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**A spectral global matrix method for modelling the response of a fluid-loaded
multilayered cylindrical shell excited by an acoustic plane wave**

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1 This paper proposes a numerically stable method for modelling a fluid-loaded mul-
2 tilayered cylindrical shell excited by a plane wave, which solves the fd instability
3 problem usually observed when using the well known transfer matrix method. In
4 the considered modelling, each layer can be either a viscoelastic coating described
5 by a general 3D elasticity model or an intermediate perfect fluid layer. The transfer
6 matrix of each layer relating the state-vector at the layer's two interfaces is estimated
7 with an appropriate standard method. Instead to multiply together the layer transfer
8 matrices in order to deduce the one of the multilayer cylinder, one writes the conti-
9 nuity relations at each interface of the considered systems yielding to build a global
10 matrix that can be solved to obtain the system response. As shown by numerical
11 applications on typical naval test cases, the proposed global matrix assembly pro-
12 cedure as opposed to the classical transfer matrix method provides both numerical
13 stability over a wide range of axial wavenumbers and circumferential orders, but also
14 the ability of considering intermediate fluid layers. Besides, this model is well-suited
15 to describe elastic solid layers of any anisotropy as illustrated by an additional case
16 considering a transverse isotropic layer.

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17 I. INTRODUCTION

18 Underwater vehicles generally include by a cylindrical shell and internal ring stiffeners
19 made of metal to resist to the hydrostatic pressure. The cylindrical shell may be coated
20 with viscoelastic layers to improve the vehicle acoustic stealth. Additionnaly sonar systems
21 may be mounted on the shell to provide for acoustic detection of other underwater and
22 surface vehicles. These sonar systems typically include several elastic and viscoelastic layers
23 to ensure sensor encapsulation, positioning, signal conditionning, and self noise rejection.
24 Incoming signals from remote vehicles is usually modeled as an incident plane wave. At
25 frequencies of interest of most sonar systems, the response of the embedded sensors to a
26 plane wave excitation only involves the local characteristics of the shell. Therefore the vehicle
27 may be modeled as an infinite cyclindrical multilayer system made of elastic, viscoelastic,
28 and fluid layers, without loss of accuracy. In the present paper, the ring stiffeners are not
29 considered but the presented model can be coupled to stiffener models subsequently.

30 Numerous works have been carried out to model the acoustic scattering and radiation of
31 multilayered plates or cylinders (see for instance ([Hull and Welch, 2010](#); [Schmidt and Jensen,](#)
32 [1985](#)) for planar geometries and ([Baron, 2011](#); [Ricks and Schmidt, 1994](#)) for cylindrical
33 geometries). For these ideal geometries, exact analytical methods have been developed
34 to save computing times compared to general element-based methods like Finite Element
35 Method and Boundary Element Method. The use of the 3-D equations of elastodynamics
36 appears as the most relevant way of modeling the layers at high wavenumber-thickness
37 product (kd), that is when layer thickness becomes large compared to structural wavelength.

38 However, specific problems and numerical instabilities can be encountered when considering
 39 the cylindrical geometry.

40 Actually, the step of integrating the displacement solution field in each layer is far less
 41 complex in the plane case than in the cylindrical case, since the equations of motion leads
 42 to a 2nd order differential system with *constant* matrix coefficients for the displacement field
 43 solution. It is not the case for cylindrical geometries for which the matrix coefficients are
 44 dependent on the radius position. Two different types of approach have been proposed to
 45 overcome this difficulty:

46 1. The use of Helmholtz decomposition allows to integrate the 2nd order differential sys-
 47 tem for the displacement field, but restrains the complexity of the considered layer to
 48 the elastic isotropic case (Ricks and Schmidt, 1994; Skelton and James, 1997). Even
 49 if the generality of this formalism can be extended to a particular case of transverse
 50 isotropy (Honarvar *et al.*, 2007; Kim and Ih, 2003; Niklasson and Datta, 1998), the di-
 51 rection of the fibers' material which can be represented is not of interest for the design
 52 of the acoustic coatings used in underwater vehicles. Moreover, the introduction of
 53 Bessel functions raises numerical problems which must be dealt with, due to the great
 54 dynamic range of these mathematical functions (Abramowitz and Stegun, 1965).

55 2. The state space approach rearranges equations of motion, Cauchy relation and Hooke's
 56 law into a 1st order differential system based on sextic Stroh's formalism, whose hybrid
 57 solution contains both displacement and normal stress field (Baron, 2011; Chen *et al.*,
 58 2004; Dutrion, 2014; Hasheminejad and Rajabi, 2007; Jamali *et al.*, 2011).

59 This approach allows to adress also the problem of elastic anisotropy, either by expand-

60 ing the matricant solution into a Peano series' decomposition (Baron, 2005, 2011), or
61 by substructuring each layer into a certain number of sublayers fine enough to neglect
62 the curvature inside each sublayer. The latter integration method is most prevalent
63 and is chosen by many authors (Chen *et al.*, 2004; Dutrion, 2014; Hasheminejad and
64 Rajabi, 2007; Jamali *et al.*, 2011).

65 However, sextic Stroh's formalism establishes transfer matrices in each layer for which the
66 classical assembly procedure is to propagate the displacement-stress field by matrix product.
67 This assembly procedure is generally called the "transfer matrix method" (TMM) and has
68 been used by many authors (see for instance(Chen *et al.*, 2004; Hasheminejad and Rajabi,
69 2007; Jamali *et al.*, 2011)). However, this one suffers from great instability at high kd , well-
70 known as the "large fd problem" and first pointed by Dunkin (Dunkin, 1965). Even if many
71 authors tried to solve these instabilities by improving the propagation scheme (Dunkin, 1965;
72 Lévesque and Piché, 1992; Rokhlin and Wang, 2002; Tan, 2005), more stable results have
73 been found when a global matrix assembly procedure is chosen (Ricks and Schmidt, 1994;
74 Schmidt and Jensen, 1985). However, this is achieved at the expense of generality for the
75 behavior of the solid layers, since it is assumed isotropic.

76 Besides, the incompatibility between transfer matrices of solid and fluid layers makes it
77 difficult to take into account intermediate fluid layers. One way of dealing with the prob-
78 lem is to ensure the consistency of the matrix product, either by condensating the transfer
79 matrices in solid layers (Cervenka and Challande, 1991) or by introducing an infinitesimal
80 shear velocity in the fluid medium then considering it as a solid layer (Lowe, 1995; Pialucha,
81 1992). An other approach is to uncouple shear stresses between the two layers of different

82 nature via an intermediate matrix ([Nayfeh and Taylor, 1988](#); [Pilarski, 1985](#)).

83 In this present paper, we propose to develop an alternative approach for modelling the multi-
84 layered shell that circumvents the drawbacks described previously with the existing method
85 (i.e. material restriction for the Helmholtz decomposition, "large fd problem" and incom-
86 patibility of the intermediate fluid layers' transfer matrices for the sextic Stroh's formalism
87 when followed by the transfer matrix method). This approach is based on the integration
88 of the displacement-stress field solution of sextic Stroh's formalism to deduce the transfer
89 matrix for each layer of the considered system. However, instead to propagate the state
90 vectors by matrix product, a global matrix assembly procedure is proposed. This approach
91 is similar to that used by authors considering Helmholtz decomposition within each layer,
92 but here it is applied to the sextic Stroh's formalism. This global matrix is built such that
93 the continuity relations at the interfaces of the different layers are satisfied.

94 This approach is specifically developed such that the broadband analysis can be carried out
95 without suffering numerical singularities at high frequencies, large radiuses of the shell or
96 high thicknesses of the layers. Unlike models that favor Helmholtz decomposition for in-
97 tegrating the solution inside each elementary layer ([Ricks and Schmidt, 1994](#); [Skelton and](#)
98 [James, 1997](#)), the proposed one is well suited for describing the full anisotropy problem.

99 To address the problem of interest for underwater applications, multilayered cylindrical
100 shells will be loaded both by internal and external fluids and will be excited by an incoming
101 acoustic plane wave excitation. The steel cylindrical shell and the acoustic coating will be
102 modelled with three-dimensional fully elastic solids, whereas internal, external and interme-

103 diate fluid layers are modelled as three-dimensional fluid domains.

104 The paper is organized as follow:

- 105 - In Sec. 2, we remind some theoretical contents related to the considered problem:
 106 first, the sextic Stroh's formalism consisting in rearranging the equations of motion,
 107 Cauchy relation and Hooke's law, is presented; Second, the 1st order differential system
 108 resulting of the sextic Stroh's formalism is integrated after partitioning each layer
 109 into several sublayers fine enough to neglect the curvature of the layer. Last, the
 110 classical assembly procedure (generally named "transfer matrix method") is presented
 111 to estimate the transfer matrix of either a given layer from that of its constitutive
 112 sublayers either or the whole multilayered cylindrical shell from the ones of the different
 113 layers;
- 114 - In Sec. 3, we highlight both kinds of singularities related to the classical assembly
 115 procedure (i.e. "large fd problem" and the management of intermediate fluid layers)
 116 on various underwater configurations. To end, the response of the immersed multilay-
 117 ered shell excited by a plane wave is computed. It is compared to that obtained from
 118 a numerically robust model limited to isotropic solid layers, and similar to the Direct
 119 Global Method (DGM) presented by Ricks ([Ricks and Schmidt, 1994](#)) in 1994;
- 120 - In Sec. 4, we describe the proposed assembly procedure that allows to overcome the
 121 numerical issues presenting previously. The performances of the improved algorithm
 122 are studied on the configurations considered in Sec. 3;

123 - Before concluding, we highlight in Sec. 5 the ability of the developed approach to
 124 take into account non-isotropic layers. The case of a shell covered by an transversely
 125 isotropic layer composed by Glass Reinforced Plastic is considered.

126 **II. SEXTIC STROH'S FORMALISM AND CLASSICAL MATRIX PRODUCT AS-**
 127 **SEMBLY**

128 **A. Description of the model**

129 An infinite thick multilayered cylindrical structure (MCS) of N layers is centred on its z
 130 axis as shown in Fig. 1a). The radial, circumferential and axial coordinates are respectively
 131 denoted by r , ϕ and z whereas its associated unit vectors are respectively mentioned by \mathbf{e}_r ,
 132 \mathbf{e}_ϕ and \mathbf{e}_z . Each layer is composed either by an homogeneous elastic material or by a perfect
 133 fluid. As described in Fig. 1b), the j^{th} layer $j \in [1, N]$ is delimited by its extremal radiuses
 134 r_{j-1} and r_j (with $r_{j-1} < r_j$). It results that r_0 and r_N denote respectively the internal and
 135 external radiuses of the MCS. The inner cavity of the MCS is supposed to be filled by air
 136 whereas the MCS is assumed to be immersed in water. By convention, the 0^{th} and $N + 1^{\text{th}}$
 137 layers respectively denote the internal air cavity and the external water domain.

138 An incoming acoustic plane wave in the external medium is considered as the only excita-
 139 tion exerted on the MCS, apart from the fluid couplings. The harmonic time dependence is
 140 chosen as $e^{-j\omega t}$ and will be suppressed throughout this paper as an algebraic convention.

142 We will develop a model for predicting the vibroacoustic response of the MCS excited by
 143 the plane wave. The latter will be characterised by the response at a given receiving point
 144 that can be placed either in the external fluid or embedded within an arbitrary layer of the
 145 MCS. The response for the point located in a fluid medium (i.e. external fluid or intermediate
 146 fluid layer) will be the acoustic pressure whereas for the point located in a solid layer, it
 147 will be the radial stress. The point of interest will be located at the position coordinates
 148 (r_A, ϕ_A, z_A) .

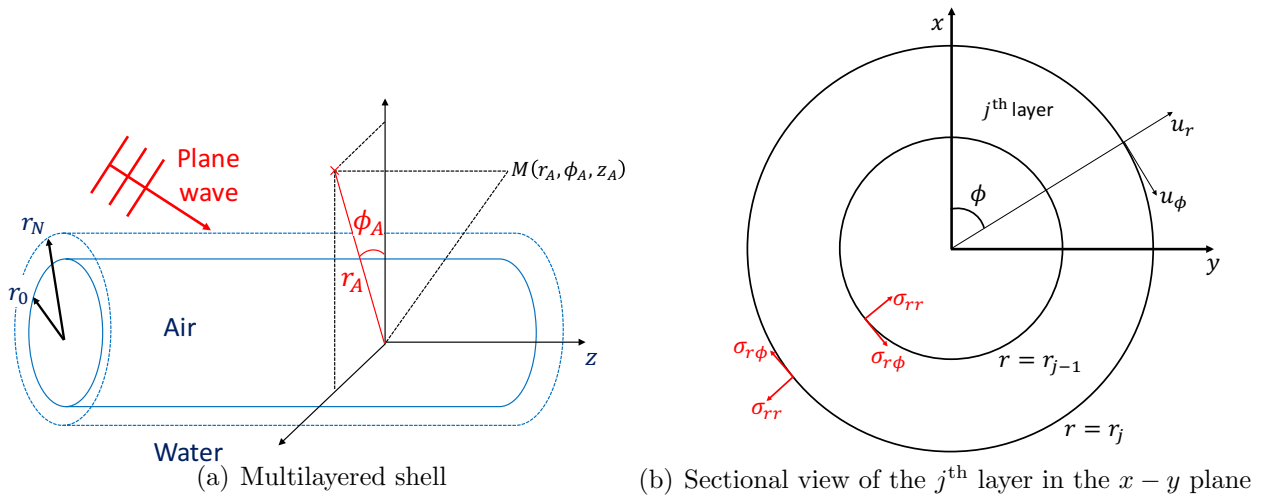


FIG. 1. Presentation of the multilayered shell impacted by an acoustic plane wave

149 **B. 3D anisotropic elasticity and state equation**

150 In 2004, Chen & al. (Chen *et al.*, 2004) derived a sixth-order state equation for a general
 151 anisotropic medium in cylindrical coordinates from the 3-dimensional fundamental equations
 152 of anisotropic elasticity. Here, one considers a particular case of their developments when
 153 no body forces nor thermal effect are taken into account.

