in the pseudogroup of area-preserving diffeomorphisms between simply connected domains in Π .

Let us pass to the statements of main results.

In our planar case in the coordinates (s, ϕ) on Π , Perline's formula [12] yields

$$v_h := (-2h(s) \cot \phi, 2h'(s)). \quad (1.1)$$

In what follows we denote

$$y := \cot \phi.$$

The vector fields v_h corresponding to all the possible C^∞ -smooth functions $h(s)$ form the vector subspace

$$L_0 := \{(-h(s)y, h'(s)) \mid h(s) \text{ is a } C^\infty \text{ - smooth function}\} \quad (1.2)$$

in the space of vector fields on Π . It is not a Lie algebra.

R.Perline [12] observed that in all dimensions the vector fields v_h are Hamiltonian and found the corresponding Hamiltonian functions. Here is his formula in two dimensions.

Proposition 1.2 *Each vector field $v_h = (-h(s)y, h'(s)) \in L_0$, $y = \cot \phi$, is Hamiltonian with the Hamiltonian function*

$$H(s, \phi) = -h(s) \sin \phi = -\frac{1}{\sqrt{1+y^2}}h(s).$$

The proposition follows from definition, see formula (2.1) for the Hamiltonian vector field with a Hamiltonian H for the symplectic form ω .

Proposition 1.3 *The commutator*

$$[L_0, L_0] = L_1$$

is a Lie algebra. It coincides with the vector space of fields of type $(h(s), yh'(s))$ where $h(s)$ runs through all the C^∞ -smooth functions of one variable.

Proposition 1.3 will be proved in Subsection 2.1.

Theorem 1.4 *The Lie algebra of vector fields generated by the vector space L_0 (which is the vector space generated by L_0 and successive Lie brackets of fields from L_0) is dense in the Lie algebra of symplectic C^∞ -smooth vector fields on Π with respect to the area form ω in the C^∞ -topology: the topology of uniform convergence with derivatives on compact subsets in Π .*

Recall the following classical definitions.

Consider a collection of C^∞ -smooth diffeomorphisms between domains on some manifold (e.g., the space of oriented lines in \mathbb{R}^2). We consider each diffeomorphism together with its domain of definition and all its restrictions to smaller domains. We will deal with those compositions of diffeomorphisms and of their inverses that are well-defined on some domains. Recall that the collection of all the above compositions is called a *pseudogroup*. (In the case when we take only compositions of mappings and not of their inverses, the above collection of compositions is called a pseudo-semigroup.)

Definition 1.5 For a given sequence of domains V_n we say that the intersections $V_n \cap V$ converge to V , if each compact subset $K \Subset V$ is contained in V_n whenever n is large enough (depending on K). Let g be a C^∞ -smooth diffeomorphism defined on a domain V : we deal with it as with the pair (g, V) . Recall that a sequence g_n of elements in the pseudogroup converges to a mapping g on V in the C^∞ -topology, if g is well-defined on V , g_n are defined on a sequence of domains V_n such that $V_n \cap V \rightarrow V$, and $g_n \rightarrow g$ uniformly on compact subsets in V with all the derivatives. The C^∞ -closure of a given pseudogroup consists of the mappings forming the pseudogroup and the limits of the above converging sequences.

For every positive-valued mapping $\delta : C^\infty(\gamma) \rightarrow \mathbb{R}_+$ set

$$\mathcal{G}(\delta) := \text{the pseudogroup generated by the collection of mappings} \quad (1.3)$$

$$\{\Delta\mathcal{T}_{\varepsilon,h} \mid h \in C^\infty(\gamma), 0 \leq \varepsilon \leq \delta(h)\}.$$

Theorem 1.6 *For every mapping $\delta : C^\infty(\gamma) \rightarrow \mathbb{R}_+$ the pseudogroup $\mathcal{G}(\delta)$ is C^∞ -dense in the pseudogroup of area-preserving mappings between simply connected domains in Π . In other words, for every simply connected domain $V \subset \Pi$ and every symplectomorphism $F : V \rightarrow W \subset \Pi$ there exists a sequence F_n of compositions of mappings $\Delta\mathcal{T}_{\varepsilon_j, h_j}$, F_n being defined on domains V_n with $V_n \cap V \rightarrow V$, such that $F_n \rightarrow F$ on V in the C^∞ -topology.*

Definition 1.7 We say that two parametrized planar curves are (α, k) -close if the distance between them in the C^k -topology is not greater than α .

Corollary 1.8 *For every $\alpha > 0$ (arbitrarily small) and $k \in \mathbb{N}$ the pseudogroup generated by reflections from curves (α, k) -close to γ is C^∞ -dense in the pseudogroup of area-preserving mappings between simply connected domains in Π .*

Theorems 1.4 will be proved in Subsection 2.1. Theorem 1.6 and Corollary 1.8 will be proved in Section 3.

1.2 Global case: strictly convex closed curves

Let now $\gamma \subset \mathbb{R}^2$ be a strictly convex planar C^∞ -smooth closed curve equipped with the natural length parameter s . The space Π of oriented lines intersecting γ transversally is a topological cylinder called the *phase cylinder*. It is identified with the space of pairs (s, ϕ) , where s is the last point of intersection of an oriented line L with γ and $\phi \in (0, \pi)$ is its angle with the tangent line $T_s\gamma$: the oriented angle, as at the beginning of Subsection 1.1.

The billiard reflection from the curve γ , which will be denoted by \mathcal{T}_γ , acts on Π as follows. Take an oriented line L intersecting γ , let A denote its last intersection point with γ . The image $\mathcal{T}_\gamma(L)$ is the line symmetric to L with respect to the tangent line $T_A\gamma$ and directed at A towards the interior: the convex domain bounded by γ . The transformation \mathcal{T}_γ is a symplectomorphism of the phase cylinder Π equipped with the symplectic form $\omega = \sin \phi ds \wedge d\phi$.

We deal with small deformations

$$\gamma_\varepsilon(s) := \gamma(s) + \varepsilon h(s) \vec{n}(s)$$

of the curve γ in the class of strictly convex closed curves depending on a small parameter ε ; here $\gamma(s)$ is a parametrization of the curve γ by its natural length parameter, $\vec{n}(s)$ is the exterior unit normal vector field and $h(s)$ is a C^∞ -smooth function on the circle parametrizing γ . Without loss of generality we can and will assume that the curve γ has length 2π , and is thus parametrized by the circle $\mathbb{R}/(2\pi\mathbb{Z})$ equipped with parameter s . We study the compositions

$$\Delta\mathcal{T}_\varepsilon = \Delta\mathcal{T}_{\varepsilon, h} := \mathcal{T}_{\gamma_\varepsilon}^{-1} \circ \mathcal{T}_\gamma.$$

The derivative in ε of the composition $\Delta\mathcal{T}_\varepsilon$ at $\varepsilon = 0$ is the symplectic vector field v_h on Π given by formula (1.1) in the coordinates (s, ϕ) on Π . Let L_0 denote the subspace of those vector fields v_h that correspond to C^∞ -smooth

functions h on the circle, see (1.2). Recall that the fields v_h are Hamiltonian with Hamiltonians

$$H(s, \phi) = -h(s) \sin \phi = -\frac{1}{\sqrt{1+y^2}}h(s), \quad y = \cot \phi,$$

and $L_1 = [L_0, L_0]$ is a Lie algebra, see Propositions 1.2 and 1.3.

