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# ON THE DERIVATIVE NONLINEAR SCHRÖDINGER EQUATION ON THE HALF LINE WITH ROBIN BOUNDARY CONDITION

PHAN VAN TIN

ABSTRACT. We consider the Schrödinger equation with nonlinear derivative term on  $[0, +\infty)$  under Robin boundary condition at 0. Using a virial argument, we obtain the existence of blowing up solutions and using variational techniques, we obtain stability and instability by blow up results for standing waves.

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## 1. INTRODUCTION

In this paper, we consider the derivative nonlinear Schrödinger equation on  $[0, +\infty)$  with Robin boundary condition at 0:

$$\begin{cases} iv_t + v_{xx} = \frac{i}{2}|v|^2v_x - \frac{i}{2}v^2\overline{v}_x - \frac{3}{16}|v|^4v & \text{for } x \in \mathbb{R}_+, \\ v(0) = \varphi, \\ \partial_x v(t, 0) = \alpha v(t, 0) \quad \forall t \in \mathbb{R}, \end{cases} \quad (1.1)$$

where  $\alpha \in \mathbb{R}$  is a given constant.

The linear parts of (1.1) can be rewritten in the following forms:

$$\begin{cases} iv_t + \tilde{H}_\alpha v = 0 & \text{for } x \in \mathbb{R}_+, \\ v(0) = \varphi, \end{cases} \quad (1.2)$$

where  $\tilde{H}_\alpha$  are self-adjoint operators defined by

$$\begin{aligned} \tilde{H}_\alpha &: D(\tilde{H}_\alpha) \subset L^2(\mathbb{R}_+) \rightarrow L^2(\mathbb{R}_+), \\ \tilde{H}_\alpha u &= u_{xx}, \quad D(\tilde{H}_\alpha) = \{u \in H^2(\mathbb{R}_+) : u_x(0^+) = \alpha u(0^+)\}. \end{aligned}$$

We call  $e^{i\tilde{H}_\alpha t} : \mathbb{R} \rightarrow \mathcal{L}(L^2(\mathbb{R}^+))$  is group defining the solution of (1.2).

The derivative nonlinear Schrödinger equation was originally introduced in Plasma Physics as a simplified model for Alfvén wave propagation. Since then, it has attracted a lot of attention from the mathematical community (see e.g [4, 5, 13, 14, 16, 17, 20, 21]).

Consider the equation (1.1), and set

$$u(t, x) = \exp\left(\frac{3i}{4} \int_\infty^x |v(t, y)|^2 dy\right) v(t, x).$$

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Using the Gauge transformation, we see that  $u$  solves

$$iu_t + u_{xx} = i\partial_x(|u|^2u), \quad t \in \mathbb{R}, x \in (0, \infty), \quad (1.3)$$

under a boundary condition  $\partial_x u(t, 0) = \alpha u(t, 0) + \frac{3i}{4}|u(t, 0)|^2 u(t, 0)$ . In all line case, there are many papers to deal with Cauchy problem of (1.3) (see e.g [15, 22, 23]). In [15], the authors establish the local well posedness in  $H^1(\mathbb{R})$  by using a Gauge transform. Indeed, since  $u$  solves (1.3) on  $\mathbb{R}$ , by setting

$$\begin{aligned} h(t, x) &= \exp\left(-i \int_{-\infty}^x |u(t, y)|^2 dy\right) u(t, x), \\ k &= h_x + \frac{i}{2}|h|^2 h, \end{aligned} \quad (1.4)$$

we have  $h, k$  solve

$$\begin{cases} ih_t + h_{xx} = -ih^2 \bar{k}, \\ ik_t + k_{xx} = ik^2 \bar{h}. \end{cases} \quad (1.5)$$

By classical arguments, we can prove that there exists a unique solution  $h, k \in C([0, T], L^2(\mathbb{R})) \cap L^4([0, T], L^\infty(\mathbb{R}))$  given  $h_0, k_0 \in L^2(\mathbb{R})$  are satisfy (1.4). To obtain the existence solution of (1.1), the authors prove that the relation (1.4) satisfies for all  $t \in [0, T]$ . Thus, since  $h, k$  solve (1.5) satisfy (1.4), if we set

$$u(t, x) = \exp\left(i \int_{-\infty}^x |h(t, y)|^2 dy\right) h(t, x),$$

then  $u \in C([0, T], H^1(\mathbb{R}))$  solves (1.1). In [1], the authors have proved the global well posedness of (1.3) given initial data in  $H^{\frac{1}{2}}(\mathbb{R})$ . In half line case, [26] Wu prove existence of blow up solution of (1.3) under Dirichlet boundary condition, given initial data in  $\Sigma := \{u_0 \in H^2(\mathbb{R}^+), xu_0 \in L^2(\mathbb{R}^+)\}$ . In this paper, we give a proof of existence of blow up solution of (1.1) under Robin boundary condition.

To study equation (1.1), we start by the definition of solution on  $H^1(\mathbb{R}^+)$ . Since (1.1) contains a Robin boundary condition, the notion of solution in  $H^1(\mathbb{R}^+)$  is not completely clear. We use the following definition. Let  $I$  be an open interval of  $\mathbb{R}$ . We say that  $v$  is a  $H^1(\mathbb{R}^+)$  solution of the problem (1.1) on  $I$  if  $v \in C(I, H^1(\mathbb{R}^+))$  satisfies the following equation

$$v(t) = e^{i\tilde{H}_\alpha t} \varphi - i \int_0^t e^{i\tilde{H}_\alpha(t-s)} g(v(s)) ds, \quad (1.6)$$

where  $g$  is the function defined by

$$g(v) = \frac{i}{2}|v|^2 v_x - \frac{i}{2}v^2 \bar{v}_x - \frac{3}{16}|v|^4 v.$$

Let  $v \in C(I, D(\tilde{H}_\alpha))$  be classical solution of (1.1). At least formally, we have

$$\frac{1}{2}\partial_t(|v|^2) = -\partial_x \mathcal{I}m(v_x \bar{v}).$$

Therefore, using the Robin boundary condition we have

$$\begin{aligned} \partial_t \left( \frac{1}{2} \int_0^\infty |v|^2 dx \right) &= -\mathcal{I}m(v_x \bar{v})(\infty) + \mathcal{I}m(v_x \bar{v})(0) \\ &= \mathcal{I}m(v_x \bar{v})(0) \\ &= \alpha \mathcal{I}m(|v(0)|^2) \\ &= 0. \end{aligned}$$

This implies the conservation of the mass. By elementary calculations, we have

$$\partial_t \left( |v_x|^2 - \frac{1}{16}|v|^6 \right) = \partial_x \left( 2\mathcal{R}e(v_x \bar{v}_t) - \frac{1}{2}|v|^2 |v_x|^2 + \frac{1}{2}v^2 \bar{v}_x^2 \right).$$

Hence, integrating the two sides in space, we obtain

$$\partial_t \left( \int_{\mathbb{R}_+} |v_x|^2 dx - \frac{1}{16}|v|^6 dx \right) = -2\mathcal{R}e(v_x(0)\bar{v}_t(0)) + \frac{1}{2}|v(0)|^2 |v_x(0)|^2 - \frac{1}{2}v(0)^2 \overline{v_x(0)}^2$$

Using the Robin boundary condition for  $v$ , we obtain

$$\partial_t \left( \int_{\mathbb{R}_+} |v_x|^2 dx - \frac{1}{16} |v|^6 dx \right) = -2\alpha \operatorname{Re}(v(0)\overline{v_t(0)}) = -\alpha \partial_t (|v(0)|^2).$$

This implies the conservation of the energy.

In this paper, we will need the following assumption.

**Assumption.** *We assume that for all  $\varphi \in H^1(\mathbb{R}^+)$  there exist a solution  $v \in C(I, H^1(\mathbb{R}^+))$  of (1.1) for some interval  $I \subset \mathbb{R}$ . Moreover,  $v$  satisfies the following conservation law:*

$$\begin{aligned} M(v) &:= \frac{1}{2} \|v\|_{H^1(\mathbb{R}^+)}^2 = M(\varphi), \\ E(v) &:= \frac{1}{2} \|v_x\|_{L^2(\mathbb{R}^+)}^2 - \frac{1}{32} \|v\|_{L^6(\mathbb{R}^+)}^6 + \frac{\alpha}{2} |v(0)|^2. \end{aligned}$$

The existence of blowing up solutions for classical nonlinear Schrödinger equations was considered by Glassey [10] in 1977. He introduced a concavity argument based on the second derivative in time of  $\|xu(t)\|_{L^2}^2$  to show the existence of blowing up solutions. In this paper, we are also interested in studying the existence of blowing-up solutions of (1.1). In the limit case  $\alpha = +\infty$ , which is formally equivalent to Dirichlet boundary condition if we write  $v(0) = \frac{1}{\alpha} v'(0) = 0$ . In [26], Wu proved the blow up in finite time of solutions of (1.1) with Dirichlet boundary condition and some conditions on the initial data. Using the method of Wu [26] we obtain the existence of blowing up solutions in the case  $\alpha > 0$ , under a weighted space condition for the initial data and negativity of the energy. Our first main result is the following.

**Theorem 1.1.** *We assume assumption 1. Let  $\alpha > 0$  and  $\varphi \in \Sigma$  where*

$$\Sigma = \left\{ u \in D(\tilde{H}_\alpha), xu \in L^2(\mathbb{R}_+) \right\}$$

*such that  $E(\varphi) < 0$ . Then the solution  $v$  of (1.1) blows-up in finite time i.e  $T_{min} > -\infty$  and  $T_{max} < +\infty$ .*

*Remark 1.2.* In general case, on the equation (1.1), if we consider the other nonlinear term instead of  $\frac{i}{2}|v|^2 v_x - \frac{i}{2}v^2 \overline{v_x} - \frac{3}{16}|v|^4 v$  then the existence of conservation of energy may be not true (see remark 2.1 (2)).

