Dynamic Utility and related nonlinear SPDE driven by Lévy Noise.

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Abstract

This work concerns the study of consistent dynamic utilities in a financial market with jumps. We extend the results established in the paper [EKM13] to this framework. The ideas are similar but the difficulties are different due to the presence of the Lévy process. An additional complexity is clearly the interpretation of the terms of jumps in the different problems primal and dual one and relate them to each other. To do, we need an extension of the Itô-Ventzel's formula to jump's frame. By verification, we show that the dynamic utility is solution of a non-linear second order stochastic partial integro-differential equation (SPIDE). The main difficulty is that this SPIDE is forward in time, so there are no results in the literature that ensure the existence of a solution or simply allow us to deduce important properties, in our study, such as concavity or monotonicity. Our approach is based on a complete study of the primal and the dual problems. This allows us, firstly, to establish a connection between the utility-SPIDE and two SDEs satisfied by the optimal processes. Based on this connection and the SDE's theory, stochastic flow technics and characteristic method allow us, secondly, to completely solve the equation; existence, uniqueness, monotony and concavity.

1 Introduction

In this work, we propose to study the consistent dynamic utilities in a financial market with jumps. These utilities have been introduced by M. Musiela and T. Zariphopoulou [MZ03, MZ] and have been the subject of several recent works in a continuous semimartingale framework [MZ10a, MZ10b, BRT09, EKM13, EKHM17, Mra20] and in [BT11, EKHM18] in a model with consumption.
These dynamic utilities aim at dealing with the multitudes of inconsistencies in the classical expected utility optimization problem. Indeed, in finance for portfolio selection, the reflection on the criterion to be optimized is rather poor: one sets an investment horizon, an increasing concave

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function (to reflect risk aversion), and one tries to maximize the expected utility of the final wealth. The optimal strategy is then highly dependent on the management horizon and the utility itself, which is set today for a future date, does not take into account the evolution of the very dynamic market. In investment banking, and in the futures markets in particular, part of the business involves "delta-hedge" strategies, i.e., strategies that are not very sensitive to market trends and that we wish to use as references. Moreover, many problems, including the financing of environmental projects, relate to very long time horizons, for which it is difficult to pretend that the market does not readjust its criteria, particularly in the case of major changes in the fundamentals of the economy.

Moreover, forward looking (see [EKM20]) is the most adapted point of view to study several problems in vast and varied fields for example e-commerce, robot advising, artificial intelligence, etc., where the issue is always about learning the utility of an agent (player, e-commerce customer, representative agent...) by observing his behavior in the face of uncertainty and also the problem of economic equilibrium as posed by He and Leland [HL93] and solved in [EKM20]. Also for modeling long term yield curves, it is important to have an adaptive criterion consistent with any maturity, otherwise we get inconsistent multi-curves, see [EKHM19, EKHM14].

In the papers by Musiela and Zariphopoulou [MZ10a] and by Berrier & ali [BRT09] the authors characterize all dynamic utilities decreasing over time, i.e. all utilities with zero volatility vector. In El Karoui and Mrad [EKM13], a more complete study is carried out. The authors characterize all dynamic utilities of Itô-type without restriction. The key point is that the authors show a strong link between the SPDE, satisfied by these utility random fields and two SDEs satisfied by the optimal processes of the primal and dual problem. The study of the solutions of such SPDEs is then reduced to those of the SDEs. With this correspondence and with the help of the abundant results in the SDE-theory, one can easily deduce (under simple assumptions of regularities of the SDE's coefficients) the existence, uniqueness, monotonicity and concavity of the solution of the utility-SPDE.

The extension of the concept of dynamic utility to the case with consumption was at first considered by Berrier & ali [BT11] and second by El Karoui & ali [EKHM18, EKHM19]. Other authors have been interested in these consistent dynamic utilities. For example, G. Zitkovic [Zit08] provides a dual formulation of consistency in addition to the primal formulation, and proves equivalence between the two. An explicit characterization is then given in a log-affine structure, where necessary and sufficient conditions for consistency are provided in this specific case. In [LZ17], the authors develop a connection between the consistent dynamic utilities and infinite horizon BSDE and a connection, for large time horizons, with a family of classical homothetic value function processes with random endowments. Further works related to this problem are Choulli, Stricker and Li (2007) [SCL07], Henderson and Hobson (2007) [HH07].

In this paper, we place ourselves in an incomplete market framework where assets are modeled as Itô processes with jumps, without restrictions on volatility vectors or jump processes. We first conduct a necessary study that will allow us to link different parameters of the primal and dual problems. The stochastic characteristics method (an extension of the classical characteristics method for the solving of PDEs) will allow us to characterize this class of utilities in any generality.

The paper is organized as follows, Sections 2.1 and 2.2 are quite general in which we set the regularity conditions necessary to carry out the calculations and particularly apply the Itô-Ventzel's formula with jumps, recalled in (2.4). Under recalled regularity conditions, we will establish the

forward dynamics of the inverse flow of a solution of a regular SDE with jumps, see Theorem 2.4. In passing, we establish some intermediate results that will be used several times in this work: for example equation (2.6) gives the dynamics of the product of two SDE's solutions XY when Corollary 2.5 yields the dynamics of the compound $Y \circ \mathcal{X}$ of Y and the inverse (with respect to the initial condition) of X.
In Section 2.3, we describe the market model with jumps, the class of admissible portfolios and characterize the dynamics of the state price density processes in this universe. In Section 3, we introduce the notions of progressive utility and market-consistent utility. In the section 3.2, we conduct a necessary study. Under the consistency condition and using the generalization of Ito-Ventzel's formula to the jump frame, we deduce the optimal strategy κ^* and then the stochastic partial integro-differential equations (SPIDEs) that necessarily satisfies the consistent utility U and its marginal utility, see Theorem 3.1 and Corollary 3.2. In Section 3.3, we study in Theorem 3.3 the dynamics of the marginal consistent utility along the optimal portfolio $U_z(t, X_t^*)$ and then show, in Corollary 3.4, that $U_z(t, X_t^*)$ is a price density process.
Section 3.4, completes this study and provides us with more details on this state price. We focus on the dual utility $\widetilde{U}$, we first establish, in Theorem 3.5, its dynamics and that of its marginal, using the results of Section 2.2. Then, we establish the main result of this section, Theorem 3.6: we study the dual optimization problem, we show that $\widetilde{U}$ is also consistent but with the class $\mathbb{Y}$ of state prices density processes and in particular the optimum is reached in $U_z(t, X_t^*)$. This last point shows us the equivalence between the monotonicity of $x \mapsto X_t^*(x)$ and that of $y \mapsto Y_t^*(y)$. Moreover if we note by $\mathcal{X}_t(z)$ the inverse of X^* then we can completely characterize the marginal utility U_z and thus its primitive U. Indeed in this case, see Theorem 3.8, we have

$$\begin{cases} U_z(t,z) = Y_t^*\big(u_z(\mathcal{X}_t(z))\big),\ u_z(.) = U_z(0,.) \\ U(t,z) = \displaystyle\int_{z_0}^{z} Y_t^*\big(u_z(\mathcal{X}_t(x))\big)dx,\ U(t,0) = 0. \end{cases}$$

In this new approach, the solution of the utility SPIDE have a pathwise representation, unlike to the characteristics method where the solutions are represented as a conditional expectation. There are several advantages of this connection between SPIDEs and SDEs due to the existence of many results in the SDE theory. To the best of our knowledge, there are no or few results that assert the monotonicity or the convexity of such solutions. Also, there may be other advantages in numerical methods and simulations of the SDE than of SPIDE as it is shown in [GM18].
In the last Section 4 of this work, we consider the inverse engineering problem. We give ourselves an initial utility function u and two monotonic processes, a wealth process $\mathcal{X}$ with inverse $\mathcal{X}$ and a state price density $\mathcal{Y}$, and we define a new utility random field U as the primitive of $\mathcal{Y}_t\big(u_z(\mathcal{X}_t(z))\big)$. We establish in Theorem 4.1 its dynamics and in Theorem 4.2, that it is $\mathbb{X}$-consistent with optimal wealth process $\mathcal{X}$ and optimal dual process $\mathcal{Y}$.

2 Formulation and spaces

All stochastic processes are defined on a standard filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ is assumed to be right continuous and complete such any semimartingale have a right continuous paths with left limits. We consider a d-dimensional Brownian motion W and an independent Lévy random measure N on $[0,\infty)\times\mathbb{R}$ with intensity measure $dt\times\nu(de)$ defined on the

probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where ν is a positive measure on $\mathbb{R}$ such that $\int_{\mathbb{R}}(1 \wedge e^2)\nu(de) < \infty$. $\tilde{N}$ denotes the compensated version of N:

$$\tilde{N}(dt, de) = N(dt, de) - \nu(de)dt.$$

We first recall some notions relative to stochastic processes depending on a spatial parameter x. For us, because of economic motivation, this parameter is the wealth of an investor, taking non negative values in $\mathbb{R}_+ = \{x \geq 0\}$. Sometimes, we will use the vocabulary of random field theory, and refer to such processes as progressive random fields. As all random fields considered in the sequel are progressive, we will often omit the mention " progressive".

(i) A progressive random field $\mathbf{X} = \{X(t,x); t \geq 0, x > 0\}$ is a random variable measurable w.r.t. $\mathcal{F}_\infty \otimes \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathbb{R}_+^0)$, which is a collection of progressive processes $t \mapsto X(t,x)$.

(ii) $\mathbf{X}$ is said to be concave, (resp. increasing) if there exists $N \in \mathcal{F}_\infty$ with $\mathbb{P}(N) = 0$, such that for any $\omega \in N^c$, and any $t \geq 0$ $x \mapsto X(t,x)(\omega)$ is concave (resp. increasing).

(iii) A random field $\mathbf{X}$ is said to be *continuous* (resp. differentiable) if there exists $N \in \mathcal{F}_\infty$ with $\mathbb{P}(N) = 0$, such that for any $\omega \in N^c$, and any $t \geq 0$ $x \mapsto X(t,x)(\omega)$ is continuous (resp. differentiable). The derivative denoted $X_x(t,x)(\omega)$ generates the so-called derivative random field $\mathbf{X}_x$. When $\mathbf{X}_x$ has a continuous version, $\mathbf{X}$ is said to be $\mathcal{C}^1$-*regular*.

2.1 Regular random fields spaces

We introduce a family of Sobolev type random semi-norms to control locally or globally the growth of the random field and its derivatives.

Definition 2.1.

- *A predictable random field $\phi : (t,x,\omega) \in \mathbb{R}_+ \times \mathbb{R} \times \Omega \to \mathbb{R}^d$ $(d \geq 1)$ is said to be in the class $\mathcal{C}^{m,\delta}$ $(m \in \mathbb{N}, \delta \in (0,1])$ if ϕ is of class $\mathcal{C}^m$ in x with locally bounded derivative such that $\partial_x^m \phi$ is δ-Hölder.*
- *A predictable random field $\psi : (t,x,e,\omega) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \times \Omega \to \mathbb{R}^d$ $(d \geq 1)$ is said to be in the class $\mathcal{C}^{m,\delta}$ $(m \in \mathbb{N}, \delta \in (0,1])$ if ψ is of class $\mathcal{C}^m$ in x, with locally bounded derivative such that $\partial_x^m \phi$ is δ-Hölder.*

2.1.1 Norms definition

Let ϕ be a continuous $\mathbb{R}^k$-valued progressive random field and let m be a non-negative integer, and δ a number in $(0,1]$. We need to control the asymptotic behavior in 0 and ∞ of ϕ, and the regularity of its Hölder derivatives when there exist. More precisely, let ϕ be in the class $\mathcal{C}^{m,\delta}(]0,+\infty[)$, i.e. (m,δ)-times continuously differentiable in x for any t, a.s.

(i) For any subset $K \subset]0,+\infty[$, we define the family of random (Hölder) K-semi-norms

$$\begin{cases} \|\phi\|_{m:K}(t,\omega) = \sup_{x\in K} \frac{\|\phi(t,x,\omega)\|}{x} + \sum_{1\leq j\leq m} \sup_{x\in K} \|\partial_x^j \phi(t,x,\omega)\| \\ \|\phi\|_{m,\delta:K}(t,\omega) = \|\phi\|_{m:K}(t,\omega) + \sup\limits_{x,y\in K} \dfrac{\|\partial_x^m \phi(t,x,\omega) - \partial_x^m \phi(t,y,\omega)\|}{|x-y|^\delta}. \end{cases} \tag{2.1}$$

The case $(m = 0, \delta = 1)$ corresponds to the local version of the Lipschitz case. When K is all the domain $]0,+\infty[$, we simply write $\|.\|_m(t,\omega)$, or $\|.\|_{m,\delta}(t,\omega)$.

(ii) The first term of these random semi-norms differs slightly from that considered by Kunita (equations (1) and (2) p.72 of [Kun97]) because instead of dividing by $1 + |x|$ we divide by x on the first terms. This does not change Kunita's results, but allows us to obtain reasonable behavior

in the neighborhood of $x = 0$ in addition to the traditional results in the neighborhood of $x = \infty$.
(iii) When the random field Φ depends on a parameter $e \in \mathbb{R}$, we define exactly in the same way $\|\phi\|_{m:K}(t,\omega,e)$ and $\|\phi\|_{m,\delta:K}(t,\omega,e)$.

2.1.2 Different spaces of regular random fields

The previous semi-norms are related to the spatial parameter. We add the temporal dimension in assuming these semi-norms (or the square of the semi-norm) to be integrable in time with respect to the Lebesgue measure on $[0,T]$ for all $T \geq 0$. Then, as in Lebesgue's Theorem, we can differentiate, pass to the limit, commute limit and integral for the random fields. Calligraphic notation recalls that these semi-norms are random.
(i) $\mathcal{K}^{m,\delta}_{loc}$ (resp. $\overline{\mathcal{K}}^{m,\delta}_{loc}$) denotes the set of all $\mathcal{C}^{m,\delta}$-random fields s.t. for any compact $K \subset]0,+\infty[$, and any T, $\int_0^T \|\phi\|_{m,\delta:K}(t,\omega) < \infty$, (resp.$\int_0^T \|\psi\|^2_{m,\delta:K}(t,\omega)dt < \infty$).
(ii) $\widehat{\mathcal{K}}^{m,\delta}_{loc,\nu}$ denotes the set of $\mathcal{C}^{m,\delta}$-random fields s.t. for any compact $K \subset]0,+\infty[$, and any T, $\int_0^T \int_{\mathbb{R}} \|\psi\|^2_{m,\delta:K}(t,e,\omega)\nu(de)dt < \infty$.
(iii) When these different norms are well-defined on the whole space $]0,+\infty[$, the derivatives (up to a certain order) are bounded in the spatial parameter, with integrable (resp. square integrable) in time random bound. In this case, we use the notations $\mathcal{K}^m_b, \overline{\mathcal{K}}^m_b$, $\mathcal{K}^{m,\delta}_b, \overline{\mathcal{K}}^{m,\delta}_b$, $\widehat{\mathcal{K}}^m_{b,\nu}$ and $\widehat{\mathcal{K}}^{m,\delta}_{b,\nu}$.
We also introduce the following spaces of processes:
(iv) $\mathcal{L}^{2,d}(\Omega\times\mathbb{R}_+)$ is the space of $\mathcal{P}$-measurable process (Z_t) with valued in $\mathbb{R}^d$ such that $\int_0^T |Z_s|^2 ds < +\infty,\ \mathbb{P}-$a.s where $\mathcal{P}$ is the σ-field of all predictable sets of $[0,T]\times\Omega,\ \forall T \geq 0$.
(v) $\mathcal{L}^{2,d}_\nu(\Omega\times\mathbb{R}_+\times\mathbb{R}$ is the space of predictable processes $(h(t,e))$ such that $\int_0^T \int_{\mathbb{R}} |h(s,e)|^2 d\nu ds < +\infty,\ \mathbb{P}-$a.s, $\forall T \geq 0$

2.2 Itô's semimartingale random fields with Jumps

2.2.1 Differentiability of Itô random fields

We shall discuss the regularity of a Itô semimartingale random field

$$\mathbf{F(t,z)} = \mathbf{F(0,z)} + \int_{\mathbf{0}}^{\mathbf{t}} \phi(\mathbf{s,z})\mathbf{ds} + \int_{\mathbf{0}}^{\mathbf{t}} \psi(\mathbf{s,z}).\mathbf{dW_s} + \int_{\mathbf{0}}^{\mathbf{t}} \int_{\mathbb{R}} \mathbf{H(t,z,e)}\tilde{\mathbf{N}}\mathbf{(dt,de)}$$

in connection with the regularity of its local characteristics (ϕ,ψ,H). By convention, an Itô random field $\mathbf{F}$ is said to be a $\mathcal{K}^{m,\delta}_{loc}$-semimartingale, whenever $F(0,z)$ is of class $\mathcal{C}^{m,\delta}$, $B^F(t,z) = \int_0^t \phi(s,z)ds$ is of class $\mathcal{K}^{m,\delta}_{loc}$, $M^F(t,z) = \int_0^t \psi(s,z).dW_s$ is of class $\overline{\mathcal{K}}^{m,\delta}_{loc}$ and $\mathcal{J}^F(t,z) = \int_0^t \int_{\mathbb{R}} H(t,z,e)\tilde{N}(dt,de)$ is of class $\widehat{\mathcal{K}}^{m,\delta}_{loc,\nu}$. The reference to $\mathcal{K}^{m,\delta}$ recall that $\mathbf{F}$ is a random field.
Notations: To fix the notations, throughout this paper we denote by $\Psi_z(t,z)$ (resp. $\Psi_{zz}(t,z)$) the first (resp. the second) derivative of a random field $\Psi(t,z)$ with respect to its spatial variable z.
As in Kunita [FK$^+$85], we are concerned both by the regularity of F from the regularity of its local characteristics (ϕ,ψ,H) and by the regularity of (ϕ,ψ,H) from that of $\mathbf{F(t,x)}$ ([FK$^+$85, Theorem 1.3]). To be concise, we also give a sufficient conditions under which we can differentiate term by term the dynamics of an Itô random field. This property is used in order to apply Itô-Ventzel's formula.