Theorem 1.9 *The Lie algebra of vector fields generated by the vector space L_0 is C^∞ -dense in the Lie algebra of Hamiltonian vector fields on Π with respect to the symplectic form ω .*

Let us recall the following well-known definition.

Definition 1.10 Let M be a symplectic manifold, and let $V \subset M$ be an open domain. A symplectomorphism $F : V \rightarrow V_1 \subset M$ is (M -) *Hamiltonian*, if there exists a smooth family $F_t : V \rightarrow V_t \subset M$ of symplectomorphisms parametrized by $t \in [0, 1]$, $V_0 = V$, $F_0 = Id$, $F_1 = F$, such that for every $t \in [0, 1]$ the derivative $\frac{dF_t}{dt}$ is a Hamiltonian vector field on V_t . (In the case when $V = V_t = M$, this definition coincides with the usual definition of Hamiltonian diffeomorphism of a manifold [3, definition 4.2.4].)

Remark 1.11 Let M be a two-dimensional topological cylinder equipped with a symplectic form, and let $V \subset M$ be a subcylinder with compact closure and smooth boundary. Further assume that V is a deformation retract of the ambient cylinder M . Then not every area-preserving map $F : V \rightarrow U \subset M$ is Hamiltonian. A necessary condition for being Hamiltonian is that for every boundary component L of the subcylinder V the self-intersecting domain bounded by L and by its image $F(L)$ has zero signed area. This follows from results presented in [3, chapters 3, 4]. The above-defined Hamiltonian symplectomorphisms between domains in a symplectic manifold form a pseudogroup: a composition of two M -Hamiltonian symplectomorphisms $U \rightarrow V$ and $V \rightarrow W$ is M -Hamiltonian.

For every positive-valued mapping $\delta : C^\infty(\gamma) \rightarrow \mathbb{R}_+$ let $\mathcal{G}(\delta)$ denote the corresponding pseudogroup introduced in (1.3), which now consists of mappings between domains in the phase cylinder Π .

Theorem 1.12 *For every mapping $\delta : C^\infty(\gamma) \rightarrow \mathbb{R}_+$ the pseudogroup $\mathcal{G}(\delta)$ is C^∞ -dense in the pseudogroup of Π -Hamiltonian symplectomorphisms between domains in Π . In other words, for every domain $V \subset \Pi$ and every Π -Hamiltonian symplectomorphism $F : V \rightarrow W \subset \Pi$ there exists a sequence*

F_n of compositions of mappings $\Delta\mathcal{T}_{\varepsilon_j, h_j}$, F_n being defined on domains V_n with $V_n \cap V \rightarrow V$, such that $F_n \rightarrow F$ on V in the C^∞ -topology.

Corollary 1.13 *For every $\alpha > 0$ (arbitrarily small) and $k \in \mathbb{N}$ the pseudogroup generated by reflections from closed curves (α, k) -close to γ is C^∞ -dense in the pseudogroup of Π -Hamiltonian symplectomorphisms between domains in Π .*

Theorem 1.9 will be proved in Subsection 2.2. Theorem 1.12 and Corollary 1.13 will be proved in Section 3.

Remark 1.14 It would be interesting to study extensions of Theorems 1.4, 1.6 1.9, 1.12 and Corollaries 1.8, 1.13 to higher dimensions. This is a work in progress.

2 Proof of Theorems 1.4 and 1.9

In Subsection 2.1 we consider the local case and prove Theorem 1.4. In Subsection 2.2 we consider the global case and prove Theorem 1.9.

In what follows in each case we denote by \mathfrak{g} the Lie algebra of vector fields on Π generated by the corresponding vector space L_0 under the Lie bracket, see Theorems 1.4 and 1.9. By \mathfrak{H} we denote the Lie algebra generated by the Hamiltonian functions of the fields in L_0 with respect to the Poisson bracket, i.e., the vector space generated by the latter Hamiltonian functions and their successive Poisson brackets.

2.1 Local case: proof of Theorem 1.4

The Lie algebra \mathfrak{g} consists of Hamiltonian vector fields, by Proposition 1.2. The Lie bracket of two Hamiltonian vector fields is the Hamiltonian vector field for the Poisson bracket of their Hamiltonians. Each divergence free vector field on Π is Hamiltonian with respect to the form ω , by simple connectivity of the domain Π . Therefore, for the proof of Theorem 1.4 it suffices to prove density of the Lie algebra \mathfrak{H} of the Hamiltonians of the vector fields in \mathfrak{g} in the space of functions of two variables (s, ϕ) with respect to the Poisson bracket.

In the proof of Theorem 1.4 we use the following formulas for a Hamiltonian vector field χ_H with Hamiltonian $H(s, \phi)$ and for the Poisson bracket in the coordinates (s, ϕ) :

$$\chi_H = \frac{1}{\sin \phi} \left(\frac{\partial H}{\partial \phi}, -\frac{\partial H}{\partial s} \right); \{F, G\} = \frac{1}{\sin \phi} \left(\frac{\partial F}{\partial s} \frac{\partial G}{\partial \phi} - \frac{\partial F}{\partial \phi} \frac{\partial G}{\partial s} \right). \quad (2.1)$$

We also use the following trigonometric formulas:

$$|\sin \phi| = \frac{1}{\sqrt{1 + \cot^2 \phi}} = \frac{1}{\sqrt{1 + y^2}}; \quad y'_\phi = \cot'(\phi) = -(1 + y^2). \quad (2.2)$$

Note that our angles ϕ always lie in the interval $(0, \pi)$. Thus, $\sin \phi > 0$, and we can omit the above module sign.

For every $d \in \mathbb{Z}_{\geq 0}$ and every function $h(s)$ set

$$H_{d,h}(s, \phi) := \frac{y^d}{\sqrt{1 + y^2}} h(s), \quad y := \cot \phi; \quad H_{-1,h}(s, \phi) := 0. \quad (2.3)$$

Recall that the vector field v_h is Hamiltonian with the Hamiltonian $-H_{0,h} = H_{0,-h}$, see Proposition 1.2.

Proposition 2.1 *For every $d, k \in \mathbb{Z}_{\geq 0}$ and any two functions $f(s), g(s)$ one has*

$$\{H_{d,f}, H_{k,g}\} = H_{d+k-1, dfg' - kf'g} + H_{d+k+1, (d-1)fg' - (k-1)f'g}. \quad (2.4)$$

Proof For every $m \in \mathbb{Z}_{\geq 0}$ one has

$$\frac{\partial}{\partial \phi} \left(\frac{y^m}{\sqrt{1 + y^2}} \right) = - \left((m-1) \frac{y^{m+1}}{\sqrt{1 + y^2}} + m \frac{y^{m-1}}{\sqrt{1 + y^2}} \right),$$

which follows from (2.2). Therefore, for every function $q(s)$ one has

$$\frac{\partial H_{m,q}}{\partial \phi} = -((m-1)H_{m+1,q} + mH_{m-1,q}).$$

The latter formula together with (2.1) imply (2.4). \square

Corollary 2.2 *For every $d \in \mathbb{Z}_{\geq 0}$ let Λ_d denote the vector space of functions of the type $H_{d,f}$, where $f(s)$ runs through all the C^∞ -smooth functions in one variable. Let*

$$\pi_k : \bigoplus_{d=0}^{+\infty} \Lambda_d \rightarrow \Lambda_k$$

denote the projection to the k -th component. One has

$$\{\Lambda_0, \Lambda_0\} = \Lambda_1, \quad \{\Lambda_1, \Lambda_1\} \subset \Lambda_1, \quad (2.5)$$

$$\{\Lambda_d, \Lambda_k\} \subset \Lambda_{d+k-1} \oplus \Lambda_{d+k+1} \text{ whenever } (d, k) \neq (1, 1), (0, 0), \quad (2.6)$$

$$\pi_{k+1}(\{\Lambda_0, \Lambda_k\}) = \Lambda_{k+1} \text{ for every } k \geq 1. \quad (2.7)$$

