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To cite this version:

Runsheng Gu, Lioudmila Vostrikova, Bruno Séjourné. Portfolio optimization of euro-denominated funds in French life insurance. 2020. hal-03025191

HAL Id: hal-03025191

https://hal.science/hal-03025191

Preprint submitted on 26 Nov 2020

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Portfolio optimization of euro-denominated funds in French life insurance

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décembre 2020

Document de travail du GRANEM n° 2020-01-058

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Document de travail du GRANEM n° 2020-01-058
décembre 2020
Classification JEL : G22, C61, G11

Mots-clés : assurance-vie, fonds en euros, optimisation du portefeuille, taux d'intérêt bas, théorie de la ruine.
Keywords: life insurance, portfolio optimization, low interest rates, ruin theory.

Résumé : Dans cet article, nous étudions un problème d'optimisation de portefeuille lié à la gestion d'actifs d'une compagnie d'assurance-vie pour son fonds euro. Dans un environnement persistant de taux bas, les conditions de fonctionnement des activités d'assurance-vie sont modifiées. Pour continuer à offrir une rémunération attractive aux assurés, les assureurs vie devraient introduire davantage d'actifs risqués dans leur portefeuille. Mais, ce faisant, ils s'exposeraient à ne pas pouvoir garantir le capital, ce qui est en contradiction avec les termes du contrat. Par ailleurs, la maturité du marché de l'assurance-vie en France crée des conditions potentielles de retraits massifs. En raison de ces multiples expositions aux risques, nous appliquons des modèles de ruine à ce problème global. Nous déterminons la formulation mathématique des deux premiers moments de la valeur d'une compagnie d'assurance-vie, en fonction de son activité et de sa stratégie d'investissement. Nous résolvons numériquement le problème d'optimisation sous contraintes. Nos résultats permettent de mieux analyser les problèmes de gestion de portefeuille des compagnies. Les stratégies d'allocation d'actifs optimales peuvent varier considérablement pour des changements minimes de certains paramètres de l'activité des assureurs : la probabilité d'insolvabilité, le niveau de capital garanti et le taux de prime.

Abstract: In this paper, we study a portfolio optimization problem related to the asset management of life insurance companies. In a persistent low-interest-rate environment, the conditions under which life insurance business operates are modified. To continue to offer a favorable return to the insured, life insurers should allocate more risky assets to their portfolio. But, doing so, they would be exposed to not being able to guarantee the capital. Besides, the maturity of the life insurance market creates potential conditions for massive withdrawals. We address those risk exposures by applying ruin models. We obtain formulae for the first two moments of the value of a life insurance company, depending on its activity and investment strategy. We show that the optimal asset allocation strategies can differ considerably for small changes in certain parameters of the insurer's business: the probability of insolvency, the level of guaranteed capital, and the premium rate.

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Portfolio optimization of euro-denominated funds in French life insurance

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Abstract

In this paper, we study a portfolio optimization problem related to the asset management of life insurance companies. In a persistent low-interest-rate environment, the conditions under which life insurance business operates are modified. To continue to offer a favorable return to the insured, life insurers should allocate more risky assets to their portfolio. But, doing so, they would be exposed to not being able to guarantee the capital. Besides, the maturity of the life insurance market creates potential conditions for massive withdrawals. We address those risk exposures by applying ruin models. We obtain formulae for the first two moments of the value of a life insurance company, depending on its activity and investment strategy. We show that the optimal asset allocation strategies can differ considerably for small changes in certain parameters of the insurer's business: the probability of insolvency, the level of guaranteed capital, and the premium rate.

Keywords: Life insurance; Portfolio optimization; Low interest rates; Ruin theory
JEL Classification: G22; C61; G11

1. Introduction

The French life insurance market is one of the largest in Europe. The outstanding life insurance contracts amounted to EUR 1.7 trillion at the end of December 2018. In the life insurance business, euro-denominated funds have been the cornerstone of any life insurance contracts in France since the mid-1980s, and they still represent 80% of the outstanding managed assets in life insurance. The capital guarantee and tax advantages provided by the life insurance product in euro-denominated funds are massively sought by households ([Cazenave-lacrouts et al., 2018]). Life insurers used to derive most of their financial income from the investment of premiums received from their policyholders in fixed-rate bonds. Besides these attractive features, the success of euro-denominated funds was due to the relative high-interest rates mainly during 1980-2010, with highly weighted default-risk-free bonds in the portfolios of life insurers.

However, the economic model of euro-denominated life insurance has sometimes experienced a real upheaval because of

· *Very low, sometimes negative, interest rates: For example, the French ten-year bond yield*

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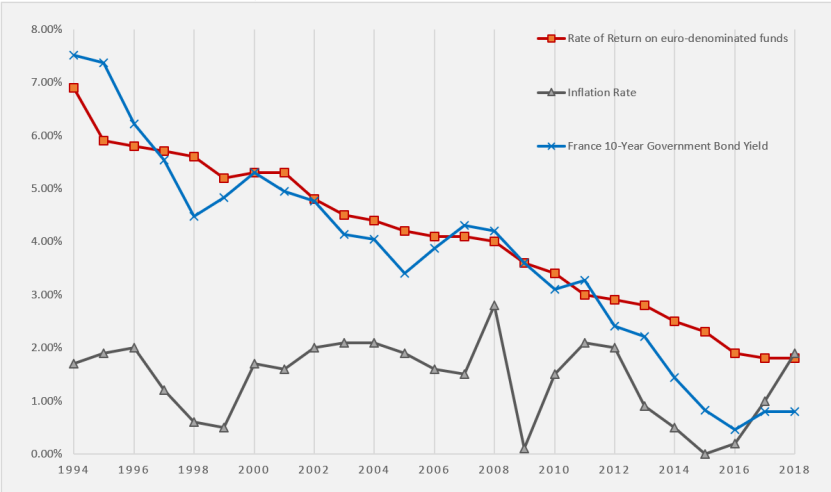
decreased to 1.00 % by 2014 and further dropped to -0.20 % by 2019.
· A rise in outflows and a decrease in inflows: not only because of the declining returns but also due to the age pyramid changes.

Fig 1: France 10-year Government Bonds Yield, and Performance of Life Insurance Products in the euro-denominated funds, 1994-2018.



Source: FFA, ACPR, Bloomberg

As shown in Figure 1, since 1990, there is a general downward trend in the real return of euro-denominated bonds from 7.00% to 5.00% in 2000 and 1.78% in 2018, which means only 1.50% net of social contributions[1]. However, as life insurers invest the bulk of insurance savings in long-term government bonds, such a decrease should not come as a surprise. Indeed, the European central bank's monetary policy decisions addressing the consequences of the 2007-2008 financial crises (both quantitative easing and the cut of the reference rates) caused a drop in the 10-year bond yields, too. This trend had already been observed after introducing the single currency, which stabilized the nominal interest rates at low levels across the Eurozone. As the inflation rate has been stable over the last two decades, around the ECB target of 2.00%, the convergence in nominal interest rates engendered a drop in real ones ([Franks et al., 2018]). Under these circumstances, the return of euro-denominated funds also plunged whenever new assets had to be added into the portfolio for replacing the maturing ones or placing the net inflows. In 2018, 10-year French government bonds yielded less than 1.00%. Moreover, when inflation is considered, the real return rates on euro-denominated funds decrease more rapidly in the past three years. If this level of bond yields persist several years more, the returns on euro-denominated funds will approach zero.

In our contention, a good benchmark to use as "secure savings" for making an informed investment choice is the Passbook A (*Livret A* in French), which is the most popular fi-

[1]The interests generated by single life insurance contracts are subject to social security contributions every year. The rate of deductions applicable is that in force on the date of acquisition of the interest. That was 15.50% in 2018. They are directly deducted by the insurer who transfers them to the tax authorities when the interest is entered into the contract.

nancial product in France. This secure and liquid savings account has a regulated ceiling deposit amount to 22,500 euros. It bears a guaranteed interest rate, which has been decreased from 0.75% in 2016 to 0.50% nowadays and is tax-exempt. Such features reveal the fierce competition that the life insurers must now face when considering the annual return. There is no limit for the amount invested in the life insurance products, and as far as life insurance has tax advantages under the French inheritance law, the French citizens still perceive the euro-denominated funds as good investments. However, the spread between the returns of those two products is diminishing and, according to the trend, is going to approach zero in the foreseeable future. Moreover, the reduction of the return on the euro-denominated funds creates a real risk of outflows to the companies since, at the same time, most contracts have become fiscally free of reinvestment. Such is the case after an 8 year holding period.

To continue offering favorable returns to their clients while guaranteeing their invested capital, the life insurance companies first decided to modify gradualy the structure of their assets in line with the emerging economic model of life insurance in euros. Secondly, they have developed unit-linked contracts in an attempt to limit the flows into euro-denominated life insurance contracts. However, in the first case, by turning to more risky assets, which potentially carry a positive risk premium, they are less likely to be able to guarantee the capital (even if they maintain a liquidity "cushion" as a cash reserve). All these factors now pose a significant risk on the life insurance model, in its euro version. We address the optimal portfolio allocation of the life insurance company among those classes of financial assets, not only on theoretical ground, by applying ruin theory and stochastic process to financial markets, but also from a statistical perspective involving simulations on real data. We contribute to the study of the ruin models with investment by obtaining formulae for the first two moments of the value of a life insurance company and then to find an optimal asset allocation for euro-denominated funds.

The rest of the paper is organized as follows. In Section 2, we analyze the life insurance business in a low-interest-rate environment and review the theoretical literature. In Section 3, we derive the explicit form of the first two moments of the value of a life insurance company with two case studies depending upon whether there is a compound Poisson process in the basic risk process, followed by an analysis. In Section 4, we find the optimal asset allocation strategy under constraints and present a numerical illustration with real data. We then examine, in Section 5, the sensitivity of our results, as well as their regulatory implications. Finally, Section 6 concludes.

2. Literature Review and Background

The phenomenon of a persistent low-interest-rate environment leading to a significant reduction in contract remuneration has already started. If the 10-year French bond remains negative for too long, it will force insurers to invest both collected savings and the proceeds from the sale of maturing bonds into well-rated lower rates bonds. What is more, we notice that the emergence of numerous old contracts will become a completely free reinvestment in other financial products thanks to the tax exemption on the benefits after the holding period of 8 years. Long-term interest rates are the valuation basis for determining premiums, policy reserves, guaranteed rates of return, and profit-sharing. As capital market rates approach the valuation interest rate, life insurers will have a problem: even if their existing

portfolios are invested in assets that yield above the valuation rate, insurers immediately lock in a loss with cash flow from new business reinvestment ([Holsboer, 2000]).

The fact is that the impact of low-interest-rate has been studied since the beginning of this century. As remarked in [Holsboer, 2000]: a milestone was reached when the long-term yield for the ten-year benchmark government bond dipped to under 4.00% for the first time in 40 years in Europe. [Boubel and Séjourné, 2001] deal with the development of European life insurance markets through diversification of life insurance products and delivery networks. Since the mid-1980s, the decline in rates of return on traditional contracts denominated in the national currency and invested in the bond markets caused by the decline in long-term interest rates was noticeable in countries with an inflationary market, which have led insurers to switch into other assets. This fall in long-term nominal rates reflects a steady decline over more than two decades in the long-term risk-free real interest rate, rather than a fall in expected inflation, which has remained broadly stable in [Bean et al., 2015] until recently. [Berdin and Gründl, 2015] showed that a prolonged period of low-interest rate would markedly affect the solvency situation of life insurers, leading to a relatively high cumulative probability of default, especially for less capitalized companies. This has become even more true since then. In a study of the secular determinants of the world's long-term real interest rate, [Rachel and Smith, 2015] attribute about two-thirds of the decline in real-world rates since the 1980s to secular factors that determine the desired saving and investment rates. [Sobrun and Turner, 2016] argue that recent estimates of unobserved concepts, such as the theoretical policy rate, the natural rate, and the long-term rate premium, suggest that the "new normal" world interest rate is lower than before. [Gründl et al., 2016] investigate the extent to which changes in macroeconomic conditions, market developments, and insurance regulation may affect the role of insurers in long-term investment financing. They conclude that regulation should neither unduly favor nor hinder long-term investment as such but place a priority on incentivizing prudent asset-and-liability management with mechanisms that allow for a "true and fair view" of insurers' risk exposures.