Theorem 2.1 (Differential Rules)**.** *Let* $\mathbf{F}$ *be an Itô semimartingale random field with local characteristics* (ϕ, ψ)*,*

$$F(t,x) = F(0,x) + \int_0^t \phi(s,x)ds + \int_0^t \psi(s,x).dW_s + \int_0^t \int_{\mathbb{R}} H(s,z,e)\tilde{N}(ds,de).$$

(i) *If* $\mathbf{F}$ *is a* $\mathcal{K}_{loc}^{m,\delta}$*-semimartingale for some* $m \geq 0,\ \delta \in (0,1]$*, its local characteristics* (ϕ, ψ, H) *are of class* $\mathcal{K}_{loc}^{m,\varepsilon} \times \overline{\mathcal{K}}_{loc}^{m,\varepsilon} \times \widehat{\mathcal{K}}_{loc,\nu}^{m,\varepsilon}$ *for any* $\varepsilon < \delta$*.*

(ii) *Conversely, if the local characteristics* $(\phi, \psi, \mathbf{H})$ *are of class* $\mathcal{K}_{loc}^{m,\delta} \times \overline{\mathcal{K}}_{loc}^{m,\delta} \times \widehat{\mathcal{K}}_{loc,\nu}^{m,\delta}$*, then* $\mathbf{F}$ *is a* $\mathcal{K}_{loc}^{m,\varepsilon}$*-semimartingale for any* $\varepsilon < \delta$*.*

(iii) *In any cases, for* $m \geq 1, \delta \in (0,1]$*, the derivative random field* $\mathbf{F}_x$ *is an Itô random field with local characteristics* (ϕ_x, ψ_x, H_x)*.*

This result was established by T. Fujiwara and H. Kunita in [FK$^+$85, Theorem 1.3]. A similar version of this result exists for SDE's solutions, see [FK$^+$85, Theorems 2.2 and 2.3].

Corollary 2.2. *Let* $\mathbf{X}$ *be an Itô semimartingale solution of a SDE*(μ, σ, h)*,*

$$X_t(x) = x + \int_0^t \mu(s, X_s(x))ds + \int_0^t \sigma(s, X_s(x)).dW_s + \int_0^t \int_{\mathbb{R}} h(s, X_{s^-}(x), e)\tilde{N}(ds,de).$$

(i) *If* $\mathbf{X}$ *is a* $\mathcal{K}_{loc}^{m,\delta}$*-semimartingale for some* $m \geq 0,\ \delta \in (0,1]$*, the SDE's coefficients* (μ, σ, h) *are of class* $\mathcal{K}_{loc}^{m,\varepsilon} \times \overline{\mathcal{K}}_{loc}^{m,\varepsilon} \times \widehat{\mathcal{K}}_{loc,\nu}^{m,\varepsilon}$ *for any* $\varepsilon < \delta$*.*

(ii) *Conversely, the SDE's coefficients* $(\mu, \sigma, \mathbf{h})$ *are of class* $\mathcal{K}_{loc}^{m,\delta} \times \overline{\mathcal{K}}_{loc}^{m,\delta} \times \widehat{\mathcal{K}}_{loc,\nu}^{m,\delta}$*, then* $\mathbf{X}$ *is a* $\mathcal{K}_{loc}^{m,\varepsilon}$*-semimartingale for any* $\varepsilon < \delta$*.*

(iii) *In any cases, for* $m \geq 1, \delta \in (0,1]$*, the derivative random field* $\mathbf{X_x}$ *is an Itô semimartingale solution of*

$$dX_x(t,x) = X_x(t^-,x)\Big[\mu_z(t, X_x(t,x))dt + \sigma_z(t, X_x(t,x))dW_t + \int_{\mathbb{R}} h_z(t, X_{t^-}(x), e)\tilde{N}(dt,de)\Big].$$

2.2.2 Itô-Ventzel's formula with jumps

In the framework of the papers [EKM13, EKHM18, MZ10b], the authors used the Itô-Ventzel's formula to study dynamic utilities, their dynamics, consistency and martingales properties... In the present paper where we have in addition a jump processes, we need an even more general Itô's formula. Luckily, an extension of the Itô-Ventzel's formula is established by B. Øksendal and T. Zhang [ØZ$^+$07][Theorem 3.1]. It gives, under regularity assumptions, the dynamics of the compound of two random fields of Itô type with jumps. The statement of this formula, essential in our framework, is the following.

Theorem 2.3. *Let* $U = \{U(t,z)\}$ *be a* $\mathcal{K}_{loc}^{2,\delta}$*-semimartingale for some* $\delta \in (0,1]$ *and* X *a semimartingale with jumps evolving as,*

$$\begin{cases} dU(t,z) = \beta(t,z)dt + \gamma(t,z)dW_t + \displaystyle\int_{\mathbb{R}} H(t,z,e)\tilde{N}(dt,de) & (2.2) \\ dX_t(x) = \mu^X(t, X_t(x))dt + \sigma^X(t, X_t(x))dW_t + \displaystyle\int_{\mathbb{R}} h^X(t, X_{t^-}(x), e)\tilde{N}(dt,de). & (2.3) \end{cases}$$

Then the compound stochastic process $U(t, X_t(x))$ is a semimartingale with dynamics,

$$\begin{aligned} dU(t,X_t(x)) &= \Big(\beta + U_z\mu^X + \tfrac{1}{2}U_{zz}\|\sigma^X\|^2 + \gamma_z.\sigma^X\Big)(t,X_t(x))dt + \big(\gamma + U_z\sigma^X\big)(t,X_t(x))dW_t \\ &+ \int_{\mathbb{R}} \Big(U\big(t,X_t(x)+h^X(t,X_t(x),e)\big) - U(t,X_t(x)) - U_z(t,X_t(x))h^X(t,X_t(x),e)\Big)\nu(de)dt \\ &+ \int_{\mathbb{R}} \Big(H(t,X_t(x)+h^X(t,X_t(x),e)) - H(t,X_t(x),e)\Big)\nu(de)dt \\ &+ \int_{\mathbb{R}} \Big(U\big(t^-,X_{t^-}(x)+h^X(t,X_{t^-}(x),e)\big) - U(t^-,X_{t^-}(x)) \\ &+ H\big(t,X_{t^-}(x)+h^X\big(t,X_{t^-}(x),e\big)\big)\Big)\tilde{N}(dt,de). \end{aligned} \tag{2.4}$$

The first line of this dynamic corresponds to the standard Itô-Ventzel's formula. The rest is only due to the presence of jumps in the dynamics of U and X.

Remark 2.1. *In the particular case where $U(t,z) = zY_t$ with*

$$dY_t = \mu^Y(t,Y_t)dt + \sigma^Y(t,Y_t)dW_t + \int_{\mathbb{R}} h^Y(t,Y_{t^-},e)\tilde{N}(dt,de). \tag{2.5}$$

The product XY follows the dynamics

$$\begin{aligned} d(X_tY_t) &= \Big(X_t\mu^Y(t,Y_t) + Y_t\mu^X(t,X_t) + \sigma^X(t,X_t)\sigma^Y(t,Y_t)\Big)dt + \Big(X_t\sigma^Y(t,Y_t) + Y_t\sigma^X(t,X_t)\Big)dW_t \\ &+ \int_{\mathbb{R}} h^Y(t,Y_t,e)h^X(t,X_t,e)\nu(de)dt \\ &+ \int_{\mathbb{R}} \Big(X_{t^-}h^Y(t,Y_{t^-},e) + Y_{t^-}h^X(t,X_{t^-},e) + h^Y(t,Y_{t^-},e)h^X(t,X_{t^-},e)\Big)\tilde{N}(dt,de). \end{aligned} \tag{2.6}$$

Therefore, the process $(X_tY_t)_t$ is a local martingale if and only if

$$X_t\mu^Y(t,Y_t) + Y_t\mu^X(t,X_t) + \sigma^X(t,X_t)\sigma^Y(t,Y_t) + \int_{\mathbb{R}} h^Y(t,Y_t,e)h^X(t,X_t,e)\nu(de) = 0,\ dt\otimes\mathbb{P}.$$

2.2.3 Homeomorphic property of Itô random fields with jumps

As previously mentioned, we need results on the existence and the regularity of one dimensional random fields which are also solutions of stochastic differential equations (SDE). The spatial parameter in this case corresponds to the initial condition. Such random fields are also called stochastic flows and are the main subject (in the multidimensional case) of the papers [Kun04, FK$^+$85] and the Kunita's book [Kun19].

It is well known in the continuous framework, under Lipschitz and regularity conditions of the SDE's coefficients, that the associated stochastic flow satisfies the homeomorphic property (see Bismut [Bis80], Kunita [Kun84] and the Kunita's book [Kun97]). But this property fails for the solution of SDE with jumps in general. P.A. Meyer in [Mey81] (Remark p.111), gave a counterexample with the following exponential equation:

$$X_t(x) = x + \int_0^t X_{s-}dZ_s,$$

where Z is semimartingale, $Z_0 = 0$, such that Z has a jump of size -1 at some stopping time τ, $\tau > 0$ a.s. Then all trajectories of X, starting at any initial value x, become zero at τ and stay there

after τ. This may be seen trivially by the explicit form of the solution given by the Doléans-Dade exponential:

$$X_t(x) = x \exp\big(Z_t - \frac{1}{2}[Z,Z]^c_t\big) \prod_{0<s\leq t} \big(1+\Delta Z_s\big)e^{-\Delta Z_s}.$$

In the general setting of non-linear SDE, at the jump time τ, the solution jumps from $X_{\tau-}(x)$ to $X_{\tau-}(x) + h(t, X_{\tau-}(x))$. Léandre [Léa85] gave a necessary and sufficient condition under which the homeomorphic property is preserved at the jump time, namely, for each $e \in \mathbb{R}$ and $t \in [0,T]$, the maps $H_e : x \mapsto x + h(t,x,e)$ should be one to one and onto. One can read also Fujiwara and Kunita [FK$^+$85], Kunita [Kun04, Kun19] and Protter [Pro05] for more details on the subject.

A first key result in this work provides, under the appropriate assumptions, the forward dynamics of the inverse flow $\mathcal{X}$ of a regular semimartingale X, monotonic with respect to its initial condition.

Theorem 2.4. *Let X be a solution of the following SDE(μ,σ,h),*

$$dX_t(x) = \mu(t,X_t(x))dt + \sigma(t,X_t(x))dW_t + \int_{\mathbb{R}} h(t,X_{t^-}(x),e)\tilde{N}(dt,de). \tag{2.7}$$

Let $\delta \in [0,1[$ and assume $(\mu,\sigma,h) \in \mathcal{K}^{2,\delta}_b \times \overline{\mathcal{K}}^{2,\delta}_b \times \widehat{\mathcal{K}}^{2,\delta}_{b,\nu}$.
(i) *Then, the SDE(μ,σ,h) admits a unique strong solution $X(z)$, starting from z at time $t=0$.*
(ii) *If in addition, the maps $z \mapsto z + h(t,z,e)$ are homeomorphic, the map $z \to X_t(z)$ is strictly increasing with inverse flow $\mathcal{X}(t,x)$ satisfying the following second order SPDE,*

$$\begin{aligned} d\mathcal{X}(t,x) &= \Big[\mathcal{X}_x(t,x)\Big(-\mu(t,x) + \int_{\mathbb{R}} h(t,x,e)\nu(de)\Big) + \frac{1}{2}\partial_x\Big(\mathcal{X}_x(t,x)\|\sigma(t,x)\|^2\Big) \\ &- \mathcal{X}(t,x) + \int_{\mathbb{R}} \psi(t,x,e)\nu(de)\Big]dt - \mathcal{X}_x(t,x)\sigma(t,x)dW_t \\ &+ \int_{\mathbb{R}} \big(-\mathcal{X}(t^-,x) + \mathcal{X}(t^-,\psi(t,x,e))\big)\tilde{N}(dt,de) \end{aligned} \tag{2.8}$$

where $\psi(t,z,e)$ is the inverse map of $z \mapsto z + h(t,z,e)$.

Remark . (i) *We can also characterize the inverse process in terms of monotonic solution of a stochastic differential equation (SDE). It suffices to use the following identities*

$$X_t(\mathcal{X}(t,x)) = x, \quad \mathcal{X}_x(t,x) = \frac{1}{X_x(t,\mathcal{X}(t,x))} \text{ and } \mathcal{X}_{xx}(t,x) = -\frac{X_{xx}(t,\mathcal{X}(t,x))}{(X_x(t,\mathcal{X}(t,x)))^3}.$$

(ii) *To our knowledge, the dynamics of the forward flow given by equation* (2.8) *has not been established in the literature. For example in H. Kunita [Kun19] Theorem 3.7.1, p.108 or [Kun04] Theorem 3.13, only the backward dynamic is given.*

Proof. Statement **(i)** is established in [Kun04, Theorem 3.1]. To show the second one, we use the results of [Kun04, Theorems 3.3, 3.5, 3.11 and 3.13] which ensure the homeomorphic and differentiability properties of the solution X (of class $\mathcal{K}^{2,\varepsilon}_{loc}, \forall \varepsilon < \delta$) and that its inverse flow $\mathcal{X}$ is also a semimartingale. We thus proceed by identification. Assume that the dynamics of $\mathcal{X}$ is the following

$$d\mathcal{X}_t(x) = \bar{\mu}(t,\mathcal{X}_t(x))dt + \bar{\sigma}(t,\mathcal{X}_t(x))dW_t + \int_{\mathbb{R}} \bar{h}(t,\mathcal{X}_{t^-}(x),e)\tilde{N}(dt,de). \tag{2.9}$$

Applying Itô-Ventzel's formula to $X(t,\mathcal{X}(t,x))=x$, one gets

$$\begin{aligned}
& dX(t,\mathcal{X}(t,x)) = \Big[\mu(t,x)+X_x(t,\mathcal{X}(t,x))\bar{\mu}(t,\mathcal{X}_t(x))+\frac{1}{2}X_{xx}(t,\mathcal{X}(t,x))\|\bar{\sigma}(t,\mathcal{X}_t(x))\|^2 \\
+ \quad & X_x(t,\mathcal{X}(t,x))\sigma_x(t,x)\bar{\sigma}(t,\mathcal{X}(t,x))\Big]dt+\big(\sigma(t,x)+X_x(t,\mathcal{X}(t,x))\bar{\sigma}(t,\mathcal{X}(t,x))\big)dW_t \\
- \quad & \int_{\mathbb{R}}\big(X_x(t,\mathcal{X}(t,x))\bar{h}(t,\mathcal{X}(t,x),e)+h(t,x,e)\big)\nu(de)dt \\
+ \quad & \int_{\mathbb{R}}\big(X_{t^-}(\mathcal{X}(t^-,x)+\bar{h}(t,\mathcal{X}(t^-,x),e))+h\big(t,X_{t^-}\big(\mathcal{X}(t^-,x)+\bar{h}(t,\mathcal{X}(t^-,x),e)\big)\big)-x\big)\tilde{N}(dt,de) \\
= \quad & 0.
\end{aligned}$$

Which is equivalent, by identification, to

$$\begin{cases}
\mu(t,x)+X_x(t,\mathcal{X}(t,x))\big(\bar{\mu}(t,\mathcal{X}_t(x))+\sigma_x(t,x)\bar{\sigma}(t,\mathcal{X}(t,x))\big)+\frac{1}{2}X_{xx}(t,\mathcal{X}(t,x))\|\bar{\sigma}(t,\mathcal{X}_t(x))\|^2 \\
\qquad -\int_{\mathbb{R}}\big(X_x(t,\mathcal{X}(t,x))\bar{h}(t,\mathcal{X}(t,x),e)+h(t,x,e)\big)\nu(de)=0, & (2.10)\\
\sigma(t,x)+X_x(t,\mathcal{X}(t,x))\bar{\sigma}(t,\mathcal{X}(t,x))=0, & (2.11)\\
X_{t^-}(\mathcal{X}(t^-,x)+\bar{h}(t,\mathcal{X}(t^-,x),e))+h\big(t,X_{t^-}\big(\mathcal{X}(t^-,x)+\bar{h}(t,\mathcal{X}(t^-,x),e)\big)\big)-x=0, & (2.12)
\end{cases}$$

which hold $\forall x,e,\ dt\otimes d\mathbb{P}$ a.s. At first, let us denote $\phi(t,z,e)$ the map $z\mapsto X(t,z)+h(t,X(t,z),e)$. As $z\mapsto z+h(t,z,e)$ is assumed to be monotonic, $\phi(t,z,e)$ is monotonic with respect to z and we denote by $\psi(t,z,e)$ its inverse. Thus equation (2.12) is equivalent to

$$\phi\big(t,X_{t^-}\big(\mathcal{X}(t^-,x)+\bar{h}(t,\mathcal{X}(t^-,x),e)\big),e\big)=x,$$

or equivalently, by inverting simultaneously ϕ and X_{t^-}

$$\bar{h}(t,\mathcal{X}(t^-,x),e)=-\mathcal{X}(t^-,x)+\mathcal{X}(t^-,\psi(t,x,e)). \tag{2.13}$$

Secondly, equation (2.11) implies $\bar{\sigma}(t,x)=-\frac{\sigma(t,X_t(x))}{X_x(t,x)}$. Subsequently, using the identities

$$X_x(t,\mathcal{X}(t,x))=\frac{1}{\mathcal{X}_x(t,x)}\ \text{and}\ \frac{X_{xx}(t,\mathcal{X}(t,x))}{X_x^3(t,x)}=-\mathcal{X}_{xx}(t,x).$$

It follows, arranging the terms

$$\begin{aligned}
& X_x(t,\mathcal{X}(t,x))\sigma_x(t,x)\bar{\sigma}(t,\mathcal{X}(t,x))+\frac{1}{2}X_{xx}(t,\mathcal{X}(t,x))\|\bar{\sigma}(t,\mathcal{X}_t(x))\|^2 \\
&= \frac{1}{\mathcal{X}_x(t,x)}\Big(-\mathcal{X}_x(t,x)\sigma_x(t,x)\sigma(t,x)+\frac{1}{2}\frac{X_{xx}(t,\mathcal{X}(t,x))}{X_x^3(t,\mathcal{X}(t,x))}\|\sigma(t,x)\|^2\Big) \\
&= -\frac{1}{\mathcal{X}_x(t,x)}\Big(\mathcal{X}_x(t,x)\sigma_x(t,x)\sigma(t,x)+\frac{1}{2}\mathcal{X}_{xx}(t,x)\|\sigma(t,x)\|^2\Big) \\
&= -\frac{1}{2}\frac{1}{\mathcal{X}_x(t,x)}\partial_x\Big(\mathcal{X}_x(t,x)\|\sigma(t,x)\|^2\Big).
\end{aligned}$$

Injecting this identity in (2.10), leads to

$$\begin{aligned}
& \mu(t,x)+\frac{1}{\mathcal{X}_x(t,x)}\bar{\mu}(t,\mathcal{X}_t(x))-\frac{1}{2}\frac{1}{\mathcal{X}_x(t,x)}\partial_x\Big(\mathcal{X}_x(t,x)\|\sigma(t,x)\|^2\Big) \\
-\frac{1}{\mathcal{X}_x(t,x)}&\int_{\mathbb{R}}\big(-\mathcal{X}(t^-,x)+\mathcal{X}(t^-,\psi(t,x,e))\big)\nu(de)-\int_{\mathbb{R}}h(t,x,e)\nu(de)=0.
\end{aligned}$$

That is,

$$\begin{aligned}\bar{\mu}(t,\mathcal{X}(t,x)) &= \mathcal{X}_x(t,x)\Big(-\mu(t,x)+\int_{\mathbb{R}} h(t,x,e)\nu(de)\Big)+\frac{1}{2}\partial_x\Big(\mathcal{X}_x(t,x)\|\sigma(t,x)\|^2\Big)\\ &+ \int_{\mathbb{R}}\big(-\mathcal{X}(t^-,x)+\mathcal{X}(t^-,\psi(t,x,e))\big))\nu(de). \qquad (2.14)\end{aligned}$$

The proof is now complete. □

The second result of this section consists in a Itô-Ventzel's formula for a compound map $Y\circ\mathcal{X}$ where Y is a regular SDE's solution and $\mathcal{X}$ is the inverse flow of some monotonic SDE's solution X.