The stability of standing waves for classical nonlinear Schrödinger equations was originally studied by Cazenave and Lions [2] with variational and compactness arguments. A second approach, based on spectral arguments, was introduced by Weinstein [24, 25] and then considerably generalized by Grillakis, Shatah and Strauss [11, 12] (see also [6], [7]). In our work, we use the variational techniques to study the stability of standing waves. First, we define

$$\begin{aligned} S_\omega(v) &:= \frac{1}{2} \left[ \|v_x\|_{L^2(\mathbb{R}^+)}^2 + \omega \|v\|_{L^2(\mathbb{R}^+)}^2 + \alpha |v(0)|^2 \right] - \frac{1}{32} \|v\|_{L^6(\mathbb{R}^+)}^6, \\ K_\omega(v) &:= \|v_x\|_{L^2(\mathbb{R}^+)}^2 + \omega \|v\|_{L^2(\mathbb{R}^+)}^2 + \alpha |v(0)|^2 - \frac{3}{16} \|v\|_{L^6(\mathbb{R}^+)}^6. \end{aligned}$$

We are interested in the following variational problem:

$$d(\omega) := \inf \{ S_\omega(v) \mid K_\omega(v) = 0, v \in H^1(\mathbb{R}^+) \setminus \{0\} \}. \quad (1.7)$$

We have the following result.

**Proposition 1.3.** *Let  $\omega, \alpha \in \mathbb{R}$  such that  $\omega > \alpha^2$ . All minimizers of (1.7) are of form  $e^{i\theta} \varphi$ , where  $\theta \in \mathbb{R}$  and  $\varphi$  is given by*

$$\varphi = 2\sqrt[4]{\omega} \operatorname{sech}^{\frac{1}{2}} \left( 2\sqrt{\omega}|x| + \tanh^{-1} \left( \frac{-\alpha}{\sqrt{\omega}} \right) \right).$$

We give the definition of stability and instability by blow up in  $H^1(\mathbb{R}^+)$ . Let  $w(t, x) = e^{i\omega t} \varphi(x)$  be a standing wave solution of (1.1).

- (1) The standing wave  $w$  is called *orbitally stable* in  $H^1(\mathbb{R}^+)$  if for all  $\varepsilon > 0$ , there exists  $\delta > 0$  such that if  $v_0 \in H^1(\mathbb{R}^+)$  satisfies

$$\|v_0 - \varphi\|_{H^1(\mathbb{R}^+)} \leq \delta,$$

then the associated solution  $v$  of (1.1) satisfies

$$\sup_{t \in \mathbb{R}} \inf_{\theta \in \mathbb{R}} \|v(t) - e^{i\theta} \varphi\|_{H^1(\mathbb{R}^+)} < \varepsilon.$$

Otherwise,  $w$  said to be *unstable*.

- (2) The standing wave  $w$  is called *unstable by blow up* if there exists a sequence  $(\varphi_n)$  such that  $\lim_{n \rightarrow \infty} \|\varphi_n - \varphi\|_{H^1(\mathbb{R}^+)} = 0$  and the associated solution  $v_n$  of (1.1) blows up in finite time for all  $n$ .

Our second main result is the following.

**Theorem 1.4.** *Let  $\alpha, \omega \in \mathbb{R}$  be such that  $\omega > \alpha^2$ . The standing wave  $e^{i\omega t} \varphi$ , where  $\varphi$  is the profile as in Proposition 1.3, solution of (1.1), satisfies the following properties.*

- (1) *If  $\alpha < 0$  then the standing wave is orbitally stable in  $H^1(\mathbb{R}^+)$ .*  
(2) *If  $\alpha > 0$  then the standing wave is unstable by blow up.*

*Remark 1.5.* To our knowledge, the conservation law play an important role to study the stability of standing waves. However, the existence of conservation of energy is not always true (see remark 1.2). Our work can only extend for the models with nonlinear terms provide the conservation law of solution.

This paper is organized as follows. First, under the assumption of local well posedness in  $H^1(\mathbb{R}^+)$ , we prove the existence of blowing up solutions using a virial argument Theorem 1.1. In section 2.1, we give the proof of Theorem 1.1. Second, in the case  $\alpha < 0$ , using similar arguments as in [3], we prove the orbital stability of standing waves of (1.1). In the case  $\alpha > 0$ , using similar arguments as in [18], we prove the instability by blow up of standing waves. The proof of Theorem 1.4 is obtained in Section 2.2.

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## 2. PROOF OF MAIN RESULTS

We consider the equation (1.1) and assume that the assumption 1 holds.

**2.1. The existence of a blowing-up solution.** In this section, we give the proof of Theorem 1.1 using a virial argument (see e.g [10] or [26] for similar arguments). Let  $\alpha > 0$ . Let  $v$  be a solution of (1.1). To prove the existence of blowing up solutions we use similar arguments as in [26]. Set

$$I(t) = \int_0^\infty x^2 |v(t)|^2 dx.$$

Let

$$u(t, x) = v(t, x) \exp\left(-\frac{i}{4} \int_x^{+\infty} |v|^2 dy\right) \quad (2.1)$$

be a Gauge transform in  $H^1(\mathbb{R}_+)$ . Then the problem (1.1) is equivalent with

$$\begin{cases} iu_t + u_{xx} = i|u|^2 u_x, \\ u_x(0) = \alpha u(0) + \frac{i}{4} |u(0)|^2 u(0). \end{cases} \quad (2.2)$$

The equation (2.2) has a simpler nonlinear form, but we pay this simplification with a nonlinear boundary condition. Observe that

$$I(t) = \int_0^\infty x^2 |u(t)|^2 dx = \int_0^\infty x^2 |v(t)|^2 dx.$$

By a direct calculation, we get

$$\partial_t I(t) = 2\operatorname{Re} \int_0^\infty x^2 \overline{u(t, x)} \partial_t u(t, x) dx = 2\operatorname{Re} \int_0^\infty x^2 \bar{u} (iu_{xx} + |u|^2 u_x) dx \quad (2.3)$$

$$= 2\operatorname{Im} \int_0^\infty 2x \bar{u} u_x dx - \frac{1}{2} \int_0^\infty 2x |u|^4 dx \quad (2.4)$$

$$= 4\operatorname{Im} \int_0^\infty x u_x \bar{u} dx - \int_0^\infty x |u|^4 dx. \quad (2.5)$$

Define

$$J(t) = \mathcal{I}m \int_0^\infty x u_x \bar{u} dx.$$

We have

$$\begin{aligned} \partial_t J(t) &= \int_0^\infty x u_x \bar{u}_t dx + \int_0^t x \bar{u} u_{xt} dx \\ &= -\mathcal{I}m \int_0^\infty x u_t \bar{u}_x dx - \mathcal{I}m \int_0^\infty (x \bar{u})_x u_t dx \\ &= -2\mathcal{I}m \int_0^\infty x u_t \bar{u}_x dx - \mathcal{I}m \int_0^\infty u_t \bar{u} dx \\ &= -2\mathcal{I}m \int_0^\infty x \bar{u}_x (i u_{xx} + |u|^2 u_x) dx - \mathcal{I}m \int_0^\infty \bar{u} (i u_{xx} + |u|^2 u_x) dx \\ &= -2\mathcal{R}e \int_0^\infty x \bar{u}_x u_{xx} dx - \mathcal{R}e \int_0^\infty \bar{u} u_{xx} dx - \mathcal{I}m \int_0^\infty |u|^2 u_x \bar{u} dx \\ &= -\int_0^\infty x \partial_x |u_x|^2 dx - \mathcal{R}e(\bar{u} u_x)(+\infty) + \mathcal{R}e(\bar{u} u_x)(0) + \mathcal{R}e \int_0^\infty \bar{u}_x u_x dx - \mathcal{I}m \int_0^\infty |u|^2 u_x \bar{u} dx \\ &= \int_0^\infty |u_x|^2 dx + \mathcal{R}e(\bar{u}(0) u_x(0)) + \int_0^\infty |u_x|^2 dx - \mathcal{I}m \int_0^\infty |u|^2 u_x \bar{u} dx \\ &= 2 \int_0^\infty |u_x|^2 dx - \mathcal{I}m \int_0^\infty |u|^2 u_x \bar{u} dx + \mathcal{R}e(\bar{u}(0) u_x(0)). \end{aligned}$$

Using the Robin boundary condition we have

$$\partial_t J(t) = 2 \int_0^\infty |u_x|^2 dx - \mathcal{I}m \int_0^\infty |u|^2 u_x \bar{u} dx + \alpha |u(0)|^2.$$

Moreover using the expression of  $v$  in term of  $u$  given in (2.1), we get

$$\begin{aligned} \partial_t J(t) &= 2 \int_0^\infty |v_x|^2 dx - \frac{1}{8} \int_0^\infty |v|^6 dx + \alpha |v(0)|^2 \\ &= 4E(v) - \alpha |v(0)|^2 \leq 4E(v) = 4E(\varphi). \end{aligned}$$

By integrating the two sides of the above inequality in time we have

$$J(t) \leq J(0) + 4E(\varphi)t. \quad (2.6)$$

Integrating the two sides of (2.3) in time we have

$$\begin{aligned} I(t) &= I(0) + 4 \int_0^t J(s) ds - \int_0^t \int_0^\infty x |u(s, x)|^4 dx ds \\ &\leq I(0) + 4 \int_0^t J(s) ds. \end{aligned}$$

Using (2.6) we have

$$\begin{aligned} I(t) &\leq I(0) + 4 \int_0^t (J(0) + 4E(\varphi)s) ds \\ &\leq I(0) + 4J(0)t + 8E(\varphi)t^2. \end{aligned}$$

From the assumption  $E(\varphi) < 0$ , there exists a finite time  $T_* > 0$  such that  $I(T_*) = 0$ ,

$$I(t) > 0 \text{ for } 0 < t < T_*.$$

Note that

$$\begin{aligned} \int_0^\infty |\varphi(x)|^2 dx &= \int_0^\infty |v(t, x)|^2 dx = -2\mathcal{R}e \int_0^\infty x v(t, x) \bar{v}_x(t, x) dx \\ &\leq 2 \|xv\|_{L_x^2(\mathbb{R}_+)} \|v_x\|_{L_x^2(\mathbb{R}_+)} = 2\sqrt{I(t)} \|v_x\|_{L_x^2(\mathbb{R}_+)}. \end{aligned}$$

Then there exists a constant  $C = C(\varphi) > 0$  such that

$$\|v_x\|_{L_x^2(\mathbb{R}_+)} \geq \frac{C}{2\sqrt{I(t)}} \rightarrow +\infty \text{ as } t \rightarrow T_*.$$

Then the solution  $v$  blows up in finite time in  $H^1(\mathbb{R}^+)$ . This complete the proof of Theorem 1.1.

