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Abstract. We consider the Schrödinger equation with derivative nonlinear term on the line. We obtain results on local well posedness under the assumption of uniqueness of weak solutions and we obtain the orbital stability of standing waves via the abstract theory of Grillakis, Shatah, Strauss. Moreover, we consider the Schrödinger equation with nonlinear derivative term on $[0,+\infty)$ under Robin boundary condition at 0. Using a virial argument, we obtain the existence of blowing up solutions and using variational techniques, we obtain stability and instability by blow up results for standing waves.

Contents

1. Introduction

We consider the singularly perturbed derivative nonlinear Schrödinger equation in two forms. First, we consider the equation with a Dirac potential:

$$\begin{cases} iu_t + u_{xx} - \gamma\delta u + i|u|^{2\sigma}u_x = 0, \\ u(0) = \varphi. \end{cases} \tag{1.1}$$

Here, $u : \mathbb{R}_t \times \mathbb{R}_x \to \mathbb{C}$, $\varphi \in H^1(\mathbb{R})$, $\gamma \in \mathbb{R}$, $\sigma > 0$, δ is the Dirac distribution at 0 defined for all $u, v \in H^1(\mathbb{R})$ by

$$\langle \delta u, v\rangle_{H^{-1},H^1} = \mathcal{R}e(u(0)\overline{v(0)}),$$

and the indices denote derivatives. Second, we consider the equation on $[0,+\infty)$ with Robin boundary condition at 0:

$$\begin{cases} iv_t + v_{xx} = \frac{i}{2}|v|^2 v_x - \frac{i}{2}v^2\overline{v_x} - \frac{3}{16}|v|^4 v \quad \text{for } x \in \mathbb{R}_+, \\ v(0) = \varphi, \\ \partial_x v(t,0) = \alpha v(t,0) \quad \forall t \in \mathbb{R}, \end{cases} \tag{1.2}$$

where $\alpha \in \mathbb{R}$ is a given constant.

The equations (1.1) and (1.2) have some relationships. First, there is a relation between the nonlinear terms. Indeed, under the transform (for $x < 0$)

$$u(t,x) = e^{-\frac{i}{4}\int_\infty^{-x}|v(t,y)|^2\,dy}v(t,-x)$$

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the equation (1.2) becomes

$$iu_t + u_{xx} + i|u|^2 u_x = 0. \tag{1.3}$$

Furthermore, the linear parts of (1.1) and (1.2) can be rewritten in the following forms:

$$\begin{cases} iu_t + H_\gamma u = 0, \\ u(0) = \varphi, \end{cases} \qquad \begin{cases} iv_t + \widetilde{H}_\alpha v = 0 \text{ for } x \in \mathbb{R}_+, \\ v(0) = \varphi, \end{cases}$$

where H_γ and $\widetilde{H}_\alpha$ are self-adjoint operators defined by

$$\begin{aligned} &\widetilde{H}_\alpha : D(\widetilde{H}_\alpha) \subset L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+), \\ &\widetilde{H}_\alpha u = u_{xx}, \quad D(\widetilde{H}_\alpha) = \left\{ u \in H^2(\mathbb{R}_+) : u_x(0^+) = \alpha u(0^+)) \right\}, \\ &H_\gamma : D(H_\gamma) \subset L^2(\mathbb{R}) \to L^2(\mathbb{R}), \\ &H_\gamma u = u_{xx}, \quad D(H_\gamma) = \left\{ u \in H^2(\mathbb{R} \setminus 0) \cap H^1(\mathbb{R}) : u_x(0^+) - u_x(0^-) = \gamma u(0) \right\}. \end{aligned}$$

For each $u \in D(\widetilde{H}_\alpha)$ set w such that $w(x) = u(\text{sign})(x)x$. Hence,

$$w \in \left\{ u \in H^2(\mathbb{R} \setminus \{0\}) \cap H^1(\mathbb{R}), u \text{ even }, u_x(0^+) - u_x(0^-) = 2\alpha u(0) \right\}$$

and $H_{2\alpha} w = w_{xx}$. This implies that $\widetilde{H}_\alpha$ can be seen as the restriction of $H_{2\alpha}$ on even functions, and we have

$$e^{i\widetilde{H}_\alpha t}\tilde{\varphi} = e^{iH_{2\alpha}t}\varphi|_{\mathbb{R}^+}, \tag{1.4}$$

where $\tilde{\varphi} \in D(\widetilde{H}_\alpha)$ and φ is the even function on $\mathbb{R}$ whose restriction on $\mathbb{R}^+$ is $\tilde{\varphi}$.

The derivative nonlinear Schrödinger equation was originally introduced in Plasma Physics as a simplified model for Alfvén wave propagation. Since then, it has attracted a lot of attention from the mathematical community (see e.g [9, 10, 22, 23, 25, 27, 33, 34]). There are also many works on the perturbed nonlinear Schrödinger equation (see e.g [1–4, 16, 18, 26, 29]). To our knowledge, there is no result combining the two difficulties. In this work, we are interested in the Cauchy problem and in the existence and stability of standing waves of (1.1) and (1.2).

The Cauchy problem for classical non linear Schrödinger equations has been studied many times before (see e.g the books of Cazenave [5] and Cazenave and Haraux [6] and the references therein). Most works use the fixed point method on Sobolev spaces. For the classical nonlinear Schrödinger equation with a Dirac potential, by using the semi-group theory it is easy to obtain the existence and uniqueness of solutions in $H^1(\mathbb{R})$. For derivative nonlinear Schrödinger equations, we cannot use directly the fixed point method. There are many other methods to deal with the local well posedness of derivative nonlinear Schrödinger equations. For example, one can use Bourgain spaces [32], the approximation method [21, 35, 36] or the Gauge transform [24]. The combination of the Dirac potential and the derivative nonlinearity renders the Cauchy problem quite difficult to solve. We use the approximate method to prove the existence of strong solution under the assumption of uniqueness of weak solution. Our first main result is the following.

Theorem 1.1. *Assume that for all initial data in $H^1(\mathbb{R})$ there exists at most one weak solution of (1.1). Let $\varphi \in H^1(\mathbb{R})$, $\sigma \geqslant 1$. Then there exists a unique maximal solution $u \in C((T_{min}, T_{max}), H^1(\mathbb{R}))$ of (1.1). Moreover, u satisfies the following conservation laws.*

(1) *Conservation of the energy:*

$$E(u(t)) := \frac{1}{2}\|u_x\|_{L^2} + \frac{\gamma}{2}|u(0)|^2 + \frac{1}{2(\sigma+1)}\mathcal{I}m \int_{\mathbb{R}} |u|^{2\sigma} u_x \overline{u}\, dx = E(\varphi). \tag{1.5}$$

(2) *Conservation of the mass*

$$M(u(t)) := \frac{1}{2}\|u\|_{L^2}^2 = M(\varphi). \tag{1.6}$$

In [24] the authors used Gauge transforms to convert the original equation into a system of two equations without derivative. In our case, it seems that we cannot use directly this method due to the perturbed term. Instead, we use the approximation method to prove the existence of solutions of (1.1).

The equation (1.1) admit special solutions called *standing waves* which are of form $e^{i\omega t}\psi$, where $\omega > 0$ and $\psi \in H^1(\mathbb{R})$ is explicitly given by

$$\psi = e^{\frac{-i}{2\sigma+2}\int_{-\infty}^{x} |\phi|^{2\sigma}\, dy}\phi(x)$$

where (see Section 2.2 for details)

$$\phi(x) = \sqrt[4\sigma]{\frac{4(\sigma+1)^2}{2\sigma+1}} \left\{ (2\sigma+1)\omega \operatorname{sech}^2\left(2\sigma\sqrt{\omega}|x| + \tanh^{-1}\left(\frac{-\gamma}{2\sqrt{\omega}}\right)\right)\right\}^{\frac{1}{4\sigma}}.$$

The stability of standing waves for classical nonlinear Schrödinger equations was originally studied by Cazenave and Lions [7] with variational and compactness arguments. A second approach, based on spectral arguments, was introduced by Weinstein [37, 38] and then considerably generalized by Grillakis, Shatah and Strauss [19, 20] (see also [11], [12]). In the case of standing waves for a Dirac potential and nonlinear terms of the form $|u|^{p-1}u$, Goodman, Holmes and Weinstein [17] proved the stability of ground states for $p = 3, \gamma > 0$ using variational methods. Using variational techniques, Fukuizumi, Ohta and Ozawa [14] have obtained the results of stability and instability for standing waves when $\gamma > 0$, $p > 1$ and Fukuizumi and Jeanjean have obtained results when $\gamma < 0$ under radial perturbations. Finally, by using the theory of Grillakis, Shatah and Strauss [19, 20], Le Coz, Fukuizumi, Fibich, Ksherim and Sivan [29] have obtained the complete picture of stability for any $p > 1, \gamma \in \mathbb{R}$. For the derivative nonlinear Schrödinger equation when $\gamma = 0$ and $\sigma = 1$ the solitons have two parameters and are stable as proved in the work of Colin and Ohta [8]. In the more general case $\sigma > 0$, Liu, Simpson and Sulem [30] have obtained stability results for solitons. In our work, we use the abstract theory of Grillakis, Shatah, Strauss [19, 20] to study the orbital stability of standing waves. Our second main result is the following.

Theorem 1.2. *Let $\omega > 0$. There exist ω_1, $\omega_2 > \frac{\gamma^2}{4}$ (explicitly known) such that the following holds. The standing wave $e^{i\omega t}\varphi_\omega$ of equation (1.1) is orbitally stable if one of the two following cases happens:*

(i) $\gamma < 0$, $\omega > \frac{\gamma^2}{4}$ *and* $0 < \sigma \leqslant 1$,
(ii) $\gamma < 0$, $\sigma > 1$ *and* $\frac{\gamma^2}{4} < \omega < \omega_1$.

The standing wave $e^{i\omega t}\varphi_\omega$ of the equation (1.1) is unstable if one of three following cases happens:

(i) $\gamma < 0$, $\omega > \omega_1$ *and* $\sigma > 1$,
(ii) $\gamma > 0$, $\omega > \frac{\gamma^2}{4}$ *and* $0 < \sigma \leqslant \frac{1}{2}$,
(iii) $\gamma > 0$, $\omega > \omega_2$ *and* $\frac{1}{2} < \sigma < 1$.

Remark 1.3. The following cases are open:
(1) $\gamma > 0$, $\omega > \frac{\gamma^2}{4}$, $\sigma \geqslant 1$.
(2) $\gamma > 0$, $\frac{\gamma^2}{4} < \omega < \omega_2$, $\frac{1}{2} < \sigma < 1$.

We now study (1.2). Since (1.2) contains a Robin boundary condition, the notion of solution in $H^1(\mathbb{R}^+)$ is not completely clear. We use the following definition. Let I be an open interval of $\mathbb{R}$. We say that v is a $H^1(\mathbb{R}^+)$ solution of the problem (1.2) on I if $v \in \mathcal{C}(I, H^1(\mathbb{R}^+))$ satisfies the following equation

$$v(t) = e^{i\widetilde{H}_\alpha t}\varphi - i\int_0^t e^{i\widetilde{H}_\alpha(t-s)}g(v(s))\,ds, \tag{1.7}$$

where $e^{i\widetilde{H}_\alpha t}\varphi$ is defined as in (1.4) and g is the function defined by

$$g(v) = \frac{i}{2}|v|^2 v_x - \frac{i}{2}v^2\overline{v}_x - \frac{3}{16}|v|^4 v.$$

To study the equation (1.2), we will need the following assumption.

Assumption. *For all $\varphi \in H^1(\mathbb{R}^+)$, there exists a unique associated maximal solution $v \in \mathcal{C}((T_{min}, T_{max}), H^1(\mathbb{R}^+))$ of (1.2). Moreover, v satisfies the blow up alternative, the solution depends continuously on the initial data, and if $\varphi \in D(\widetilde{H}_\alpha)$ then for all time t, we have $v(t) \in D(\widetilde{H}_\alpha)$.*

The existence of blowing up solutions for classical nonlinear Schrödinger equations was considered by Glassey [15] in 1977. He introduced a concavity argument based on the second derivative in time of $\|xu(t)\|_{L^2}^2$ to show the existence of blowing up solutions. In this paper, we are also interested in studying the existence of blowing-up solutions of (1.2). In the limit case $\alpha = +\infty$, which is formally equivalent to Dirichlet boundary condition if we write $v(0) = \frac{1}{\alpha}v'(0) = 0$, the equation (1.2) transforms into the equation (1.1) with $\gamma = 0$ by using Gauge transform. In [39], Wu

proved the blow up in finite time of solutions of (1.2) with Dirichlet boundary condition and some conditions on the initial data. Using the method of Wu [39] we obtain the existence of blowing up solutions in the case $\alpha > 0$, under a weighted space condition for the initial data and negativity of the energy. Our third main result is the following.