Corollary 2.5. *Let Y be a $\mathcal{K}^{2,\delta}_{loc}$-semimartingale and X a $\mathcal{K}^{3,\delta}_{loc}$-semimartingale for some $\delta\in]0,1[$, satisfying the dynamics*

$$\begin{cases} dX_t(x)=\mu(t,X_t(x))dt+\sigma(t,X_t(x))dW_t+\displaystyle\int_{\mathbb{R}} h(t,X_{t^-}(x),e)\tilde{N}(dt,de), & (2.15)\\ dY_t(y)=\mu^Y(t,Y_t(y))dt+\sigma^Y(t,Y_t(y))dW_t+\displaystyle\int_{\mathbb{R}} h^Y(t,Y_{t^-}(y),e)\tilde{N}(dt,de). & (2.16)\end{cases}$$

Assume X to be monotonic with respect to its initial condition and let $\mathcal{X}$ denotes its inverse flow. Then the compound map $\Phi(t,x):=Y_t(\mathcal{X}(t,x))$ satisfies the following second order SPDE

$$\begin{aligned}d\Phi(t,x)=&\Big(\mu^Y(t,\Phi(t,x))-\Phi_x(t,x)\big[\mu(t,x)+\sigma^Y_y(t,\Phi(t,x))\sigma(t,x)\big]\\ &+\tfrac{1}{2}\partial_x\Big(\Phi_x(t,x)\|\sigma(t,x)\|^2\Big)\Big)dt\\ &+\int_{\mathbb{R}}\Big(H(t,x,e)-h^Y(t,\Phi(t,x),e)+\Phi_x(t,x)h(t,x,e)\Big)\nu(de)dt\\ &+\big(\sigma^Y(t,\Phi(t,x))-\Phi_x(t,x)\sigma(t,x)\big)dW_t+\int_{\mathbb{R}}H(t,x,e)\tilde{N}(dt,de)\end{aligned}$$

with $H(t,x,e):=\Phi\big(t^-,\psi(t,x,e)\big)+h^Y(t,\Phi(t^-,\psi(t,x,e)),e)-\Phi(t^-,x)$.

This result, established here in a general way, will be very useful to us in the sequel and will be used several times.

Proof. From (2.8), we know that the inverse flot $\mathcal{X}$ evolves as

$$\begin{aligned}d\mathcal{X}(t,x) &= \Big[\mathcal{X}_x(t,x)\Big(-\mu(t,x)+\int_{\mathbb{R}}h(t,x,e)\nu(de)\Big)+\frac{1}{2}\partial_x\Big(\mathcal{X}_x(t,x)\|\sigma(t,x)\|^2\Big)\\ &- \mathcal{X}(t,x)+\int_{\mathbb{R}}\psi(t,x,e)\nu(de)]dt-\mathcal{X}_x(t,x)\sigma(t,x)dW_t\\ &+ \int_{\mathbb{R}}\big(-\mathcal{X}(t^-,x)+\mathcal{X}(t^-,\psi(t,x,e))\big)\tilde{N}(dt,de).\end{aligned}$$

It suffices to apply the generalized Itô-Ventzel's formula (2.4), to get

$$\begin{aligned}
dY(t,\mathcal{X}(t,x)) = \Big(\mu^Y(t,Y(t,\mathcal{X}(t,x))) + Y_y(t,\mathcal{X}(t,x))\Big[\mathcal{X}_x(t,x)\Big(-\mu(t,x)+\int_{\mathbb{R}} h(t,x,e)\nu(de)\Big)\\
+\tfrac{1}{2}\partial_x\Big(\mathcal{X}_x(t,x)\|\sigma(t,x)\|^2\Big) - \mathcal{X}(t,x) + \int_{\mathbb{R}}\psi(t,x,e)\nu(de)\Big] + \frac{1}{2}Y_{yy}(t,\mathcal{X}(t,x))\mathcal{X}_x^2(t,x)\|\sigma(t,x)\|^2\\
-\mathcal{X}_x(t,x)Y_y(t,\mathcal{X}(t,x))\sigma_y^Y(t,Y(t,\mathcal{X}(t,x))).\sigma(t,x)\Big)dt\\
+\big(\sigma^Y(t,Y(t,\mathcal{X}(t,x))) - \mathcal{X}_x(t,x)Y_y(t,\mathcal{X}(t,x))\sigma(t,x)\big)dW_t\\
+\int_{\mathbb{R}}\Big(Y\big(t,\psi(t,x,e)\big) - Y(t,\mathcal{X}(t,x)) - Y_y(t,\mathcal{X}(t,x))(-\mathcal{X}(t,x)+\mathcal{X}(t,\psi(t,x,e)))\Big)\nu(de)dt\\
+\int_{\mathbb{R}}\Big(h^Y(t,Y(t,\mathcal{X}(t,\psi(t,x,e)),e))) - h^Y(t,Y(t,\mathcal{X}(t,x)),e)\Big)\nu(de)dt\\
+\int_{\mathbb{R}}\Big(Y_{t^-}\big(\mathcal{X}(t^-,\psi(t,x,e))\big) - Y_{t^-}(\mathcal{X}(t^-,x)) + h^Y(t,Y_{t^-}(\mathcal{X}(t^-(\psi(t,x,e))),e)\Big)\tilde{N}(dt,de).
\end{aligned}$$

Now using the identities,

$$\begin{aligned}
\mathcal{X}_x(t,x)Y_y(t,\mathcal{X}(t,x)) = \partial_x(Y(t,\mathcal{X}(t,x))),\\
Y_{yy}(t,\mathcal{X}(t,x))\mathcal{X}_x^2(t,x) + Y_y(t,\mathcal{X}(t,x))\mathcal{X}_{xx}(t,x) = \partial_{xx}^2(Y(t,\mathcal{X}(t,x))),
\end{aligned}$$

and the notation $H(t,x,e) := \Phi\big(t^-,\psi(t,x,e),e\big) - \Phi(t^-,x) + h^Y(t,\Phi(t^-,\psi(t,x,e)),e)$, the dynamics of the random map $\Phi(t,x) := Y(t,\mathcal{X}(t,x))$, becomes, after simplifications

$$\begin{aligned}
d\Phi(t,x) = \Big(\mu^Y(t,\Phi(t,x)) - \Phi_x(t,x)\big[\mu(t,x)+\sigma_y^Y(t,\Phi(t,x))\sigma(t,x)\big]\\
+\tfrac{1}{2}\partial_x\Big(\Phi_x(t,x)\|\sigma(t,x)\|^2\Big)\Big)dt\\
+\int_{\mathbb{R}}\Big(H(t,x,e) - h^Y(t,\Phi(t,x),e) + \Phi_x(t,x)h(t,x,e)\Big)\nu(de)dt\\
+\big(\sigma^Y(t,\Phi(t,x)) - \Phi_x(t,x)\sigma(t,x)\big)dW_t + \int_{\mathbb{R}} H(t,x,e)\tilde{N}(dt,de).
\end{aligned}$$

□

2.3 The market model with jumps

The securities market which consists of $d+1$ assets, one of them being riskless. Their prices $(S^i)_{i=0}^d$ are assumed to be positive semimartingales defined on the filtered probability space $(\Omega,\mathcal{F}_{t\geq 0},\mathbb{P})$. The riskless asset evolves as $dS_t^0 = S_t^0 r_t dt$ where $r_\cdot$ is the short rate of the financial market. The risky assets are also denoted by $\mathbf{S} = (S^i)_{i=1}^d$.

A *self-financing* portfolio is a pair (x,ϕ), where x is the positive wealth invested in the portfolio at time 0, and the column vector $\phi = (\phi^i)_{1\leq i\leq d}$ is a predictable $\mathbf{S}$-integrable process specifying the amount of each asset held in the portfolio. The value process, also called wealth process, $X^\phi = (X_t^\phi)_{t\geq 0}$ of such portfolio ϕ is determined by the self-financing constraint,

$$\frac{X_t^\phi}{S_t^0} = \frac{x}{S_0} + \int_0^t \frac{\phi_\alpha}{S_\alpha^0}.d(\frac{S_\alpha}{S_\alpha^0}), \tag{2.17}$$

In the following, to limit the purely technical difficulties, we only consider portfolios with positive value. This naturally leads us to characterize portfolios by means of relative weights π in place of the amounts ϕ. The relation between these two notions is easy since $\phi_t = (\pi_t^1 X_t^\phi(x),..,\pi_t^d X_t^\phi(x))^T$,

where the transpose operator is denoted by ^{T}. The advantage of the second formulation is that the assumption of positive wealth is automatically satisfied, since the previous equation becomes with the notation X^π in place of X^ϕ,

$$\frac{dX_t^\pi}{X_t^\pi} = r_t dt + \pi_t.\big(\frac{dS_t}{S_t} - r_t \mathbf{1} dt\big), \tag{2.18}$$

where the d-dimensional vector denoted by $\mathbf{1}$ is such all components are equal to 1.
Let us now recall that a probability measure $\mathbb{Q} \sim \mathbb{P}$ is called an equivalent local martingale measure if, for any positive wealth process, $\frac{X}{S^0}$ is a local martingale under $\mathbb{Q}$. To ensure the absence of arbitrage opportunities, we postulate that the family of equivalent local martingale measures is not empty, (see [DS94] and [DS98] for a precise statement and references). The existence of many equivalent martingale measures implies the incompleteness of the market.

Itô's Market Using vector and matrix notation, the dynamics of the vector prices $S = (S^i)_{i=1,\dots,d}$ under this probability measure is described by the following equation:

$$dS_t = S_{t^-}\big[b_t dt + \sigma_t dW_t + \int_{\mathbb{R}} \sigma_t h^S(t,e)\tilde{N}(dt,de)\big]$$

$b = (b^i)_{i=1,\dots,d}$ is a column vector $d \times 1$ which belongs to $\mathcal{L}^{2,d}(\Omega \times \mathbb{R}_+)$, the coefficient b^i represents the appreciation rate by time unit of the asset i. σ is the volatility matrix $n \times d$ (n lines d columns) which belongs to $\mathcal{L}^{2,n\times d}(\Omega \times \mathbb{R}_+)$, whose i^{th} line is the vector $\sigma^i \in \mathbb{R}^d$ of of the asset i. The jump coefficient h^S is a column vector $d \times 1$ s.t. $\sigma h^S \in \mathcal{L}^{2,d}_\nu(\Omega \times \mathbb{R}_+ \times \mathbb{R})$ and $h^S(t,.) \in \mathcal{L}^{1,d}_\nu(\mathbb{R})$ for any t.
In this model setup, the wealth process dynamics (2.18) become,

$$dX_t^\pi = X_{t^-}^\pi \Big[\big(r_t + \pi_t.(b_t - r_t)\mathbf{1}\big)dt + \sigma_t\pi_t.\big(dW_t + \int_{\mathbb{R}} h^S(t,e)\tilde{N}(dt,de)\big)\Big]. \tag{2.19}$$

As usual, the basic risky assets family $\mathbf{S}$ is assumed to be *non-redundant*, (that is for any $i \in 1..d$, the asset S^i can not be replicated by an admissible portfolio). Therefore the matrix $(\sigma\sigma^T)(t,\omega)$ is *non-singular*.

The existence of an equivalent local martingale measure in this framework implies that the excess of return vector belongs to the range of volatility matrix: in other words, there exists a $\mathcal{F}$-non anticipating process $\eta \in \mathbb{R}^n$ such that $b_t - r_t\mathbf{1} = \sigma_t^T\eta_t$. Additional integrability assumptions are necessary to ensure that the exponential martingale generated by $\eta.W$ is the density of some probability measure. So, we assume that $\eta \in \mathcal{L}^{2,d}(\Omega \times \mathbb{R}_+)$ and $r \in \mathcal{L}^{1,1}(\Omega \times \mathbb{R}_+)$.
Herein, the dynamics of portfolios is

$$dX_t^\pi = X_{t^-}^\pi\big[r_t dt + \sigma_t\pi_t.(dW_t + \eta_t dt) + \sigma_t\pi_t \int_{\mathbb{R}} h^S(t,e)\tilde{N}(dt,de)\big].$$

As the key role is played by the volatility vector $\sigma\pi$, in order to facilitate the exposition, we denote it by $\kappa := \sigma\pi$. To fix the notation, let $\mathcal{R}_t \subset \mathbb{R}^n$ the range of σ_t, and $\mathcal{R}_t^\perp$ the orthogonal vector subspace. By assumption, κ_t is required to lie at any time t in $\mathcal{R}_t$. Note that under market assumptions ($\sigma_t^T\sigma_t$ non-singular) there exists a unique vector π_t such that $\kappa_t = \sigma_t\pi_t$. Replacing X^π by X^κ, the above equation becomes

$$dX_t^\kappa = X_{t^-}^\kappa\big[r_t dt + \kappa_t.(dW_t + \eta_t dt + \int_{\mathbb{R}} h^S(t,e)\tilde{N}(dt,de))\big],\ \kappa_t \in \mathcal{R}_t. \tag{2.20}$$

Definition 2.2. **(i)** *A positive process X^κ satisfying (2.20) with $\kappa_t \in \mathcal{R}_t$ for any t a.s., is said an admissible wealth process and $x\kappa$ the associated admissible strategy if $\kappa \in \mathcal{L}^{2,d}(\Omega \times \mathbb{R}_+)$ and $\kappa h^S \in \mathcal{L}^{2,d}_\nu(\Omega \times \mathbb{R}_+ \times \mathbb{R})$.*
(ii) *The set of all admissible portfolios will be denoted by $\mathbb{X}$.*

The following short notations will be used extensively. Let $\mathcal{R}$ be a vector subspace of $\mathbb{R}^n$. For any $\alpha \in \mathbb{R}^n$, $\alpha^{\mathcal{R}}$ is the orthogonal projection of the vector α onto $\mathcal{R}$ and $\alpha^\perp$ is the orthogonal projection onto $\mathcal{R}^\perp$, the orthogonal linear space of $\mathcal{R}$.

Remark . *Without loss of generalities, since $\kappa.\eta = \kappa.\eta^{\mathcal{R}}$ and $\kappa.h^S = \kappa.h^{S,\mathcal{R}}$, we can assume in all the following that η and h^S are in $\mathcal{R}$.*

State price density processes: In any portfolio optimization problem, adjoint processes, also called state price density processes, play an important role. We give a definition and then characterize all these processes by identifying their dynamics.

Definition 2.3. *A positive semimartingale Y is a state price density process if and only if for any portfolio process X^κ, $X^\kappa Y_t$ is a local martingale. The set of such processes is denoted by $\mathbb{Y}$.*

Next result, characterizes the dynamics of any state price density process.

Lemma 2.6. *Any state price density process $Y \in \mathbb{Y}$ is necessarily a solution of a dynamic of the form,*

$$\frac{dY_t^{\nu,h}}{Y_{t^-}^{\nu,h}} = -r_t dt + \Big(\nu_t - \eta_t - \int_{\mathbb{R}} h^Y(t,e)h^S(t,e)\nu(de)\Big)dW_t + \int_{\mathbb{R}} h^Y(t,e)\tilde{N}(dt,de), \tag{2.21}$$

for some random vector $\nu \in \mathcal{R}^\perp$ and random jump $h^Y \in \mathbb{R}$ such that $\nu \in \mathcal{L}^{2,d}_\nu(\Omega \times \mathbb{R}_+)$, $h^Y h^S, \in \mathcal{L}^{2,d}_\nu(\Omega \times \mathbb{R}_+ \times \mathbb{R})$ and $h^Y \in \mathcal{L}^{2,1}_\nu(\Omega \times \mathbb{R}_+ \times \mathbb{R})$.

We will come back to this dynamic in the continuation of this work and more particularly when studying the dual problem.

Remark . *If $h^Y \equiv 0$, we find the dynamics of standard states price density processes in a market without jumps.*

Proof. Assume the semimartingale Y to satisfies the dynamic,

$$dY_t = \mu_t^Y dt + \sigma_t^Y dW_t + \int_{\mathbb{R}} h^Y(t,e)\nu(de),$$

and let X^κ be an admissible wealth process satisfying the dynamics (2.20),

$$dX_t^\kappa = X_{t^-}^\kappa \big[r_t dt + \kappa_t.(dW_t + \eta_t dt + \int_{\mathbb{R}} h^S(t,e)\tilde{N}(dt,de))\big],\ \kappa_t \in \mathcal{R}_t.$$

According to Remark 2.1, YX^κ is a local martingale is equivalent to

$$\begin{aligned} &X_t^\kappa \mu_t^Y + Y_t(X_t^\kappa r_t + X_t^\kappa \kappa(t,X_t)\eta_t) + X_t^\kappa \kappa(t,X_t)\sigma_t^Y \\ &+ \int_{\mathbb{R}} (X_t^\kappa \kappa(t,X_t)h^S(t,e))h^Y(t,e)\nu(de) = 0,\ \forall \kappa \in \mathcal{R}\ dt \otimes \mathbb{P}. \end{aligned} \tag{2.22}$$

Differentiating with respect to the vector $X^\kappa \kappa$, follows

$$< L, Y_t\eta_t + \sigma_t^Y + \int_{\mathbb{R}} h^S(t,e)h^Y(t,e)\nu(de) >= 0,\ \forall L \in \mathcal{R}\ dt \otimes \mathbb{P}. \tag{2.23}$$

So that, for any t, $\sigma_t^Y + Y_t\eta_t + \int_{\mathbb{R}} h^S(t,e)h^Y(t,e)\nu(de) \in \mathcal{R}_t^{\perp}$. In other words, there exists an optional process $\nu \in \mathcal{R}^{\perp}$ such that;

$$\sigma_t^Y = Y_t(\nu_t - \eta_t) - \int_{\mathbb{R}} h^S(t,e)h^Y(t,e)\nu(de). \tag{2.24}$$

As a consequence we have necessarily, $\mu_t^Y = -Y_t r_t$. □

3 Progressive and consistent dynamic utilities

In a dynamic and stochastic environment, the classical notion of utility is not flexible enough to help us to make good choices in the long run. M. Musiela and T. Zariphopoulou [MZ03, MZ10b, MZ10a, MZ] were the first to suggest to use instead of the classical criterion the concept of progressive dynamic utility, consistent with respect to a given investment universe in a sense specified below. The concept of progressive utility gives an adaptative way to include new information on environment evolution available to economic agents. More recently, Fritelli [FM11] introduced very closed notion called stochastic dynamic utility, in view of study the certainty equivalent. Since these utility functions are stochastic, time dependent and moving forward, we consider them as a family of Itô's semimartingales depending on a parameter, the wealth of the agent in our economic context.