*Remark 2.1.* In (1.1), if we consider nonlinear term  $i|v|^2v_x$  instead of  $\frac{i}{2}|v|^2v_x - \frac{i}{2}v^2\overline{v}_x - \frac{3}{16}|v|^4v$  then there is no conservation of energy of solution. Indeed, set

$$u(t, x) = v(t, x) \exp\left(-\frac{i}{4} \int_{-\infty}^x |v(t, y)|^2 dy\right).$$

If  $v$  solves

$$\begin{cases} iv_t + v_{xx} = i|v|^2v_x, \\ \partial_x v(t, 0) = \alpha v(t, 0) \end{cases}$$

then  $u$  solves

$$\begin{cases} iu_t + u_{xx} = \frac{i}{2}|u|^2u_x - \frac{i}{2}u^2\overline{u}_x - \frac{3}{16}|u|^4u, \\ \partial_x u(t, 0) = \alpha u(t, 0) - \frac{i}{4}|u(t, 0)|^2u(t, 0). \end{cases} \quad (2.7)$$

By similar arguments in proof of Theorem 1.1, since  $u$  solves (2.7), we have

$$\partial_t \left( |u_x|^2 - \frac{1}{16}|u|^6 \right) = \partial_x \left( 2\mathcal{R}e(u_x\overline{u}_t) - \frac{1}{2}|u|^2|u_x|^2 + \frac{1}{2}u^2\overline{u}_x^2 \right).$$

Integrating the two sides in space, we obtain

$$\partial_t \left( \int_{\mathbb{R}^+} |u_x|^2 - \frac{1}{16}|u|^6 dx \right) = -2\mathcal{R}e(u_x(0)\overline{u}_t(0)) + \frac{1}{2}|u(0)|^2|u_x(0)|^2 - \frac{1}{2}u(0)^2\overline{u}_x(0)^2.$$

Using the boundary condition of  $u$ , we obtain

$$\begin{aligned} \partial_t \left( \int_{\mathbb{R}^+} |u_x|^2 - \frac{1}{16}|u|^6 dx \right) &= -2\alpha\mathcal{R}e(u(0)\overline{u}_t(0)) - \frac{1}{2}\mathcal{I}m(u(0)|u(0)|^2\overline{u}_t(0)) \\ &\quad + \frac{1}{2}|u(0)|^4 \left( \alpha^2 + \frac{1}{16}|u(0)|^4 - \left( \alpha + \frac{i}{4}|u(0)|^2 \right)^2 \right) \\ &= -\alpha\partial_t(|u(0)|^2) + A, \end{aligned}$$

where  $A = -\frac{1}{2}\mathcal{I}m(u(0)|u(0)|^2\overline{u}_t(0)) + \frac{1}{2}|u(0)|^4 \left( \alpha^2 + \frac{1}{16}|u(0)|^4 - \left( \alpha + \frac{i}{4}|u(0)|^2 \right)^2 \right)$ . Moreover, we can not write  $A$  in form  $\partial_t B(u(0))$ , for some function  $B : \mathbb{C} \rightarrow \mathbb{C}$ . Then, there is no conservation of energy of  $u$  and hence, there is no conservation of energy of  $v$ .

**2.2. Stability and instability of standing waves.** In this section, we give the proof of Theorem 1.4. First, we find the form of the standing waves of (1.1).

2.2.1. *Standing waves.* Let  $v = e^{i\omega t}\varphi$  be a solution of (1.1). Then  $\varphi$  solves

$$\begin{cases} 0 = \varphi_{xx} - \omega\varphi + \frac{1}{2}\mathcal{I}m(\varphi_x\overline{\varphi})\varphi + \frac{3}{16}|\varphi|^4\varphi, & \text{for } x > 0 \\ \varphi_x(0) = \alpha\varphi(0), \\ \varphi \in H^2(\mathbb{R}^+). \end{cases} \quad (2.8)$$

Set

$$A := \omega - \frac{1}{2}\mathcal{I}m(\varphi_x\overline{\varphi}) - \frac{3}{16}|\varphi|^4$$

By writing  $\varphi = f + ig$  for  $f$  and  $g$  real valued functions, for  $x > 0$ , we have

$$\begin{aligned} f_{xx} &= Af, \\ g_{xx} &= Ag. \end{aligned}$$

Thus,

$$\partial_x(f_xg - g_xf) = f_{xx}g - g_{xx}f = 0 \text{ when } x \neq 0.$$

Hence, by using  $f, g \in H^2(\mathbb{R}^+)$ , we have

$$f_x(x)g(x) - g_x(x)f(x) = 0 \text{ when } x \neq 0.$$

Then, for all  $x \neq 0$ , we have

$$\mathcal{I}m(\varphi_x(x)\overline{\varphi(x)}) = g_x(x)f(x) - f_x(x)g(x) = 0,$$

hence, (2.8) is equivalent to

$$\begin{cases} 0 = \varphi_{xx} - \omega\varphi + \frac{3}{16}|\varphi|^4\varphi, & \text{for } x > 0 \\ \varphi_x(0) = \alpha\varphi(0), \\ \varphi \in H^2(\mathbb{R}^+). \end{cases} \quad (2.9)$$

We have the following description of the profile  $\varphi$ .

**Proposition 2.2.** *Let  $\omega > \alpha^2$ . There exists a unique (up to phase shift) solution  $\varphi$  of (2.9), which is of the form*

$$\varphi = 2\sqrt[4]{\omega} \operatorname{sech}^{\frac{1}{2}} \left( 2\sqrt{\omega}|x| + \tanh^{-1} \left( \frac{-\alpha}{\sqrt{\omega}} \right) \right), \quad (2.10)$$

for all  $x > 0$ .

*Proof.* Let  $w$  be the even function defined by

$$w(x) = \begin{cases} \varphi(x) & \text{if } x \geq 0, \\ \varphi(-x) & \text{if } x \leq 0. \end{cases}$$

Then  $w$  solves

$$\begin{cases} 0 = -w_{xx} + \omega w - \frac{3}{16}|w|^4w, & \text{for } x \neq 0, \\ w_x(0^+) - w_x(0^-) = 2\alpha w(0), \\ w \in H^2(\mathbb{R}) \setminus \{0\} \cap H^1(\mathbb{R}). \end{cases} \quad (2.11)$$

Using the results of Fukuizumi and Jeanjean [8], we obtain that

$$w(x) = 2\sqrt[4]{\omega} \operatorname{sech}^{\frac{1}{2}} \left( 2\sqrt{\omega}|x| + \tanh^{-1} \left( \frac{-\alpha}{\sqrt{\omega}} \right) \right)$$

up to phase shift provided  $\omega > \alpha^2$ . Hence, for  $x > 0$  we have

$$\varphi(x) = 2\sqrt[4]{\omega} \operatorname{sech}^{\frac{1}{2}} \left( 2\sqrt{\omega}|x| + \tanh^{-1} \left( \frac{-\alpha}{\sqrt{\omega}} \right) \right)$$

up to phase shift. This implies the desired result.  $\square$

**2.2.2. The variational problems.** In this section, we give the proof of Proposition 1.3.

First, we introduce another variational problem:

$$\tilde{d}(\omega) := \inf \left\{ \tilde{S}_\omega(v) \mid v \text{ even}, \tilde{K}_\omega(v) = 0, v \in H^1(\mathbb{R}) \setminus \{0\} \right\}, \quad (2.12)$$

where  $\tilde{S}_\omega, \tilde{K}_\omega$  are defined for all  $v \in H^1(\mathbb{R})$  by

$$\begin{aligned} \tilde{S}_\omega(v) &:= \frac{1}{2} \left[ \|v_x\|_{L^2(\mathbb{R})}^2 + \omega \|v\|_{L^2(\mathbb{R})}^2 + 2\alpha |v(0)|^2 \right] - \frac{1}{32} \|v\|_{L^6(\mathbb{R})}^6, \\ \tilde{K}_\omega(v) &:= \|v_x\|_{L^2(\mathbb{R})}^2 + \omega \|v\|_{L^2(\mathbb{R})}^2 + 2\alpha |v(0)|^2 - \frac{3}{16} \|v\|_{L^6(\mathbb{R})}^6. \end{aligned}$$

The functional  $\tilde{K}_\omega$  is called Nehari functional. The following result has proved in [8, 9].

**Proposition 2.3.** *Let  $\omega > \alpha^2$  and  $\varphi$  satisfies*

$$\begin{cases} -\varphi_{xx} + 2\alpha\delta\varphi + \omega\varphi - \frac{3}{16}|\varphi|^4\varphi = 0, \\ \varphi \in H^1(\mathbb{R}) \setminus \{0\}. \end{cases} \quad (2.13)$$

*Then, there exists a unique positive solution  $\varphi$  of (2.13). This solution is the unique positive minimizer of (2.12). Furthermore, we have an explicit formula for  $\varphi$*

$$\varphi(x) = 2\sqrt[4]{\omega} \operatorname{sech}^{\frac{1}{2}} \left( 2\sqrt{\omega}|x| + \tanh^{-1} \left( \frac{-\alpha}{\sqrt{\omega}} \right) \right).$$

We have the following relation between the variational problems.