154 This formalism rests on the assumption of infinitesimal strain theory in any j^{th} layer, and

155 the rearrangement of equations of motion, Cauchy relation and Hooke's law. Let us consider
 156 the state vector defined by:

$$\mathbf{V} = \left[u_z \quad u_\phi \quad u_r \quad \sigma_{rr} \quad \sigma_{r\phi} \quad \sigma_{rz} \right]^T \quad (1)$$

157 where \mathbf{u} and $\boldsymbol{\sigma}$ represent the displacement and the stress, respectively.

158 After algebraic manipulations of the previously named equations (see (Chen *et al.*, 2004)),
 159 one can obtain the so-called state equation for any anisotropic solid layer j (index j is
 160 omitted on purpose):

$$\frac{\partial \mathbf{V}}{\partial r}(r, \phi, z) = [\mathcal{Q}] \mathbf{V}(r, \phi, z) \quad (2)$$

161 where $[\mathcal{Q}]$ is the Stroh's operator being dependent on the elastic constants of the considered
 162 layer (Chen *et al.*, 2004).

163 For the sake of brevity and clarity, in the following one only considers the orthotropic elastic
 164 materials although the presented numerical process could be also applied to any anisotropic
 165 material. In the specific orthotropic case, the stiffness tensor shows only 9 independent
 166 elastic constants and Hooke's law takes the expression (47) given in Appendix.

167 Thus, for this case, the $[\mathcal{Q}]$ Stroh's operator reduces to (48) given once again in Appendix.

168 C. Fourier transforms and Stroh's differential system

169 Operator $[\mathcal{Q}]$ in equation (2) includes partial derivatives along the circumferential and
 170 axial directions which make the integration of the solution not immediately tractable. Thus,
 171 the field quantities are represented by Fourier integral transforms in the axial direction and
 172 Fourier series transforms in the circumferential direction. Actually, any $F(r, \phi, z)$ physical

173 field in the cylindrical coordinate system can be expressed as

$$F(r, \phi, z) = \frac{1}{2\pi} \sum_{n=-\infty}^{+\infty} e^{in\phi} \int_{-\infty}^{+\infty} \tilde{F}(r, n, \alpha) e^{i\alpha z} d\alpha \quad (3)$$

174 where the associated spectral quantity $\tilde{F}(r, n, \alpha)$ is given by the Fourier transform:

$$\tilde{F}(r, n, \alpha) = \frac{1}{2\pi} \int_0^{2\pi} e^{-in\phi} \int_{-\infty}^{+\infty} F(r, \phi, z) e^{-i\alpha z} dz d\phi \quad (4)$$

175 Solving equation (2) in the spectral domain by using Fourier transforms allows us to reduce
 176 the linear partial differential equations in the variables (r, ϕ, z) into ordinary differential
 177 equations in the radial variable r . Hence, equation (2) and the Stroh's matrix $[\mathbf{Q}]$ for the
 178 orthotropic case become in the spectral domain:

$$\frac{d\tilde{\mathbf{V}}}{dr}(r, n, \alpha) = [\mathbf{Q}(r)] \tilde{\mathbf{V}}(r, n, \alpha) \quad (5)$$

179 The latter equation is a 1st order differential system for the radial variable r , in which the
 180 matrix $[\mathbf{Q}(r)]$ is given by (50) in [Appendix](#) and is also dependent of the radial coordinate.
 181 This is the main reason why the derived system differs from the one obtained in cartesian
 182 coordinates for plane layers, and why the solution is not directly integrable in the same way.
 183 However, the decomposition of the layer in thin sublayers will allow us to neglect locally the
 184 effect of the curvature of the cylindrical sublayer and to approximate the matrix $[\mathbf{Q}(r)]$ as
 185 a constant in the equation system (5) for each thin sublayer. This approach is described in
 186 the following subsection.

187

D. Transfer matrices

188

1. Case of an elastic solid layer

189

Let us consider a solid layer partitioned into P different sublayers of equal thickness as

190

represented in Fig. 2.

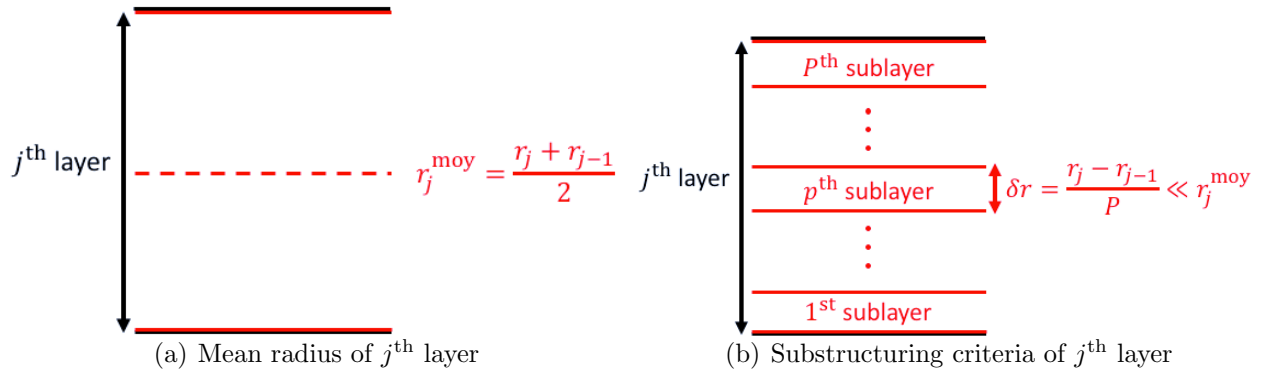


FIG. 2. Substructuring approach for the integration of the Stroh's system

191

Its extremal radiuses and mean radius are noted respectively r_{j-1} , r_j and $r_j^{\text{moy}} =$

192

$\frac{r_j + r_{j-1}}{2}$. Each sublayer has a constant thickness $\delta r = \frac{r_j - r_{j-1}}{P}$ which is assumed much

193

lower than the mean radius (i.e. typically $\frac{\delta r}{r_j^{\text{moy}}} \leq 0.1$) in order to neglect the curvature

194

locally inside each sublayer. The sublayer p of the j^{th} layer is then delimited between

195

its extremal radiuses $r_{j,p}^{\text{inf}}$ and $r_{j,p}^{\text{sup}}$, and its mean radius is noted $r_{j,p}^{\text{moy}}$. These geometrical

196

parameters verify for $p \in [1, P]$:

$$r_{j,p}^{\text{inf}} = r_{j-1} + (p - 1) \delta r, \quad r_{j,p}^{\text{moy}} = r_{j,p}^{\text{inf}} + \frac{\delta r}{2}, \quad r_{j,p}^{\text{sup}} = r_{j,p}^{\text{inf}} + \delta r \quad (6)$$

197

Inside each sublayer p of the j^{th} layer, the state vector \mathbf{V}_j^p is the solution of a Stroh's system

198

only valid along the thickness δr . As this latter parameter is assumed small compared to

199 the mean radius, one can approximate the Stroh's matrix by its value at mean radius $r_{j,p}^{\text{moy}}$:

$$\begin{aligned} \frac{d\tilde{\mathbf{V}}_j^p}{dr}(r, n, \alpha) &= [\mathbf{Q}_j^p(r_{j,p}^{\text{moy}})] \tilde{\mathbf{V}}_j^p(r, n, \alpha), \quad \forall r \in [r_{j,p}^{\text{inf}}, r_{j,p}^{\text{sup}}] \\ \tilde{\mathbf{V}}_j^p(r_{j,p}^{\text{inf}}, n, \alpha) &= \tilde{\mathbf{V}}_{j,0}^p \end{aligned} \quad (7)$$

200 According to Picard-Lindelöf's theorem, uniqueness of previous system (7) is ensured by the
 201 initial data of the state vector at the inferior interface of the sublayer which is noted $\tilde{\mathbf{V}}_{j,0}^p$.
 202 Besides, operator $[\mathbf{Q}_j^p(r_{j,p}^{\text{moy}})]$ in Eq. (7) is assumed constant across the entire thickness δr
 203 of the sublayer, and is evaluated at its mean radius $r_{j,p}^{\text{moy}}$. The solution is expressed with a
 204 matrix exponential as:

$$\mathbf{V}_j^p(r, n, \alpha) = e^{(r-r_{j,p}^{\text{inf}})[\mathbf{Q}_j^p(r_{j,p}^{\text{moy}})]} \mathbf{V}_{j,0}^p \quad (8)$$

205 The transfer matrix of the sublayer p is defined as the matrix that relates the state vector
 206 at the inferior radius to the one at the superior radius of the sublayer p . Eq. (8) evaluated
 207 for $r = r_{j,p}^{\text{sup}}$ enables to establish this transfer matrix noted $[\mathbf{T}_j]^p$:

$$\begin{aligned} \tilde{\mathbf{V}}_j^p(r_{j,p}^{\text{sup}}, n, \alpha) &= [\mathbf{T}_j]^p \tilde{\mathbf{V}}_j^p(r_{j,p}^{\text{inf}}, n, \alpha) \\ \text{with } [\mathbf{T}_j]^p &= e^{(r_{j,p}^{\text{sup}}-r_{j,p}^{\text{inf}})[\mathbf{Q}_j^p(r_{j,p}^{\text{moy}})]} \end{aligned} \quad (9)$$

208 **2. Case of an acoustic fluid layer**

209 Let us consider an acoustic fluid layer j . Only the acoustic pressure p_a and the radial
 210 displacement u_a are the components of the state vector $\mathbf{V}_j = \begin{bmatrix} u_a & p_a \end{bmatrix}^T$. Considering the
 211 Fourier transforms of the acoustic wave equation, one can deduce its solutions in terms of
 212 Bessel functions. It can then be shown (Ricks and Schmidt, 1994) that the 2×1 spectral
 213 state vectors at both extremal interfaces of the fluid layer j can be related by a 2×2 transfer

214 matrix as:

$$\tilde{\mathbf{V}}_j(r_j, n, \alpha) = [\mathbf{T}]_j \tilde{\mathbf{V}}_j(r_{j-1}, n, \alpha) \quad (10)$$

$$\text{with } [\mathbf{T}]_j = [\mathbf{F}(r_j, n, \alpha)] [\mathbf{F}(r_{j-1}, n, \alpha)]^{-1}$$

215 where $[\mathbf{F}(r, n, \alpha)]$ is called the elementary fluid matrix, and has the following coefficients:

$$[\mathbf{F}(r, n, \alpha)] = \begin{bmatrix} \gamma_L J'_n(\gamma_L r) & \gamma_L H_n^{(1)'}(\gamma_L r) \\ -\rho \omega^2 J_n(\gamma_L r) & -\rho \omega^2 H_n^{(1)}(\gamma_L r) \end{bmatrix} \quad (11)$$

216 where J_n and $H_n^{(1)}$ are respectively Bessel and Hankel functions of the first kind. These
 217 functions are evaluated at $\gamma_L r$, where $\gamma_L = \sqrt{k_L^2 - \alpha^2}$ is the projected wavenumber along
 218 the radial coordinate, $k_L = \frac{\omega}{c_L}$ is the acoustic wavenumber in the considered fluid, and c_L
 219 and ρ are respectively the sound velocity and the mass density in this same medium.

220 E. Classical matrix product assembly: the Transfer Matrix Method (TMM)

221 This section presents the classical assembly procedure of the different layers of the struc-
 222 ture which leads to numerical singularities as we will see in Sec. 3. It is based on the writing
 223 of a matrix product which expresses implicitly the continuity relations at each interface be-
 224 tween state vectors of two consecutive layers. This approach can be applied indifferently to
 225 the transfer matrices of the sublayers of a given layer to obtain its transfer matrix or to the
 226 transfer matrices of the different layers of the MCS to obtain the global transfer matrix of
 227 the MCS.