Theorem 1.4. *Assume Assumption A. Let $\varphi \in \Sigma$ where*

$$\Sigma = \left\{u \in H^1(\mathbb{R}^+), xu \in L^2(R_+))\right\}.$$

Then there exists a unique solution $v \in C((T_{min}, T_{max}), \Sigma)$ of (1.2). Furthermore, v satisfies the conservation laws of the mass and the energy as follows:

$$M(v) := \frac{1}{2}\|v\|^2_{L^2(\mathbb{R}^+)} = M(\varphi) \quad \text{for all } t \in (T_{min}, T_{max}),$$
$$E(v) := \frac{1}{2}\|v_x\|^2_{L^2(\mathbb{R}^+)} - \frac{1}{32}\|v\|^6_{L^6(\mathbb{R}^+)} + \frac{\alpha}{2}|v(0)|^2 = E(\varphi).$$

Assume $\alpha \geqslant 0$ and $E(\varphi) < 0$. Then the solution v of (1.2) blows-up in finite time i.e $T_{min} > -\infty$ and $T_{max} < +\infty$.

Stability and instability of standing waves are obtained by variational arguments. Define

$$S_\omega(v) := \frac{1}{2}\left[\|v_x\|^2_{L^2(\mathbb{R}^+)} + \omega\|v\|^2_{L^2(\mathbb{R}^+)} + \alpha|v(0)|^2\right] - \frac{1}{32}\|v\|^6_{L^6(\mathbb{R}^+)},$$
$$K_\omega(v) := \|v_x\|^2_{L^2(\mathbb{R}^+)} + \omega\|v\|^2_{L^2(\mathbb{R}^+)} + \alpha|v(0)|^2 - \frac{3}{16}\|v\|^6_{L^6(\mathbb{R}^+)}.$$

We are interested in the following variational problem:

$$d(\omega) := \inf\left\{S_\omega(v) \mid K_\omega(v) = 0, v \in H^1(\mathbb{R}^+) \setminus \{0\}\right\}. \tag{1.8}$$

Proposition 1.5. *Let $\omega, \alpha \in \mathbb{R}$ such that $\omega > \alpha^2$. All minimizers of (1.8) are of form $e^{i\theta}\varphi$, where $\theta \in \mathbb{R}$ and φ is given by*

$$\varphi = 2\sqrt[4]{\omega}\,\mathrm{sech}^{\frac{1}{2}}\left(2\sqrt{\omega}|x| + \tanh^{-1}\left(\frac{-\alpha}{\sqrt{\omega}}\right)\right).$$

We give the definition of stability and instability by blow up in $H^1(\mathbb{R}^+)$. Let $w(t,x) = e^{i\omega t}\varphi(x)$ be a standing wave solution of (1.2).

(1) The standing wave w is called *orbitally stable* in $H^1(\mathbb{R}^+)$ if for all $\varepsilon > 0$, there exists $\delta > 0$ such that if $v_0 \in H^1(\mathbb{R}^+)$ satisfies
$$\|v_0 - \varphi\|_{H^1(\mathbb{R}^+)} \leqslant \delta,$$
then the associated solution v of (1.2) satisfies
$$\sup_{t\in\mathbb{R}} \inf_{\theta\in\mathbb{R}} \|v(t) - e^{i\theta}\varphi\|_{H^1(\mathbb{R}^+)} < \varepsilon.$$
Otherwise, w said to be *instable*.

(2) The standing wave w is called *instable by blow up* if there exists a sequence (φ_n) such that $\lim_{n\to\infty} \|\varphi_n - \varphi\|_{H^1(\mathbb{R}^+)} = 0$ and the associated solution v_n of (1.2) blows up in finite time for all n.

Using the variational characterization Proposition 1.5 and the method of Colin and Ohta [8], we obtain the orbital stability of these standing wave solutions if $\alpha < 0$. When $\alpha > 0$, we obtain instability by blow up.

Theorem 1.6. *Let $\alpha, \omega \in \mathbb{R}$ be such that $\omega > \alpha^2$. The standing wave $e^{i\omega t}\varphi$, where φ is the profile as in Proposition 1.5, solution of (1.2), satisfies the following properties.*

(1) *If $\alpha < 0$ then the standing wave is orbitally stable in $H^1(\mathbb{R}^+)$.*
(2) *If $\alpha > 0$ then the standing wave is instable by blow up.*

This paper is organized as follows. In Section 3, we consider the first model (1.1) of a derivative nonlinear Schrödinger equation on the line with a Dirac potential. We prove the local wellposedness result Theorem 1.1 and the stability results of the standing waves Theorem 1.2. In Section 4, we consider the second model (1.2) of a derivative nonlinear Schrödinger equation on $[0, \infty)$ under Robin boundary condition at 0. First, under the assumption of local well posedness in $H^1(\mathbb{R}^+)$, we

prove the existence of blowing up solutions using a virial argument Theorem 1.4. Second, in the case $\alpha < 0$, using variational techniques, we prove the orbital stability of standing waves of (1.2). Finally, in the case $\alpha > 0$, using similar arguments as in [28], we prove the instability by blow up of standing waves.

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2. On the singularly perturbed derivative Schrödinger equation on the line

2.1. **The Cauchy problem.** In this section we give a proof of Theorem 1.1.

For $a > 0$, consider

$$V^a(x) = \gamma a e^{-\pi a^2 x^2}.$$

Then $V^a \rightharpoonup \gamma\delta$ weakly-* in $H^{-1}(\mathbb{R})$ as $a \to \infty$. We consider the approximated problem

$$\begin{cases} iu_t + u_{xx} - V^a u + i|u|^{2\sigma}u_x = 0, \\ u(0) = \varphi. \end{cases} \tag{2.1}$$

Using the similar arguments in the proof of [21, Theorem 2.1.4], using the assumption $\sigma \geqslant 1$, there exist (T_a^-, T_a^+) and a unique solution $u^a \in C((T_a^-, T_a^+), H^1(\mathbb{R}))$ of (2.1) such that for all $t \in (T_a^-, T_a^+)$ we have

$$E^a(u^a(t)) = E^a(\varphi), \quad M(u^a(t)) = M(\varphi),$$

where the approximated energy E^a is defined by

$$E^a(v) := \frac{1}{2}\|v_x\|_{L^2(\mathbb{R})}^2 + \frac{1}{2}\int_{\mathbb{R}} V^a|v|^2\,dx + \frac{1}{2(\sigma+1)}\mathcal{I}m\int_{\mathbb{R}} |v|^{2\sigma}v_x\overline{v}\,dx.$$

The proof of Theorem 1.1 is divided into three steps. First, we give an estimate on the sequence (u^a). Second, under the assumption of uniqueness of weak solutions, we prove that the sequence (u^a) converges (up to subsequence) to a function u strongly in $H^1(\mathbb{R})$. It implies that u is a strong solution of (1.1) and u satisfies the conservation laws. Finally, we prove the continuous dependence on the initial data of the solution u.

2.1.1. *Step 1: Estimate on the sequence* (u^a). In this section we prove the following result.

Proposition 2.1. *There exist $T_0 > 0$ independent of a and a constant M such that*

$$\sup_{a>1}\left(\|u^a\|_{L^\infty([-T_0,T_0],H^1(\mathbb{R}))} + \|u_t^a\|_{L^\infty([-T_0,T_0],H^{-1}(\mathbb{R}))}\right) \leqslant M. \tag{2.2}$$

We rely on the arguments of Hayashi and Ozawa [21] to show the boundedness of the sequence u^a in $H^1(\mathbb{R})$. We need the following preliminary results.

Lemma 2.2. *For all $a > 0$ and for all $u \in H^1(\mathbb{R})$ we have*

$$\left|\int_{\mathbb{R}} V^a|u|^2\,dx\right| \leqslant 2|\gamma|\|u\|_{L^2}\|\partial_x u\|_{L^2}.$$

Proof. Recall first the well known inequality

$$\|v\|_{L^\infty}^2 \leqslant 2\|v\|_{L^2}\|\partial_x v\|_{L^2}. \tag{2.3}$$

Using (2.3) and the remark that $\|V^a\|_{L^1} = |\gamma|$ we have

$$\left|\int_{\mathbb{R}} V^a|u|^2\,dx\right| \leqslant \|V^a\|_{L^1}\|u\|_{L^\infty}^2 = |\gamma|\|u\|_{L^\infty}^2 \leqslant 2|\gamma|\|u\|_{L^2}\|\partial_x u\|_{L^2}.$$

This finishes the proof. □

Lemma 2.3. *Let $a > 0$ and u^a be a solution of (2.1). For all $r \geqslant 1$ there exists $C > 0$ such that*

$$\frac{d}{dt}\int_{\mathbb{R}} |u^a|^{2r}\,dx \leqslant C\|u^a\|_{H^1}^{2r}. \tag{2.4}$$

Proof. We first assume that $u^a \in H^2(\mathbb{R})$. Since u^a solves 2.1 we have

$$\begin{aligned}
\frac{d}{dt}\int_{\mathbb{R}} |u^a|^{2r} &= \int_{\mathbb{R}} 2r|u^a|^{2(r-1)}\mathcal{R}e(u_t^a\overline{u^a})\,dx \\
&= \int_{\mathbb{R}} 2r|u^a|^{2(r-1)}\mathcal{R}e((iu_{xx}^a - iV^a u^a - |u^a|^{2\sigma}u_x^a)\overline{u^a})\,dx \\
&= \int_{\mathbb{R}} 2r|u^a|^{2(r-1)}\left(-\mathcal{I}m(u_{xx}^a\overline{u^a}) - \frac{1}{2}|u^a|^{2\sigma}\partial_x|u^a|^2\right)dx \\
&= \mathcal{I}m\int_{\mathbb{R}} 2ru_x^a\partial_x(|u^a|^{2(r-1)})\overline{u^a}\,dx - \frac{r}{\sigma+r}\int_{\mathbb{R}}\partial_x(|u^a|^{2r+2\sigma})\,dx \\
&= \mathcal{I}m\int_{\mathbb{R}} 2ru_x^a\partial_x(|u^a|^{2(r-1)})\overline{u^a}\,dx \\
&\leqslant C\|u^a\|_{L^\infty}^{2(r-1)}\|u_x^a\|_{L^2}^2 \\
&\leqslant C\|u^a\|_{H^1(\mathbb{R})}^{2r}
\end{aligned}$$

The above calculation is valid in $H^2(\mathbb{R})$ but we can obtain the result in $H^1(\mathbb{R})$ by density using the continuous dependence on the initial data property. □

Proof. Proof of Proposition 2.1. We now come back to the boundedness property of u^a in $H^1(\mathbb{R})$. Using Lemma 2.2 we have

$$\begin{aligned}
\|u^a\|_{H^1}^2 &= \|u^a\|_{L^2}^2 + \|u_x^a\|_{L^2}^2 \\
&= \|u^a\|_{L^2}^2 + 2E^a(u^a) - \int_{\mathbb{R}} V^a|u^a|^2\,dx - \frac{1}{\sigma+1}\mathcal{I}m\int_{\mathbb{R}}|u^a|^{2\sigma}u_x^a\overline{u^a}\,dx \\
&\leqslant \|u^a\|_{L^2}^2 + 2E^a(u^a) + 2|\gamma|\|u^a\|_{L^2(\mathbb{R})}\|u_x^a\|_{L^2} + \frac{1}{\sigma+1}\|u_x^a\|_{L^2}\|(u^a)^{2\sigma+1}\|_{L^2} \\
&\leqslant \|u^a\|_{L^2}^2 + 2E^a(u^a) + 4\gamma^2\|u^a\|_{L^2}^2 + \frac{1}{4}\|u_x^a\|_{L^2}^2 + \frac{1}{\sigma+1}\|u_x^a\|_{L^2}\|u^a\|_{L^{4\sigma+2}}^{2\sigma+1} \\
&\leqslant (1+4\gamma^2)\|u^a\|_{L^2}^2 + 2E^a(u^a) + \frac{1}{4}\|u_x^a\|_{L^2}^2 + \frac{1}{2\sigma+2}\|u_x^a\|_{L^2}^2 + \frac{1}{2\sigma+2}\|u^a\|_{L^{4\sigma+2}}^{4\sigma+2}.
\end{aligned}$$

It implies that

$$\begin{aligned}
\left(1 - \frac{1}{4} - \frac{1}{2\sigma+2}\right)\|u^a\|_{H^1}^2 &\leqslant (1+4\gamma^2)\|u^a\|_{L^2}^2 + 2E^a(u^a) + \frac{1}{2\sigma+2}\|u^a\|_{L^{4\sigma+2}}^{4\sigma+2} \\
&= 2(1+4\gamma^2)M(u^a) + 2E^a(u^a) + \frac{1}{2\sigma+2}\|u^a\|_{L^{4\sigma+2}}^{4\sigma+2}.
\end{aligned}$$

Set $C = 1 - \frac{1}{4} - \frac{1}{2\sigma+2}$ and

$$\varepsilon^a(u^a) := \frac{1}{C}\left(2(1+4\gamma^2)M(u^a) + 2E^a(u^a) + \frac{1}{2\sigma+2}\|u^a\|_{L^{4\sigma+2}}^{4\sigma+2}\right),$$

then we have

$$\|u^a\|_{H^1(\mathbb{R})}^2 \leqslant \varepsilon^a(u^a). \tag{2.5}$$

By conservation of energy and mass, we have

$$\frac{d}{dt}\varepsilon^a(u^a) = \frac{1}{C(2\sigma+2)}\frac{d}{dt}\|u^a\|_{L^{4\sigma+2}}^{4\sigma+2}.$$

Hence, by applying Lemma 2.3 for $r = 2\sigma+1$ we obtain

$$\frac{d}{dt}\varepsilon^a(u^a) \leqslant C\|u^a\|_{H^1}^{2(2\sigma+1)} \leqslant C(\varepsilon^a(u^a))^{2\sigma+1}.$$

It is equivalent to

$$\frac{\frac{d}{dt}\varepsilon^a(u^a)}{(\varepsilon^a(u^a))^{2\sigma+1}} \leqslant C.$$

Integral two sides of the above inequality from 0 to t we have

$$\frac{-1}{2\sigma}\left(\varepsilon^a(u^a)^{-2\sigma}(t)-\varepsilon^a(u^a)^{-2\sigma}(0)\right)\leqslant Ct.$$

Hence,

$$\varepsilon^a(u^a)^{-2\sigma}(t)\geqslant \varepsilon^a(u^a)^{-2\sigma}(0)-2\sigma Ct. \tag{2.6}$$

Moreover, it is easy to see that $E^a(\varphi)$ is uniformly bounded by some constant independent of a, and we obtain

$$\varepsilon^a(u^a)(0)=\frac{1}{C}\left(2(1+4\gamma^2)M(\varphi)+2E^a(\varphi)+\frac{1}{2\sigma+2}\|\varphi\|^{4\sigma+2}_{L^{4\sigma+2}}\right)<D \tag{2.7}$$

for D independent of a. Let T_0 such that

$$|T_0|\leqslant\frac{D^{-2\sigma}}{2C\sigma}. \tag{2.8}$$

Combined (2.6) and (2.7), there exists $K>0$ independent of a such that for all $t\in[-T_0,T_0]$ we have

$$\varepsilon^a(u^a)(t)\leqslant K.$$

Hence, since (2.5) holds, we have

$$\sup_{a>1}\|u^a\|_{L^\infty([-T_0,T_0],H^1)}\leqslant K.$$

Since u^a verifies 2.1, this implies that there exists a constant N such that

$$\sup_{a>1}\|u^a_t\|_{L^\infty([-T_0,T_0],H^{-1})}\leqslant N.$$

Choosing $M=K+N$, we obtain the desired result for Proposition 2.1. □

2.1.2. *Step 2: The convergence of the sequence* (u^a).