**Proposition 2.4.** *Let  $\omega > \alpha^2$ . We have*

$$d(\omega) = \frac{1}{2} \tilde{d}(\omega).$$



*Proof.* Assume  $v$  is a minimizer of (1.7), define the  $H^1(\mathbb{R})$  function  $w$  by

$$w(x) = \begin{cases} v(x) & \text{if } x > 0, \\ v(-x) & \text{if } x < 0. \end{cases}$$

The function  $w \in H^1(\mathbb{R}) \setminus \{0\}$  verifies

$$\begin{aligned} \tilde{S}_\omega(w) &= 2S_\omega(v) = 2d(\omega), \\ \tilde{K}_\omega(w) &= 2K_\omega(v) = 0. \end{aligned}$$

This implies that

$$\tilde{d}(\omega) \leq \tilde{S}_\omega(w) = 2d(\omega). \quad (2.14)$$

Now, assume  $v$  is a minimizer of (2.12). Let  $w$  be the restriction of  $v$  on  $\mathbb{R}^+$ , then,

$$K_\omega(w) = \frac{1}{2}\tilde{K}_\omega(v) = 0.$$

Hence, we obtain

$$\tilde{d}(\omega) = \tilde{S}_\omega(v) = 2S_\omega(w) \geq 2d(\omega). \quad (2.15)$$

Combining (2.14) and (2.15) we have

$$\tilde{d}(\omega) = 2d(\omega).$$

This implies the desired result.  $\square$

*Proof of Theorem 1.3.* Let  $v$  be a minimizer of (1.7). Define  $w(x) \in H^1(\mathbb{R})$  by

$$w(x) = \begin{cases} v(x) & \text{if } x > 0, \\ v(-x) & \text{if } x < 0. \end{cases}$$

Then,  $w$  is an even function. Moreover,  $w$  satisfies

$$\begin{aligned} \tilde{K}_\omega(w) &= 2K_\omega(v) = 0, \\ \tilde{S}_\omega(w) &= 2S_\omega(v) = 2d(\omega) = \tilde{d}(\omega). \end{aligned}$$

Hence,  $w$  is a minimizer of (2.12). From Propositions 2.3, 2.4,  $w$  is of the form  $e^{i\theta}\varphi$ , where  $\theta \in \mathbb{R}$  is a constant and  $\varphi$  is of the form

$$2\sqrt[4]{\omega} \operatorname{sech}^{\frac{1}{2}} \left( 2\sqrt{\omega}|x| + \tanh^{-1} \left( \frac{-\alpha}{\sqrt{\omega}} \right) \right).$$

Hence,  $v = w|_{\mathbb{R}^+}$  satisfies

$$v(x) = e^{i\theta}\varphi(x),$$

for  $x > 0$ . This completes the proof of Proposition 1.3.  $\square$

**2.2.3. Stability and instability of standing waves.** In this section, we give the proof of Theorem 1.4. We use the notations  $\tilde{S}_\omega$  and  $\tilde{K}_\omega$  as in Section 2.2.2. First, we define

$$N(v) := \frac{3}{16} \|v\|_{L^6(\mathbb{R}^+)}^6, \quad (2.16)$$

$$L(v) := \|v_x\|_{L^2(\mathbb{R}^+)}^2 + \omega \|v\|_{L^2(\mathbb{R}^+)}^2 + \alpha |v(0)|^2. \quad (2.17)$$

We can rewrite  $S_\omega, K_\omega$  as follows

$$\begin{aligned} S_\omega &= \frac{1}{2}L - \frac{1}{6}N, \\ K_\omega &= L - N. \end{aligned}$$

We have the following classical properties of the above functions.

**Lemma 2.5.** *Let  $(\omega, \alpha) \in \mathbb{R}^2$  such that  $\omega > \alpha^2$ . The following assertions hold.*

(1) *There exists a constant  $C > 0$  such that*

$$L(v) \geq C \|v\|_{H^1(\mathbb{R}^+)}^2 \quad \forall v \in H^1(\mathbb{R}^+).$$

(2) *We have  $d(\omega) > 0$ .*

(3) *If  $v \in H^1(\mathbb{R}^+)$  satisfies  $K_\omega(v) < 0$  then  $L(v) > 3d(\omega)$ .*

*Proof.* We have

$$\begin{aligned} |v(0)|^2 &= - \int_0^\infty \partial_x (|v(x)|^2) dx = -2\operatorname{Re} \int_0^\infty v(x) \bar{v}_x(x) dx \\ &\leq 2\|v\|_{L^2(\mathbb{R}^+)} \|v_x\|_{L^2(\mathbb{R}^+)}. \end{aligned}$$

Hence,

$$\begin{aligned} L(v) &= \|v_x\|_{L^2(\mathbb{R}^+)}^2 + \omega \|v\|_{L^2(\mathbb{R}^+)}^2 + \alpha |v(0)|^2 \\ &\geq \|v_x\|_{L^2(\mathbb{R}^+)}^2 + \omega \|v\|_{L^2(\mathbb{R}^+)}^2 - 2|\alpha| \|v\|_{L^2(\mathbb{R}^+)} \|v_x\|_{L^2(\mathbb{R}^+)} \\ &\geq C \|v\|_{H^1(\mathbb{R}^+)}^2 + (1-C) \|v_x\|_{L^2(\mathbb{R}^+)}^2 + (\omega - C) \|v\|_{L^2(\mathbb{R}^+)}^2 - 2|\alpha| \|v\|_{L^2(\mathbb{R}^+)} \|v_x\|_{L^2(\mathbb{R}^+)} \\ &\geq C \|v\|_{H^1(\mathbb{R}^+)}^2 + (2\sqrt{(1-C)(\omega - C)} - 2|\alpha|) \|v\|_{L^2(\mathbb{R}^+)} \|v_x\|_{L^2(\mathbb{R}^+)}. \end{aligned}$$

From the assumption  $\omega > \alpha^2$ , we can choose  $C \in (0, 1)$  such that

$$2\sqrt{(1-C)(\omega - C)} - 2|\alpha| > 0.$$

This implies (1). Now, we prove (2). Let  $v$  be an element of  $H^1(\mathbb{R}^+)$  satisfying  $K_\omega(v) = 0$ . We have

$$C \|v\|_{H^1(\mathbb{R}^+)}^2 \leq L(v) = N(v) \leq C_1 \|v\|_{H^1(\mathbb{R}^+)}^6.$$

Then,

$$\|v\|_{H^1(\mathbb{R}^+)}^2 \geq \sqrt[4]{\frac{C}{C_1}}.$$

From the fact that, for  $v$  satisfying  $K_\omega(v) = 0$ , we have  $S_\omega(v) = S_\omega(v) - \frac{1}{6}K_\omega(v) = \frac{1}{3}L(v)$ , this implies that

$$d(\omega) = \frac{1}{3} \inf \{L(v) : v \in H^1(\mathbb{R}^+), K_\omega(v) = 0\} \geq \frac{C}{3} \sqrt[4]{\frac{C}{C_1}} > 0.$$

Finally, we prove (3). Let  $v \in H^1(\mathbb{R}^+)$  satisfying  $K_\omega(v) < 0$ . Then, there exists  $\lambda_1 \in (0, 1)$  such that  $K_\omega(\lambda_1 v) = \lambda_1^2 L(v) - \lambda_1^6 N(v) = 0$ . Since  $v \neq 0$ , we have  $3d(\omega) \leq L(\lambda_1 v) = \lambda_1^2 L(v) < L(v)$ .  $\square$

Define

$$\tilde{N}(v) := \frac{3}{16} \|v\|_{L^6}^6, \quad (2.18)$$

$$\tilde{L}(v) := \|v_x\|_{L^2}^2 + \omega \|v\|_{L^2}^2 + 2\alpha |v(0)|^2. \quad (2.19)$$

We can rewrite  $S_\omega, K_\omega$  as follows

$$\begin{aligned} \tilde{S}_\omega &= \frac{1}{2} \tilde{L} - \frac{1}{6} \tilde{N}, \\ \tilde{K}_\omega &= \tilde{L} - \tilde{N}. \end{aligned}$$

As consequence of the previous lemma, we have the following result.

**Lemma 2.6.** *Let  $(\omega, \alpha) \in \mathbb{R}^2$  such that  $\omega > \alpha^2$ . The following assertions hold.*

(1) *There exists a constant  $C > 0$  such that*

$$\tilde{L}(v) \geq C \|v\|_{H^1}^2 \quad \forall v \in H^1(\mathbb{R}).$$

(2) *We have  $\tilde{d}(\omega) > 0$ .*

(3) *If  $v \in H^1$  satisfies  $\tilde{K}_\omega(v) < 0$  then  $\tilde{L}(v) > 3\tilde{d}(\omega)$ .*

We introduce the following properties.

**Lemma 2.7.** *Let  $2 \leq p < \infty$  and  $(f_n)$  be a bounded sequence in  $L^p(\mathbb{R})$ . Assume that  $f_n \rightarrow f$  a.e in  $\mathbb{R}$ . Then we have*

$$\|f_n\|_{L^p}^p - \|f_n - f\|_{L^p}^p - \|f\|_{L^p}^p \rightarrow 0.$$

**Lemma 2.8.** *The following minimization problem is equivalent to the problem (2.12) i.e same minimum and the minimizers:*

$$d := \inf \left\{ \frac{1}{16} \|u\|_{L^6}^6 : u \text{ even}, u \in H^1(\mathbb{R}) \setminus \{0\}, \tilde{K}_\omega(u) \leq 0 \right\}. \quad (2.20)$$

*Proof.* We see that the minimizer problem (2.12) is equivalent to following problem:

$$\inf \left\{ \frac{1}{16} \|u\|_{L^6}^6 : u \text{ even } u \in H^1(\mathbb{R}) \setminus \{0\}, \tilde{K}_\omega(u) = 0 \right\}. \quad (2.21)$$

Let  $v$  be a minimizer of (2.12) then  $\tilde{K}_\omega(v) \leq 0$ , hence,  $\tilde{d}(\omega) = \frac{1}{16} \|v\|_{L^6}^6 \geq d$ . Now, let  $v$  be a minimizer of (2.20). We prove that  $\tilde{K}_\omega(v) = 0$ . Indeed, assuming  $\tilde{K}_\omega(v) < 0$ , we have

$$\tilde{K}_\omega(\lambda v) = \lambda^2 \left( \|v_x\|_{L^2}^2 + \omega \|v\|_{L^2}^2 + 2\alpha |v(0)|^2 - \frac{3\lambda^4}{16} \|v\|_{L^6}^6 \right) \leq 0,$$

as  $0 < \lambda$  is small enough. Thus, by continuity, there exists a  $\lambda_0 \in (0, 1)$  such that  $\tilde{K}_\omega(\lambda_0 v) = 0$ . We have  $d < \tilde{d}(\omega) \leq \frac{1}{16} \|\lambda_0 v\|_{L^6}^6 < \frac{1}{16} \|v\|_{L^6}^6 = d$ . Which is a contradiction. It implies that  $\tilde{K}_\omega(v) = 0$  and  $v$  is a minimizer of (2.21), hence  $v$  is a minimizer of (2.12). This completes the proof.  $\square$

Now, using the similar arguments in [9, Proof of Proposition 2], we have the following result.