228 It has to be noted than when the structure has intermediate fluid layers, a condensation of
 229 the state vector is required as the state vectors of the structure layers have 6 components

230 whereas there are only 2 for the fluid layers. The following sections presents first the case
 231 with no condensation and then the case with condensation.

232 **1. Without intermediate fluid layers**

233 When no fluid layers are considered in the structure, consistency in the matrix product
 234 is obtained naturally between all elastic solid layers.

235 *a. Case 1: Assembly of sublayers to calculate the transfer matrix of the considered layer.*

236 In Sec. 2.4.1., the elastic solid layer j has been partitioned in P fictive sublayers and the
 237 transfer matrix of each sublayer has been determined by Eq. (9). At the interface between
 238 each sublayer, the displacement continuity and the force equilibrium should be fulfilled.

239 Actually, for any given sublayers $p - 1$ and p , $p \in \llbracket 2; P \rrbracket$, one has at the shared interface of
 240 equation $r = r_{j,p}^{\text{inf}} = r_{j,p-1}^{\text{sup}}$:

$$\forall p \in \llbracket 2; P \rrbracket, \mathbf{V}_j^p(r_{j,p}^{\text{inf}}, n, \alpha) = \mathbf{V}_j^{p-1}(r_{j,p-1}^{\text{sup}}, n, \alpha) \quad (12)$$

241 Hence, the transfer matrix for the j^{th} layer is built as the product of the different sublayer
 242 transfer matrices, given that the state vector may be propagated across the entire thickness
 243 of the layer:

$$\mathbf{V}_j^P(r_{j,P}^{\text{sup}}, n, \alpha) = \prod_{p=1}^P [\mathbf{T}_j]^{P+1-p} \mathbf{V}_1^1(r_{j,1}^{\text{inf}}, n, \alpha) \quad (13)$$

244 By identifying that $r_{j,1}^{\text{inf}} = r_{j-1}$ and $r_{j,P}^{\text{sup}} = r_j$, the transfer matrix for the j^{th} layer is easily
 245 recognizable:

$$\mathbf{V}_j(r_j, n, \alpha) = [\mathbf{T}]_j \mathbf{V}_j(r_{j-1}, n, \alpha), \text{ with } [\mathbf{T}]_j = \prod_{p=1}^P [\mathbf{T}_j]^{P+1-p} \quad (14)$$

246 *b. Case 2: Assembly of layers to calculate the transfer matrix of the multilayered cylin-*
 247 *drical shell.*

248 For interfaces separating two consecutive elastic solid layers, the same principle can be
 249 applied, assuming a rigid junction between the two layers. Thus, by considering the $j + 1^{\text{th}}$
 250 interface separating the layers j and $j + 1$ connected at $r = r_j$, we have:

$$\mathbf{V}_j(r_j, n, \alpha) = \mathbf{V}_{j+1}(r_j, n, \alpha) \quad (15)$$

251 Propagating the displacement and stress fields across the physical layers is then achieved the
 252 same way as across the different sublayers of a given material, and a global transfer matrix
 253 $[\mathbf{T}]_{\text{struc}}$ for the whole multilayered structure can be defined:

$$\mathbf{V}_N(r_N, n, \alpha) = [\mathbf{T}]_{\text{struc}} \mathbf{V}_1(r_0, n, \alpha), \quad [\mathbf{T}]_{\text{struc}} = \prod_{j=1}^N [\mathbf{T}]_{N+1-j} \quad (16)$$

254 **2. With at least one intermediate fluid layer**

255 In the case where the MCS presents at least one intermediate fluid layer, one notes n_f
 256 the number of intermediate fluid layers. This implies that the number of *groups* of solid
 257 layers is $n_f + 1$, a group being a stack of elastic solid layers framed by two fluid layers.

258 Let us consider now a specific group g of elastic solid layers separated by two fluid layers at
 259 its lower and upper interfaces, noted respectively r_g^{lower} and r_g^{upper} . Then state vectors at both
 260 sides of fluid-solid interfaces are no longer consistent, and we define for the group of elastic
 261 solid layers a reduced state vector $\mathbf{V}_g^r = \begin{bmatrix} u_r^g \\ \sigma_{rr}^g \end{bmatrix}^T$ which contains the radial (i.e. normal
 262 to the interface) displacement and the radial stress evaluated at any radial coordinate of the

263 multilayered system, giving for the full state vector the following form:

$$\mathbf{V} = \left[\begin{array}{c} u_z^g \quad u_\phi^g \quad \mathbf{V}_g^r \quad \sigma_{r\phi}^g \quad \sigma_{rz}^g \end{array} \right]^T \quad (17)$$

264 Hence, the continuity relation is only about normal displacement and normal stress, and
 265 entails nullity of shear stresses. For the upper interface of the solid group, this leads to:

$$\begin{aligned} \mathbf{V}_g^r(r_g^{\text{upper}}, n, \alpha) &= \mathbf{V}_{g+1}^r(r_g^{\text{upper}}, n, \alpha) \\ \left[\begin{array}{c} \sigma_{r\phi} \quad \sigma_{rz} \end{array} \right]^T (r_g^{\text{upper}}, n, \alpha) &= \mathbf{0}_2 \end{aligned} \quad (18)$$

266 and the same goes for the lower interface.

267 A usual way for dealing with the latter continuity relations consists in condensing the trans-
 268 fer matrix of the group of solid layers.

269 The g^{th} solid group's transfer matrix $[\mathbf{T}]_g$, obtained by matrix product of the transfer ma-
 270 trices of each layer of the group, can be partitioned in 2×2 blocks as follows:

$$\left(\begin{array}{c} \left(\begin{array}{c} u_z^g \\ u_\phi^g \end{array} \right) \\ \mathbf{V}_g^r \\ \left(\begin{array}{c} 0 \\ 0 \end{array} \right) \end{array} \right) (r_g^{\text{upper}}, n, \alpha) = \left(\begin{array}{ccc} [\mathbf{T}_{11}]_g & [\mathbf{T}_{12}]_g & [\mathbf{T}_{13}]_g \\ [\mathbf{T}_{21}]_g & [\mathbf{T}_{22}]_g & [\mathbf{T}_{23}]_g \\ [\mathbf{T}_{31}]_g & [\mathbf{T}_{32}]_g & [\mathbf{T}_{33}]_g \end{array} \right) \left(\begin{array}{c} \left(\begin{array}{c} u_z^g \\ u_\phi^g \end{array} \right) \\ \mathbf{V}_g^r \\ \left(\begin{array}{c} 0 \\ 0 \end{array} \right) \end{array} \right) (r_g^{\text{lower}}, n, \alpha) \quad (19)$$

271 The last four equations of system (19) give the following 2×1 equations:

$$\left\{ \begin{array}{l} \mathbf{V}_g^r(r_g^{\text{upper}}, n, \alpha) = [\mathbf{T}_{21}]_g \left(\begin{array}{c} u_z^g \\ u_\phi^g \end{array} \right) \Big|_{(r_g^{\text{lower}}, n, \alpha)} + [\mathbf{T}_{22}]_g \mathbf{V}_g^r(r_g^{\text{lower}}, n, \alpha) \\ \left(\begin{array}{c} 0 \\ 0 \end{array} \right) = [\mathbf{T}_{31}]_g \left(\begin{array}{c} u_z^g \\ u_\phi^g \end{array} \right) \Big|_{(r_g^{\text{lower}}, n, \alpha)} + [\mathbf{T}_{32}]_g \mathbf{V}_g^r(r_g^{\text{lower}}, n, \alpha) \end{array} \right. \quad (20)$$

272 By rearranging the set of equations (20) and eliminating the axial and circumferential dis-
 273 placements, one can establish a *reduced* transfer matrix which propagates the reduced state
 274 vector from the lower to the upper interface of g^{th} group:

$$\begin{aligned} \mathbf{V}_g^r(r_g^{\text{upper}}, n, \alpha) &= [\mathbf{T}]_g^r \mathbf{V}_g^r(r_g^{\text{lower}}, n, \alpha) \\ \text{with } [\mathbf{T}]_g^r &= \left[[\mathbf{T}_{22}]_g - [\mathbf{T}_{21}]_g [\mathbf{T}_{31}]_g^{-1} [\mathbf{T}_{32}]_g \right] \end{aligned} \quad (21)$$

275 Once every group of solid layers has been condensated, one can propagate the reduced
 276 state vector across the MCS via the following equation:

$$\mathbf{V}_N^r(r_N, n, \alpha) = [\mathbf{T}]_{\text{struc}}^r \mathbf{V}_1^r(r_0, n, \alpha), \quad [\mathbf{T}]_{\text{struc}}^r = \prod_{k=1}^{n_f+1} [\mathbf{T}]_{n_f+2-k}^r [\mathbf{T}]_{n_f+2-k}^f \quad (22)$$

277 where $[\mathbf{T}]_k^f$ is the transfer matrix of the intermediate fluid layer framed by the $k-1^{\text{th}}$ and
 278 k^{th} groups of solid elastic layers.

279 F. Global calculation and evaluation of the spectral pressure at the receiving point

280 In the previous section, one has determined the global transfer matrix of the MCS. In
 281 order to calculate the global response of the fluid loaded MCS, it is necessary to write
 282 the boundary conditions at the internal and external radius of the MCS. In the case of an

283 incident plane wave in the external medium exciting the MCS, the spectral pressure at the
 284 external radius r_N can be split into the blocked pressure p_b and the scattered pressure p_e
 285 (Maxit, 2014):

$$p(r_N, n, \alpha) = p_b(r_N, n, \alpha) + p_e(r_N, n, \alpha). \quad (23)$$

286 The scattered pressure p_e can be expressed via the external fluid impedance Z_{fe} and the
 287 normal displacement at $r = r_N$ of the N^{th} layer of the MCS (after writing the continuity of
 288 the normal displacements at the interface between the N^{th} layer and the external fluid):

$$p_e(r_N, n, \alpha) = Z_{fe} \mathbf{u}_N(r_N, n, \alpha) \cdot \mathbf{e}_r \quad (24)$$

289 where $Z_{fe}(n, \alpha) = \frac{\rho_e \omega^2 H_n^{(1)}(\gamma_e r_N)}{\gamma_e H_n^{(1)'(\gamma_e r_N)}$, with $\gamma_e = \sqrt{k_e^2 - \alpha^2}$, $k_e = \frac{\omega}{c_e}$ being the acoustic
 290 wavenumber in the external fluid, and c_e and ρ_e being the sound velocity and the mass
 291 density in this same medium.

292 Supposing that the acoustic wavevector characterizing the incident wave is in the plane
 293 $\phi = 0$ and that it makes an angle ψ_i with the axial axis, the blocked pressure induced by
 294 the acoustic plane wave can be expressed by (Skelton and James, 1997):

$$p_b(r_N, n, \alpha) = \frac{2(-i)^{n+1}}{\pi \gamma_e r_N H_n^{(1)}(\gamma_e r_N)} 2\pi \delta(\alpha - \alpha^i) \quad (25)$$

295 where $\alpha^i = \cos(\psi_i) k_e$ is the projection of the acoustic wavevector onto the axial axis.

296 On another hand, the pressure exerted by the internal air cavity can be related to the normal
 297 displacement of the first layer at $r = r_0$:

$$p_{1e}(r_0, n, \alpha) = Z_{fi} \mathbf{u}_1(r_0, n, \alpha) \cdot \mathbf{e}_r \quad (26)$$

where

$$Z_{fi}(r, n, \alpha) = \frac{\rho_i \omega^2 J_n(\gamma_i(\alpha) r)}{\gamma_i(\alpha) J_n'(\gamma_i(\alpha) r_0)},$$

with $\gamma_i = \sqrt{k_i^2 - \alpha^2}$, $k_i = \frac{\omega}{c_i}$ being the acoustic wavenumber in the external fluid, and c_i and ρ_i being the sound velocity and the mass density in this same medium.

Using a vector-matrix form to be compliant with the transfer matrix, the stress fields applied at the external and internal radiuses on the MCS can be written:

$$\boldsymbol{\sigma}_N(r_N, n, \alpha) = -\mathbf{P}_b(r_N, n, \alpha) - [\mathbf{T}_{fe}] \mathbf{u}_N(r_N, n, \alpha) \quad (27)$$

$$\boldsymbol{\sigma}_1(r_0, n, \alpha) = [\mathbf{T}_{fi}] \mathbf{u}_1(r_0, n, \alpha) \quad (28)$$

298 with

$$\begin{aligned} \mathbf{P}_b(r_N, n, \alpha) &= p_b(r_N, n, \alpha) \mathbf{e}_r \\ [\mathbf{T}_{fi}] &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & Z_{fi} \end{pmatrix}, [\mathbf{T}_{fe}] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & Z_{fe} \end{pmatrix} \end{aligned} \quad (29)$$

299 G. Resolution of the mathematical problem

300 Considering the transfer matrix of the MCS given by Eq. (16) and the boundary condi-
301 tions defined by Eq. (27)-(28), one can deduce the displacement and the stress fields of the
302 MCS at the external and internal radiuses.