[Gollier, 2015] pointed out the possibility of the insurance crisis: should the interest rates start to rise in the euro area, in particular rapidly, insurers would end up with a considerable stock of bonds showing an unrealized loss. Policyholders would be drawn to other products on the market that would be more attractive than today. Moreover, as mentioned above, the long-standing policyholders should also potentially exercise their exit option when the tax advantage is over after 8 years.

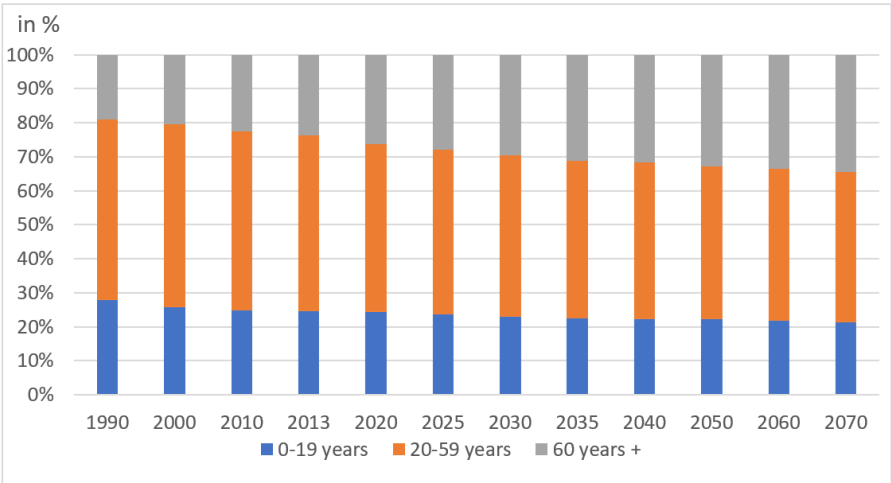
In a low-interest-rate environment, insurers have been changing their investment policy towards higher-paying investments by increasing the proportion of shares (at the cost of raising additional capital), private bond investments, or even real estate in their investments, or by accepting more geographical diversification for example, even if this is accompanied by greater risk-taking. The European Insurance and Occupational Pensions Authority (EIOPA)[2] investigated both a quantitative and qualitative section focusing on the asset side of the balance sheet at a European level. Thirteen French groups participated in this survey. Several trends are identified: a small decrease in the debt portfolio against a small increase in other investments; a trend towards lower credit rating quality fixed income securities with downgrades of a large number of sovereign and corporate bonds; a trend towards more illiquid investments such as non-listed equity and loans excluding mortgages;

[2]Investment behavior report, EIOPA, November 2017

the increased average maturity of the bond portfolio; the tendency to invest into new asset classes such as infrastructure, mortgages, loans and real estate. Natixis Investment Managers commissioned a global survey[3] of 200 Chief Investment Officers (CIOs) at insurers in Europe, North America, and Asia. The survey results reveal three key trends driving investment strategy for insurance CIO teams: Three-quarters of insurers rank interest rates as key portfolio risk. 89 % of insurers globally say regulations deter them from investing in higher-risk assets. Two-thirds of insurers outsource at least some of their portfolio, mainly to gain access to expertise.

It also shows challenges that the life insurance sector will have to solve in the coming years from the holding rate of life insurance contracts by age and the population in France. According to [Cazenave-lacrouts et al., 2018], the holding rate of life insurance contracts increases with age: the holding rate of life insurance contracts by households aged 60 and more (44.30%) is far more significant than those aged under 30 (23.70%) in France. Life insurance is attractive because it not only allows holders to accumulate wealth during their life, for example for retirement purposes, but it can also transmit it in succession under favorable tax conditions. Both would explain why the holding rate increases with age.

Fig 2: Evolution of the population projections of France, from 1990 to 2070.

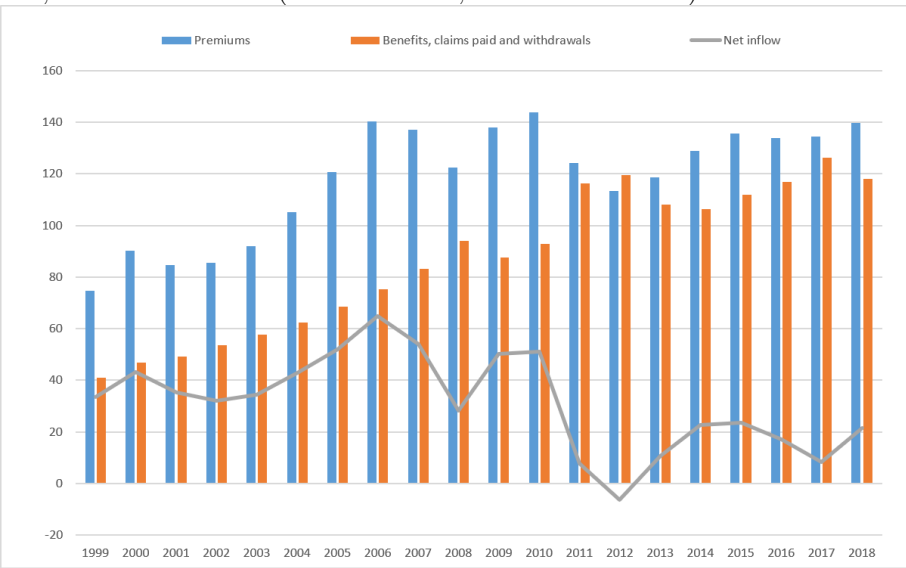


Source: INSEE – Population estimates

From the Population Projections in 2070 ([Blanpain and Buisson, 2016]) in Figure 2, we notice that the proportion of older adults will increase gradually by 2070. Both the retirement of many generations and the higher holding rate of life insurance contracts by those generations indicate the higher probability of redemption of life insurance contracts and more significant outflows from the asset portfolios of life insurance business by older adults based on their needs, in case of financing their retirement.

[3]Insurance survey: Rates, liabilities, and regulation put CIOs between a rock and a hard place, Natixis investment managers, November 2019

Fig 3: Evolution of the premiums, withdrawals and net inflows of life insurance contracts in France, from 1999 to 2018 (direct business, in billions of euros).

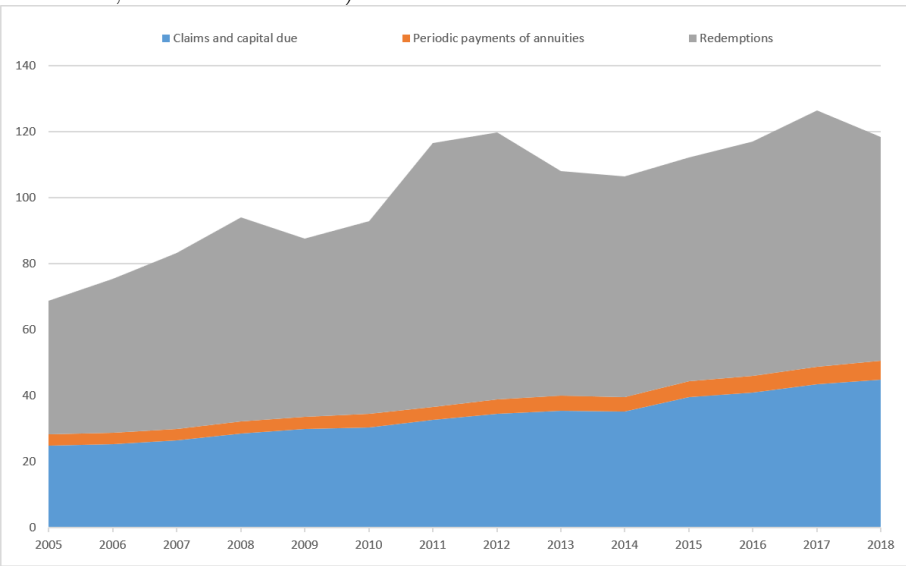


Source: FFA

When observing the economic activity of life insurers in France, the trend for benefits, withdrawals, and claims paid has been increasing in the life insurance sector, along with the decrease in net inflow (Figure 3). At the same time, the premiums remain at around 120 - 140 billion euros.

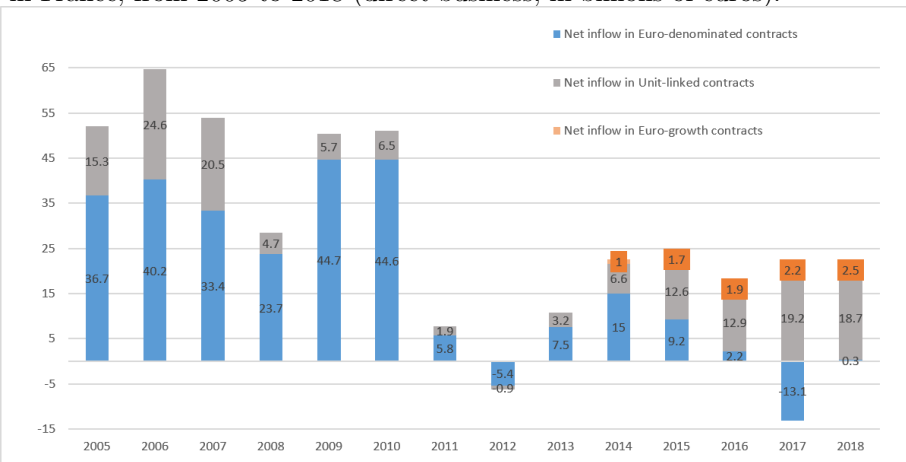
Firstly, we decompose the benefits, claims paid, and withdrawals to analyze the breakdown of withdrawals of life insurance contracts in France. It shows that the decrease in net inflow since 2006 is due to the increase in redemptions, and mainly to the jumps in redemptions during the periods of the 2007-08 financial crisis and 2011-12 public debt crisis, as shown in Figure 4. Secondly, we focus on the net inflows from three different life insurance contracts in Figure 5. The net inflows from the unit-linked contracts have been increasing before the 2007-08 financial crisis and since 2012. This trend is the same as when the stock market is bullish, where we show the stock index of the CAC 40 as an example in Figure 6. The gradual decline in the return on euro-denominated contracts, related to the fall in long-term interest rates, was spreading throughout Europe. The excellent performance of the stock markets allowed unit-linked products to present as an attractive alternative that enabled investors to benefit at least partly (according to the nature of the contract) from the rise in stock market prices while remaining within the attractive taxation framework of life insurance. This phenomenon was noticed for the first time at the end of the 1990s, when the inflow of unit-linked products exceeded the inflow of euro-denominated products ([Boubel and Séjourné, 2001]).

Fig 4: Breakdown of withdrawals of life insurance contracts in France, from 2005 to 2018 (direct business, in billions of euros).



Source: FFA

Fig 5: Life insurance net inflow from euro-denominated, unit-linked and euro-growth contracts in France, from 2005 to 2018 (direct business, in billions of euros).



Source: FFA

The net inflow from euro-denominated contracts has been decreasing since the early 2010s. A partial explanation lies in the issuance of euro-growth contracts in 2014. Since the objective of euro-growth funds is to generate more efficient returns than those of euro-denominated funds, without taking excessive risk in long-term (at least eight years), it

is reasonable that euro-growth contracts absorb one part of the shares in the net inflows from euro-denominated contracts. What is more, the approval of the Pact law [4] for the transformation and growth of businesses, which modified the terms of investment in life insurance contracts, provides for the possibility of transforming part or total of the existing contracts into a euro-growth contract within the same insurer, while retaining the tax precedence of the policyholders. This law will also potentially increase the net inflows into euro-growth contracts in the future. However, these contracts remain largely unknown, and a bigger part of the inflows is invested in traditional unit-linked contracts.