Progressive utility and its Fenchel conjugate We start with the definition of a progressive utility as progressive random field with concavity property.

Definition 3.1 (Progressive Utility)**.** *A* progressive utility *is a càdlàg progressive random field on* $\mathbb{R}_+^0$, $\mathbf{U} = \{U(t,z); t \geq 0, z > 0\}$ *such that,*

(i) Utility property: $\mathbf{U}$ *is strictly concave, strictly increasing, and non negative*

(ii) Regularity property: $\mathbf{U}$ *is a* $\mathcal{C}^2$*-random field, with continuous first and second derivatives random fields* $\mathbf{U_z}$ *and* $\mathbf{U_{zz}}$.

(ii) Inada conditions: $\mathbf{U}$ *goes to* 0 *when* x *goes to* 0 *and the derivative* $\mathbf{U_z}$ *goes to* ∞ *when* z *goes to* 0*, and to* 0 *when* z *goes to* ∞.

Given its importance in convex analysis, we introduce together with any progressive utility $\mathbf{U}$, its convex conjugate $\widetilde{\mathbf{U}}$ (also called conjugate progressive utility (CPU)), that is the Fenchel-Legendre transform of the convex random field $-\mathbf{U}(,-.)$.

Definition 3.2 (Progressive conjugate utility)**.** *The convex conjugate of the progressive utility* $\mathbf{U}$ *is the progressive random field* $\widetilde{\mathbf{U}}$ *defined on* $\mathbb{R}_+^0$ *by* $\widetilde{\mathbf{U}} = \{\widetilde{U}(t,y); t \geq 0, y > 0\}$*, where*

$$\widetilde{U}(t,y) \stackrel{def}{=} \max_{z>0, z\in Q^+} \big(U(t,z) - z\,y\big).$$

(i) *Under Inada condition,* $\widetilde{\mathbf{U}}$ *is twice continuously differentiable, strictly convex, strictly decreasing, with* $\widetilde{U}(.,0^+) = U(+\infty)$, $\widetilde{U}(.,+\infty) = U(0^+), a.s.$

(ii) *The marginal utility* $\mathbf{U}_z$ *is the inverse of the opposite of the marginal conjugate utility* $\widetilde{\mathbf{U}}_y$*, that is* $U_x(t,.)^{-1}(y) = -\widetilde{U}_y(t,y)$*, with Inada conditions* $\widetilde{U}_y(.,0^+) = -\infty$, $\widetilde{U}_y(.,+\infty) = 0$.

(iii) *The bi-dual relation holds true* $U(t,z) = \inf_{y>0, y\in Q^+} \big(\widetilde{U}(t,y) + z\,y\big)$.

Moreover $\widetilde{U}(t,y) = U\big(t, -\widetilde{U}(t,y)\big) + \widetilde{U}_y(t,y)\,y$*, and* $U(t,z) = \widetilde{U}\big(t, U_z(t,z)\big) + z\,U_z(t,z)$.

Consistent dynamic utility In the following, we are interested in the progressive utilities which are consistent with the class $\mathbb{X}$ of admissible wealth processes. The definition of these utilities is the same as in the continuous case.

Definition 3.3 ($\mathbb{X}$-consistent dynamic utility)**.** *A $\mathbb{X}$-consistent dynamic utility $\mathbf{U} = \{U(t,z); t \geq 0, z > 0\}$ is a progressive utility with the following additional properties:*

Consistency with the test-class: *For any admissible wealth process $X \in \mathbb{X}$,*

$$\mathbb{E}(U(t, X_t)/\mathcal{F}_s) \leq U(s, X_s), \quad \forall s \leq t \quad a.s.$$

Existence of optimal wealth: *For any initial wealth $x > 0$, there exists an optimal wealth process $X^* \in \mathbb{X}$ such that $X_0^* = x$, and for all $s \leq t$,*

$$U(s, X_s^*) = \mathbb{E}(U(t, X_t^*)/\mathcal{F}_s) \ \ \forall s \leq t \quad a.s.$$

In short for any admissible wealth $X \in \mathbb{X}$, $U(., X_.)$ is a positive supermartingale and a martingale for the optimal-benchmark wealth X^*.

3.1 Itô Progressive Utility

In this paper, we focus on progressive consistent utilities $\mathbf{U}$ which are a collection of Itô's semimartingales: for any z, $U(.,z)$ is a Itô semimartingale, driven by the d-dimensional Brownian motion W and the Lévy random measure N defined on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ (see Section 2). In other words, for a given initial condition $u(z) = U(0,z)$, U is assumed to satisfies the dynamics

$$dU(t,z) = \beta(t,z)dt + \gamma(t,z).dW_t + \int_{\mathbb{R}} H(t,z,e)\tilde{N}(dt,de), \quad U(0,z) = u(z), \tag{3.1}$$

where $(\beta, \gamma, H) = \{(\beta(t,z), \sigma(t,z), H(t,z,e)); t \geq 0, z > 0, e \in \mathbb{R}\}$ are the local characteristics of U assumed to be progressive random fields, with values in $\mathbb{R}$, $\mathbb{R}^d$ and $\mathbb{R}^d$ respectively. β is called the drift characteristic, γ the diffusion characteristic and H the jumps characteristic. We refer to the books of H. Kunita [Kun97, Kun19] for all technical results concerning the theory of semimartingale random fields with and without jumps. The assumption of finite dimensional Brownian motion greatly simplifies the theory.

3.2 Consistency and the utility HJB-SPDE

In this Section, we are concerned by establishing a sufficient condition on the characteristics (β, γ, H) of the utility random field U ensuring the consistency property. For this, we proceed by verification in order to establish a HJB-type condition.

To begin, let us apply at first the Itô-Venzel's formula. The process $U(t, X_t^\kappa)$ evolves as follows,

$$\begin{aligned}
dU(t,X_t^\kappa) &= \Big(\beta(t,X_t^\kappa) + X_t^\kappa U_z(t,X_t^\kappa) r_t + \frac{1}{2} U_{zz}(t,X_t^\kappa)\|X_t^\kappa \kappa_t\|^2 \\
&+ X_t^\kappa \big[\gamma_z(t,X_t^\kappa) + U_z(t,X_t^\kappa)\eta_t\big].\kappa_t\Big)dt \\
&+ \int_{\mathbb{R}} \Big(U\big(t, X_t^\kappa(1+\kappa_t.h^S(t,e))\big) - U(t,X_t^\kappa) - U_z(t,X_t^\kappa)X_t^\kappa\kappa_t.h^S(t,e)\Big)\nu(de)dt \\
&+ \int_{\mathbb{R}} \Big(H\big(t, X_t^\kappa(1+\kappa_t.h^S(t,e))\big) - H(t,X_t^\kappa,e)\Big)\nu(de)dt + \big(\gamma + X_t^\kappa U_z\kappa\big)(t,X_t)dW_t \\
&+ \int_{\mathbb{R}} \Big(U\big(t, X_{t^-}^\kappa(1+\kappa_t.h^S(t,e))\big) - U(t,X_{t^-}^\kappa) + H(t, X_{t^-}^\kappa(1+\kappa_t.h^S(t,e)))\Big)\tilde{N}(dt,de) \\
&= \Big(\beta(t,X_t^\kappa) + X_t^\kappa U_z(t,X_t^\kappa) r_t - \int_{\mathbb{R}} \Big(U(t,X_t^\kappa) + H\big(t,X_t^\kappa,e\big)\Big)\nu(de) + \mathcal{Q}(t,X_t^\kappa,\kappa_t)\Big)dt \\
&+ \big(\gamma(t,X_t) + X_t^\kappa U_z(t,X_t)\kappa_t\big)dW_t \\
&+ \int_{\mathbb{R}} \Big(U\big(t, X_{t^-}^\kappa(1+\kappa_t.h^S(t,e))\big) - U(t,X_{t^-}^\kappa) + H\big(t, X_{t^-}^\kappa(1+\kappa_t.h^S(t,e))\big)\Big)\tilde{N}(dt,de).
\end{aligned}$$

Where, denoting by $\alpha_t := \int_{\mathbb{R}} h^S(t,e)\nu(de)$, the quantity $\mathcal{Q}$ is defined by

$$\begin{aligned}
\mathcal{Q}(t,z,\kappa) &:= \int_{\mathbb{R}} \Big(U+H\Big)\big(t, z(1+\kappa_t.h^S(t,e))\big)\nu(de) + \frac{1}{2}U_{zz}(t,z)\|z\kappa_t\|^2 \\
&+ z\kappa_t.\big(\gamma_z + U_z(t,z)(\eta_t - \alpha_t)\big),
\end{aligned} \tag{3.2}$$

which rewrites

$$\begin{aligned}
\mathcal{Q}(t,z,\kappa) &:= \int_{\mathbb{R}} \Big(U+H\Big)\big(t, z(1+\kappa_t.h^S(t,e))\big)\nu(de) \\
&+ \frac{1}{2}U_{zz}(t,z)\Big[\|z\kappa_t + \frac{\gamma_z(t,z) + U_z(t,z)(\eta_t-\alpha_t)}{U_{zz}(t,z)}\|^2 - \|\frac{\gamma_z(t,z) + U_z(t,z)(\eta_t-\alpha_t)}{U_{zz}(t,z)}\|^2\Big].
\end{aligned}$$

From this, observe that Optimizing over the class $\mathcal{R}$ of admissible policies $\kappa \in \mathcal{R}$, consists on maximizing the quantity $\mathcal{Q}$.

To ensure the existence of a finite maximum of this form, we make the following assumption for the rest of this work.

Assumption 3.1. *For any $t \in \mathbb{R}_+$ and $e \in \mathbb{R}$, $U(t,.) + H(t,.,e)$ is strictly concave and there exist two positive random fields $C^1(t,e)$ and $C^2(t,e)$ and a real number $p > -1$ such that*

$$\begin{cases}
U_z(t,z) + H_z(t,z,e) \le C_1(t,e)z^{-p} + C_2(t,e), \ \forall z, & (3.3) \\
\displaystyle\int_{\mathbb{R}\times[0,T]} \big(C_1(t,e) + C_2(t,e)\big)\nu(de)dt < +\infty, a.s.\ \forall T. & (3.4)
\end{cases}$$

This assumption implies that $\mathcal{Q}$ is a strictly concave function of $x\kappa$ and goes to $-\infty$ when $\|x\kappa\| \to \infty$. Indeed, this hypothesis implies that $U(t,z) + H(t,z,e) \le \frac{1}{1-p}C_1(t,e)z^{1-p} + C_2(t,e)z + C_3(t)$, $\forall z$ and then the behavior of $\mathcal{Q}$ in the neighborhood of infinity, because $1-p<2$, is that of $\frac{1}{2}U_{zz}\|z\kappa\|^2$, $U_{zz} < 0$.

Consequently, the supremum of $\mathcal{Q}(t,z,\kappa)$ in the above expression is reached at the optimal policy κ^* characterized by the following formula,

$$\begin{aligned}
< \nabla_\kappa \mathcal{Q}(t, zh^S(t,e), \kappa^*), L > &:= < \int_{\mathbb{R}} \Big(U_z + H_z\Big)\big(t, z(1+\kappa_t^*.h^S(t,e)\big)h^S(t,e)\nu(de), L > \\
&+ < U_{zz}(t,z)z\kappa_t^* + \gamma_z(t,z) + U_z(t,z)(\eta_t - \alpha_t), L >= 0, \ \forall L \in \mathcal{R},
\end{aligned}$$

which holds only in the case where the orthogonal projection onto $\mathcal{R}$ of the vector $\nabla_\kappa \mathcal{Q}(t,z,\kappa^*)$ is the null vector. In other words, the optimal policy is given by

$$z\kappa_t^*(z) = -\frac{\gamma_z^{\mathcal{R}}(t,z) + U_z(t,z)(\eta_t - \alpha_t)}{U_{zz}(t,z)} - \frac{\int_{\mathbb{R}} (U_z + H_z)\big(t, z(1+\kappa_t^*(z).h^S(t,e)\big).h^S(t,e)\nu(de)}{U_{zz}(t,z)}. \quad (3.5)$$

We recognize here the policy $-\frac{\gamma_z^{\mathcal{R}}(t,z)+U_z(t,z)\eta_t}{U_{zz}(t,z)}$ which corresponds to the optimal strategy in case without Jumps, see [EKM13]. Plugging expression (3.5), we get after easy calculations

$$\begin{aligned}\mathcal{Q}(t,z,\kappa^*) &= \int_{\mathbb{R}} (U+H)\big(t, z(1+\kappa_t^*(z).h^S(t,e)\big)\nu(de) - \frac{1}{2U_{zz}}\|\gamma_z^{\mathcal{R}} + U_z(t,z)(\eta_t-\alpha_t)\|^2 \\ &+ \frac{1}{2U_{zz}(t,z)}\|\int_{\mathbb{R}} (U_z+H_z)\big(t, z(1+\kappa_t^*(z).h^S(t,e)\big)h^S(t,e)\nu(de)\|^2. \quad (3.6)\end{aligned}$$

We are now able to state the following verification Theorem.

Theorem 3.1. *Assume U to be a $\mathbb{X}$-consistent dynamic utility of class $\mathcal{K}^{2,\delta}_{loc}$, $\delta \in]0,1[$ and let (β,γ,H) denote its local characteristics. Suppose Assumption 3.1 holds, then*

(i) *The characteristic coefficient β is given by the following HJB-condition*

$$\beta(t,z) = -zU_z(t,z)r_t + \int_{\mathbb{R}} (U(t,z) + H(t,z,e))\nu(de) - \mathcal{Q}(t,z,\kappa^*). \quad (3.7)$$

(ii) *The optimal policy is unique and is given by*

$$z\kappa^*(z) = -\frac{\gamma_z^{\mathcal{R}}(t,z) + U_z(t,z)(\eta_t - \alpha_t)}{U_{zz}(t,z)} - \frac{\int_{\mathbb{R}} (U_z + H_z)\big(t, z(1+\kappa_t^*(z).h^S(t,e)\big)h^S(t,e)\nu(de)}{U_{zz}(t,z)}. \quad (3.8)$$

(iii) *Furthermore, its optimal portfolio is solution of the following SDE(κ^*) with jumps:*

$$dX_t^* = X_{t^-}^*\big[r_t dt + \kappa_t^*(X_t^*).(dW_t + \eta_t dt + \int_{\mathbb{R}} h^S(t,e)\tilde{N}(dt,de))\big]. \quad (3.9)$$

As a direct consequence of this Theorem, we have easily the following result.

Corollary 3.2. *Under Assumptions of Theorem 3.1, a market-consistent dynamic U is solution of the following SPDE with Jumps,*

$$\begin{aligned}dU(t,z) &= \Big(-zU_z(t,z)r_t + \int_{\mathbb{R}} (U(t,z)+H(t,z,e))\nu(de) - \mathcal{Q}(t,z,\kappa^*)\Big)dt \\ &+ \gamma(t,z)dW_t + \int_{\mathbb{R}} H(t,z,e)\tilde{N}(dt,de). \quad (3.10)\end{aligned}$$

Moreover, by regularity of U, its derivative, by Theorem 2.1, satisfies

$$\begin{aligned}dU_z(t,z) &= \Big(-(U_z(t,z) + zU_{zz}(t,z))r_t + \int_{\mathbb{R}} (U_z(t,z) + H_z(t,z,e))\nu(de) - \mathcal{Q}_z(t,z,\kappa^*)\Big)dt \\ &+ \gamma_z(t,z)dW_t + \int_{\mathbb{R}} H_z(t,z,e)\tilde{N}(dt,de). \quad (3.11)\end{aligned}$$

Remark . *Note that the equation* (3.10) *is a fully nonlinear SPDEs driven by a Lévy noise., see [PZ07]. There is few studies in the literature of this type of fairly complex equations, especially concerning the existence, uniqueness and regularity properties of solutions. However, using similar ideas as the characteristics method, we give, in what follows, sufficient conditions ensuring these properties and the characterization of solutions for this class of SPIDEs. The key point is to link this equations to two SDEs with jumps.*

3.3 Marginal dynamic utility dynamics

In this section, we are particularly interested in the dynamics of marginal utility along the optimal portfolio. We show in particular that the process $U_z(t, X_t^*)$ is a state price density process (Definition 2.3). This allows us later to establish the link with the dual problem and then a connection between the utility-SPIDE (3.11) and two SDEs satisfied by the optimal processes (dual and primal one) and finally to characterize the class of solutions, under some regularity conditions.
Let us at first study the dynamics of marginal utility along the optimal portfolio X^*.