303 First, the global transfer matrix is partitioned into 3×3 blocks as follows, in the case where
304 the MCS does not exhibit intermediate fluid layers:

$$\mathbf{V}_N(r_N, n, \alpha) = \begin{pmatrix} [\mathbf{T}]_{u_N u_1} & [\mathbf{T}]_{u_N \sigma_1} \\ [\mathbf{T}]_{\sigma_N u_1} & [\mathbf{T}]_{\sigma_N \sigma_1} \end{pmatrix} (n, \alpha) \mathbf{V}_1(r_0, n, \alpha), \quad (30)$$

where

$$\mathbf{V}_N = \begin{bmatrix} \mathbf{u}_N & \boldsymbol{\sigma}_N \end{bmatrix}^T \quad \text{and} \quad \mathbf{V}_1 = \begin{bmatrix} \mathbf{u}_1 & \boldsymbol{\sigma}_1 \end{bmatrix}^T .$$

305 Then the displacement field at the internal radius of the MCS can be written:

$$\mathbf{u}_1(r_0, n, \alpha) = - [\mathbf{R}]_{\mathbf{u}_1 \mathbf{F}_N} \mathbf{P}_b(r_N, n, \alpha) \quad (31)$$

306 where

$$[\mathbf{R}]_{\mathbf{u}_1 \mathbf{F}_N} = [[\mathbf{T}_{\text{fe}}] [\mathbf{T}]_{\mathbf{u}_N \mathbf{u}_1} + [\mathbf{T}]_{\boldsymbol{\sigma}_N \mathbf{u}_1} + [[\mathbf{T}]_{\boldsymbol{\sigma}_N \boldsymbol{\sigma}_1} + [\mathbf{T}_{\text{fe}}] [\mathbf{T}]_{\mathbf{u}_N \boldsymbol{\sigma}_1}] [\mathbf{T}_{\text{fl}}]^{-1} \quad (32)$$

307 An explicit expression for the state vector $\mathbf{V}_1 = \begin{bmatrix} \mathbf{u}_1 & \boldsymbol{\sigma}_1 \end{bmatrix}^T$ at the internal radius of the MCS

308 can then be established. This state vector can be propagated to the radius of the receiving

309 point using a transfer matrix between the internal radius and the radius of the receiving

310 point. This latter can be estimated by the same way that the global transfer matrix of the

311 MCS has been obtained in Sec. 2.5. From the state vector at the radius of the receiving point

312 expressed in the wavenumber domain (n, α) , one can deduce the response at the receiving

313 point by the inverse Fourier transforms (3).

314 The same way is considered in the case where the structure has at least one intermediate

315 fluid layer. However, as it has been described in Sec. II E 2, only the normal displacement

316 and the normal stress are considered in the reduced matrix transfer. Thus, the latter is

317 partitioned into blocks of four scalar $T_{u_N u_1}^r$, $T_{\boldsymbol{\sigma}_N u_1}^r$, $T_{\boldsymbol{\sigma}_N \boldsymbol{\sigma}_1}^r$ and $T_{u_N \boldsymbol{\sigma}_1}^r$, as follows:

$$\mathbf{V}_N^r(r_N, n, \alpha) = \begin{pmatrix} T_{u_N u_1}^r & T_{u_N \boldsymbol{\sigma}_1}^r \\ T_{\boldsymbol{\sigma}_N u_1}^r & T_{\boldsymbol{\sigma}_N \boldsymbol{\sigma}_1}^r \end{pmatrix} (n, \alpha) \mathbf{V}_1^r(r_0, n, \alpha), \quad (33)$$

where

$$\mathbf{V}_N^r = \begin{bmatrix} u_N & \boldsymbol{\sigma}_N \end{bmatrix}^T \quad \text{and} \quad \mathbf{V}_1^r = \begin{bmatrix} u_1 & \boldsymbol{\sigma}_1 \end{bmatrix}^T .$$

318 Finally, the normal displacement at the internal radius can be written:

$$u_1(r_0, n, \alpha) \cdot \mathbf{e}_r = -R_{u_1 \mathbf{F}_N} p_b(r_N, n, \alpha) \quad (34)$$

319 with

$$R_{u_1 \mathbf{F}_N} = \frac{1}{Z_{\text{fe}} T_{u_N u_1}^r + T_{\sigma_N u_1}^r + [T_{\sigma_N \sigma_1}^r + Z_{\text{fe}} T_{u_N \sigma_1}^r] Z_{\text{fi}}}. \quad (35)$$

320 III. NUMERICAL CONVERGENCE OF THE TMM APPROACH

321 A. Validation test cases

322 In this section, we are going to highlight some numerical instabilities of the classical
 323 assembly procedure described previously. Numerical singularities compared to a reference
 324 are going to be shown on test cases related to underwater applications. In the section 4, we
 325 will propose an alternative assembly procedure avoiding these instabilities. Let us consider
 326 the four different multilayered cylindrical structures shown on Fig. 3.

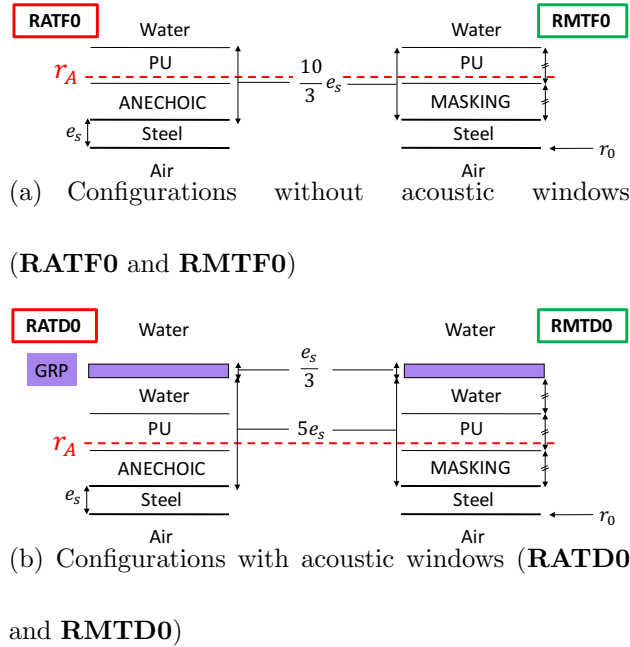


FIG. 3. Schematic representation of the multilayered cross-section of the different test cases

- 327 - the **RATF0** configuration corresponds to a cylindrical steel shell of thickness e_s and
- 328 of internal radius r_0 covered by an acoustic coating composed by a layer of anechoic
- 329 material and a layer of polyurethane (PU);
- 330 - the **RMTF0** configuration is similar to the **RATF0** at the difference that the anechoic
- 331 material is replaced by a masking material;
- 332 - the **RATD0** and **RMTD0** configurations correspond to **RATF0** and **RMTF0**, re-
- 333 spectively, with a supplementary layer of glass reinforced polymer material (GRP) at
- 334 a given distance of the outer surface of the PU layer. It results in an intermediate
- 335 layer of water between the PU and GRP layer. For an application point of view, this
- 336 supplementary GRP layer may represent the acoustic window used in a Sonar flank
- 337 array to keep the sensors away from the turbulent flow. In this section, the GRP ma-

338 material is assumed isotropic to allow us to obtain a reference result with the Helmholtz
 339 decomposition (Ricks and Schmidt, 1994).

340 For these 4 configurations, the MCS is immersed in water and the internal cavity is filled
 341 with air. The MCS is excited by an acoustic plane wave of magnitude A_0 . The mechanical
 342 properties of the different materials and the ratios of the layer thicknesses to the steel shell
 343 thickness e/e_s are shown in Table I.

TABLE I. Material properties. Every solid material is assumed isotropic.

	Steel	PU	ANECHOIC	MASKING	Water	GRP (assumed isotropic)
c_L (m.s ⁻¹)	6020	1600	900	300	1500	3000
η_L	0,01	0,02	0,25	0,25	0	0,05
c_T (m.s ⁻¹)	3220	150	150	100	0	2000
η_T	0,01	0,1	0,35	0,45	0	0,1
ρ (kg.m ⁻³)	7800	1100	1400	1100	1000	2000
e/e_s	1	$\frac{5}{3}$	$\frac{5}{3}$	$\frac{5}{3}$	$\frac{5}{3}$	$\frac{1}{3}$

344 The calculations will be carried out for a receiving point located at a radius $r_A = r_0 + \frac{10}{3}e_s$.
 345 This receiving point is inside the PU layer mimicking an hydrophone sensor embedded in
 346 the PU of a Sonar. The response at this point will be evaluated in term of the radial stress
 347 σ_{rr} .

348 The TMM calculations have been performed as described in the section 2. The material
 349 properties are used to evaluate the Stroh's matrix (Eq. (5)) reminding that the stiffness
 350 constants are related to the longitudinal and transverse velocities by: $c_{11} = c_{22} = c_{33} =$
 351 ρc_L^2 , $c_{44} = c_{55} = c_{66} = \rho c_T^2$ and $c_{12} = c_{13} = c_{23} = \rho(c_L^2 - 2c_T^2)$. As detailed in the section
 352 [IID 1](#), the layer can be decomposed in several sublayers in order to suppose that the Stroh's
 353 matrix is independent on the radius in the sublayer. Calculations considering different
 354 numbers of sublayers for each layer were carried out and they showed very close results even
 355 when considering one single sublayer. One concludes that for the considered configurations
 356 and the considered frequency range of interest, it is not necessary to decompose the layers
 357 in different sublayers. This is due to the fact that the criterion $\frac{\delta r}{r_j^{\text{moy}}} \leq 0.1$ is sufficiently
 358 respected for the considered cases. In the following subsections, the obtained TMM results
 359 are going to be compared to the reference results obtained with the so called Dynamic
 360 Global Method based on Helmholtz Decomposition (DGM-HD). This latter is based on the
 361 DGM method introduced by Ricks in 1994 ([Ricks and Schmidt, 1994](#)), namely a Helmholtz
 362 decomposition performed in each elementary layer, followed by a global matrix assembly
 363 procedure. The DGM-HD calculations have been previously validated by comparison with
 364 other models, in particular finite element models. We can underline that the TMM and
 365 DGM-HD approaches differ, both in the elementary layer description (Stroh's formalism for
 366 TMM, Helmholtz decomposition for DGM-HD) and in the assembly procedure (propagation
 367 of the state vector for TMM, global assembly matrix for DGM-HD).

368

B. Large fd problem

369 In Fig. 4, one presents the radial stress response at the receiving point as a function of
370 the normalized frequency $k_e r_0$ for the **RATF0** and **RMTF0** configurations. These con-
371 figurations do not include any intermediate fluid layer (see Fig. 3(a)). A good agreement
372 between the TMM and DGM-HD calculations is only observed for normalized frequencies
373 below around 100. Above this frequency limit, large singularities between the two calcula-
374 tions can be observed and increase roughly with frequency. This numerical instability can
375 be attributed to the so-called "large fd problem" (Dunkin, 1965; Lévesque and Piché, 1992).
376 It results from the TMM indiscrimination of all exponential partial waves within the layer
377 in the case of a large thickness-frequency combination (Lowe, 1995). For underwater appli-
378 cations considering layers of acoustic materials of several centimeters and a large frequency
379 band of interest, the TMM approach is then not clearly appropriate as highlighted by the
380 results of Figure 4.

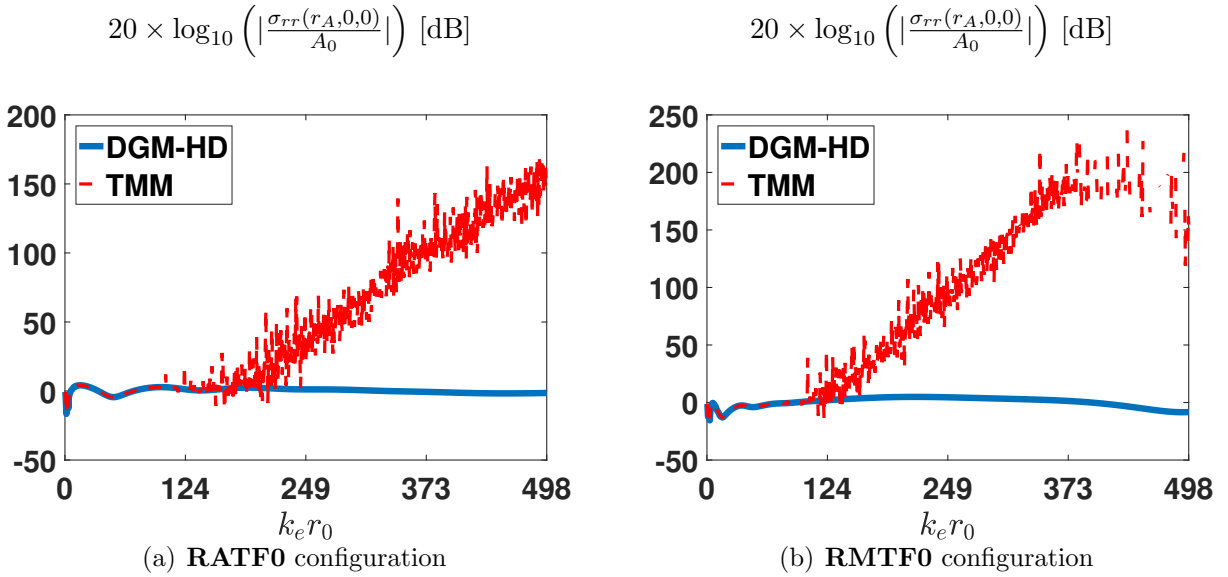


FIG. 4. Highlighting instability known as the *large fd problem*: Radial stress at the receiving point for an incident plane wave.