Fig 6: The CAC 40 index, historical data of monthly price.

Source: Bloomberg

Nevertheless, on the brighter side, life insurance remains the preferred financial investment of households after the passbook: 36.50% of metropolitan households own at least one contract ([Cazenave-lacrouts et al., 2018]). Despite the significant drop in yields of euro-denominated funds, some households still seem to favor the security of this investment rather than its yield. And the portfolio management of life insurers remains then the big question.

Next, we review the literature related to the ruin theory we apply to address the optimal asset allocation problem of euro-denominated life insurance. The ruin models with investment income have a long history, going back to [Lundberg, 1903]. In Lundberg's model, the company does not earn any investment income on its capital. [Cramér, 1938] modeled the value of an insurance company using a compound Poisson process with drift and estimated the probability of ruin of insurance companies. In the first half of the 20th century, the stochastic process theory was far less developed, and the first attempt that incorporates investment incomes was undertaken by [Segerdahl, 1942] with the assumption that capital earns interest at a fixed rate r_f, which can be understood as the risk-free rate. Half a cen-

[4] In French, La Loi Pacte: "Loi n. 2019-486 du 22 mai 2019 relative à la croissance et la transformation des entreprises"

tury later, with the inspiration from mathematical finance, [Paulsen, 1993] suggested that capital is allowed to be invested in risky assets, with a risky remuneration R.

Several studies focused on the risk process and the ruin probability of the life insurance company. [Harrison, 1977] considers a generalization of the classical model of collective risk theory. He obtains a general upper bound for the probability of ruin, a general solution for the case where the cumulative income process of an insurance company has no jumps. The results show that if the income process is well approximated by Brownian motion with drift, then the process of the asset is well approximated by a certain diffusion process, which he calls compounding Brownian motion, and the probability of ruin is well approximated by a corresponding first passage probability. [Delbaen and Haezendonck, 1987] give a general description of the classical risk process when macro-economic factors such as interest and inflation are taken into account, and they study the effects of factors on bounds on ruin probabilities. The problem of ruin in a risk model when assets earn investment income is treated in [Paulsen, 1993], [Paulsen, 1998], [Paulsen, 2008] and [Paulsen and Gjessing, 1997]. Their studies cover presentations of the relevant integro-differential equations, exact and numerical solutions, asymptotic results, bounds on the ruin probability, and the possibility of minimizing the ruin probability by investment and possibly reinsurance control. They mainly focus on continuous-time models, but discrete-time models are also considered. [Morales and Schoutens, 2003] present a risk model achieved by incorporating a Lévy process when the aggregate claims and premium fluctuations evolve by jumps. They show how the infinite activity feature of such a family of processes can be used to account for discrete premium fluctuations as well as for semi-heavy tailed claims. [Vostrikova and Spielmann, 2020] study the ruin problem with investment where the business part X is a Lévy process, and the return on investment R is a semi-martingale.

In recent years, several investigations on simulations and empirical analysis have been developed, trying to find the optimal asset allocation for an insurance company. [Wang et al., 2007] study the optimal investment problem for an insurer through the martingale approach. When the insurer's risk process is modeled by a Lévy process and the security market is described by the standard Black-Scholes model, closed-form solutions to the problems of mean-variance efficient investment and constant absolute risk aversion (CARA) utility maximization are obtained. They analyze the effect of the claim process on the mean-variance efficient investment using their explicit solutions. They find that the mean-variance efficient strategies do depend on the claim process. [Brokate et al., 2008] focus on asymptotic tail estimation and apply the numerical methods to find the distribution tail. Determine the optimal investment by maximizing the expected wealth subject to a risk bound given in terms of a Value-at-risk, measuring risk in terms of a high quantile of an appropriate risk process. Their method shows higher accuracy when the risky investment is not too small. [Huang and Lee, 2010] use a multi-asset model to investigate the optimal asset allocation of life insurance reserves and obtain formulae for the first two moments of the accumulated asset value. They provide a new perspective for solving both single-period and multi-period asset allocation problems in application to life insurance policies. [Yu et al., 2010] apply the simulation optimization approach to the multi-period asset allocation problem of property-casualty insurers. They construct a simulation model to simulate operations of a property-casualty insurer and develop multi-phase evolution strategies (MPES) to be used with the simulation model to search for promising asset allocations for the insurer. They find that the re-allocation strategy resulting from MPES

outperforms re-balancing strategies. Their simulation optimization approach to the asset allocation decisions for better investment performance is also applicable to other financial institutions, such as life insurance companies. [Fidan Neslihan et al., 2016] do in-sample and out-of-sample simulations for portfolios of stocks from the Dow Jones, S&P 100, and DAX indices to compare portfolio optimization with the Second-Order Stochastic Dominance (SSD) constraints with mean-variance and minimum variance portfolio optimization. Their results show a superior performance of portfolios with SSD constraints.

3. Modelization

To make the ideas of the life insurance business transparent, we introduce the risk process employing two basic processes following the survey of [Paulsen, 2008], i.e.,
· a basic risk process X with $X_0 = 0$,
· a return on investment generating process R with $R_0 = 0$.

Suppose that the basic risk process of an insurance company can be described by $X = (X_t)_{t\geq 0}$ such that

$$X_t = a_X t + \sigma_X W_t + \sum_{k=1}^{N_t} Z_k. \tag{3.1}$$

with $a_X \neq 0$, $\sigma_X \neq 0$, and $t \geq 0$, where $W = (W_t)_{t\geq 0}$ is a standard Brownian Motion, $N = (N_t)_{t\geq 0}$ is a nonnegative integer-valued Poisson process with mean λ, and $Z_k = (Z_k)_{k\in N}$, a sequence of independent and identically distributed random variables with mean $\mathbb{E}[Z_k] = \beta_Z$ and finite variance $Var(Z_k) = \sigma_Z^2$. Suppose that N_t is independent of the sequence $(Z_k)_{k\in N}$, W, N and $(Z_k)_{k\in N}$ are mutually independent. In this modelization, a_X is the premium rate, λ is the number of withdrawals, β_Z is the average sizes in each withdrawal, σ_Z^2 is the variance of withdrawals, while σ_X represents fluctuations in premium income and maybe also small withdrawals.

Before the mid-1990s, the life insurance business in France has been developing prosperously and smoothly. Households have invested massively in euro-denominated funds. And the withdrawals were not so numerous. In such situation the basic risk process $(X_t)_{t\geq 0}$ is given by

$$X_t = a_X t + \sigma_X W_t.$$

The life insurance company invests the proportion $(1-\gamma)$ in a non-risky asset with an interest rate of $r > 0$ and the proportion $(0 < \gamma < 1)$ in a risky asset with a return of R_t, so that the risk process $(Y_t)_{t\geq 0}$ of this company verifies: for $t > 0$:

$$dY_t = dX_t + (1-\gamma)Y_t r dt + \gamma Y_t dR_t, \tag{3.2}$$

with $Y_0 = y$, corresponding to the initial capital of an insurance company.

Suppose that the risky return $R = (R_t)_{t\geq 0}$ corresponds to a Black-Scholes model: for $t > 0$,

$$dR_t = a_R dt + \sigma_R dB_t,$$

with $R_0 = 0$, where a_R is a constant, describing the drift; $\sigma_R > 0$ is a constant, describing the volatility; $B = (B_t)_{t\geq 0}$ is a standard Brownian Motion independent of $(X_t)_{t\geq 0}$.

Then, the return on investment generating process is

$$R_t^{(\gamma)} = [\gamma a_R + (1-\gamma)r]t + \gamma\sigma_R B_t.$$

To simplify the notations of the investment generating process, let

$$\mu_\gamma = \gamma a_R + (1-\gamma)r, \tag{3.3a}$$
$$\sigma_\gamma = \gamma\sigma_R, \tag{3.3b}$$

Then, we have

$$R_t^{(\gamma)} = \mu_\gamma t + \sigma_\gamma B_t. \tag{3.3c}$$

Therefore, (3.2) is equivalent to:

$$dY_t = dX_t + Y_t dR_t^{(\gamma)}. \tag{3.4}$$

As known, the solution of (3.4) is given by [Paulsen, 1996] (Theorem 11.3):

$$Y_t^{(\gamma)} = \mathcal{E}(R^{(\gamma)})_t \left[y + \int_0^t \frac{dX_s}{\mathcal{E}(R^{(\gamma)})_s}\right],$$

where $\mathcal{E}(R^{(\gamma)})$ is Doléans-Dade's exponential,

$$\mathcal{E}(R^{(\gamma)})_t = exp\left(\mu_\gamma t + \sigma_\gamma B_t - \frac{1}{2}\sigma_\gamma^2 t\right),$$

in the case when $\gamma = 0$,

$$Y_t^{(0)} = exp(rt)\left[y + \int_0^t \frac{dX_s}{exp(rs)}\right],$$

which means a completely risk-free investment.

3.1. Two case studies

We propose two case studies depending upon whether there is a compound Poisson process in the basic risk process, X, to obtain the first two moments of the value of a life insurance company.

Case 1. when both the basic risk process X_t and the return on investment generating process $R_t^{(\gamma)}$ of the insurance company are modeled by two independent Brownian Motions,

$$X_t = a_X t + \sigma_X W_t,$$

and

$$R_t^{(\gamma)} = \mu_\gamma t + \sigma_\gamma B_t;$$

Case 2. when the basic risk process X_t is modeled by the sum of a Brownian Motion and a compound Poisson process, and the return on investment generating process $R_t^{(\gamma)}$ of the insurance company is modeled by a Brownian Motion,

$$X_t = a_X t + \sigma_X W_t + \sum_{k=1}^{N_t} Z_k,$$

and

$$R_t^{(\gamma)} = \mu_\gamma t + \sigma_\gamma B_t.$$

We give the formulae of the first two moments, $\mathbb{E}\left[Y_t^{(\gamma)}(y)\right]$ and $\mathbb{E}\left\{\left[Y_t^{(\gamma)}(y)\right]^2\right\}$, in *Case 1* and *Case 2*, where the proofs of propositions are in **Appendix A** and **Appendix B** respectively.

In *Case 1*: we obtain the following results.

Proposition 1. For $0 < \gamma < 1$ and $t \geq 0$:

$$\mathbb{E}\left[Y_t^{(\gamma)}(y)\right] = \begin{cases} e^{\mu_\gamma t}\left(y + \frac{a_X}{\mu_\gamma}\right) - \frac{a_X}{\mu_\gamma}, & if\ \mu_\gamma \neq 0, \\ y + a_X t, & if\ \mu_\gamma = 0. \end{cases}$$

Proposition 2. For $0 < \gamma < 1$ and $t \geq 0$,
when $\mu_\gamma > 0$:

$$\begin{aligned} &\mathbb{E}\left\{\left[Y_t^{(\gamma)}(y)\right]^2\right\} \\ =&y^2 e^{\left(2\mu_\gamma + \sigma_\gamma^2\right)t} + 2a_X\left(y + \frac{a_X}{\mu_\gamma}\right)\frac{e^{\left(2\mu_\gamma+\sigma_\gamma^2\right)t} - e^{\mu_\gamma t}}{\mu_\gamma + \sigma_\gamma^2} + \left(\sigma_X^2 - \frac{2a_X^2}{\mu_\gamma}\right)\frac{e^{\left(2\mu_\gamma+\sigma_\gamma^2\right)t} - 1}{2\mu_\gamma + \sigma_\gamma^2}; \end{aligned}$$

when $\mu_\gamma = 0$:

$$\mathbb{E}\left\{\left[Y_t^{(\gamma)}(y)\right]^2\right\} = y^2 e^{\sigma_\gamma^2 t} + 2a_X y\frac{e^{\sigma_\gamma^2 t} - 1}{\sigma_\gamma^2} + 2a_X^2\frac{e^{\sigma_\gamma^2 t} - 1 - \sigma_\gamma^2 t}{\sigma_\gamma^4} + \sigma_X^2\frac{e^{\sigma_\gamma^2 t} - 1}{\sigma_\gamma^2},$$

where μ_γ and σ_γ are defined in (3.3a) and (3.3b).