**Proposition 2.9.** *Let  $(\omega, \alpha) \in \mathbb{R}^2$  be such that  $\alpha < 0$ ,  $\omega > \alpha^2$  and  $(w_n) \subset H^1(\mathbb{R})$  be a even sequence satisfying the following properties*

$$\begin{aligned} \tilde{S}_\omega(w_n) &\rightarrow \tilde{d}(\omega), \\ \tilde{K}_\omega(w_n) &\rightarrow 0. \end{aligned}$$

*as  $n \rightarrow \infty$ . Then, there exists a minimizer  $w$  of (2.12) such that  $w_n \rightarrow w$  strongly in  $H^1(\mathbb{R})$  up to subsequence.*

*Proof.* In what follows, we shall often extract subsequence without mentioning this fact explicitly. We divide the proof into two steps.

**Step 1. Weakly convergence to a nonvanishing function of minimizer sequence** We have

$$\frac{1}{3} \tilde{L}(w_n) = \tilde{S}_\omega(w_n) - \frac{1}{6} \tilde{K}_\omega(w_n) \rightarrow \tilde{d}(\omega),$$

as  $n \rightarrow \infty$ . Then,  $(w_n)$  is bounded in  $H^1(\mathbb{R})$  and there exists  $w \in H^1(\mathbb{R})$  even such that  $w_n \rightharpoonup w$  in  $H^1(\mathbb{R})$  up to subsequence. We prove  $w \neq 0$ . Assume that  $w \equiv 0$ . Define, for  $u \in H^1(\mathbb{R})$ ,

$$\begin{aligned} S_\omega^0(u) &= \frac{1}{2} \|u_x\|_{L^2}^2 + \frac{\omega}{2} \|u\|_{L^2}^2 - \frac{1}{32} \|u\|_{L^6}^6, \\ K_\omega^0(u) &= \|u_x\|_{L^2}^2 + \omega \|u\|_{L^2}^2 - \frac{3}{16} \|u\|_{L^6}^6. \end{aligned}$$

Let  $\psi_\omega$  be minimizer of following problem

$$\begin{aligned} d^0(\omega) &= \inf \{ S_\omega^0(u) : u \text{ even}, u \in H^1(\mathbb{R}) \setminus \{0\}, K_\omega^0(u) = 0 \} \\ &= \inf \left\{ \frac{1}{16} \|u\|_{L^6}^6 : u \text{ even}, u \in H^1(\mathbb{R}) \setminus \{0\}, K_\omega^0(u) \leq 0 \right\}. \end{aligned}$$

We have  $K_\omega^0(w_n) = \tilde{K}_\omega(w_n) - 2\alpha |w_n(0)|^2 \rightarrow 0$ , as  $n \rightarrow \infty$ . Since,  $\alpha < 0$ . we have  $\tilde{K}_\omega(\psi_\omega) < 0$  and hence we obtain

$$\tilde{d}(\omega) < \frac{1}{16} \|\psi_\omega\|_{L^6}^6 = d^0(\omega) \quad (2.22)$$

We set

$$\lambda_n = \left( \frac{\|\partial_x w_n\|_{L^2}^2 + \omega \|w_n\|_{L^2}^2}{\frac{3}{16} \|w_n\|_{L^6}^6} \right)^{\frac{1}{4}}.$$

We here remark that  $0 < \tilde{d}(\omega) = \lim_{n \rightarrow \infty} \frac{1}{16} \|w_n\|_{L^6}^6$ . It follows that

$$\lambda_n^4 - 1 = \frac{K_\omega^0(w_n)}{\frac{3}{16} \|w_n\|_{L^6}^6} \rightarrow 0,$$

as  $n \rightarrow \infty$ . We see that  $K_\omega^0(\lambda_n w_n) = 0$  and  $\lambda_n w_n \neq 0$ . By the definition of  $d^0(\omega)$ , we have

$$d^0(\omega) \leq \frac{1}{16} \|\lambda_n w_n\|_{L^6}^6 \rightarrow \tilde{d}(\omega) \text{ as } n \rightarrow \infty.$$

This contradicts to (2.22). Thus,  $w \neq 0$ .

**Step 2. Conclude the proof** Using Lemma 2.7 we have

$$\tilde{K}_\omega(w_n) - \tilde{K}_\omega(w_n - w) - \tilde{K}_\omega(w) \rightarrow 0, \quad (2.23)$$

$$\tilde{L}(w_n) - \tilde{L}(w_n - w) - \tilde{L}(w) \rightarrow 0. \quad (2.24)$$

Now, we prove  $\tilde{K}_\omega(w) \leq 0$  by contradiction. Suppose that  $\tilde{K}_\omega(w) > 0$ . By the assumption  $\tilde{K}_\omega(w_n) \rightarrow 0$  and (2.23), we have

$$\tilde{K}_\omega(w_n - w) \rightarrow -\tilde{K}_\omega(w) < 0.$$

Thus,  $\tilde{K}_\omega(w_n - w) < 0$  for  $n$  large enough. By Lemma 2.6 (3), we have  $\tilde{L}(w_n - w) \geq 3\tilde{d}(\omega)$ . Since  $\tilde{L}(w_n) \rightarrow 3\tilde{d}(\omega)$ , by (2.24), we have

$$\tilde{L}(w) = \lim_{n \rightarrow \infty} (\tilde{L}(w_n) - \tilde{L}(w_n - w)) \leq 0.$$

Moreover,  $w \neq 0$  and by Lemma 2.6 (1), we have  $\tilde{L}(w) > 0$ . This is a contradiction. Hence,  $\tilde{K}_\omega(w) < 0$ . By Lemma 2.6 (2), (3) and weakly lower semicontinuity of  $\tilde{L}$ , we have

$$3\tilde{d}(\omega) \leq \tilde{L}(w) \leq \liminf_{n \rightarrow \infty} \tilde{L}(w_n) = 3\tilde{d}(\omega).$$

Thus,  $\tilde{L}(w) = 3\tilde{d}(\omega)$ . Combining with (2.24), we have  $\tilde{L}(w_n - w) \rightarrow 0$ , as  $n \rightarrow \infty$ . By Lemma 2.6 (1), we have  $w_n \rightarrow w$  strongly in  $H^1(\mathbb{R})$ . Hence,  $w$  is a minimizer of (2.12). This completes the proof.  $\square$

To prove the stability statement (1) for  $\alpha < 0$  in Theorem 1.4, we will use similar arguments as in the work of Colin and Ohta [3]. We need the following property.

**Lemma 2.10.** *Let  $\alpha < 0$ ,  $\omega > \alpha^2$ . If a sequence  $(v_n) \subset H^1(\mathbb{R}^+)$  satisfies*

$$S_\omega(v_n) \rightarrow d(\omega), \quad (2.25)$$

$$K_\omega(v_n) \rightarrow 0, \quad (2.26)$$

*then there exist a constant  $\theta_0 \in \mathbb{R}$  such that  $v_n \rightarrow e^{i\theta_0}\varphi$ , up to subsequence, where  $\varphi$  is defined as in Proposition 1.3.*

*Proof.* Define the sequence  $(w_n) \subset H^1(\mathbb{R})$  as follows,

$$w_n(x) = \begin{cases} v_n(x) & \text{for } x > 0, \\ v_n(-x) & \text{for } x < 0. \end{cases}$$

We can check that

$$\begin{aligned} \tilde{S}_\omega(w_n) &= 2S_\omega(v_n) \rightarrow 2d(\omega) = \tilde{d}(\omega), \\ \tilde{K}_\omega(w_n) &= 2K_\omega(v_n) \rightarrow 0, \end{aligned}$$

as  $n \rightarrow \infty$ . Using Proposition 2.9, there exists a minimizer  $w_0$  of (2.12) such that  $w_n \rightarrow w_0$  strongly in  $H^1(\mathbb{R})$ , up to subsequence. For convenience, we assume that  $w_n \rightarrow w_0$  strongly in  $H^1(\mathbb{R})$ . By Proposition 2.3, there exists a constant  $\theta_0 \in \mathbb{R}$  such that

$$w_0 = e^{i\theta_0}\tilde{\varphi},$$

where  $\tilde{\varphi}$  is defined by

$$\tilde{\varphi}(x) = \begin{cases} \varphi(x) & \text{for } x > 0, \\ \varphi(-x) & \text{for } x < 0. \end{cases} \quad (2.27)$$

Hence, the sequence  $(v_n)$  is the restriction of the sequence  $(w_n)$  on  $\mathbb{R}^+$ , and satisfies

$$v_n \rightarrow e^{i\theta_0}\varphi, \text{ strongly in } H^1(\mathbb{R}^+),$$

up to subsequence. This completes the proof.  $\square$

Define

$$\begin{aligned}\mathcal{A}_\omega^+ &= \{v \in H^1(\mathbb{R}^+) \setminus \{0\} : S_\omega(v) < d(\omega), K_\omega(v) > 0\}, \\ \mathcal{A}_\omega^- &= \{v \in H^1(\mathbb{R}^+) \setminus \{0\} : S_\omega(v) < d(\omega), K_\omega(v) < 0\}, \\ \mathcal{B}_\omega^+ &= \{v \in H^1(\mathbb{R}^+) \setminus \{0\} : S_\omega(v) < d(\omega), N(v) < 3d(\omega)\}, \\ \mathcal{B}_\omega^- &= \{v \in H^1(\mathbb{R}^+) \setminus \{0\} : S_\omega(v) < d(\omega), N(v) > 3d(\omega)\}.\end{aligned}$$

We have the following result.