381 **C. Cases with intermediate fluid layers**

382 Now, let us focus on the **RATD0** and **RMTD0** configurations which exhibit an interme-
 383 diate fluid layer between the PU and GRP layer. As stated previously, the incompatibility
 384 of the state vectors between a solid layer and a fluid layer requires a specific treatment in
 385 the TMM approach. Two approaches are studied numerically in this subsection, namely the
 386 condensation of the solid matrices as presented in the section II E 2 and the introduction of
 387 a transverse velocity for mimicking a viscous fluid.

388 **1. Condensation of the solid matrices**

389 The results of the TMM approach using the condensation of the transfer matrices of the
 390 solid layers (see section II E 2) are compared in Fig.5 with the DGM-HD results for the two
 391 configurations presenting an intermediate fluid layer.

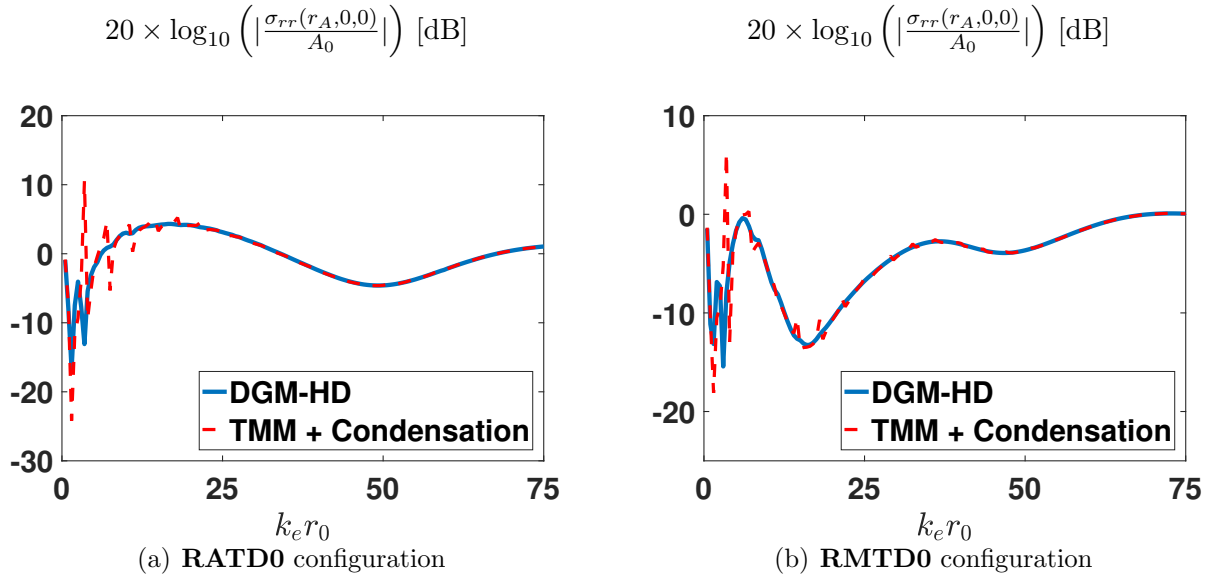


FIG. 5. Highlighting the numerical instability in the low frequency domain when using the condensation of the solid transfer matrices: Radial stress at the receiving point for a normal incident plane wave.

392 These comparisons are performed at low normalized frequencies (i.e. $k_e r_0 < 75$) for which
 393 the TMM approach gave satisfactory results for configurations without intermediate fluid
 394 layer as shown in Fig. 4. One can observe significant singularities at the lower part of the
 395 considered frequency band. This results from the poor conditioning of the matrix inverted
 396 in the condensation procedure and the propagation of numerical errors with the classical
 397 assembly procedure of the TMM approach.

398 **2. *Fluid layers with transverse velocities***

399 Instead of using the condensation procedure, the fluid layer is represented by an isotropic
400 solid layer for which the longitudinal velocity C_L corresponds to the acoustic velocity and
401 the transverse velocity C_T is "small" but not null. This allows to prevent the inconsistency
402 of the matrix product between a solid layer and a fluid layer. This approach is easy to
403 implement but its difficulty lies in the choice of the transverse velocity. The results of the
404 TMM approach with two different values of the transverse velocity are compared in Fig. 6
405 with the DGM-HD results used as reference. One can observe that the TMM results depend
406 highly on the values of the transverse velocity loss factor. For $C_T = 1 \text{ m.s}^{-1}$ with a loss
407 factor $\eta_T = 0.05$, the TMM results diverge for normalised frequencies above around 25. On
408 the contrary, a good agreement is observed when $C_T = 1 \text{ m.s}^{-1}$ without damping. However,
409 this last encouraging result obtained for a normal incident plane wave is not general. For
410 instance, for an incident angle of 45° , one can notice in Fig. 6 that the TMM results oscillate
411 around the reference results and the singularities become significant in the higher part of
412 the frequency range. Clearly, this approach for taking the intermediate fluid layer into
413 account is not optimal. It appears difficult to establish a reliable criterion for choosing these
414 parameters, since the two different sets of parameters considered here show very different
415 results, and no good correlation is guaranteed for a wide frequency range and various incident
416 angles.

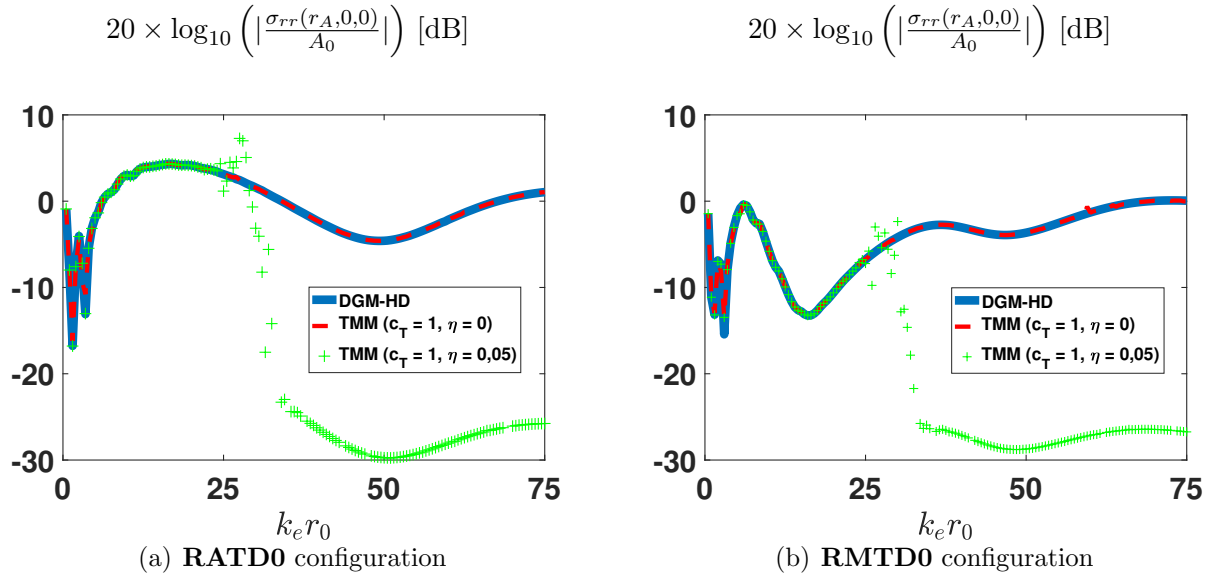


FIG. 6. Highlighting of the responses obtained with the TMM approach when the intermediate fluid layer is represented by an isotropic layer with a small transverse velocity: Radial stress at the receiving point for a normal incident plane wave.

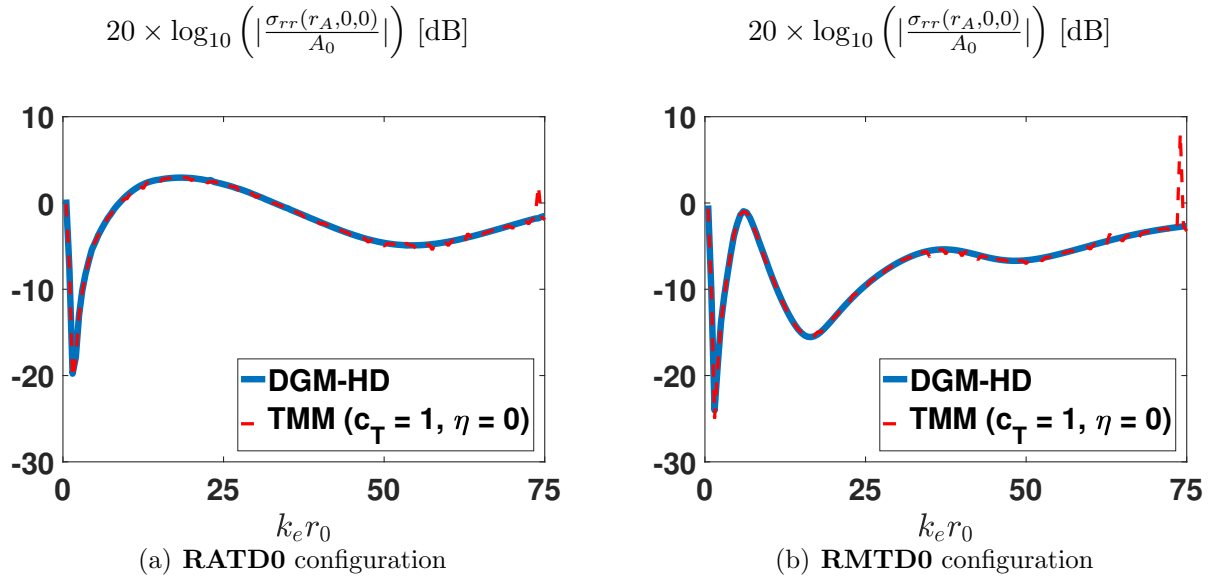


FIG. 7. As the previous figure but for an incident angle of 45° .

417 **IV. SOLVING THE NUMERICAL INSTABILITIES WITH THE PROPOSED AS-**
 418 **SEMBLY PROCEDURE**

419 To solve the numerical instabilities related to the "large fd problem" and to overcome the
 420 difficulty for the TMM approach to take into account intermediate fluid layers, we propose
 421 an alternative approach for assembling the transfer matrix. This procedure is based on
 422 building a global matrix similar to the one built in the Dynamic Global Method (DGM)
 423 which uses the Helmholtz decomposition in elementary layers (Ricks and Schmidt, 1994).
 424 As the transfer matrices of the different layers will be extracted from the Stroh's formalism
 425 (as presented in Sec. 2) and the assembly approach will mimic the DGM, we will refer to it
 426 as the DGM-SF method in the following whereas DGM-HD will always denote the classical
 427 Dynamic Global Method(DGM) using the Helmholtz decomposition in each layer.

428 The proposed assembly approach is based on the writing in the spectral domain of a global
 429 matrix $[\mathbf{G}]_{\text{DGM-SF}}(n, \alpha)$ which relates the second member vector $\mathbf{E}(n, \alpha)$ characterizing the
 430 excitation and a global vector $\{\mathbf{V}\}_{\text{GLOB}}(n, \alpha)$ containing all the state vectors evaluated at
 431 every lower interface of each layer of the structure:

$$[\mathbf{G}]_{\text{DGM-SF}}(n, \alpha)\{\mathbf{V}\}_{\text{GLOB}}(n, \alpha) = \mathbf{E}(n, \alpha) \quad (36)$$

432 In the following, we are going to describe how the global matrix $[\mathbf{G}]_{\text{DGM-SF}}$ should be built
 433 so that the continuity relations on the displacement and stress fields are fulfilled for every
 434 interface of the MCS. This assembly procedure depends on the type of interface, and is
 435 detailed in the different subsections.

436 The principle consists in rewriting the continuity relations such that they relate the state

437 vectors at the lower radius of the two layers sharing the considered interface. It permits
 438 to deduce the blocks of the global matrix $[\mathbf{G}]_{\text{DGM-SF}}$ related to the concatenated subvector
 439 composed of state vectors at the lower radius of the layers that share the considered interface.

