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On the derivative nonlinear Schrödinger equation on the half line with Robin boundary condition

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We consider the Schrödinger equation with nonlinear derivative term on $[0, +\infty)$ under Robin boundary condition at 0. Using a virial argument, we obtain the existence of blowing up solutions and using variational techniques, we obtain stability and instability by blow up results for standing waves.

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I. INTRODUCTION

In this paper, we consider the derivative nonlinear Schrödinger equation on $[0, +\infty)$ with Robin boundary condition at 0:

$$\begin{cases} iv_t + v_{xx} = \frac{i}{2}|v|^2 v_x - \frac{i}{2}v^2 \overline{v}_x - \frac{3}{16}|v|^4 v & \text{for } x \in \mathbb{R}^+, \\ v(0, x) = v_0(x), \\ \partial_x v(t, 0) = \alpha v(t, 0) \quad \forall t \in \mathbb{R}, \end{cases} \quad (1.1)$$

where $\alpha \in \mathbb{R}$ is a given constant. The particular choice for the nonlinearity will become clear after Remark 1.1.

The linear parts of (1.1) can be rewritten in the following forms:

$$\begin{cases} iv_t + \tilde{H}_\alpha v = 0 & \text{for } x \in \mathbb{R}^+, \\ v(0) = v_0, \end{cases} \quad (1.2)$$

where \tilde{H}_α are self-adjoint operators defined by

$$\begin{aligned} \tilde{H}_\alpha : D(\tilde{H}_\alpha) \subset L^2(\mathbb{R}^+) &\rightarrow L^2(\mathbb{R}^+), \\ \tilde{H}_\alpha u &= u_{xx}, \quad D(\tilde{H}_\alpha) = \{u \in H^2(\mathbb{R}^+) : u_x(0^+) = \alpha u(0^+)\}. \end{aligned}$$

We recall that $e^{i\tilde{H}_\alpha t} : \mathbb{R} \rightarrow \mathcal{L}(L^2(\mathbb{R}^+))$ is a group which defines the solution of (1.2).

The derivative nonlinear Schrödinger equation was originally introduced in Plasma Physics as a simplified model for Alfvén wave propagation. Since then, it has attracted a lot of attention from the mathematical community (see Refs. 1–3).

Robin boundary conditions are a weighted combination of Dirichlet boundary conditions and Neumann boundary conditions. Robin boundary conditions are also called *impedance boundary conditions*, from their application in *electromagnetic* problems, or *convective boundary conditions*, from their application in *heat transfer* problems. In mathematics, there are many works studying equations under the Robin boundary condition (see e.g Ref. 4 and 5).

Consider the equation (1.1), and set

$$u(t, x) = \exp\left(\frac{3i}{4} \int_0^x |v(t, y)|^2 dy\right) v(t, x).$$

Using the Gauge transformation, we see that u solves

$$iu_t + u_{xx} = i\partial_x(|u|^2 u), \quad t \in \mathbb{R}, x \in (0, \infty), \quad (1.3)$$

under a boundary condition $\partial_x u(t, 0) = \alpha u(t, 0) + \frac{3i}{4}|u(t, 0)|^2 u(t, 0)$. In the case on all line, there are many papers dealing with the Cauchy problem of (1.3). In Ref. 6, the authors established the local well posedness in $H^1(\mathbb{R})$ by using a Gauge transform. Indeed, since u solves (1.3) on \mathbb{R} , by setting

$$\begin{aligned} h(t, x) &= \exp\left(-i \int_{-\infty}^x |u(t, y)|^2 dy\right) u(t, x), \\ k &= h_x + \frac{i}{2}|h|^2 h, \end{aligned} \quad (1.4)$$

we have h, k solve

$$\begin{cases} ih_t + h_{xx} = -ih^2 \bar{k}, \\ ik_t + k_{xx} = ik^2 \bar{h}. \end{cases} \quad (1.5)$$

By classical arguments, we can prove that given $h_0, k_0 \in L^2(\mathbb{R})$ satisfying (1.4) there exists a unique solution $h, k \in C([0, T], L^2(\mathbb{R})) \cap L^4([0, T], L^\infty(\mathbb{R}))$. To obtain the existence of solution of (1.1), the authors proved that the relation (1.4) is satisfied for all $t \in [0, T]$. Thus, since h, k solve (1.5) and satisfy (1.4), if we set

$$u(t, x) = \exp\left(i \int_{-\infty}^x |h(t, y)|^2 dy\right) h(t, x),$$

then $u \in C([0, T], H^1(\mathbb{R}))$ solves (1.1). In Ref.7, the author proved the global well posedness on $H^1(\mathbb{R})$ under a $L^2(\mathbb{R})$ norm bound for the initial data (see also Refs. 8 and 9). In Ref. 10, the authors proved the global well posedness of (1.3) given initial data in $H^{\frac{1}{2}}(\mathbb{R})$ and that furthermore the $H^{\frac{1}{2}}(\mathbb{R})$ norm is globally bounded in time. This result closes the discussion in the setting of the Sobolev spaces $H^s(\mathbb{R})$. In the half line case, Wu¹¹ proved existence of blow up solution of (1.3) under Dirichlet boundary condition, given initial data in $\Sigma := \{u_0 \in H^2(\mathbb{R}^+), xu_0 \in L^2(\mathbb{R}^+)\}$. In this paper, we give a proof of existence of blow up solutions of (1.1) under Robin boundary condition.

To study equation (1.1), we start by the definition of solution on $H^1(\mathbb{R}^+)$. Since (1.1) contains a Robin boundary condition, the notion of solution in $H^1(\mathbb{R}^+)$ is not completely clear. We use the following definition. Let I be an open interval of \mathbb{R} . We say that v is a $H^1(\mathbb{R}^+)$ solution of the problem (1.1) on I if $v \in C(I, H^1(\mathbb{R}^+))$ satisfies the following equation

$$v(t) = e^{i\tilde{H}\alpha t} \varphi - i \int_0^t e^{i\tilde{H}\alpha(t-s)} g(v(s)) ds, \quad (1.6)$$

where g is defined by

$$g(v) = \frac{i}{2}|v|^2 v_x - \frac{i}{2}v^2 \bar{v}_x - \frac{3}{16}|v|^4 v.$$

Let $v \in C(I, D(\tilde{H}_\alpha))$ be a classical solution of (1.1). At least formally, we have

$$\frac{1}{2} \partial_t (|v|^2) = -\partial_x \mathcal{I}m(v_x \bar{v}).$$

Therefore, using the Robin boundary condition we have

$$\begin{aligned} \partial_t \left(\frac{1}{2} \int_0^\infty |v|^2 dx \right) &= -\mathcal{I}m(v_x \bar{v})(t, \infty) + \mathcal{I}m(v_x \bar{v})(t, 0) \\ &= \mathcal{I}m(v_x \bar{v})(t, 0) \\ &= \alpha \mathcal{I}m(|v(t, 0)|^2) \\ &= 0. \end{aligned}$$

This implies the conservation of the mass. By elementary calculations, we have

$$\partial_t \left(|v_x|^2 - \frac{1}{16} |v|^6 \right) = \partial_x \left(2\mathcal{R}e(v_x \bar{v}_t) - \frac{1}{2} |v|^2 |v_x|^2 + \frac{1}{2} v^2 \bar{v}_x^2 \right).$$

Hence, integrating the two sides in space, we obtain

$$\partial_t \left(\int_{\mathbb{R}^+} |v_x|^2 dx - \frac{1}{16} |v|^6 dx \right) = -2\mathcal{R}e(v_x(t, 0) \bar{v}_t(t, 0)) + \frac{1}{2} |v(t, 0)|^2 |v_x(t, 0)|^2 - \frac{1}{2} v(t, 0)^2 \overline{v_x(t, 0)}^2.$$

Using the Robin boundary condition for v , we obtain

$$\partial_t \left(\int_{\mathbb{R}^+} |v_x|^2 dx - \frac{1}{16} |v|^6 dx \right) = -2\alpha \mathcal{R}e(v(t, 0) \overline{v_t(t, 0)}) = -\alpha \partial_t (|v(t, 0)|^2).$$

This implies the conservation of the energy which is defined as in (1.7).

In this paper, we will need the following assumption.

Assumption 1. We assume that for all $v_0 \in H^1(\mathbb{R}^+)$ there exists a solution $v \in C(I, H^1(\mathbb{R}^+))$ of (1.1) for some interval $I \subset \mathbb{R}$. Moreover, v satisfies the following conservation laws:

$$\begin{aligned} M(v) &:= \frac{1}{2} \|v\|_{H^1(\mathbb{R}^+)}^2 = M(v_0), \\ E(v) &:= \frac{1}{2} \|v_x\|_{L^2(\mathbb{R}^+)}^2 - \frac{1}{32} \|v\|_{L^6(\mathbb{R}^+)}^6 + \frac{\alpha}{2} |v(t, 0)|^2 = E(v_0). \end{aligned} \tag{1.7}$$

Remark I.1. In (1.1), if we consider the nonlinear term $i|v|^2 v_x$ instead of $\frac{i}{2} |v|^2 v_x - \frac{i}{2} v^2 \bar{v}_x - \frac{3}{16} |v|^4 v$ then there is no conservation of energy of solution. Indeed, set

$$u(t, x) = v(t, x) \exp \left(-\frac{i}{4} \int_\infty^x |v(t, y)|^2 dy \right).$$

If v solves

$$\begin{cases} iv_t + v_{xx} = i|v|^2 v_x, \\ \partial_x v(t, 0) = \alpha v(t, 0) \end{cases}$$

then u solves

$$\begin{cases} iu_t + u_{xx} = \frac{i}{2}|u|^2 u_x - \frac{i}{2}u^2 \overline{u_x} - \frac{3}{16}|u|^4 u, \\ \partial_x u(t, 0) = \alpha u(t, 0) - \frac{i}{4}|u(t, 0)|^2 u(t, 0). \end{cases} \quad (1.8)$$

By elementary calculations, since u solves (1.8), we have

$$\partial_t \left(|u_x|^2 - \frac{1}{16}|u|^6 \right) = \partial_x \left(2\Re(u_x \overline{u_t}) - \frac{1}{2}|u|^2 |u_x|^2 + \frac{1}{2}u^2 \overline{u_x^2} \right).$$

Integrating two sides in space, we obtain

$$\partial_t \left(\int_{\mathbb{R}^+} |u_x|^2 - \frac{1}{16}|u|^6 dx \right) = -2\Re(u_x(t, 0) \overline{u_t(t, 0)}) + \frac{1}{2}|u(t, 0)|^2 |u_x(t, 0)|^2 - \frac{1}{2}u(t, 0)^2 \overline{u_x(t, 0)^2}.$$

Using the boundary condition of u , we obtain

$$\begin{aligned} \partial_t \left(\int_{\mathbb{R}^+} |u_x|^2 - \frac{1}{16}|u|^6 dx \right) &= -2\alpha \Re(u(t, 0) \overline{u_t(t, 0)}) - \frac{1}{2} \Im(u(t, 0) |u(t, 0)|^2 \overline{u_t(t, 0)}) \\ &\quad + \frac{1}{2}|u(t, 0)|^4 \left(\alpha^2 + \frac{1}{16}|u(t, 0)|^4 - \left(\alpha + \frac{i}{4}|u(t, 0)|^2 \right)^2 \right) \\ &= -\alpha \partial_t (|u(t, 0)|^2) + A, \end{aligned}$$

where $A = -\frac{1}{2} \Im(u(t, 0) |u(t, 0)|^2 \overline{u_t(t, 0)}) + \frac{1}{2}|u(t, 0)|^4 \left(\alpha^2 + \frac{1}{16}|u(t, 0)|^4 - \left(\alpha + \frac{i}{4}|u(t, 0)|^2 \right)^2 \right)$. Moreover, we cannot write A in form $\partial_t B(u(t, 0))$, for some function $B : \mathbb{C} \rightarrow \mathbb{C}$. Then, there is no conservation of energy of u and hence, there is no conservation of energy of v .