In *Case 2*: we get the following formulas.

Proposition 3. For $0 < \gamma < 1$ and $t \geq 0$:

$$\mathbb{E}\left[Y_t^{(\gamma)}(y)\right] = \begin{cases} e^{\mu_\gamma t}\left(y + \frac{a_X + \lambda\beta_Z}{\mu_\gamma}\right) - \frac{a_X + \lambda\beta_Z}{\mu_\gamma}, & if\ \mu_\gamma \neq 0, \\ y + \left(a_X + \lambda\beta_Z\right)t, & if\ \mu_\gamma = 0. \end{cases}$$

Proposition 4. For $0 < \gamma < 1$ and $t \geq 0$,
when $\mu_\gamma > 0$:

$$\begin{aligned} &\mathbb{E}\left\{\left[Y_t^{(\gamma)}(y)\right]^2\right\} \\ =&y^2 e^{\left(2\mu_\gamma + \sigma_\gamma^2\right)t} + 2a_{\lambda,\beta}\left(y + \frac{a_{\lambda,\beta}}{\mu_\gamma}\right)\frac{e^{\left(2\mu_\gamma+\sigma_\gamma^2\right)t} - e^{\mu_\gamma t}}{\mu_\gamma + \sigma_\gamma^2} + \left(\sigma_{\lambda,\beta}^2 - \frac{2a_{\lambda,\beta}^2}{\mu_\gamma}\right)\frac{e^{\left(2\mu_\gamma+\sigma_\gamma^2\right)t} - 1}{2\mu_\gamma + \sigma_\gamma^2}; \end{aligned}$$

when $\mu_\gamma = 0$:

$$\mathbb{E}\left\{\left[Y_t^{(\gamma)}(y)\right]^2\right\} = y^2 e^{\sigma_\gamma^2 t} + 2a_{\lambda,\beta} y\frac{e^{\sigma_\gamma^2 t} - 1}{\sigma_\gamma^2} + 2a_{\lambda,\beta}^2\frac{e^{\sigma_\gamma^2 t} - 1 - \sigma_\gamma^2 t}{\sigma_\gamma^4} + \sigma_{\lambda,\beta}^2\frac{e^{\sigma_\gamma^2 t} - 1}{\sigma_\gamma^2},$$

where $a_{\lambda,\beta} = a_X + \lambda\beta_Z$, $\sigma^2_{\lambda,\beta} = \sigma^2_X + \lambda\left(\beta^2_Z + \sigma^2_Z\right)$, μ_γ and σ_γ are defined in (3.3a) and (3.3b).
In addition, the variance of the income of the life insurance company can be calculated by:

$$Var\left[Y_t^{(\gamma)}(y)\right] = \mathbb{E}\left\{\left[Y_t^{(\gamma)}(y)\right]^2\right\} - \mathbb{E}\left[Y_t^{(\gamma)}(y)\right]^2.$$

The obtained formulae enable the analysis of portfolio problems and a first approximation of optimal investment strategies for life insurance companies.

3.2. Analysis

By calculating the first-order partial derivatives, we obtain the relationships between each variable and the first two moments of the income of the life insurance company in both cases, in Table 1 and Table 2, respectively. The impact of each variable on the expectation of the value of a life insurance company is shown in Table 1. It shows that in *Case 1*, the expected value of a life insurance company is positively related to the initial capital, the premium rate, and the investment drift when the investment return is nonnegative. In *Case 2*, the basic risk process also has impacts on the expected value of a life insurance company. When both the expected value of the number of claim events and the average size in claim events are positive, there will be a positive impact on the expected value of a life insurance company, indicating that there are net inflows into the life insurance business.

Table 1
The impact of each variable on the expected value of a life insurance company

Variables	**Case 1**		**Case 2**	
	$\mu_\gamma \neq 0$	$\mu_\gamma = 0$	$\mu_\gamma \neq 0$	$\mu_\gamma = 0$
the initial capital, y	positive	positive	positive	positive
the premium rate, a_X	positive	positive	positive	positive
fluctuations in premium income, σ_X	none	none	none	none
the investment return, μ_γ	positive	none	positive	none
the investment volatility, σ_γ	none	none	none	none
the expected value of the number of claim events, λ	none	none	both are positive when $\beta_Z > 0$	
the average size in claim events, β_Z	none	none	positive	positive
the variance of claim sizes, σ_Z	none	none	none	none

When the investment return is positive, the impact on the second moment of the value of the life insurance company is determined by a mixture of the premium rate, the investment return, the investment volatility, the expected value of the number of claim events, and the average size in claim events. The complex impact of the premium rate on the second moment can be explained as when the premium rate increases, the expectation of the income of the insurance company increases, and the uncertainty of the second moment of the income of the insurance company also increases. However, it is not clear whether the second moment increases, decreases, or remain unchanged. The optimal premium rate may not be the case when the premium rate is the highest. In other words, increasing the premium rate does not necessarily lead to the optimal Sharpe ratio.

Compared with *Case 1*, the difference in *Case 2* is that the compound Poisson process also impacts the income of the life insurance company through the expected value of the number of claimed events, the mean of claim sizes, and the variance of claim sizes.

Table 2
The impact of each variable on the second moment of the value of a life insurance company

Variables	**Case 1**		**Case 2**	
	$\mu_\gamma > 0$	$\mu_\gamma = 0$	$\mu_\gamma > 0$	$\mu_\gamma = 0$
the initial capital, y	positive	positive	positive	positive
the premium rate, a_X	complex	positive	complex	positive
fluctuations in premium income, σ_X	positive	positive	positive	positive
the investment return, μ_γ	complex	none	complex	none
the investment volatility, σ_γ	complex	positive	complex	positive
the expected value of the number of claim events, λ	none	none	complex	positive
the average size in claim events, β_Z	none	none	complex	positive
the variance of claim sizes, σ_Z	none	none	positive	positive

3.3. Modelization with multiple risky assets

Suppose that there are multiple risky assets in the investment. Let $n \in \mathbb{N}^*$ the number of risky assets modeled by the Brownian Motion with drifts:

$$R_t^{(i)} = a_R^{(i)} t + \sigma_R^{(i)} B_t^{(i)},\ 1 \leq i \leq n,$$

with dependent Brownian Motions $\left(B_t^{(i)}\right)_{t\geq 0}$.

We denote by $\mathbb{C}$ the covariance matrix of n-dimensional Brownian Motions $\left(\overrightarrow{B_t}\right)_{t\geq 0}$, $\overrightarrow{B_t} = \left(B_t^{(1)}, B_t^{(2)}, ..., B_t^{(n)}\right)^T$,

$$\mathbb{C} = (c_{i,j})_{\substack{1\leq i\leq n\\ 1\leq j\leq n}} = \begin{bmatrix} c_{11}, & \cdots, & c_{1j}, & \cdots, & c_{1n} \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ c_{i1}, & \vdots, & c_{ij}, & \vdots, & c_{in} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{n1}, & \cdots, & c_{nj}, & \cdots, & c_{nn} \end{bmatrix},$$

with $c_{i,i} = Var\left(B_1^{(i)}\right) = \left(\sigma^{(i)}\right)^2$, $c_{i,j} = cov\left(B_1^{(i)}, B_1^{(j)}\right)$.

In this case $Y_t^{(\gamma)}$ verifies the equation:

$$dY_t^{(\gamma)} = dX_t + Y_t^{(\gamma)}(1-\gamma)rdt + Y_t^{(\gamma)} \sum_{i=1}^{n} \gamma_i dR_t^{(i)}, \tag{3.5}$$

where $\gamma = \gamma_1 + \gamma_2 + ... + \gamma_n$ and $(\gamma_i)_{1\leq i\leq n}$ are the proportions of the investment in the i-th risky asset respectively.

From the properties of Brownian Motions, $\left(\sum_{i=1}^{n} \gamma_i dR_t^{(i)}\right)_{t\geq 0}$ is a Brownian Motion with the drift:

$$a_R^{(\gamma)} = \sum_{i=1}^{n} \gamma_i a_R^{(i)},$$

and the variation:

$$\left(\sigma_R^{(\gamma)}\right)^2 = \langle \mathbb{C}\overrightarrow{\gamma}\overrightarrow{\gamma}\rangle = \sum_{i=1}^{n}\sum_{j=1}^{n}\gamma_i\gamma_j c_{i,j} = \sum_{i=1}^{n}\sum_{j=1}^{n}\gamma_i\gamma_j\sigma_R^{(i)}\sigma_R^{(j)}\rho_{i,j}, \tag{3.6}$$

where $\overrightarrow{\gamma} = (\gamma_1, \gamma_2, ..., \gamma_n)^T$.

Then (3.5) is equivalent to:

$$dY_t^{(\gamma)} = dX_t + Y_t^{(\gamma)} dR_t^{(\gamma)},$$

where

$$R_t^{(\gamma)} = \left[(1-\gamma)r + a_R^{(\gamma)}\right] t + \sigma_R^{(\gamma)} B_t,$$

and $(B_t)_{t\geq 0}$ is a *new Brownian Motion* obtained by linear combination of the previous Brownian Motions $B^{(i)}$, $1 \leq i \leq n$. More precisely,

$$\left(\sum_{i=1}^{n}\gamma_i\sigma_R^{(i)}B_t^{(i)}\right)_{t\geq 0} \overset{\mathcal{L}}{=} \left(\sigma_R^{(\gamma)}B_t\right)_{t\geq 0}.$$

To simplify the notations, let

$$\mu_\gamma = (1-\gamma)r + a_R^{(\gamma)}, \tag{3.7a}$$

$$\sigma_\gamma = \sigma_R^{(\gamma)}, \tag{3.7b}$$

Then:

$$R_t^{(\gamma)} = \mu_\gamma t + \sigma_\gamma B_t. \tag{3.7c}$$

Finally, by the same reasoning as the proofs of *Proposition 1* and *Proposition 2* in *Case 1* and the same reasoning as the proofs of *Proposition 3* and *Proposition 4* in *Case 2*, we can get the expressions for the first two moments of the value of a life insurance company when there are multiple risky assets. The expressions in the model with multiple risky assets are the same as when there is one risky asset, except that μ_γ and σ_γ are defined in (3.7a) and (3.7b), respectively.