**Lemma 2.11.** *Let  $\omega, \alpha \in \mathbb{R}^2$  such that  $\alpha < 0$  and  $\omega > \alpha^2$ .*

- (1) *The sets  $\mathcal{A}_\omega^+$  and  $\mathcal{A}_\omega^-$  are invariant under the flow of (1.1).*
- (2)  *$\mathcal{A}_\omega^+ = \mathcal{B}_\omega^+$  and  $\mathcal{A}_\omega^- = \mathcal{B}_\omega^-$ .*

*Proof.* (1) Let  $u_0 \in \mathcal{A}_\omega^+$  and  $u(t)$  the associated solution for (1.1) on  $(T_{min}, T_{max})$ . By  $u_0 \neq 0$  and the conservation laws, we see that  $S_\omega(u(t)) = S_\omega(u_0) < d(\omega)$  for  $t \in (T_{min}, T_{max})$ . Moreover, by definition of  $d(\omega)$  we have  $K_\omega(u(t)) \neq 0$  on  $(T_{min}, T_{max})$ . Since the function  $t \mapsto K_\omega(u(t))$  is continuous, we have  $K_\omega(u(t)) > 0$  on  $(T_{min}, T_{max})$ . Hence,  $\mathcal{A}_\omega^+$  is invariant under flow of (1.1). By the same way,  $\mathcal{A}_\omega^-$  is invariant under flow of (1.1).

(2) If  $v \in \mathcal{A}_\omega^+$  then by (2.19), (2.18) we have  $N(v) = 3S_\omega(v) - 2K_\omega(v) < 3d(\omega)$ , which shows  $v \in \mathcal{B}_\omega^+$ , hence  $\mathcal{A}_\omega^+ \subset \mathcal{B}_\omega^+$ . Now, let  $v \in \mathcal{B}_\omega^+$ . We show  $K_\omega(v) > 0$  by contradiction. Suppose that  $K_\omega(v) \leq 0$ . Then, by Lemma 2.6 (3),  $L(v) \geq 3d(\omega)$ . Thus, by (2.19) and (2.18), we have

$$S_\omega(v) = \frac{1}{2}L(v) - \frac{1}{6}N(v) \geq d(\omega),$$

which contradicts  $S_\omega(v) < d(\omega)$ . Therefore, we have  $K_\omega(v) > 0$ , which shows  $v \in \mathcal{A}_\omega^+$  and  $\mathcal{B}_\omega^+ \subset \mathcal{A}_\omega^+$ . Next, if  $v \in \mathcal{A}_\omega^-$ , then by Lemma 2.6 (3),  $L(v) > 3d(\omega)$ . Thus, by (2.19) and (2.18), we have  $N(v) = L(v) - K_\omega(v) > 3d(\omega)$ , which shows  $v \in \mathcal{B}_\omega^-$ . Thus,  $\mathcal{A}_\omega^- \subset \mathcal{B}_\omega^-$ . Finally, if  $v \in \mathcal{B}_\omega^-$ , then by (2.19) and (2.18), we have  $2K_\omega(v) = 3S_\omega(v) - N(v) < 3d(\omega) - 3d(\omega) = 0$ , which shows  $v \in \mathcal{A}_\omega^-$ , hence,  $\mathcal{B}_\omega^- \subset \mathcal{A}_\omega^-$ . This completes the proof.  $\square$

From Proposition 1.3, we have

$$d(\omega) = S_\omega(\varphi).$$

Since  $\alpha < 0$ , we see that

$$d''(\omega) = \partial_\omega \|\varphi\|_{L^2(\mathbb{R}^+)}^2 = \frac{1}{2} \partial_\omega \|\tilde{\varphi}\|_{L^2(\mathbb{R})}^2 > 0,$$

where  $\tilde{\varphi}$  is defined as (2.27) and we know from [9], [8] that

$$\partial_\omega \|\tilde{\varphi}\|_{L^2(\mathbb{R})}^2 > 0,$$

for  $\alpha < 0$ . We define the function  $h : (-\varepsilon_0, \varepsilon_0) \rightarrow \mathbb{R}$  by

$$h(\tau) = d(\omega \pm \tau),$$

for  $\varepsilon_0 > 0$  sufficiently small such that  $h''(\tau) > 0$  and the sign  $+$  or  $-$  is selected such that  $h'(\tau) > 0$  for  $\tau \in (-\varepsilon_0, \varepsilon_0)$ . Without loss of generality, we can assume

$$h(\tau) = d(\omega + \tau).$$

**Lemma 2.12.** *Let  $(\omega, \alpha) \in \mathbb{R}^2$  such that  $\omega > \alpha^2$  and let  $h$  be defined as above. Then, for any  $\varepsilon \in (0, \varepsilon_0)$ , there exists  $\delta > 0$  such that if  $v_0 \in H^1(\mathbb{R}^+)$  satisfies  $\|v_0 - \varphi\|_{H^1(\mathbb{R}^+)} < \delta$ , then the solution  $v$  of (1.1) with  $v(0) = v_0$  satisfies  $3h(-\varepsilon) < N(v(t)) < 3h(\varepsilon)$  for all  $t \in (T_{min}, T_{max})$ .*

*Proof.* The proof of the above lemma is similar to the one of [3] or [19]. Let  $\varepsilon \in (0, \varepsilon_0)$ . Since  $h$  is increasing, we have  $h(-\varepsilon) < h(0) < h(\varepsilon)$ . Moreover, by  $K_\omega(\varphi) = 0$  and (2.18), (2.19), we see that  $3h(0) = 3d(\omega) = 3S_\omega(\varphi) = N(\varphi)$ . Thus, if  $u_0 \in H^1(\mathbb{R}^+)$  satisfies  $\|u_0 - \varphi\|_{H^1(\mathbb{R}^+)} < \delta$  then we have  $3h(0) = N(u_0) + O(\delta)$  and  $3h(-\varepsilon) < N(u_0) < 3h(\varepsilon)$  for sufficiently small  $\delta > 0$ . Since  $h(\pm\varepsilon) = d(\omega \pm \varepsilon)$  and the set  $\mathcal{B}_\omega^\pm$  are invariant under the flow of (1.1) by Lemma 2.11, to conclude the proof, we only have to show that there exists  $\delta > 0$  such that if  $u_0 \in H^1(\mathbb{R}^+)$  satisfies

$\|u_0 - \varphi\|_{H^1(\mathbb{R}^+)} < \delta$  then  $S_{\omega \pm \varepsilon}(u_0) < h(\pm \varepsilon)$ . Assume that  $u_0 \in H^1(\mathbb{R}^+)$  satisfies  $\|u_0 - \varphi\|_{H^1(\mathbb{R}^+)} < \delta$ . We have

$$\begin{aligned} S_{\omega \pm \varepsilon}(u_0) &= S_{\omega \pm \varepsilon}(\varphi) + O(\delta) \\ &= S_{\omega}(\varphi) \pm \varepsilon M(\varphi) + O(\delta) \\ &= h(0) \pm \varepsilon h'(0) + O(\delta). \end{aligned}$$

On the other hand, by the Taylor expansion, there exists  $\tau_1 = \tau_1(\varepsilon) \in (-\varepsilon_0, \varepsilon_0)$  such that

$$h(\pm \varepsilon) = h(0) \pm \varepsilon h'(0) + \frac{\varepsilon^2}{2} h''(\tau_1).$$

Since  $h''(\tau_1) > 0$  by definition of  $h$ , we see that there exists  $\delta > 0$  such that if  $u_0 \in H^1(\mathbb{R}^+)$  satisfies  $\|u_0 - \varphi\|_{H^1(\mathbb{R}^+)} < \delta$  then  $S_{\omega \pm \varepsilon}(u_0) < h(\pm \varepsilon)$ . This completes the proof.  $\square$

*Proof of Theorem 1.4 (1).* Assume that  $e^{i\omega t}\varphi$  is not stable for (1.1). Then, there exists a constant  $\varepsilon_1 > 0$ , a sequence of solutions  $(v^n)$  to (1.1), and a sequence  $\{t_n\} \in (0, \infty)$  such that

$$v_n(0) \rightarrow \varphi \text{ in } H^1(\mathbb{R}^+), \quad \inf_{\theta \in \mathbb{R}} \|v_n(t_n) - e^{i\theta}\varphi\|_{H^1(\mathbb{R}^+)} \geq \varepsilon_1. \quad (2.28)$$

By using the conservation laws of solutions of (1.1), we have

$$S_{\omega}(v_n(t_n)) = S_{\omega}(v_n(0)) \rightarrow S_{\omega}(\varphi) = d(\omega). \quad (2.29)$$

Using Lemma 2.12, we have

$$N(v_n(t_n)) \rightarrow 3d(\omega). \quad (2.30)$$

Combined (2.29) and (2.30), we have

$$K_{\omega}(v_n(t_n)) = 2S_{\omega}(v_n(t_n)) - \frac{2}{3}N(v_n(t_n)) \rightarrow 0.$$

Therefore, using Lemma 2.10, there exists  $\theta_0 \in \mathbb{R}$  such that  $(v_n(t_n, \cdot))$  has a subsequence (we denote it by the same letter) that converges to  $e^{i\theta_0}\varphi$  in  $H^1(\mathbb{R}^+)$ , where  $\varphi$  is defined as in Proposition 1.3. Hence, we have

$$\inf_{\theta \in \mathbb{R}} \|v_n(t_n) - e^{i\theta}\varphi\|_{H^1(\mathbb{R}^+)} \rightarrow 0, \quad (2.31)$$

as  $n \rightarrow \infty$ , this contradicts (2.28). Hence, we obtain the desired result.  $\square$

Next, we give the proof of Theorem 1.4 (2), using similar arguments as in [18].