The existence of blow-up solutions for classical nonlinear Schrödinger equations was considered by Glassey¹² in 1977. He introduced a concavity argument based on the second derivative in time of $\|xu(t)\|_{L^2}^2$ to show the existence of blow up solutions. In this paper, we are also interested in studying the existence of blow-up solutions of (1.1). The limit case $\alpha = +\infty$ is formally equivalent to Dirichlet boundary condition if we write $v(t, 0) = \frac{1}{\alpha} v_x(t, 0) = 0$. In Ref. 11, the author proved the existence of blow up solutions of (1.1) with Dirichlet boundary condition and some conditions on the initial data. Using similar arguments as in Ref. 11, we obtain the existence of blow up solutions in the case $\alpha \geq 0$, under a weighted space condition for the initial data and negativity of the energy. Our first main result is the following.

Theorem I.2. *We assume Assumption 1. Let $\alpha \geq 0$ and $v_0 \in \Sigma$ where*

$$\Sigma = \left\{ u \in D(\tilde{H}_\alpha), xu \in L^2(\mathbb{R}_+) \right\}$$

such that $E(v_0) < 0$. Then the associated solution v of (1.1) blows up in finite time i.e $T_{min} > -\infty$ and $T_{max} < +\infty$.

The stability of standing waves for classical nonlinear Schrödinger equations was originally studied by Cazenave and Lions¹³ with variational and compactness arguments. A second approach, based on spectral arguments, was introduced by Weinstein^{14,15} and then considerably generalized by Grillakis, Shatah and Strauss^{16,17} (see also Refs. 18, 19). The derivative nonlinear Schrödinger equation has a two-parameter family of solitary waves solutions. The stability of these particular solutions was studied in many works before (see e.g Refs. 20–22). In our work, we use the variational techniques to study the stability of standing waves. First, we define

$$\begin{aligned} S_\omega(v) &:= \frac{1}{2} \left[\|v_x\|_{L^2(\mathbb{R}^+)}^2 + \omega \|v\|_{L^2(\mathbb{R}^+)}^2 + \alpha |v(0)|^2 \right] - \frac{1}{32} \|v\|_{L^6(\mathbb{R}^+)}^6, \\ K_\omega(v) &:= \|v_x\|_{L^2(\mathbb{R}^+)}^2 + \omega \|v\|_{L^2(\mathbb{R}^+)}^2 + \alpha |v(0)|^2 - \frac{3}{16} \|v\|_{L^6(\mathbb{R}^+)}^6, \end{aligned}$$

where $v(0)$ is value of function v at $x = 0$. We are interested in the following variational problem:

$$d(\omega) := \inf \{ S_\omega(v) \mid K_\omega(v) = 0, v \in H^1(\mathbb{R}^+) \setminus \{0\} \}. \quad (1.9)$$

We have the following result.

Proposition I.3. *Let $\omega, \alpha \in \mathbb{R}$ such that $\omega > \alpha^2$. All minimizers of (1.9) are of form $e^{i\theta} \varphi_\omega$, where $\theta \in \mathbb{R}$ and φ_ω is given by*

$$\varphi_\omega = 2\sqrt[4]{\omega} \operatorname{sech}^{\frac{1}{2}} \left(2\sqrt{\omega}|x| + \tanh^{-1} \left(\frac{-\alpha}{\sqrt{\omega}} \right) \right).$$

We give the definition of stability and instability by blow up in $H^1(\mathbb{R}^+)$. Let $w(t, x) = e^{i\omega t} \varphi_\omega(x)$ be a standing wave solution of (1.1).

- (1) The standing wave w is called *orbitally stable* in $H^1(\mathbb{R}^+)$ if for all $\varepsilon > 0$, there exists $\delta > 0$ such that if $v_0 \in H^1(\mathbb{R}^+)$ satisfies

$$\|v_0 - \varphi_\omega\|_{H^1(\mathbb{R}^+)} \leq \delta,$$

then the associated solution v of (1.1) satisfies

$$\sup_{t \in \mathbb{R}} \inf_{\theta \in \mathbb{R}} \|v(t) - e^{i\theta} \varphi_\omega\|_{H^1(\mathbb{R}^+)} < \varepsilon.$$

Otherwise, w said to be *unstable*.

- (2) The standing wave w is called *unstable by blow up* if there exists a sequence (φ_n) such that $\lim_{n \rightarrow \infty} \|\varphi_n - \varphi_\omega\|_{H^1(\mathbb{R}^+)} = 0$ and the associated solution v_n of (1.1) blows up in finite time for all n .

Our second main result is the following.

Theorem I.4. *Let $\alpha, \omega \in \mathbb{R}$ be such that $\omega > \alpha^2$. The standing wave $e^{i\omega t} \varphi_\omega$, where φ_ω is the profile as in Proposition I.3, solution of (1.1), satisfies the following properties.*

- (1) *If $\alpha < 0$ then the standing wave is orbitally stable in $H^1(\mathbb{R}^+)$.*
- (2) *If $\alpha \geq 0$ then the standing wave is unstable by blow up.*

Remark I.5. The conservation laws play an important role to study the stability of standing waves. However, the existence of conservation of energy is not always true (see remark I.1). Our work can only extend for the models with nonlinear terms such that the energy is conserved.

Remark I.6. The blow up of solution is true in the case $\alpha = 0$ i.e Neumann boundary condition. In addition, there exist standing waves in this case. The formula of standing waves in this case is only a special case of the general case $\alpha \in \mathbb{R}$. Using similar arguments as in Ref. 23 (see also Ref. 24, Theorem 5.2), we obtain that in case $\alpha = 0$, the standing waves is unstable by blow up.

This paper is organized as follows. First, under the assumption of local well posedness in $H^1(\mathbb{R}^+)$, we prove the existence of blow-up solutions using a virial argument: In Section II A, we give the proof of Theorem I.2. Second, in the case $\alpha < 0$, using similar arguments as in Ref. 25, we prove the orbital stability of standing waves of (1.1). In the case $\alpha \geq 0$, using similar arguments as in Ref. 24, we prove the instability by blow up of standing waves. The proof of Theorem I.4 is obtained in Section II B. Finally, in Section III, we prove the invariance of a set under the flow of equation (1.1) which gives us another proof of the instability of standing waves. In this section, we also prove that $\partial_\omega \|\tilde{\varphi}_\omega\|_{L^2(\mathbb{R})}^2 > 0$ provided $\alpha < 0$, which is important in the proof of stability of standing waves.

II. PROOF OF THE MAIN RESULTS

We consider the equation (1.1) and assume that Assumption 1 holds.

A. The existence of blow-up solutions

In this section, we give the proof of Theorem 1.2 using a virial argument. Let $\alpha \geq 0$ and v be a solution of (1.1). To prove the existence of blow-up solutions, we use similar arguments as in Ref.

11. Set

$$I(t) = \int_0^\infty x^2 |v(t)|^2 dx.$$

Let

$$u(t, x) = v(t, x) \exp \left(-\frac{i}{4} \int_x^\infty |v|^2 dy \right) \quad (2.10)$$

be a Gauge transform in $H^1(\mathbb{R}^+)$. Then the problem (1.1) is equivalent with

$$\begin{cases} iu_t + u_{xx} = i|u|^2 u_x, \\ u_x(t, 0) = \alpha u(t, 0) + \frac{i}{4} |u(t, 0)|^2 u(t, 0). \end{cases} \quad (2.11)$$

The equation (2.11) has a simpler nonlinear form, but we pay this simplification with a nonlinear boundary condition. Observe that

$$I(t) = \int_0^\infty x^2 |u(t)|^2 dx = \int_0^\infty x^2 |v(t)|^2 dx.$$

By a direct calculation, we get

$$\partial_t I(t) = 2\Re \int_0^\infty x^2 \overline{u(t, x)} \partial_t u(t, x) dx = 2\Re \int_0^\infty x^2 \bar{u} (iu_{xx} + |u|^2 u_x) dx \quad (2.12)$$

$$= 2\Im \int_0^\infty 2x \bar{u} u_x dx - \frac{1}{2} \int_0^\infty 2x |u|^4 dx \quad (2.13)$$

$$= 4\Im \int_0^\infty x u_x \bar{u} dx - \int_0^\infty x |u|^4 dx. \quad (2.14)$$