Proposition 2.4. *There exists $u\in C([-T_0,T_0],H^1(\mathbb{R}))$ such that up to subsequence for all $t\in[-T_0,T_0]$, the sequence $u^a(t)$ converges strongly to $u(t)$ in $H^1(\mathbb{R})$.*

We introduce the following convergence lemma.

Lemma 2.5. *Let $(v^n)_{n\in\mathbb{N}}\subset H^1(\mathbb{R})$ be a bounded sequence in $H^1(\mathbb{R})$ such that $v^n\rightharpoonup v$ weakly in $H^1(\mathbb{R})$. Then*

$$i|v^n|^2\partial_x v^n\rightharpoonup i|v|^2\partial_x v$$

weakly- in $H^{-1}(\mathbb{R})$.*

Proof. Let $M>0$ be such that $\sup_n\|v^n\|_{H^1}<M$. Let $\psi\in H^1(\mathbb{R})$. We have

$$\begin{aligned}\left\langle i|v^n|^2\partial_x v^n-i|v|^2\partial_x v,\psi\right\rangle_{H^{-1},H^1}&=\langle i(v^n-v)\overline{v^n}\partial_x v^n,\psi\rangle_{H^{-1},H^1}+\langle iv(\overline{v^n-v})\partial_x v^n,\psi\rangle_{H^{-1},H^1}\\&\quad+\left\langle i|v|^2\partial_x(v^n-v),\psi\right\rangle_{H^{-1},H^1}\\&=:J_1+J_2+J_3.\end{aligned}$$

Let $\psi_k\in\mathcal{D}(\mathbb{R})$ such that $\|\psi_k-\psi\|_{H^1}\to 0$. We have

$$\begin{aligned}J_1&=\int_{\mathbb{R}} i(v^n-v)\overline{v^n}\partial_x v^n(\psi-\psi_k)\,dx+\int_{\mathbb{R}} i(v^n-v)\overline{v^n}\partial_x v^n\psi_k\,dx\\&\leqslant\|v^n-v\|_{L^\infty}\|v^n\|_{L^\infty}\|\partial_x v^n\|_{L^2}\|\psi-\psi_k\|_{L^2}+\|\partial_x v^n\|_{L^2}\|v^n\|_{L^\infty}\|(v^n-v)\psi_k\|_{L^2}\\&\leqslant C(M)(\|\psi-\psi_k\|_{L^2}+\|(v^n-v)\psi_k\|_{L^2}).\end{aligned}$$

Let $\varepsilon>0$. Fix k_0 large enough such that $C(M)\|\psi-\psi_{k_0}\|_{L^2}<\frac{\varepsilon}{2}$. From the fact that

$$\|(v^n-v)\psi_{k_0}\|_{L^2}\to 0\text{ as }n\to+\infty,$$

there exists N_0 such that for $n>N_0$ we have $C(M)\|(v^n-v)\psi_k\|_{L^2}<\frac{\varepsilon}{2}$. Hence for $n>N_0$ we have $J_1<\varepsilon$. It follows that $J_1\to 0$ as $n\to+\infty$. By a similar argument, we have $J_2\to 0$ as $n\to+\infty$. For the term J_3, we remark that $\partial_x(v^n-v)\rightharpoonup 0$ weakly in $L^2(\mathbb{R})$, therefore $J_3\to 0$ as $n\to\infty$. This gives the desired result. □

Proof. Proof of Proposition 2.4. Recall from Proposition 2.1 that for all $t \in [-T_0, T_0]$, u^a is uniformly bounded in $H^1(\mathbb{R})$ and u^a_t is uniformly bounded in $H^{-1}(\mathbb{R})$. It is classical to see that (for details, see [5, Proposition 1.3.14]) for $u^a \in L^\infty([-T_0, T_0], H^1(\mathbb{R})) \cap W^{1,\infty}([-T_0, T_0], H^{-1}(\mathbb{R}))$ there exists $u \in L^\infty([-T_0, T_0], H^1(\mathbb{R})) \cap W^{1,\infty}([-T_0, T_0], H^{-1}(\mathbb{R}))$ such that up to subsequence for all $t \in [-T_0, T_0]$,

$$u^a(t) \rightharpoonup u(t) \tag{2.9}$$

weakly in $H^1(\mathbb{R})$ when $a \to +\infty$. Without loss of generality we assume that the convergence holds for all a. Using Lemma 2.5 we obtain

$$i|u^a|^2 u^a_x \rightharpoonup i|u|^2 u_x \text{ weakly-* in } H^{-1}(\mathbb{R}). \tag{2.10}$$

Moreover, thanks to Sobolev embeddings, we have

$$u^a \to u \quad \text{ a.e and uniformly on compact sets of } \mathbb{R}$$

and it is not hard to see that this permits us to show

$$V^a u^a \rightharpoonup \gamma \delta u \quad \text{ weakly-* in } H^{-1}(\mathbb{R}). \tag{2.11}$$

Since u^a satisfies (2.1), it follows from (2.10), (2.11) and (2.9) that u satisfies (1.1). We have

$$\|u(t)\|_{L^2(\mathbb{R})} = \|\varphi\|_{L^2(\mathbb{R})} = \|u^a(t)\|_{L^2}.$$

Combined with $u^a(t) \rightharpoonup u(t)$ weakly in $H^1(\mathbb{R})$, this implies that $u^a(t) \to u(t)$ strongly in $L^2(\mathbb{R})$ for all $t \in [-T_0, T_0]$. In addition, we have $u^a(t) \to u(t)$ in L^r for all $r \in [2, \infty]$. We obtain as $a \to +\infty$

$$\mathcal{I}m \int_{\mathbb{R}} |u^a|^{2\sigma} u^a_x \overline{u^a}\, dx \to \mathcal{I}m \int_{\mathbb{R}} |u|^{2\sigma} u_x \overline{u}\, dx, \tag{2.12}$$

and

$$E(u(t)) \leqslant \lim_{a \to +\infty} E^a(u^a(t)) = \lim_{a \to +\infty} E^a(\varphi) = E(\varphi). \tag{2.13}$$

Thus we have shown that for each $\varphi \in H^1(\mathbb{R})$ there exists a weak solution $u \in L^\infty([-T_0, T_0], H^1(\mathbb{R})) \cap W^{1,\infty}([-T_0, T_0], H^{-1}(\mathbb{R}))$ of (1.1) and u satisfies

$$\|u(t)\|^2_{L^2} = \|\varphi\|^2_{L^2}, \quad E(u(t)) \leqslant E(\varphi).$$

Let us show that $E(u(t))$ is constant on $[-T_0, T_0]$ under the assumption of uniqueness of weak solutions. Indeed, let $t_0, t_1 \in [-T_0, T_0]$. Let $\varphi = u(t_0)$ and v be the associated solution given by the above arguments. The solution $v(. - t_0)$ is defined on $[-T_0, T_0]$, by the assumption of uniqueness, $v(. - t_0) = u(.)$ on $[-T_0, T_0]$. Hence,

$$E(u(t_1)) \leqslant E(u(t_0)).$$

By similar arguments we also obtain

$$E(u(t_0)) \leqslant E(u(t_1)).$$

It follows that for all $t_0, t_1 \in [-T_0, T_0]$

$$E(u(t_0)) = E(u(t_1)).$$

We have

$$E^a(u^a) = E^a(\varphi) \to E(\varphi) = E(u(t)) \text{ as } a \to +\infty. \tag{2.14}$$

Combined (2.12) with (2.14), it is easily seen that $u^a(t) \to u(t)$ strongly in $H^1(\mathbb{R})$ for all $t \in [-T_0, T_0]$. We have for all $t \in [-T_0, T_0]$

$$E(u(t)) = E(\varphi), \quad M(u(t)) = M(\varphi), \tag{2.15}$$

hence by [5, Lemma 3.3.6], $u \in C^{0,1/2}([-T_0, T_0], L^2(\mathbb{R}))$, so the function

$$t \to \frac{1}{2(\sigma+1)} \mathcal{I}m \int_{\mathbb{R}} |u(t)|^{2\sigma} u_x(t) \overline{u(t)}\, dx$$

is continuous $[-T_0, T_0] \to \mathbb{R}$. In view of (2.15), this implies that $\|u(t)\|_{H^1}$ is continuous from $[-T_0, T_0]$ to $\mathbb{R}$. Therefore, $u \in C([-T_0, T_0], H^1(\mathbb{R}))$. This concludes the proof of existence of a unique strongly solution of 1.1.

□

2.2. **The orbital stability of standing waves.** In this section, we are interested in the orbital stability of the standing wave solutions of the problem (1.1).

2.2.1. *Standing waves.* A standing wave is a solution of (1.1) of the form

$$u(t,x) = e^{i\omega t}\varphi(x).$$

The profile $\varphi \in H^1(\mathbb{R})$ is a solution of

$$-\varphi_{xx} + \gamma\delta\varphi + \omega\varphi - i|\varphi|^{2\sigma}\varphi_x = 0. \tag{2.16}$$

We have the following description for the solution of (2.16) (see [13, Lemma 25] for a proof).

Lemma 2.6. *Let $\gamma \in \mathbb{R}$ and $\omega > \frac{\gamma^2}{4}$. Then any solution $\varphi \in H^1(\mathbb{R})$ of (2.16) verifies the following:*

$$\begin{cases} \varphi \in \mathcal{C}(\mathbb{R}) \cap \mathcal{C}^2(\mathbb{R}\setminus 0), \\ -\varphi_{xx}(x) + \omega\varphi(x) - i|\varphi(x)|^{2\sigma}\varphi_x(x) = 0 \quad \text{for } x \neq 0, \\ \varphi_x(0^+) - \varphi_x(0^-) = \gamma\varphi(0), \\ \varphi_x(x),\ \varphi(x) \to 0, \quad \text{as } |x| \to \infty. \end{cases} \tag{2.17}$$

Let $\varphi \in H^1(\mathbb{R})$ be a solution of (2.17). As in [30], set

$$\theta(x) := \frac{-1}{2\sigma+2}\int_{-\infty}^{x} |\varphi|^{2\sigma}\, dy, \quad \phi(x) := \varphi(x)e^{-i\theta(x)}. \tag{2.18}$$

Since φ satisfies (2.17) we have $\phi \in H^1(\mathbb{R})$ and

$$\phi_x(0^+) - \phi_x(0^-) = e^{-i\theta(0)}(\varphi_x(0^+) - \varphi_x(0^-)) = \gamma\phi(0). \tag{2.19}$$

Moreover, by elementary calculations, we can verify that ϕ solves

$$-\phi_{xx} + \omega\phi - \frac{2\sigma+1}{4(\sigma+1)^2}|\phi|^{4\sigma}\phi - \frac{\sigma}{\sigma+1}|\phi|^{2\sigma-2}\phi\mathcal{I}m(\phi\overline{\phi_x}) = 0 \quad \text{for } x \neq 0. \tag{2.20}$$

Set

$$A := \omega - \frac{2\sigma+1}{4(\sigma+1)^2}|\phi|^{4\sigma} - \frac{\sigma}{\sigma+1}|\phi|^{2\sigma-2}\mathcal{I}m(\phi\overline{\phi_x})$$

By writing $\phi = f + ig$ for f and g are real valued functions, for all $x \neq 0$, we have

$$f_{xx} = Af, \tag{2.21}$$

$$g_{xx} = Ag. \tag{2.22}$$

Thus,

$$\partial_x(f_xg - g_xf) = f_{xx}g - g_{xx}f = 0 \quad \text{when } x \neq 0.$$

Hence, by using $f, g \in H^2(\mathbb{R}\setminus 0)$ we have

$$f_x(x)g(x) - g_x(x)f(x) = 0 \quad \text{when } x \neq 0.$$

Then, for all $x \neq 0$ we have

$$\mathcal{I}m(\phi(x)\overline{\phi_x(x)}) = f_x(x)g(x) - g_x(x)f(x) = 0.$$

Combined with (2.20) we have

$$-\phi_{xx} + \omega\phi - \frac{2\sigma+1}{4(\sigma+1)^2}|\phi|^{4\sigma}\phi = 0 \quad \text{for } x \neq 0. \tag{2.23}$$

The solutions of (2.23) with jump condition (2.19) are well known (see e.g [13]):

$$\phi(x) = \sqrt[4\sigma]{\frac{4(\sigma+1)^2}{2\sigma+1}}\left((2\sigma+1)\omega\ \mathrm{sech}^2\left(2\sigma\sqrt{\omega}|x| + \tanh^{-1}\left(\frac{-\gamma}{2\sqrt{\omega}}\right)\right)\right)^{\frac{1}{4\sigma}}$$

up to phase shift. Then all standing wave solutions of (1.1) are of the form

$$e^{ic_0}e^{i\omega t}e^{-\frac{i}{2\sigma+2}\int_{-\infty}^{x}|\phi|^{2\sigma}\,dy}\phi(x),$$

where $c_0 \in \mathbb{R}$ and ϕ is defined as above.