The Lie algebra \mathfrak{S} generated by Λ_0 under the Poisson bracket coincides with

$$\mathfrak{S}_{loc} := \bigoplus_{d \geq 0} \Lambda_d.$$

Proof Inclusion (2.6) and the right inclusion in (2.5) follow immediately from (2.4). Let us prove the left formula in (2.5). One has

$$\{H_{0,f}, H_{0,g}\} = H_{1,f'g-g'f}, \quad (2.8)$$

by (2.4). It is clear that each function $\eta(s)$ can be represented by an expression $f'g - g'f$, since the functions in question are defined on an interval. For example, one can take $f = \int_{s_0}^s \eta(\tau) d\tau$ and $g \equiv 1$. This proves the left formula in (2.5). The proof of statement (2.7) is analogous. The last statement of the corollary follows from (2.5)-(2.7). \square

Proof of Proposition 1.3. The space L_1 consists of Hamiltonian vector fields with Hamiltonians from the space Λ_1 . This follows from definition, the first formula in (2.5) and the fact that taking Lie brackets of Hamiltonian vector fields corresponds to taking Poisson brackets of their Hamiltonians. This together with the right inclusion in (2.5) implies that L_1 is a Lie algebra. The vector field with a Hamiltonian function

$$H_{1,h} = \frac{y}{\sqrt{1+y^2}} h(s) = \cos \phi h(s) \in \Lambda_1$$

takes the form

$$\frac{1}{\sin \phi} \left(\frac{\partial H_{1,h}}{\partial \phi}, -\frac{\partial H_{1,h}}{\partial s} \right) = -(h(s), yh'(s)).$$

Proposition 1.3 is proved. \square

Proposition 2.3 *The Lie algebra $\mathfrak{G}_{loc} = \bigoplus_{d \geq 0} \Lambda_d$ is dense in the space of C^∞ -smooth functions on Π in the C^∞ -topology.*

Proof The polynomials in two variables (s, y) divided by $\sqrt{1+y^2}$ are dense, by Weierstrass Theorem and since the mapping $\phi \mapsto y = \cot \phi$ is a diffeomorphism $(0, \pi) \rightarrow \mathbb{R}$. Therefore, finite sums

$$\sum_{d=0}^m \frac{y^d}{\sqrt{1+y^2}} f_d(s) = \sum_{d=0}^m H_{d,f_d},$$

where f_d are arbitrary smooth functions, are also dense. This implies the statement of the proposition. \square

Proof of Theorem 1.4. The Lie algebra \mathfrak{g} generated by the vector space L_0 is dense in the space of Hamiltonian vector fields on Π in the C^∞ topology, by Corollary 2.2 and Proposition 2.3. All the area-preserving (i.e., divergence-free) vector fields on Π are Hamiltonian, since Π is a topological disk. This implies the statement of Theorem 1.4. \square

2.2 Global case: proof of Theorem 1.9

Let $H_{d,h}$ be functions on the phase cylinder Π given by formula (2.3); now $h(s)$ being C^∞ -smooth functions on the circle. Let Λ_d be the vector space of all the functions $H_{d,h}$. Set

$$\Lambda_{d,0} := \{H_{d,h} \in \Lambda_d \mid \int_0^{2\pi} h(s)ds = 0\}.$$

For every odd polynomial vanishing at zero with derivative,

$$P(y) = \sum_{j=1}^k a_j y^{2j+1}, \quad (2.9)$$

set

$$\tilde{P}(x) := x^{-\frac{1}{2}} P(x^{\frac{1}{2}}) = \sum_{j=1}^k a_j x^j. \quad (2.10)$$

Lemma 2.4 *The Lie algebra \mathfrak{S} of functions on Π generated by the vector space Λ_0 under the Poisson bracket coincides with*

$$\mathfrak{S}_{glob} = \Lambda_1 \oplus (\oplus_{d \in 2\mathbb{Z}_{\geq 0}} \Lambda_d) \oplus (\oplus_{d \in 2\mathbb{Z}_{\geq 1}+1} \Lambda_{d,0}) \oplus \Psi, \quad (2.11)$$

$$\Psi := \left\{ \frac{P(y)}{\sqrt{1+y^2}} \mid P(y) \text{ is a polynomial as in (2.9) with } \tilde{P}'(-1) = 0 \right\}. \quad (2.12)$$

For every C^∞ -smooth function $h : S^1 = \mathbb{R}/2\pi\mathbb{Z} \rightarrow \mathbb{R}$ set

$$\hat{h} := \int_0^{2\pi} h(s)ds.$$

In the proof of Lemma 2.4 we use the following four propositions.

Proposition 2.5 *For every $d, k \in \mathbb{Z}_{\geq 0}$ and every pair of smooth functions $f(s)$ and $g(s)$ on the circle one has*

$$\begin{aligned} \{H_{d,f}, H_{k,g}\} &= H_{d+k-1, h_{d+k-1}(s)} + H_{d+k+1, h_{d+k+1}(s)}, \\ (d+k-2)\hat{h}_{d+k-1} &= (d+k)\hat{h}_{d+k+1}. \end{aligned} \quad (2.13)$$

Proof The first formula in (2.13) holds with

$$h_{d+k-1} = (d+k)fg' - k(fg)', \quad h_{d+k+1} = (d+k-2)fg' - (k-1)(fg)', \quad (2.14)$$

by (2.4). This together with the fact that the derivative $(fg)'$ has zero average implies that the ratio of averages of the functions $h_{d+k\mp 1}$ is equal to $\frac{d+k}{d+k-2}$ and proves (2.13). \square

It is clear that the Lie algebra \mathfrak{H} is contained in the direct sum of the subspaces Λ_j , by (2.4). Recall that for every $j \in \mathbb{Z}_{\geq 0}$ by π_j we denote the projection of the latter direct sum to the j -th component Λ_j .