Define

$$J(t) = \Im \int_0^\infty x u_x \bar{u} dx.$$

We have

$$\begin{aligned}
\partial_t J(t) &= \mathcal{I}m \int_0^\infty x u_x \bar{u}_t dx + \mathcal{I}m \int_0^\infty x \bar{u} u_{xt} dx \\
&= -\mathcal{I}m \int_0^\infty x u_t \bar{u}_x dx - \mathcal{I}m \int_0^\infty (x \bar{u})_x u_t dx \\
&= -2\mathcal{I}m \int_0^\infty x u_t \bar{u}_x dx - \mathcal{I}m \int_0^\infty u_t \bar{u} dx \\
&= -2\mathcal{I}m \int_0^\infty x \bar{u}_x (i u_{xx} + |u|^2 u_x) dx - \mathcal{I}m \int_0^\infty \bar{u} (i u_{xx} + |u|^2 u_x) dx \\
&= -2\mathcal{R}e \int_0^\infty x \bar{u}_x u_{xx} dx - \mathcal{R}e \int_0^\infty \bar{u} u_{xx} dx - \mathcal{I}m \int_0^\infty |u|^2 u_x \bar{u} dx \\
&= -\int_0^\infty x \partial_x |u_x|^2 dx - \mathcal{R}e(\bar{u} u_x)(t, +\infty) + \mathcal{R}e(\bar{u} u_x)(t, 0) + \mathcal{R}e \int_0^\infty \bar{u}_x u_x dx - \mathcal{I}m \int_0^\infty |u|^2 u_x \bar{u} dx \\
&= \int_0^\infty |u_x|^2 dx + \mathcal{R}e(\bar{u}(t, 0) u_x(t, 0)) + \int_0^\infty |u_x|^2 dx - \mathcal{I}m \int_0^\infty |u|^2 u_x \bar{u} dx \\
&= 2 \int_0^\infty |u_x|^2 dx - \mathcal{I}m \int_0^\infty |u|^2 u_x \bar{u} dx + \mathcal{R}e(\bar{u}(t, 0) u_x(t, 0)).
\end{aligned}$$

Using the Robin boundary condition we have

$$\partial_t J(t) = 2 \int_0^\infty |u_x|^2 dx - \mathcal{I}m \int_0^\infty |u|^2 u_x \bar{u} dx + \alpha |u(t, 0)|^2.$$

Moreover using the expression of v in term of u given in (2.10), we get

$$\begin{aligned}
\partial_t J(t) &= 2 \int_0^\infty |v_x|^2 dx - \frac{1}{8} \int_0^\infty |v|^6 dx + \alpha |v(t, 0)|^2 \\
&= 4E(v) - \alpha |v(t, 0)|^2 \leq 4E(v) = 4E(v_0).
\end{aligned}$$

By integrating the two sides of the above inequality in time we have

$$J(t) \leq J(0) + 4E(v_0)t. \quad (2.15)$$

Integrating the two sides of (2.12) in time we have

$$\begin{aligned}
I(t) &= I(0) + 4 \int_0^t J(s) ds - \int_0^t \int_0^\infty x |u(s, x)|^4 dx ds \\
&\leq I(0) + 4 \int_0^t J(s) ds.
\end{aligned}$$

Using (2.15) we have

$$\begin{aligned}
I(t) &\leq I(0) + 4 \int_0^t (J(0) + 4E(v_0)s) ds \\
&\leq I(0) + 4J(0)t + 8E(v_0)t^2.
\end{aligned}$$

From the assumption $E(v_0) < 0$, there exists a finite time $T_* > 0$ such that $I(T_*) = 0$,

$$I(t) > 0 \text{ for } 0 < t < T_*.$$

Note that

$$\begin{aligned} \int_0^\infty |v_0(x)|^2 dx &= \int_0^\infty |v(t, x)|^2 dx = -2\Re \int_0^\infty x v(t, x) \overline{v_x}(t, x) dx \\ &\leq 2\|xv\|_{L_x^2(\mathbb{R}^+)} \|v_x\|_{L_x^2(\mathbb{R}^+)} = 2\sqrt{I(t)} \|v_x\|_{L_x^2(\mathbb{R}^+)}. \end{aligned}$$

Then there exists a constant $C = C(v_0) > 0$ such that

$$\|v_x\|_{L_x^2(\mathbb{R}^+)} \geq \frac{C}{2\sqrt{I(t)}} \rightarrow +\infty \text{ as } t \rightarrow T_*.$$

Then the solution v blows up in finite time in $H^1(\mathbb{R}^+)$. This complete the proof of Theorem [1.2](#).

B. Stability and instability of standing waves

In this section, we give the proof of Theorem [1.4](#). To avoid confusion, throughout of this section, we denote $v(0)$ is value of function v at point $x = 0$. First, we find the form of the standing waves of [\(1.1\)](#).

1. Standing waves

Let $v = e^{i\omega t} \varphi_\omega$ be a solution of [\(1.1\)](#). Then φ_ω solves

$$\begin{cases} 0 = \varphi_{xx} - \omega \varphi + \frac{1}{2} \Im m(\varphi_x \overline{\varphi}) \varphi + \frac{3}{16} |\varphi|^4 \varphi, & \text{for } x > 0 \\ \varphi_x(0) = \alpha \varphi(0), \\ \varphi \in H^2(\mathbb{R}^+). \end{cases} \quad (2.16)$$

Set

$$A := \omega - \frac{1}{2} \Im m(\varphi'_\omega \overline{\varphi_\omega}) - \frac{3}{16} |\varphi_\omega|^4$$

By writing $\varphi_\omega = f + ig$ for f and g real valued functions, for $x > 0$, we have

$$f_{xx} = Af,$$

$$g_{xx} = Ag.$$

Thus,

$$\partial_x(f_x g - g_x f) = f_{xx}g - g_{xx}f = 0 \text{ when } x > 0.$$

Hence, by using $f, g \in H^2(\mathbb{R}^+)$, we have

$$f_x(x)g(x) - g_x(x)f(x) = 0 \text{ when } x > 0.$$

Then, for all $x > 0$, we have

$$\mathcal{I}m(\varphi'_\omega(x)\overline{\varphi_\omega(x)}) = g_x(x)f(x) - f_x(x)g(x) = 0,$$

hence, φ_ω solves

$$\begin{cases} 0 = \varphi_{xx} - \omega\varphi + \frac{3}{16}|\varphi|^4\varphi, & \text{for } x > 0 \\ \varphi_x(0) = \alpha\varphi(0), \\ \varphi \in H^2(\mathbb{R}^+). \end{cases} \quad (2.17)$$

We have the following result.

Proposition II.1. *Let $\omega > \alpha^2$. There exists a unique (up to phase shift) solution φ_ω of (2.17), which is of the form*

$$\varphi_\omega = 2\sqrt[4]{\omega} \operatorname{sech}^{\frac{1}{2}} \left(2\sqrt{\omega}|x| + \tanh^{-1} \left(\frac{-\alpha}{\sqrt{\omega}} \right) \right), \quad (2.18)$$

for all $x > 0$.

Proof. Let φ_ω be a solution of (2.17) and w be the even function defined by

$$w(x) = \begin{cases} \varphi_\omega(x) & \text{if } x \geq 0, \\ \varphi_\omega(-x) & \text{if } x \leq 0. \end{cases}$$

Then w solves

$$\begin{cases} 0 = -w_{xx} + \omega w - \frac{3}{16}|w|^4w, & \text{for } x \neq 0, \\ w_x(0^+) - w_x(0^-) = 2\alpha w(0), \\ w \in H^2(\mathbb{R}) \setminus \{0\} \cap H^1(\mathbb{R}). \end{cases} \quad (2.19)$$

Using the results of Fukuizumi and Jeanjean²⁶ (see Proposition II.2), we obtain that

$$w(x) = 2\sqrt[4]{\omega} \operatorname{sech}^{\frac{1}{2}} \left(2\sqrt{\omega}|x| + \tanh^{-1} \left(\frac{-\alpha}{\sqrt{\omega}} \right) \right)$$

up to phase shift provided $\omega > \alpha^2$. Hence, for $x > 0$ we have

$$\varphi_\omega(x) = 2\sqrt[4]{\omega} \operatorname{sech}^{\frac{1}{2}} \left(2\sqrt{\omega}|x| + \tanh^{-1} \left(\frac{-\alpha}{\sqrt{\omega}} \right) \right)$$

up to phase shift. This implies the desired result. \square

2. The variational problems

In this section, we give the proof of Proposition I.3.

First, we introduce another variational problem:

$$\tilde{d}(\omega) := \inf \left\{ \tilde{S}_\omega(v) \mid v \text{ even}, \tilde{K}_\omega(v) = 0, v \in H^1(\mathbb{R}) \setminus \{0\} \right\}, \quad (2.20)$$

where $\tilde{S}_\omega, \tilde{K}_\omega$ are defined for all $v \in H^1(\mathbb{R})$ by

$$\begin{aligned} \tilde{S}_\omega(v) &:= \frac{1}{2} \left[\|v_x\|_{L^2(\mathbb{R})}^2 + \omega \|v\|_{L^2(\mathbb{R})}^2 + 2\alpha |v(0)|^2 \right] - \frac{1}{32} \|v\|_{L^6(\mathbb{R})}^6, \\ \tilde{K}_\omega(v) &:= \|v_x\|_{L^2(\mathbb{R})}^2 + \omega \|v\|_{L^2(\mathbb{R})}^2 + 2\alpha |v(0)|^2 - \frac{3}{16} \|v\|_{L^6(\mathbb{R})}^6. \end{aligned}$$

The functional \tilde{K}_ω is called Nehari functional. The following result was proved in Refs. 26 and 27.

Proposition II.2. *Let $\omega > \alpha^2$. Consider the following equation*

$$\begin{cases} -\varphi_{xx} + 2\alpha\delta\varphi + \omega\varphi - \frac{3}{16}|\varphi|^4\varphi = 0, \\ \varphi \in H^1(\mathbb{R}) \setminus \{0\}. \end{cases} \quad (2.21)$$

There exists a unique positive solution $\tilde{\varphi}_\omega$ of (2.21). This solution is the unique positive minimizer of (2.20). Furthermore, we have an explicit formula for $\tilde{\varphi}_\omega$

$$\tilde{\varphi}_\omega(x) = 2\sqrt[4]{\omega} \operatorname{sech}^{\frac{1}{2}} \left(2\sqrt{\omega}|x| + \tanh^{-1} \left(\frac{-\alpha}{\sqrt{\omega}} \right) \right). \quad (2.22)$$

We have the following relation between the variational problems.