2.2.2. *Orbital stability of standing waves.* In this section, we give the proof of Theorem 1.2. Recall that the energy and mass are defined in (1.5), (1.6). We have

$$E'(u) = -u_{xx} + \gamma\delta u - i|u|^{2\sigma}u_x, \quad M'(u) = u.$$

Hence, φ solves

$$E'(\varphi) + \omega M'(\varphi) = 0.$$

Set

$$Hu := E''(\varphi)u + \omega M''(\varphi)u \quad \text{and} \quad d(\omega) = E(\varphi) + \omega M(\varphi).$$

Let $n(H)$ be number of negative eigenvalue of the operator H and define

$$p(d''(\omega)) = \begin{cases} 1 \text{ if } \partial_\omega M(\phi) > 0, \\ 0 \text{ if } \partial_\omega M(\phi) < 0. \end{cases}$$

We use the following theory of Grillakis, Shatah and Strauss [19, 20] to obtain the desired result. In our case, the results of [19, 20] boil down to the following.

Proposition 2.7. *The following is true.*

(1) *The standing wave $e^{i\omega t}\varphi$ is orbitally stable if $n(H) = p(d''(\omega))$.*
(2) *The standing wave $e^{i\omega t}\varphi$ is unstable if $n(H) - p(d''(\omega))$ is odd.*

Using similar arguments as in the proof of [30, Lemma 3.3] we have the following result.

Lemma 2.8. *Let*

$$u = e^{i\theta}(u_1 + iu_2)$$

where θ is defined by (2.18), u_1, u_2 are the real part and the imaginary part of $e^{-i\theta}u$. Then

$$\langle Hu, u\rangle = \left\langle \widetilde{L}_{11}u_1, u_1 \right\rangle + \int_{\mathbb{R}} \left[\phi(\phi^{-1}u_2)_x + \frac{\sigma}{\sigma+1}\phi^{2\sigma}u_1\right]^2 dx. \tag{2.24}$$

where

$$\widetilde{L}_{11} \equiv -\partial_x^2 + \gamma\delta + \omega - \frac{8\sigma^2 + 6\sigma + 1}{4(\sigma+1)^2}\phi^{4\sigma}.$$

Proof. By elementary calculations, we have

$$\langle Hu, u\rangle = \langle L_{11}u_1, u_1\rangle + \langle L_{21}u_2, u_1\rangle + \langle L_{12}u_1, u_2\rangle + \langle L_{22}u_2, u_2\rangle,$$

where

$$\begin{aligned}
L_{11} &:= -\partial_x^2 + \gamma\delta + \omega - \frac{4\sigma^2 + 6\sigma + 1}{4(\sigma+1)^2}\phi^{4\sigma}, \\
L_{21} &:= -\frac{\sigma}{\sigma+1}\phi^{2\sigma-1}\phi_x + \frac{\sigma}{\sigma+1}\phi^{2\sigma}\partial_x, \\
L_{12} &:= -\frac{(2\sigma+1)\sigma}{\sigma+1}\phi^{2\sigma-1}\phi_x - \frac{\sigma}{\sigma-1}\phi^{2\sigma}\partial_x, \\
L_{22} &:= -\partial_x^2 + \gamma\delta + \omega - \frac{2\sigma+1}{4(\sigma+1)^2}\phi^{4\sigma}.
\end{aligned}$$

We see that

$$L_{11} = \widetilde{L}_{11} + \frac{\sigma^2}{(\sigma+1)^2}\phi^{4\sigma}. \tag{2.25}$$

Moreover, we can check that

$$\langle L_{12}u_1, u_2\rangle = \left\langle -\frac{2\sigma+1}{2(\sigma+1)}\partial_x(\phi^{2\sigma}), u_1u_2 \right\rangle - \left\langle \frac{\sigma}{\sigma+1}\phi^{2\sigma}\partial_x u_1, u_2 \right\rangle \tag{2.26}$$

$$= \frac{2\sigma+1}{2(\sigma+1)}\left\langle \phi^{2\sigma}u_{2x}, u_1\right\rangle + \frac{1}{2(\sigma+1)}\left\langle \phi^{2\sigma}u_{1x}, u_2\right\rangle. \tag{2.27}$$

Similarly, we have

$$\langle L_{21}u_2, u_1\rangle = -\frac{1}{2(\sigma+1)}\left\langle \partial_x(\phi^{2\sigma}), u_1u_2\right\rangle + \frac{\sigma}{\sigma+1}\left\langle \phi^{2\sigma}u_{2x}, u_1\right\rangle \tag{2.28}$$

$$= \frac{2\sigma+1}{2(\sigma+1)}\left\langle \phi^{2\sigma}u_{2x}, u_1\right\rangle + \frac{1}{2(\sigma+1)}\left\langle \phi^{2\sigma}u_{1x}, u_2\right\rangle. \tag{2.29}$$

Combining (2.27), (2.29) we have

$$\langle L_{12}u_1, u_2\rangle + \langle L_{21}u_2, u_1\rangle = \frac{2\sigma+1}{\sigma+1}\left\langle \phi^{2\sigma}u_{2x}, u_1\right\rangle + \frac{1}{\sigma+1}\left\langle \phi^{2\sigma}u_{1x}, u_2\right\rangle.$$

Introducing $\widetilde{u_2} = \phi^{-1}u_2$ into the above expression, and integrating by parts, we obtain

$$\langle L_{12}u_1, u_2\rangle + \langle L_{21}u_2, u_1\rangle = \frac{2\sigma}{\sigma+1}\left\langle \phi^{2\sigma}\widetilde{u}_{2x}, u_1\right\rangle. \tag{2.30}$$

We have

$$\begin{aligned}\langle L_{22}u_2, u_2\rangle &= \left\langle -\partial_x^2 u_2, u_2\right\rangle + \left\langle \left(\omega + \gamma\delta - \frac{2\sigma+1}{4(\sigma+1)^2}\phi^{4\sigma}\right)u_2, u_2\right\rangle \\ &= \int_{\mathbb{R}} u_{2x}^2\,dx + \left\langle \omega\phi + \gamma\delta\phi - \frac{2\sigma+1}{4(\sigma+1)^2}\phi^{4\sigma+1}, \widetilde{u}_2u_2\right\rangle \\ &= \int_{\mathbb{R}} (\phi\widetilde{u}_{2x} + \phi_x\widetilde{u}_2)^2\,dx + \left\langle \omega\phi + \gamma\delta\phi - \frac{2\sigma+1}{4(\sigma+1)^2}\phi^{4\sigma+1}, \widetilde{u}_2u_2\right\rangle.\end{aligned}$$

Using the fact that

$$-\phi_{xx} + \gamma\delta\phi + \omega - \frac{2\sigma+1}{4(\sigma+1)^2}\phi^{4\sigma+1} = 0 \quad \text{in } H^{-1}(\mathbb{R}),$$

we have

$$\begin{aligned}&\langle L_{22}u_2, u_2\rangle \\ &= \int_{\mathbb{R}} (\phi\widetilde{u}_{2x} + \phi_x\widetilde{u}_2)^2\,dx + \langle \phi_{xx}, \widetilde{u}_2u_2\rangle \\ &= \int_{\mathbb{R}} (\phi\widetilde{u}_{2x} + \phi_x\widetilde{u}_2)^2\,dx - \int_{\mathbb{R}} \phi_x\partial_x(\widetilde{u}_2^2\phi)\,dx \\ &= \int_{\mathbb{R}} \phi^2\widetilde{u}_{2x}^2 + 2\phi\phi_x\widetilde{u}_2\widetilde{u}_{2x} + \phi_x^2\widetilde{u}_2^2\,dx - \int_{\mathbb{R}} \phi_x(\phi_x\widetilde{u}_2^2 + 2\phi\widetilde{u}_2\widetilde{u}_{2x})\,dx \\ &= \int_{\mathbb{R}} \phi^2\widetilde{u}_{2x}^2\,dx.\end{aligned}$$

Combined the above expression with (2.25), (2.30) we have

$$\begin{aligned}&\langle Hu, u\rangle \\ &= \left\langle \widetilde{L}_{11}u_1, u_1\right\rangle + \left\langle \frac{\sigma^2}{(\sigma+1)^2}\phi^{4\sigma}u_1, u_1\right\rangle + \frac{2\sigma}{\sigma+1}\left\langle \phi^{2\sigma}\widetilde{u}_{2x}, u_1\right\rangle + \int_{\mathbb{R}} \phi^2\widetilde{u}_{2x}^2\,dx \\ &= \left\langle \widetilde{L}_{11}u_1, u_1\right\rangle + \int_{\mathbb{R}} \left(\phi\widetilde{u}_{2x} + \frac{\sigma}{\sigma+1}\phi^{2\sigma}u_1\right)^2 dx \\ &= \left\langle \widetilde{L}_{11}u_1, u_1\right\rangle + \int_{\mathbb{R}} \left(\phi(\phi^{-1}u_2)_x + \frac{\sigma}{\sigma+1}\phi^{2\sigma}u_1\right)^2 dx.\end{aligned}$$

This completes the proof. □

With the help of the previous lemma we will prove the following result.

Proposition 2.9. *For all values of $\sigma > 0$ and $\omega > 0$ we have*

$$n(H) = n(\widetilde{L}_{11}).$$

Proof. First, since $\widetilde{L}_{11}$ is a self adjoint operator, the spectrum of $\widetilde{L}_{11}$ satisfies

$$\sigma(\widetilde{L}_{11}) \subset \mathbb{R}.$$

Moreover, the essential spectrum of $\widetilde{L}_{11}$ satisfies

$$\sigma_{ess}(\widetilde{L}_{11}) = \sigma_{ess}(-\partial_x^2 + \omega) = [\omega, \infty).$$

In addition, $\widetilde{L}_{11}$ is bounded from below, hence, there is a finite number of negative eigenvalues. In particular, $n(\widetilde{L}_{11})$ is well defined. Now, let $u \in H^1(\mathbb{R})$ be a function such that

$$\langle Hu, u\rangle < 0.$$

Hence, from the assumption $\gamma < 0$ and (2.24) we have

$$\left\langle \widetilde{L}_{11}u_1, u_1 \right\rangle < 0.$$

It implies that

$$n(H) \leqslant n(\widetilde{L}_{11}).$$

Moreover, if $u_1 \in H^1(\mathbb{R})$ is a function such that

$$\left\langle \widetilde{L}_{11}u_1, u_1 \right\rangle < 0,$$

then by choosing $u_2 \in H^1(\mathbb{R})$ such that

$$u_2(x) = \frac{-\sigma}{\sigma+1}\phi(x)\int_0^x \phi(y)^{2\sigma-1}u_1(y)\,dy,$$

we can check that $u_2 \in H^1(\mathbb{R})$ and

$$\phi(\phi^{-1}u_2)_x + \frac{\sigma}{\sigma+1}\phi^{2\sigma}u_1 = 0.$$

The function $u = e^{i\theta}(u_1 + iu_2)$ then satisfies

$$\langle Hu, u\rangle < 0.$$

Thus,

$$n(H) \geqslant n(\widetilde{L}_{11}).$$

This implies the desired result. □

By using [29, Lemma 12] we obtain the following result.

Lemma 2.10. *The following assertions hold.*

(i) *If $\gamma > 0$ then $n(H) = n(\widetilde{L}_{11}) = 2$.*
(ii) *If $\gamma < 0$ then $n(H) = n(\widetilde{L}_{11}) = 1$.*

We now turn our attention toward $d(\omega)$. We have

$$d''(\omega) = \partial_\omega M(\varphi) = \partial_\omega M(\phi).$$

Using [29, Lemma 8, Lemma 18] we have the following result.