4. Optimal asset allocation

4.1. Practical problem

In order to illustrate the portfolio optimization problem, we choose to consider the four main categories of assets included in the French life insurance balance sheets : government bonds, corporate bonds, stocks and real estate. The goal of the optimal asset allocation is to maximize μ_γ subjected to the condition that the probability that the result of the exercise at the end of the period T is less than saving capital must be less or equal to α, i.e.,

$$P\left(Y_T^{(\gamma)} \leq -c\right) \leq \alpha,$$

where c is a saving capital and α is the probability of insolvency, $\alpha = 1\%$ or 5%.
We have

$$\mu_\gamma = \mu_1\gamma_1 + \mu_2\gamma_2 + \mu_3\gamma_3 + \mu_4\gamma_4,$$

where μ_1, μ_2, μ_3, μ_4 are the returns for the assets number $1, 2, 3, 4$, and γ_1, γ_2, γ_3, γ_4 are the proportions of each asset, respectively,

$$\sum_{i=1}^{4} \gamma_i = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 \leq 1,$$

$\gamma_1 \geq 0$, $\gamma_2 \geq 0$, $\gamma_3 \geq 0$, $\gamma_4 \geq 0$. From (3.6), we also have:

$$\sigma_\gamma^2 = \sum_{i=1}^{4}\sum_{j=1}^{4} \gamma_i\gamma_j\sigma_i\sigma_j\rho_{i,j},$$

where σ_i^2, $i = 1, 2, 3, 4$, are the variations of assets and $\rho_{i,j}$ are the correlation coefficients between assets i and j, $i, j \in \{1, 2, 3, 4\}$.
The objective is to find:

$$\max_{\gamma \in J} \mathbb{E}\left(Y_t^{(\gamma)}\right),$$

subject to the condition

$$P\left(-Y_T^{(\gamma)} \geq c\right) \leq \alpha,$$

where $J = \left\{(\gamma_1, \gamma_2, \gamma_3, \gamma_4) \middle| \sum_{i=1}^{4} \gamma_i \leq 1,\ \gamma_i \leq \gamma_{\max}^{(i)}\right\}$.
Since the calculation of this probability is very complicated, we use upper estimation of this probability, namely

$$P\left(-Y_T^{(\gamma)} \geq c\right) \leq \frac{\mathbb{E}\left[\left(Y_T^{(\gamma)}\right)^2\right]}{c^2}.$$

We recall that the case with $\mu_\gamma > 0$ and $a_X \geq 0$ is more common, so that up to now, we consider this situation. Moreover, the initial capital, y, can be put into the saving capital. We take the formulas corresponding to $y = 0$ and $\mu_\gamma > 0$.
For *Case 1*, we get:

$$\mathbb{E}\left(Y_T^{(\gamma)}\right) = \frac{a_X}{\mu_\gamma}\left(e^{\mu_\gamma T} - 1\right),$$

$$\mathbb{E}\left[\left(Y_T^{(\gamma)}\right)^2\right] = \frac{2a_X^2}{\mu_\gamma}\frac{e^{\left(2\mu_\gamma + \sigma_\gamma^2\right)T} - e^{\mu_\gamma T}}{\mu_\gamma + \sigma_\gamma^2} + \left(\sigma_X^2 - \frac{2a_X^2}{\mu_\gamma}\right)\frac{e^{\left(2\mu_\gamma + \sigma_\gamma^2\right)T} - 1}{2\mu_\gamma + \sigma_\gamma^2}.$$

If we consider *Case 2*, a_X and σ_X^2 should be simply replaced by $a_X + \lambda\beta_Z$ and $\sigma_X^2 + \lambda(\beta_Z^2 + \sigma_Z^2)$ with λ the intensity of the Poisson process and β_Z, σ_Z^2 being the mean and the variance of the independent and identically distributed random variance $(Z_k)_{k\in N}$. So, for the maximization problem, we need to consider only the first case for the modeling of X_t

because other cases can be obtained by changing the parameters.
We introduce the function

$$f(x) = \frac{e^x - 1}{x}$$

and

$$g(x) = \frac{e^x}{x}.$$

Then, given c, α, T, a_X, σ_X^2, $\gamma_{\max}^{(1)}$, $\gamma_{\max}^{(2)}$, $\gamma_{\max}^{(3)}$, $\gamma_{\max}^{(4)}$ and r, we solve numerically the maximization problem to find

$$\max_{\gamma \in J} \left[a_X T \cdot f(\mu_\gamma T) \right],$$

subject to the condition:

$$2a_X^2 T^2 \cdot g(\mu_\gamma T) \cdot f\left[(\mu_\gamma + \sigma_\gamma^2)T\right] + (\sigma_X^2 - \frac{2a_X^2}{\mu_\gamma})T \cdot f\left[(2\mu_\gamma + \sigma_\gamma^2)T\right] \leq \alpha \cdot c^2.$$

When we find $\vec{\gamma} = (\gamma_1^*, \gamma_2^*, \gamma_3^*, \gamma_4^*)$ which gives the optimal allocation of the maximization and the corresponding μ_* and σ_*^2, the maximum of the expectation will be equal to $\frac{a_X}{\mu_*}\left(e^{\mu_* T} - 1\right)$, and the mean interest rate for the unit period of time will be equal to $\frac{1}{T}\ln\left[\frac{a_X}{\mu_*}\left(e^{\mu_* T} - 1\right)\right]$.

4.2. Numerical illustration

To limit the impact of both the market risks and the liquidity risk, the French Insurance Code specifies certain constraints on the structure of the asset portfolio of life insurance companies. The main limits of the composition of euro-denominated funds in terms of investment in percentage are (maximum investment ratios by asset class):
· 100% for bonds and bond funds;
· 65% for equities and equity funds;
· 40% for real estate.

Then, we have $\gamma_{\max}^{(1)} = \gamma_{\max}^{(2)} = 100\%$, $\gamma_{\max}^{(3)} = 65\%$ and $\gamma_{\max}^{(4)} = 40\%$.

For the parameters in the basic risk process, we obtain a premium of 85.7 billion euros and the redemption of 53.6 billion euros, from the ACPR 2018 Insurance Market Figures.

The datasets used for the return on investment generating process include the France 10-year Government Bond (1, government bond), the Bloomberg Barclays Euro Aggregate Corporate Total Return Index Value Unhedged EU (2, corporate bond), the EURO STOXX 50 Index (3, stock) and the Euronext IEIF REIT Europe Index (4, real estate) from December 31, 2018, to December 31, 2019, obtained from Bloomberg terminal. The government bond is the proxy for the risk-free asset since it is well rated by agencies - AA for S&P and Fitch, and the corporate bond, stock and real estate are used as risky assets. The expected return and volatility are approximated by the annualized average return and standard deviation of the daily returns in the period, respectively.

The following set of parameters has been used in the numerical solution of the maximization problem. The basic risk process X_t:

$$a_X = 85.7, \text{ and } \sigma_X = 53.6.$$

The return on investment generating process R_t:

$$\mu_1 = 0.1263\%, \mu_2 = 6.2396\%, \mu_3 = 24.7671\% \text{ and } \mu_4 = 29.0565\%,$$
$$\sigma_1 = 0.0199\%, \sigma_2 = 1.8270\%, \sigma_3 = 13.1638\% \text{ and } \sigma_4 = 13.4955\%,$$

$$(\rho_{i,j})_{\substack{1\le i\le 4\\ 1\le j\le 4}} = \begin{bmatrix} 1 & 0.0935 & 0.0467 & 0.0109 \\ 0.0935 & 1 & -0.0056 & 0.0251 \\ 0.0467 & -0.0056 & 1 & 0.4669 \\ 0.0109 & 0.0251 & 0.4669 & 1 \end{bmatrix}.$$

We set the period, $T = 1$, the probability of insolvency, $\alpha = 0.01$ and the saving capital, $c = 1050$.

Given all the parameters above, we find the optimal allocation of the maximization,

$$\vec{\gamma}^* = (0.0979, 0.8115, 0.0407, 0.0499),$$

and the corresponding $\mu_* = 3.79\%$ and $\sigma_*^2 = 3.53\%$.

5. Sensitivity analysis, discussion and implications

The optimal asset allocations depend on the selection of a range of parameters. Then, we examine the sensitivity of the optimal investment strategy to some parameters.

5.1. Sensitivity of the optimal asset allocation to different constraints

In Table 3, we show the optimal asset allocation for $\alpha = 0.0095$, 0.01, and 0.0105. We also explore the optimal asset allocation for $c = 1000, 1050$, and 1100 in Table 4. Both the coefficient α and c can be viewed as the risk tolerance measure of the life insurance company, such that as α or c increases, the insurer's risk-averse attitude decreases, and the investment strategy becomes more aggressive. Small variations in α and c would lead to huge shifts in the portfolio.

Table 3
Optimal asset allocation of the portfolio under different probabilities of insolvency

α	**Government bonds**	**Corporate bonds**	**Stocks**	**Real estate**
0.0095	0.7172	0.2532	0.0139	0.0155
0.01	0.0979	0.8115	0.0407	0.0499
0.0105	0.0000	0.7140	0.1141	0.1719

5.2. Sensitivity of the optimal asset allocation to the parameters of the premium rate

We investigate the sensitivity of the optimal asset allocation to the parameter, a_X, the premium rate in Table 5. The results show that as a_X increases, the investment strategy becomes more conservative. In other words, the investment will be more conservative to meet the constraints when increasing the premium without enhancing the saving capital.

Table 4
Optimal asset allocation of the portfolio under different saving capitals

c	**Government bonds**	**Corporate bonds**	**Stocks**	**Real estate**
1000	1.0000	0.0000	0.0000	0.0000
1050	0.0979	0.8115	0.0407	0.0499
1100	0.0000	0.5153	0.1882	0.2965

Table 5
Optimal asset allocation of the portfolio under different values of the premium rates

a_X	**Government bonds**	**Corporate bonds**	**Stocks**	**Real estate**
80	0.0000	0.4981	0.1946	0.3073
85.7	0.0979	0.8115	0.0407	0.0499
90	0.9722	0.0193	0.0041	0.0044

5.3. Discussion and implications

1. The reason why we choose the value of 1050 for the saving capital, c, is to make the calculation results of different cases in the sensitivity analysis comparable, reduce the occurrence of 0 proportion in the allocation.
2. The allocation of classic euro-denominated funds of French companies at the end of 2019 is 32.50% in government bonds, 47.50% in corporate bonds, 8.60% in stocks, and 7.20% in real estate[5]. There is a large difference in the allocation of bonds when we compare this allocation to the theoretical optimal allocation that we found. There are several explanations for this. Firstly, our database includes four assets. In reality, there are thousands of assets available for asset allocation by life insurers. There is a higher probability that the result of our model will be affected by the performance of one asset. Secondly, our model looks for asset allocation that maximizes the return on investment subject to the constraint. Among the four assets, the performance of the corporate bond index best meets the requirements of the model, 81.15% is allocated to the corporate bond index. Thirdly, in reality, the portfolio is more stable and does not change much compared with the previous year. There are neither transaction costs nor rebalancing costs in our model since we do not consider the initial portfolio. Moreover, it is now in an environment of ever-decreasing interest rates, and the historical interest rate of the government bonds is higher, which offers higher returns for older bonds still held by the French life insurance companies. Then, the allocation ratio to the government bond is higher in reality (32.50%) than in our model (9.79%).

Solvency II includes a solvency capital requirement (SCR) coverage ratio of more than 200% and a minimum capital requirement (MCR) coverage ratio of more than 550% under life and mixed insurers, providing ample protection against the risks to which they are exposed.
3. A minimal change in the probability of insolvency and the saving capital will make a larger difference in the optimal asset allocation. The results shown are in line with the requirements of the Solvency II framework.
4. Keeping other variables unchanged, only the increase in the net inflow of investable funds following the increase in the premium rate will lead to a more conservative investment

[5]Newsletter 42: Composition moyenne des fonds en euros à fin 2019, Good Value for Money

strategy to make the portfolio meet the optimal return-risk conditions. If the premium rate rises, the corresponding increase in the saving capital will enable the portfolio to maintain the original return-risk performance or even better performance.

6. Conclusion

In this paper, we study a portfolio optimization problem related to the management of life insurance business – euro-denominated funds. In a persistently very low-interest-rates environment, the euro-denominated life insurance business faces multiple risks: diminishing returns on euro-denominated funds, increasing risk exposures in asset allocations, and unpredicted dramatic redemptions of the contracts. We apply a risk model based on this background.

Then, we obtain the expressions on the first two moments of the income of a life insurance company with investments in two cases:
1. when both the basic risk process X_t and the return on investment generating process $R_t^{(\gamma)}$ of the insurance company are modeled by two independent Brownian Motions;
2. when the basic risk process X_t is modeled by the sum of a Brownian Motion and a compound Poisson process, and the return on investment generating process $R_t^{(\gamma)}$ of the insurance company is modeled by a Brownian Motion.