Proposition II.3. *Let $\omega > \alpha^2$. We have*

$$d(\omega) = \frac{1}{2} \tilde{d}(\omega).$$

Proof. Assume v is a minimizer of (1.9), define the $H^1(\mathbb{R})$ function w by

$$w(x) = \begin{cases} v(x) & \text{if } x > 0, \\ v(-x) & \text{if } x < 0. \end{cases}$$

The function $w \in H^1(\mathbb{R}) \setminus \{0\}$ verifies

$$\begin{aligned} \tilde{S}_\omega(w) &= 2S_\omega(v) = 2d(\omega), \\ \tilde{K}_\omega(w) &= 2K_\omega(v) = 0. \end{aligned}$$

This implies that

$$\tilde{d}(\omega) \leq \tilde{S}_\omega(w) = 2d(\omega). \quad (2.23)$$

Now, assume v is a minimizer of (2.20). Let w be the restriction of v on \mathbb{R}^+ , then,

$$K_\omega(w) = \frac{1}{2} \tilde{K}_\omega(v) = 0.$$

Hence, we obtain

$$\tilde{d}(\omega) = \tilde{S}_\omega(v) = 2S_\omega(w) \geq 2d(\omega). \quad (2.24)$$

Combining (2.23) and (2.24) we have

$$\tilde{d}(\omega) = 2d(\omega).$$

This implies the desired result. □

Proof of Theorem 1.3. Let v be a minimizer of (1.9). Define $w(x) \in H^1(\mathbb{R})$ by

$$w(x) = \begin{cases} v(x) & \text{if } x > 0, \\ v(-x) & \text{if } x < 0. \end{cases}$$

Then, w is an even function. Moreover, w satisfies

$$\begin{aligned} \tilde{K}_\omega(w) &= 2K_\omega(v) = 0, \\ \tilde{S}_\omega(w) &= 2S_\omega(v) = 2d(\omega) = \tilde{d}(\omega). \end{aligned}$$

Hence, w is a minimizer of (2.20). From Proposition II.2, w is of the form $e^{i\theta} \tilde{\varphi}_\omega$. Hence, $v = w|_{\mathbb{R}^+}$ satisfies

$$v = e^{i\theta} \tilde{\varphi}_\omega|_{\mathbb{R}^+} = e^{i\theta} \varphi_\omega.$$

This completes the proof of Proposition 1.3. □

3. Stability and instability of standing waves

In this section, we give the proof of Theorem 1.4. We use the notations \tilde{S}_ω and \tilde{K}_ω as in Section II B 2. First, we define

$$N(v) := \frac{3}{16} \|v\|_{L^6(\mathbb{R}^+)}^6, \quad (2.25)$$

$$L(v) := \|v_x\|_{L^2(\mathbb{R}^+)}^2 + \omega \|v\|_{L^2(\mathbb{R}^+)}^2 + \alpha |v(0)|^2. \quad (2.26)$$

We can rewrite S_ω, K_ω as follows

$$S_\omega = \frac{1}{2}L - \frac{1}{6}N,$$

$$K_\omega = L - N.$$

We have the following classical properties of the above functions.

Lemma II.4. *Let $(\omega, \alpha) \in \mathbb{R}^2$ such that $\omega > \alpha^2$. The following assertions hold.*

(1) *There exists a constant $C > 0$ such that*

$$L(v) \geq C\|v\|_{H^1(\mathbb{R}^+)}^2 \quad \forall v \in H^1(\mathbb{R}^+).$$

(2) *We have $d(\omega) > 0$.*

(3) *If $v \in H^1(\mathbb{R}^+)$ satisfies $K_\omega(v) < 0$ then $L(v) > 3d(\omega)$.*

Proof. We have

$$|v(0)|^2 = - \int_0^\infty \partial_x(|v(x)|^2) dx = -2\Re \int_0^\infty v(x) \bar{v}_x(x) dx$$

$$\leq 2\|v\|_{L^2(\mathbb{R}^+)}\|v_x\|_{L^2(\mathbb{R}^+)}.$$

Hence,

$$\begin{aligned} L(v) &= \|v_x\|_{L^2(\mathbb{R}^+)}^2 + \omega\|v\|_{L^2(\mathbb{R}^+)}^2 + \alpha|v(0)|^2 \\ &\geq \|v_x\|_{L^2(\mathbb{R}^+)}^2 + \omega\|v\|_{L^2(\mathbb{R}^+)}^2 - 2|\alpha|\|v\|_{L^2(\mathbb{R}^+)}\|v_x\|_{L^2(\mathbb{R}^+)} \\ &\geq C\|v\|_{H^1(\mathbb{R}^+)}^2 + (1-C)\|v_x\|_{L^2(\mathbb{R}^+)}^2 + (\omega-C)\|v\|_{L^2(\mathbb{R}^+)}^2 - 2|\alpha|\|v\|_{L^2(\mathbb{R}^+)}\|v_x\|_{L^2(\mathbb{R}^+)} \\ &\geq C\|v\|_{H^1(\mathbb{R}^+)}^2 + (2\sqrt{(1-C)(\omega-C)} - 2|\alpha|)\|v\|_{L^2(\mathbb{R}^+)}\|v_x\|_{L^2(\mathbb{R}^+)}. \end{aligned}$$

From the assumption $\omega > \alpha^2$, we can choose $C \in (0, 1)$ such that

$$2\sqrt{(1-C)(\omega-C)} - 2|\alpha| > 0.$$

This implies (1). Now, we prove (2). Let v be an element of $H^1(\mathbb{R}^+)$ satisfying $K_\omega(v) = 0$. We have

$$C\|v\|_{H^1(\mathbb{R}^+)}^2 \leq L(v) = N(v) \leq C_1\|v\|_{H^1(\mathbb{R}^+)}^6.$$

Then,

$$\|v\|_{H^1(\mathbb{R}^+)}^2 \geq \sqrt[4]{\frac{C}{C_1}}.$$

From the fact that, for v satisfying $K_\omega(v) = 0$, we have $S_\omega(v) = S_\omega(v) - \frac{1}{6}K_\omega(v) = \frac{1}{3}L(v)$, this implies that

$$d(\omega) = \frac{1}{3} \inf \{L(v) : v \in H^1(\mathbb{R}^+), K_\omega(v) = 0\} \geq \frac{C}{3} \sqrt[4]{\frac{C}{C_1}} > 0.$$

Finally, we prove (3). Let $v \in H^1(\mathbb{R}^+)$ satisfying $K_\omega(v) < 0$. Then, there exists $\lambda_1 \in (0, 1)$ such that $K_\omega(\lambda_1 v) = \lambda_1^2 L(v) - \lambda_1^6 N(v) = 0$. Since $v \neq 0$, we have $3d(\omega) \leq L(\lambda_1 v) = \lambda_1^2 L(v) < L(v)$. \square

Define

$$\tilde{N}(v) := \frac{3}{16} \|v\|_{L^6(\mathbb{R})}^6, \quad (2.27)$$

$$\tilde{L}(v) := \|v_x\|_{L^2(\mathbb{R})}^2 + \omega \|v\|_{L^2(\mathbb{R})}^2 + 2\alpha |v(0)|^2. \quad (2.28)$$

We can rewrite $\tilde{S}_\omega, \tilde{K}_\omega$ as follows

$$\tilde{S}_\omega = \frac{1}{2} \tilde{L} - \frac{1}{6} \tilde{N},$$

$$\tilde{K}_\omega = \tilde{L} - \tilde{N}.$$

As consequence of the previous lemma, we have the following result.

Lemma II.5. *Let $(\omega, \alpha) \in \mathbb{R}^2$ such that $\omega > \alpha^2$. The following assertions hold.*

(1) *There exists a constant $C > 0$ such that*

$$\tilde{L}(v) \geq C \|v\|_{H^1}^2 \quad \forall v \in H^1(\mathbb{R}).$$

(2) *We have $\tilde{d}(\omega) > 0$.*

(3) *If $v \in H^1(\mathbb{R})$ satisfies $\tilde{K}_\omega(v) < 0$ then $\tilde{L}(v) > 3\tilde{d}(\omega)$.*

We introduce the following properties.

Lemma II.6. *Let $2 \leq p < \infty$ and (f_n) be a bounded sequence in $L^p(\mathbb{R})$. Assume that $f_n \rightarrow f$ a.e in \mathbb{R} . Then we have*

$$\|f_n\|_{L^p}^p - \|f_n - f\|_{L^p}^p - \|f\|_{L^p}^p \rightarrow 0.$$

For the proof of Lemma II.6, see Ref. 28.

Lemma II.7. *The following minimization problem is equivalent to the problem (2.20) i.e they have the same minimum and the same minimizers:*

$$d := \inf \left\{ \frac{1}{16} \|u\|_{L^6}^6 : u \text{ even}, u \in H^1(\mathbb{R}) \setminus \{0\}, \tilde{K}_\omega(u) \leq 0 \right\}. \quad (2.29)$$

Proof. We see that the minimizer problem (2.20) is equivalent to following problem:

$$\inf \left\{ \frac{1}{16} \|u\|_{L^6}^6 : u \text{ even } u \in H^1(\mathbb{R}) \setminus \{0\}, \tilde{K}_\omega(u) = 0 \right\}. \quad (2.30)$$

Let v be a minimizer of (2.20) then $\tilde{K}_\omega(v) = 0$, hence, $\tilde{d}(\omega) = \frac{1}{16} \|v\|_{L^6}^6 \geq d$. Now, let v be a minimizer of (2.29). We prove that $\tilde{K}_\omega(v) = 0$. Indeed, assuming $\tilde{K}_\omega(v) < 0$, we have

$$\tilde{K}_\omega(\lambda v) = \lambda^2 \left(\|v_x\|_{L^2}^2 + \omega \|v\|_{L^2}^2 + 2\alpha |v(0)|^2 - \frac{3\lambda^4}{16} \|v\|_{L^6}^6 \right) > 0,$$

as $0 < \lambda$ is small enough. Thus, by continuity, there exists a $\lambda_0 \in (0, 1)$ such that $\tilde{K}_\omega(\lambda_0 v) = 0$. We have $d \leq \tilde{d}(\omega) \leq \frac{1}{16} \|\lambda_0 v\|_{L^6}^6 < \frac{1}{16} \|v\|_{L^6}^6 = d$, which is a contradiction. It implies that $\tilde{K}_\omega(v) = 0$ and v is a minimizer of (2.30), hence v is a minimizer of (2.20). This completes the proof. \square

Now, using similar arguments as in Ref. 27, Proof of Proposition 2, we have the following result.

Proposition II.8. *Let $(\omega, \alpha) \in \mathbb{R}^2$ be such that $\alpha < 0$, $\omega > \alpha^2$ and $(w_n) \subset H^1(\mathbb{R})$ be a even sequence satisfying the following properties as $n \rightarrow \infty$.*

$$\begin{aligned} \tilde{S}_\omega(w_n) &\rightarrow \tilde{d}(\omega), \\ \tilde{K}_\omega(w_n) &\rightarrow 0. \end{aligned}$$

Then, there exists a minimizer w of (2.20) such that $w_n \rightarrow w$ strongly in $H^1(\mathbb{R})$ up to subsequence.

Proof. In what follows, we shall often extract subsequence without mentioning this fact explicitly. We divide the proof into two steps.