440 **A. Continuity relations at internal interfaces**

441 **1. Interface between two solid layers**

442 Let consider the $j + 1^{\text{th}}$ interface ($j \in \llbracket 2; N - 1 \rrbracket$) of the structure, separating j^{th} and
 443 $j + 1^{\text{th}}$ elastic solid layers and having for equation $r = r_j$. The displacement and stress fields
 444 are continuous across this interface as it can be described by Eq. (15). Using this equation
 445 with Eq. (14), one can relate both state vectors at the bottom of j^{th} and $j + 1^{\text{th}}$ layers via
 446 the transfer matrix of j^{th} layer:

$$\mathbf{V}_{j+1}(r_j, n, \alpha) = [\mathbf{T}]_j \mathbf{V}_j(r_{j-1}, n, \alpha) \quad (37)$$

447 For convenience, this equation can be rewritten on the form:

$$\begin{pmatrix} [\mathbf{T}]_j & -[\mathbf{I}_{6 \times 6}] \end{pmatrix} \begin{pmatrix} \mathbf{V}_j(r_{j-1}, n, \alpha) \\ \mathbf{V}_{j+1}(r_j, n, \alpha) \end{pmatrix} = \begin{pmatrix} \mathbf{0}_{6 \times 1} \\ \mathbf{0}_{6 \times 1} \end{pmatrix} \quad (38)$$

448 This equation that expresses the continuity conditions at the interface between the two
 449 solid layers is used in practice for filling the global matrix during the assembly procedure as
 450 represented in Fig. 8:

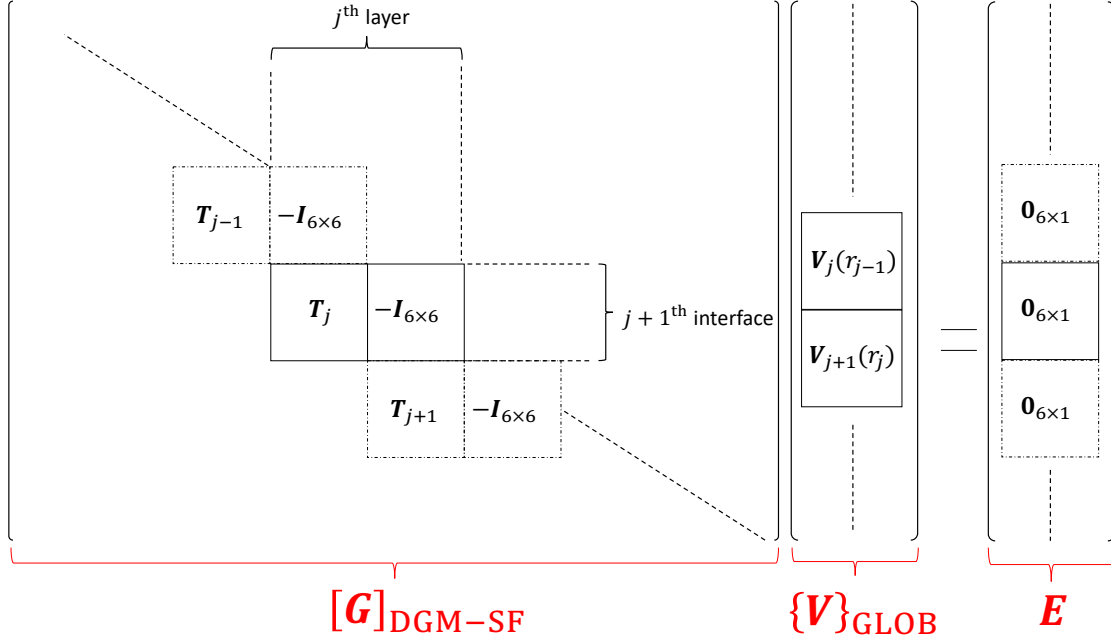


FIG. 8. Global matrix assembly procedure for *solid-solid* interfaces

451 **2. Interface between a solid and a fluid layer**

452 Let consider now an intermediate fluid layer j between two elastic solid layers $j - 1$ and
 453 $j + 1$, then only the normal displacement and the normal stress are continuous across both
 454 extremal interfaces of j^{th} layer shared with layers $j - 1$ and $j + 1$. Since \mathbf{V}_{j-1} and \mathbf{V}_{j+1} are
 455 6×1 vectors both larger than the 2×1 state vector \mathbf{V}_j in the j^{th} layer, one is brought to
 456 refer to specific rows of \mathbf{V}_{j-1} , \mathbf{V}_{j+1} , and of the transfer matrices of layers $j - 1$ and $j + 1$ in
 457 order to write the continuity relations. The exponents [3; 4] make reference to the normal
 458 displacement and normal stress (i.e. u_r , σ_{rr}) whereas the exponents [5; 6] make reference to
 459 the tangential stresses (i.e. $\sigma_{r\phi}$, σ_{rz}).

460 The continuity relations can then be written:

461 - for the $j - 1^{\text{th}}$ interface:

$$\begin{cases} \mathbf{V}_j(r_{j-1}, n, \alpha) = [\mathbf{T}^{[3;4]}]_{j-1} \mathbf{V}_{j-1}(r_{j-2}, n, \alpha) \\ \mathbf{V}_{j-1}^{[5;6]}(r_{j-1}, n, \alpha) = [\mathbf{T}^{[5;6]}]_{j-1} \mathbf{V}_{j-1}(r_{j-2}, n, \alpha) = \mathbf{0}_{2 \times 1} \end{cases} \quad (39)$$

462 - for the j^{th} interface:

$$\begin{cases} \mathbf{V}_{j+1}^{[3;4]}(r_j, n, \alpha) = \mathbf{V}_j(r_j, n, \alpha) = [\mathbf{T}]_j \mathbf{V}_j(r_{j-1}, n, \alpha) \\ \mathbf{V}_{j+1}^{[5;6]}(r_j, n, \alpha) = \mathbf{0}_{2 \times 1} \end{cases} \quad (40)$$

463 where exponents [3; 4] and [5; 6] remind the corresponding rows in state vectors of the con-
464 sidered solid layers.

465 This set of 8 equations can be rewritten on the following form:

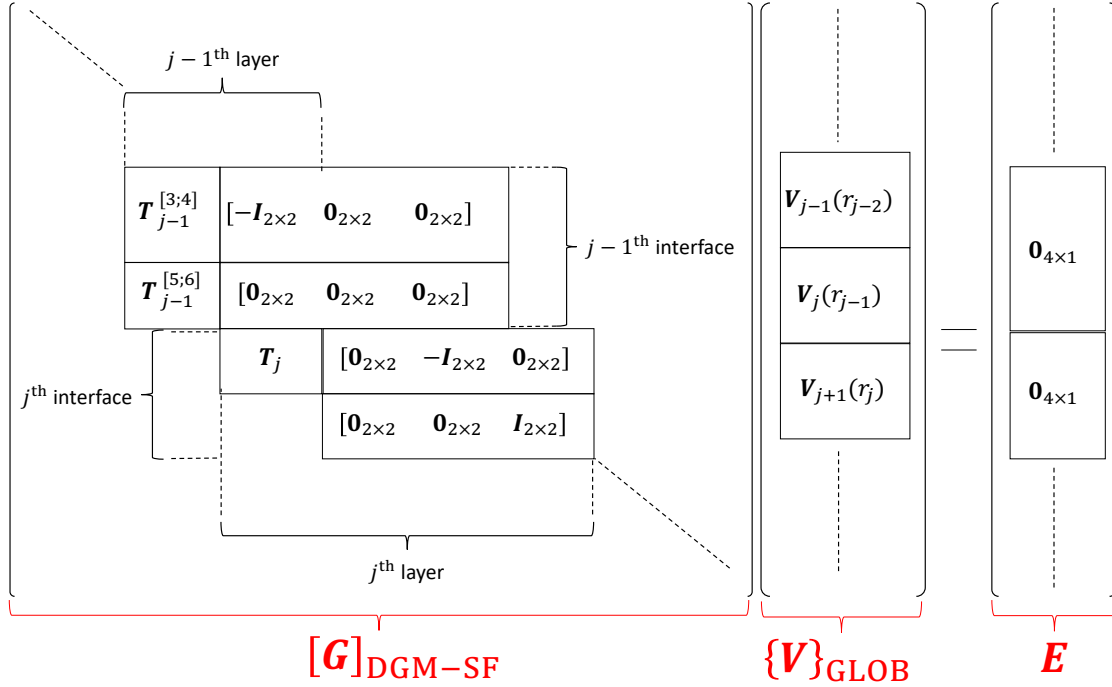
$$\begin{pmatrix} [\mathbf{T}^{[3;4]}]_{j-1} & -[\mathbf{I}_{2 \times 2}] \\ [\mathbf{T}^{[5;6]}]_{j-1} & [\mathbf{0}_{2 \times 2}] \end{pmatrix} \begin{pmatrix} \mathbf{V}_{j-1}(r_{j-2}, n, \alpha) \\ \mathbf{V}_j(r_{j-1}, n, \alpha) \end{pmatrix} = \begin{pmatrix} \mathbf{0}_{2 \times 1} \\ \mathbf{0}_{2 \times 1} \end{pmatrix} \quad (41)$$

466

$$\begin{pmatrix} [\mathbf{T}]_j & [[\mathbf{0}_{2 \times 2}] & -[\mathbf{I}_{2 \times 2}] & [\mathbf{0}_{2 \times 2}]] \\ [\mathbf{0}_{2 \times 2}] & [[\mathbf{0}_{2 \times 2}] & [\mathbf{0}_{2 \times 2}] & [\mathbf{I}_{2 \times 2}]] \end{pmatrix} \begin{pmatrix} \mathbf{V}_j(r_{j-1}, n, \alpha) \\ \mathbf{V}_{j+1}(r_j, n, \alpha) \end{pmatrix} = \begin{pmatrix} \mathbf{0}_{2 \times 1} \\ \mathbf{0}_{2 \times 1} \end{pmatrix} \quad (42)$$

467 Once again, these equations are used to fill the global matrix by blocks as represented in

468 Fig. 9.


 FIG. 9. Global matrix assembly procedure for *solid-fluid* interfaces

469 3. Continuity relations at extremal interfaces

470 The continuity relations at both internal and extremal radiuses provide 6 equations in
 471 total, which take the following form in the case of the plane wave excitation:

$$\begin{aligned}
 \mathbf{V}_1^{[4;6]}(r_0, n, \alpha) &= \begin{bmatrix} [\mathbf{T}_{\text{fi}}] & [\mathbf{0}_{3 \times 3}] \end{bmatrix} \mathbf{V}_1(r_0, n, \alpha) \\
 \mathbf{V}_N^{[4;6]}(r_N, n, \alpha) &= -\mathbf{P}_b(r_N, n, \alpha) - \begin{bmatrix} [\mathbf{T}_{\text{fe}}] & [\mathbf{0}_{3 \times 3}] \end{bmatrix} \mathbf{V}_N(r_N, n, \alpha)
 \end{aligned} \tag{43}$$

472 where the \mathbf{P}_b vector and the $[\mathbf{T}_{\text{fi}}]$, $[\mathbf{T}_{\text{fe}}]$ matrices have been defined in section II F. These
 473 equations can be rewritten on the form:

$$\begin{bmatrix} -[\mathbf{T}_{\text{fi}}] & [\mathbf{I}_{3 \times 3}] \end{bmatrix} \mathbf{V}_1(r_0, n, \alpha) = \mathbf{0}_3 \tag{44}$$

474

$$\begin{bmatrix} [\mathbf{T}_{fe}] & [\mathbf{I}_{3 \times 3}] \end{bmatrix} \mathbf{V}_N(r_N, n, \alpha) = -\mathbf{P}_b(r_N, n, \alpha) \quad (45)$$

475 However, the state vector at the top of the last layer $\mathbf{V}_N(r_N, n, \alpha)$ is not stored in the global
 476 vector $\{\mathbf{V}\}_{\text{GLOB}}$. Hence, Eq. (45) is rewritten in function of state vector at the bottom of
 477 the last layer $\mathbf{V}_N(r_{N-1}, n, \alpha)$ by using the transfer matrix of the last layer $[\mathbf{T}_N]$:

$$\begin{bmatrix} [\mathbf{T}_{fe}] & [\mathbf{I}_{3 \times 3}] \end{bmatrix} [\mathbf{T}_N] \mathbf{V}_N(r_{N-1}, n, \alpha) = -\mathbf{P}_b(r_N, n, \alpha) \quad (46)$$

478 The equations 44 and 46 are used to fill by blocks the global matrix as represented in Fig.

479 10.