By calculating the first-order partial derivatives, we show the relationships between each variable and the first two moments of the income of a life insurance company in two cases. We investigate the optimal asset allocation on the market basis with numerical methods. By demonstrating one example of data in 2019, we conduct an application to one case analysis, with the numerical illustration and the sensitivity analysis of this strategy.

The result verifies the assumptions and analysis of the modelization. The impact of each variable on the optimal asset allocation is analyzed, which is in line with the requirements of the Solvency II framework. The result also shows certain implications for the industry regulation and the insurance business and investment business of life insurance companies.

We conclude that the optimal asset allocation of the investment of the life insurance company depends on both the basic risk process X_t and the return on investment generating process $R_t^{(\gamma)}$.

Acknowledgement

The authors would like to acknowledge financial support from the PANORisk project of the "Région Pays de la Loire".

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Appendix A: Case 1

A.1. Proof of Proposition 1

Proof. First, we calculate conditional expectation of Y_t conditionally to $\mathcal{E}(R^{(\gamma)})_t = h_t$. Namely, we show that:

$$\mathbb{E}\left[Y_t^{(\gamma)}(y)\middle|\mathcal{E}\left(R^{(\gamma)}\right)_{0\le s\le t} = (h_s)_{0\le s\le t}\right] = h_t\left(y + a_X\int_0^t \frac{ds}{h_s}\right). \tag{A.1}$$

Since W and B are independent Brownian Motions, and hence, $R^{(\gamma)}$ and X are independent,

$$\mathbb{E}\left[Y_t^{(\gamma)}(y)\middle|\mathcal{E}\left(R^{(\gamma)}\right)_{0\le s\le t} = (h_s)_{0\le s\le t}\right] = h_t\left(y + \mathbb{E}\int_0^t \frac{dX_s}{h_s}\right). \tag{A.2}$$

Taking into account *Case 1*:

$$\int_0^t \frac{dX_s}{h_s} = a_X\int_0^t \frac{ds}{h_s} + \sigma_X\int_0^t \frac{dW_s}{h_s}. \tag{A.3}$$

We note that $(\int_0^t \frac{dW_s}{h_s})$ is a local martingale, and by localisation procedure, we show that:

$$\mathbb{E}\left(\int_0^t \frac{dW_s}{h_s}\right) = 0. \tag{A.4}$$

From (A.2), (A.2) and (A.3) we get (A.1).
We know from the properties of conditional expectation that:

$$\mathbb{E}\left[Y_t^{(\gamma)}(y)\right] = \mathbb{E}\left\{\mathbb{E}\left[Y_t^{(\gamma)}(y)\middle|\mathcal{E}\left(R^{(\gamma)}\right)_{0\le s\le t} = (h_s)_{0\le s\le t}\right]\right\}.$$

So, we have:

$$\begin{aligned}\mathbb{E}\left[Y_t^{(\gamma)}(y)\right] &= \mathbb{E}\left[\mathcal{E}(R^{(\gamma)})_t\left(y + a_X\int_0^t \frac{ds}{\mathcal{E}(R^{(\gamma)})_s}\right)\right]\\ &= y\,\mathbb{E}\left[\mathcal{E}(R^{(\gamma)})_t\right] + a_X\,\mathbb{E}\left[\int_0^t \frac{\mathcal{E}(R^{(\gamma)})_t ds}{\mathcal{E}(R^{(\gamma)})_s}\right].\end{aligned} \tag{A.5}$$

Now, for the first term on the right-hand part of (A.5)

$$\mathbb{E}\left[\mathcal{E}(R^{(\gamma)})_t\right] = \mathbb{E}\left[exp\left(\mu_\gamma t + \sigma_\gamma B_t - \frac{1}{2}\sigma_\gamma^2 t\right)\right] = exp\left(\mu_\gamma t\right), \tag{A.6}$$

since $\mathbb{E}\left[exp\left(\sigma_\gamma B_t - \frac{1}{2}\sigma_\gamma^2\right)\right] = 1$.
For the second part of (A.5) we get:

$$\begin{aligned}&\mathbb{E}\left[\int_0^t \frac{\mathcal{E}(R^{(\gamma)})_t ds}{\mathcal{E}(R^{(\gamma)})_s}\right]\\ =&\mathbb{E}\left\{\int_0^t exp\left[(\mu_\gamma(t-s) + \sigma_\gamma(B_t - B_s) - \frac{1}{2}\sigma_\gamma^2(t-s)\right]\right\}.\end{aligned}$$

It is known that: $B_t - B_s \overset{\mathcal{L}}{=} B_{t-s}$, so that doing time change $u = t - s$ we have:

$$\mathbb{E}\left[\int_0^t \frac{\mathcal{E}(R^{(\gamma)})_t ds}{\mathcal{E}(R^{(\gamma)})_s}\right] = \mathbb{E}\left[\int_0^t exp\left(\mu_\gamma u + \sigma_\gamma B_u - \frac{1}{2}\sigma_\gamma^2 u\right) du\right].$$

Using Fubini Theorem, we exchange the expectation and integration over $[0, t]$ and since

$$\mathbb{E}\left[exp\left(\mu_\gamma u + \sigma_\gamma B_u - \frac{1}{2}\sigma_\gamma^2 u\right)\right] = exp\left(\mu_\gamma u\right),$$

we get:

$$\mathbb{E}\left[\int_0^t \frac{\mathcal{E}(R^{(\gamma)})_t ds}{\mathcal{E}(R^{(\gamma)})_s}\right] = \int_0^t exp(\mu_\gamma u) du.$$

During financial crisis or even more extreme market conditions, for instance, when $a_R \leq 0 < r$, there will be one allocation that $\mu_\gamma = \gamma a_R + (1-\gamma) r = 0$. Then there should be two expressions for $\mathbb{E}\left[\int_0^t \frac{\mathcal{E}(R^{(\gamma)})_t ds}{\mathcal{E}(R^{(\gamma)})_s}\right]$:

$$\mathbb{E}\left[\int_0^t \frac{\mathcal{E}(R^{(\gamma)})_t ds}{\mathcal{E}(R^{(\gamma)})_s}\right] = \begin{cases} \left.\frac{e^{\mu_\gamma u}}{\mu_\gamma}\right|_0^t = \frac{e^{\mu_\gamma t}-1}{\mu_\gamma}, & if\ \mu_\gamma \neq 0, \\ t, & if\ \mu_\gamma = 0. \end{cases} \tag{A.7}$$

Finally, from (A.5), (A.6) and (A.7):

$$\mathbb{E}\left[Y_t^{(\gamma)}(y)\right] = \begin{cases} e^{\mu_\gamma t}\left(y + \frac{a_X}{\mu_\gamma}\right) - \frac{a_X}{\mu_\gamma}, & if\ \mu_\gamma \neq 0, \\ y + a_X t, & if\ \mu_\gamma = 0. \end{cases} \tag{A.8}$$

□

A.2. Proof of Proposition 2

Before proving Proposition 2, we propose and prove Lemma 1:

Lemma 1. Suppose that $(g_t)_{t\geq 0}$ is a deterministic continuous function verifying the integral equation

$$g_t = g_0 + f_t + k\int_0^t g_s ds, \tag{A.9}$$

with continuous differentiable deterministic function $(f_t)_{t\geq 0}$. Then

$$g_t = e^{kt}\left(g_0 + \int_0^t e^{-ks} f_s^{'} ds\right). \tag{A.10}$$

Proof. The relation in (A.9) is equivalent to:

$$g_t^{'} = f_t^{'} + k g_t,\ g_0 = 0. \tag{A.11}$$

Differentiating (A.10) we get (A.11). And since the solution of the equation is unique, the result claimed follows. □

Proof. To calculate $\mathbb{E}\left\{\left[Y_t^{(\gamma)}(y)\right]^2\right\}$, we use the Itô formula. To avoid the complicated notations, we omit for a moment the index (γ). From (3.2), (3.3c), (3.4) and *Case 1* it follows that:

$$dY_t = a_X dt + \sigma_X dW_t + Y_t \mu_\gamma dt + Y_t \sigma_\gamma dB_t.$$

The Ito formula with the function $f(x) = x^2$ gives:

$$Y_t^2 = Y_0^2 + 2\int_0^t Y_s dY_s + \frac{1}{2}\int_0^t 2d\langle Y^c\rangle_s,$$

where Y^c is the continuous martingale part of the process Y, and $\langle Y^c\rangle$ is predictable quadratic variation of Y^c. Since $Y_0^2 = y^2$ and (A.11) we get:

$$\begin{aligned} Y_t^2 =& y^2 + 2a_X \int_0^t Y_s ds + 2\sigma_X \int_0^t Y_s dW_s + 2\mu_\gamma \int_0^t Y_s^2 ds \\ &+ 2\sigma_\gamma \int_0^t Y_s^2 dB_s + \langle Y^c\rangle_t. \end{aligned} \tag{A.12}$$

Since $\langle Y^c\rangle_t = \langle \sigma_X \int_0^t dW_s + \sigma_\gamma \int_0^t Y_s dB_s\rangle$, with independent W and B, we get

$$\langle Y^c\rangle_t = \sigma_X^2 t + \sigma_\gamma^2 \int_0^t Y_s^2 ds. \tag{A.13}$$

Let $\tau_n = \inf\{t \geq 0 :| Y_t |> n\}$ with $\inf\{\emptyset\} = +\infty$. Then $\left(\int_0^{t\wedge\tau_n} Y_s dW_s\right)_{t\geq 0}$ and $\left(\int_0^{t\wedge\tau_n} Y_s^2 dB_s\right)_{t\geq 0}$ are local martingales. It implies that there exists a sequence of stopping times $(s_n)_{n\geq 1}$ going to $+\infty$ such that $\left(\int_0^{t\wedge\tau_n\wedge s_n} Y_s dW_s\right)_{t\geq 0}$ and $\left(\int_0^{t\wedge\tau_n\wedge s_n} Y_s^2 dB_s\right)_{t\geq 0}$ are martingales.
We put $\tau_n^{'} = \tau_n \wedge s_n$, then from (A.12) and (A.13):

$$\begin{aligned} Y_{t\wedge\tau_n^{'}}^2 =& y^2 + 2a_X \int_0^{t\wedge\tau_n^{'}} Y_s ds + 2\sigma_X \int_0^{t\wedge\tau_n^{'}} Y_s dW_s + 2\mu_\gamma \int_0^{t\wedge\tau_n^{'}} Y_s^2 ds \\ &+ 2\sigma_\gamma \int_0^{t\wedge\tau_n^{'}} Y_s^2 dB_s + \sigma_X^2\left(t\wedge\tau_n^{'}\right) + \sigma_\gamma^2 \int_0^{t\wedge\tau_n^{'}} Y_s^2 ds. \end{aligned} \tag{A.14}$$

We take mathematical expectation in (A.14), and since $\left(\int_0^{t\wedge\tau_n^{'}} Y_s dW_s\right)_{t\geq 0}$ and $\left(\int_0^{t\wedge\tau_n^{'}} Y_s^2 dB_s\right)_{t\geq 0}$ are martingales, we get that:

$$\begin{aligned} \mathbb{E}\left[Y_{t\wedge\tau_n^{'}}^2\right] =& y^2 + 2a_X \int_0^{t\wedge\tau_n^{'}} Y_s ds + 2\mu_\gamma \int_0^{t\wedge\tau_n^{'}} Y_s^2 ds \\ &+ \sigma_X^2\left(t\wedge\tau_n^{'}\right) + \sigma_\gamma^2 \int_0^{t\wedge\tau_n^{'}} Y_s^2 ds. \end{aligned}$$

Since $Y_s \geq 0$, we can do limit passage $\lim\limits_{n\to+\infty}$ in each term in the right-hand side by Lebesgue's Monotone Convergence Theorem. We prove that $\left[Y_{t\wedge\tau_n^{'}}^2\right]_{n\in\mathbb{N}}$ is uniformly integrable and we pass to the limit in the left-hand side.