Theorem 3.3. *Under Assumptions of Theorem 3.1, the marginal utility along the optimal choice X^* is a state price density process.*

$$\begin{cases} dU_z(t,X_t^*) = -U_z(t,X_t^*)r_t dt + \Big(\gamma_z(t,X_t^*) + U_{zz}(t,X_t^*)X_t^*\kappa_t^*(X_t^*)\Big)dW_t & (3.12)\\ \qquad\qquad + \displaystyle\int_{\mathbb{R}} \bar{H}_z(t,X_{t-}^*,e)\tilde{N}(dt,de), & (3.13)\\ \bar{H}_z(t,z,e) = U_z\big(t, z(1+\kappa_{t-}^*(z)h^S(t,e))\big) - U_z(t,z) + H_z(t, z(1+\kappa_{t-}^*(z).h^S(t,e)),e). \end{cases}$$

Proof. The Itô-Ventzel's formula, combined with (3.11) and (2.20) ($\kappa = \kappa^*$), implies

$$\begin{aligned} dU_z(t,X_t^*) = \Big[& - U_z(t,X_t^*)r_t - X_t^*U_{zz}(t,X_t^*)r_t + \int_{\mathbb{R}}(U_z(t,X_t^*) + H_z(t,X_t^*,e))\nu(de)\\ - \quad & \mathcal{Q}_x(t,X_t^*,\kappa^*) + X_t^*U_{zz}(t,X_t^*)r_t + \frac{1}{2}U_{zzz}(t,X_t^*)\|X_t^*\kappa_t^*(X_t^*)\|^2\\ + \quad & X_t^*\kappa_t^*(X_t^*)\big[\gamma_{xx}(t,X_t^*) + U_{zz}(t,X_t^*)\eta_t\big]\Big]dt + \Big(\gamma_z(t,X_t^*) + U_{zz}(t,X_t^*)X_t^*\kappa_t^*(X_t^*)\Big)dW_t\\ + \quad & \int_{\mathbb{R}}\Big[U_z\big(t,X_t^*(1+\kappa_t^*(X_t^*)h^S(t,e))\big) - U_z(t,X_t^*) - U_{zz}(t,X_t^*(x))X_t^*\kappa_t^*(X_t^*)h^S(t,e)\\ + \quad & H_z(t,X_t^*(1+\kappa_t^*(X_t^*)h^S(t,e)) - H_z(t,X_t^*,e)\Big]\nu(de)dt\\ + \quad & \int_{\mathbb{R}}\Big(\big(U_z + H_z(.,.,e)\big)\big(t,X_{t-}^*(1+\kappa_t^*(X_{t-}^*)h^S(t,e)\big) - U_z(t,X_{t-}^*)\Big)\tilde{N}(dt,de). \end{aligned}$$

Which becomes after a few simplifications and by arranging the terms

$$\begin{aligned} dU_z(t,X_t^*) = \Big[& - U_z(t,X_t^*)r_t - \mathcal{Q}_x(t,X_t^*,\kappa^*)\\ + \quad & \frac{1}{2}U_{zzz}(t,X_t^*)\|X_t^*\kappa_t^*(X_t^*)\|^2 + X_t^*\kappa_t^*(X_t^*).\big(\gamma_{xx}(t,X_t^*) + U_{zz}(t,X_t^*)(\eta_t - \alpha_t)\big)\Big]dt\\ + \quad & \Big(\gamma_z(t,X_t^*) + U_{zz}(t,X_t^*)X_t^*\kappa_t^*(X_t^*)\Big)dW_t\\ + \quad & \int_{\mathbb{R}}\Big(U_z\big(t,X_t^*(1+\kappa_t^*(X_t^*)h^S(t,e))\big) + H_z\big(t,X_t^*(1+\kappa_t^*(X_t^*)h^S(t,e))\big)\Big)\nu(de)dt\\ + \quad & \int_{\mathbb{R}}\Big(\big(U_z + H_z(.,.,e)\big)\big(t,X_{t-}^*(1+\kappa_t^*(X_{t-}^*).h^S(t,e))\big) - U_z(t,X_{t-}^*)\Big)\tilde{N}(dt,de) \end{aligned}$$

where we have used the fact that

$$-\int_{\mathbb{R}} U_{zz}(t,X_t^*)X_t^*\kappa_t^*(X_t^*).h^S(t,e)\nu(de) = -U_{zz}(t,X_t^*)X_t^*\kappa_t^*(X_t^*).\alpha_t.$$

To go further in the calculations, we now calculate the quantity $\mathcal{Q}_x(t,X_t^*,\kappa^*)$. To do this, we start from the definition of this operator;

$$\begin{aligned} \mathcal{Q}(t,x,\kappa^*) \quad := \quad & \int_{\mathbb{R}}\Big(U + H\Big)\big(t,x(1+\kappa_t^*(x)h^S(t,e))\big)\nu(de)\\ + \quad & \frac{1}{2}U_{zz}(t,x)\|x\kappa_t^*(x)\|^2 + x\kappa_t^*(x)(\gamma_z(t,x) + U_z(t,x)(\eta_t - \alpha_t)). \end{aligned}$$

Differentiating with respect to x, one gets

$$\begin{aligned}\mathcal{Q}_x(t,x,\kappa^*) &= \int_{\mathbb{R}}\Big(U_z+H_z\Big)\big(t,x(1+\kappa_t^*(x)h^S(t,e))\big)(1+(x\kappa_t^*(x))_x h^S(t,e))\nu(de)\\ &+ \frac{1}{2}U_{zzz}(t,x)\|x\kappa_t^*(x)\|^2 + x\kappa_t^*(x)\big(\gamma_{zz}^{\mathcal{R}}+U_{zz}(t,x)(\eta_t-\alpha_t)\big)\\ &+ (x\kappa_t^*(x))_x\Big(U_{zz}(t,x)x\kappa_t^*(x)+\gamma_z^{\mathcal{R}}(t,x)+U_z(t,x)(\eta_t-\alpha_t)\Big),\end{aligned}$$

which equivalent, using (3.8), to

$$\begin{aligned}&- \mathcal{Q}_x(t,x,\kappa^*) + \int_{\mathbb{R}}\Big(U_z+H_z\Big)\big(t,x(1+\kappa_t^*(x).h^S(t,e))\big)\nu(de)\\ &+ \frac{1}{2}U_{zzz}(t,x)\|x\kappa_t^*(x)\|^2 + x\kappa_t^*(x)(\gamma_{zz}^{\mathcal{R}}+U_{zz}(t,x)(\eta_t-\alpha_t))\\ &= (x\kappa_t^*(x))_x\Big(U_{zz}(t,x)x\kappa_t^*(x)+\gamma_z^{\mathcal{R}}(t,x)+U_z(t,x)(\eta_t-\alpha_t)\\ &+ \int_{\mathbb{R}}\Big(U_z+H_z\Big)\big(t,x(1+\kappa_t^*(x).h^S(t,e))\big)\nu(de)\Big).\end{aligned}$$

Now according to Theorem 3.1, the quantity in parenthesis in the last term is zero. So,

$$\begin{aligned}&- \mathcal{Q}_x(t,x,\kappa^*) + \int_{\mathbb{R}}\Big(U_z+H_z\Big)\big(t,x(1+\kappa_t^*(x).h^S(t,e))\big)\nu(de)\\ &+ \frac{1}{2}U_{zzz}(t,x)\|x\kappa_t^*(x)\|^2 + x\kappa_t^*(x).\big(\gamma_{zz}^{\mathcal{R}}+U_{zz}(t,x)(\eta_t-\alpha_t)\big)=0.\end{aligned}$$

Injecting this identity in the dynamics of $U_z(t,X_t^*)$ above, follows

$$\begin{aligned}dU_z(t,X_t^*) &= -U_z(t,X_t^*)r_t dt + \Big(\gamma_z(t,X_t^*)+U_{zz}(t,X_t^*)X_t^*\kappa_t^*(X_t^*)\Big)dW_t\\ &+ \int_{\mathbb{R}}\Big(U_z\big(t,X_{t^-}^*(1+\kappa_t^*(X_{t^-}^*).h^S(t,e)\big)-U_z(t^-,X_{t^-}^*)\\ &+ H_z(t,X_{t^-}^*(1+\kappa_t^*(X_{t^-}^*).h^S(t,e),e)\Big)\tilde{N}(dt,de).\end{aligned}$$

This achieves the proof. □

We now interpret the dynamics (3.13) as that of a state price density process by identifying it with that of the form (2.21). We have the following characterization of $U_z(t,X_t^*)$.

Corollary 3.4. *Under Assumptions of Theorem 3.3, the positive process $U_z(t,X_t^*(x))$, is a state price density process Y^U, starting from the initial condition $u_z(x)$, satisfying the dynamics*

$$\begin{cases}\dfrac{dY_t^U}{Y_{t^-}^U} = -r_t dt + \Big(\nu^U(t,Y_t^U)-\eta_t-\displaystyle\int_{\mathbb{R}}h^S(t,e)h^U(t,Y_t^U,e)\nu(de)\Big)dW_t + \int_{\mathbb{R}}h^U(t,Y_t^U,e)\tilde{N}(dt,de)),\\ \nu^U(t,y) = \dfrac{\gamma_z^{\perp}(t,-\widetilde{U}_y(t,y))}{y},\ \textit{where}\ -\widetilde{U}_y(t,.) = (U_z)^{-1}(t,.),\\ h^U(t,y,e) = \bar{H}_z(t,-\widetilde{U}_y(t,y),e).\end{cases} \tag{3.14}$$

Proof. To show this corollary, simply combine the results of Theorem 3.3 with those of Theorem 3.1. Indeed, from this last one, equation (3.8), the optimal strategy, recalling the notation $\alpha_t = \int_{\mathbb{R}}h^S(t,e)\nu(de)$, is given by

$$x\kappa_t^*(x) = -\frac{\gamma_z^{\mathcal{R}}(t,x)+U_z(t,x)(\eta-\alpha)}{U_{zz}(t,x)} - \frac{\displaystyle\int_{\mathbb{R}}(U_z+H_z)\big(t,x(1+\kappa_t^*(x)h^S(t,e)\big).h^S(t,e)\nu(de)}{U_{zz}(t,x)}.$$

That is, the volatility vector of $U_z(t, X_t^*)$ in the dynamics (3.13) is

$$\begin{aligned}
& \gamma_z(t, X_t^*) + U_{zz}(t, X_t^*) X_t^* \kappa_t^*(X_t^*) \\
= \;& \gamma_z^{\perp}(t, X_t^*) - U_z(t, X_t^*)\eta_t \\
- \;& \int_{\mathbb{R}} \Big[\Big(U_z + H_z \Big)\big(t, X_t^* + \kappa_t^*(X_t^*).h^S(t,e)\big) - U_z(t, X_t^*) \Big] h^S(t,e)\nu(de) \\
= \;& \gamma_z^{\perp}(t, X_t^*) - U_z(t, X_t^*)\eta_t - \int_{\mathbb{R}} \bar{H}_z(t, X_t^*, e) h^S(t,e)\nu(de) \\
= \;& \gamma_z^{\perp}(t, -\widetilde{U}_y(t, U_z(t, X_t^*)) - U_z(t, X_t^*)\eta_t - \int_{\mathbb{R}} \bar{H}_z\big(t, -\widetilde{U}_y(t, U_z(t, X_t^*), e)\big) h^S(t,e)\nu(de),
\end{aligned}$$

where we have used the identity $X_t^* = -\widetilde{U}_y(t, U_z(t, X_t^*))$ in the last line. □

3.4 Duality

In this section we are interested in the Fenchel-Transform $\widetilde{U}$ of the consistent utility U. It is a complementary but necessary study, since it enables us to better understand the role of the volatility γ and the jump characteristics H of U. Indeed, we know from the equation (3.8) that the orthogonal projection $\gamma_z^{\mathcal{R}}$ of γ_z partly characterizes the optimal strategy κ^*, but we know little about the role of the orthogonal component $\gamma_z^{\perp}$. In [EKM13], the duality has allowed the authors to show that $\gamma_z^{\perp}$ characterizes in turn the optimal dual and thus they have succeeded in characterizing a large class of these utilities. The approach is thus similar to the one developed in this last paper. First, we establish the dynamics of the marginal dual utility $\widetilde{U}_y$ using the fact that $-\widetilde{U}_y$ is the inverse map of U_z and then integrate to get that of $\widetilde{U}$. Second, we study the dual optimization program and the we show the main result of this section that states that the optimal dual process is $U_z(t, X_t^*(x))$.

3.4.1 Dynamics of the conjugate of consistent utility

In this section, we are concerned with the dynamics of the $\widetilde{U}$ conjugate of consistent utility. This allows us in the following to establish a link between the primal optimization problem and the dual one. The proof of the following Theorem is based on the results of Theorem 2.4, since $-\widetilde{U}_y$ is the inverse flow of U_z whose dynamic is given by (3.11). We first establish the dynamics of $\widetilde{U}_y$ and then that of $\widetilde{U}$ by a simple integration.

Theorem 3.5. *Let U be a regular consistent dynamic utilities of class $\mathcal{K}_{loc}^{3,\delta}$ with characteristics (β, γ, H) satisfying the HJB-SPDE* (3.10) *and Assumption 3.1. Then*
(i) *The maps $z \to U_z(t,z) + H_z(t,z,e)$ is invertible for any $t \in \mathbb{R}_+$ and $e \in \mathbb{R}$ a.s. Let $\Psi(t,.,e)$ denotes it inverse.*
(ii) *Let $\Phi(t,y,e)$ denotes the primitive of $-\Psi(t,y,e)$ and $\mathcal{Q}$ the quadratic operator defined by* (3.6). *The marginal dual utility $\widetilde{U}_y$ satisfies the SPIDE*

$$\begin{aligned}
d\widetilde{U}_y(t,y) &= \partial_y \Big[y\widetilde{U}_y r_t - \mathcal{Q}(t, -\widetilde{U}_y(t,y), \kappa^*) + \int_{\mathbb{R}} \big(\Phi(t,y,e) - y\widetilde{U}_y(t,y) \big)\nu(de) \Big] dt \\
&- \widetilde{U}_{yy}(t,y)\gamma_z(t, -\widetilde{U}(t,y))dW_t + \int_{\mathbb{R}} \partial_y \big[\Phi(t,y,e) - \widetilde{U}(t,y) \big] \tilde{N}(dt, de). \qquad (3.15)
\end{aligned}$$

(iii) *The dual utility $\widetilde{U}$ satisfies the SPDE*

$$\begin{aligned} d\widetilde{U}(t,y) &= \Big[y\widetilde{U}_y r_t - \mathcal{Q}(t,-\widetilde{U}_y(t,y),\kappa^*) + \int_{\mathbb{R}}(\Phi(t,y,e) - y\widetilde{U}_y(t,y))\nu(de)\Big]dt \\ &+ \gamma(t,-\widetilde{U}(t,y))dW_t + \int_{\mathbb{R}}\big(\Phi(t,y,e) - \widetilde{U}(t,y)\big)\tilde{N}(dt,de), \end{aligned} \tag{3.16}$$

Proof. From the definition of the Fenchel-transform, the marginal dual utility $-\widetilde{U}_y$ is the inverse maps (with respect to the spatial parameter z) of the marginal utility U_z which satisfies, from Corollary 3.2,

$$\begin{aligned} dU_z(t,z) &= \Big(-(U_z(t,z) + zU_{zz}(t,z))r_t + \int_{\mathbb{R}}(U_z(t,z) + H_z(t,z,e))\nu(de) - \mathcal{Q}_x(t,z,\kappa^*)\Big)dt \\ &+ \gamma_z(t,z)dW_t + \int_{\mathbb{R}} H_z(t,z,e)\tilde{N}(dt,de). \end{aligned} \tag{3.17}$$

It follows, using the fact that $z = -\widetilde{U}_y(t,U_z(t,z))$, from Theorem 2.4, equation (2.8), that the volatility vector and the jump coefficient of $-\widetilde{U}_y$ are given respectively by $\widetilde{U}_{yy}(t,y)\gamma_y(-\widetilde{U}_y(t,y)$ and $\widetilde{U}_y + \Psi_z(t,z,e)$ a.s. for any t and y.
Let's now turn to the drift $-\tilde{\beta}_y$ of $-\widetilde{U}_y$. Still according to Theorem 2.4, (2.8) combined with the identity $U_{zz}(t,-\widetilde{U}_y(t,y)) = -\frac{1}{\widetilde{U}_{yy}(t,y)}$ gives, omitting for simplicity the dependance on (t,y)

$$\begin{aligned} -\tilde{\beta}_y(t,y) &= -\widetilde{U}_{yy}(t,y)\Big[-\Big(-(y + \frac{\widetilde{U}_y}{\widetilde{U}_{yy}}(t,y))r_t + \int_{\mathbb{R}}(y + H_z(t,-\widetilde{U}_y(t,y),e))\nu(de) \\ &- (\partial_x\mathcal{Q})(t,-\widetilde{U}_y(t,y),\kappa^*)\Big) + \int_{\mathbb{R}} H_z(t,-\widetilde{U}_y,e))\nu(de)\Big] + \frac{1}{2}\partial_y\Big(-\widetilde{U}_{yy}\|\gamma_z(t,-\widetilde{U}_y)\|^2\Big) \\ &+ \widetilde{U}_y + \int_{\mathbb{R}}\Psi(t,y,e)\nu(de) \\ &= -(y\widetilde{U}_{yy}(t,y) + \widetilde{U}_y(t,y))r_t + y\widetilde{U}_{yy}(t,y) + \widetilde{U}_y(t,y) - \widetilde{U}_{yy}(t,y)(\partial_x\mathcal{Q})(t,-\widetilde{U}_y(t,y),\kappa^*) \\ &- \frac{1}{2}\partial_y\Big(\widetilde{U}_{yy}(t,y)\|\gamma_z(t,-\widetilde{U}_y(t,y))\|^2\Big) + \int_{\mathbb{R}}\Psi(t,y,e)\nu(de). \end{aligned}$$

If we now denote by $\Phi(t,.,e)$ a primitive of $-\Psi(t,.,e)$, it follows arranging the terms

$$\begin{aligned} -\tilde{\beta}_y(t,y) &= -\partial_y\Big[y\widetilde{U}_y(t,y)r_t - y\widetilde{U}_y(t,y) - \mathcal{Q}(t,-\widetilde{U}_y(t,y),\kappa^*) + \frac{1}{2}\widetilde{U}_{yy}(t,y)\|\gamma_z(t,-\widetilde{U}_y(t,y))\|^2 \\ &+ \int_{\mathbb{R}}\Phi(t,y,e)\nu(de)\Big]. \end{aligned}$$

This achieves the proof of (3.15). The dynamics (3.16) is obtained by a simple integration with respect to y. □

3.4.2 The dual optimisation problem

Now we establish the main result of this section. It shows that the dual utility is consistent with the class $\mathbb{Y}$ of state price density processes. For any $Y \in \mathbb{Y}$, the process $\widetilde{U}(t,Y_t)$ is a submartingale and martingale for a certain Y^*. The most important point is that this optimal process is characterized by $Y_t^* = U_z(t,X_t^*)$. Recall, from Lemma 2.6, that any $Y \in \mathbb{Y}$ satisfies the dynamics

$$\frac{dY_t^{\nu,h}}{Y_{t^-}^{\nu,h}} = -r_t dt + \Big(\nu_t - \eta_t - \int_{\mathbb{R}} h^S(t,e)h(t,e)\nu(de)\Big)dW_t + \int_{\mathbb{R}} h^Y(t,e)\tilde{N}(dt,de)), \tag{3.18}$$

with $\nu \in \mathcal{R}^\perp,\ h^Y \in \mathbb{R}$.