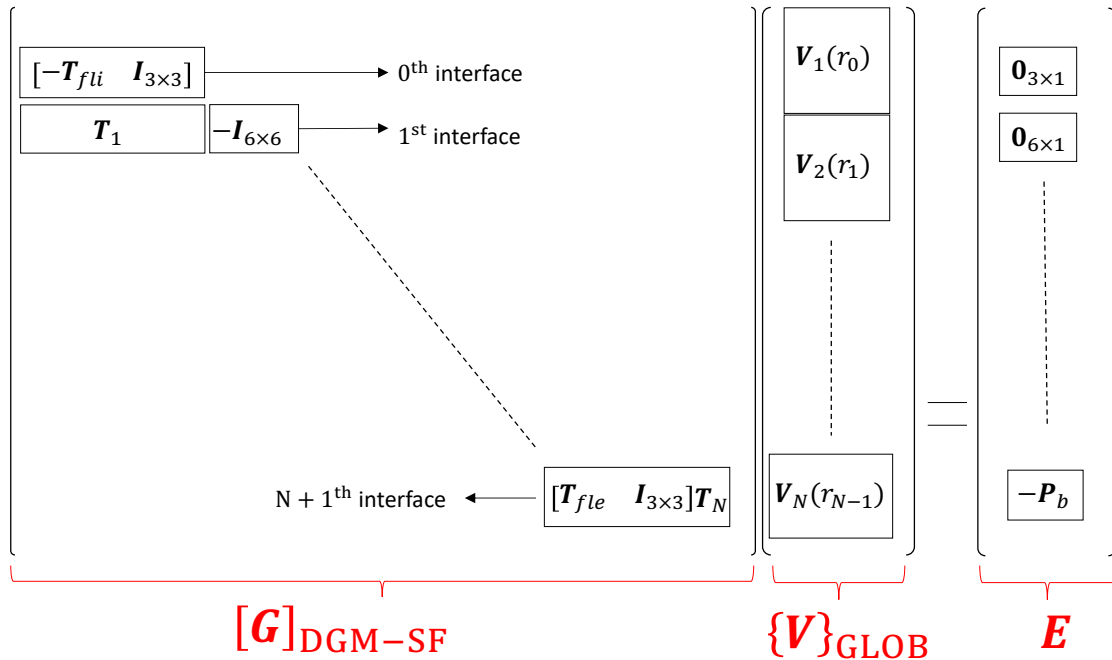


FIG. 10. Global matrix assembly procedure for extremal interfaces.

480

4. Summarize

481

The DGM-SF procedure can be decomposed in three steps:

- 482 - the transfer matrix of each layer (or each sublayer) is estimated accordingly to the
- 483 developments of the section 2.4;
- 484 - the global matrix $\{\mathbf{V}\}_{\text{GLOB}}(n, \alpha)$ and the second member vector $\mathbf{E}(n, \alpha)$ are filled by
- 485 blocks depending on the type of layer (or sublayer) interface as summarized by Figures
- 486 7, 8 and 9;
- 487 - the global matrix system (36) is solved. The state vector at the bottom of the layer
- 488 containing the receiving point, is then propagated to the radius of the receiving point
- 489 using the transfer matrix between the internal radius and the radius of the receiving
- 490 point. The response at the receiving point is then deduced from the inverse Fourier
- 491 transforms (3) of the state vector in the wavenumber domain (n, α) .

492 **B. Numerical stability enhancement provided by the DGM-SF**

493 In this section, we highlight the numerical stability improvements provided by the pro-

494 posed assembly procedure, both on the "large fd problem" and the ability to take into

495 account intermediate fluid layers.

496 *1. About the "large fd problem"*

497 Figure 11 represents the modulus of the spectral radial displacement at the inner radius

498 of the steel shell in the normalized frequency-circumferential order $(k_e r_0, n)$ space for the

499 **RMTF0** configuration. The results obtained by the two assembly procedures are com-

500 pared. One notices a good agreement between the two calculations only for both the low

501 circumferential orders and the low frequency of the considered frequency range. Outside
 502 this domain, the conventional TMM assembly procedure exhibits results that are noisy and
 503 largely different from those of the proposed DGM-SF assembly procedure. For validation,
 504 the latter compared with the DGM-HD reference results on Figure 12 for both **RATF0**
 505 and **RMTF0** configurations. One recalls that the TMM approach gave poor results on
 506 Figure 4 for the same configurations. A very good agreement can be observed on Figure 12
 507 between the DGM-HD et DGM-SF calculations, even for the highest part of the frequency
 508 range of interest. Other comparisons (not presented here) for different incident angles of the
 509 acoustic waves show also a good agreement between DGM-HD et DGM-SF. This shows that
 510 the DGM-SF procedure allows to overcome the numerical instability of the TMM assembly
 511 procedure related to the "large fd problem".

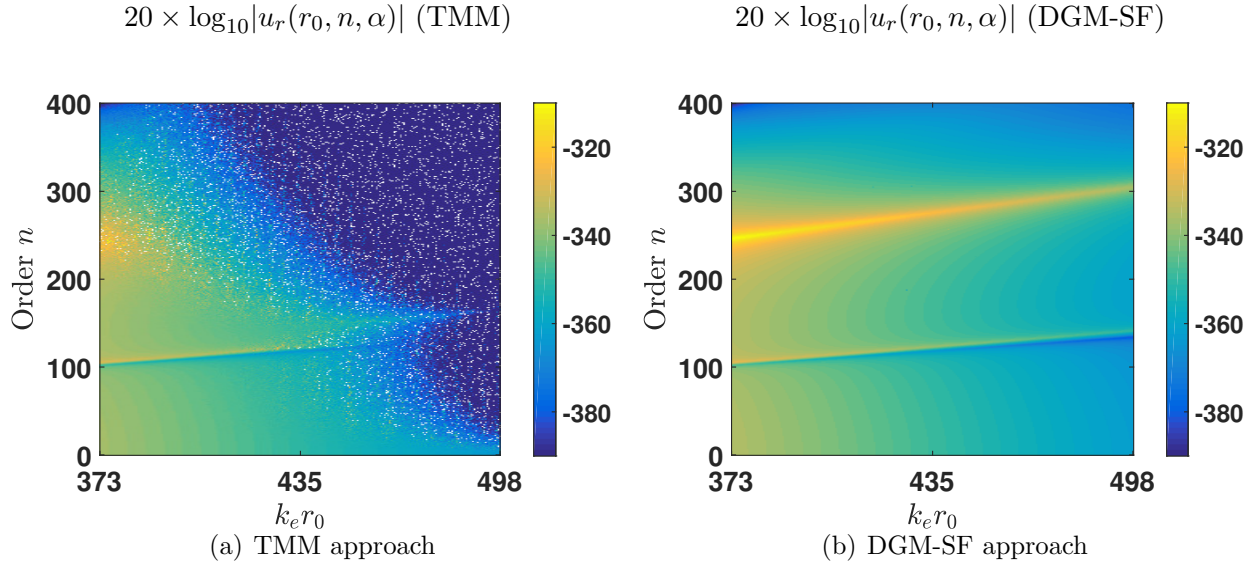


FIG. 11. Comparison of the spectral displacements at the inner radius calculated by the TMM and DGM-SF approaches for the **RMTF0** configuration.

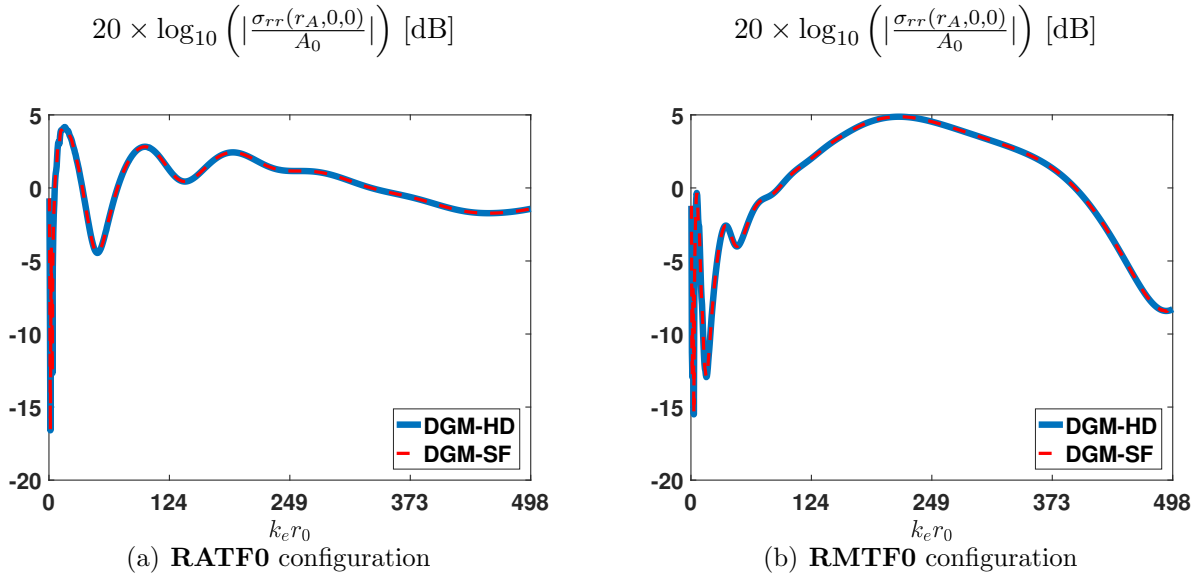


FIG. 12. Validation of DGM-SF assembly procedure for the cases without intermediate fluid layers.

Radial stress at the receiving point for a normal incident plane wave.

512 2. *About the difficulty of taking into account intermediate fluid layers*

513 In this section, one highlights the ability of the DGM-SF assembly procedure to take
 514 into account intermediate fluid layers. As seen earlier on Fig. 5 and Fig. 6, classical
 515 TMM assembly procedure is not well suited to describe intermediate fluid layers, and very
 516 poor correlation can be noted at low frequency regime with the DGM-HD reference model,
 517 either when condensating transfer matrices of solid layers or when introducing low transverse
 518 velocities into intermediate fluid layers.

519 The comparison of DGM-HD and DGM-SF results is proposed in Fig. 13 for **RATD0** and
 520 **RMTD0** configurations which both exhibit an intermediate fluid layer. Again, one can
 521 notice a good agreement between the two approaches.

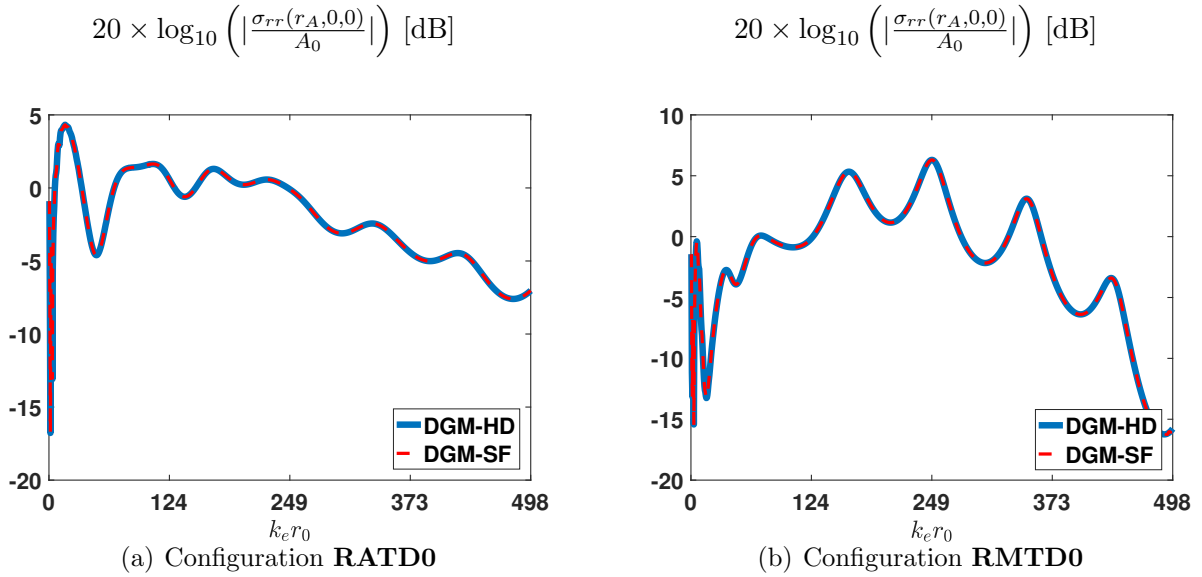


FIG. 13. Validation of DGM-SF assembly procedure for the cases with an intermediate fluid layer.

Radial stress at the receiving point for a normal incident plane wave.

522 **V. ABILITY OF THE PROPOSED DGM-SF PROCEDURE TO MODEL THE**
 523 **ANISOTROPY OF SOLID LAYERS**

524 Before concluding, we illustrate on an application case the interest of the DGM-SF ap-
 525 proach compared to the DGM-HD one. Unlike the DGM-HD method that performs a
 526 Helmholtz decomposition in each elementary solid layer assuming particular material sym-
 527 metries (as isotropic), the proposed DGM-SF procedure integrates the Stroh's first-order
 528 differential system without making any assumptions on the material symmetries (see sec-
 529 tion 2).

530 For underwater applications, the GRP layer that may represent an acoustic dome of a Sonar
 531 flank array and that has been considered in the **RATD0** and **RMTD0** configurations is

532 typically an anisotropic layer. As the GRP material can be manufactured by adding layers
 533 of woven roving and resin, the mechanical properties are different in the fiber plane from
 534 transversely to the plane. For the considered cylindrical GRP layer, one can suppose that
 535 the woven rovings are rolled around a cylinder with different directions of the fibers for each
 536 roving. It results that the fibers are randomly oriented in the plane $(\mathbf{e}_\phi, \mathbf{e}_z)$. The GRP is
 537 thus well represented by a transversely isotropic model, and Young moduli in the directions
 538 of the fibers (i.e. $(\mathbf{e}_\phi, \mathbf{e}_z)$) are higher than the transverse Young modulus (i.e. in the radial
 539 direction \mathbf{e}_r).