This gives using Fubini theorem that:

$$\mathbb{E}\left(Y_t^2\right) = y^2 + 2a_X \int_0^t \mathbb{E}\left(Y_s\right) ds + 2\mu_\gamma \int_0^t \mathbb{E}\left(Y_s^2\right) ds + \sigma_X^2 t + \sigma_\gamma^2 \int_0^t \mathbb{E}\left(Y_s^2\right) ds.$$

We get with $g_t = \mathbb{E}\left(Y_t^2\right)$, $f_t = 2a_X \int_0^t \mathbb{E}\left(Y_s\right) ds + \sigma_X^2 t$ and $k = 2\mu_\gamma + \sigma_\gamma^2$, from Proposition 1 and Lemma 1 that:

$$\mathbb{E}\left(Y_t^2\right) = e^{\left(2\mu_\gamma + \sigma_\gamma^2\right)t} \left\{ y^2 + \int_0^t e^{-\left(2\mu_\gamma + \sigma_\gamma^2\right)s} \left[2a_X \mathbb{E}\left(Y_s\right) + \sigma_X^2\right] ds \right\}. \tag{A.15}$$

The integral part in the right-hand side of (A.15) is:

$$\begin{aligned}
&\int_0^t e^{-\left(2\mu_\gamma + \sigma_\gamma^2\right)s} \left[2a_X \mathbb{E}\left(Y_s\right) + \sigma_X^2\right] ds \\
=&\int_0^t e^{-\left(2\mu_\gamma + \sigma_\gamma^2\right)s} \left[2a_X \left(y e^{\mu_\gamma s} + a_X \frac{e^{\mu_\gamma s} - 1}{\mu_\gamma}\right) + \sigma_X^2\right] ds \\
=&\int_0^t \left[2a_X y e^{-\left(\mu_\gamma + \sigma_\gamma^2\right)s} + \frac{2a_X^2}{\mu_\gamma} e^{-\left(\mu_\gamma + \sigma_\gamma^2\right)s}\right. \\
&\left. + \left(\sigma_X^2 - \frac{2a_X^2}{\mu_\gamma}\right) e^{-\left(2\mu_\gamma + \sigma_\gamma^2\right)s}\right] ds.
\end{aligned} \tag{A.16}$$

To perform the integration in (A.16), it is necessary to consider: whether $\mu_\gamma = -\frac{1}{2}\sigma_\gamma^2$ or $\mu_\gamma = -\sigma_\gamma^2$.
When $\mu_\gamma \neq 0$, $\mu_\gamma \neq -\frac{1}{2}\sigma_\gamma^2$ and $\mu_\gamma \neq -\sigma_\gamma^2$, the integration in (A.16) is:

$$\begin{aligned}
&\int_0^t \left[2a_X y e^{-\left(\mu_\gamma + \sigma_\gamma^2\right)s} + \frac{2a_X^2}{\mu_\gamma} e^{-\left(\mu_\gamma + \sigma_\gamma^2\right)s}\right. \\
&\left. + \left(\sigma_X^2 - \frac{2a_X^2}{\mu_\gamma}\right) e^{-\left(2\mu_\gamma + \sigma_\gamma^2\right)s}\right] ds \\
=&\left[-2a_X y \frac{e^{-\left(\mu_\gamma + \sigma_\gamma^2\right)t}}{\mu_\gamma + \sigma_\gamma^2}\Bigg|_0^t\right] + \left[-\frac{2a_X^2}{\mu_\gamma} \frac{e^{-\left(\mu_\gamma + \sigma_\gamma^2\right)t}}{\mu_\gamma + \sigma_\gamma^2}\Bigg|_0^t\right] \\
&+ \left[-\left(\sigma_X^2 - \frac{2a_X^2}{\mu_\gamma}\right) \frac{e^{-\left(2\mu_\gamma + \sigma_\gamma^2\right)t}}{2\mu_\gamma + \sigma_\gamma^2}\Bigg|_0^t\right] \\
=&2a_X \left(y + \frac{a_X}{\mu_\gamma}\right) \frac{1 - e^{-\left(\mu_\gamma + \sigma_\gamma^2\right)t}}{\mu_\gamma + \sigma_\gamma^2} + \left(\sigma_X^2 - \frac{2a_X^2}{\mu_\gamma}\right) \frac{1 - e^{-\left(2\mu_\gamma + \sigma_\gamma^2\right)t}}{2\mu_\gamma + \sigma_\gamma^2}.
\end{aligned}$$

This gives, when $\mu_\gamma \neq 0$, $\mu_\gamma \neq -\frac{1}{2}\sigma_\gamma^2$ and $\mu_\gamma \neq -\sigma_\gamma^2$, the following relation

$$\begin{aligned}
&\mathbb{E}\left(Y_t^2\right) \\
=&e^{\left(2\mu_\gamma + \sigma_\gamma^2\right)t} \left[y^2 + 2a_X \left(y + \frac{a_X}{\mu_\gamma}\right) \frac{1 - e^{-\left(\mu_\gamma + \sigma_\gamma^2\right)t}}{\mu_\gamma + \sigma_\gamma^2}\right.
\end{aligned}$$

$$
\begin{aligned}
&+\left(\sigma_X^2-\frac{2a_X^2}{\mu_\gamma}\right)\frac{1-e^{-\left(2\mu_\gamma+\sigma_\gamma^2\right)t}}{2\mu_\gamma+\sigma_\gamma^2}\Bigg] \qquad (A.17)\\
=&y^2e^{\left(2\mu_\gamma+\sigma_\gamma^2\right)t}+2a_X\left(y+\frac{a_X}{\mu_\gamma}\right)\frac{e^{\left(2\mu_\gamma+\sigma_\gamma^2\right)t}-e^{\mu_\gamma t}}{\mu_\gamma+\sigma_\gamma^2}\\
&+\left(\sigma_X^2-\frac{2a_X^2}{\mu_\gamma}\right)\frac{e^{\left(2\mu_\gamma+\sigma_\gamma^2\right)t}-1}{2\mu_\gamma+\sigma_\gamma^2}.
\end{aligned}
$$

When $\mu_\gamma=-\frac{1}{2}\sigma_\gamma^2$, the integration in (A.16) is:

$$
\begin{aligned}
&\int_0^t\left[\left(2a_Xy+\frac{2a_X^2}{\mu_\gamma}\right)e^{-\frac{1}{2}\sigma_\gamma^2 s}+\sigma_X^2-\frac{2a_X^2}{\mu_\gamma}\right]ds\\
=&\left[-\left(2a_Xy+\frac{2a_X^2}{\mu_\gamma}\right)\frac{e^{-\frac{1}{2}\sigma_\gamma^2 t}}{\frac{1}{2}\sigma_\gamma^2}\Bigg|_0^t\right]+\left[\left(\sigma_X^2-\frac{2a_X^2}{\mu_\gamma}\right)t\Bigg|_0^t\right]\\
=&4a_X\left(y+\frac{a_X}{\mu_\gamma}\right)\frac{1-e^{-\frac{1}{2}\sigma_\gamma^2 t}}{\sigma_\gamma^2}+\left(\sigma_X^2-\frac{2a_X^2}{\mu_\gamma}\right)t.
\end{aligned}
$$

This gives, when $\mu_\gamma\neq-\frac{1}{2}\sigma_\gamma^2$, the following result

$$
\begin{aligned}
&\mathbb{E}\left(Y_t^2\right)\\
=&y^2+4a_X\left(y-\frac{2a_X}{\sigma_\gamma^2}\right)\frac{1-e^{-\frac{1}{2}\sigma_\gamma^2 t}}{\sigma_\gamma^2}+\left(\sigma_X^2+\frac{4a_X^2}{\sigma_\gamma^2}\right)t. \qquad (A.18)
\end{aligned}
$$

When $\mu_\gamma=-\sigma_\gamma^2$, the integration in (A.16) is:

$$
\begin{aligned}
&\int_0^t\left[2a_Xy+\frac{2a_X^2}{\mu_\gamma}+\left(\sigma_X^2-\frac{2a_X^2}{\mu_\gamma}\right)e^{\sigma_\gamma^2 s}\right]ds\\
=&\left[\left(2a_Xy+\frac{2a_X^2}{\mu_\gamma}\right)t\Bigg|_0^t\right]+\left[\left(\sigma_X^2-\frac{2a_X^2}{\mu_\gamma}\right)\frac{e^{\sigma_\gamma^2 t}}{\sigma_\gamma^2}\Bigg|_0^t\right]\\
=&2a_X\left(y+\frac{a_X}{\mu_\gamma}\right)t+\left(\sigma_X^2-\frac{2a_X^2}{\mu_\gamma}\right)\frac{e^{\sigma_\gamma^2 t}-1}{\sigma_\gamma^2}.
\end{aligned}
$$

This gives, when $\mu_\gamma=-\sigma_\gamma^2$, the following formula

$$
\begin{aligned}
&\mathbb{E}\left(Y_t^2\right)\\
=&e^{-\sigma_\gamma^2 t}\left[y^2+2a_X\left(y+\frac{a_X}{\mu_\gamma}\right)t+\left(\sigma_X^2-\frac{2a_X^2}{\mu_\gamma}\right)\frac{e^{\sigma_\gamma^2 t}-1}{\sigma_\gamma^2}\right] \qquad (A.19)\\
=&y^2e^{-\sigma_\gamma^2 t}+2a_X\left(y-\frac{a_X}{\sigma_\gamma^2}\right)te^{-\sigma_\gamma^2 t}+\left(\sigma_X^2+\frac{2a_X^2}{\sigma_\gamma^2}\right)\frac{1-e^{-\sigma_\gamma^2 t}}{\sigma_\gamma^2}.
\end{aligned}
$$

When $\mu_\gamma=0$, the integral part in the right-hand side of (A.15) is:

$$
\int_0^t e^{-\left(2\mu_\gamma+\sigma_\gamma^2\right)s}\left[2a_X\,\mathbb{E}\left(Y_s\right)+\sigma_X^2\right]ds
$$

$$
\begin{aligned}
&= \int_0^t e^{-\sigma_\gamma^2 s}\left[2a_X\left(y+a_X s\right)+\sigma_X^2\right]ds \\
&= \int_0^t \left(2a_X y e^{-\sigma_\gamma^2 s}+2a_X^2 s e^{-\sigma_\gamma^2 s}+\sigma_X^2 e^{-\sigma_\gamma^2 s}\right)ds \\
&= -2a_X y\frac{e^{-\sigma_\gamma^2 t}}{\sigma_\gamma^2}\Big|_0^t - 2a_X^2\frac{e^{-\sigma_\gamma^2 t}\left(\sigma_\gamma^2 t+1\right)}{\sigma_\gamma^4}\Big|_0^t - \sigma_X^2\frac{e^{-\sigma_\gamma^2 t}}{\sigma_\gamma^2}\Big|_0^t \\
&= 2a_X y\frac{1-e^{-\sigma_\gamma^2 t}}{\sigma_\gamma^2}+2a_X^2\frac{1-e^{-\sigma_\gamma^2 t}-\sigma_\gamma^2 t e^{-\sigma_\gamma^2 t}}{\sigma_\gamma^4}+\sigma_X^2\frac{1-e^{-\sigma_\gamma^2 t}}{\sigma_\gamma^2}.
\end{aligned}
$$