Step 1. Weak convergence to a nonvanishing function for the minimizing sequence. We have

$$\frac{1}{3} \tilde{L}(w_n) = \tilde{S}_\omega(w_n) - \frac{1}{6} \tilde{K}_\omega(w_n) \rightarrow \tilde{d}(\omega),$$

as $n \rightarrow \infty$. Then, (w_n) is bounded in $H^1(\mathbb{R})$ and there exists $w \in H^1(\mathbb{R})$ even such that $w_n \rightharpoonup w$ in $H^1(\mathbb{R})$ up to subsequence. We prove $w \neq 0$. Assume that $w \equiv 0$. Define, for $u \in H^1(\mathbb{R})$,

$$\begin{aligned} S_\omega^0(u) &= \frac{1}{2} \|u_x\|_{L^2}^2 + \frac{\omega}{2} \|u\|_{L^2}^2 - \frac{1}{32} \|u\|_{L^6}^6, \\ K_\omega^0(u) &= \|u_x\|_{L^2}^2 + \omega \|u\|_{L^2}^2 - \frac{3}{16} \|u\|_{L^6}^6. \end{aligned}$$

Let ψ_ω be minimizer of the following problems

$$\begin{aligned} d^0(\omega) &= \inf \{ S_\omega^0(u) : u \text{ even}, u \in H^1(\mathbb{R}) \setminus \{0\}, K_\omega^0(u) = 0 \} \\ &= \inf \left\{ \frac{1}{16} \|u\|_{L^6}^6 : u \text{ even}, u \in H^1(\mathbb{R}) \setminus \{0\}, K_\omega^0(u) \leq 0 \right\}. \end{aligned}$$

We have $K_\omega^0(w_n) = \tilde{K}_\omega(w_n) - 2\alpha|w_n(0)|^2 \rightarrow 0$, as $n \rightarrow \infty$. Since $\alpha < 0$, we have $\tilde{K}_\omega(\psi_\omega) < 0$ and hence we obtain

$$\tilde{d}(\omega) < \frac{1}{16} \|\psi_\omega\|_{L^6}^6 = d^0(\omega) \quad (2.31)$$

We set

$$\lambda_n = \left(\frac{\|\partial_x w_n\|_{L^2}^2 + \omega \|w_n\|_{L^2}^2}{\frac{3}{16} \|w_n\|_{L^6}^6} \right)^{\frac{1}{4}}.$$

We here remark that $0 < \tilde{d}(\omega) = \lim_{n \rightarrow \infty} \frac{1}{16} \|w_n\|_{L^6}^6$. It follows that

$$\lambda_n^4 - 1 = \frac{K_\omega^0(w_n)}{\frac{3}{16} \|w_n\|_{L^6}^6} \rightarrow 0,$$

as $n \rightarrow \infty$. We see that $K_\omega^0(\lambda_n w_n) = 0$ and $\lambda_n w_n \neq 0$. By the definition of $d^0(\omega)$, we have

$$d^0(\omega) \leq \frac{1}{16} \|\lambda_n w_n\|_{L^6}^6 \rightarrow \tilde{d}(\omega) \text{ as } n \rightarrow \infty.$$

This contradicts (2.31). Thus, $w \neq 0$.

Step 2. Conclusion of the proof. Using Lemma II.6 we have

$$\tilde{K}_\omega(w_n) - \tilde{K}_\omega(w_n - w) - \tilde{K}_\omega(w) \rightarrow 0, \quad (2.32)$$

$$\tilde{L}(w_n) - \tilde{L}(w_n - w) - \tilde{L}(w) \rightarrow 0. \quad (2.33)$$

Now, we prove $\tilde{K}_\omega(w) \leq 0$ by contradiction. Suppose that $\tilde{K}_\omega(w) > 0$. By the assumption $\tilde{K}_\omega(w_n) \rightarrow 0$ and (2.32), we have

$$\tilde{K}_\omega(w_n - w) \rightarrow -\tilde{K}_\omega(w) < 0.$$

Thus, $\tilde{K}_\omega(w_n - w) < 0$ for n large enough. By Lemma II.5 (3), we have $\tilde{L}(w_n - w) \geq 3\tilde{d}(\omega)$. Since $\tilde{L}(w_n) \rightarrow 3\tilde{d}(\omega)$, by (2.33), we have

$$\tilde{L}(w) = \lim_{n \rightarrow \infty} (\tilde{L}(w_n) - \tilde{L}(w_n - w)) \leq 0.$$

Moreover, $w \neq 0$ and by Lemma II.5 (1), we have $\tilde{L}(w) > 0$. This is a contradiction. Hence, $\tilde{K}_\omega(w) < 0$. By Lemma II.5 (2), (3) and weak lower semicontinuity of \tilde{L} , we have

$$3\tilde{d}(\omega) \leq \tilde{L}(w) \leq \liminf_{n \rightarrow \infty} \tilde{L}(w_n) = 3\tilde{d}(\omega).$$

Thus, $\tilde{L}(w) = 3\tilde{d}(\omega)$. Combining with (2.33), we have $\tilde{L}(w_n - w) \rightarrow 0$, as $n \rightarrow \infty$. By Lemma II.5 (1), we have $w_n \rightarrow w$ strongly in $H^1(\mathbb{R})$. Hence, w is a minimizer of (2.20). This completes the proof. \square

To prove the stability statement (1) for $\alpha < 0$ in Theorem I.4, we will use similar arguments as in the work of Colin and Ohta²⁵. We need the following property.

Lemma II.9. *Let $\alpha < 0$, $\omega > \alpha^2$. If a sequence $(v_n) \subset H^1(\mathbb{R}^+)$ satisfies*

$$S_\omega(v_n) \rightarrow d(\omega), \quad (2.34)$$

$$K_\omega(v_n) \rightarrow 0, \quad (2.35)$$

then there exists a constant $\theta_0 \in \mathbb{R}$ such that $v_n \rightarrow e^{i\theta_0}\varphi_\omega$, up to subsequence, where φ_ω is defined as in Proposition I.3.

Proof. Define the sequence $(w_n) \subset H^1(\mathbb{R})$ as follows,

$$w_n(x) = \begin{cases} v_n(x) & \text{for } x > 0, \\ v_n(-x) & \text{for } x < 0. \end{cases}$$

We can check that

$$\tilde{S}_\omega(w_n) = 2S_\omega(v_n) \rightarrow 2d(\omega) = \tilde{d}(\omega),$$

$$\tilde{K}_\omega(w_n) = 2K_\omega(v_n) \rightarrow 0,$$

as $n \rightarrow \infty$. Using Proposition II.8, there exists a minimizer w_0 of (2.20) such that $w_n \rightarrow w_0$ strongly in $H^1(\mathbb{R})$, up to subsequence. For convenience, we assume that $w_n \rightarrow w_0$ strongly in $H^1(\mathbb{R})$. By Proposition II.2, there exists a constant $\theta_0 \in \mathbb{R}$ such that

$$w_0 = e^{i\theta_0}\tilde{\varphi}_\omega,$$

where $\tilde{\varphi}$ is defined as in (2.22). Hence,

$$v_n \rightarrow e^{i\theta_0}\tilde{\varphi}_\omega|_{\mathbb{R}^+} = e^{i\theta}\varphi_\omega, \text{ strongly in } H^1(\mathbb{R}^+),$$

up to subsequence. This completes the proof. \square

Define

$$\begin{aligned}\mathcal{A}_\omega^+ &= \{v \in H^1(\mathbb{R}^+) \setminus \{0\} : S_\omega(v) < d(\omega), K_\omega(v) > 0\}, \\ \mathcal{A}_\omega^- &= \{v \in H^1(\mathbb{R}^+) \setminus \{0\} : S_\omega(v) < d(\omega), K_\omega(v) < 0\}, \\ \mathcal{B}_\omega^+ &= \{v \in H^1(\mathbb{R}^+) \setminus \{0\} : S_\omega(v) < d(\omega), N(v) < 3d(\omega)\}, \\ \mathcal{B}_\omega^- &= \{v \in H^1(\mathbb{R}^+) \setminus \{0\} : S_\omega(v) < d(\omega), N(v) > 3d(\omega)\}.\end{aligned}$$

We have the following result.

Lemma II.10. *Let $\omega, \alpha \in \mathbb{R}^2$ such that $\alpha < 0$ and $\omega > \alpha^2$. The following assertions holds.*

(1) *The sets \mathcal{A}_ω^+ and \mathcal{A}_ω^- are invariant under the flow of (1.1).*

(2) *$\mathcal{A}_\omega^+ = \mathcal{B}_\omega^+$ and $\mathcal{A}_\omega^- = \mathcal{B}_\omega^-$.*

Proof. (1) Let $u_0 \in \mathcal{A}_\omega^+$ and $u(t)$ the associated solution for (1.1) on (T_{min}, T_{max}) . By $u_0 \neq 0$ and the conservation laws, we see that $S_\omega(u(t)) = S_\omega(u_0) < d(\omega)$ for $t \in (T_{min}, T_{max})$. Moreover, by definition of $d(\omega)$ we have $K_\omega(u(t)) \neq 0$ on (T_{min}, T_{max}) . Since the function $t \mapsto K_\omega(u(t))$ is continuous, we have $K_\omega(u(t)) > 0$ on (T_{min}, T_{max}) . Hence, \mathcal{A}_ω^+ is invariant under flow of (1.1). By the same way, \mathcal{A}_ω^- is invariant under flow of (1.1).

(2) If $v \in \mathcal{A}_\omega^+$ then by (2.28), (2.27) we have $N(v) = 3S_\omega(v) - 2K_\omega(v) < 3d(\omega)$, which shows $v \in \mathcal{B}_\omega^+$, hence $\mathcal{A}_\omega^+ \subset \mathcal{B}_\omega^+$. Now, let $v \in \mathcal{B}_\omega^+$. We show $K_\omega(v) > 0$ by contradiction. Suppose that $K_\omega(v) \leq 0$. Then, by Lemma II.5 (3), $L(v) \geq 3d(\omega)$. Thus, by (2.28) and (2.27), we have

$$S_\omega(v) = \frac{1}{2}L(v) - \frac{1}{6}N(v) \geq d(\omega),$$

which contradicts $S_\omega(v) < d(\omega)$. Therefore, we have $K_\omega(v) > 0$, which shows $v \in \mathcal{A}_\omega^+$ and $\mathcal{B}_\omega^+ \subset \mathcal{A}_\omega^+$. Next, if $v \in \mathcal{A}_\omega^-$, then by Lemma II.5 (3), $L(v) > 3d(\omega)$. Thus, by (2.28) and (2.27), we have $N(v) = L(v) - K_\omega(v) > 3d(\omega)$, which shows $v \in \mathcal{B}_\omega^-$. Thus, $\mathcal{A}_\omega^- \subset \mathcal{B}_\omega^-$. Finally, if $v \in \mathcal{B}_\omega^-$, then by (2.28) and (2.27), we have $2K_\omega(v) = 3S_\omega(v) - N(v) < 3d(\omega) - 3d(\omega) = 0$, which shows $v \in \mathcal{A}_\omega^-$, hence, $\mathcal{B}_\omega^- \subset \mathcal{A}_\omega^-$. This completes the proof. \square

From Proposition I.3, we have

$$d(\omega) = S_\omega(\varphi_\omega).$$

Since $\alpha < 0$, using Proposition III.5, we have

$$d''(\omega) = \partial_\omega \|\varphi_\omega\|_{L^2(\mathbb{R}^+)}^2 = \frac{1}{2} \partial_\omega \|\tilde{\varphi}_\omega\|_{L^2(\mathbb{R})}^2 > 0,$$

We define the function $h : (-\varepsilon_0, \varepsilon_0) \rightarrow \mathbb{R}$ by

$$h(\tau) = d(\omega + \tau).$$

Since $h'(0) = d'(\omega) = \|\varphi_\omega\|_{L^2(\mathbb{R}^+)}^2 > 0$ and $h''(0) = d''(\omega) > 0$, by choosing ε_0 small enough, we can assume that $h'(\tau) > 0$ and $h''(\tau) > 0$ for $\tau \in (-\varepsilon_0, \varepsilon_0)$.