Theorem 3.6. *Let Assumptions of Theorem 3.5 hold and let $D_t(Y_t^{\nu,h},\nu,h)$ denotes the drift coefficient of the semimartingale $\widetilde{U}(t,Y_t^{\nu,h})$, $(\nu,h)\in\mathcal{R}^{\perp}\times\mathbb{R}$.*

(i) *The optimization program*

$$\inf_{(\nu,h)\in\mathcal{R}^{\perp}\times\mathbb{R}} D_t(y,\nu,h),$$

is achieved at the following dual policies

$$\begin{cases} y\nu_t^*(y)=\gamma_z^{\perp}(t,-\widetilde{U}_y(t,y)), & (3.19)\\ y+yh^*(t,y,e)=(U_z+H_z)\Big(\xi(t,y)+\widetilde{U}_{yy}(t,y)\int_{\mathbb{R}} yh^*(t,y,e)h^S(t,e)\nu(de)\Big)\Big), & (3.20)\\ \xi(t,y):=-\widetilde{U}_y(t,y)+\widetilde{U}_{yy}(t,y)\big(y\eta+\gamma_z^{\mathcal{R}}(t,-\widetilde{U}_y(t,y))\big). & (3.21)\end{cases}$$

(ii) *The pair (ν^*,h^*) is linked with the optimal strategy κ^* (Theorem 3.1, (3.8)) as follows*

$$\begin{cases} y\Big(\nu_t^*-\eta_t-\int_{\mathbb{R}}h^S(t,e)h^*(t,y,e)\nu(de)\Big)=\Big[\gamma_z(t,z)+z\kappa_t^*(z)\Big]_{z=-\widetilde{U}_y(t,y)},\\ y+yh^*(t,y,e)=\big(U_z+H_z\big)\big(z(1+\kappa_t^*(z)h^S(t,e)\big)_{z=-\widetilde{U}_y(t,y)}.\end{cases}$$

(iii) *The dual process $Y_t^*:=Y_t^{\nu^*,h^*}$ and $U_z(t,X_t^*)$ satisfy the same dynamics as i.e. (3.14), since $h^*=h^U$ and $\nu^*=\nu^U$. Moreover, $Y_t^*(u_z(x))=U_z(t,X_t^*(x))$ if uniqueness of the solution holds.*

(iv) Consistency with the dual class $\mathbb{Y}$: *For any $Y^{\nu,h}\in\mathbb{Y}$, the process $\widetilde{U}(t,Y_t^{\nu,h})$ is a submartingale and local martingale for $Y^{\nu,h}=Y^*$. The martingale property of $\widetilde{U}_y(t,Y_t^*)$, holds under integrability conditions.*

Proof. Applying the generalized Itô-Ventzel's formula (2.4) to the compound $\widetilde{U}(t,Y_t)$, the drift term is given by

$$\begin{aligned} D_t(Y_t^{\nu,h},\nu,h) &= Y_t^{\nu,h}\widetilde{U}_y(t,Y_t^{\nu,h})r_t-Y_t^{\nu,h}\widetilde{U}_y(t,Y_t^{\nu,h})-\mathcal{Q}(t,-\widetilde{U}_y(t,Y_t^{\nu,h}),\kappa^*)\\ &+ \frac{1}{2}\widetilde{U}_{yy}(t,Y_t^{\nu,h})\|\gamma_z(t,-\widetilde{U}_y(t,Y_t^{\nu,h}))\|^2+\int_{\mathbb{R}}\Phi(t,Y_t^{\nu,h},e)\nu(de)-Y_t^{\nu,h}\widetilde{U}_y(t,Y_t^{\nu,h})r_t\\ &+ \frac{1}{2}(Y_t^{\nu,h})^2\widetilde{U}_{yy}(t,Y_t^{\nu,h})\|\nu_t-\eta_t-\int_{\mathbb{R}}h^S(t,e)h^Y(t,e)\nu(de)\|^2\\ &- Y_t^{\nu,h}\widetilde{U}_{yy}(t,Y_t^{\nu,h})\gamma_z(t,-\widetilde{U}_y(t,Y_t^{\nu,h}).\Big(\nu_t-\eta_t-\int_{\mathbb{R}}h^S(t,e)h(t,e)\nu(de)\Big)\\ &+ \int_{\mathbb{R}}\Big(\widetilde{U}(t,Y_t^{\nu,h}+Y_t^{\nu,h}h^Y(t,e)-\widetilde{U}(t,Y_t^{\nu,h})-Y_t\widetilde{U}_y(t,Y_t^{\nu,h})h(t,e)\Big)\nu(de)\\ &+ \int_{\mathbb{R}}\Big(\Phi(t,Y_t^{\nu,h}+Y_t^{\nu,h}h^Y(t,e),e)+\widetilde{U}(t,Y_t^{\nu,h})-\Phi(t,Y_t^{\nu,h},e)\Big)\nu(de).\end{aligned}$$

Which becomes, after simplifications

$$D_t(Y_t^{\nu,h},\nu,h) = -Y_t^{\nu,h}\widetilde{U}_y(t,Y_t^{\nu,h})-\mathcal{Q}(t,-\widetilde{U}_y(t,Y_t^{\nu,h}),\kappa^*)+\tilde{\mathcal{Q}}(t,Y_t^{\nu,h},\nu,h).$$

With the notation,

$$\begin{aligned}\tilde{\mathcal{Q}}(t,y,\nu,h) &:= \int_{\mathbb{R}}\Big(\Phi(t,y+yh(t,e),e)-y\widetilde{U}_y(t,y)h(t,e)\Big)\nu(de)\\ &+ \frac{1}{2}\widetilde{U}_{yy}(t,y)\|y\big(\nu_t-\eta_t-\int_{\mathbb{R}}h^S(t,e)h(t,e)\nu(de)\big)-\gamma_z(t,-\widetilde{U}_y(t,y))\|^2. \quad (3.22)\end{aligned}$$

Therefore,

$$\inf_{(\nu,h)\in\mathcal{R}^\perp\times\mathbb{R}} D_t(y,\nu,h) = -y\widetilde{U}_y(t,y) - \mathcal{Q}(t,-\widetilde{U}_y(t,y),\kappa^*) + \inf_{(\nu,h)\in\mathcal{R}^\perp\times\mathbb{R}} \tilde{\mathcal{Q}}(t,y,\nu,h).$$

So, let us focus on $\inf_{(\nu,h)\in\mathcal{R}^\perp\times\mathbb{R}} \tilde{\mathcal{Q}}(t,y,\nu,h)$. To do, the key point is to remark that $\Phi(t,.,e)$, defined as the primitive of $-\Psi(t,.,e)$ (where $\Psi(t,.,e)$ is the inverse of $U_z(t,.)+H_z(t,.,e)$), is the Fenchel transform's of the strictly concave (Assumption 3.1) random field $U(t,.)+H(t,.,e)$. So $\Phi(t,.,e)$ is strictly convex and consequently $\tilde{\mathcal{Q}}(t,Y_t,\nu,h)$ is strictly convex in (ν,h). In addition, by Assumption 3.1, $\tilde{\mathcal{Q}}$ goes to $+\infty$ if $\|(\nu,h)\|\to\infty$. To summarize, there exist a unique pair (ν^*,h^*) achieving the infimum of $\tilde{\mathcal{Q}}(t,y,\nu,h)$ and satisfies $\nabla_{(\nu,h)}\tilde{\mathcal{Q}}(t,y,\nu^*,h^*)=0$. So, a basic differentiation with respect to the pair (ν,h), implies the characterizations given by equations (3.19) and (3.20).

(ii) Let now recall the optimal strategy κ^* given by (3.8) in Theorem 3.1, as the unique solution of

$$x\kappa^*(t,x) = -\frac{\gamma_z^{\mathcal{R}}(t,x)+U_z(t,x)(\eta-\alpha)}{U_{zz}(t,x)} - \frac{\int_{\mathbb{R}}(U_z+H_z)\big(t,x(1+\kappa^*(t,x).h^S(t,e)\big).h^S(t,e)\nu(de)}{U_{zz}(t,x)},$$

with $\alpha_t=\int_{\mathbb{R}}h^S(t,e)\nu(de)$. Let us observe on the one hand that, using $U_z(t,-\widetilde{U}_y(t,y))=y$ and $U_{zz}(t,-\widetilde{U}_y(t,y))=-\frac{1}{\widetilde{U}_{yy}(t,y)}$, that $-\widetilde{U}_y(t,y)\kappa^*(t,-\widetilde{U}_y(t,y))$ is the unique solution of

$$\begin{aligned}-\widetilde{U}_y(t,y)\kappa^*(t,-\widetilde{U}_y(t,y)) &= -\widetilde{U}_y(t,y)\kappa^0(t,-\widetilde{U}_y(t,y))\\ &+ \widetilde{U}_{yy}(t,y)\int_{\mathbb{R}}\big(U_z+H_z\big)\Big(t,-\widetilde{U}_y(t,y)-\widetilde{U}_y(t,y)\kappa^*(t,-\widetilde{U}_y(t,y)),e\Big)\nu(de),\end{aligned}$$

where we have used the notation,

$$-\widetilde{U}_y(t,y)\kappa^0(t,-\widetilde{U}_y(t,y)) := \widetilde{U}_{yy}(t,y)\big(y\eta+\gamma_z(t,-\widetilde{U}_y(t,y))\big).$$

On the other hand, observe that the volatility $\tilde{\sigma}^*(t,y) := y\Big(\nu_t-\eta_t-\int_{\mathbb{R}}h^S(t,e)h^*(t,y,e)\nu(de)\Big)$ of $Y^*:=Y^{\nu^*,h^*}$,

$$\begin{aligned}-\ & \widetilde{U}_{yy}(t,y)\big(\tilde{\sigma}^*(t,y)-\gamma_z(t,-\widetilde{U}_y(t,y))\big) = -\widetilde{U}_y(t,y)\kappa^0(t,-\widetilde{U}_y(t,y))\\ +\ & \widetilde{U}_{yy}(t,y)\int_{\mathbb{R}}\big(U_z+H_z\big)\Big(t,-\widetilde{U}_y(t,y)-\widetilde{U}_{yy}(t,y)\big(\tilde{\sigma}^*(t,y)-\gamma_z(t,-\widetilde{U}_y(t,y))\big),e\Big)\nu(de)\end{aligned}$$

The processes $-\widetilde{U}_y(t,y)\kappa^*(t,-\widetilde{U}_y(t,y))$ and $-\widetilde{U}_{yy}(t,y)\big(\tilde{\sigma}^*(t,y)-\gamma_z(t,-\widetilde{U}_y(t,y))\big)$ are therefore solutions of the same equation which admits a unique solution. They are therefore equal.

$$-\widetilde{U}_y(t,y)\kappa^*(t,-\widetilde{U}_y(t,y)) = -\widetilde{U}_{yy}(t,y)\big(\tilde{\sigma}^*(t,y)-\gamma_z(t,-\widetilde{U}_y(t,y))\big)$$

which implies

$$\begin{cases} y\Big(\nu_t^*-\eta_t-\displaystyle\int_{\mathbb{R}}h^S(t,e)h^*(t,y,e)\nu(de)\Big) = \Big[\gamma_z(t,z)+z\kappa_t^*(z)\Big]_{z=-\widetilde{U}_y(t,y)}, & (3.23)\\ y+yh^*(t,y,e) = \big(U_z+H_z\big)\big(\big[z(1+\kappa_t^*(z).h^S(t,e)\big]_{z=-\widetilde{U}_y(t,y)}\big). & (3.24)\end{cases}$$

This shows (**ii**).

To show (**iii**), it suffices to remark that (3.24) is equivalent to

$$yh^*(t,y,e) = \Big[\big(U_z+H_z\big)\big(z(1+\kappa_t^*(z)h^S(t,e)\big)-U_z(t,z)\Big]_{z=-\widetilde{U}_y(t,y)}.$$

Compare this with the dynamics of $U_z(t, X_t^*)$ of Corollary 3.4, we conclude that $Y^* = Y^{\nu^*,h^*}$ and $U_z(t, X_t^*)$ satisfies the same dynamics.

(iv) Let now show the martingale property of $\widetilde{U}(t, Y_t^*)$. To do, we focus on the drift coefficient $D_t(Y_t^*, \nu^*, h^*)$, which is from the previous calculations

$$D_t(Y_t^*, \nu^*, h^*) = -Y_t^* \widetilde{U}_y(t, Y_t^*) - \mathcal{Q}(t, -\widetilde{U}_y(t, Y_t^*), \kappa^*) + \tilde{\mathcal{Q}}(t, Y_t^*, \nu^*, h^*),$$

which rewrites, from equations (3.2) and (3.22),

$$\begin{aligned} D_t(y, \nu^*, h^*) &= -y\widetilde{U}_y(t, y) - \int_{\mathbb{R}} \Big(U + H\Big)\big(t, z(1 + \kappa_t^*(z).h^S(t, e))|_{z=-\widetilde{U}_y(t,y)}\big)\nu(de) \\ &- \frac{1}{2}\big(U_{zz}(t, x)\|z\kappa_t^*(z)\|^2\big)_{z=-\widetilde{U}_y(t,y)} - \Big(z\kappa_t^*.(\gamma_z + U_z(t, z)(\eta_t - \alpha_t))\Big)_{z=-\widetilde{U}_y(t,y)} \\ &+ \int_{\mathbb{R}} \Big(\Phi(t, y + yh^*(t, y, e), e) - y\widetilde{U}_y(t, y)h^*(t, y, e)\Big)\nu(de) \\ &+ \frac{1}{2}\widetilde{U}_{yy}(t, y)\|y\big(\nu_t^* - \eta_t - \int_{\mathbb{R}} h^S(t, e)h^*(t, e)\nu(de)\big) - \gamma_z(t, -\widetilde{U}_y(t, y))\|^2. \end{aligned}$$

Thanks to equations (3.8), (3.23) and (3.24), we have

$$\begin{aligned} -z\kappa_t^*(z).(\gamma_z + U_z(t, z)(\eta_t - \alpha_t)) &= U_{zz}(t, z)\|z\kappa_t^*(z)\|^2 \\ &+ z\kappa_t^*(z)\big(\int_{\mathbb{R}} \Big(U_z + H_z\Big)\big(t, z(1 + \kappa_t^*(z).h^S(t, e))\big)h^S(t, e)\nu(de), \end{aligned}$$

and

$$\int_{\mathbb{R}} y\widetilde{U}_y(t, y)h^*(t, y, e)\nu(de) = \widetilde{U}_y(t, y)\Big(-y + \int_{\mathbb{R}} (U_z + H_z)\big(t, z(1 + \kappa_t^*(z).h^S(t, e)|_{z=-\widetilde{U}_y(t,y)})\big)\nu(de)\Big)\Big).$$

That is

$$\begin{aligned} D_t(y, \nu^*, h^*) &= -y\widetilde{U}_y(t, y) - \int_{\mathbb{R}} \Big(U + H\Big)\big(t, z(1 + \kappa_t^*(z).h^S(t, e))|_{z=-\widetilde{U}_y(t,y)}\big)\nu(de) \\ &- \frac{1}{2}\big(U_{zz}(t, x)\|z\kappa_t^*(z)\|^2\big)_{z=-\widetilde{U}_y(t,y)} + \big(U_{zz}(t, x)\|z\kappa_t^*(z)\|^2\big)_{z=-\widetilde{U}_y(t,y)} \\ &+ \Big[z\kappa_t^*(z)\big(\int_{\mathbb{R}} \Big(U_z + H_z\Big)\big(t, z(1 + \kappa_t^*(z).h^S(t, e))\big)h^S(t, e)\nu(de)\Big]_{z=-\widetilde{U}_y(t,y)} \\ &+ \int_{\mathbb{R}} \Big(\Phi(t, z(1 + \kappa_t^*(z).h^S(t, e))|_{z=-\widetilde{U}_y(t,y)}, e)\Big)\nu(de) - y\widetilde{U}_y(t, y) \\ &- \widetilde{U}_y(t, y)\int_{\mathbb{R}} (U_z + H_z)\big(t, z(1 + \kappa_t^*(z).h^S(t, e)|_{z=-\widetilde{U}_y(t,y)})\big)\nu(de) \\ &- \frac{1}{2}\big(U_{zz}(t, x)\|z\kappa_t^*(z)\|^2\big)_{z=-\widetilde{U}_y(t,y)}, \end{aligned}$$

which after simplifications, reduces to

$$\begin{aligned} D_t(y, \nu^*, h^*) &= -\int_{\mathbb{R}} \Big(U + H\Big)\big(t, z(1 + \kappa_t^*(z).h^S(t, e))|_{z=-\widetilde{U}_y(t,y)}\big)\nu(de) \\ &+ \Big[z\kappa_t^*(z)\big(\int_{\mathbb{R}} \Big(U_z + H_z\Big)\big(t, z(1 + \kappa_t^*(z).h^S(t, e))\big)h^S(t, e)\nu(de)\Big]_{z=-\widetilde{U}_y(t,y)} \\ &+ \int_{\mathbb{R}} \Big(\Phi(t, z(1 + \kappa_t^*(z).h^S(t, e))|_{z=-\widetilde{U}_y(t,y)}, e)\Big)\nu(de) \\ &- \widetilde{U}_y(t, y)\int_{\mathbb{R}} (U_z + H_z)\big(t, z(1 + \kappa_t^*(z).h^S(t, e)|_{z=-\widetilde{U}_y(t,y)})\big)\nu(de), \end{aligned}$$

or equivalently,

$$\begin{cases} D_t(y,\nu^*,h^*) = \displaystyle\int_{\mathbb{R}} \Big(\Phi(t,\zeta_t(y),e) - (U+H)(t,\zeta_t(y),e) + \zeta_t(y)(U_z+H_z)(t,\zeta_t(y),e)\Big)\nu(de) \\ \zeta_t(y) := z(1+\kappa_t^*(z).h^S(t,e))|_{z=-\widetilde{U}_y(t,y)}. \end{cases}$$

The key point to conclude is to note, as outlined above, that Φ is the Fenchel-Transform of the concave function $U+H$, so we have necessarily the master equation

$$\Phi(t,y,e) = (U+H)(t,y,e) - y(U_z+H_z)(t,y,e),\ \forall t,y,e.$$

Replace y by $\zeta_t(y)$ in the last identity, we conclude that

$$D_t(y,\nu^*,h^*) \equiv 0,\ \forall y \text{ a.s. for any } t.$$

So, under integrability condition, if the solution $Y^* := Y^{\nu^*,h^*}$ exists, the process $\widetilde{U}(t,Y_t^*)$ is a martingale. Moreover for any $Y^{\nu,h} \in \mathbb{Y}$, $\widetilde{U}(t,Y^{\nu,h})$ is a submartingale since from the above $D_t(y,\nu,h) \geq D_t(y,\nu^*,h^*) = 0$. □

3.5 Utility characterization

Until then, we have carried out a necessary study. Given a consistent regular utility U, we have characterized the optimal processes of both problems, dual and primal one in terms of U, its derivatives and its characteristics (β,γ,H). Conversely, the following Corollary gives a characterization of these characteristics (β,γ,H), in terms of the optimal dual and primal policies. Based on this result, we give in the following theorem a complete characterization of the $\mathbb{X}$-consistent utilities in terms of optimal processes. This result is the culmination of all the preceding results. It also shows us how such utilities can be constructed (learned) if we observe the behavior of an agent. This inverse problem will be the subject of the last section.