Lemma 2.11. *Let $\gamma \in \mathbb{R}$ and $\omega > \frac{\gamma^2}{4}$. There exist ω_1, ω_2 such that*

(i) *$\partial_\omega M(\phi) > 0$ in one of four following cases*
  - *$\gamma > 0$, $\omega > \frac{\gamma^2}{4}$, $0 < \sigma \leqslant \frac{1}{2}$,*
  - *$\gamma > 0$, $\omega > \omega_2$, $\frac{1}{2} < \sigma < 1$,*
  - *$\gamma < 0$, $0 < \sigma \leqslant 1$,*
  - *$\gamma < 0$, $\sigma > 1$, $\omega_1 > \omega > \frac{\gamma^2}{4}$.*

(ii) *$\partial_\omega M(\phi) < 0$ in one of three following cases*
  - *$\gamma > 0$, $\omega_2 > \omega > \frac{\gamma^2}{4}$, $\frac{1}{2} < \sigma < 1$,*
  - *$\gamma > 0$, $\omega > \frac{\gamma^2}{4}$, $\sigma \geqslant 1$,*
  - *$\gamma < 0$, $\sigma > 1$, $\omega > \omega_1$.*

The constants ω_1, ω_2 are defined as in [13,14] as follow

$$\frac{\sigma-1}{\sigma}J(\omega_1) = \frac{-\gamma}{2\sqrt{\omega_1}}\left(1 - \frac{\gamma^2}{4\omega_1}\right)^{-\frac{(2\sigma-1)}{2\sigma}}, \quad J(\omega_1) = \int_{\tanh^{-1}\left(\frac{-\gamma}{2\sqrt{\omega_1}}\right)}^{\infty} \operatorname{sech}^{1/\sigma} y\,dy.$$

for $\gamma < 0$ and

$$\frac{\sigma-1}{\sigma}J(\omega_2) = \frac{-\gamma}{2\sqrt{\omega_2}}\left(1 - \frac{\gamma^2}{4\omega_2}\right)^{-\frac{(2\sigma-1)}{2\sigma}}, \quad J(\omega_2) = \int_{\tanh^{-1}\left(\frac{-\gamma}{2\sqrt{\omega_2}}\right)}^{\infty} \operatorname{sech}^{1/\sigma} y\,dy.$$

for $\gamma > 0$.

Combining Lemma 2.11 and Proposition 2.7 we obtain the results of Theorem 1.2.

3. On the derivative Schrödinger equation on the half line with Robin boundary condition

In this section we are studying in (1.2). All along this section, we suppose that Assumption 1 holds.

3.1. **The existence of a blowing-up solution.** In this section, we give the proof of Theorem 1.4 using a virial argument (see e.g [15] or [39] for similar arguments). Let $\alpha > 0$. Let v be a solution of (1.2). By density of $D(\widetilde{H}_\alpha)$ in $H^1(\mathbb{R}^+)$, we can assume that $v \in D(\widetilde{H}_\alpha)$. Recall that since $v \in H^2(\mathbb{R}^+)$, we have

$$v(x), v_x(x) \to 0 \text{ as } x \to \infty. \tag{3.1}$$

Formally, since v solves (1.2), we have

$$\frac{1}{2}\partial_t(|v|^2) = -\partial_x \mathcal{I}m(v_x\overline{v}).$$

Therefore, using the Robin boundary condition we have

$$\begin{aligned}\partial_t\left(\frac{1}{2}\int_0^\infty |v|^2\,dx\right) &= -\mathcal{I}m(v_x\overline{v})(\infty) + \mathcal{I}m(v_x\overline{v})(0)\\ &= \mathcal{I}m(v_x\overline{v})(0)\\ &= \alpha\mathcal{I}m(|v(0)|^2)\\ &= 0.\end{aligned}$$

This implies the conservation of the mass. By elementary calculations, we have

$$\partial_t\left(|v_x|^2 - \frac{1}{16}|v|^6\right) = \partial_x\left(2\mathcal{R}e(v_x\overline{v}_t) - \frac{1}{2}|v|^2|v_x|^2 + \frac{1}{2}v^2\overline{v}_x^2\right).$$

Hence, integrating the two sides in space, we obtain

$$\partial_t\left(\int_{\mathbb{R}_+}|v_x|^2\,dx - \frac{1}{16}|v|^6\,dx\right) = -2\mathcal{R}e(v_x(0)\overline{v}_t(0)) + \frac{1}{2}|v(0)|^2|v_x(0)|^2 - \frac{1}{2}v(0)^2\overline{v_x(0)}^2$$

Using the Robin boundary condition for v, we obtain

$$\partial_t\left(\int_{\mathbb{R}_+}|v_x|^2\,dx - \frac{1}{16}|v|^6\,dx\right) = -2\alpha\mathcal{R}e(v(0)\overline{v_t(0)}) = -\alpha\partial_t(|v(0)|^2).$$

This implies the conservation of the energy.

To prove the existence of blowing up solutions we use similar arguments as in [39]. Set

$$I(t) = \int_0^\infty x^2|v(t)|^2\,dx.$$

Let

$$u(t,x) = v(t,x)\exp\left(-\frac{i}{4}\int_x^{+\infty}|v|^2\,dy\right) \tag{3.2}$$

be a Gauge transform in $H^1(\mathbb{R}_+)$. Then the problem (1.2) is equivalent with

$$\begin{cases} iu_t + u_{xx} = i|u|^2u_x,\\ u_x(0) = \alpha u(0) + \frac{i}{4}|u(0)|^2u(0).\end{cases} \tag{3.3}$$

The equation (3.3) has a simpler nonlinear form, but we pay this simplification with a nonlinear boundary condition. Observe that

$$I(t) = \int_0^\infty x^2|u(t)|^2\,dx = \int_0^\infty x^2|v(t)|^2\,dx.$$

By a direct calculation, we get

$$\partial_t I(t) = 2\mathcal{R}e\int_0^\infty x^2\overline{u(t,x)}\partial_t u(t,x)\,dx = 2\mathcal{R}e\int_0^\infty x^2\overline{u}(iu_{xx}+|u|^2u_x)\,dx \tag{3.4}$$

$$= 2\mathcal{I}m\int_0^\infty 2x\overline{u}u_x\,dx - \frac{1}{2}\int_0^\infty 2x|u|^4\,dx \tag{3.5}$$

$$= 4\mathcal{I}m\int_0^\infty xu_x\overline{u}\,dx - \int_0^\infty x|u|^4\,dx. \tag{3.6}$$

Define

$$J(t) = \mathcal{I}m\int_0^\infty xu_x\overline{u}\,dx.$$

We have

$$\begin{aligned}
\partial_t J(t) &= \int_0^\infty xu_x\overline{u}_t\,dx + \int_0^t x\overline{u}u_{xt}\,dx\\
&= -\mathcal{I}m\int_0^\infty xu_t\overline{u}_x\,dx - \mathcal{I}m\int_0^\infty (x\overline{u})_x u_t\,dx\\
&= -2\mathcal{I}m\int_0^\infty xu_t\overline{u}_x\,dx - \mathcal{I}m\int_0^\infty u_t\overline{u}\,dx\\
&= -2\mathcal{I}m\int_0^\infty x\overline{u}_x(iu_{xx}+|u|^2u_x)\,dx - \mathcal{I}m\int_0^\infty \overline{u}(iu_{xx}+|u|^2u_x)\,dx\\
&= -2\mathcal{R}e\int_0^\infty x\overline{u}_x u_{xx}\,dx - \mathcal{R}e\int_0^\infty \overline{u}u_{xx}\,dx - \mathcal{I}m\int_0^\infty |u|^2u_x\overline{u}\,dx\\
&= -\int_0^\infty x\partial_x|u_x|^2\,dx - \mathcal{R}e(\overline{u}u_x)(+\infty) + \mathcal{R}e(\overline{u}u_x)(0) + \mathcal{R}e\int_0^\infty \overline{u_x}u_x\,dx - \mathcal{I}m\int_0^\infty |u|^2u_x\overline{u}\,dx\\
&= \int_0^\infty |u_x|^2\,dx + \mathcal{R}e(\overline{u}(0)u_x(0)) + \int_0^\infty |u_x|^2\,dx - \mathcal{I}m\int_0^\infty |u|^2u_x\overline{u}\,dx\\
&= 2\int_0^\infty |u_x|^2\,dx - \mathcal{I}m\int_0^\infty |u|^2u_x\overline{u}\,dx + \mathcal{R}e(\overline{u}(0)u_x(0)).
\end{aligned}$$

Using the Robin boundary condition we have

$$\partial_t J(t) = 2\int_0^\infty |u_x|^2\,dx - \mathcal{I}m\int_0^\infty |u|^2u_x\overline{u}\,dx + \alpha|u(0)|^2.$$

Moreover using the expression of v in term of u given in (3.2), we get

$$\begin{aligned}
\partial_t J(t) &= 2\int_0^\infty |v_x|^2\,dx - \frac{1}{8}\int_0^\infty |v|^6\,dx + \alpha|v(0)|^2\\
&= 4E(v) - \alpha|v(0)|^2 \leqslant 4E(v) = 4E(\varphi).
\end{aligned}$$

By integrating the two sides of the above inequality in time we have

$$J(t) \leqslant J(0) + 4E(\varphi)t. \tag{3.7}$$

Integrating the two sides of (3.4) in time we have

$$\begin{aligned}
I(t) &= I(0) + 4\int_0^t J(s)\,ds - \int_0^t\int_0^\infty x|u(s,x)|^4\,dx\,ds\\
&\leqslant I(0) + 4\int_0^t J(s)\,ds.
\end{aligned}$$

Using (3.7) we have

$$\begin{aligned}
I(t) &\leqslant I(0) + 4\int_0^t (J(0) + 4E(\varphi)s)\,ds\\
&\leqslant I(0) + 4J(0)t + 8E(\varphi)t^2.
\end{aligned}$$

From the assumption $E(\varphi) < 0$, there exists a finite time $T_* > 0$ such that $I(T_*) = 0$,

$$I(t) > 0 \text{ for } 0 < t < T_*.$$

Note that

$$\begin{aligned}\int_0^\infty |\varphi(x)|^2\,dx = \int_0^\infty |v(t,x)|^2\,dx &= -2\mathcal{R}e\int_0^\infty xv(t,x)\overline{v_x}(t,x)\,dx\\ &\leqslant 2\|xv\|_{L^2_x(\mathbb{R}_+)}\|v_x\|_{L^2_x(\mathbb{R}_+)} = 2\sqrt{I(t)}\|v_x\|_{L^2_x(\mathbb{R}_+)}.\end{aligned}$$

Then there exists a constant $C = C(\varphi) > 0$ such that

$$\|v_x\|_{L^2_x(\mathbb{R}_+)} \geqslant \frac{C}{2\sqrt{I(t)}} \to +\infty \text{ as } t \to T_*.$$

Then the solution v blows up in finite time in $H^1(\mathbb{R}^+)$. This complete the proof of Theorem 1.4.

Remark 3.1. We can prove that if $\|\varphi\|^2_{L^2(\mathbb{R}^+)} < \pi$ then the associated solution v of (1.2) is bounded in $H^1(\mathbb{R}^+)$ by conservation laws and the Gagliardo-Nirenberg inequality. This implies that the Cauchy problem (1.2) is globally wellposed in $H^1(\mathbb{R}^+)$ when the initial data is small enough in $L^2(\mathbb{R}^+)$.

3.2. **Stability and instability of standing waves.** In this section, we give the proof of Theorem 1.6. First, we find the form of the standing waves of (1.2).

3.2.1. *Standing waves.* Let $v = e^{i\omega t}\varphi$ be a solution of (1.2). Then φ solves

$$\begin{cases} 0 = \varphi_{xx} - \omega\varphi + \frac{1}{2}\mathcal{I}m(\varphi_x\overline{\varphi})\varphi + \frac{3}{16}|\varphi|^4\varphi, \quad \text{for } x > 0\\ \varphi_x(0) = \alpha\varphi(0),\\ \varphi \in H^2(\mathbb{R}^+). \end{cases} \tag{3.8}$$

Set

$$A := \omega - \frac{1}{2}\mathcal{I}m(\varphi_x\overline{\varphi}) - \frac{3}{16}|\varphi|^4$$

By writing $\varphi = f + ig$ for f and g real valued functions, for $x > 0$, we have

$$\begin{aligned} f_{xx} &= Af,\\ g_{xx} &= Ag. \end{aligned}$$

Thus,

$$\partial_x(f_xg - g_xf) = f_{xx}g - g_{xx}f = 0 \text{ when } x \neq 0.$$

Hence, by using $f, g \in H^2(\mathbb{R}^+)$, we have

$$f_x(x)g(x) - g_x(x)f(x) = 0 \text{ when } x \neq 0.$$

Then, for all $x \neq 0$, we have

$$\mathcal{I}m(\varphi_x(x)\overline{\varphi(x)}) = g_x(x)f(x) - f_x(x)g(x) = 0,$$

hence, (3.8) is equivalent to

$$\begin{cases} 0 = \varphi_{xx} - \omega\varphi + \frac{3}{16}|\varphi|^4\varphi, \quad \text{for } x > 0\\ \varphi_x(0) = \alpha\varphi(0),\\ \varphi \in H^2(\mathbb{R}^+). \end{cases} \tag{3.9}$$

We have the following description of the profile φ.

Proposition 3.2. *Let $\omega > \alpha^2$. There exists a unique (up to phase shift) solution φ of (3.9), which is of the form*

$$\varphi = 2\sqrt[4]{\omega}\,\mathrm{sech}^{\frac{1}{2}}\left(2\sqrt{\omega}|x| + \tanh^{-1}\left(\frac{-\alpha}{\sqrt{\omega}}\right)\right), \tag{3.10}$$

for all $x > 0$.