Lemma II.11. *Let $(\omega, \alpha) \in \mathbb{R}^2$ such that $\alpha < 0$ and $\omega > \alpha^2$ and let h be defined as above. Then, for any $\varepsilon \in (0, \varepsilon_0)$, there exists $\delta > 0$ such that if $v_0 \in H^1(\mathbb{R}^+)$ satisfies $\|v_0 - \varphi_\omega\|_{H^1(\mathbb{R}^+)} < \delta$, then the solution v of (1.1) with $v(0, x) = v_0(x)$ satisfies $3h(-\varepsilon) < N(v(t)) < 3h(\varepsilon)$ for all $t \in (T_{\min}, T_{\max})$.*

Proof. The proof of the above lemma is similar to the one on Ref.25. Let $\varepsilon \in (0, \varepsilon_0)$. Since h is increasing, we have $h(-\varepsilon) < h(0) < h(\varepsilon)$. Moreover, by $K_\omega(\varphi_\omega) = 0$ and (2.27), (2.28), we see that $3h(0) = 3d(\omega) = 3S_\omega(\varphi_\omega) = N(\varphi_\omega)$. Thus, if $u_0 \in H^1(\mathbb{R}^+)$ satisfies $\|u_0 - \varphi_\omega\|_{H^1(\mathbb{R}^+)} < \delta$ then we have $3h(0) = N(u_0) + O(\delta)$ and $3h(-\varepsilon) < N(u_0) < 3h(\varepsilon)$ for sufficiently small $\delta > 0$. Since $h(\pm\varepsilon) = d(\omega \pm \varepsilon)$ and the set \mathcal{B}_ω^\pm are invariant under the flow of (1.1) by Lemma II.10, to conclude the proof, we only have to show that there exists $\delta > 0$ such that if $u_0 \in H^1(\mathbb{R}^+)$ satisfies $\|u_0 - \varphi_\omega\|_{H^1(\mathbb{R}^+)} < \delta$ then $S_{\omega \pm \varepsilon}(u_0) < h(\pm\varepsilon)$. Assume that $u_0 \in H^1(\mathbb{R}^+)$ satisfies $\|u_0 - \varphi_\omega\|_{H^1(\mathbb{R}^+)} < \delta$. We have

$$\begin{aligned} S_{\omega \pm \varepsilon}(u_0) &= S_{\omega \pm \varepsilon}(\varphi_\omega) + O(\delta) \\ &= S_\omega(\varphi_\omega) \pm \varepsilon M(\varphi_\omega) + O(\delta) \\ &= h(0) \pm \varepsilon h'(0) + O(\delta). \end{aligned}$$

On the other hand, by the Taylor expansion, there exists $\tau_1 = \tau_1(\varepsilon) \in (-\varepsilon_0, \varepsilon_0)$ such that

$$h(\pm\varepsilon) = h(0) \pm \varepsilon h'(0) + \frac{\varepsilon^2}{2} h''(\tau_1).$$

Since $h''(\tau_1) > 0$ by definition of h , we see that there exists $\delta > 0$ such that if $u_0 \in H^1(\mathbb{R}^+)$ satisfies $\|u_0 - \varphi_\omega\|_{H^1(\mathbb{R}^+)} < \delta$ then $S_{\omega \pm \varepsilon}(u_0) < h(\pm\varepsilon)$. This completes the proof. \square

Proof of Theorem I.4 (I). Assume that $e^{i\omega t} \varphi_\omega$ is not stable for (1.1). Then, there exists a constant $\varepsilon_1 > 0$, a sequence of solutions (v^n) to (1.1), and a sequence $\{t_n\} \in (0, \infty)$ such that

$$v_n(0, x) \rightarrow \varphi_\omega(x) \text{ in } H^1(\mathbb{R}^+), \quad \inf_{\theta \in \mathbb{R}} \|v_n(t_n) - e^{i\theta} \varphi_\omega\|_{H^1(\mathbb{R}^+)} \geq \varepsilon_1. \quad (2.36)$$

Using the conservation laws of (1.1), we have

$$S_\omega(v_n(t_n)) = S_\omega(v_n(0)) \rightarrow S_\omega(\varphi_\omega) = d(\omega). \quad (2.37)$$

Using Lemma II.11, we have

$$N(v_n(t_n)) \rightarrow 3d(\omega). \quad (2.38)$$

Combining (2.37) and (2.38), we have

$$K_\omega(v_n(t_n)) = 2S_\omega(v_n(t_n)) - \frac{2}{3}N(v_n(t_n)) \rightarrow 0.$$

Therefore, using Lemma II.9, there exists $\theta_0 \in \mathbb{R}$ such that $(v_n(t_n, \cdot))$ has a subsequence (we denote it by the same letter) that converges to $e^{i\theta_0}\varphi_\omega$ in $H^1(\mathbb{R}^+)$, where φ_ω is defined as in Proposition I.3. Hence, we have

$$\inf_{\theta \in \mathbb{R}} \|v_n(t_n) - e^{i\theta}\varphi_\omega\|_{H^1(\mathbb{R}^+)} \rightarrow 0, \quad (2.39)$$

as $n \rightarrow \infty$, this contradicts (2.36). Hence, we obtain the desired result. \square

Next, we give the proof of Theorem I.4 (2). We divide the proof in two cases.

First, let $\alpha = 0$. In this case, we use similar arguments as in Ref. 23. We have

$$\begin{aligned} E_\omega(v) &= \frac{1}{2}\|v_x\|_{L^2(\mathbb{R}^+)}^2 - \frac{1}{32}\|v\|_{L^6(\mathbb{R}^+)}^6, \\ P(v) &= \|v_x\|_{L^2(\mathbb{R}^+)}^2 - \frac{1}{16}\|v\|_{L^6(\mathbb{R}^+)}^6. \end{aligned}$$

Thus, $E(\varphi_\omega) = P(\varphi_\omega) = 0$. Let $\varepsilon > 0$ and $\varphi_{\omega,\varepsilon} = (1 + \varepsilon)\varphi_\omega$. We have

$$E(\varphi_{\omega,\varepsilon}) = (1 + \varepsilon)^2 \frac{1}{2}\|\varphi_\omega\|_{L^2(\mathbb{R}^+)}^2 - (1 + \varepsilon)^6 \frac{1}{32}\|\varphi_\omega\|_{L^6(\mathbb{R}^+)}^6 = ((1 + \varepsilon)^2 - (1 + \varepsilon)^6) \frac{1}{2}\|\varphi_\omega\|_{L^2(\mathbb{R}^+)}^2 < 0.$$

In the addition, $|x|\varphi_{\omega,\varepsilon}(x) \in L^2(\mathbb{R}^+)$ by exponential decay of φ_ω . Using Theorem I.2, the solution associated to $\varphi_{\omega,\varepsilon}$ blows up in finite time. As $\varphi_{\omega,\varepsilon} \rightarrow \varphi_\omega$ in $H^1(\mathbb{R}^+)$, we obtain the instability by blow-up of standing waves.

Now, let $\alpha > 0$. We use similar arguments as in Ref. 24. Let $e^{i\omega t}\varphi_\omega$ be the standing wave solution of (1.1). Introduce the scaling

$$v_\lambda(x) = \lambda^{\frac{1}{2}}v(\lambda x).$$

Let S_ω, K_ω be defined as in Proposition I.3, for convenience, we will remove the index ω . Define

$$P(v) := \frac{\partial}{\partial \lambda} S(v_\lambda)|_{\lambda=1} = \|v_x\|_{L^2(\mathbb{R}^+)}^2 - \frac{1}{16}\|v\|_{L^6(\mathbb{R}^+)}^6 + \frac{\alpha}{2}|v(0)|^2.$$

In the following lemma, we investigate the behaviour of the above functional under scaling.

Lemma II.12. Assume $\alpha > 0$. Let $v \in H^1(\mathbb{R}^+) \setminus \{0\}$ be such that $v(0) \neq 0$, $P(v) \leq 0$. Then there exists $\lambda_0 \in (0, 1]$ such that

- (i) $P(v_{\lambda_0}) = 0$,
- (ii) $\lambda_0 = 1$ if only if $P(v) = 0$,
- (iii) $\frac{\partial}{\partial \lambda} S(v_\lambda) = \frac{1}{\lambda} P(v_\lambda)$,
- (iv) $\frac{\partial}{\partial \lambda} S(v_\lambda) > 0$ on $(0, \lambda_0)$ and $\frac{\partial}{\partial \lambda} S(v_\lambda) < 0$ on (λ_0, ∞) ,
- (v) The function $\lambda \rightarrow S(v_\lambda)$ is concave on (λ_0, ∞) .