Proposition 2.6 *For every $d, k \in \mathbb{Z}_{\geq 0}$ one has*

$$\{\Lambda_d, \Lambda_k\} \subset \Lambda_{d+k-1} \oplus \Lambda_{d+k+1}, \quad (2.15)$$

$$\pi_{d+k+1}(\{\Lambda_d, \Lambda_k\}) = \pi_{d+k+1}(\{\Lambda_{d,0}, \Lambda_{k,0}\}) = \begin{cases} 0 & \text{if } d = k = 1, \\ \Lambda_{d+k+1}, & \text{if } d + k \neq 2, \\ \Lambda_{3,0}, & \text{if } \{d, k\} = \{0, 2\}. \end{cases} \quad (2.16)$$

In particular,

$$\{\Lambda_0, \Lambda_0\} = \Lambda_1. \quad (2.17)$$

Proof For $d = k = 1$ formula (2.16) follows from (2.4). For $d = 0, k = 2$ one has $\pi_3(\{\Lambda_0, \Lambda_2\}) \subset \Lambda_{3,0}$, by (2.13): the left-hand side in (2.13) vanishes, hence, $\widehat{h}_{d+k+1} = \widehat{h}_3 = 0$. Let us prove that in fact, the latter inclusion is equality and moreover,

$$\pi_3(\{\Lambda_{0,0}, \Lambda_{2,0}\}) = \Lambda_{3,0}. \quad (2.18)$$

Indeed, for every two functions f and g on the circle one has

$$\{H_{0,f}, H_{2,g}\} = H_{1,-2f'g} + H_{3,-(fg)'},$$

by (2.4). It is clear that every function h on the circle with zero average is a derivative of some function $-f$ on the circle (we choose f with zero average). Hence, $h = -f' = (fg)'$ with $g \equiv 1$. This already implies the formula $\pi_3(\{\Lambda_0, \Lambda_2\}) = \Lambda_{3,0}$, but not the above formula (2.18): the function $g \equiv 1$ does not have zero average. To prove (2.18), let us show that every function f with zero average can be represented as a sum $\sum_{j=1}^4 f_j g_j$ with f_j and g_j being smooth functions of zero average. Indeed, f is the sum of a linear combination $f_{=1}(s) = ae^{is} + \bar{a}e^{-is}$, $a \in \mathbb{C}$, and a Fourier series $f_{\geq 2}$

containing only e^{ins} with $|n| \geq 2$. It is clear that $f_{=1}(s) = e^{3is}(e^{-3is}f_{=1}(s))$ and $f_{\geq 2} = e^{is}(e^{-is}f_{\geq 2})$, and the latter are products of two complex functions with zero average. Their real parts are obviously sums of pairs of such products. Therefore, f can be represented as a sum of four such products. This together with the above discussion implies that for every function h with zero average the function $H_{3,h}$ is the π_3 -projection of a sum of four Poisson brackets $\{H_{0,f_j}, H_{2,g_j}\}$ with f_j, g_j being of zero average. This proves (2.18) and the third formula in (2.16).

Let us now treat the remaining middle case: $d+k \neq 2$. To do this, it suffices to show that every smooth function $h(s)$ on the circle can be represented as a finite sum

$$h = \sum_{l=1}^N h_{d+k+1,l}, \quad h_{d+k+1,l} := (d-1)f_l g_l' - (k-1)f_l' g_l, \quad (2.19)$$

see (2.13) and (2.4), where $f_l(s)$ and $g_l(s)$ are smooth functions on the circle with zero average. Moreover, it suffices to prove the same statement for complex-valued functions. Indeed, if (2.19) holds for a complex function $h(s)$ and a finite collection of pairs $(f_l(s), g_l(s))$ of complex functions, $l = 1, \dots, N$, then the similar equality holds for the function $\operatorname{Re} h_l(s)$ and the collection of pairs $(\operatorname{Re} f_l, \operatorname{Re} g_l)$, $(-\operatorname{Im} f_l, \operatorname{Im} g_l)$ taken for all l . Let, say, $d \neq 1$. Let us write a complex function $h(s)$ as a Fourier series

$$h(s) = \sum_{n \in \mathbb{Z}} a_n e^{ins}.$$

Set

$$f_1(s) = e^{is}, \quad g_1(s) = \sum_{n \in \mathbb{Z}_{\neq 1}} b_n e^{i(n-1)s}.$$

Set $h_{d+k+1,1} := (d-1)f_1 g_1' - (k-1)f_1' g_1$. One has

$$h(s) - h_{d+k+1,1}(s) = a_1 e^{is} + \sum_{n \neq 1} (a_n - ib_n((d-1)(n-1) - (k-1))) e^{ins}. \quad (2.20)$$

We would like to make the above difference zero. For each individual $n \neq 1$ one can solve the equation

$$a_n - ib_n((d-1)(n-1) - (k-1)) = 0,$$

provided that $(d-1)(n-1) \neq k-1$, i.e., $n \neq n(d,k) := \frac{k-1}{d-1} + 1$. Take b_n found from the above equation for all $n \neq 1, n(d,k)$. They yield a converging

and C^∞ -smooth Fourier series

$$g_1(s) = \sum_{n \neq 1, n(d,k)} b_n e^{i(n-1)s},$$

since so is $h(s) = \sum_{n \in \mathbb{Z}} a_n e^{ins}$ and $b_n = o(a_n)$, as $n \rightarrow \infty$. The corresponding function $h_{d+k+1,1}$, see (2.19), satisfies the equality

$$h(s) - h_{d+k+1,1}(s) = a_1 e^{is} + a_{n(d,k)} e^{isn(d,k)}. \quad (2.21)$$

Now we set $f_2(s) = e^{(p+1)is}$ with some $p \in \mathbb{Z} \setminus \{0, -1, n(d,k) - 1\}$, and we would like to find a function

$$g_2(s) = c_1 e^{-pis} + c_2 e^{i(n(d,k)-(p+1))s}$$

such that $h = h_{d+k+1,1} + h_{d+k+1,2}$, see (2.19). The latter equation is equivalent to the equation

$$a_1 e^{is} + a_{n(d,k)} e^{isn(d,k)} = (d-1)f_2(s)g_2'(s) - (k-1)f_2'(s)g_2(s), \quad (2.22)$$

by (2.21). Its right-hand side divided by i equals $c_1(-p(d-1) - (p+1)(k-1))e^{is} + c_2((d-1)(n(d,k) - (p+1)) - (k-1)(p+1))e^{isn(d,k)}$. Therefore, one can find constant coefficients c_1, c_2 in the definition of the function g_2 such that equation (2.22) holds, if

$$(n(d,k)-(p+1))(d-1)-(p+1)(k-1) \neq 0, \quad p(d-1)+(p+1)(k-1) \neq 0. \quad (2.23)$$

The left-hand sides of inequalities (2.23) are linear non-homogeneous functions in p with coefficients at p being equal to $\mp(d+k-2) \neq 0$. Hence, choosing appropriate $p \in \mathbb{Z} \setminus \{0, -1, n(d,k) - 1\}$ one can achieve that inequalities (2.23) hold and hence, equation (2.22) can be solved in c_1, c_2 . Finally, we have solved equation (2.19) with an arbitrary complex function h and $N = 2$, in complex functions $f_l, g_l, l = 1, 2$ with zero averages. This together with the above discussion finishes the proof of statement (2.16). Statement (2.17) follows from the second statement in (2.16). \square

Proposition 2.7 *The Lie algebra \mathfrak{H} contains the direct sum*

$$\mathfrak{G}_{glob,0} := \Lambda_1 \oplus (\oplus_{d \in 2\mathbb{Z}_{\geq 0}} \Lambda_d) \oplus (\oplus_{d \in 2\mathbb{Z}_{\geq 1}+1} \Lambda_{d,0}).$$

Proof The algebra \mathfrak{H} contains Λ_0 and $\Lambda_1 = \{\Lambda_0, \Lambda_0\}$, see (2.17). It also contains Λ_2 , by the latter statement and since $\{\Lambda_0, \Lambda_1\} \subset \Lambda_0 \oplus \Lambda_2$, see