Proof. Let w be the even function defined by

$$w(x) = \begin{cases} \varphi(x) \text{ if } x \geqslant 0,\\ \varphi(-x) \text{ if } x \leqslant 0. \end{cases}$$

Then w solves

$$\begin{cases} 0 = -w_{xx} + \omega w - \frac{3}{16}|w|^4 w, \text{ for } x \neq 0, \\ w_x(0^+) - w_x(0^-) = 2\alpha w(0), \\ w \in H^2(\mathbb{R}) \setminus \{0\} \cap H^1(\mathbb{R}). \end{cases} \tag{3.11}$$

Using the results of Fukuizumi and Jeanjean [13], we obtain that

$$w(x) = 2\sqrt[4]{\omega}\,\mathrm{sech}^{\frac{1}{2}}\left(2\sqrt{\omega}|x| + \tanh^{-1}\left(\frac{-\alpha}{\sqrt{\omega}}\right)\right)$$

up to phase shift provided $\omega > \alpha^2$. Hence, for $x > 0$ we have

$$\varphi(x) = 2\sqrt[4]{\omega}\,\mathrm{sech}^{\frac{1}{2}}\left(2\sqrt{\omega}|x| + \tanh^{-1}\left(\frac{-\alpha}{\sqrt{\omega}}\right)\right)$$

up to phase shift. This implies the desired result. □

3.2.2. *The variational problems.* In this section, we give the proof of Proposition 1.5.

First, we introduce another variational problem:

$$\widetilde{d}(\omega) := \inf\left\{\widetilde{S}_\omega(v) \mid v \text{ even}, \widetilde{K}_\omega(v) = 0, v \in H^1(\mathbb{R}) \setminus \{0\}\right\}, \tag{3.12}$$

where $\widetilde{S}_\omega$, $\widetilde{K}_\omega$ are defined for all $v \in H^1(\mathbb{R})$ by

$$\widetilde{S}_\omega(v) := \frac{1}{2}\left[\|v_x\|^2_{L^2(\mathbb{R})} + \omega\|v\|^2_{L^2(\mathbb{R})} + 2\alpha|v(0)|^2\right] - \frac{1}{32}\|v\|^6_{L^6(\mathbb{R})},$$
$$\widetilde{K}_\omega(v) := \|v_x\|^2_{L^2(\mathbb{R})} + \omega\|v\|^2_{L^2(\mathbb{R})} + 2\alpha|v(0)|^2 - \frac{3}{16}\|v\|^6_{L^6(\mathbb{R})}.$$

The functional $\widetilde{K}_\omega$ is called Nehari functional. The following result has proved in [13, 14].

Proposition 3.3. *Let $\omega > \alpha^2$ and φ satisfies*

$$\begin{cases} -\varphi_{xx} + 2\alpha\delta\varphi + \omega\varphi - \frac{3}{16}|\varphi|^4\varphi = 0, \\ \varphi \in H^1(\mathbb{R}) \setminus \{0\}. \end{cases} \tag{3.13}$$

Then, there exists a unique positive solution φ of (3.13). This solution is the unique positive minimizer of (3.12). Furthermore, we have an explicit formula for φ

$$\varphi(x) = 2\sqrt[4]{\omega}\,\mathrm{sech}^{\frac{1}{2}}\left(2\sqrt{\omega}|x| + \tanh^{-1}\left(\frac{-\alpha}{\sqrt{\omega}}\right)\right).$$

We have the following relation between the variational problems.

Proposition 3.4. *Let $\omega > \alpha^2$. We have*

$$d(\omega) = \frac{1}{2}\widetilde{d}(\omega).$$

Proof. Assume v is a minimizer of (1.8), define the $H^1(\mathbb{R})$ function w by

$$w(x) = \begin{cases} v(x) \text{ if } x > 0, \\ v(-x) \text{ if } x < 0. \end{cases}$$

The function $w \in H^1(\mathbb{R}) \setminus \{0\}$ verifies

$$\widetilde{S}_\omega(w) = 2S_\omega(v) = 2d(\omega),$$
$$\widetilde{K}_\omega(w) = 2K_\omega(v) = 0.$$

This implies that

$$\widetilde{d}(\omega) \leqslant \widetilde{S}_\omega(w) = 2d(\omega). \tag{3.14}$$

Now, assume v is a minimizer of (3.12). Let w be the restriction of v on $\mathbb{R}^+$, then,

$$K_\omega(w) = \frac{1}{2}\widetilde{K}_\omega(v) = 0.$$

Hence, we obtain

$$\widetilde{d}(\omega) = \widetilde{S}_\omega(v) = 2S_\omega(w) \geqslant 2d(\omega). \tag{3.15}$$

Combining (3.14) and (3.15) we have

$$\widetilde{d}(\omega) = 2d(\omega).$$

This implies the desired result. □

Proof of Theorem 1.5. Let v be a minimizer of (1.8). Define $w(x) \in H^1(\mathbb{R})$ by

$$w(x) = \begin{cases} v(x) \text{ if } x > 0, \\ v(-x) \text{ if } x < 0. \end{cases}$$

Then, w is an even function. Moreover, w satisfies

$$\widetilde{K}_\omega(w) = 2K_\omega(v) = 0,$$
$$\widetilde{S}_\omega(w) = 2S_\omega(v) = 2d(\omega) = \widetilde{d}(\omega).$$

Hence, w is a minimizer of (3.12). From Propositions 3.3, 3.4, w is of the form $e^{i\theta}\varphi$, where $\theta \in \mathbb{R}$ is a constant and φ is of the form

$$2\sqrt[4]{\omega}\,\mathrm{sech}^{\frac{1}{2}}\left(2\sqrt{\omega}|x| + \tanh^{-1}\left(\frac{-\alpha}{\sqrt{\omega}}\right)\right).$$

Hence, $v = w|_{\mathbb{R}^+}$ satisfies

$$v(x) = e^{i\theta}\varphi(x),$$

for $x > 0$. This completes the proof of Proposition 1.5. □

Remark 3.5. Let $\omega, \alpha \in \mathbb{R}$ such that $\omega > \alpha^2$. Define

$$\mathcal{H}_\omega = \left\{v \in H^1(\mathbb{R}^+), S_\omega(v) \leqslant d(\omega), K_\omega(v) \geqslant 0\right\}.$$

If $\varphi \in \mathcal{H}_\omega$ then the associated solution $v \in \mathcal{C}((T_{min}, T_{max}), H^1(\mathbb{R}^+))$ of (1.2) is global. Indeed, assume that $\varphi \in \mathcal{H}_\omega$. Let $t \in (T_{min}, T_{max})$, using the conservation laws, we have

$$S_\omega(v(t)) = E(v(t)) + \omega M(v(t)) = E(\varphi) + \omega M(\varphi) = S_\omega(\varphi) \leqslant d(\omega).$$

Now, assume there exists $t_0 > 0$ such that $K_\omega(v(t_0)) < 0$, then, there exists $t_1 \in [0, t_0)$ such that $K_\omega(v(t_1)) = 0$. Hence, $v(t_1)$ is a minimizer of (1.8) and $v(t_1) = e^{i\theta}\varphi$. This implies that v is a standing wave solution of (1.2), hence, $K_\omega(v(t)) = 0$ for all t. This contradicts the assumption. Hence, for all $t \in (T_{min}, T_{max})$, $v(t) \in \mathcal{H}_\omega$ and

$$\frac{1}{2}\|v_x\|^2_{L^2(\mathbb{R}^+)} + \frac{1}{2}\omega\|v\|^2_{L^2(\mathbb{R}^+)} + \frac{1}{2}\alpha|v(0)|^2 - \frac{1}{32}\|v\|^6_{L^6(\mathbb{R}^+)} \leqslant \widetilde{d}(\omega),$$
$$\|v_x\|^2_{L^2(\mathbb{R}^+)} + \omega\|v\|^2_{L^2(\mathbb{R}^+)} + \alpha|v(0)|^2 - \frac{3}{16}\|v\|^6_{L^6(\mathbb{R}^+)} \geqslant 0.$$

It implies that $\left(\frac{3}{16} - \frac{2}{32}\right)\|v\|^6_{L^6(\mathbb{R}^+)}$ bounded. Thus, $\|v_x\|_{L^2(\mathbb{R}^+)}$ is uniformly bounded in (T_{min}, T_{max}) and we obtain the desired result.

3.2.3. *Stability and instability of standing waves.* In this section, we give the proof of Theorem 1.6. We use the notations $\widetilde{S}_\omega$ and $\widetilde{K}_\omega$ as in Section 3.2.2. First, we recall a property, which is proved in [14, Proof of Proposition 2].

Proposition 3.6. *Let $(w_n) \subset H^1(\mathbb{R})$ satisfies the following properties*

$$\widetilde{S}_\omega(w_n) \to \tilde{d}(\omega),$$
$$\widetilde{K}_\omega(w_n) \to 0.$$

as $n \to \infty$. Then, there exists a minimizer w of (3.12) such that $w_n \to w$ strongly in $H^1(\mathbb{R})$ up to subsequence.

To prove the stability statement (1) for $\alpha < 0$ in Theorem 1.6, we will use similar arguments as in the work of Colin and Ohta [8]. We need the following property.

Lemma 3.7. *Let $\alpha < 0$, $\omega > \alpha^2$. If a sequence $(v_n) \subset H^1(\mathbb{R}^+)$ satisfies*

$$S_\omega(v_n) \to d(\omega), \tag{3.16}$$
$$K_\omega(v_n) \to 0, \tag{3.17}$$

then there exist a constant $\theta_0 \in \mathbb{R}$ such that $v_n \to e^{i\theta_0}\varphi$, up to subsequence, where φ is defined as in Proposition 1.5.

Proof. Define the sequence $(w_n) \subset H^1(\mathbb{R})$ as follows,

$$w_n(x) = \begin{cases} v_n(x) \text{ for } x > 0, \\ v_n(-x) \text{ for } x < 0. \end{cases}$$

We can check that

$$\widetilde{S}_\omega(w_n) = 2S_\omega(v_n) \to 2d(\omega) = \tilde{d}(\omega),$$
$$\widetilde{K}_\omega(w_n) = 2K_\omega(v_n) \to 0,$$

as $n \to \infty$. Using Proposition 3.6, there exists a minimizer w_0 of (3.12) such that $w_n \to w_0$ strongly in $H^1(\mathbb{R})$, up to subsequence. For convenience, we assume that $w_n \to w_0$ strongly in $H^1(\mathbb{R})$. By Proposition 3.3, there exists a constant $\theta_0 \in \mathbb{R}$ such that

$$w_0 = e^{i\theta_0}\tilde{\varphi},$$

where $\tilde{\varphi}$ is defined by

$$\tilde{\varphi}(x) = \begin{cases} \varphi(x) \text{ for } x > 0, \\ \varphi(-x) \text{ for } x < 0. \end{cases} \tag{3.18}$$

Hence, the sequence (v_n) is the restriction of the sequence (w_n) on $\mathbb{R}^+$, and satisfies

$$v_n \to e^{i\theta_0}\varphi, \text{ strongly in } H^1(\mathbb{R}^+),$$

up to subsequence. This completes the proof. □

Now, we define

$$N(v) := \frac{3}{16}\|v\|^6_{L^6(\mathbb{R}^+)}, \tag{3.19}$$
$$L(v) := \|v_x\|^2_{L^2(\mathbb{R}^+)} + \omega\|v\|^2_{L^2(\mathbb{R}^+)} + \alpha|v(0)|^2. \tag{3.20}$$

We can rewrite S_ω, K_ω as follows

$$S_\omega = \frac{1}{2}L - \frac{1}{6}N,$$
$$K_\omega = L - N.$$

We have the following classical properties of the above functions.