Proof. A simple calculation leads to

$$P(v_\lambda) = \lambda^2 \|v_x\|_{L^2(\mathbb{R}^+)}^2 - \frac{\lambda^2}{16} \|v\|_{L^6(\mathbb{R}^+)}^6 + \frac{\lambda \alpha}{2} |v(0)|^2.$$

Then, for $\lambda > 0$ small enough, we have

$$P(v_\lambda) > 0.$$

By continuity of P , there exists $\lambda_0 \in (0, 1]$ such that $P(v_{\lambda_0}) = 0$. Hence (i) is proved. If $\lambda_0 = 1$ then $P(v) = 1$. Conversely, if $P(v) = 0$ then

$$0 = P(v_{\lambda_0}) = \lambda_0^2 P(v) + \frac{\lambda_0 - \lambda_0^2}{2} \alpha |v(0)|^2 = \frac{\lambda_0 - \lambda_0^2}{2} \alpha |v(0)|^2.$$

By the assumption $v(0) \neq 0$, we have $\lambda_0 = 1$, hence (ii) is proved. Item (iii) is obtained by a simple calculation. To obtain (iv), we use (iii). We have

$$\begin{aligned} P(v_\lambda) &= \lambda^2 \lambda_0^{-2} P(v_{\lambda_0}) + \left(\frac{\lambda \alpha}{2} - \frac{\lambda^2 \lambda_0^{-1} \alpha}{2} \right) |v(0)|^2 \\ &= \frac{\lambda \alpha (\lambda_0 - \lambda)}{2 \lambda_0} |v(0)|^2. \end{aligned}$$

Hence, $P(v_\lambda) > 0$ if $\lambda < \lambda_0$ and $P(v_\lambda) < 0$ if $\lambda > \lambda_0$. This proves (iv). Finally, we have

$$\frac{\partial^2}{\partial^2 \lambda} S(v_\lambda) = P(v) - \frac{\alpha}{2} |v(0)|^2 < 0.$$

This proves (v). □

In the case of functions such that $v(0) = 0$, we have the following lemma.

Lemma II.13. *Let $v \in H^1(\mathbb{R}^+) \setminus \{0\}$, $v(0) = 0$ and $P(v) = 0$ then we have*

$$S(v_\lambda) = S(v) \quad \text{for all } \lambda > 0.$$

Proof. The proof is simple, using the fact that

$$\frac{\partial}{\partial \lambda} S(v_\lambda) = \frac{1}{\lambda} P(v_\lambda) = \lambda P(v) = 0.$$

Hence, we obtain the desired result. \square

Now, consider the minimization problems

$$d_{\mathcal{M}} := \inf \{S(v) : v \in \mathcal{M}\}, \quad (2.40)$$

$$m := \inf \{S(v), v \in H^1(\mathbb{R}^+) \setminus 0, S'(v) = 0\}, \quad (2.41)$$

where

$$\mathcal{M} = \{v \in H^1(\mathbb{R}^+) \setminus 0, P(v) = 0, K(v) \leq 0\}.$$

By classical arguments, we can prove the following property.

Proposition II.14. *Let m be defined as above. Then, we have*

$$m = \inf \{S(v) : v \in H^1(\mathbb{R}^+) \setminus 0, K(v) = 0\}.$$

We have the following relation between the minimization problems m and $d_{\mathcal{M}}$.

Lemma II.15. *Let m and $d_{\mathcal{M}}$ be defined as above. We have*

$$m = d_{\mathcal{M}}.$$

Proof. Let \mathcal{G} be the set of all minimizers of (2.41). If $\varphi \in \mathcal{G}$ then $S'(\varphi) = 0$. By the definition of S, P, K we have $P(\varphi) = 0$ and $K(\varphi) = 0$. Hence, $\varphi \in \mathcal{M}$, this implies $S(\varphi) \geq d_{\mathcal{M}}$. Thus, $m \geq d_{\mathcal{M}}$.

Conversely, let $v \in \mathcal{M}$. If $K(v) = 0$ then $S(v) \geq m$, using Proposition II.14. Otherwise, $K(v) < 0$. Using the scaling $v_\lambda(x) = \lambda^{\frac{1}{2}} v(\lambda x)$, we have

$$K(v_\lambda) = \lambda^2 \|v_x\|_{L^2(\mathbb{R}^+)}^2 - \frac{3\lambda^2}{16} \|v\|_{L^6(\mathbb{R}^+)}^6 + \omega \|v\|_{L^2(\mathbb{R}^+)}^2 + \frac{\alpha\lambda}{2} |v(0)|^2 \rightarrow \omega \|v\|_{L^2(\mathbb{R}^+)}^2 > 0,$$

as $\lambda \rightarrow 0$. Hence, $K(v_\lambda) > 0$ as $\lambda > 0$ is small enough. Thus, there exists $\lambda_1 \in (0, 1)$ such that $K(v_{\lambda_1}) = 0$. Using Proposition II.14, $S(v_{\lambda_1}) \geq m$. We consider two cases. First, if $v(0) = 0$ then using Lemma II.13, we have $S(v) = S(v_{\lambda_1}) \geq m$. Second, if $v(0) \neq 0$ then using Lemma II.12, we have $S(v) \geq S(v_{\lambda_1}) \geq m$. In any case, $S(v) \geq m$. This implies $d_{\mathcal{M}} \geq m$, and completes the proof. \square

Define

$$\mathcal{V} := \{v \in H^1(\mathbb{R}^+) \setminus \{0\} : K(v) < 0, P(v) < 0, S(v) < m\}.$$

We have the following important lemma.

Lemma II.16. *If $v_0 \in \mathcal{V}$ then the solution v of (1.1) associated with v_0 satisfies $v(t) \in \mathcal{V}$ for all t in the time interval of existence.*

Proof. Since $S(v_0) < 0$, by conservation of the energy and the mass we have

$$S(v(t)) = E(v(t)) + \omega M(v(t)) = E(v_0) + \omega M(v_0) = S(v_0) < m. \quad (2.42)$$

If there exists $t_0 > 0$ such that $K(v(t_0)) \geq 0$ then by continuity of K and v , there exists $t_1 \in (0, t_0]$ such that $K(v(t_1)) = 0$. This implies $S(v(t_1)) \geq m$, using Proposition II.14. This contradicts (2.42). Hence, $K(v(t)) < 0$ for all t in the time of existence of v . Now, we prove $P(v(t)) < 0$ for all t in the time of existence of v . Assume that there exists $t_2 > 0$ such that $P(v(t_2)) \geq 0$, then, there exists $t_3 \in (0, t_2]$ such that $P(v(t_3)) = 0$. Using the previous lemma, $S(v(t_3)) \geq m$, which contradicts (2.42). This completes the proof. \square

Using the above lemma, we have the following property of solutions of (1.1) when the initial data lies on \mathcal{V} .

Lemma II.17. *Let $v_0 \in \mathcal{V}$, v be the corresponding solution of (1.1) in (T_{min}, T_{max}) . There exists $\delta > 0$ independent of t such that $P(v(t)) < -\delta$, for all $t \in (T_{min}, T_{max})$.*

Proof. Let $t \in (T_{min}, T_{max})$, $u = v(t)$ and $u_\lambda(x) = \lambda^{\frac{1}{2}}u(\lambda x)$. Using Lemma II.12, there exists $\lambda_0 \in (0, 1)$ such that $P(u_{\lambda_0}) = 0$. If $K(u_{\lambda_0}) \leq 0$ then we keep λ_0 . Otherwise, $K(u_{\lambda_0}) > 0$, then, there exists $\tilde{\lambda}_0 \in (\lambda_0, 1)$ such that $K(u_{\tilde{\lambda}_0}) = 0$. We replace λ_0 by $\tilde{\lambda}_0$. In any case, we have

$$S(u_{\lambda_0}) \geq m. \quad (2.43)$$

By (v) of Proposition II.12 we have

$$S(u) - S(u_{\lambda_0}) \geq (1 - \lambda_0) \frac{\partial}{\partial \lambda} S(u_\lambda)|_{\lambda=1} = (1 - \lambda_0)P(u).$$

In addition $P(u) < 0$, we obtain

$$S(u) - S(u_{\lambda_0}) \geq (1 - \lambda_0)P(u) > P(u). \quad (2.44)$$

Combined (2.43) and (2.44), we obtain

$$S(v_0) - m = S(v(t)) - m = S(u) - m \geq S(u) - S(u_{\lambda_0}) > P(u) = P(v(t)).$$

Setting

$$-\delta := S(v_0) - m,$$

we obtain the desired result. \square

Using the previous lemma, if the initial data lies on \mathcal{V} and satisfies a weight condition then the associated solution blows up in finite time on $H^1(\mathbb{R}^+)$. More precisely, we have the following result.

Proposition II.18. *Let $\phi \in \mathcal{V}$ such that $|x|\phi \in L^2(\mathbb{R}^+)$. Then the corresponding solution v of (1.1) blows up in finite time on $H^1(\mathbb{R}^+)$.*

Proof. By Lemma II.17, there exists $\delta > 0$ such that $P(v(t)) < -\delta$ for $t \in (T_{min}, T_{max})$. Remember that

$$\frac{\partial}{\partial t} \|xv(t)\|_{L^2(\mathbb{R}^+)}^2 = J(t) - \int_{\mathbb{R}^+} x|v|^4 dx, \quad (2.45)$$

where $J(t)$ satisfies

$$\partial_t J(t) = 4 \left(2\|v_x\|_{L^2(\mathbb{R}^+)}^2 - \frac{1}{8}\|v\|_{L^6(\mathbb{R}^+)}^6 + \alpha|v(t,0)|^2 \right) = 8(P(v(t))) < -8\delta.$$

This implies that

$$J(t) = J(0) + 8 \int_0^t P(v(s)) ds < J(0) - 8\delta t.$$

Hence, from (2.45), we have

$$\begin{aligned} \|xv(t)\|_{L^2(\mathbb{R}^+)}^2 &= \|xv_0\|_{L^2(\mathbb{R}^+)}^2 + \int_0^t J(s) ds - \int_0^t \int_{\mathbb{R}^+} x|v|^4 dx ds \\ &\leq \|xv_0\|_{L^2(\mathbb{R}^+)}^2 + \int_0^t (J(0) - 8\delta s) ds \\ &\leq \|xv_0\|_{L^2(\mathbb{R}^+)}^2 + J(0)t - 4\delta t^2. \end{aligned}$$

Thus, for t sufficiently large, there is a contradiction with $\|xv(t)\|_{L^2(\mathbb{R}^+)} \geq 0$. Hence, $T_{max} < \infty$ and $T_{min} > -\infty$. By the blow up alternative, we have

$$\lim_{t \rightarrow T_{max}} \|v_x\|_{L^2(\mathbb{R}^+)} = \lim_{t \rightarrow T_{min}} \|v_x\|_{L^2(\mathbb{R}^+)} = \infty.$$

This completes the proof. \square

Proof of Theorem I.4 (2). Using Proposition II.18, we need to construct a sequence $(\varphi_n) \subset \mathcal{V}$ such that φ_n converges to φ_ω in $H^1(\mathbb{R}^+)$. Define

$$\varphi_\lambda(x) = \lambda^{\frac{1}{2}} \varphi_\omega(\lambda x).$$

We have

$$S(\varphi_\omega) = m, \quad P(\varphi_\omega) = K(\varphi_\omega) = 0, \quad \varphi_\omega(0) \neq 0.$$