(2.15), and $\pi_2(\{\Lambda_0, \Lambda_1\}) = \Lambda_2$, by (2.16). Hence, $\mathfrak{H} \supset \bigoplus_{j=0}^2 \Lambda_j$. Analogously one has $\mathfrak{H} \supset \Lambda_{3,0}$, by the latter statement, and (2.15), (2.16) applied to $(d, k) = (0, 2)$. Hence, $\mathfrak{H} \supset (\bigoplus_{j=0}^2 \Lambda_j) \oplus \Lambda_{3,0}$. One has $\mathfrak{H} \supset \Lambda_4$, by the latter statement and (2.15), (2.16) applied to $(d, k) = (0, 3)$. Hence, $\mathfrak{H} \supset (\bigoplus_{0 \leq j \leq 4, j \neq 3} \Lambda_j) \oplus \Lambda_{3,0}$. Let us show that $\mathfrak{H} \supset \Lambda_5$. Indeed, the bracket $\{\Lambda_0, \Lambda_4\}$ is contained in $\Lambda_3 \oplus \Lambda_5$, see (2.15). For every functions f and g on the circle one has $\{H_{0,f}, H_{4,g}\} = H_{3,h_3} + H_{5,h_5}$, where the ratio of averages of the functions h_3 and h_5 is equal to 2, see (2.13). Therefore, if the average of the function h_5 vanishes, then so does the average of the function h_3 . This implies that the subspace of those elements in $\{\Lambda_0, \Lambda_4\}$ whose projections to Λ_5 have zero averages coincides with the subspace with the analogous property for the projection π_3 . This together with the above statement implies that \mathfrak{H} contains $\Lambda_{5,0}$. Applying the above argument successively to Poisson brackets $\{\Lambda_0, \Lambda_n\}$, $n \geq 5$, we get the statement of Proposition 2.7. \square

Proposition 2.8 *The Lie algebra \mathfrak{H} is the direct sum of the subspace $\mathfrak{G}_{glob,0}$ and a vector subspace Ψ in*

$$\mathcal{P} = \left\{ \frac{P(y)}{\sqrt{1+y^2}} \mid P \text{ is an odd polynomial, } P'(0) = 0 \right\}.$$

The corresponding subspace $\sqrt{1+y^2}\Psi$ in the space of polynomials is generated by the polynomials

$$R_j(y) := jy^{2j-1} + (j-1)y^{2j+1}, \quad j \in \mathbb{N}, \quad j \geq 2. \quad (2.24)$$

Proof The direct sum $\bigoplus_{j \geq 0} \Lambda_j \supset \mathfrak{H}$ is the direct sum of the spaces $\mathfrak{G}_{glob,0}$ and \mathcal{P} , by definition. This together with Proposition 2.7 implies that \mathfrak{H} is the direct sum of the subspace $\mathfrak{G}_{glob,0}$ and a subspace $\Psi \subset \mathcal{P}$. Let us describe the subspace Ψ . To do thus, consider the projection

$$\pi_{\text{odd}>1} : \bigoplus_{j \geq 0} \Lambda_j \rightarrow \bigoplus_{j \in 2\mathbb{Z}_{\geq 1}+1} \Lambda_j.$$

Claim 1. *The projection $\pi_{\text{odd}>1}\mathfrak{H}$ lies in \mathfrak{H} . It is spanned as a vector space over \mathbb{R} by the subspace $\Lambda_{3,0}$ and some Poisson brackets $\{H_{d,a(s)}, H_{k,b(s)}\}$ with $d+k \geq 4$ being even. All the above brackets with all the functions $a(s)$, $b(s)$ with zero average lie in $\pi_{\text{odd}>1}\mathfrak{H}$.*

Proof The inclusion $\pi_{\text{odd}>1}\mathfrak{H} \subset \mathfrak{H}$ follows from the fact that $\pi_{\text{odd}>1}$ is the projection along the vector subspace $\Lambda_1 \oplus (\bigoplus_{j \in 2\mathbb{Z}_{\geq 0}} \Lambda_j) \subset \mathfrak{G}_{glob,0} \subset \mathfrak{H}$. Each

element of the Lie algebra \mathfrak{H} is represented as a sum of a vector in Λ_0 and a linear combination of Poisson brackets $\{H_{d,a}, H_{k,b}\}$, by definition and (2.4). The latter Poisson brackets lie in $\Lambda_{d+k-1} \oplus \Lambda_{d+k+1}$, by (2.15), and thus, have components of the same parity $d+k+1 \pmod{2}$. Note that if $d+k-1=1$, then the above bracket lies in $\Lambda_1 \oplus \Lambda_{3,0} \subset \mathfrak{G}_{glob,0} \subset \mathfrak{H}$, by (2.16), and its $\pi_{odd>1}$ -projection lies in $\Lambda_{3,0}$. The two last statements together imply the second statement of the claim. If $a(s)$ and $b(s)$ have zero average, then $H_{d,a}, H_{k,b} \in \mathfrak{G}_{glob,0} \subset \mathfrak{H}$, thus, $\{H_{d,a}, H_{k,b}\} \in \mathfrak{H}$. Hence, they lie in $\pi_{odd>1}\tilde{\mathfrak{H}}$, if $d+k$ is even and no less than 4. This proves the claim. \square

Taking projection π_Ψ of a vector $w \in \mathfrak{H}$ to Ψ consists of first taking its projection

$$\pi_{odd>1}w = \sum_{j=1}^k H_{2j+1, f_j(s)} = \frac{1}{\sqrt{1+y^2}} \sum_{j=1}^k f_j(s) y^{2j+1}$$

and then replacing each $f_j(s)$ in the above right-hand side by its average \widehat{f}_j :

$$\pi_\Psi w = \frac{1}{\sqrt{1+y^2}} P_w(y), \quad P_w(y) = \sum_{j=1}^k \widehat{f}_j y^{2j+1}.$$

If $w = \{H_{d,a}, H_{k,b}\}$ with $d+k = 2j \geq 4$, then $P_w(y) = cR_j(y)$, see (2.24), $c \in \mathbb{R}$, which follows from (2.13). This together with Claim 1 implies that the vector space $\frac{1}{\sqrt{1+y^2}}\Psi$ is contained in the vector space spanned by the polynomials R_j . Now it remains to prove the converse: each R_j is contained in Ψ . To this end, we have to find the above functions a and b with zero average so that $P_w(y) \neq 0$, i.e., so that the above constant factor c be non-zero. Let $b(s)$ be an arbitrary smooth non-constant function on the circle with zero average. Set $a(s) = b'(s)$. Let $d+k = 2j \geq 4$. Then

$$w = \{H_{d,a}, H_{k,b}\} = \frac{1}{\sqrt{1+y^2}} (f_{j-1}(s) y^{2j-1} + f_j(s) y^{2j+1}),$$

$$f_{j-1} = 2j(b')^2 - k(b'b)',$$

by (2.14). The function $f_{j-1}(s)$ has positive average, since so does its first term, while its second term has zero average. Therefore, $P_w = cR_j$, $c > 0$. Proposition 2.8 is proved. \square

Lemma 2.9 *The vector subspace generated by the polynomials R_j from (2.24) coincides with the space of odd polynomials $P(y)$ with $P'(0) = 0$ such that the corresponding polynomial $\tilde{P}(x) = x^{-\frac{1}{2}}P(x^{\frac{1}{2}})$ has vanishing derivative at -1 .*

As it is shown below, the lemma is implied by the following proposition.