This gives, when $\mu_\gamma = 0$:

$$
\begin{aligned}
&\mathbb{E}\left(Y_t^2\right) \\
&= e^{\sigma_\gamma^2 t}\left(y^2+2a_X y\frac{1-e^{-\sigma_\gamma^2 t}}{\sigma_\gamma^2}+2a_X^2\frac{1-e^{-\sigma_\gamma^2 t}-\sigma_\gamma^2 t e^{-\sigma_\gamma^2 t}}{\sigma_\gamma^4}\right. \\
&\quad \left.+\sigma_X^2\frac{1-e^{-\sigma_\gamma^2 t}}{\sigma_\gamma^2}\right) \qquad \text{(A.20)} \\
&= y^2 e^{\sigma_\gamma^2 t}+2a_X y\frac{e^{\sigma_\gamma^2 t}-1}{\sigma_\gamma^2}+2a_X^2\frac{e^{\sigma_\gamma^2 t}-1-\sigma_\gamma^2 t}{\sigma_\gamma^4}+\sigma_X^2\frac{e^{\sigma_\gamma^2 t}-1}{\sigma_\gamma^2}.
\end{aligned}
$$

□

Appendix B: Case 2

B.1. Proof of Proposition 3

Proof. First, we calculate conditional expectation of Y_t conditionally to $\mathcal{E}(R^{(\gamma)})_t = h_t$. Namely, we show that:

$$
\begin{aligned}
&\mathbb{E}\left[Y_t^{(\gamma)}(y)\middle|\mathcal{E}\left(R^{(\gamma)}\right)_{0\le s\le t}=(h_s)_{0\le s\le t}\right] \\
&= h_t\left[y+\left(a_X+\lambda\beta_Z\right)\int_0^t\frac{ds}{h_s}\right].
\end{aligned} \qquad \text{(B.1)}
$$

Since W and B are independent Brownian Motions, W, N and $(Z_k)_{k\in N}$ are independent, and hence, $R^{(\gamma)}$ and X are independent,

$$
\mathbb{E}\left[Y_t^{(\gamma)}(y)\middle|\mathcal{E}\left(R^{(\gamma)}\right)_{0\le s\le t}=(h_s)_{0\le s\le t}\right] = h_t\left(y+\mathbb{E}\int_0^t\frac{dX_s}{h_s}\right). \qquad \text{(B.2)}
$$

Taking into account *Case 2* we have:

$$
\mathbb{E}\left(\int_0^t\frac{dX_s}{h_s}\right) = \mathbb{E}\left(a_X\int_0^t\frac{ds}{h_s}\right)+\mathbb{E}\left(\sigma_X\int_0^t\frac{dW_s}{h_s}\right)+\mathbb{E}\left(\int_0^t\frac{dQ_s}{h_s}\right). \qquad \text{(B.3)}
$$

We note that $(\int_0^t \frac{dW_s}{h_s})$ is a local martingale, and by localisation procedure, we show that:

$$\mathbb{E}\left(\sigma_X \int_0^t \frac{dW_s}{h_s}\right) = 0, \tag{B.4}$$

and that

$$\mathbb{E}\left(\int_0^t \frac{dQ_s}{h_s}\right) = \lambda\beta_Z \int_0^t \frac{ds}{h_s}. \tag{B.5}$$

From the *Case 1* and the relation in (B.1), we conclude substituting a_X by $a_x + \lambda\beta_Z$, then the expectation in this case is:

$$\mathbb{E}\left[Y_t^{(\gamma)}(y)\right] = \begin{cases} e^{\mu_\gamma t}\left(y + \frac{a_X+\lambda\beta_Z}{\mu_\gamma}\right) - \frac{a_X+\lambda\beta_Z}{\mu_\gamma}, & if\ \mu_\gamma \neq 0 \\ y + (a_X + \lambda\beta_Z)\, t, & if\ \mu_\gamma = 0. \end{cases} \tag{B.6}$$

□

B.2. Proof of Proposition 4

Proof. We omit $^{(\gamma)}$ for the simplicity of the notation.
We do the calculus using the Itô Formula:

$$\begin{aligned} & f\left(Y(t)\right) \\ & = f\left(Y(0)\right) + \int_0^t f'\left(Y(s)\right) dY^c(s) + \frac{1}{2}\int_0^t f''\left(Y(s)\right) d\langle Y^c, Y^c\rangle_s \\ & + \sum_{0<s\leq t} \left[f\left(Y(s)\right) - f\left(Y(s-)\right)\right], \end{aligned} \tag{B.7}$$

where Y^c is continuous martingale part of Y. In our case,

$$dY_t^c = \sigma_X dW_t + \sigma_R^\gamma Y_{t-} dB_t.$$

The equation (B.7) in the differential form is:

$$\begin{aligned} dY_t &= dX_t + Y_{t-} dR_t \\ &= a_X dt + \sigma_X dW_t + dQ_t + \mu_\gamma Y_{t-} dt + \sigma_\gamma Y_{t-} dB_t, \end{aligned} \tag{B.8}$$

The Ito formula with the function $f(x) = x^2$ in integral form gives:

$$\begin{aligned} Y_t^2 =& y^2 + 2a_X \int_0^t Y_s ds + 2\sigma_X \int_0^t Y_s dW_s + 2\mu_\gamma \int_0^t Y_s^2 ds \\ & + 2\sigma_\gamma \int_0^t Y_s^2 dB_s + \langle Y^c\rangle_t + \sum_{0<s\leq t} \left[Y^2(s) - Y^2(s-)\right]. \end{aligned} \tag{B.9}$$

The last term in the right-side hand of (B.7) in our case is:

$$\sum_{0<s\leq t} \left[Y^2(s) - Y^2(s-)\right]$$

$$
\begin{aligned}
&= \sum_{0<s\leq t} \left[(Y(s-) + \Delta Y_s)^2 - (Y(s-))^2 \right] \\
&= \sum_{0<s\leq t} \left[(2Y(s-) + \Delta Q_s) \cdot (\Delta Q_s)^2 \right] \\
&= \sum_{0<s\leq t} \left[2Y(s-) \cdot \Delta Q_s + (\Delta Q_s)^2 \right] \\
&= [Q,Q]_t + 2\int_0^t Y_{s-} dQ_s.
\end{aligned}
$$

Moreover, $\langle Y^c \rangle_t = \langle \sigma_X \int_0^t dW_s + \sigma_\gamma \int_0^t Y_s dB_s \rangle$, with independent W and B, and it gives

$$
\langle Y^c \rangle_t = \sigma_X^2 t + \sigma_\gamma^2 \int_0^t Y_s^2 ds. \tag{B.10}
$$

Let $\tau_n = \inf\{t \geq 0 :| Y_t |> n\}$ with $\inf\{\emptyset\} = +\infty$. Then $\left(\int_0^{t\wedge\tau_n} Y_s dW_s\right)_{t\geq 0}$ and $\left(\int_0^{t\wedge\tau_n} Y_s^2 dB_s\right)_{t\geq 0}$ are local martingales. It implies that there exists a sequence of stopping times $(s_n)_{n\geq 1}$ going to $+\infty$ such that $\left(\int_0^{t\wedge\tau_n\wedge s_n} Y_s dW_s\right)_{t\geq 0}$ and $\left(\int_0^{t\wedge\tau_n\wedge s_n} Y_s^2 dB_s\right)_{t\geq 0}$ are martingales.
We put $\tau_n^{'} = \tau_n \wedge s_n$, then from (B.9) and (B.10):

$$
\begin{aligned}
&Y^2_{t\wedge\tau_n'} \\
=&y^2 + 2a_X \int_0^{t\wedge\tau_n^{'}} Y_s ds + 2\sigma_X \int_0^{t\wedge\tau_n^{'}} Y_s dW_s + 2\mu_\gamma \int_0^{t\wedge\tau_n^{'}} Y_s^2 ds \\
&+ 2\sigma_\gamma \int_0^{t\wedge\tau_n^{'}} Y_s^2 dB_s + \sigma_X^2 \left(t\wedge\tau_n^{'}\right) + \sigma_\gamma^2 \int_0^{t\wedge\tau_n^{'}} Y_s^2 ds \\
&+ [Q,Q]_{t\wedge\tau_n'} + 2\int_0^{t\wedge\tau_n^{'}} Y_{s-} dQ_s.
\end{aligned} \tag{B.11}
$$

We take mathematical expectation in (B.11), and since $\left(\int_0^{t\wedge\tau_n^{'}} Y_s dW_s\right)_{t\geq 0}$ and $\left(\int_0^{t\wedge\tau_n^{'}} Y_s^2 dB_s\right)_{t\geq 0}$ are martingales, we get that:

$$
\begin{aligned}
&\mathbb{E}\left[Y^2_{t\wedge\tau_n'}\right] \\
=&y^2 + 2a_X\,\mathbb{E}\int_0^{t\wedge\tau_n^{'}} Y_s ds + 2\mu_\gamma\,\mathbb{E}\int_0^{t\wedge\tau_n^{'}} Y_s^2 ds + \sigma_X^2 \left(t\wedge\tau_n^{'}\right) \\
&+ \sigma_\gamma^2\,\mathbb{E}\int_0^{t\wedge\tau_n^{'}} Y_s^2 ds + \mathbb{E}\,[Q,Q]_{t\wedge\tau_n'} + 2\,\mathbb{E}\int_0^{t\wedge\tau_n^{'}} Y_{s-} dQ_s.
\end{aligned} \tag{B.12}
$$

Then we calculate

$$
\mathbb{E}\,[Q,Q]_t = \mathbb{E}\int_0^t \int_{\mathbb{R}} x^2\, d\mu_Q(x) = \mathbb{E}\int_0^t \int_{\mathbb{R}} x^2\, d\nu_Q(x,t)
$$

$$= \mathbb{E}\int_0^t \int_{\mathbb{R}} x^2\, dF_Z(x)\lambda ds = Var(Q_t)$$
$$= \lambda t\left(\beta_Z^2 + \sigma_Z^2\right).$$

Furthermore, the integral of a continuous stochastic process with respect to the compound Poisson process,

$$2\,\mathbb{E}\int_0^t Y_{s-} dQ_s = 2\,\mathbb{E}\int_0^t Y_{s-}\int_{\mathbb{R}} x\, F_Z\left(dx\right)\lambda ds$$
$$= 2\lambda\beta_Z \int_0^t \mathbb{E}\left(Y_s\right) ds.$$

Since $Y_s \geq 0$, we can do limit passage, $\lim\limits_{n\to+\infty}$, in each term in the right-hand side by Lebesgue's Monotone Convergence Theorem. We prove that $\left[Y^2_{t\wedge\tau'_n}\right]_{n\in\mathbb{N}}$ is uniformly integrable and we pass to the limit in the left-hand side.
This gives using Fubini Theorem that:

$$\mathbb{E}\left(Y_t^2\right) = y^2 + 2a_X\int_0^t \mathbb{E}\left(Y_s\right) ds + 2\mu_\gamma\int_0^t \mathbb{E}\left(Y_s^2\right) ds + \sigma_X^2 t + \sigma_\gamma^2\int_0^t \mathbb{E}\left(Y_s^2\right) ds$$
$$+ \lambda t\left(\beta_Z^2 + \sigma_Z^2\right) + 2\lambda\beta_Z\int_0^t \mathbb{E}\left(Y_s\right) ds.$$

We see that with

$$g_t = \mathbb{E}\left(Y_t^2\right),$$
$$k = 2\mu_\gamma + \sigma_\gamma^2 + 2\lambda\beta_Z, \text{ and}$$
$$f_t = 2a_X\int_0^t \mathbb{E}\left(Y_s\right) ds + \sigma_X^2 t + \lambda t\left(\beta_Z^2 + \sigma_Z^2\right) + 2\lambda\beta_Z\int_0^t \mathbb{E}\left(Y_s\right),$$

the equations for $\mathbb{E}\left\{\left[Y_t^{(\gamma)}(y)\right]^2\right\}$ can be obtained from *Case 1*, replacing a_X by $a_X + \lambda\beta_Z$ and σ_X^2 by $\sigma_X^2 + \lambda(\beta_Z^2 + \sigma_Z^2)$. □

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