Assume  $\alpha > 0$ . Let  $e^{i\omega t}\varphi$  be the standing wave solution of (1.1). Introduce the scaling

$$v_{\lambda}(x) = \lambda^{\frac{1}{2}}v(\lambda x).$$

Let  $S_{\omega}$ ,  $K_{\omega}$  be defined as in Proposition 1.3, for convenience, we will remove the index  $\omega$ . Define

$$P(v) := \frac{\partial}{\partial \lambda} S(v_{\lambda})|_{\lambda=1} = \|v_x\|_{L^2(\mathbb{R}^+)}^2 - \frac{1}{16}\|v\|_{L^6(\mathbb{R}^+)}^6 + \frac{\alpha}{2}|v(0)|^2.$$

In the following lemma, we investigate the behaviour of the above functional under scaling.

**Lemma 2.13.** *Let  $v \in H^1(\mathbb{R}^+) \setminus \{0\}$  be such that  $v(0) \neq 0$ ,  $P(v) \leq 0$ . Then there exists  $\lambda_0 \in (0, 1]$  such that*

- (i)  $P(v_{\lambda_0}) = 0$ ,
- (ii)  $\lambda_0 = 1$  if only if  $P(v) = 0$ ,
- (iii)  $\frac{\partial}{\partial \lambda} S(v_{\lambda}) = \frac{1}{\lambda}P(v_{\lambda})$ ,
- (iv)  $\frac{\partial}{\partial \lambda} S(v_{\lambda}) > 0$  on  $(0, \lambda_0)$  and  $\frac{\partial}{\partial \lambda} S(v_{\lambda}) < 0$  on  $(\lambda_0, \infty)$ ,
- (v) The function  $\lambda \rightarrow S(v_{\lambda})$  is concave on  $(\lambda_0, \infty)$ .

*Proof.* A simple calculation leads to

$$P(v_{\lambda}) = \lambda^2\|v_x\|_{L^2(\mathbb{R}^+)}^2 - \frac{\lambda^2}{16}\|v\|_{L^6(\mathbb{R}^+)}^6 + \frac{\lambda\alpha}{2}|v(0)|^2.$$

Then, for  $\lambda > 0$  small enough, we have

$$P(v_{\lambda}) > 0.$$

By continuity of  $P$ , there exists  $\lambda_0 \in (0, 1]$  such that  $P(v_{\lambda_0}) = 0$ . Hence (i) is proved. If  $\lambda_0 = 1$  then  $P(v) = 1$ . Conversely, if  $P(v) = 0$  then

$$0 = P(v_{\lambda_0}) = \lambda_0^2 P(v) + \frac{\lambda_0 - \lambda_0^2}{2} \alpha |v(0)|^2 = \frac{\lambda_0 - \lambda_0^2}{2} \alpha |v(0)|^2.$$

By the assumption  $v(0) \neq 0$ , we have  $\lambda_0 = 1$ , hence (ii) is proved. Item (iii) is obtained by a simple calculation. To obtain (iv), we use (iii). We have

$$\begin{aligned} P(v_\lambda) &= \lambda^2 \lambda_0^{-2} P(v_{\lambda_0}) + \left( \frac{\lambda \alpha}{2} - \frac{\lambda^2 \lambda_0^{-1} \alpha}{2} \right) |v(0)|^2 \\ &= \frac{\lambda \alpha (\lambda_0 - \lambda)}{2 \lambda_0} |v(0)|^2. \end{aligned}$$

Hence,  $P(v_\lambda) > 0$  if  $\lambda < \lambda_0$  and  $P(v_\lambda) < 0$  if  $\lambda > \lambda_0$ . This proves (iv). Finally, we have

$$\frac{\partial^2}{\partial^2 \lambda} S(v_\lambda) = P(v) - \frac{\alpha}{2} |v(0)|^2 < 0.$$

This proves (v).  $\square$

In the case of functions such that  $v(0) = 0$ , we have the following lemma.

**Lemma 2.14.** *Let  $v \in H^1(\mathbb{R}^+) \setminus \{0\}$ ,  $v(0) = 0$  and  $P(v) = 0$  then we have*

$$S(v_\lambda) = S(v) \quad \text{for all } \lambda > 0.$$

*Proof.* The proof is simple, using the fact that

$$\frac{\partial}{\partial \lambda} S(v_\lambda) = \frac{1}{\lambda} P(v_\lambda) = \lambda P(v) = 0.$$

Hence, we obtain the desired result.  $\square$

Now, consider the minimization problems

$$d_{\mathcal{M}} := \inf \{S(v) : v \in \mathcal{M}\}, \quad (2.32)$$

$$m := \inf \{S(v), v \in H^1(\mathbb{R}^+) \setminus 0, S'(v) = 0\}, \quad (2.33)$$

where

$$\mathcal{M} = \{v \in H^1(\mathbb{R}^+) \setminus 0, P(v) = 0, K(v) \leq 0\}.$$

By classical arguments, we can prove the following property.

**Proposition 2.15.** *Let  $m$  be defined as above. Then, we have*

$$m = \inf \{S(v) : v \in H^1(\mathbb{R}^+) \setminus 0, K(v) = 0\}.$$

We have the following relation between the minimization problems  $m$  and  $d_{\mathcal{M}}$ .

**Lemma 2.16.** *Let  $m$  and  $d_{\mathcal{M}}$  be defined as above. We have*

$$m = d_{\mathcal{M}}.$$

*Proof.* Let  $\mathcal{G}$  be the set of all minimizers of (2.33). If  $\varphi \in \mathcal{G}$  then  $S'(\varphi) = 0$ . By the definition of  $S$ ,  $P$ ,  $K$  we have  $P(\varphi) = 0$  and  $K(\varphi) = 0$ . Hence,  $\varphi \in \mathcal{M}$ , this implies  $S(\varphi) \geq d_{\mathcal{M}}$ . Thus,  $m \geq d_{\mathcal{M}}$ .

Conversely, let  $v \in \mathcal{M}$ . If  $K(v) = 0$  then  $S(v) \geq m$ , using Proposition 2.15. Otherwise,  $K(v) < 0$ . Using the scaling  $v_\lambda(x) = \lambda^{\frac{1}{2}} v(\lambda x)$ , we have

$$K(v_\lambda) = \lambda^2 \|v_x\|_{L^2(\mathbb{R}^+)}^2 - \frac{3\lambda^2}{16} \|v\|_{L^6(\mathbb{R}^+)}^6 + \omega \|v\|_{L^2(\mathbb{R}^+)}^2 + \frac{\alpha\lambda}{2} |v(0)|^2 \rightarrow \omega \|v\|_{L^2(\mathbb{R}^+)}^2 > 0,$$

as  $\lambda \rightarrow 0$ . Hence,  $K(v_\lambda) > 0$  as  $\lambda > 0$  is small enough. Thus, there exists  $\lambda_1 \in (0, 1)$  such that  $K(v_{\lambda_1}) = 0$ . Using Proposition 2.15,  $S(v_{\lambda_1}) \geq m$ . We consider two cases. First, if  $v(0) = 0$  then using Lemma 2.14, we have  $S(v) = S(v_{\lambda_1}) \geq m$ . Second, if  $v(0) \neq 0$  then using Lemma 2.13, we have  $S(v) \geq S(v_{\lambda_1}) \geq m$ . In any case,  $S(v) \geq m$ . This implies  $d_{\mathcal{M}} \geq m$ , and completes the proof.  $\square$

Define

$$\mathcal{V} := \{v \in H^1(\mathbb{R}^+) \setminus \{0\} : K(v) < 0, P(v) < 0, S(v) < m\}.$$

We have the following important lemma.

**Lemma 2.17.** *If  $v_0 \in \mathcal{V}$  then the solution  $v$  of (1.1) associated with  $v_0$  satisfies  $v(t) \in \mathcal{V}$  for all  $t$  in the time of existence.*

*Proof.* Since  $S(v_0) < 0$ , by conservation of the energy and the mass we have

$$S(v) = E(v) + \omega M(v) = E(v_0) + \omega M(v_0) = S(v_0) < m. \quad (2.34)$$

If there exists  $t_0 > 0$  such that  $K(v(t_0)) \geq 0$  then by continuity of  $K$  and  $v$ , there exists  $t_1 \in (0, t_0]$  such that  $K(v(t_1)) = 0$ . This implies  $S(v(t_1)) \geq m$ , using Proposition 2.15. This contradicts (2.34). Hence,  $K(v(t)) < 0$  for all  $t$  in the time of existence of  $v$ . Now, we prove  $P(v(t)) < 0$  for all  $t$  in the time of existence of  $v$ . Assume that there exists  $t_2 > 0$  such that  $P(v(t_2)) \geq 0$ , then, there exists  $t_3 \in (0, t_2]$  such that  $P(v(t_3)) = 0$ . Using the previous lemma,  $S(v(t_3)) \geq m$ , which contradicts (2.34). This completes the proof.  $\square$

Using the above lemma, we have the following property of solutions of (1.1) when the initial data lies on  $\mathcal{V}$ .

**Lemma 2.18.** *Let  $v_0 \in \mathcal{V}$ ,  $v$  be the corresponding solution of (1.1) in  $(T_{min}, T_{max})$ . There exists  $\delta > 0$  independent of  $t$  such that  $P(v(t)) < -\delta$ , for all  $t \in (T_{min}, T_{max})$ .*

*Proof.* Let  $t \in (T_{min}, T_{max})$ ,  $u = v(t)$  and  $u_\lambda(x) = \lambda^{\frac{1}{2}}u(\lambda x)$ . Using Lemma 2.13, there exists  $\lambda_0 \in (0, 1)$  such that  $P(u_{\lambda_0}) = 0$ . If  $K(u_{\lambda_0}) \leq 0$  then we keep  $\lambda_0$ . Otherwise,  $K(u_{\lambda_0}) > 0$ , then, there exists  $\tilde{\lambda}_0 \in (\lambda_0, 1)$  such that  $K(u_{\tilde{\lambda}_0}) = 0$ . We replace  $\lambda_0$  by  $\tilde{\lambda}_0$ . In any case, we have

$$S(u_{\lambda_0}) \geq m. \quad (2.35)$$

By (v) of Proposition 2.13 we have

$$S(u) - S(u_{\lambda_0}) \geq (1 - \lambda_0) \frac{\partial}{\partial \lambda} S(u_\lambda)|_{\lambda=1} = (1 - \lambda_0)P(u).$$

In addition  $P(u) < 0$ , we obtain

$$S(u) - S(u_{\lambda_0}) \geq (1 - \lambda_0)P(u) > P(u). \quad (2.36)$$

Combined (2.35) and (2.36), we obtain

$$S(v_0) - m = S(v(t)) - m = S(u) - m \geq S(u) - S(u_{\lambda_0}) > P(u) = P(v(t)).$$

Setting

$$-\delta := S(v_0) - m,$$

we obtain the desired result.  $\square$

Using the previous lemma, if the initial data lies on  $\mathcal{V}$  and satisfies a weight condition then the associated solution blows up in finite time on  $H^1(\mathbb{R}^+)$ . More precisely, we have the following result.