Proposition 2.10 *An odd polynomial*

$$P_k(y) = \sum_{j=1}^k a_j y^{2j+1} \quad (2.25)$$

is a linear combination of polynomials $R_d(y)$, see (2.24), if and only if

$$\sum_{j=1}^k (-1)^j j a_j = 0. \quad (2.26)$$

Proof The polynomials (2.24) obviously satisfy (2.26). In the space of odd polynomials $P(y)$ with $P'(0) = 0$ of any given degree $d \geq 5$ equation (2.26) defines a hyperplane. The polynomials (2.24) of degree no greater than d also generate a hyperplane there. Hence, these two hyperplanes coincide. The proposition is proved. \square

Proof of Lemma 2.9. For every odd polynomial $P(y)$ as in (2.25) one has

$$\tilde{P}(x) = \sum_{j=1}^k a_j x^j.$$

Hence, equation (2.26) is equivalent to the equation $\tilde{P}'(-1) = 0$. Lemma 2.9 is proved. \square

Proof of Lemma 2.4. Lemma 2.4 follows from Proposition 2.8 and Lemma 2.9. \square

Lemma 2.11 *The Lie algebra $\mathfrak{G}_{glob} = \mathfrak{H}$ is C^∞ -dense in the space of smooth functions on the phase cylinder Π .*

Proof Let us multiply each function from \mathfrak{G}_{glob} by $\sqrt{1+y^2}$; then each function becomes a polynomial in y with coefficients being smooth functions on a circle. All the polynomials in y with coefficients as above are C^∞ -dense in the space of functions on $\Pi = S_s^1 \times (0, \pi)_\phi$, by Weierstrass Theorem and

since the mapping $\phi \mapsto y = \cot \phi$ is a diffeomorphism $(0, \pi) \rightarrow \mathbb{R}$. The polynomials realized by functions from \mathfrak{H} in the above way are exactly the polynomials that are represented in unique way as sums of at most four polynomials of the following types:

- 1) any polynomial of degree at most 2;
- 2) any even polynomial containing only monomials of degree at least 4;
- 3) any odd polynomial of type $P(y; s) = \sum_{j=1}^k a_j(s)y^{2j+1}$ with coefficients $a_j(s)$ being of zero average;
- 4) any odd polynomial of type $P(y) = \sum_{j=1}^k b_j y^{2j+1}$ with constant coefficients b_j and $\tilde{P}'(-1) = 0$.

This follows from Lemma 2.4.

For the proof of Lemma 2.11 it suffices to show that the odd polynomials of type 4) are C^∞ -dense in the space of odd polynomials in y with constant coefficients and vanishing derivative at 0.

Take an arbitrary odd polynomial of type $Q(y) = \sum_{j=1}^k b_j y^{2j+1}$. The polynomial $\tilde{Q}(x) := x^{-\frac{1}{2}}Q(x^{\frac{1}{2}}) = \sum_{j=1}^k b_j x^k$ can be approximated in the topology of uniform convergence with derivatives on segments $[0, A]$ with A arbitrarily large by polynomials $\tilde{R}(x)$ with $\tilde{R}(0) = 0$ and $\tilde{R}'(-1) = 0$. Indeed, let us extend the restriction $\tilde{Q}|_{\{x \geq 0\}}$ to a C^∞ -smooth function on the semi-interval $[-1, +\infty)$ with vanishing derivative at -1 . Thus extended function can be approximated by polynomials $\tilde{H}_n(x)$. One can normalize the above polynomials \tilde{H}_n to vanish at 0 and to have zero derivative at -1 by adding a small linear non-homogeneous function $a_n x + b_n$. Then the corresponding polynomials $H_n(y) := y\tilde{H}_n(y^2)$ are of type 4) and approximate $Q(y)$. This together with the above discussion proves Lemma 2.11. \square

Proof of Theorem 1.9. Theorem 1.9 follows from Lemmas 2.4 and 2.11. \square

3 Density of pseudo-groups. Proofs of Theorems 1.6, 1.12 and Corollaries 1.8, 1.13

Let γ be either a germ of planar curve, or a strictly convex planar closed curve; it is supposed to be C^∞ -smooth. In what follows Π will denote an open subset in the space of oriented lines in \mathbb{R}^2 where the reflection from the curve γ is well-defined: it is either a topological disk as in Subsection 1.1, or the phase cylinder as in Subsection 1.2.

Definition 3.1 Let M be a manifold, $V \subset M$ be an open subset, v be a

smooth vector field on M , and $t \in \mathbb{R}$, $t \neq 0$. We say that the time t flow map g_v^t is *well-defined on V* , if all the flow maps g_v^τ with $\tau \in (0, t]$ ($\tau \in [t, 0)$, if $t < 0$) are well-defined on V , that is, the corresponding differential equation has a well-defined solution for every initial condition in V for all the time values $\tau \in (0, t]$ (respectively, $\tau \in [t, 0)$).

Recall that \mathfrak{H} denotes the Lie algebra of functions on Π generated by the space Λ_0 of functions of type $H(s, \phi) = -h(s) \sin \phi$ with respect to the Poisson bracket.

Proposition 3.2 *Let \mathcal{F} denote the pseudogroup generated by flow maps (well-defined on domains in the sense of the above definition) of the Hamiltonian vector fields with Hamiltonian functions from the Lie algebra \mathfrak{H} . For every mapping $\delta : C^\infty(\gamma) \rightarrow \mathbb{R}_+$ the C^∞ -closure $\overline{\mathcal{G}(\delta)}$ of the corresponding pseudogroup $\mathcal{G}(\delta)$, see (1.3), contains \mathcal{F} .*

Proof The Lie algebra of Hamiltonian vector fields with Hamiltonians in \mathfrak{H} consists of finite linear combinations of successive commutators of vector fields with Hamiltonians in Λ_0 , which form the vector space L_0 .

Claim 1. *The well-defined flow maps (on domains) of each iterated commutator of a collection of vector fields in L_0 are contained in $\overline{\mathcal{G}(\delta)}$.*

Proof Induction in the number of Lie brackets in the iterated commutator.

Basis of induction. The space L_0 consists of the derivatives $v_h := \frac{d}{d\varepsilon} \Delta \mathcal{T}_{\varepsilon, h}$, $\Delta \mathcal{T}_{\varepsilon, h} = \mathcal{T}_{\gamma_{\varepsilon, h}}^{-1} \circ \mathcal{T}_\gamma$, for all C^∞ -smooth functions h . Fix an arbitrary function $h \in C^\infty(\gamma)$, a domain $W \subset \Pi$ and a $t \in \mathbb{R} \setminus \{0\}$, say $t > 0$, such that the flow map $g_{v_h}^t$ is well-defined on W . Note that the mapping $F_\varepsilon := (\Delta \mathcal{T}_{\varepsilon, h})^{[\frac{t}{\varepsilon}]}$ converges to $g_{v_h}^t$, as $\varepsilon \rightarrow 0$, whenever it is defined. Since $g_{v_h}^t$ is well-defined on W , we therefore see that F_ε is well-defined on a domain $W_\varepsilon \subset \Pi$ such that $W_\varepsilon \cap W \rightarrow W$, as $\varepsilon \rightarrow 0$, and F_ε converges to $g_{v_h}^t$ on W in the C^∞ topology. Note that $F_\varepsilon \in \mathcal{G}(\delta)$. Therefore, $g_{v_h}^t|_W \in \overline{\mathcal{G}(\delta)}$. Thus, the flow maps of the vector fields in L_0 are contained in $\overline{\mathcal{G}(\delta)}$.