Lemma 3.8. *Let $(\omega, \alpha) \in \mathbb{R}^2$ such that $\omega > \alpha^2$. The following assertions hold.*

(1) *There exists a constant $C > 0$ such that*
$$L(v) \geqslant C\|v\|^2_{H^1(\mathbb{R}^+)} \quad \forall v \in H^1(\mathbb{R}^+).$$
(2) *We have $d(\omega) > 0$.*
(3) *If $v \in H^1(\mathbb{R}^+)$ satisfies $K_\omega(v) < 0$ then $L(v) > 3d(\omega)$.*

Proof. We have

$$|v(0)|^2 = -\int_0^\infty \partial_x(|v(x)|^2)\,dx = -2\mathcal{R}e\int_0^\infty v(x)\overline{v}_x(x)\,dx$$
$$\leqslant 2\|v\|_{L^2(\mathbb{R}^+)}\|v_x\|_{L^2(\mathbb{R}^+)}.$$

Hence,

$$\begin{aligned} L(v) &= \|v_x\|^2_{L^2(\mathbb{R}^+)} + \omega\|v\|^2_{L^2(\mathbb{R}^+)} + \alpha|v(0)|^2 \\ &\geqslant \|v_x\|^2_{L^2(\mathbb{R}^+)} + \omega\|v\|^2_{L^2(\mathbb{R}^+)} - 2|\alpha|\|v\|_{L^2(\mathbb{R}^+)}\|v_x\|_{L^2(\mathbb{R}^+)} \\ &\geqslant C\|v\|^2_{H^1(\mathbb{R}^+)} + (1-C)\|v_x\|^2_{L^2(\mathbb{R}^+)} + (\omega - C)\|v\|^2_{L^2(\mathbb{R}^+)} - 2|\alpha|\|v\|_{L^2(\mathbb{R}^+)}\|v_x\|_{L^2(\mathbb{R}^+)} \\ &\geqslant C\|v\|^2_{H^1(\mathbb{R}^+)} + (2\sqrt{(1-C)(\omega - C)} - 2|\alpha|)\|v\|_{L^2(\mathbb{R}^+)}\|v_x\|_{L^2(\mathbb{R}^+)}. \end{aligned}$$

From the assumption $\omega > \alpha^2$, we can choose $C \in (0,1)$ such that

$$2\sqrt{(1-C)(\omega - C)} - 2|\alpha| > 0.$$

This implies (1). Now, we prove (2). Let v be an element of $H^1(\mathbb{R}^+)$ satisfying $K_\omega(v) = 0$. We have

$$C\|v\|^2_{H^1(\mathbb{R}^+)} \leqslant L(v) = N(v) \leqslant C_1\|v\|^6_{H^1(\mathbb{R}^+)}.$$

Then,

$$\|v\|^2_{H^1(\mathbb{R}^+)} \geqslant \sqrt[4]{\frac{C}{C_1}}.$$

From the fact that, for v satisfying $K_\omega(v) = 0$, we have $S_\omega(v) = S_\omega(v) - \frac{1}{6}K_\omega(v) = \frac{1}{3}L(v)$, this implies that

$$d(\omega) = \frac{1}{3}\inf\left\{L(v) : v \in H^1(\mathbb{R}^+), K_\omega(v) = 0\right\} \geqslant \frac{C}{3}\sqrt[4]{\frac{C}{C_1}} > 0.$$

Finally, we prove (3). Let $v \in H^1(\mathbb{R}^+)$ satisfying $K_\omega(v) < 0$. Then, there exists $\lambda_1 \in (0,1)$ such that $K_\omega(\lambda_1 v) = \lambda_1^2 L(v) - \lambda_1^6 N(v) = 0$. Since $v \neq 0$, we have $3d(\omega) \leqslant L(\lambda_1 v) = \lambda_1^2 L(v) < L(v)$. □

Define

$$\begin{aligned}
\mathcal{A}^+_\omega &= \left\{v \in H^1(\mathbb{R}^+) \setminus \{0\} : S_\omega(v) < d(\omega), K_\omega(v) > 0\right\},\\
\mathcal{A}^-_\omega &= \left\{v \in H^1(\mathbb{R}^+) \setminus \{0\} : S_\omega(v) < d(\omega), K_\omega(v) < 0\right\},\\
\mathcal{B}^+_\omega &= \left\{v \in H^1(\mathbb{R}^+) \setminus \{0\} : S_\omega(v) < d(\omega), N(v) < 3d(\omega)\right\},\\
\mathcal{B}^-_\omega &= \left\{v \in H^1(\mathbb{R}^+) \setminus \{0\} : S_\omega(v) < d(\omega), N(v) > 3d(\omega)\right\}.
\end{aligned}$$

We have the following result.

Lemma 3.9. *Let $\omega, \alpha \in \mathbb{R}^2$ such that $\alpha < 0$ and $\omega > \alpha^2$.*

(1) *The sets $\mathcal{A}^+_\omega$ and $\mathcal{A}^-_\omega$ are invariant under the flow of (1.2).*
(2) *$\mathcal{A}^+_\omega = \mathcal{B}^+_\omega$ and $\mathcal{A}^-_\omega = \mathcal{B}^-_\omega$.*

Proof. (1) Let $u_0 \in \mathcal{A}^+_\omega$ and $u(t)$ the associated solution for (1.2) on (T_{min}, T_{max}). By $u_0 \neq 0$ and the conservation laws, we see that $S_\omega(u(t)) = S_\omega(u_0) < d(\omega)$ for $t \in (T_{min}, T_{max})$. Moreover, by definition of $d(\omega)$ we have $K_\omega(u(t)) \neq 0$ on (T_{min}, T_{max}). Since the function $t \mapsto K_\omega(u(t))$ is continuous, we have $K_\omega(u(t)) > 0$ on (T_{min}, T_{max}). Hence, $\mathcal{A}^+_\omega$ is invariant under flow of (1.2). By the same way, $\mathcal{A}^-_\omega$ is invariant under flow of (1.2).
(2) If $v \in \mathcal{A}^+_\omega$ then by (3.20), (3.19) we have $N(v) = 3S_\omega(v) - 2K_\omega(v) < 3d(\omega)$, which shows $v \in \mathcal{B}^+_\omega$, hence $\mathcal{A}^+_\omega \subset \mathcal{B}^+_\omega$. Now, let $v \in \mathcal{B}^+_\omega$. We show $K_\omega(v) > 0$ by contradiction. Suppose that $K_\omega(v) \leqslant 0$. Then, by Lemma 3.8 (3), $L(v) \geqslant 3d(\omega)$. Thus, by (3.20) and (3.19), we have

$$S_\omega(v) = \frac{1}{2}L(v) - \frac{1}{6}N(v) \geqslant d(\omega),$$

which contradicts $S_\omega(v) < d(\omega)$. Therefore, we have $K_\omega(v) > 0$, which shows $v \in \mathcal{A}^+_\omega$ and $\mathcal{B}^+_\omega \subset \mathcal{A}^+_\omega$. Next, if $v \in \mathcal{A}^-_\omega$, then by Lemma 3.8 (3), $L(v) > 3d(\omega)$. Thus, by (3.20) and (3.19), we have $N(v) = L(v) - K_\omega(v) > 3d(\omega)$, which shows $v \in \mathcal{B}^-_\omega$. Thus, $\mathcal{A}^-_\omega \subset \mathcal{B}^-_\omega$. Finally, if $v \in \mathcal{B}^-_\omega$, then by (3.20) and (3.19), we have $2K_\omega(v) = 3S_\omega(v) - N(v) < 3d(\omega) - 3d(\omega) = 0$, which shows $v \in \mathcal{A}^-_\omega$, hence, $\mathcal{B}^-_\omega \subset \mathcal{A}^-_\omega$. This completes the proof. □

From Proposition 1.5, we have

$$d(\omega) = S_\omega(\varphi).$$

Since $\alpha < 0$, we see that

$$d''(\omega) = \partial_\omega\|\varphi\|^2_{L^2(\mathbb{R}^+)} = \frac{1}{2}\partial_\omega\|\tilde{\varphi}\|^2_{L^2(\mathbb{R})} > 0,$$

where $\tilde{\varphi}$ is defined as (3.18) and we know from [14], [13] that

$$\partial_\omega\|\tilde{\varphi}\|^2_{L^2(\mathbb{R})} > 0,$$

for $\alpha < 0$. We define the function $h : (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}$ by

$$h(\tau) = d(\omega \pm \tau),$$

for $\varepsilon_0 > 0$ sufficiently small such that $h''(\tau) > 0$ and the sign $+$ or $-$ is selected such that $h'(\tau) > 0$ for $\tau \in (-\varepsilon_0, \varepsilon_0)$. Without loss of generality, we can assume

$$h(\tau) = d(\omega + \tau).$$

Lemma 3.10. *Let $(\omega, \alpha) \in \mathbb{R}^2$ such that $\omega > \alpha^2$ and let h be defined as above. Then, for any $\varepsilon \in (0, \varepsilon_0)$, there exists $\delta > 0$ such that if $v_0 \in H^1(\mathbb{R}^+)$ satisfies $\|v_0 - \varphi\|_{H^1(\mathbb{R}^+)} < \delta$, then the solution v of (1.2) with $v(0) = v_0$ satisfies $3h(-\varepsilon) < N(v(t)) < 3h(\varepsilon)$ for all $t \in (T_{min}, T_{max})$.*

Proof. The proof of the above lemma is similar to the one of [8] or [31]. Let $\varepsilon \in (0, \varepsilon_0)$. Since h is increasing, we have $h(-\varepsilon) < h(0) < h(\varepsilon)$. Moreover, by $K_\omega(\varphi) = 0$ and (3.19), (3.20), we see that $3h(0) = 3d(\omega) = 3S_\omega(\varphi) = N(\varphi)$. Thus, if $u_0 \in H^1(\mathbb{R}^+)$ satisfies $\|u_0 - \varphi\|_{H^1(\mathbb{R}^+)} < \delta$ then we have $3h(0) = N(u_0) + O(\delta)$ and $3h(-\varepsilon) < N(u_0) < 3h(\varepsilon)$ for sufficiently small $\delta > 0$. Since $h(\pm\varepsilon) = d(\omega \pm \varepsilon)$ and the set $\mathcal{B}_\omega^\pm$ are invariant under the flow of (1.2) by Lemma 3.9, to conclude the proof, we only have to show that there exists $\delta > 0$ such that if $u_0 \in H^1(\mathbb{R}^+)$ satisfies $\|u_0 - \varphi\|_{H^1(\mathbb{R}^+)} < \delta$ then $S_{\omega\pm\varepsilon}(u_0) < h(\pm\varepsilon)$. Assume that $u_0 \in H^1(\mathbb{R}^+)$ satisfies $\|u_0 - \varphi\|_{H^1(\mathbb{R}^+)} < \delta$. We have

$$\begin{aligned} S_{\omega\pm\varepsilon}(u_0) &= S_{\omega\pm\varepsilon}(\varphi) + O(\delta) \\ &= S_\omega(\varphi) \pm \varepsilon M(\varphi) + O(\delta) \\ &= h(0) \pm \varepsilon h'(0) + O(\delta). \end{aligned}$$

On the other hand, by the Taylor expansion, there exists $\tau_1 = \tau_1(\varepsilon) \in (-\varepsilon_0, \varepsilon_0)$ such that

$$h(\pm\varepsilon) = h(0) \pm \varepsilon h'(0) + \frac{\varepsilon^2}{2} h''(\tau_1).$$

Since $h''(\tau_1) > 0$ by definition of h, we see that there exists $\delta > 0$ such that if $u_0 \in H^1(\mathbb{R}^+)$ satisfies $\|u_0 - \varphi\|_{H^1(\mathbb{R}^+)} < \delta$ then $S_{\omega\pm\varepsilon}(u_0) < h(\pm\varepsilon)$. This completes the proof. □

Proof of Theorem 1.6 (1). Assume that $e^{i\omega t}\varphi$ is not stable for (1.2). Then, there exists a constant $\varepsilon_1 > 0$, a sequence of solutions (v^n) to (1.2), and a sequence $\{t_n\} \in (0, \infty)$ such that

$$v_n(0) \to \varphi \text{ in } H^1(\mathbb{R}^+), \ \inf_{\theta\in\mathbb{R}} \|v_n(t_n) - e^{i\theta}\varphi\|_{H^1(\mathbb{R}^+)} \geqslant \varepsilon_1. \tag{3.21}$$

By using the conservation laws of solutions of (1.2), we have

$$S_\omega(v_n(t_n)) = S_\omega(v_n(0)) \to S_\omega(\varphi) = d(\omega). \tag{3.22}$$

Using Lemma 3.10, we have

$$N(v_n(t_n)) \to 3d(\omega). \tag{3.23}$$

Combined (3.22) and (3.23), we have

$$K_\omega(v_n(t_n)) = 2S_\omega(v_n(t_n)) - \frac{2}{3} N(v_n(t_n)) \to 0.$$

Therefore, using Lemma 3.7, there exists $\theta_0 \in \mathbb{R}$ such that $(v_n(t_n, .))$ has a subsequence (we denote it by the same letter) that converges to $e^{i\theta_0}\varphi$ in $H^1(\mathbb{R}^+)$, where φ is defined as in Proposition 1.5. Hence, we have

$$\inf_{\theta\in\mathbb{R}} \|v_n(t_n) - e^{i\theta}\varphi\|_{H^1(\mathbb{R}^+)} \to 0, \tag{3.24}$$

as $n \to \infty$, this contradicts (3.21). Hence, we obtain the desired result. □

Next, we give the proof of Theorem 1.6 (2), using similar arguments as in [28].

Assume $\alpha > 0$. Let $e^{i\omega t}\varphi$ be the standing wave solution of (1.2). Introduce the scaling

$$v_\lambda(x) = \lambda^{\frac{1}{2}} v(\lambda x).$$

Let S_ω, K_ω be defined as in Proposition 1.5, for convenience, we will remove the index ω. Define

$$P(v) := \frac{\partial}{\partial\lambda} S(v_\lambda)|_{\lambda=1} = \|v_x\|^2_{L^2(\mathbb{R}^+)} - \frac{1}{16}\|v\|^6_{L^6(\mathbb{R}^+)} + \frac{\alpha}{2}|v(0)|^2.$$

In the following lemma, we investigate the behaviour of the above functional under scaling.

Lemma 3.11. *Let $v \in H^1(\mathbb{R}^+) \setminus \{0\}$ be such that $v(0) \neq 0$, $P(v) \leqslant 0$. Then there exists $\lambda_0 \in (0, 1]$ such that*

(i) $P(v_{\lambda_0}) = 0$,

(ii) $\lambda_0 = 1$ *if only if* $P(v) = 0$,
(iii) $\frac{\partial}{\partial\lambda}S(v_\lambda) = \frac{1}{\lambda}P(v_\lambda)$,
(iv) $\frac{\partial}{\partial\lambda}S(v_\lambda) > 0$ *on* $(0, \lambda_0)$ *and* $\frac{\partial}{\partial\lambda}S(v_\lambda) < 0$ *on* (λ_0, ∞),
(v) *The function* $\lambda \to S(v_\lambda)$ *is concave on* (λ_0, ∞).