540 To illustrate the effect of this anisotropy, one has started from the characteristics of the
 541 supposed isotropic GRP layer defined in Table I and one has modified the Young modulus
 542 in the \mathbf{e}_r direction to define one transverse isotropic configuration as presented in Table II. A
 543 distinction of the material properties has been made between the radial direction numbered
 544 1 and those of the fiber "plane" numbered 2 and 3. The Young's modulus in the radial
 545 direction has been softened by imposing the value of $\frac{E_2}{E_1}$ ratio, where GRP 3/1 stands for
 546 $\frac{E_2}{E_1} = 3$. These characteristics are used to calculate the stiffness constants of the Hooke law
 547 (defined with Eq. (47)) and then the Stroh's matrix with Eq. (5). The integration of the
 548 Stroh's differential system can be achieved as described in the section IID 1 in order to de-
 549 duce the transfer matrix of the transverse isotropic GRP layer. The radial stress responses
 551 at the receiving point have been compared for the **RATD0** and **RMTD0** configurations
 552 considering both isotropic and transverse isotropic GRP cases defined in Table II. Fig. 14
 553 shows the radial stress responses at the receiving point r_A for both **RATD0** and **RMTD0**

TABLE II.

Mechanical properties, isotropic (GRP ISO) versus transverse isotropic case (GRP 3/1).

	GRP ISO	GRP 3/1
E_1 (GPa)	17,7	6,00
$E_2 = E_3$ (GPa)	17,7	17,7
$G_{12} = G_{13}$ (GPa)	7,92	3,00
G_{23} (GPa)	7,92	7,92
$\nu_{12} = \nu_{13}$	0,2	0,2
ν_{23}	0,1	0,1
ρ (kg.m ⁻³)	2000	2000

554 configurations. On top are represented the responses for normal incidence, and below those
 555 for an oblique incidence of (45°) of the acoustic wave.

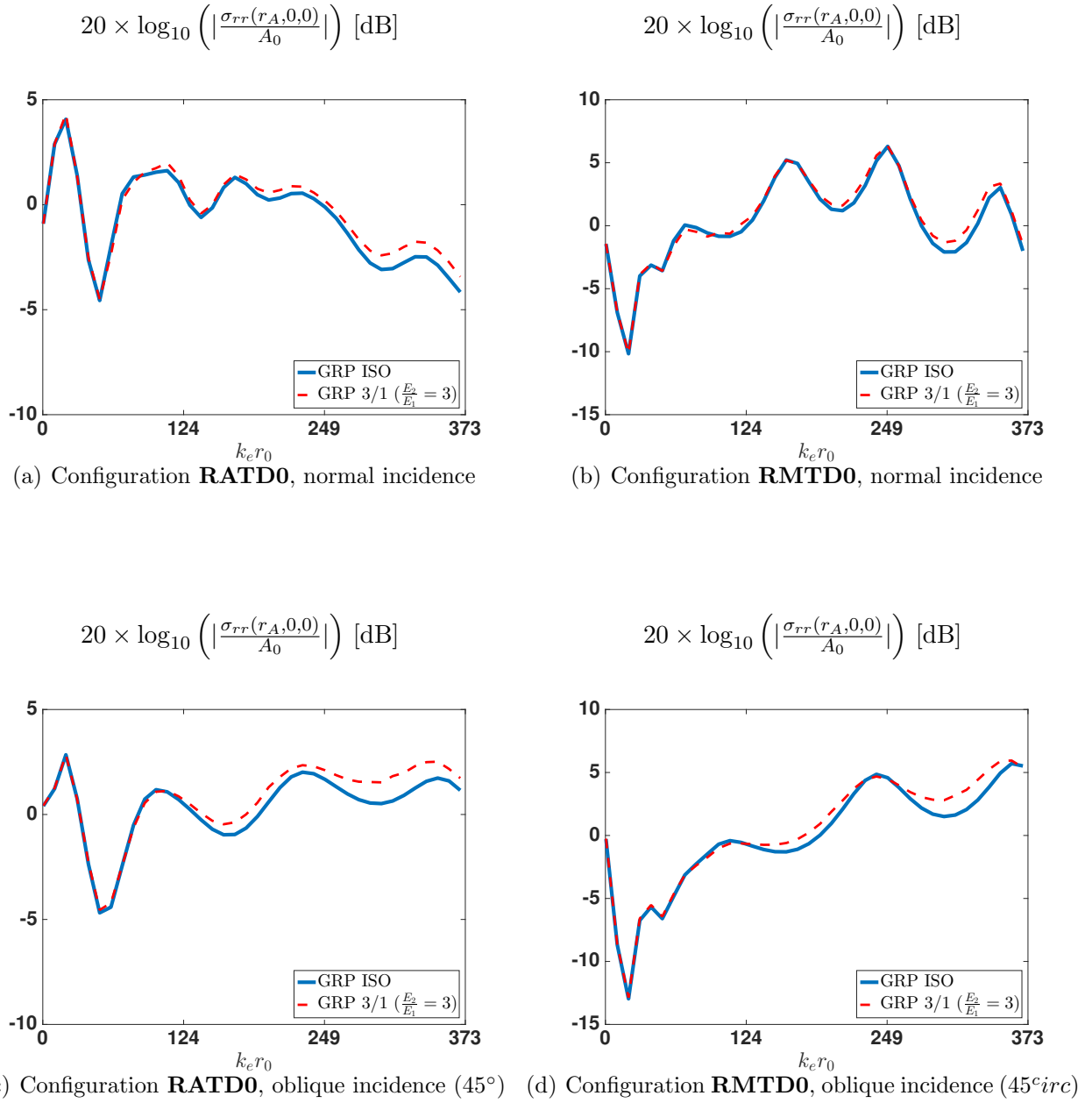


FIG. 14. Influence of the transverse isotropic model for the GRP layer: Radial stress at the receiving point for normal and oblique (45°) incident plane wave.

556 Differences are observed on each excitation case between both GRP models at high fre-
 557 quencies. Larger differences are observed for **RATD0** configuration at both incidences.
 558 For the higher frequencies of the considered frequency range, one can notice that the re-

559 sponses obtained with the transverse anisotropic GRP model have greater levels than with
560 the isotropic GRP model. This is explained by the fact that the Young modulus has been
561 softened in the transverse anisotropic model, increasing the transmission coefficient of the
562 GRP, and increasing the level at the sensor radius.

563 This result allows us illustrating the interest of the DGM-SF approach. It should however
564 not be considered as a general statement. A detailed study should be carried out in the fu-
565 ture to understand the effect of transverse isotropic layer on the response of a multilayered
566 cylindrical shell impacted by a plane wave. It is however outside the scope of the present
567 paper.

568 VI. CONCLUSION

569 A semi-analytical model of a multilayered cylindrical shell immersed in water and excited
570 by an acoustic plane wave has been developed for underwater applications. One has shown
571 that the classical transfer matrix method (TMM) allowing to assembly the transfer matrix of
572 each layer lead to numerical instabilities. These have been highlighted on typical underwater
573 configurations. The first instability is related to the well-known "large fd problem" whereas
574 the second one is induced by the matrix condensation used to take into account intermediate
575 fluid layers. An alternative assembly procedure based on the building of a global matrix
576 has been proposed to overcome these instabilities. It leads to the DGM-SF approach that
577 allows us to describe each solid layer with the Stroh's formalism and to assemble them by a
578 global matrix. It results in an efficient and numerical stable process that has been validated

579 on different underwater configurations. Moreover, the process is well suited to take into
 580 account anisotropic layers such as composite acoustic windows.

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585 **APPENDIX: STROH'S OPERATOR FOR AN ORTHOTROPIC LAYER**

586 For an orthotropic layer, the Hooke's law can be written as:

$$\begin{pmatrix} \sigma_{rr} \\ \sigma_{\phi\phi} \\ \sigma_{zz} \\ \sigma_{\phi z} \\ \sigma_{rz} \\ \sigma_{r\phi} \end{pmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{22} & c_{23} & 0 & 0 & 0 \\ c_{13} & c_{23} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66} \end{bmatrix} \begin{pmatrix} \epsilon_{rr} \\ \epsilon_{\phi\phi} \\ \epsilon_{zz} \\ \gamma_{\phi z} \\ \gamma_{rz} \\ \gamma_{r\phi} \end{pmatrix} \tag{47}$$

587 where c_{ij} are the elastic constants.

588 The Stroh's operator can then be written:

$$[\mathcal{Q}] = \begin{bmatrix} 0 & 0 & -\frac{\partial}{\partial z} & 0 & 0 & \frac{1}{c_{55}} \\ 0 & \frac{1}{r} & -\frac{1}{r} \frac{\partial}{\partial \theta} & 0 & \frac{1}{c_{66}} & 0 \\ -\frac{c_{13}}{c_{11}} \frac{\partial}{\partial z} & -\frac{c_{12}}{c_{11}r} \frac{\partial}{\partial \theta} & -\frac{c_{12}}{c_{11}r} & \frac{1}{c_{11}} & 0 & 0 \\ \frac{e_{23}}{r} \frac{\partial}{\partial z} & \frac{e_{22}}{r^2} \frac{\partial}{\partial \theta} & \frac{e_{22}}{r^2} - \rho\omega^2 \frac{1}{r} \left(\frac{c_{12}}{c_{11}} - 1 \right) & -\frac{1}{r} \frac{\partial}{\partial \theta} & -\frac{\partial}{\partial z} \\ -\frac{1}{r} (c_{44} + e_{23}) \frac{\partial^2}{\partial \theta \partial z} & -\left[\rho\omega^2 + \frac{e_{22}}{r^2} \frac{\partial^2}{\partial \theta^2} + c_{44} \frac{\partial^2}{\partial z^2} \right] & -\frac{e_{22}}{r^2} \frac{\partial}{\partial \theta} & -\frac{c_{12}}{c_{11}r} \frac{\partial}{\partial \theta} & -\frac{2}{r} & 0 \\ -\left[\rho\omega^2 + \frac{c_{44}}{r^2} \frac{\partial^2}{\partial \theta^2} + e_{33} \frac{\partial^2}{\partial z^2} \right] & -\frac{1}{r} (c_{44} + e_{23}) \frac{\partial^2}{\partial \theta \partial z} & -\frac{e_{23}}{r} \frac{\partial}{\partial z} & -\frac{c_{13}}{c_{11}} \frac{\partial}{\partial z} & 0 & -\frac{1}{r} \end{bmatrix} \quad (48)$$

589 where ρ is the mass density of the considered layer and e_{ij} are derived from the elastic

590 constants c_{ij} :

$$e_{22} = \left(c_{22} - \frac{c_{12}^2}{c_{11}} \right), e_{23} = \left(c_{23} - \frac{c_{12}c_{13}}{c_{11}} \right), e_{33} = \left(c_{33} - \frac{c_{13}^2}{c_{11}} \right) \quad (49)$$

591 Applying the Fourier transform defined by Eq. (4) to the Stroh operator gives the Stroh
 592 matrix:

$$\begin{aligned}
 [\mathbf{Q}(r)] = & \begin{bmatrix}
 0 & 0 & -i\alpha & 0 & 0 & 0 & \frac{1}{c_{55}} \\
 0 & \frac{1}{r} & -\frac{in}{r} & 0 & \frac{1}{c_{66}} & 0 & 0 \\
 -i\alpha \frac{c_{13}}{c_{11}} & -\frac{in}{r} \frac{c_{12}}{c_{11}} & -\frac{c_{12}}{c_{11}r} & \frac{1}{c_{11}} & 0 & 0 & 0 \\
 i\alpha \frac{e_{23}}{r} & \frac{in}{r^2} e_{22} & \frac{e_{22}}{r^2} - \rho\omega^2 & \frac{1}{r} \left(\frac{c_{12}}{c_{11}} - 1 \right) & -\frac{in}{r} & -i\alpha & 0 \\
 \alpha \frac{n}{r} (c_{44} + e_{23}) & \frac{n^2}{r^2} e_{22} + \alpha^2 c_{44} - \rho\omega^2 & -\frac{in}{r^2} e_{22} & -\frac{in}{r} \frac{c_{12}}{c_{11}} & -\frac{2}{r} & 0 & 0 \\
 \frac{n^2}{r^2} c_{44} + \alpha^2 e_{33} - \rho\omega^2 & \frac{\alpha n}{r} (c_{44} + e_{23}) & -\frac{i\alpha}{r} e_{23} & -i\alpha \frac{c_{13}}{c_{11}} & 0 & -\frac{1}{r} & 0
 \end{bmatrix} \\
 & (50)
 \end{aligned}$$

593

594

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