Induction step. Let v and w be vector fields on Π such that their well-defined flow maps lie in $\overline{\mathcal{G}(\delta)}$. Let us prove the same statement for their commutator $[v, w]$. Note that for small τ the flow maps g_v^τ , g_w^τ are well-defined on domains in Π converging to Π , as $\tau \rightarrow 0$. Let $t \in \mathbb{R} \setminus \{0\}$, say $t > 0$, and a domain $D \subset \Pi$ be such that the flow map $g_{[v, w]}^t$ be well-defined on D . Then $g_{[v, w]}^t|_D \in \overline{\mathcal{G}(\delta)}$, by a classical argument: the composition $(g_v^{\frac{t}{N}} \circ g_w^{\frac{t}{N}} \circ g_v^{-\frac{t}{N}} \circ g_w^{-\frac{t}{N}})^{N^2}$ is well-defined on a domain W_N with $W_N \cap D \rightarrow D$,

as $N \rightarrow \infty$; it belongs to $\overline{\mathcal{G}(\delta)}$ and converges to $g_{[v,w]}^t$ on D in the C^∞ -topology. The induction step is done. The claim is proved. \square

Claim 2. *Let well-defined flow maps of vector fields v and w lie in $\overline{\mathcal{G}(\delta)}$. Then well-defined flow maps of all their linear combinations also lie in $\overline{\mathcal{G}(\delta)}$.*

Proof It suffices to prove the claim for the sum $v + w$, since $g_{cv}^t = g_v^{ct}$. The composition $(g_v^{\frac{t}{N}} g_w^{\frac{t}{N}})^N$ obviously converges to g_{v+w}^t on every domain where the latter flow map is well-defined. This proves the claim. \square

Claims 1 and 2 together imply the statement of Proposition 3.2. \square

Recall the following well-known notion.

Definition 3.3 We say that a symplectomorphism F of a symplectic manifold M has *compact support* if it is equal to the identity outside some compact subset. We say that it is *Hamiltonian with compact support*, if it can be connected to the identity by a smooth path F_t in the group of symplectomorphisms $M \rightarrow M$, $F_0 = Id$, $F_1 = F$, such that the vector fields $\frac{dF_t}{dt}$ are Hamiltonian with compact supports contained in one and the same compact subset.

Proposition 3.4 *The group of Hamiltonian symplectomorphisms with compact support is dense in the pseudogroup of M -Hamiltonian symplectomorphisms between domains in the ambient manifold M .*

Proof Consider the Hamiltonian vector fields $\frac{dH_t}{dt}$ on domains V_t from the definition of a M -Hamiltonian symplectomorphism $V \rightarrow U \subset M$ (see Definition 1.10); let $g_t : V_t \rightarrow \mathbb{R}$ denote the corresponding Hamiltonian functions. We may identify V_t with V by the mappings H_t and consider g_t as one function g on $V \times [0, 1]_t$. We can approximate it by functions g_n with compact supports; $g_n \rightarrow g$ in the C^∞ -topology, i.e., the convergence is uniform with all derivatives on compact subsets. The approximating functions yield a family of globally defined functions $g_{t,n} : M \rightarrow \mathbb{R}$ with compact supports contained in V_t . Let $v_{n,t}$ denote the Hamiltonian vector fields with Hamiltonians $g_{t,n}$. Then the time 1 map of the non-autonomous differential equation defined by $v_{t,n}$ converges to F on V in the C^∞ -topology, as $n \rightarrow \infty$. \square

Proposition 3.5 *The whole group of Hamiltonian symplectomorphisms $\Pi \rightarrow \Pi$ with compact support lies in the C^∞ -closure of the pseudogroup generated by well-defined flow maps of Hamiltonian vector fields with Hamiltonian functions from the Lie algebra \mathfrak{H} .*

Proof Consider a Hamiltonian symplectomorphism $F : \Pi \rightarrow \Pi$ with compact support, let $v(x, t) = \frac{dF_t}{dt}$ be the corresponding family of Hamiltonian vector fields with compact supports. The map F is the C^∞ -limit of compositions of time $\frac{1}{n}$ flow maps of the autonomous Hamiltonian vector fields $v(x, \frac{k}{n})$, $k = 0, \dots, n-1$, as $n \rightarrow \infty$. Each above Hamiltonian vector field $v(x, \frac{k}{n})$ can be approximated by Hamiltonian vector fields with Hamiltonians in \mathfrak{H} , by Theorems 1.4 and 1.9. Then F becomes approximated by products of their flows. This implies the statement of Proposition 3.5. \square

Proof of Theorem 1.12. Each Π -Hamiltonian symplectomorphism between domains in Π is a limit of a converging sequence of Hamiltonian symplectomorphisms $\Pi \rightarrow \Pi$ with compact supports in the C^∞ -topology, by Proposition 3.4. The latter symplectomorphisms are, in their turn, limits of compositions of well-defined flow maps of vector fields with Hamiltonians from the algebra \mathfrak{H} , by Proposition 3.5. For every mapping $\delta : C^\infty(\gamma) \rightarrow \mathbb{R}_+$ the flow maps under question lie in $\overline{\mathcal{G}(\delta)}$, by Proposition 3.2. This proves Theorem 1.12. \square

To prove Theorem 1.6, which states similar approximability of (a priori non-Hamiltonian) symplectomorphisms in the local case, we use the following three well-known propositions communicated to the author by Felix Schlenk.

Proposition 3.6 *Let M be a manifold. Let $V \subset M$ be an open subset with compact closure and smooth boundary that is C^∞ -smoothly diffeomorphic to a closed ball. Then every diffeomorphism $F : \overline{V} \rightarrow F(\overline{V}) \Subset M$ extends to a C^∞ -smooth diffeomorphism $F : M \rightarrow M$ isotopic to identity with compact support. The isotopy can be chosen C^∞ -smooth and so that its diffeomorphisms coincide with the identity outside some compact subset.*

Proof (Felix Schlenk). **Step 1.** It suffices to show that

Lemma 3.7 *The space of smooth embeddings $\varphi : \overline{B}^n \rightarrow M$ is connected.*

Indeed, given V as in the proposition, choose a diffeomorphism $\varphi : \overline{B}^n \rightarrow V$. By the lemma, we find a smooth family $\varphi_t : \overline{B}^n \rightarrow M$ of smooth embeddings, $t \in [0, 1]$, with $\varphi_0 = \varphi$ and $\varphi_1 = F \circ \varphi$. Consider the vector field $v(x, t) = \frac{d}{dt} \varphi_t(x)$ that generates this isotopy. The vector field v is defined on the compact subset $\{(x, t) \mid x \in \varphi_t(\overline{B}^n)\}$ of $M \times [0, 1]$. Choose any smooth extension \tilde{v} of v to $M \times [0, 1]$ that vanishes outside a compact set. Then the flow maps of \tilde{v} , $t \in [0, 1]$, form the desired isotopy.