Proof. A simple calculation leads to

$$P(v_\lambda) = \lambda^2\|v_x\|^2_{L^2(\mathbb{R}^+)} - \frac{\lambda^2}{16}\|v\|^6_{L^6(\mathbb{R}^+)} + \frac{\lambda\alpha}{2}|v(0)|^2.$$

Then, for $\lambda > 0$ small enough, we have

$$P(v_\lambda) > 0.$$

By continuity of P, there exists $\lambda_0 \in (0, 1]$ such that $P(v_{\lambda_0}) = 0$. Hence (i) is proved. If $\lambda_0 = 1$ then $P(v) = 1$. Conversely, if $P(v) = 0$ then

$$0 = P(v_{\lambda_0}) = \lambda_0^2 P(v) + \frac{\lambda_0 - \lambda_0^2}{2}\alpha|v(0)|^2 = \frac{\lambda_0 - \lambda_0^2}{2}\alpha|v(0)|^2.$$

By the assumption $v(0) \neq 0$, we have $\lambda_0 = 1$, hence (ii) is proved. Item (iii) is obtained by a simple calculation. To obtain (iv), we use (iii). We have

$$\begin{aligned} P(v_\lambda) &= \lambda^2\lambda_0^{-2}P(v_{\lambda_0}) + \left(\frac{\lambda\alpha}{2} - \frac{\lambda^2\lambda_0^{-1}\alpha}{2}\right)|v(0)|^2 \\ &= \frac{\lambda\alpha(\lambda_0 - \lambda)}{2\lambda_0}|v(0)|^2. \end{aligned}$$

Hence, $P(v_\lambda) > 0$ if $\lambda < \lambda_0$ and $P(v_\lambda) < 0$ if $\lambda > \lambda_0$. This proves (iv). Finally, we have

$$\frac{\partial^2}{\partial^2\lambda}S(v_\lambda) = P(v) - \frac{\alpha}{2}|v(0)|^2 < 0.$$

This proves (v). □

In the case of functions such that $v(0) = 0$, we have the following lemma.

Lemma 3.12. *Let* $v \in H^1(\mathbb{R}^+) \setminus \{0\}$, $v(0) = 0$ *and* $P(v) = 0$ *then we have*

$$S(v_\lambda) = S(v) \quad \text{for all } \lambda > 0.$$

Proof. The proof is simple, using the fact that

$$\frac{\partial}{\partial\lambda}S(v_\lambda) = \frac{1}{\lambda}P(v_\lambda) = \lambda P(v) = 0.$$

Hence, we obtain the desired result. □

Now, consider the minimization problems

$$d_{\mathcal{M}} := \inf\{S(v) : v \in \mathcal{M}\}, \tag{3.25}$$

$$m := \inf\left\{S(v), v \in H^1(\mathbb{R}^+) \setminus 0, S'(v) = 0\right\}, \tag{3.26}$$

where

$$\mathcal{M} = \left\{v \in H^1(\mathbb{R}^+) \setminus 0, P(v) = 0, K(v) \leqslant 0\right\}.$$

By classical arguments, we can prove the following property.

Proposition 3.13. *Let* m *be defined as above. Then, we have*

$$m = \inf\left\{S(v) : v \in H^1(\mathbb{R}^+) \setminus 0, K(v) = 0\right\}.$$

We have the following relation between the minimization problems m and $d_{\mathcal{M}}$.

Lemma 3.14. *Let* m *and* $d_{\mathcal{M}}$ *be defined as above. We have*

$$m = d_{\mathcal{M}}.$$

Proof. Let $\mathcal{G}$ be the set of all minimizers of (3.26). If $\varphi \in \mathcal{G}$ then $S'(\varphi) = 0$. By the definition of S, P, K we have $P(\varphi) = 0$ and $K(\varphi) = 0$. Hence, $\varphi \in \mathcal{M}$, this implies $S(\varphi) \geqslant d_{\mathcal{M}}$. Thus, $m \geqslant d_{\mathcal{M}}$.

Conversely, let $v \in \mathcal{M}$. If $K(v) = 0$ then $S(v) \geqslant m$, using Proposition 3.13. Otherwise, $K(v) < 0$. Using the scaling $v_\lambda(x) = \lambda^{\frac{1}{2}} v(\lambda x)$, we have

$$K(v_\lambda) = \lambda^2 \|v_x\|^2_{L^2(\mathbb{R}^+)} - \frac{3\lambda^2}{16}\|v\|^6_{L^6(\mathbb{R}^+)} + \omega\|v\|^2_{L^2(\mathbb{R}^+)} + \frac{\alpha\lambda}{2}|v(0)|^2 \to \omega\|v\|^2_{L^2(\mathbb{R}^+)} > 0,$$

as $\lambda \to 0$. Hence, $K(v_\lambda) > 0$ as $\lambda > 0$ is small enough. Thus, there exists $\lambda_1 \in (0,1)$ such that $K(v_{\lambda_1}) = 0$. Using Proposition 3.13, $S(v_{\lambda_1}) \geqslant m$. We consider two cases. First, if $v(0) = 0$ then using Lemma 3.12, we have $S(v) = S(v_{\lambda_1}) \geqslant m$. Second, if $v(0) \neq 0$ then using Lemma 3.11, we have $S(v) \geqslant S(v_{\lambda_1}) \geqslant m$. In any case, $S(v) \geqslant m$. This implies $d_{\mathcal{M}} \geqslant m$, and completes the proof. □

Define

$$\mathcal{V} := \left\{v \in H^1(\mathbb{R}^+) \setminus \{0\} : K(v) < 0, P(v) < 0, S(v) < m\right\}.$$

We have the following important lemma.

Lemma 3.15. *If $v_0 \in \mathcal{V}$ then the solution v of (1.2) associated with v_0 satisfies $v(t) \in \mathcal{V}$ for all t in the time of existence.*

Proof. Since $S(v_0) < 0$, by conservation of the energy and the mass we have

$$S(v) = E(v) + \omega M(v) = E(v_0) + \omega M(v_0) = S(v_0) < m. \tag{3.27}$$

If there exists $t_0 > 0$ such that $K(v(t_0)) \geqslant 0$ then by continuity of K and v, there exists $t_1 \in (0, t_0]$ such that $K(v(t_1)) = 0$. This implies $S(v(t_1)) \geqslant m$, using Proposition 3.13. This contradicts (3.27). Hence, $K(v(t)) < 0$ for all t in the time of existence of v. Now, we prove $P(v(t)) < 0$ for all t in the time of existence of v. Assume that there exists $t_2 > 0$ such that $P(v(t_2)) \geqslant 0$, then, there exists $t_3 \in (0, t_2]$ such that $P(v(t_3)) = 0$. Using the previous lemma, $S(v(t_3)) \geqslant m$, which contradicts (3.27). This completes the proof. □

Using the above lemma, we have the following property of solutions of (1.2) when the initial data lies on $\mathcal{V}$.

Lemma 3.16. *Let $v_0 \in \mathcal{V}$, v be the corresponding solution of (1.2) in (T_{min}, T_{max}). There exists $\delta > 0$ independent of t such that $P(v(t)) < -\delta$, for all $t \in (T_{min}, T_{max})$.*

Proof. Let $t \in (T_{min}, T_{max})$, $u = v(t)$ and $u_\lambda(x) = \lambda^{\frac{1}{2}} u(\lambda x)$. Using Lemma 3.11, there exists $\lambda_0 \in (0,1)$ such that $P(u_{\lambda_0}) = 0$. If $K(u_{\lambda_0}) \leqslant 0$ then we keep λ_0. Otherwise, $K(u_{\lambda_0}) > 0$, then, there exists $\widetilde{\lambda}_0 \in (\lambda_0, 1)$ such that $K(u_{\widetilde{\lambda}_0}) = 0$. We replace λ_0 by $\widetilde{\lambda}_0$. In any case, we have

$$S(u_{\lambda_0}) \geqslant m. \tag{3.28}$$

By (v) of Proposition 3.11 we have

$$S(u) - S(u_{\lambda_0}) \geqslant (1 - \lambda_0)\frac{\partial}{\partial\lambda} S(u_\lambda)|_{\lambda=1} = (1 - \lambda_0)P(u).$$

In addition $P(u) < 0$, we obtain

$$S(u) - S(u_{\lambda_0}) \geqslant (1 - \lambda_0)P(u) > P(u). \tag{3.29}$$

Combined (3.28) and (3.29), we obtain

$$S(v_0) - m = S(v(t)) - m = S(u) - m \geqslant S(u) - S(u_{\lambda_0}) > P(u) = P(v(t)).$$

Setting

$$-\delta := S(v_0) - m,$$

we obtain the desired result. □

Using the previous lemma, if the initial data lies on $\mathcal{V}$ and satisfies a weight condition then the associated solution blows up in finite time on $H^1(\mathbb{R}^+)$. More precisely, we have the following result.

Proposition 3.17. *Let $\varphi \in \mathcal{V}$ such that $|x|\varphi \in L^2(\mathbb{R}^+)$. Then the corresponding solution v of (1.2) blows up in finite time on $H^1(\mathbb{R}^+)$.*

Proof. By Lemma 3.16, there exists $\delta > 0$ such that $P(v(t)) < -\delta$ for $t \in (T_{min}, T_{max})$. Remember that

$$\frac{\partial}{\partial t}\|xv(t)\|^2_{L^2(\mathbb{R}^+)} = J(t) - \int_{\mathbb{R}^+} x|v|^4\,dx, \tag{3.30}$$

where $J(t)$ satisfies

$$\partial_t J(t) = 4\left(2\|v_x\|^2_{L^2(\mathbb{R}^+)} - \frac{1}{8}\|v\|^6_{L^6(\mathbb{R}^+)} + \alpha|v(0)|^2\right) = 8(P(v(t))) < -8\delta.$$

This implies that

$$J(t) = J(0) + 8\int_0^t P(v(s))\,ds < J(0) - 8\delta t.$$

Hence, from (3.30), we have

$$\begin{aligned}\|xv(t)\|^2_{L^2(\mathbb{R}^+)} &= \|xv(0)\|^2_{L^2(\mathbb{R}^+)} + \int_0^t J(s)\,ds - \int_0^t\int_{\mathbb{R}^+} x|v|^4\,dx\,ds\\ &\leqslant \|xv(0)\|^2_{L^2(\mathbb{R}^+)} + \int_0^t (J(0) - 8\delta s)\,ds\\ &\leqslant \|xv(0)\|^2_{L^2(\mathbb{R}^+)} + J(0)t - 4\delta t^2.\end{aligned}$$

Thus, for t sufficiently large, there is a contradiction with $\|xv\|_{L^2(\mathbb{R}^+)} \geqslant 0$. Hence, $T_{max} < \infty$ and $T_{min} > -\infty$. By the blow up alternative, we have

$$\lim_{t\to T_{max}} \|v_x\|_{L^2(\mathbb{R}^+)} = \lim_{t\to T_{min}} \|v_x\|_{L^2(\mathbb{R}^+)} = \infty.$$

This completes the proof. □

Proof of Theorem 1.6 (2). Using Proposition 3.17, we need to construct a sequence $(\varphi_n) \subset \mathcal{V}$ such that φ_n converges to φ in $H^1(\mathbb{R}^+)$. Define

$$\varphi_\lambda(x) = \lambda^{\frac{1}{2}}\varphi(\lambda x).$$

We have

$$S(\varphi) = m, \quad P(\varphi) = K(\varphi) = 0, \quad \varphi(0) \neq 0.$$

By (iv) of Proposition 3.11,

$$S(\varphi_\lambda) < m \text{ for all } \lambda > 0.$$

In the addition,

$$P(\varphi_\lambda) < 0 \text{ for all } \lambda > 1.$$

Moreover,

$$\begin{aligned}\frac{\partial}{\partial\lambda}K(\varphi_\lambda) &= 2\lambda\left(\|\varphi_x\|^2_{L^2(\mathbb{R}^+)} - \frac{3}{16}\|\varphi\|^6_{L^6(\mathbb{R}^+)}\right) + \alpha|\varphi(0)|^2\\ &= 2\lambda(K(\varphi) - \omega\|\varphi\|^2_{L^2(\mathbb{R}^+)} - \alpha|\varphi(0)|^2) + \alpha|\varphi(0)|^2\\ &= -2\omega\lambda\|\varphi\|^2_{L^2(\mathbb{R}^+)} - \alpha(2\lambda - 1)|\varphi(0)|^2\\ &< 0,\end{aligned}$$

when $\lambda > 1$. Thus, $K(\varphi_\lambda) < K(\varphi) = 0$ when $\lambda > 1$. This implies $\varphi_\lambda \in \mathcal{V}$ when $\lambda > 1$. Let $\lambda_n > 1$ such that $\lambda_n \to 1$ as $n \to \infty$. Define, for $n \in \mathbb{N}^*$

$$\varphi_n = \varphi_{\lambda_n},$$

then, the sequence (φ_n) satisfies the desired property. This completes the proof of Theorem 1.6. □

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