Corollary 3.7. *Assume U to be a $\mathbb{X}$-consistent dynamic utility of class $\mathcal{K}^{2,\delta}$, $\delta \in]0,1[$ and let (β,γ,H) denote its local characteristics. Suppose Assumption 3.1 holds, then*
(i) *The drift characteristic β is given by the following HJB-condition*

$$\beta(t,z) = -zU_z(t,z)r_t + \int_{\mathbb{R}} (U(t,z)+H(t,z,e))\nu(de) - \mathcal{Q}(t,z,\kappa^*). \tag{3.25}$$

(ii) *Assume the $z \mapsto z + z\kappa_t^*(z).\kappa_t^*(z)h^S(t,e)$ to be homeomorhic and denote by $\psi(t,z,e)$ its inverse map. The diffusion and the jump characteristics γ_z and H_z are explained by the optimal policies of the primal and dual problems*

$$\begin{cases} H_z(t,z,e) = U_z(t,\psi(t,z,e))\big(1+h^*(t,U_z(t,\psi(t,z,e)),e)\big) - U_z(t,z), \\ \gamma_z(t,z) = -U_{zz}(t,z)z\kappa_t^*(z) + U_z(t,z)\Big(-\eta_t + \alpha_t + \nu^*(t,U_z(t,z)) \\ \qquad\qquad + \displaystyle\int_{\mathbb{R}} h^*(t,U_z(t,z),e)h^S(t,e)\nu(de)\Big). \end{cases} \tag{3.26}$$

The following theorem gives a trajectory characterization, under some regularity conditions, of the dynamics consistent utilities in terms of their optimal processes.

Theorem 3.8 (Main Result)**.** *Assume that $(\nu^*,h^*) \in \overline{\mathcal{K}}_b^{0,1} \times \widehat{\mathcal{K}}_{b,\nu}^{0,1}$, s.t. $\forall t,e,\ y \mapsto y + h^*(t,y,e)$ is homeomorphic a.s.*

(i) *Then the conjugate SDE*(ν^*, h^*),

$$\begin{aligned}\frac{dY_t(y)}{Y_{t^-}(y)} &= -r_t dt + \Big(\nu^*(t, Y_t(y)) - \eta_t - \int_{\mathbb{R}} h^*(t, Y_t(y), e) h^S(t,e)\nu(de)\Big) dW_t \\ &+ \int_{\mathbb{R}} h^*(t, Y_{t^-}(y), e)\tilde{N}(dt, de)),\ Y_0^*(y) = y. \end{aligned} \tag{3.27}$$

is uniformly Lipschitz and has a unique strong solution $Y^*(y)$*, which is strictly positive, and strictly monotonic, with range* $[0, \infty)$.

(ii) *The optimal wealth process is the unique solution of the SDE*(κ^*) (3.9),

$$dX_t^*(x) = X_{t^-}^*(x)\big[r_t dt + \kappa^*(t, X_t^*(x)).(dW_t + \eta_t dt + \int_{\mathbb{R}} h^S(t,e)\tilde{N}(dt,de))\big],\ X_0^*(x) = x. \tag{3.28}$$

Moreover $x \mapsto X_t^*(x)$ *is also strictly increasing.*

(iii) *Denote* $\mathcal{X}(x)$ *the inverse flow of* X^**, the dynamic utility* U *is the unique solution of the SPIDE* (3.10) *and is characterized as follows*

$$\begin{cases} U_z(t,z) = Y_t^*(u_z(\mathcal{X}_t(z))),\ u_z(.) = U_z(0,.), \\ U(t,z) = \displaystyle\int_{z_0}^{z} Y_t^*(u_z(\mathcal{X}_t(x)))dx,\ U(t,0) = 0. \end{cases}$$

Comments: Note that this result allows us to characterize these utilities if the optimal SDE (3.27) admits a monotonic solution. In [EKM13], frame without jumps, we just have to assume, a.s., that $\|\gamma_z^{\perp}(t,z)\| \leq K_t U_z(t,z)$ and $\|\gamma_{zz}^{\perp}(t,z)\| \leq K_t|U_{zz}(t,z)|$ for some process $K \in \mathcal{L}^2(dt)$, because this implies that $\nu^* \in \overline{\mathcal{K}}_b^{0,1}$. This can always be taken as a hypothesis, but the problem in this framework is to find an explicit condition on h^* which is not obvious because it is defined from an integro-differential equation.

Note that this result also characterizes a class of solutions of the SPIDE (3.10), making a connection with those of two SDEs (3.28) and (3.27). From a numerical point of view we can then propose a simple scheme to solve the SPIDE, by combining two Euler schemes to solve the optimal SDEs, as established in the paper [GM18].

Assumptions of this result give us the existence of a monotonic strictly positive strong solution Y^* of SDE(ν^*, h^*), and by local regularity, the existence of a monotonic solution X^* of SDE(κ^*). Obviously, we can interchange the roles of Y^* and X^*. Indeed, in line with Theorem 3.3, Corollary 3.4 and **(iii)** of Theorem 3.6, given the existence of a solution of the SPIDE (3.10), the existence of a solution $X^*(x)$ starting from x to the SDE(κ^*) implies that $U_z(t, X_t^*(x))$ is solution of the SDE(ν^*, h^*) and so is a state price density process $Y_t^*(u_z(x))$, starting from $u_z(x)$. That means in our context that the existence of a solution to SDE(κ^*) is equivalent to the existence of a solution to SDE(ν^*, h^*).

Otherwise, if we give ourselves X^* monotonic solution of SDE(κ^*) with a semimartingale inverse $\mathcal{X}$ and Y^* a $\mathcal{K}^2$-regular solution to SDE(ν^*, h^*) then the compound process $Y_t^*(u_z\mathcal{X}(t,x)$ is an obvious solution of the SPIDE (3.10), see Theorem 4.1 below.

Proof. **(i)** is a classical result in the SDE's theory, see Kunita's book [Kun19].

(ii) From Theorem 3.6, the process $U_z(t, X_t^*(x))$ is also solution of the SDE(ν^*, h^*). By uniqueness of the solution we have necessarily $U_z(t, X_t^*(x)) = Y_t^*(u_z(x))$. Assume that there is a second solution to the SDE(κ^*) than X^* which we denote by $\bar{X}$. Applying the generalized Itô-Ventzel's formula, one can shows that $U_z(t, \bar{X}_t(x))$ is also solution of the SDE(ν^*, h^*), which have a unique

solution, then by monotonicity of U_z we have that $\bar{X}_t(x) = X^*(x)$ a.s. for any t. Now as $Y^*(u_z(x) = U_z(t, X_t^*(x))$ is a not explosive, then obviously X^* is a strong solution (non-explosive).
Still using the first-order optimality relation $Y^*(u_z(x)) = U_z(t, X_t^*(x))$, the monotonicity of Y^* combined with that of U_z and u_z, implies that also X^* is strictly increasing. The rest of the proof is now obvious. □

3.6 Example

Now we give an example of these consistent dynamic utilities in the well known case where they are of power type. We give a necessary and sufficient condition to guarantee the consistency property. This is a very special case, but it shows the restriction imposed by this last property.

Proposition 3.9. *Consider the dynamic utility of power-type, defined by*

$$\begin{cases} U^\theta(t,z) = Z_t \dfrac{z^{1-\theta}}{1-\theta}, \ \theta \in]0,1[, & (3.29) \\ dZ_t = Z_{t^-}\Big(\mu_t^Z dt + \gamma_t^Z dW_t + \displaystyle\int_{\mathbb{R}} h^Z(t,e)\nu(de)\Big). & (3.30) \end{cases}$$

(i) *Its local characteristics $(\beta^\theta, \gamma^\theta, H^\theta)$ are, by Itô's formula*

$$\beta^\theta(t,z) = U^\theta(t,z)\mu_t^Z, \ \gamma^\theta(t,z) = U^\theta(t,z)\gamma_t^Z, \ H^\theta(t,z,e) = U^\theta(t,z)h_t^Z(e). \tag{3.31}$$

(ii) *U^θ is $\mathbb{X}$-consistent if and only if the SDE's coefficient of Z depend on the risk aversion θ and are linked as follows*

$$\begin{aligned} \mu_t^Z &= -(1-\theta)r_t + \int_{\mathbb{R}} (1+h^Z(t,e))\Big[1-\big(1+\kappa_t^{*,\theta}.h^S(t,e)\big)^{1-\theta}\Big]\nu(de) & (3.32) \\ &- < \kappa_t^{*,\theta}, \frac{\theta(1-\theta)}{2}\kappa_t^{*,\theta} + \gamma_t^Z + (\eta_t - \alpha_t) > . & (3.33) \end{aligned}$$

(ii) *The optimal policies are given by*

$$\begin{cases} z\kappa_t^{*,\theta} = \dfrac{z}{\theta}\big(\gamma_t^{Z,\mathcal{R}} + \eta_t - \alpha_t\big) + \dfrac{z}{\theta}\displaystyle\int_{\mathbb{R}} (1+h^Z(t,e))(1+\kappa_t^{*,\theta}.h^S(t,e))^{-\theta}h^S(t,e)\nu(de) & (3.34) \\ y\nu_t^{*,\theta}(y) = y\gamma_t^{Z,\perp}, & (3.35) \end{cases}$$

in particular $\kappa^{*,\theta}$ and $\nu^{*,\theta}$ are independent on z and thus the optimal processes $X^{*\theta}(x)$ and $Y^{*,\theta}(y)$ are strictly increasing since they are linear with respect to their initial conditions.

Proof. Statement **(i)** being obvious, we focus on the second one. From the previous results, equation (3.25), U is consistent if and only if its drift characteristics $\beta^\theta(t,z) = U^\theta(t,z)\mu_t^Z$ satisfies

$$\beta^\theta(t,z) = U^\theta(t,z)\mu_t^Z = -zU_z(t,z)r_t + \int_{\mathbb{R}} (U(t,z) + H(t,z,e))\nu(de) - \mathcal{Q}(t,z,\kappa^{*,\theta}). \tag{3.36}$$

Where, from (3.2),

$$\begin{aligned} \mathcal{Q}(t,z,\kappa^{*,\theta}) &:= \int_{\mathbb{R}} \Big(U^\theta + H^\theta\Big)\big(t, z(1+\kappa_t^{*,\theta}.h^S(t,e))\big)\nu(de) + \frac{1}{2}U_{zz}^\theta(t,z)\|z\kappa_t^{*,\theta}\|^2 \\ &+ z\kappa_t^{*,\theta}.(\gamma_z^\theta(t,z) + U_z^\theta(t,z)(\eta_t - \alpha_t)). \end{aligned} \tag{3.37}$$

Observe at first, that by definition we have $U+H = U(1+h^Z)$, $U_z+H_z = U_z(1+h^Z) = z^{-\theta}(1+h^Z)$. Thus, if we focus on the optimal policy $\kappa^{*,\theta}$ associated with U^θ (assumed to be consistent), then by (3.8) and identities $zU_z = (1-\theta)U(t,z)$, $zU_{zz} = -\theta U_z$, it is necessarily given by

$$z\kappa_t^{*,\theta} = \frac{z}{\theta}\big(\gamma_t^{Z,\mathcal{R}} + \eta_t - \alpha_t\big) + \frac{z}{\theta}\int_{\mathbb{R}}(1+h^Z(t,e))(1+\kappa_t^{*,\theta}.h^S(t,e))^{-\theta}h^S(t,e)\nu(de). \quad (3.38)$$

This enables us to calculate the quantity $\mathcal{Q}(t,z,\kappa^{*,\theta})$,

$$\begin{aligned}\mathcal{Q}(t,z,\kappa^{*,\theta}) &= U^\theta(t,z)\int_{\mathbb{R}}(1+h^Z(t,e))\big(1+\kappa_t^{*,\theta}.h^S(t,e)\big)^{1-\theta}\nu(de) + \frac{\theta(1-\theta)}{2}U(t,z)\|\kappa_t^{*,\theta}\|^2\\ &+ U^\theta(t,z)\kappa_t^{*,\theta}.(\gamma_t^Z + \eta_t - \alpha_t) \quad (3.39)\\ &= U^\theta(t,z)\Big[\int_{\mathbb{R}}(1+h^Z(t,e))\big(1+\kappa_t^{*,\theta}.h^S(t,e)\big)^{1-\theta}\nu(de) + \frac{\theta(1-\theta)}{2}\|\kappa_t^{*,\theta}\|^2\\ &+ \kappa_t^{*,\theta}.(\gamma_t^Z + \eta_t - \alpha_t)\Big]. \quad (3.40)\end{aligned}$$

Finally, the consistency equation (3.36), becomes

$$\begin{aligned}U^\theta(t,z)\mu_t^Z &= -zU_z^\theta(t,z)r_t + U^\theta(t,z)\int_{\mathbb{R}}(1+h^Z(t,e))\nu(de) - \mathcal{Q}(t,z,\kappa^{*,\theta})\\ &= -(1-\theta)U^\theta(t,z)r_t + U^\theta(t,z)\int_{\mathbb{R}}(1+h^Z(t,e))\nu(de)\\ &- U^\theta(t,z)\Big[\int_{\mathbb{R}}(1+h^Z(t,e))\big(1+\kappa^{*,\theta}.h^S(t,e)\big)^{1-\theta}\nu(de) + \frac{\theta(1-\theta)}{2}\|\kappa_t^{*,\theta}\|^2\\ &+ \kappa_t^{*,\theta}.(\gamma_t^Z + \eta_t - \alpha_t)\Big]\\ &= U^\theta(t,z)\Big[-(1-\theta)r_t + \int_{\mathbb{R}}(1+h^Z(t,e))\Big[1-\big(1+\kappa^{*,\theta}.h^S(t,e)\big)^{1-\theta}\Big]\nu(de)\\ &- <\kappa^{*,\theta}, \frac{\theta(1-\theta)}{2}\kappa_t^{*,\theta} + \gamma_t^Z + \eta_t - \alpha_t>\Big].\end{aligned}$$

Thus simplifying by U^θ, the necessary condition of consistency is

$$\begin{aligned}\mu_t^Z &= -(1-\theta)r_t + \int_{\mathbb{R}}(1+h^Z(t,e))\Big[1-\big(1+\kappa^{*,\theta}.h^S(t,e)\big)^{1-\theta}\Big]\nu(de)\\ &- <\kappa_t^{*,\theta}, \frac{\theta(1-\theta)}{2}\kappa_t^{*,\theta} + \gamma_t^Z + \eta_t - \alpha_t>.\end{aligned}$$

The proof is now achieved. □

4 Reverse Engineering

In view of the results of the previous section (Theorem 3.3 and Corollary 3.4), given an initial utility function u, the marginal dynamic utility along the optimal portfolio $U_z(t, X_t^*(x))$ is a state price process Y^* starting from the initial condition $u_z(x)$, i.e. $U_z(t, X_t^*(x)) = Y_t^*(u_z(x))$.
If now, we assume that the optimal portfolio $X_t^*(x)$ is a monotonic random map with respect to the initial wealth x, then necessarily $y \mapsto Y_t^*(y)$ must be also monotonic given the monotonicity of U_z and that of X^*. More interesting, denoting $\mathcal{X}(x)$ the inverse of $X(x)$, one can characterizes le marginale utility by the identity $U_z(t,x) = Y_t^*(u_z(X_t^*(x))$.
In this section, we consider the reverse problem, i.e. we give ourselves two monotonic processes $\mathcal{X}$ and $\mathcal{Y}$ in their initial conditions and we seek to build a dynamic utility consistent with our market

model. $\mathcal{X}$ will play the role of the optimal portfolio X^* and $\mathcal{Y}$ will play the role of the state price density process Y^*. In other words, we propose an explicit way to recover all consistent utilities U generating this wealth as optimal process. In the classical expected utility framework, this reverse engineering problem has been considered by He and Huang (1992) [HH94] in a complete market and then in the framework of dynamic utilities in [EKM13] in an incomplete financial market without jumps.

In what follows, we propose to study this problem within the framework of this paper. We are concerned with two processes $\mathcal{X}$ and $\mathcal{Y}$ solutions of two regular stochastic differential equations.

$$\begin{cases} \dfrac{d\mathcal{X}_t(x)}{\mathcal{X}_{t^-}(x)} = r_t dt + \bar{\kappa}_t(\mathcal{X}_t(x)).(dW_t + \eta_t dt + \displaystyle\int_{\mathbb{R}} h^S(t,e)\tilde{N}(dt,de)), & (4.1) \\ \dfrac{d\mathcal{Y}_t(y)}{\mathcal{Y}_{t^-}(y)} = -r_t dt + \Big(\bar{\nu}(t,\mathcal{Y}_t(y)) - \eta_t - \displaystyle\int_{\mathbb{R}} \bar{h}(t,\mathcal{Y}_t(y),e)h^S(t,e)\nu(de)\Big)dW_t \\ \qquad\qquad + \displaystyle\int_{\mathbb{R}} \bar{h}(t,\mathcal{Y}_{t^-}(y),e)\tilde{N}(dt,de)). & (4.2) \end{cases}$$

for some random vectors $\bar{\nu} \in \mathcal{R}^\perp$ and $\bar{\kappa} \in \mathcal{R}$.

The advantage of starting from these equations is the abundance of results in the SDE theory (see precisely [Kun19]). We can therefore use the existing results to put sufficient regularity assumptions on the SDE's coefficients which ensure the existence, regularity, uniqueness, monotonicity of a strong solutions and particularly to ensure the semimartingale property of $\mathcal{X}$ the inverse flow of the solution $\mathcal{X}$ of the SDE($\bar{\kappa}$) .

The following result shows that the compound maps $\Phi(t,z) := \mathcal{Y}_t(v_z(\mathcal{X}(t,z))$ ($\mathcal{X}$ for the inverse flow of $\mathcal{X}$) is the marginal of a dynamic utility (with initial data v) and satisfies the same dynamics as the derivative of a consistent utility.