Step 2. Proof of the lemma.

Let $\varphi_0, \varphi_1: \overline{B}^n \rightarrow M$ be two smooth embeddings. Take a smooth isotopy $g_t, t \in [0, 1]$, of M that moves $\varphi_0(0)$ to $\varphi_1(0)$. Choose $\varepsilon > 0$ such that

$$g_1(\varphi_0(\overline{B}_\varepsilon^n)) \subset \varphi_1(\overline{B}^n).$$

Consider the smooth family of embeddings $\overline{B}_1^n \rightarrow M$ defined by

$$\varphi_0^s(x) = \varphi_0(sx), \quad s \in [\varepsilon, 1]. \quad (3.1)$$

By definition, φ_1 is the restriction of a smooth embedding $\tilde{\varphi}_1$ of the open ball $B_{1+\delta}^n$. We now have two smooth embeddings $g_1 \circ \varphi_0^\varepsilon, \varphi_1: \overline{B}_1^n \rightarrow \tilde{\varphi}_1(B_{1+\delta}^n)$. Since $\tilde{\varphi}_1(B_{1+\delta}^n)$ is diffeomorphic to \mathbb{R}^n , the classical Alexander trick shows that the embeddings $g_1 \circ \varphi_0^\varepsilon, \varphi_1$ can be connected, see e.g. [14, Appendix A, formula (A.1)]. Since $g_1 \circ \varphi_0^\varepsilon$ is connected to φ_0^ε by $g_t \circ \varphi_0^\varepsilon$, and since φ_0^ε is connected to φ_0 by (3.1), the lemma and the proposition are proven. \square

Proposition 3.8 *Let M be a two-dimensional symplectic manifold, and let $V \subset M$ be an open subset with compact closure and smooth boundary. Let \overline{V} be C^∞ -smoothly diffeomorphic to a disk. Then every symplectomorphism $F: \overline{V} \rightarrow F(\overline{V}) \Subset M$ can be extended to a symplectomorphism $F: M \rightarrow M$ with compact support. Moreover, F can be chosen isotopic to the identity via symplectomorphisms with compact supports contained in one and the same compact subset in M .*

Proof (Felix Schlenk). Consider an arbitrary extension G of F to a diffeomorphism $G: M \rightarrow M$ with compact support that is isotopic to the identity through diffeomorphisms $G_s: M \rightarrow M$ with compact supports contained in a common compact subset $K \subset M, K \supset (\overline{V} \cup F(\overline{V}))$; $s \in [0, 1]$, $G_0 = Id$, $G_1 = G$. It exists by Proposition 3.6. Set $W := M \setminus K$. Let ω be the symplectic form of the manifold M . Consider its pullback $\alpha := G^*(\omega)$, which is a symplectic form on M . The forms $\alpha_t := t\alpha + (1-t)\omega, t \in [0, 1]$, are also symplectic: they are obviously closed, and they are positive area forms, since M is two-dimensional. This is the place we use two-dimensionality. The forms α_t are cohomologous to ω , the corresponding areas of the manifold M are equal, and $\alpha_t \equiv \omega$ on $\overline{V} \cup W$. One has $\alpha_0 = \omega, \alpha_1 = \alpha$. There exists a family of diffeomorphisms $S_t: M \rightarrow M, t \in [0, 1]$, $S_0 = Id$, such that $S_t^*(\alpha_t) = \omega$ and S_t coincide with the identity on the set $V \cup W$, where $\omega = \alpha_t$. This follows from the relative version of Moser's deformation argument [8, exercise 3.18]; see also [11], [5, p.11]. The composition $\Phi := G \circ S_1: M \rightarrow M$ preserves the symplectic form ω and coincides with G on $V \cup W$. Hence, $\Phi|_V = F, \Phi|_W = Id$.

Applying the above construction of the symplectomorphism Φ to the diffeomorphisms G_s instead of G yields a smooth isotopy Φ_s of the symplectomorphism $\Phi = \Phi_1$ to the identity via symplectomorphisms with supports in K . (But now G_s is not necessarily symplectic on V , and $(\Phi_s)|_V$ is not necessarily the identity.) The proposition is proved. \square

Proposition 3.9 *Every symplectomorphism with compact support of an open topological disk equipped with a symplectic structure is a Hamiltonian symplectomorphism with compact support.*

Proof Every symplectomorphism in question is isotopic to the identity via a smooth family F_t of symplectomorphisms with compact support, by Proposition 3.8. The derivatives $\frac{dF_t}{dt}$ form a t -dependent family of divergence-free vector fields. Each divergence-free vector field X on a topological disk equipped with a symplectic form ω is symplectic. Hence, $L_X\omega = d(i_X\omega) = 0$, by Cartan's formula. Thus, $i_X\omega$ is a closed 1-form and hence, exact, $i_X\omega = dH$, since the underlying manifold is a topological disk. This proves the proposition. \square

Remark 3.10 Proposition 3.9 is a part of the following well-known fact: *the group of compactly-supported symplectomorphisms of an open symplectic topological disk is contractible and path-connected by smooth paths.* The same statement holds for symplectomorphisms of \mathbb{R}^4 (Gromov's theorem [9, p.345, theorem 9.5.2]). In higher dimensions it is not known whether a similar statement is true. For every $k \in \mathbb{N}$ there exists an exotic symplectic structure on \mathbb{R}^{4k} admitting a symplectomorphism $\mathbb{R}^{4k} \rightarrow \mathbb{R}^{4k}$ with compact support that is not smoothly isotopic to the identity in the class of (not necessarily symplectic) diffeomorphisms [4, theorem 1.1].

Proof of Theorem 1.6. Let $V \subset \Pi$ be an arbitrary simply connected domain. Let us show that each symplectomorphism $F : V \rightarrow F(V) \subset \Pi$ can be approximated by elements of the pseudogroup $\mathcal{G}(\delta)$ for every δ . Consider an exhaustion $V_1 \Subset V_2 \Subset \dots$ of the domain $V = V_\infty$ by simply connected domains with compact closures in V and smooth boundaries. Each restriction $F|_{V_n}$ extends to a symplectomorphism with compact support, by Proposition 3.8. The latter extension is Hamiltonian with compact support, by Proposition 3.9. The elements of the pseudogroup $\mathcal{G}(\delta)$ accumulate to the restrictions to V of all the Hamiltonian symplectomorphisms $\Pi \rightarrow \Pi$ with compact support, see the above proof of Theorem 1.12. Therefore, some

sequence of elements in $\mathcal{G}(\delta)$ converges to F on V in the C^∞ -topology. This proves Theorem 1.6. \square

Proof of Corollaries 1.8 and 1.13. Fix some $\alpha > 0$ and $k \in \mathbb{N}$. We can choose a mapping $\delta : C^\infty(\gamma) \rightarrow \mathbb{R}_+$ so that for every $h \in C^\infty(\gamma)$ and every $\varepsilon \in [0, \delta(h))$ the curve $\gamma_{\varepsilon, h}(s) = \gamma(s) + \varepsilon \vec{n}(s)h(s)$ is (α, k) -close to γ . Then the statements of Corollaries 1.8 and 1.13 follow immediately from Theorems 1.6 and 1.12 respectively. \square

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