**Proposition 2.19.** *Let  $\varphi \in \mathcal{V}$  such that  $|x|\varphi \in L^2(\mathbb{R}^+)$ . Then the corresponding solution  $v$  of (1.1) blows up in finite time on  $H^1(\mathbb{R}^+)$ .*

*Proof.* By Lemma 2.18, there exists  $\delta > 0$  such that  $P(v(t)) < -\delta$  for  $t \in (T_{min}, T_{max})$ . Remember that

$$\frac{\partial}{\partial t} \|xv(t)\|_{L^2(\mathbb{R}^+)}^2 = J(t) - \int_{\mathbb{R}^+} x|v|^4 dx, \quad (2.37)$$

where  $J(t)$  satisfies

$$\partial_t J(t) = 4 \left( 2\|v_x\|_{L^2(\mathbb{R}^+)}^2 - \frac{1}{8}\|v\|_{L^6(\mathbb{R}^+)}^6 + \alpha|v(0)|^2 \right) = 8(P(v(t))) < -8\delta.$$

This implies that

$$J(t) = J(0) + 8 \int_0^t P(v(s)) ds < J(0) - 8\delta t.$$



Hence, from (2.37), we have

$$\begin{aligned} \|xv(t)\|_{L^2(\mathbb{R}^+)}^2 &= \|xv(0)\|_{L^2(\mathbb{R}^+)}^2 + \int_0^t J(s) ds - \int_0^t \int_{\mathbb{R}^+} x|v|^4 dx ds \\ &\leq \|xv(0)\|_{L^2(\mathbb{R}^+)}^2 + \int_0^t (J(0) - 8\delta s) ds \\ &\leq \|xv(0)\|_{L^2(\mathbb{R}^+)}^2 + J(0)t - 4\delta t^2. \end{aligned}$$

Thus, for  $t$  sufficiently large, there is a contradiction with  $\|xv\|_{L^2(\mathbb{R}^+)} \geq 0$ . Hence,  $T_{max} < \infty$  and  $T_{min} > -\infty$ . By the blow up alternative, we have

$$\lim_{t \rightarrow T_{max}} \|v_x\|_{L^2(\mathbb{R}^+)} = \lim_{t \rightarrow T_{min}} \|v_x\|_{L^2(\mathbb{R}^+)} = \infty.$$

This completes the proof.  $\square$

*Proof of Theorem 1.4 (2).* Using Proposition 2.19, we need to construct a sequence  $(\varphi_n) \subset \mathcal{V}$  such that  $\varphi_n$  converges to  $\varphi$  in  $H^1(\mathbb{R}^+)$ . Define

$$\varphi_\lambda(x) = \lambda^{\frac{1}{2}} \varphi(\lambda x).$$

We have

$$S(\varphi) = m, \quad P(\varphi) = K(\varphi) = 0, \quad \varphi(0) \neq 0.$$

By (iv) of Proposition 2.13,

$$S(\varphi_\lambda) < m \text{ for all } \lambda > 0.$$

In the addition,

$$P(\varphi_\lambda) < 0 \text{ for all } \lambda > 1.$$

Moreover,

$$\begin{aligned} \frac{\partial}{\partial \lambda} K(\varphi_\lambda) &= 2\lambda \left( \|\varphi_x\|_{L^2(\mathbb{R}^+)}^2 - \frac{3}{16} \|\varphi\|_{L^6(\mathbb{R}^+)}^6 \right) + \alpha |\varphi(0)|^2 \\ &= 2\lambda (K(\varphi) - \omega \|\varphi\|_{L^2(\mathbb{R}^+)}^2 - \alpha |\varphi(0)|^2) + \alpha |\varphi(0)|^2 \\ &= -2\omega \lambda \|\varphi\|_{L^2(\mathbb{R}^+)}^2 - \alpha (2\lambda - 1) |\varphi(0)|^2 \\ &< 0, \end{aligned}$$

when  $\lambda > 1$ . Thus,  $K(\varphi_\lambda) < K(\varphi) = 0$  when  $\lambda > 1$ . This implies  $\varphi_\lambda \in \mathcal{V}$  when  $\lambda > 1$ . Let  $\lambda_n > 1$  such that  $\lambda_n \rightarrow 1$  as  $n \rightarrow \infty$ . Define, for  $n \in \mathbb{N}^*$

$$\varphi_n = \varphi_{\lambda_n},$$

then, the sequence  $(\varphi_n)$  satisfies the desired property. This completes the proof of Theorem 1.4.  $\square$

## REFERENCES

- [1] H. Bahouri and G. Perelman. Global well-posedness for the derivative nonlinear schrödinger equation, 2020.
- [2] T. Cazenave and P.-L. Lions. Orbital stability of standing waves for some nonlinear Schrödinger equations. *Comm. Math. Phys.*, 85(4):549–561, 1982.
- [3] M. Colin and M. Ohta. Stability of solitary waves for derivative nonlinear Schrödinger equation. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 23(5):753–764, 2006.
- [4] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao. Global well-posedness for Schrödinger equations with derivative. *SIAM J. Math. Anal.*, 33(3):649–669, 2001.
- [5] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao. A refined global well-posedness result for Schrödinger equations with derivative. *SIAM J. Math. Anal.*, 34(1):64–86, 2002.
- [6] S. De Bièvre, F. Genoud, and S. Rota Nodari. Orbital stability: analysis meets geometry. In *Nonlinear optical and atomic systems*, volume 2146 of *Lecture Notes in Math.*, pages 147–273. Springer, Cham, 2015.
- [7] S. De Bièvre and S. Rota Nodari. Orbital stability via the energy-momentum method: the case of higher dimensional symmetry groups. *Arch. Ration. Mech. Anal.*, 231(1):233–284, 2019.
- [8] R. Fukuizumi and L. Jeanjean. Stability of standing waves for a nonlinear Schrödinger equation with a repulsive Dirac delta potential. *Discrete Contin. Dyn. Syst.*, 21(1):121–136, 2008.
- [9] R. Fukuizumi, M. Ohta, and T. Ozawa. Nonlinear Schrödinger equation with a point defect. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 25(5):837–845, 2008.
- [10] R. T. Glassey. On the blowing up of solutions to the Cauchy problem for nonlinear Schrödinger equations. *J. Math. Phys.*, 18(9):1794–1797, 1977.
- [11] M. Grillakis, J. Shatah, and W. Strauss. Stability theory of solitary waves in the presence of symmetry. I. *J. Funct. Anal.*, 74(1):160–197, 1987.

- [12] M. Grillakis, J. Shatah, and W. Strauss. Stability theory of solitary waves in the presence of symmetry. II. *J. Funct. Anal.*, 94(2):308–348, 1990.
- [13] N. Hayashi. The initial value problem for the derivative nonlinear Schrödinger equation in the energy space. *Nonlinear Anal.*, 20(7):823–833, 1993.
- [14] N. Hayashi and T. Ozawa. On the derivative nonlinear Schrödinger equation. *Phys. D*, 55(1-2):14–36, 1992.
- [15] N. Hayashi and T. Ozawa. Finite energy solutions of nonlinear Schrödinger equations of derivative type. *SIAM J. Math. Anal.*, 25(6):1488–1503, 1994.
- [16] S. Herr. On the Cauchy problem for the derivative nonlinear Schrödinger equation with periodic boundary condition. *Int. Math. Res. Not.*, pages Art. ID 96763, 33, 2006.
- [17] D. J. Kaup and A. C. Newell. An exact solution for a derivative nonlinear Schrödinger equation. *J. Mathematical Phys.*, 19(4):798–801, 1978.
- [18] S. Le Coz. Standing waves in nonlinear Schrödinger equations. In *Analytical and numerical aspects of partial differential equations*, pages 151–192. Walter de Gruyter, Berlin, 2009.
- [19] J. Shatah. Stable standing waves of nonlinear Klein-Gordon equations. *Comm. Math. Phys.*, 91(3):313–327, 1983.
- [20] H. Takaoka. Global well-posedness for Schrödinger equations with derivative in a nonlinear term and data in low-order Sobolev spaces. *Electron. J. Differential Equations*, pages No. 42, 23, 2001.
- [21] H. Takaoka. A priori estimates and weak solutions for the derivative nonlinear Schrödinger equation on torus below  $H^{1/2}$ . *J. Differential Equations*, 260(1):818–859, 2016.
- [22] M. Tsutsumi and I. Fukuda. On solutions of the derivative nonlinear Schrödinger equation. Existence and uniqueness theorem. *Funkcial. Ekvac.*, 23(3):259–277, 1980.
- [23] M. Tsutsumi and I. Fukuda. On solutions of the derivative nonlinear Schrödinger equation. II. *Funkcial. Ekvac.*, 24(1):85–94, 1981.
- [24] M. I. Weinstein. Modulational stability of ground states of nonlinear Schrödinger equations. *SIAM J. Math. Anal.*, 16(3):472–491, 1985.
- [25] M. I. Weinstein. Lyapunov stability of ground states of nonlinear dispersive evolution equations. *Comm. Pure Appl. Math.*, 39(1):51–67, 1986.
- [26] Y. Wu. Global well-posedness for the nonlinear Schrödinger equation with derivative in energy space. *Anal. PDE*, 6(8):1989–2002, 2013.

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