Theorem 4.1. *Let $\delta \in]0,1[$ and assume the pair of processes $(\mathcal{X},\mathcal{Y})$ belongs to $\mathcal{K}^{3,\delta}_{loc} \times \mathcal{K}^{2,\delta}_{loc}$ s.t. $x \mapsto x + x\bar{\kappa}_t(x).h^S(t,e)$ and $y \mapsto y + y\bar{h}(t,y,e)$ are homeomorphic. For a given initial utility function $v \in \mathcal{C}^{2,\delta}$, let V be the strictly concave random field satisfying $V_z(t,z) := \mathcal{Y}_t(v_z(\mathcal{X}(t,z))$. Then V is a $\mathcal{K}^{3,\delta}_{loc}$-semimartingale and its marginal V_z is solution of the second order SPIDE*

$$\begin{cases} dV_z(t,z) = -\partial_z\Big(zV_z(t,z))r_t + \mathcal{Q}^V(t,z,\bar{\kappa}) - \displaystyle\int_{\mathbb{R}} \big(V(t,z) + H^V(t,z,e)\big)\nu(de)\Big)dt \\ \qquad\qquad +\gamma_z(t,z)dW_t + \displaystyle\int_{\mathbb{R}} H^V_z(t,z,e)\tilde{N}(dt,de), & (4.3) \\ H^V_z(t,z,e) = V_z\big(t^-,\psi(t,z,e)\big) + V_z\big(t^-,\psi(t,z,e)\big)\bar{h}\big(t,V_z\big(t^-,\psi(t,z,e)\big)\big),e\big) - V_z(t^-,z), \\ \gamma^V_z(t,z) = \bar{\sigma}^Y(t,V_z(t,z)) - V_{zz}(t,z)z\bar{\kappa}_t(z), & (4.4) \\ \bar{\sigma}^Y(t,y) := y\Big(\bar{\nu}(t,y) - \eta_t - \displaystyle\int_{\mathbb{R}} h^S(t,e)\bar{h}(t,y,e)\nu(de)\Big). & (4.5) \end{cases}$$

where we recall, according to Theorem 2.4 that $\psi(t,z,e)$ is the inverse map of $z \to z\big(1+\bar{\kappa}_t(z)h^S(t,e)\big)$ and $\mathcal{Q}^V$ the quadratic form

$$\begin{aligned} \mathcal{Q}^V(t,z,\bar{\kappa}) &:= \int_{\mathbb{R}} \Big(V + H^V\Big)\big(t,z(1+\bar{\kappa}_t(z)h^S(t,e))\big)\nu(de) \\ &+ \frac{1}{2}V_{zz}(t,z)\|z\bar{\kappa}_t(z)\|^2 + z\bar{\kappa}_t(z).(\gamma^V_z(t,z) + V_z(t,z)(\eta_t - \alpha_t)). \end{aligned} \quad (4.6)$$

Proof. To simplify the notations we use $\bar{\sigma} := z\bar{\kappa}$. Since $\mathcal{Y}(.)$ and $\mathcal{Y}(u_z(.))$ obey the same dynamics, one can apply the result of Corollary 2.5, and gets the dynamics of the compound map $V_z(t,z) :=$

$\mathcal{Y}(t, u_z(\mathcal{X}(t,z)))$,

$$\begin{aligned} dV_z(t,z) = \Big(& - V_z(t,z)r_t - V_{zz}(t,z)\big[zr_t + \bar\sigma_t(z).\eta_t + \bar\sigma_y^Y(t,V_z(t,z)).\bar\sigma_t(z)\big] \\ & + \tfrac{1}{2}\partial_z\Big(V_{zz}(t,z)\|\bar\sigma_t(z)\|^2\Big)\Big)dt \\ & + \int_{\mathbb{R}}\Big(H_z^V(t,z,e) - \bar h(t,V_z(t,z),e) + V_{zz}(t,z)\bar\sigma_t(z).h^S(t,e)\Big)\nu(de)dt \\ & + \gamma_z^V(t,z).dW_t + \int_{\mathbb{R}} H_z^V(t,z,e)\tilde N(dt,de), \end{aligned}$$

where, also by Corollary 2.5, the jumps and volatility characteristics (denoted resp. H_z and γ_z) are given by the following identities

$$\begin{cases} H_z^V(t,z,e) = V_z\big(t^-,\phi(t,z,e)\big) + V_z\big(t^-,\phi(t,z,e)\big)\bar h\big(t,V_z\big(t^-,\phi(t,z,e)\big),e\big) - V_z(t^-,z), \\ \gamma_z^V(t,z) := \bar\sigma_t^Y(V_z(t,z)) - V_{zz}(t,z)\bar\sigma(t,z). \end{cases}$$

Let us focus on the drift term, which is, using the notation $\alpha_t = \int_{\mathbb{R}} h^S(t,e)\nu(de)$, rewrites

$$\begin{aligned} D(t,z) := \; & - \; \partial_z(zV_z(t,z))r_t - V_{zz}(t,z)\bar\sigma(t,z)\big[\eta_t - \alpha_t + \bar\sigma_y^Y(t,V_z(t,z))\big] \\ & + \; \frac{1}{2}\partial_z\Big(V_{zz}(t,z)\|\bar\sigma_t(z)\|^2\Big) + \int_{\mathbb{R}}\Big(H_z^V(t,z,e) - \bar h(t,V_z(t,z),e)\Big)\nu(de). \end{aligned} \tag{4.7}$$

Differentiating with respect to z the identity;

$$\gamma_z^V(t,z) = \bar\sigma_t^Y(V_z(t,z)) - V_{zz}(t,z)\bar\sigma_t(z),$$

it follows,

$$V_{zz}(t,z)\bar\sigma_y^Y(t,V_z(t,z)) = \gamma_{zz}^V(t,z) + V_{zzz}(t,z)\bar\sigma_t(z) + V_{zz}(t,z)\bar\sigma_z(t,z).$$

Injecting this identity in (4.7), one gets, on the one hand

$$\begin{aligned} D(t,z) = \; & - \; \partial_z(zV_z(t,z))r_t - \bar\sigma_t(z)\Big[V_{zz}(t,z)(\eta_t-\alpha_t) + \gamma_{zz}^V(t,z) + V_{zzz}(t,z)\bar\sigma_t(z) + V_{zz}(t,z)\bar\sigma_z(t,z)\Big] \\ & + \; \frac{1}{2}V_{zzz}(t,z)\|\bar\sigma_t(z)\|^2 + V_{zz}(t,z)\bar\sigma_t(z).\bar\sigma_z(t,z) + \int_{\mathbb{R}}\Big(H_z^V(t,z,e) - \bar h(t,V_z,e)\Big)\nu(de) \\ = \; & -\partial_z(zV_z)r_t - \bar\sigma_t(z).\big[V_{zz}(t,z)(\eta_t-\alpha_t) + \gamma_{zz}^V(t,z)\big] - \frac{1}{2}\partial_z\Big(V_{zz}(t,z)\|\bar\sigma_t(z)\|^2\Big) \\ & + \; V_{zz}(t,z)\bar\sigma_z(t,z)\bar\sigma_t(z) + \int_{\mathbb{R}}\Big(H_z^V(t,z,e) - \bar h(t,V_z(t,z),e)\Big)\nu(de) \end{aligned}$$

On the other hand, as

$$\begin{cases} \gamma_z^V(t,z) = \bar\sigma_t^Y(V_z(t,z)) - V_{zz}(t,z)\bar\sigma_t(z), \\ \bar\sigma_t^Y(V_z(t,z) = V_z(t,z)\bar\nu_t(V_z(t,z)) - V_z(t,z)\eta_t - V_z(t,z)\displaystyle\int_{\mathbb{R}}\bar h(t,V_z(t,z),e)h^S(t,e)\nu(de), \\ \text{with } \bar\nu_t(V_z(t,z)).\bar\sigma_t(z) = 0,\ dt\otimes\mathbb{P}, \end{cases}$$

one can observes that

$$\begin{aligned} V_{zz}(t,z)\bar\sigma_z(t,z).\bar\sigma_t(z) \; &= \; -\bar\sigma_z(t,z)\big[V_z(t,z)(\eta_t-\alpha_t) + \gamma_z^V(t,z)\big] \\ & - \; V_z(t,z)\bar\sigma_z(t,z).\int_{\mathbb{R}} h^S(t,e)\big(1+\bar h(t,V_z(t,z),e)\big)\nu(de). \end{aligned}$$

Consequently, it becomes obvious that

$$\begin{aligned} D(t,z) \; &= \; -\partial_z\Big(zV_z(t,z)r_t + \bar\sigma_t(z).\big[\gamma_z^V(t,z) + V_z(t,z)(\eta_t-\alpha_t)\big] + \frac{1}{2}V_{zz}(t,z)\|\bar\sigma_t(z)\|^2\Big) \\ & + \; \int_{\mathbb{R}}\Big(H_z^V(t,z,e) - \bar h(t,V_z(t,z),e) - V_z(t,z)\bar\sigma_z(t,z).h^S(t,e)\big(1+\bar h(t,V_z(t,z),e)\big)\Big)\nu(de) \end{aligned}$$

The idea from now, is to bring up the derivative of a quadratic form $\mathcal{Q}^\Phi$ equivalent to that appearing in the optimization program of Section 3.2, equation (3.2), that is

$$\begin{aligned}\partial_z \mathcal{Q}^V(t,z,\bar\kappa) &:= \int_{\mathbb{R}} \Big(V_z + H^V_z\Big)\big(t,(z+\bar\sigma_t(z).h^S(t,e))\big)(1+\bar\sigma_z(t,z).h^S(t,e))\nu(de)\\ &+ \partial_z\Big(\frac{1}{2}V_{zz}(t,z)\|\bar\sigma_t(z)\|^2 + \bar\sigma_t(z).(\gamma^V_z(t,z)+V_z(t,z)(\eta_t-\alpha_t))\Big). \end{aligned} \tag{4.8}$$

or equivalently, since $\bar\sigma = z\bar\kappa$

$$\begin{aligned}\mathcal{Q}^V(t,z,\bar\kappa) &:= \int_{\mathbb{R}} \Big(V + H^V\Big)\big(t,(z+\bar\sigma_t(z).h^S(t,e))\big)\nu(de)\\ &+ \frac{1}{2}V_{zz}(t,z)\|\bar\sigma_t(z)\|^2 + \bar\sigma_t(z).(\gamma^V_z(t,z)+V_z(t,z)(\eta_t-\alpha_t)). \end{aligned} \tag{4.9}$$

Adding and subtracting the quantity $\int_{\mathbb{R}} \Big(V_z+H^V_z\Big)\big(t,(z+\bar\sigma_t(z).h^S(t,e))\big)(1+\bar\sigma_z(t,z).h^S(t,e))\nu(de)$, leads to

$$\begin{aligned}D(t,z) &= -\partial_z(zV_z(t,z))r_t - \partial_z\mathcal{Q}^V(t,z,\bar\kappa)\\ &+ \int_{\mathbb{R}}\Big(H^V_z(t,z,e) - \bar h(t,V_z(t,z),e) - V_z(t,z)\bar\sigma_z(t,z).(h^S(t,e)\big(1+\bar h(t,V_z(t,z),e)\big)\big)\\ &+ \big(V_z+H^V_z\big)\big(t,(z+\bar\sigma_t(z).h^S(t,e))\big)(1+\bar\sigma_z(t,z).h^S(t,e))\Big)\nu(de). \end{aligned} \tag{4.10}$$

To simplify, observe, from the definition of the process $\psi(t,z,e)$ as the inverse map of $z\to X(t,z)+\bar\sigma(t,z)h^S(t,,e)$, that

$$\psi(t,x,z+\bar\sigma_t(z)) = \mathcal{X}(t,z), \tag{4.11}$$

which implies, by definition of H^Φ, that

$$V_z(t,z+\bar\sigma_t(z)) + H^V_z(t,x,z+\bar\sigma_t(z)) = V_z(t,z) + \bar h(t,V_z(t,z),e). \tag{4.12}$$

Injecting this identity in (4.10), we get after simplifications

$$D(t,z) = -\partial_z(zV_z(t,z))r_t - \partial_z\mathcal{Q}^V(t,z,\bar\kappa) + \int_{\mathbb{R}}\big(V_z(t,z)+H^V_z(t,z,e)\big)\nu(de),$$

which achieves the proof. □

Remark . *Note that, according to the notations of this theorem we have by projecting on the space of constraints $\mathcal{R}$,*

$$z\bar\kappa_t(z) = -\frac{\gamma^{V,\mathcal{R}}_z(t,z)+V_z(t,z)(\eta_t-\alpha_t)}{V_{zz}(t,z)} - \frac{\int_{\mathbb{R}} V_z(t,z)\big(1+\bar h(t,V_z(t,z),e)\big)h^S(t,e)\nu(de)}{V_{zz}(t,z)}$$

Let us then observe that, the identity

$$H^V_z(t,z,e) = V_z\big(t^-,\psi(t,z,e)\big) - V_z(t^-,z) + V_z\big(t^-,\psi(t,z,e)\big)\bar h\big(t,V_z\big(t^-,\psi(t,z,e)\big)\big),e\big)$$

is equivalent since, $\phi(t,x,e)$ is the inverse map of $z\to z\big(1+\bar\kappa_t(z).h^S(t,e)\big)$, to

$$H^V_z(t,z\big(1+\bar\kappa_t(z)h^S(t,e)\big),e) + V_z\big(t^-,z(1+\bar\kappa_t(z).h^S(t,e))\big) = V_z\big(t^-,z\big)\big(1+\bar h\big(t,V_z(t^-,z),e\big)\big)$$

So that,

$$z\bar\kappa_t(z) = -\frac{\gamma^{V,\mathcal{R}}_z(t,z)+V_z(t,z)(\eta_t-\alpha_t)}{V_{zz}(t,z)} - \frac{\int_{\mathbb{R}}\Big(V_z+H^V_z\Big)(t,z\big(1+\bar\kappa_t(z).h^S(t,e)\big),e)h^S(t,e)\nu(de)}{V_{zz}(t,z)}. \tag{4.13}$$

This identity is the key point in to prove the next result.

To conclude, the following theorem shows that V (constructed in the previous result) is indeed a $\mathbb{X}$-consistent utility.

Theorem 4.2. *Under Assumptions of Theorem 4.1, the random field $V(t,z)$ defined by its marginal, $V_z(t,z) := \mathcal{Y}_t(u_z(\mathcal{X}(t,z))$ is a $\mathbb{X}$-consistent dynamic utility.*

(i) UTILITY-SPIDE: *Using notations of Theorem 4.1, V satisfies*

$$\begin{aligned} dV(t,z) &= -\Big(zV_z(t,z))r_t + \mathcal{Q}^V(t,z,\bar{\kappa}) - \int_{\mathbb{R}} \big(V(t,z) + H^V(t,z,e)\big)\nu(de)\Big)dt \\ &+ \gamma^V(t,z)dW_t + \int_{\mathbb{R}} H^V(t,z,e)\tilde{N}(dt,de). \end{aligned}$$

(ii) CONSISTENCY: *For any admissible wealth process $X \in \mathbb{X}$, $V(t,X_t)$ is a supermartingale and martingale for $X = \mathcal{X}$.*

Proof. **(i)** is immediate from Theorem 4.1. Let focus on the second statement **(ii)**, for this we are concerned with the drift characteristics of the process $V(t,X_t^\kappa)$, which, by the generalized Itô-Ventzel's formula, is given by

$$\begin{aligned} & \beta^V(t,X_t^\kappa) + X_t^\kappa V_z(t,X_t^\kappa)r_t - \int_{\mathbb{R}} \Big(V(t,X_t^\kappa) + H^V(t,X_t^\kappa,e)\Big)\nu(de) + \mathcal{Q}(t,X_t^\kappa,\kappa_t) \\ = \;& -\mathcal{Q}^V(t,X_t^\kappa,\bar{\kappa}_t) + \int_{\mathbb{R}} \Big(V + H^V\Big)\big(t,z(1+\kappa_t.h^S(t,e))\big)\nu(de) + \frac{1}{2}V_{zz}(t,z)\|z\kappa_t\|^2 \\ + \;& z\kappa_t.(\gamma_z^V + V_z(t,z)(\eta_t - \alpha_t)) \\ \overset{(4.6)}{=} \;& \int_{\mathbb{R}} \Big[\Big(V + H^V\Big)\big(t,z(1+\kappa_t.h^S(t,e))\big) - \Big(V + H^V\Big)\big(t,z(1+\bar{\kappa}_t(z).h^S(t,e))\big)\Big]\nu(de) \\ + \;& \frac{1}{2}V_{zz}(t,z)\|z\kappa_t\|^2 - \frac{1}{2}V_{zz}(t,z)\|z\bar{\kappa}_t(z)\|^2 + (z\kappa_t - z\bar{\kappa}_t(z)).(\gamma_z^V + V_z(t,z)(\eta_t - \alpha_t)). \end{aligned}$$

Now from the identity (4.13), the quantity $\gamma_z^V(t,z) + V_z(t,z)(\eta_t - \alpha_t)$ in the last term is,

$$\gamma_z^{V,\mathcal{R}}(t,z) + V_z(t,z)(\eta_t - \alpha_t) = -V_{zz}(t,z)z\bar{\kappa}_t(z) - \int_{\mathbb{R}} \Big(V_z + H_z^V\Big)\big(t,z\big(1+\bar{\kappa}_t(z).h^S(t,e)\big),e\big)h^S(t,e)\nu(de)$$

Injecting this in the last identity implies that the drift term in the dynamics of $V(t,X_t^\kappa)$ is given, after arranging the terms, by

$$\begin{aligned} & \int_{\mathbb{R}} \Big[\Big(V + H^V\Big)\big(t,z(1+\kappa_t.h^S(t,e))\big) - \Big(V + H^V\Big)\big(t,z(1+\bar{\kappa}_t(z).h^S(t,e))\big)\Big]\nu(de) \\ - \;& (z\kappa_t - z\bar{\kappa}_t(z)).\int_{\mathbb{R}} \Big(V_z + H_z^V\Big)\big(t,z\big(1+\bar{\kappa}_t(z).h^S(t,e)\big),e\big)h^S(t,e)\nu(de) + \frac{1}{2}V_{zz}(t,z)\|z\kappa_t - z\bar{\kappa}_t(z)\|^2 \end{aligned}$$

For reasons of clarity, if we denote $a_t := z(1+\kappa_t.h^S(t,e))$ and $b_t := z(1+\bar{\kappa}_t(z).h^S(t,e))$ this quantity is rewritten

$$\int_{\mathbb{R}} \Big\{\Big(V + H^V\Big)(t,a_t,e) - \Big[\Big(V + H^V\Big) + (a_t - b_t)\Big(V_z + H_z^V\Big)\Big](t,b_t,e)\Big\}\nu(de) + \frac{1}{2}V_{zz}(t,z)\|a_t - b_t\|^2$$

By concavity of $V + H^V$ the integral is negative and by concavity of V the last term is also negative. Thus the drift term in the dynamics of $V(t,X_t^\kappa)$ is negative and is equal to zero if and only if $z\kappa_t = z\bar{\kappa}_t(z)$. This is equivalent to: For any admissible wealth process $X \in \mathbb{X}$, $V(t,X_t)$ is a supermartingale and martingale for $X = \mathcal{X}$. The proof is then complete. □

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