By (iv) of Proposition II.12,

$$S(\varphi_\lambda) < m \text{ for all } \lambda > 0.$$

In the addition,

$$P(\varphi_\lambda) < 0 \text{ for all } \lambda > 1.$$

Moreover,

$$\begin{aligned} \frac{\partial}{\partial \lambda} K(\varphi_\lambda) &= 2\lambda \left(\|\varphi'_\omega\|_{L^2(\mathbb{R}^+)}^2 - \frac{3}{16} \|\varphi_\omega\|_{L^6(\mathbb{R}^+)}^6 \right) + \alpha |\varphi_\omega(0)|^2 \\ &= 2\lambda (K(\varphi_\omega) - \omega \|\varphi_\omega\|_{L^2(\mathbb{R}^+)}^2 - \alpha |\varphi_\omega(0)|^2) + \alpha |\varphi_\omega(0)|^2 \\ &= -2\omega \lambda \|\varphi_\omega\|_{L^2(\mathbb{R}^+)}^2 - \alpha (2\lambda - 1) |\varphi_\omega(0)|^2 \\ &< 0, \end{aligned}$$

when $\lambda > 1$. Thus, $K(\varphi_\lambda) < K(\varphi_\omega) = 0$ when $\lambda > 1$. This implies $\varphi_\lambda \in \mathcal{V}$ when $\lambda > 1$. Let $\lambda_n > 1$ such that $\lambda_n \rightarrow 1$ as $n \rightarrow \infty$. Define, for $n \in \mathbb{N}^*$

$$\varphi_n = \varphi_{\lambda_n},$$

then, the sequence (φ_n) satisfies the desired property. This completes the proof of Theorem I.4. □

III. SOME TECHNICAL LEMMAS

Let φ_ω be defined as in (2.18). Recall that

$$\begin{aligned} S_\omega(v) &= \frac{1}{2} \|v_x\|_{L^2(\mathbb{R}^+)}^2 + \frac{\omega}{2} \|v\|_{L^2(\mathbb{R}^+)}^2 + \frac{\alpha}{2} |v(0)|^2 - \frac{1}{32} \|v\|_{L^6(\mathbb{R}^+)}^6, \\ K_\omega(v) &= \|v_x\|_{L^2(\mathbb{R}^+)}^2 + \omega \|v\|_{L^2(\mathbb{R}^+)}^2 + \alpha |v(0)|^2 - \frac{3}{16} \|v\|_{L^6(\mathbb{R}^+)}^6, \\ P(v) &= \|v_x\|_{L^2(\mathbb{R}^+)}^2 + \frac{\alpha}{2} |v(0)|^2 - \frac{1}{16} \|v\|_{L^6(\mathbb{R}^+)}^6, \end{aligned}$$

for all $v \in H^1(\mathbb{R}^+)$. We have the following result.

Lemma III.1. *The minimizers of two following variational problems are same:*

$$\begin{aligned} d(\omega) &= \inf\{S_\omega(v) \mid K_\omega(v) = 0, v \in H^1(\mathbb{R}^+) \setminus \{0\}\}, \\ \mu(\omega) &= \inf\{S_\omega(v) \mid S'_\omega(v) = 0, v \in H^1(\mathbb{R}^+) \setminus \{0\}\}. \end{aligned}$$

Moreover, $d(\omega) = \mu(\omega)$ and each minimizer is of form $e^{i\theta}\varphi_\omega$, where $\theta \in \mathbb{R}$ and φ_ω is defined as in (2.18)

Proof. This is a classical result (see e.g Ref. 25, Lemma 10). □

Lemma III.2. *Let $\alpha > 0$ and $v \in H^1(\mathbb{R}^+)$ satisfies $v \neq 0$. Assume that $P(v) \leq 0$. Then the following holds*

$$\frac{1}{2}P(v) \leq S_\omega(v) - \mu(\omega).$$

Proof. We use similar arguments as in Ref.29, proof of Lemma 4.3. Recall that

$$v_\lambda(x) = \lambda^{\frac{1}{2}}v(\lambda x).$$

Define

$$f(\lambda) = S_\omega(v_\lambda) - \frac{\lambda^2}{2}P(v) = \frac{\omega}{2}\|v\|_{L^2(\mathbb{R}^+)}^2 + \frac{\alpha}{2}|v(0)|^2 \left(\lambda - \frac{\lambda^2}{2} \right).$$

We have

$$f'(\lambda) = \frac{\alpha}{2}|v(0)|^2(1 - \lambda).$$

Hence, in case $v(0) \neq 0$ we have

$$f(1) = \max_{\lambda > 0} f(\lambda).$$

Moreover, in case $v(0) = 0$ we have $f(\lambda) \equiv \frac{\omega}{2}\|v\|_{L^2(\mathbb{R}^+)}^2$. Thus, in all case we have

$$f(1) = \max_{\lambda > 0} f(\lambda).$$

We have

$$K_\omega(v_\lambda) = \lambda^2 \left(\|v_x\|_{L^2(\mathbb{R}^+)}^2 - \frac{3}{16}\|v\|_{L^6(\mathbb{R}^+)}^6 \right) + \alpha\lambda|v(0)|^2 + \omega\|v\|_{L^2(\mathbb{R}^+)}^2.$$

Thus, for $\lambda > 0$ small enough we have $K_\omega(v_\lambda) > 0$. For $\lambda > 0$ large enough, using $P(v) \leq 0$ and $\alpha > 0$ (hence $\|v_x\|_{L^2(\mathbb{R}^+)}^2 \leq \frac{1}{16}\|v\|_{L^6(\mathbb{R}^+)}^6$) we have $K_\omega(v_\lambda) < 0$. It follows that there exists a $\lambda_0 > 0$ such that $K_\omega(v_{\lambda_0}) = 0$. By the definition of $\mu(\omega)$, $P(v) \leq 0$ and $f(1) \geq f(\lambda_0)$, we have

$$\mu(\omega) \leq S_\omega(v_{\lambda_0}) \leq S_\omega(v_{\lambda_0}) - \frac{\lambda_0^2}{2}P(v) \leq S_\omega(v) - \frac{1}{2}P(v).$$

This completes the proof. □

Next, using the result of the previous lemmas we have the following.

Lemma III.3. *The following set is invariant under flow of (1.1)*

$$\mathcal{H} := \{S_\omega(v) < d(\omega), P(v) < 0, v \in H^1(\mathbb{R}^+)\}.$$

Proof. Let $v_0 \in \mathcal{H}$ and $v \in C(I, H^1(\mathbb{R}^+))$ be the associated solution of (1.1). By the conservation law of (1.1), we have $S_\omega(v(t)) = S_\omega(v_0) < d(\omega)$. It remains to prove that $P(v(t)) < 0$ on I . By continuity of function $t \rightarrow P(v(t))$, we only need to prove that $P(v(t)) \neq 0$ for all $t \in I$. On the contrary, suppose that $P(v(t_0)) = 0$ for some $t_0 \in I$. Using Lemma III.2, we have

$$d(\omega) = \mu(\omega) \leq S_\omega(v(t_0)) - \frac{1}{2}P(v(t_0)) = S_\omega(v(t_0)),$$

which contradicts the fact that $S_\omega(v(t)) < d(\omega)$ on I . This completes the proof. \square

The following lemma is a consequence of the above lemma.

Lemma III.4. *Let $v_0 \in \mathcal{H}$. Then the associated solution of (1.1) blows up in finite time.*

Proof. Let $v(t)$ be the associated solution of (1.1). Using Lemma III.3, we have $v(t) \in \mathcal{H}$ for all t . In the addition, using Lemma III.2, we have

$$\frac{1}{2}P(v(t)) \leq S_\omega(v(t)) - \mu(\omega) = S_\omega(v_0) - d(\omega) < 0.$$

By classical arguments, we have that $v(t)$ blows up in finite time. \square

The above lemma gives another proof of instability of standing waves. The following result is important in the proof of stability of standing waves.

Proposition III.5. *Let $\alpha < 0$ and $\tilde{\varphi}_\omega$ be defined as in (2.22). Then, we have*

$$\partial_\omega \|\tilde{\varphi}_\omega\|_{L^2(\mathbb{R})}^2 > 0.$$

Proof. Recall the formula of $\tilde{\varphi}_\omega$ as follows

$$\tilde{\varphi}_\omega = 2\sqrt[4]{\omega} \operatorname{sech}^{\frac{1}{2}} \left(2\sqrt{\omega}|x| + \tanh^{-1} \left(\frac{-\alpha}{\sqrt{\omega}} \right) \right).$$

Thus,

$$\begin{aligned}
\|\tilde{\Phi}_\omega\|_{L^2(\mathbb{R})}^2 &= \int_{\mathbb{R}} \tilde{\Phi}_\omega^2 dx \\
&= \int_{\mathbb{R}} 4\sqrt{\omega} \operatorname{sech} \left(2\sqrt{\omega}|x| + \tanh^{-1} \left(\frac{-\alpha}{\sqrt{\omega}} \right) \right) dx \\
&= 8\sqrt{\omega} \int_{\mathbb{R}^+} \operatorname{sech} \left(2\sqrt{\omega}x + \tanh^{-1} \left(\frac{-\alpha}{\sqrt{\omega}} \right) \right) dx \\
&= 4 \int_{\tanh^{-1} \left(\frac{-\alpha}{\sqrt{\omega}} \right)}^{\infty} \operatorname{sech}(y) dy \quad \text{for } y = 2\sqrt{\omega}x + \tanh^{-1} \left(\frac{-\alpha}{\sqrt{\omega}} \right) \\
&:= h(\omega),
\end{aligned}$$

Define the following functions

$$\begin{aligned}
h(z) &= 4 \int_z^{\infty} \operatorname{sech}(y) dy, \\
z(y) &= \tanh^{-1}(y), \\
y(\omega) &= \frac{-\alpha}{\sqrt{\omega}} = -\alpha \omega^{-\frac{1}{2}}.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
h'(\omega) &= h'(z)z'(y)y'(\omega) \\
&= -4 \operatorname{sech}(z) \frac{1}{\tanh'(\tanh^{-1}(y))} (-\alpha) \frac{-1}{2} \omega^{-\frac{3}{2}} \\
&= -2\alpha \operatorname{sech}(z) \frac{1}{\tanh'(\tanh^{-1}(y))} \omega^{-\frac{3}{2}}.
\end{aligned}$$

Since, $\alpha < 0$ and

$$\tanh'(x) = \frac{4e^{2x}}{(e^{2x} + 1)^2} > 0, \text{ for all } x,$$

we have

$$h'(\omega) > 0.$$

which completes the proof. □

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DATA AVAILABILITY STATEMENT

The data that supports the findings of this study are available within the article.

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